COMPUTING EIGENPAIRS

AP

EIGENPAIRS

STUDY MATERIALS

I. Goodfellow, Y. Bengio and A. Courville, ch. 2:

Deep Learning, MIT Press, 2016.

J. Lescovec, A. Rajaraman, J. Ullmann, ch. 11:

Mining of Massive datasets, MIT Press, 2016.

SPECTRAL ANALYSIS

EIGENPAIRS

If, given a matrix A we find a scalar λ and a vector $\vec{e}$ s.t.

$$A\vec{e} = \lambda\vec{e}$$

then λ and $\vec{e}$ will be an eigenpair of A.

If $rank(A) = n$ then there could be up to n such pairs.

In practice, eigenpairs

- are always *costly* to find.
- they might have $\lambda = 0$: no information, or
- λ might not be a real number: no interpretation.

CONDITIONS FOR *GOOD* EIGEN-

A square matrix A is called *positive semidefinite* when for any $\vec{x}$ we have

$$\vec{x}^T A\vec{x} \geq 0$$

In such case its eigenvalues are non-negative: $\lambda_i \geq 0$.

UNDERLYING IDEA, I

In Geometry, applying a matrix to a vector, $A\vec{x}$, creates all sorts of alteration to the space, e.g,

- rotation
- deformation

Eigenvectors, i.e., solutions to $A\vec{e} = \lambda\vec{e}$

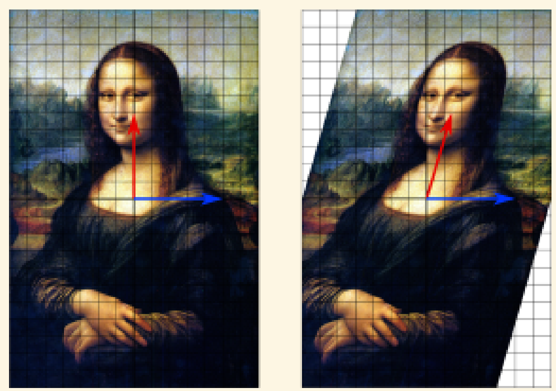
describe the direction along which matrix A operates an **expansion**

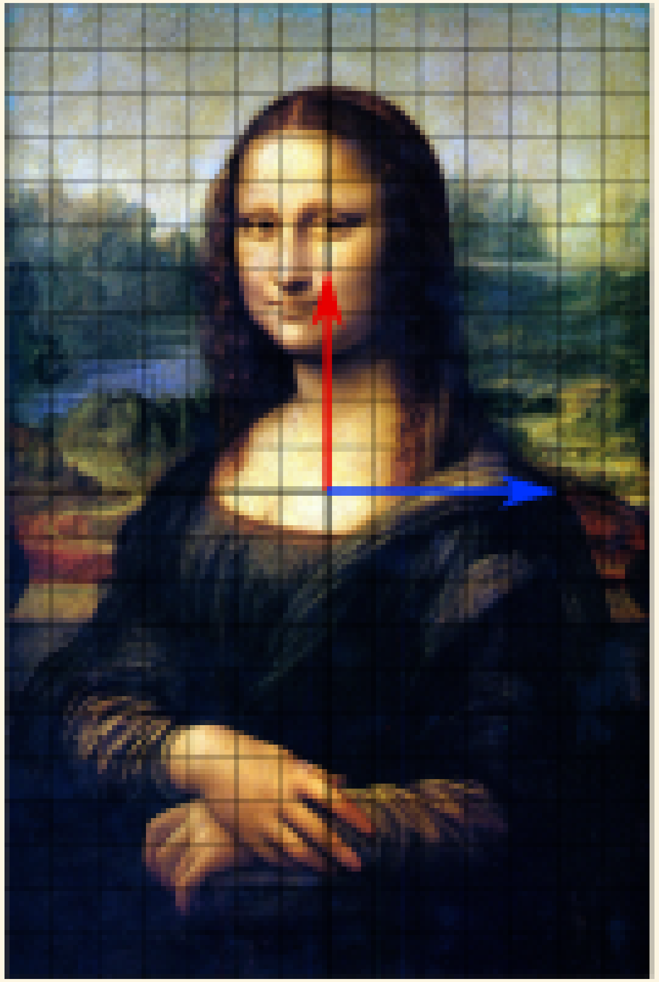
EXAMPLE: SHEAR MAPPING

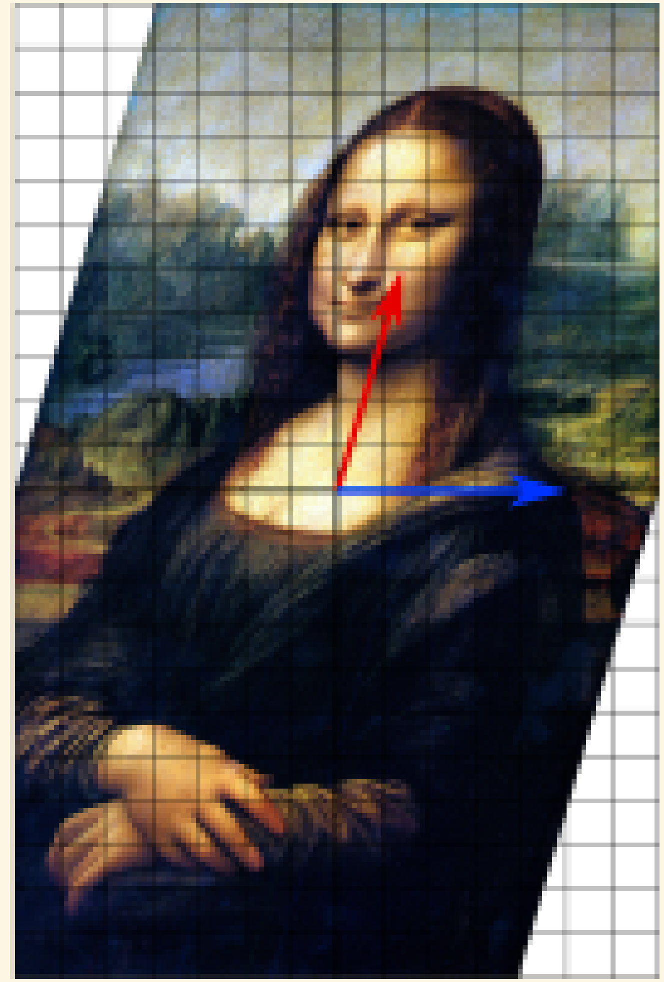
```
A = [[1, .27],
     [0,   1]
    ]
```

deforms a vector by increading the first dimension by a quantity proportional to the value of the second dimension:

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x + \frac{3}{11} y \\ y \end{bmatrix}$$







The blue line is unchanged:

- an $[x, 0]^T$ eigenvector
- corresponding to $\lambda = 1$

ACTIVITY MATRICES, I

Under certains conditions:

-the eigenpairs exists,

-e-values are real, non-negative numbers (0 is ok), and

-e-vectors are orthogonal with each other:

User-activity matrices normally meet those conditions!

ACTIVITY MATRICES, II

If an activity matrix has *good* eigenpairs,

each e-vector represents a *direction*

we interpret those directions as *topics* that hidden (latent) within the data.

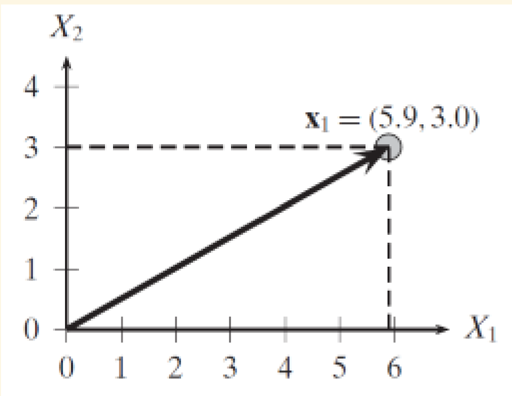
e-values *expand* one's affiliation to a specific *topic*.

NORMS AND DISTANCES

EUCLIDEAN NORM

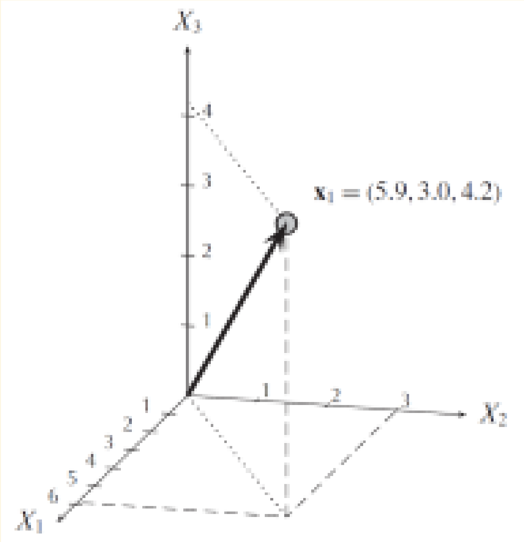
Pythagora's theorem, essentially.

$$||\vec{x}|| = \sqrt{\vec{x}^T \vec{x}} = \sqrt{\sum_{i=1}^{m} x_i^2}$$



Generalisation:

$$||\vec{x}||_p = \left(|x_1|^p + |x_1|^p + \ldots |x_m|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{m} |x_i|^p\right)^{\frac{1}{p}}$$



The Frobenius norm $||\cdot||_F$ extends $||\cdot||_2$ to matrices:

$$||\vec{A}||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

Also used in practice:

$||\vec{x}||_0$ = # of non-zero scalar values in $\vec{x}$

$$||\vec{x}||_\infty = max\{|x_i|\}$$

NORMALIZATION

The *unit* or *normalized* vector of $\vec{x}$

$$\vec{u} = \frac{\vec{x}}{||\vec{x}||} = (\frac{1}{||\vec{x}||})\vec{x}$$

- has the same direction of the original
- its norm is constructed to be 1.

COMPUTING EIGENPAIRS

WITH MATHS

$$M\vec{e} = \lambda\vec{e}$$

Handbook solution: solve the equivalent system

$$(M - \lambda\vec{I})\vec{e} = \vec{0}$$

Either of the two factors should be 0. Hence, a non-zero vector **e** is associated to a solution of

$$|M - \lambda\vec{I}| = 0$$

$$|M - \lambda \vec{I}| = 0$$

In Numerical Analysis many methods are available.

Their general algorithmic structure:

-find the λs that make $|\ldots| = 0$, then

-for each λ find its associated vector **e**.

WITH COMPUTER SCIENCE

At the scale of the Web, few methods will still work!

Ideas:

1. find the e-vectors first, with an iterated method.
2. interleave iteration with control on the *expansion in value*

$$\overrightarrow{x_0} = [1, 1, \dots 1]^T$$

$$\overrightarrow{x_{k+1}} = \frac{M\vec{x}_k}{||M\vec{x}_k||}$$

until an approximate fix point: $x_{l+1} \approx x_l$.

Now, eliminate the contribution of the first eigenpair:

$$M^* = M - \lambda_1' \vec{x}_1 \vec{x}_1^T$$

(since $\vec{x}_1$ is a column vector, $\vec{x}_1^T \vec{x}_1$ will be a scalar: its norm. Vice versa, $\vec{x}_1 \vec{x}_1^T$ will be a matrix)

Now, we repeat the iteration on M^* to find the second eigenpair.

Times are in $\Theta(dn^2)$.

For better scalability, we will cover Pagerank later.

EIGENPAIRS IN PYTHON

E-PAIRS WITH NUMPY

```python
import numpy as np

# this is the specific submodule
from numpy import linalg as la
```

```python
# create a 'blank' matrix
m = np.zeros([7, 5])

m = [[1, 1, 1, 0, 0],
     [3, 3, 3, 0, 0],
     [4, 4, 4, 0, 0],
     [5, 5, 5, 0, 0],
     [0, 0, 0, 4, 4],
     [0, 0, 0, 5, 5],
     [0, 0, 0, 2, 2]
    ]
```

```python
def find_eigenpairs(mat):
    """Test the quality of Numpy eigenpairs"""
    n = len(mat)

    # is it squared?
    m = len(mat[0])

    if n==m:
      eig_vals, eig_vects = la.eig(mat)
    else:
      # force to be squared
      eig_vals, eig_vects = la.eig(mat@mat.T)

    # they come in ascending order, take the last one on the right
    dominant_eig = abs(eig_vals[-1])
    return dominant_eig
```

OLDER VERSIONS:

E-values come normalized: $\sqrt{\lambda_1^2 + \ldots \lambda_n^2} = 1$; hence we later multiply them by $\frac{1}{\sqrt{n}}$

```
# lambda_1 = find_eigenpairs(m)

# lambda_1
```