RANKING AND RATING: MARKOV CHAINS

AP

Ch. 6 of Langville-Meyer’s textbook is dedicated to Markov chains in sport prediction

We learn a new key concept of Data Science

NOTATION

A STOCHASTIC MATRIX S

describes the probab. of a *transition* of some sort between places or states etc.

$$s_{ij} = Pr[\text{the system goes from i to j}]$$

As a result:

$$\mathbf{S} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

NOTATION OF THE CHAPTER

Notation for the Markov Rating Method

k — number of statistics to incorporate into the Markov model

$\mathbf{V}_{stat1}, \mathbf{V}_{stat2}, \ldots, \mathbf{V}_{statk}$ — raw voting matrix for each game statistic k
$[\mathbf{V}_{stat}]_{ij}$ = number of votes team i casts for team j using statistic $stat$

$\mathbf{S}_{stat1}, \mathbf{S}_{stat2}, \ldots, \mathbf{S}_{statk}$ — stochastic matrices built from corresponding voting matrices $\mathbf{V}_{stat1}, \mathbf{V}_{stat2}, \ldots, \mathbf{V}_{statk}$

$\mathbf{S}$ — final stochastic matrix built from $\mathbf{S}_{stat1}, \mathbf{S}_{stat2}, \ldots, \mathbf{S}_{statk}$;
$\mathbf{S} = \alpha_1 \mathbf{S}_{stat1} + \alpha_2 \mathbf{S}_{stat2} + \cdots + \alpha_k \mathbf{S}_{statk}$

α_i — weight associated with game statistic i; $\sum_{i=1}^{k} \alpha_i = 1$ and $\alpha_i \geq 0$.

$\bar{\mathbf{S}}$ — stochastic Markov matrix that is guaranteed to be irreducible;
$\bar{\mathbf{S}} = \beta \mathbf{S} + (1-\beta)/n\, \mathbf{E}, \quad 0 < \beta < 1$

$\mathbf{r}$ — Markov rating vector; stationary vector (i.e., dominant eigenvector) of $\bar{\mathbf{S}}$

n — number of teams in the league = order of $\bar{\mathbf{S}}$

THE MARKOV METHOD

THE FAIRWHEATHER FAN

switches their allegiance to the winning team **of the moment.**

If they support i, what is the prob. that they switch to j?

$$\mathbf{S} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

How did we obtain this matrix?

Input: the win-loss data:

$$
\mathbf{V} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Rows normalised to 1:

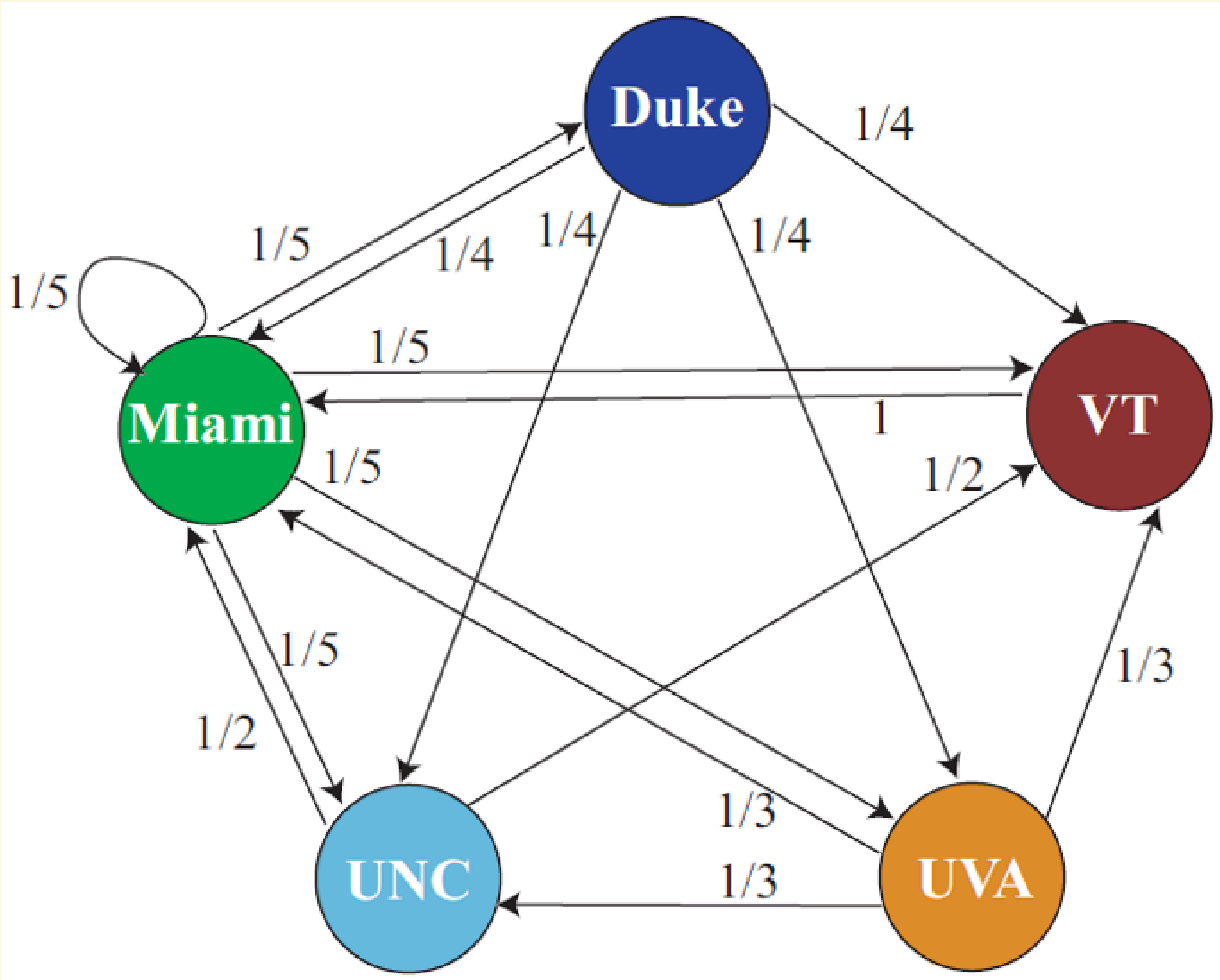
$$\mathbf{N} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The Miami row sums to 0: not stochastic!

As with PageRank, substitue all $\mathbf{0}^T$ rows with $\frac{1}{n}\mathbf{1}^T$

$$\mathbf{S} = \begin{array}{c|ccccc} & \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \\ \text{Duke} & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ \text{Miami} & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ \text{UNC} & 0 & 1/2 & 0 & 0 & 1/2 \\ \text{UVA} & 0 & 1/3 & 1/3 & 0 & 1/3 \\ \text{VT} & 0 & 1 & 0 & 0 & 0 \end{array}.$$

Now the fair-wheather fan takes a long, random walk along this *Markov graph:*



We record the number of times the random walker passess each vertex.

After a while, the proportion of visits to each node stabiles.

The vector $\mathbf{r}$ with the frequencies is a *stationary vector*

$\mathbf{r}$ corresponds to the dominant e-vector of the Markov-chain matrix!

Team	$\mathbf{r}$	Rank
Duke	.087	5th
Miami	.438	1st
UNC	.146	3rd
UVA	.110	4th
VT	.219	2nd

HOW TO CREATE THE BASE MATRIX

WITH POINTS DIFFERENTIAL

$$\mathbf{V} = \begin{array}{c|ccccc} & \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \\ \hline \text{Duke} & 0 & 45 & 3 & 31 & 45 \\ \text{Miami} & 0 & 0 & 0 & 0 & 0 \\ \text{UNC} & 0 & 18 & 0 & 0 & 27 \\ \text{UVA} & 0 & 8 & 2 & 0 & 38 \\ \text{VT} & 0 & 20 & 0 & 0 & 0 \end{array}$$

$$
\mathbf{S} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array}
\begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\
\left(\begin{array}{ccccc}
0 & 45/124 & 3/124 & 31/124 & 45/124 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 18/45 & 0 & 0 & 27/45 \\
0 & 8/48 & 2/48 & 0 & 38/48 \\
0 & 1 & 0 & 0 & 0
\end{array} \right) \end{array}
$$

Team	**r**	Rank
Duke	.088	5th
Miami	.442	1st
UNC	.095	4th
UVA	.110	3rd
VT	.265	2nd

WINNERS/LOSERS WITH POINTS

$$\mathbf{S}_{point} = \begin{array}{c|c} & \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{array}{c} \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} & \begin{pmatrix} 0 & 52/159 & 24/159 & 38/159 & 45/159 \\ 7/47 & 0 & 16/47 & 17/47 & 7/47 \\ 21/90 & 34/90 & 0 & 5/90 & 30/90 \\ 7/91 & 25/91 & 7/91 & 0 & 52/91 \\ 0 & 27/44 & 3/44 & 14/44 & 0 \end{pmatrix} \end{array}$$

$$
\mathbf{S}_{point} = \begin{array}{c c} & \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{array}{l} \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} & \begin{pmatrix} 0 & 52/159 & 24/159 & 38/159 & 45/159 \\ 7/47 & 0 & 16/47 & 17/47 & 7/47 \\ 21/90 & 34/90 & 0 & 5/90 & 30/90 \\ 7/91 & 25/91 & 7/91 & 0 & 52/91 \\ 0 & 27/44 & 3/44 & 14/44 & 0 \end{pmatrix} \end{array}
$$

Team	**r**	Rank
Duke	.095	5th
Miami	.296	1st
UNC	.149	4th
UVA	.216	3rd
VT	.244	2nd

WITH YARDAGE

	Duke	Miami	UNC	UVA	VT
Duke		100-557	209-357	207-315	35-362
Miami	557-100		321-188	500-452	304-167
UNC	357-209	188-321		270-199	196-338
UVA	315-207	452-500	199-270		334-552
VT	362-35	167-304	338-196	552-334	

$$\mathbf{V}_{yardage} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 577 & 357 & 315 & 362 \\ 100 & 0 & 188 & 452 & 167 \\ 209 & 321 & 0 & 199 & 338 \\ 207 & 500 & 270 & 0 & 552 \\ 35 & 304 & 196 & 334 & 0 \end{pmatrix} \end{array}$$

$$
\mathbf{S}_{yardage} = \begin{array}{c c} & \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{array}{l} \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} & \begin{pmatrix} 0 & 577/1611 & 357/1611 & 315/1611 & 362/1611 \\ 100/907 & 0 & 188/907 & 452/907 & 167/907 \\ 209/1067 & 321/1067 & 0 & 199/1067 & 338/1067 \\ 207/1529 & 500/1529 & 270/1529 & 0 & 552/1529 \\ 35/869 & 304/869 & 196/869 & 334/869 & 0 \end{pmatrix} \end{array}
$$

Team	**r**	Rank
Duke	.105	5th
Miami	.249	2nd
UNC	.170	4th
UVA	.260	1st
VT	.216	3rd

WITH TURNOVER

$$\mathbf{S}_{turnover} = \begin{array}{c c} & \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{array}{c} \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} & \begin{pmatrix} 0 & 1/9 & 3/9 & 4/9 & 1/9 \\ 3/9 & 0 & 4/9 & 1/9 & 1/9 \\ 2/7 & 3/7 & 0 & 1/7 & 1/7 \\ 1/5 & 0 & 1/5 & 0 & 3/5 \\ 1/10 & 6/10 & 2/10 & 1/10 & 0 \end{pmatrix} \end{array}$$

WITH POSSESSION

$$\mathbf{S}_{poss} = \begin{array}{c c} & \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{array}{c} \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} & \begin{pmatrix} 0 & 29.7/118.6 & 30.8/118.6 & 28/118.6 & 30.1/118.6 \\ 30.3/123.6 & 0 & 36.3/123.6 & 31/123.6 & 26/123.6 \\ 29.1/111.6 & 23.7/111.6 & 0 & 27.5/111.6 & 31.3/111.6 \\ 32/131.9 & 29/131.9 & 32.5/131.9 & 0 & 38.4/131.9 \\ 29.9/114.2 & 34/114.2 & 28.7/114.2 & 21.6/114.2 & 0 \end{pmatrix} \end{array}$$

WITH LINEAR COMBINATIONS OF FEATURES

$$\mathbf{S} = \alpha_1 \mathbf{S_{points}} + \alpha_2 \mathbf{S_{yard.}} + \alpha_3 \mathbf{S_{turn.}} + \alpha_4 \mathbf{S_{poss.}}$$

If weights are all non-negative and sum to 1, also $\mathbf{S}$ will be stocastic.

Weights are assigned by experts or…

could be learned by an outer ML system running on historical data.

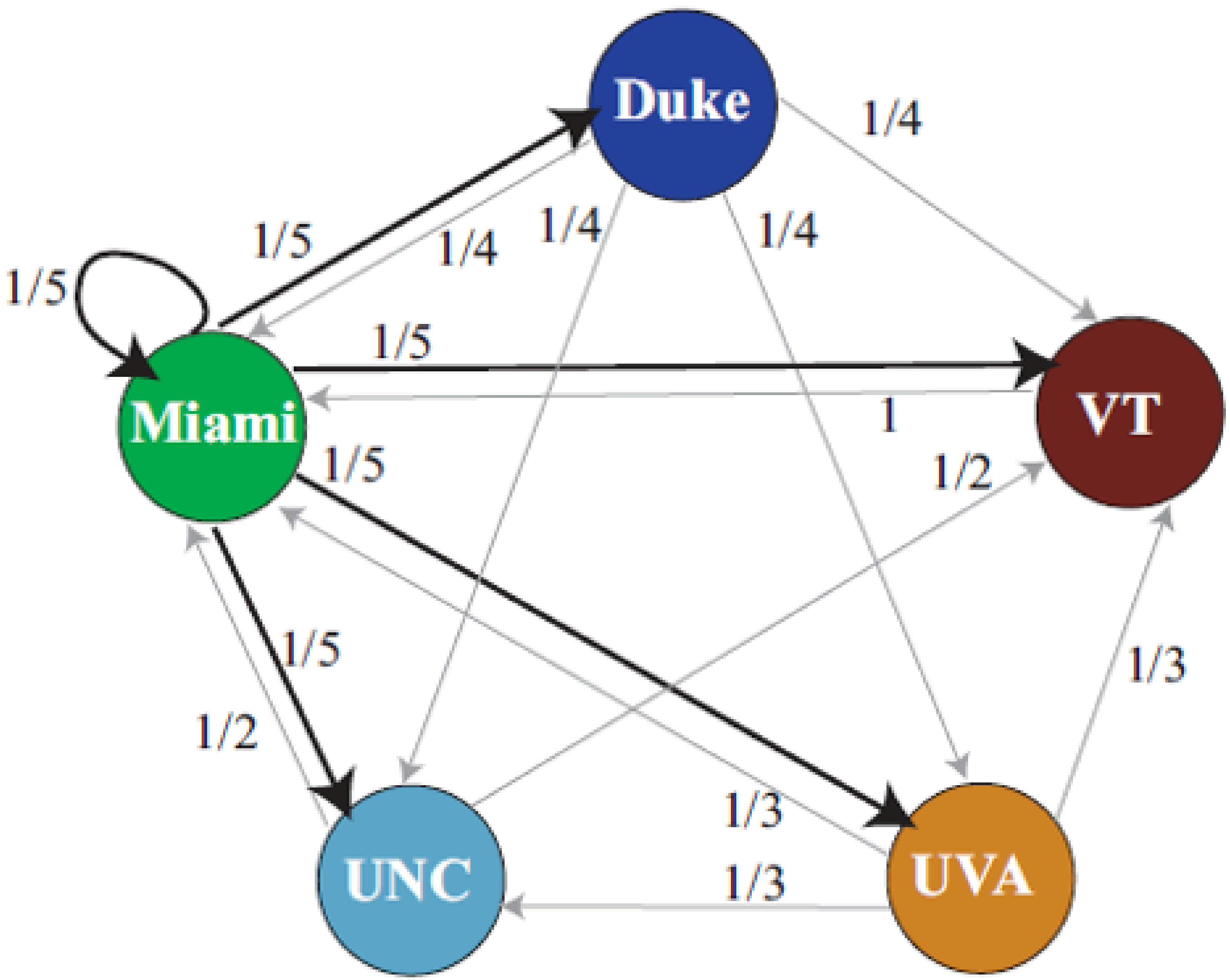
By default, let's set all 4 α weights to $\frac{1}{4}$:

$$\mathbf{S} = \begin{array}{c} \\ \text{Duke} \\ \text{Miami} \\ \text{UNC} \\ \text{UVA} \\ \text{VT} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Duke} & \text{Miami} & \text{UNC} & \text{UVA} & \text{VT} \end{array} \\ \begin{pmatrix} 0 & 0.2617 & 0.2414 & 0.2788 & 0.2182 \\ 0.2094 & 0 & 0.3215 & 0.3055 & 0.1636 \\ 0.2439 & 0.3299 & 0 & 0.1578 & 0.2684 \\ 0.1637 & 0.2054 & 0.1750 & 0 & 0.4559 \\ 0.1005 & 0.4653 & 0.1863 & 0.2479 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} .15 \\ .24 \\ .19 \\ .20 \\ .21 \end{pmatrix}$$

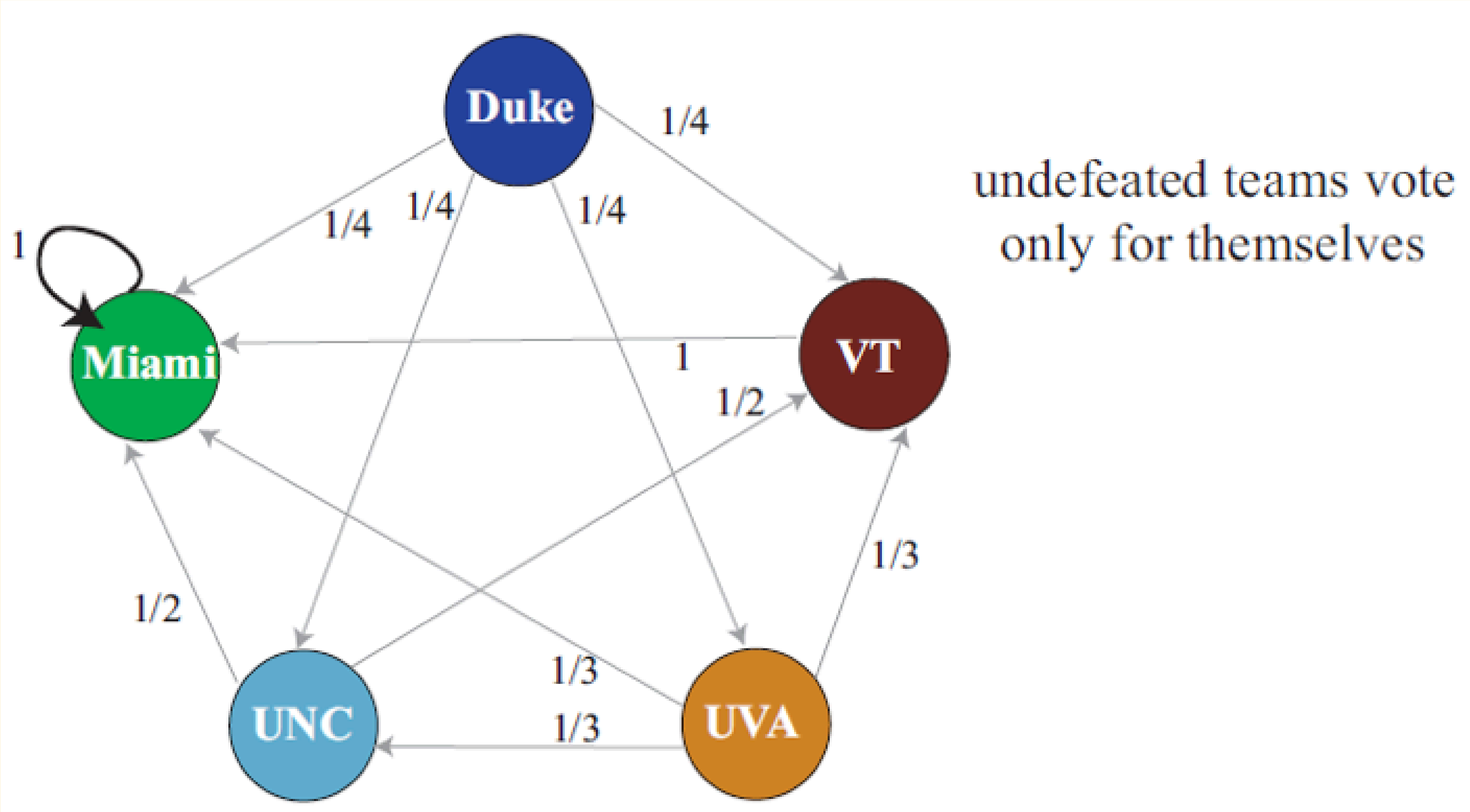
(rating compression starts manifesting)

ISSUES AT THE EXTREMES

HANDLING UNDEFEATED TEAMS



undefeated teams vote equally for all teams



A random walker soon get stuck with Miami!

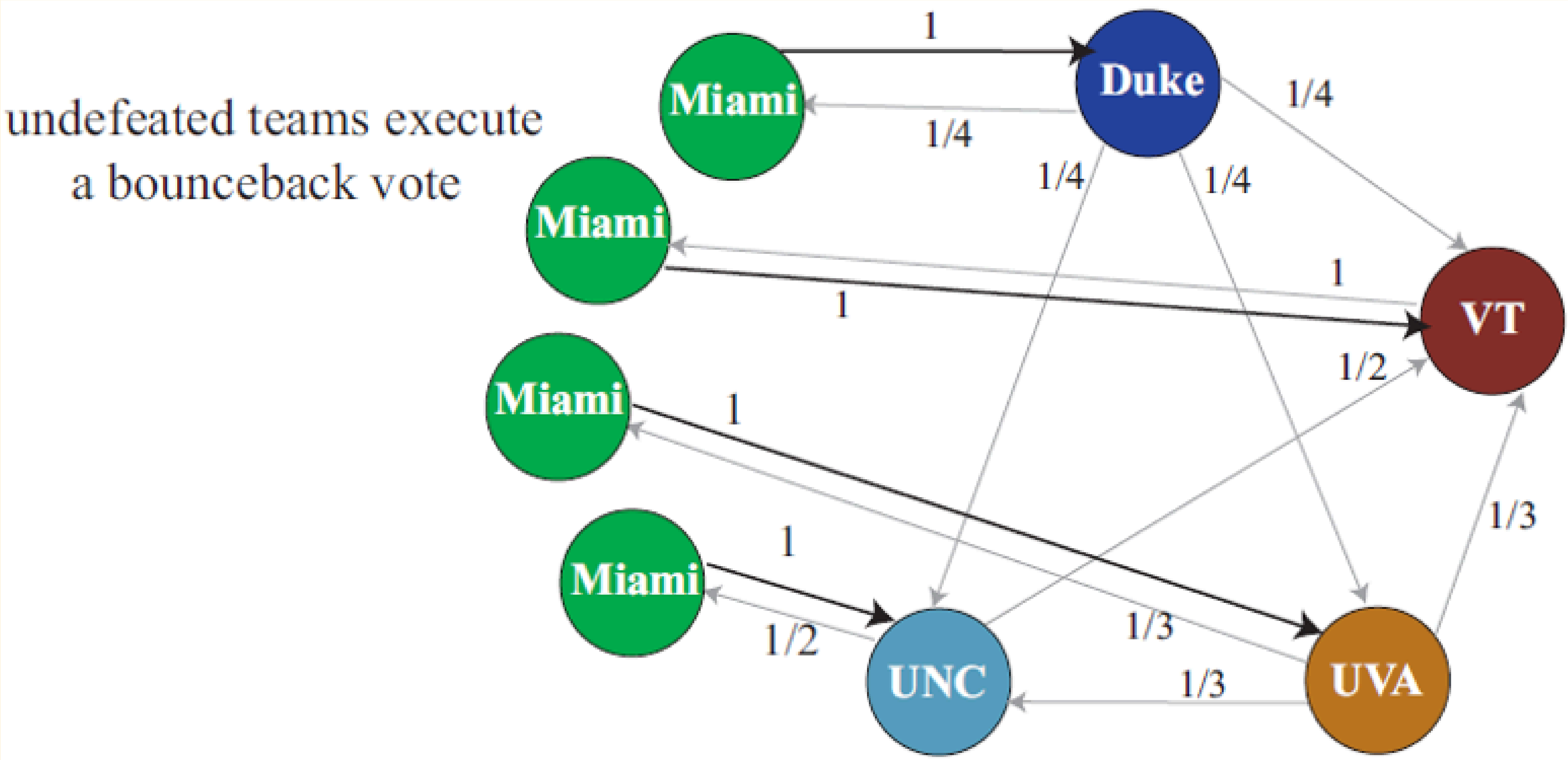
Assign a probability to escape:

$\overline{\mathbf{S}} = \beta \mathbf{S} + \frac{(1-\beta)}{n} \mathbf{1}$ (1 everywhere)

PageRank: $\beta = 0.85$

NFL: $\beta = 0.6$

NCAA: $\beta = 0.5$



A better example: modeling the ‘Back’ button of the browser when we visit a dead-end page.

SUMMARY OF THE METHOD

THE ALGORITHM

MARKOV METHOD FOR RATING TEAMS

1. Form $\mathbf{S}$ using voting matrices for the k game statistics of interest.

$$\mathbf{S} = \alpha_1 \mathbf{S}_{stat1} + \alpha_2 \mathbf{S}_{stat2} + \ldots + \alpha_k \mathbf{S}_{statk},$$

   where $\alpha_i \geq 0$ and $\sum_{i=1}^{k} \alpha_i = 1$.

2. Compute $\mathbf{r}$, the stationary vector or dominant eigenvector of $\mathbf{S}$. (If $\mathbf{S}$ is reducible, use the irreducible $\bar{\mathbf{S}} = \beta \mathbf{S} + (1-\beta)/n\, \mathbf{E}$ instead, where $0 < \beta < 1$.)

COMPARISON WITH MASSEY'S

The point-differential M. chain:

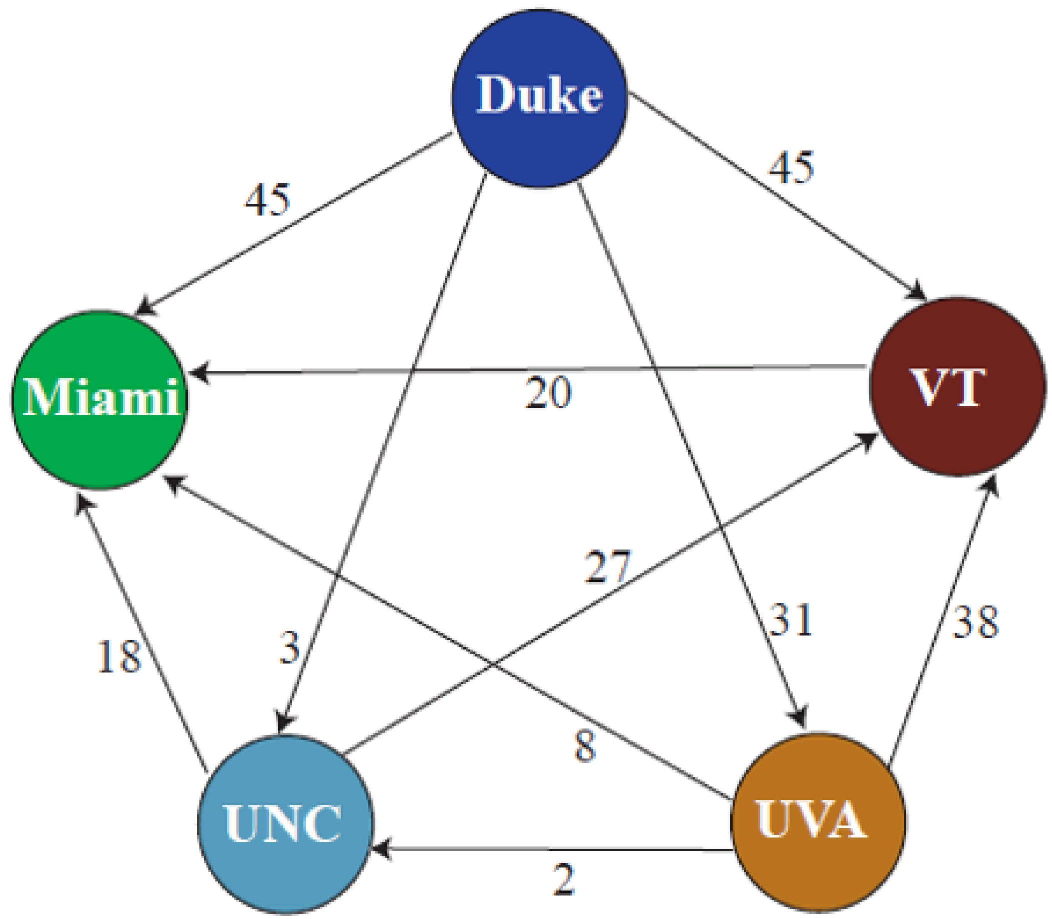


Figure 6.2 Markov graph of point differential voting matrix

Massey graph for the same season

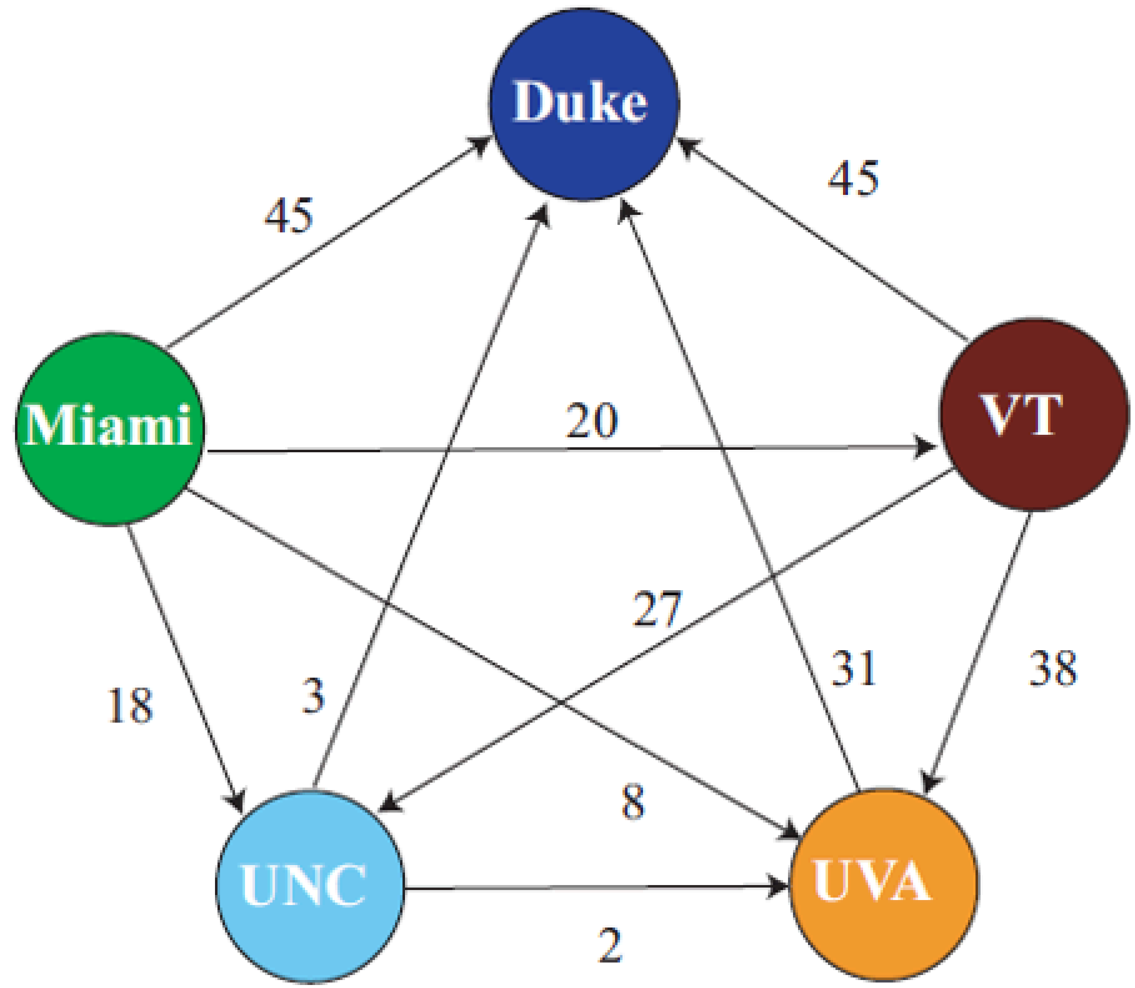


Figure 6.3 Massey graph for the same five team example as Figure 6.2

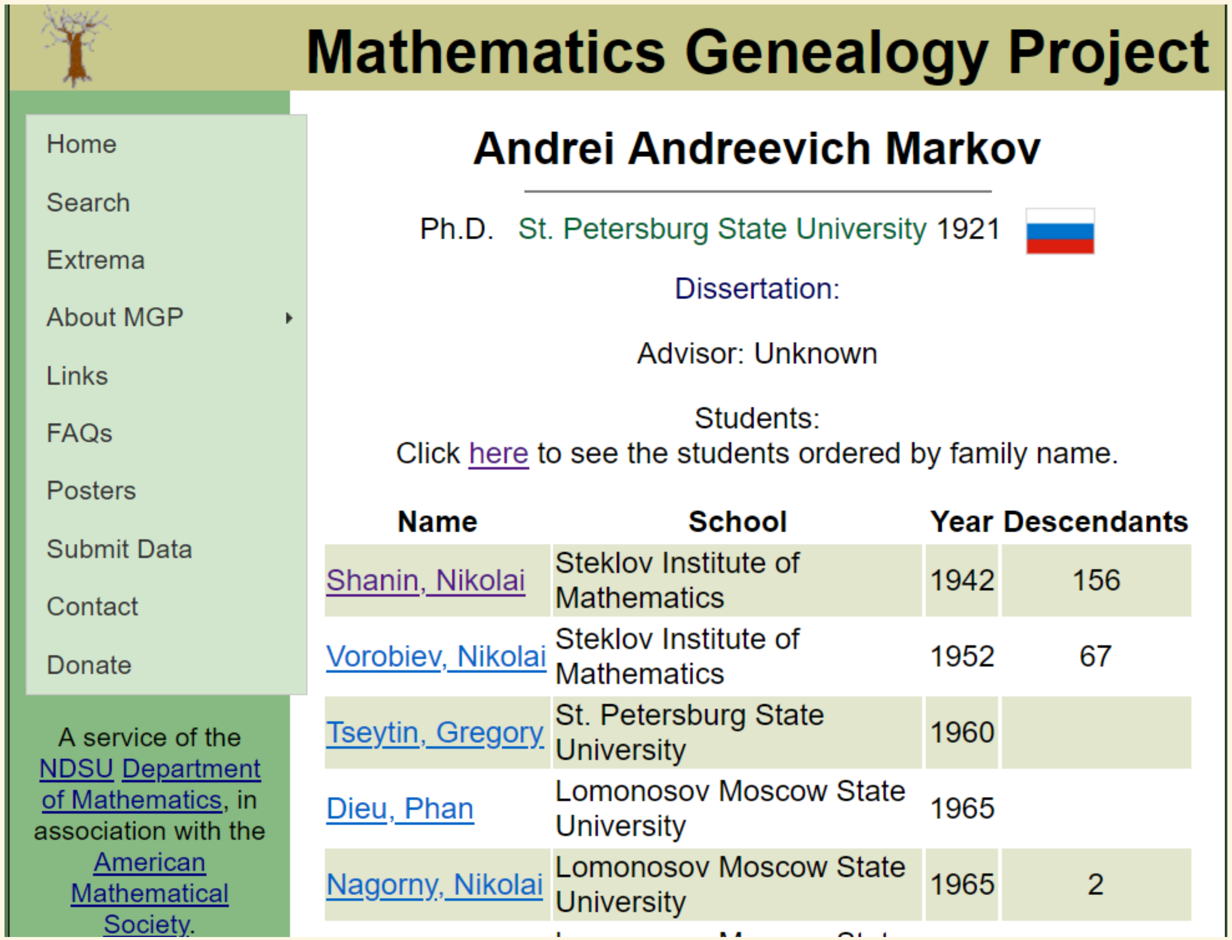
FURTHER APPLICATIONS

Let’s hire fairwheater fans to do random walks:

by accumulation and stabilisation of the frequencies we will find out the dominant e-vector of $\mathbf{S}$ *without engaging in matrix operations.*

TRIVIA: THE MATHS GENALOGY PROJECT:

Markov begot Shanin, Shanin begot Gelfond, and Gelfond begot me.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A service of the NDSU Department of Mathematics, in association with the American Mathematical Society.

Andrei Andreevich Markov

Ph.D. St. Petersburg State University 1921

Dissertation:

Advisor: Unknown

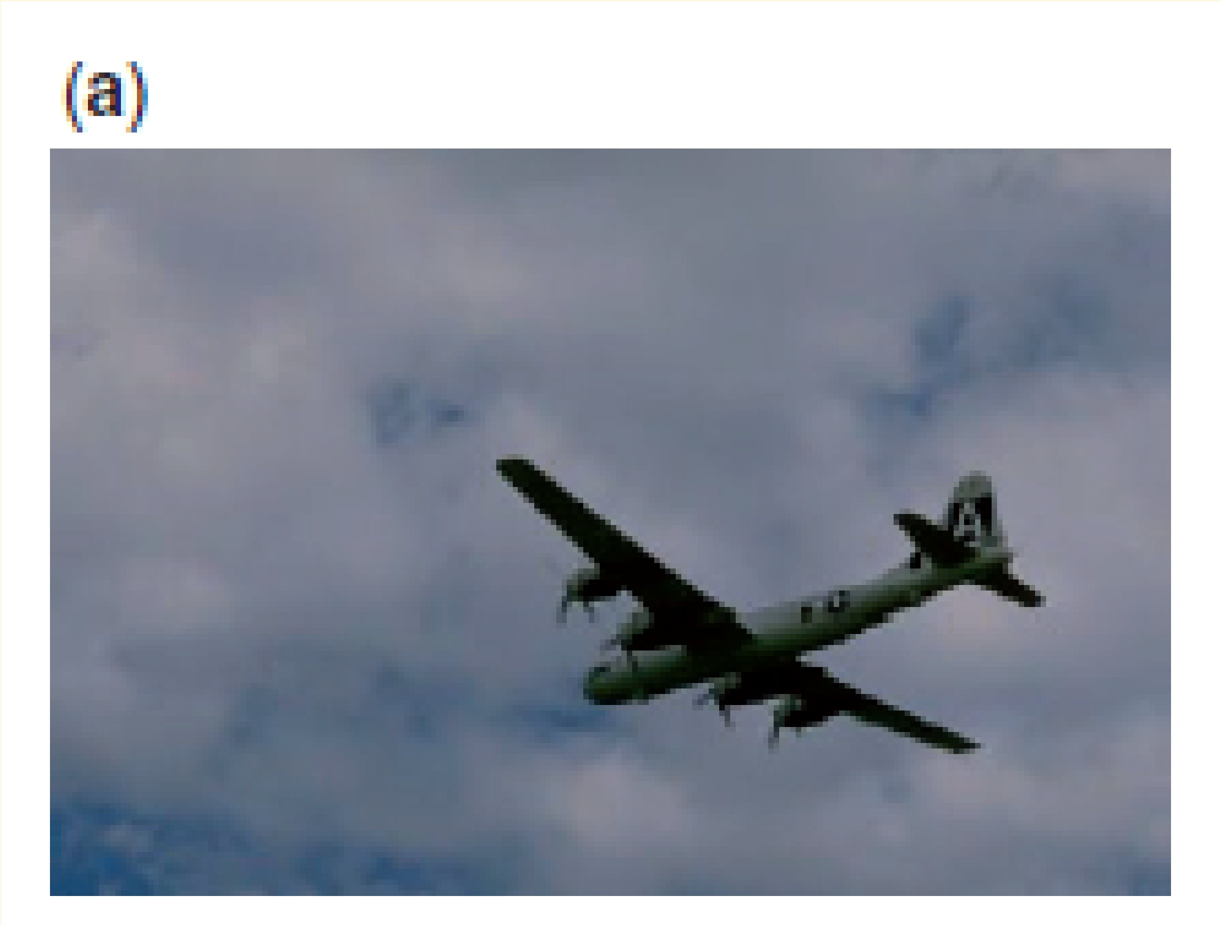
Students:
Click here to see the students ordered by family name.

Name	School	Year	Descendants
Shanin, Nikolai	Steklov Institute of Mathematics	1942	156
Vorobiev, Nikolai	Steklov Institute of Mathematics	1952	67
Tseytin, Gregory	St. Petersburg State University	1960	
Dieu, Phan	Lomonosov Moscow State University	1965	
Nagorny, Nikolai	Lomonosov Moscow State University	1965	2

I begot Han, Prifti and Matuozzo who ...

CODA: RANDOM WALKS FOR MACHINE VISION

IMAGE SEGMENTATION



Find objects inside a picture

Could random walker discover the perimeter of objects by walking *around* them?

THE DATA

A photos (bitmap) can be seen as

- a m x n matrix, each value, the pixel being an RGB encoding over [0..255]
- a m x n x 3 tensor where each layer, sometimes called *channel* containe [0..255] intensitites of the respective color

a network of pixel nodes joint in a mesh: each node is connected rectilinearly with 2 (corner), 3 (border) or 4 (inner) neighbour pixels.

MAPPING

RGB values can be normalised to [0..1] by mapping the three values into *intensities*, i.e, the lenght of the vector over N^3

- total black: $[0][0][0] \rightarrow 0$
- total white: $[255][255][255] \rightarrow 1$
- total red: $[255][0][0] \rightarrow ?$

The normalised norm:

$$|p_{ij}|_3 = \frac{1}{\sqrt{3}} \sqrt{\frac{p_{ij}^{red} + p_{ij}^{green} + p_{ij}^{blue}}{255}}$$

THE NORM IN ACTION

So, for a total-red pixel:

$$|p_{ij}|_3 = \frac{1}{\sqrt{3}}\sqrt{\frac{255+0+0}{255}} = \frac{1}{\sqrt{3}} \approx \frac{1}{1.732} \approx 0.57735.$$

For a total-brown pixel:

$$|p_{ij}|_3 = \frac{1}{\sqrt{3}}\sqrt{\frac{255+0+255}{255}} = \frac{1}{\sqrt{3}}\sqrt{2} \approx \frac{1.4142}{1.732} \approx 0.8165.$$

THE RANDOM WALK MODEL

Let random walkers to prefer to remain on the same likely surface/object, i.e., not cross-through density *slopes*

make the prob. to move to a neighbour pixel inverse-proportional to the difference in intensity between the origin and destination pixels.