Lecture 16: Shortest Paths II - Dijkstra

Lecture Overview

- Review
- Shortest paths in DAGs
- Shortest paths in graphs without negative edges
- Dijkstra's Algorithm

Readings

CLRS, Sections 24.2-24.3

Review

$d[v]$ is the length of the current shortest path from starting vertex s. Through a process of relaxation, $d[v]$ should eventually become $\delta(s, v)$, which is the length of the shortest pathfrom s to v. $\Pi[v]$ is the predecessor of v in the shortest path from s to v.

Basic operation in shortest path computation is the *relaxation operation*

RELAX(u, v, w)
  if $d[v] > d[u] + w(u, v)$
    then $d[v] \leftarrow d[u] + w(u, v)$
      $\Pi[v] \leftarrow u$

Relaxation is Safe

Lemma: The relaxation algorithm maintains the invariant that $d[v] \geq \delta(s, v)$ for all $v \in V$.

Proof: By induction on the number of steps.

Consider $RELAX(u, v, w)$. By induction $d[u] \geq \delta(s, u)$. By the triangle inequality, $\delta(s, v) \leq \delta(s, u) + \delta(u, v)$. This means that $\delta(s, v) \leq d[u] + w(u, v)$, since $d[u] \geq \delta(s, u)$ and $w(u, v) \geq \delta(u, v)$. So setting $d[v] = d[u] + w(u, v)$ is safe. $\square$

DAGs:

Can't have negative cycles because there are no cycles!

1. Topologically sort the DAG. Path from u to v implies that u is before v in the linear ordering.

2. One pass over vertices in topologically sorted order relaxing each edge that leaves each vertex.
   $\Theta(V + E)$ time

Example:

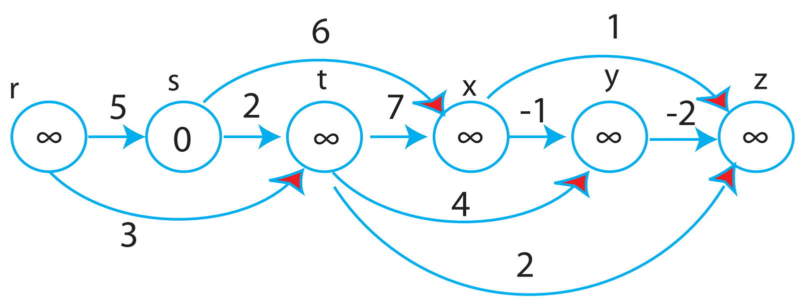


Figure 1: Shortest Path using Topological Sort.

Vertices sorted left to right in topological order

Process r: stays ∞. All vertices to the left of s will be ∞ by definition

Process s: $t : \infty \to 2$ $\quad$ $x : \infty \to 6$ (see top of Figure 2)

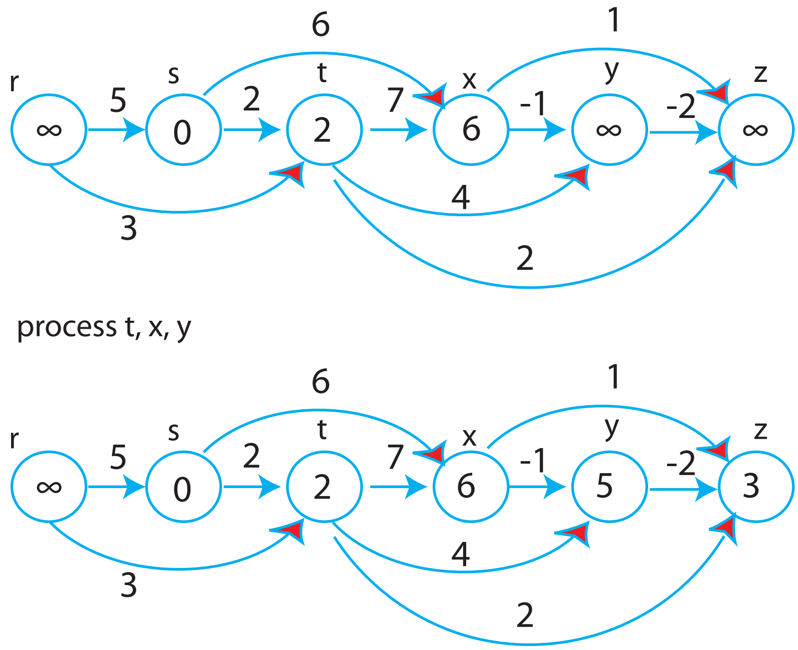


Figure 2: Preview of Dynamic Programming

DIJKSTRA Demo

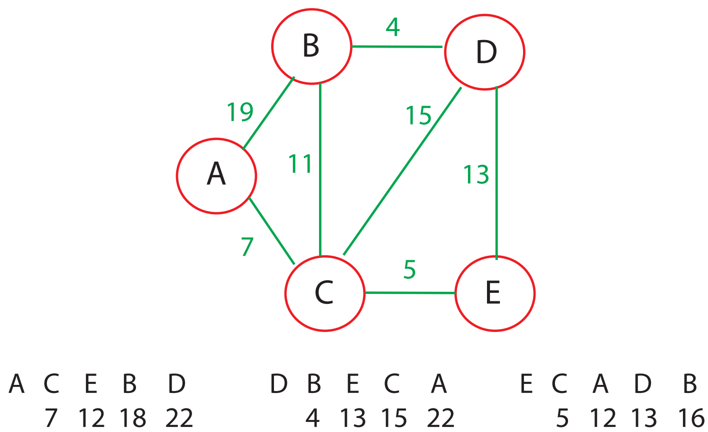


Figure 3: Dijkstra Demonstration with Balls and String.

Dijkstra's Algorithm

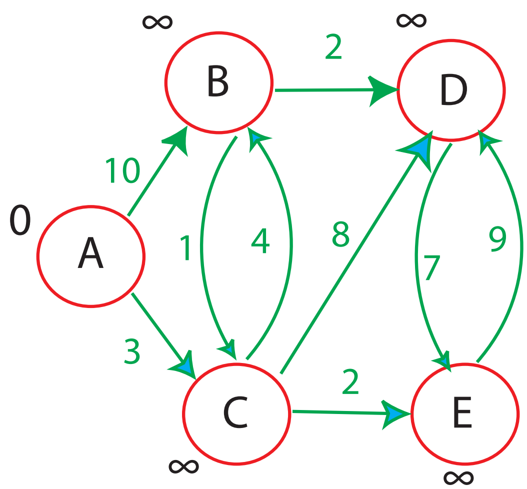
For each edge $(u, v) \epsilon E$, assume $w(u, v) \geq 0$, maintain a set S of vertices whose final shortest path weights have been determined. Repeatedly select $u \epsilon V - S$ with <u>minimum</u> shortest path estimate, add u to S, relax all edges out of u.

Pseudo-code

Dijkstra (G, W, s) //uses priority queue Q

    Initialize (G, s)
    $S \leftarrow \phi$
    $Q \leftarrow V[G]$ //Insert into Q
    while $Q \neq \phi$
        do $u \leftarrow$ EXTRACT-MIN(Q) //deletes u from Q
        $S = S \cup \{u\}$
        for each vertex $v \epsilon$ Adj$[u]$
            do RELAX (u, v, w) $\leftarrow$ this is an implicit DECREASE_KEY operation

Example



S = { }	{ A	B	C	D	E }	=	Q
S = { A }	0	∞	∞	∞	∞		
S = { A, C }	0	10	3	∞	∞	←	after relaxing edges from A
S = { A, C }	0	7	3	11	5	←	after relaxing edges from C
S = { A, C, E }	0	7	3	11	5		
S = { A, C , E, B}	0	7	3	9	5	←	after relaxing edges from B

Figure 4: Dijkstra Execution

Strategy: Dijkstra is a greedy algorithm: choose closest vertex in $V - S$ to add to set S.

Correctness: We know relaxation is safe. The key observation is that each time a vertex u is added to set S, we have $d[u] = \delta(s, u)$.

Dijkstra Complexity

$\Theta(v)$ inserts into priority queue
$\Theta(v)$ EXTRACT_MIN operations
$\Theta(E)$ DECREASE_KEY operations

Array impl:

$\Theta(v)$ time for extra min
$\Theta(1)$ for decrease key
Total: $\Theta(V.V + E.1) = \Theta(V^2 + E) = \Theta(V^2)$

Binary min-heap:

$\Theta(\lg V)$ for extract min
$\Theta(\lg V)$ for decrease key
Total: $\Theta(V \lg V + E \lg V)$

Fibonacci heap (not covered in 6.006):

$\Theta(\lg V)$ for extract min
$\Theta(1)$ for decrease key
amortized cost
Total: $\Theta(V \lg V + E)$