7.1

Fundamental Principle of Counting

Learning objectives:

- To study the Product rule and its Extended Version.
- To study the Sum rule and its Extended Version.
  AND
- To practice the related problems.

The mathematical theory of counting is formally known as *combinatorial analysis*. The principle of counting is basic to the combinatorial analysis.
We present two basic counting principles, the ***Product Rule*** and the ***Sum Rule***.

Product Rule:

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Proof:

The product rule may be proved by enumerating all the possible ways of the two tasks as follows:

$$\begin{array}{l} (1,1),(1,2), \quad \cdots \quad ,(1,n) \\ (2,1),(2,2), \quad \cdots \quad ,(2,n) \\ \vdots \\ (m,1),(m,2), \quad \cdots \quad ,(m,n) \end{array}$$

We say that the outcome is (i, j) if task 1 results in its i^{th} possible way and task 2 then results in the j^{th} of its possible ways. Hence the set of possible outcomes consists of m rows, each row containing n elements, which proves the result.

Example 1:

A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as *mother and child of the year*, how many different choices are possible?

Solution:

By regarding the choice of the woman as the outcome of the first task and the subsequent choice of one of her children as the outcome of the second task, we see from the product rule that there are $10 \times 3 = 30$ possible choices.

When there are more than two tasks to be performed, the product rule can be extended as follows:

Extended Version of the Product Rule:

Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \dots, T_m$ in sequence. If each task T_i, $i = 1, 2, \dots, m$ can be performed in n_i ways, regardless of how the previous task were done, then there are $n_1 . n_2 . \dots . n_m$ ways to carried out the procedure.

This can be proved by mathematical induction and the product rule for two tasks.

Example 2:

An engineering college planning committee consists of 3 first year students, 4 second year students, 5 pre-final year students, and 2 final year students. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution:

We may regard the choice of a subcommittee as the combined outcome of the four separate tasks of choosing a single representative from each of the classes. Hence, it follows from the extended version of the product rule that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees.

Example 3:

How many different 7-place license plates are possible if the first three places are to be occupied by letters and the later 4 by numbers? How many license plates would be possible if repetition among letters or numbers were prohibited?

Solution:

By the extended version of the product rule, the number of license plates is

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 17{,}57{,}60{,}000$$

In the second case there would be

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 7{,}86{,}24{,}000$$

possible license plates.

Example 4:

How many functions defined on n points, are possible if each function value is either 0 or 1?

Solution:

Let the points be $1, 2, \dots, n$. Since $f(i)$ must be either 0 or 1 for each $i = 1, 2, \cdots, n$, it follows that there are

$$\underbrace{2 \times 2 \times 2 \times \dots \times 2}_{n\ times} = 2^n$$

possible functions.

We now introduce the Sum Rule

The Sum Rule:

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example5:

Suppose there are 6 male professors and 3 female Professors teaching calculus. In how many ways a student can choose a calculus Professor?

Solution:

By sum rule, a student can choose a calculus professor in $6 + 3 = 9$ ways.

We can extend the Sum Rule to more than two tasks.

Extended Version of Sum Rule:

Suppose that a task can be done in one of n_1 ways, in one of n_2 ways, … , or in one of n_m ways, where none of the set n_i ways of doing the tasks is the same as any of the set n_j ways, for all pairs i and j with $1 \le i < j \le m$. Then the number of ways to do the task is $n_1 + n_2 + \dots + n_m$.

The extended version of the sum rule can be proved using mathematical induction from the sum rule of two sets.

Example6:

A student can choose a computer project from one of three lists. The three lists contain 13, 5 and 9 possible projects and no project is on more than one list. In how many ways a student can choose a computer project?

Solution:

A student can choose a project by selecting a project from the first list, the second list or the third list. Since no project is on more than one list, by extended sum rule a student can choose $13 + 5 + 9 = 27$ ways.

IP1.

Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution:

There are as many words as there are ways of filling in 4 vacant places by the 4 letters without repetition as follows.

The first place can be filled in 4 different ways by anyone of the 4 letters.

The second place can be filled by anyone of the remaining 3 letters in 3 different ways

The third place can be filled by anyone of the remaining 2 letters in 2 different ways

The fourth place can be filled in 1 way. Hence by the extended product rule, the required number of words is

$$= 4 \times 3 \times 2 \times 1 = 24$$

IP2.

Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution:

A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3flags, 4 flags and 5 flags separately.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 different flags available.

5
4

By product rule, we have $5 \times 4 = 20$

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 different flags.

5
4
3

By product rule, we have $5 \times 4 \times 3 = 60$

Continuing the same way, we find that

The number of 4 flag signals$= 5 \times 4 \times 3 \times 2 = 120$ and the number of 5 flag signals$= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Now, by sum rule, the number of desired signals

$$= 20 + 60 + 120 + 120 = 320$$

IP3.

In forming 5 letter (with distinct letters) words using the letters of the word EQUATIONS.

a. How many begin with a consonant
b. How many in which the vowels and consonants alternate.
c. How many in which Q is immediately followed by U.

Solution:

The given word EQUATIONS contains totally 9 letters in which we have 5 vowels and 4 consonants.

a.

C				
4	8	7	6	5

After filling the first position with any one of the 4 consonants, there are 8 letters remaining.
By extended product rule, the number of 5 letter words that begin with consonants is

$$= 4 \times 8 \times 7 \times 6 \times 5 = 6720$$

b. By extended product rule, the number of 5 letter words that contains the vowels and consonants alternately is

V	C	V	C	V
5	4	4	3	3

OR

C	V	C	V	C
4	5	3	4	2

$$= (5 \times 4 \times 4 \times 3 \times 3) + (4 \times 5 \times 3 \times 4 \times 2) = 1200$$

c. First we place Q so that U may follow it (Q may occupy any one of the first four positions but not the last). Next we place U (in only 1 way), and then we fill the three other positions from among 7 letters remaining.

Q	U			

By the extended product rule, the number of 5 letter words in which Q is immediately followed by U is

$$= (4 \times 1 \times 7 \times 6 \times 5) = 840$$

IP4.

If repetitions are not allowed.

a. How many 3-digit numbers can be formed with the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$?
b. How many of these are odd numbers?
c. How many of these are even numbers?
d. How many are divisible by 5?
e. How many are greater than 600?

Solution:

a.

$\neq 0$		

The first position can be filled in 9 ways, since 0 cannot be used. The middle position can be filled in 9 ways since 0 can be used and the last position can filled in 8 ways.

Thus, by extended product rule, the required number of 3-digit numbers is $9 \times 9 \times 8 = 648$

b. We have

$\neq 0$		Odd

First, we fill the last position with the 5 digits 1,3,5,7,9 in 5 ways.
Now, we fill the first position with the remaining 8 digits in 8 ways.(since one odd digit and 0 are excluded)
The middle position can be filled with the remaining 8 digits in 8 ways (since two digits are now excluded)
Thus, the number of 3-digit odd numbers is

$$= 8 \times 8 \times 5 = 320$$

c. The number of 3-digit even numbers

$$= 648 - 320 = 328 \quad (How?)$$

(*The students are encouraged to do it by the direct method*)

d. A number is divisible by 5 if and only if it ends with 0 or 5

First we form all numbers ending in 0 as follows

9	8	1
		0

Next, we form all the numbers ending in 5 as follows

8	8	1
		5
$\neq 0$		

Hence the 3-digit numbers that are divisible by 5 is

$$(9 \times 8 \times 1) + (8 \times 8 \times 1) = 72 + 64 = 136$$

e. To get the numbers greater than 600,

4	9	8
6		

First, we fill the first position with any one of the digits 6, 7, 8 or 9 in 4 ways.
The middle position can be filled with the remaining 9 digits in 9 ways and the last position can be filled with remaining 8 digits in 8 ways.
Thus, the number of 3-digit numbers greater than 600 is
$= 4 \times 9 \times 8 = 288.$

P1.

A code word consists of two *distinct* English alphabets followed by two *distinct* numbers between 1 and 9. For example C A 2 3.

I. How many such code words are there?

II. How many such code words end with an even integer?

Solution:

I. By the hypothesis, a code word consists of two distinct English alphabets followed by two distinct numbers from 1 to 9.

We have 26 English alphabets and 9 digits (1 to 9).

To form such a code word, we have to choose the first alphabet in 26 ways and the second alphabet in 25 ways.

Again out of 9 digits, first digit can be chosen in 9 ways and the second digit can be chosen in 8 ways.

Hence by extended product rule, the number of such distinct codes $= 26 \times 25 \times 9 \times 8 = 46800$

II. Two distinct alphabets can be selected in 26×25 ways and the Unit's place can be filled in 4 ways (i.e., by $2, 4, 6, 8$). Tenth place can be filled in 8 ways (since one of the digits is already used).

Thus, the number of desired codes is

$$= 26 \times 25 \times 4 \times 8 = 20800.$$

P2.

How many 3-digit even numbers can be formed by the digits $1, 2, 3, 4, 5$ without the repetition of the digits?

Solution:

The unit's place can be filled either by 2 or by 4 to get 3-digit even number with digits 1, 2, 3, 4, 5.

Suppose the unit's place is filled by 2. Since the digits cannot be repeated the ten's place can be filled in 4 ways and having filled ten's place, the hundred's place can be filled by the remaining 3 digits. Thus, by the product rule, the number of 3-digit even numbers ending with 2 formed from the digits 1, 2, 3, 4, 5 is $4 \times 3 = 12$.

Similarly, the number of 3-digit even numbers ending with 4 formed from the digits 1, 2, 3, 4, 5 is 12.

By sum rule, the number of 3-digit even numbers formed by the digits 1, 2, 3, 4, 5 without the repetition of the digits is

$$12 + 12 = 24$$

P3.

In forming 5 letter words (with distinct letters) using the letters of the word EQUATIONS?

a. How many consists only of vowels.
b. How many contain all of the consonants
c. How many begin with E and end in S

Solution:

The given word EQUATIONS contains totally 9 letters in which we have 5 vowels and 4 consonants.

a. There are five places to be filled and 5 vowels are at our disposal.

5	4	3	2	1

By extended product rule, the number of 5 letter words that consists only vowels is

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120$$

b. Each word is to contain 4 consonants and one of the 5 vowels. A vowel (V) has 5 choices. Now, the vowel can be placed in any one of the 5 places and we can fill the remaining 4 positions with consonants (C).

For example:

V	C	C	C	C

By extended product rule, the number of 5 letter words that consists of all consonants is

$$= 5(5 \times 4 \times 3 \times 2 \times 1) = 600$$

c.

E				S
	7	6	5	

Now, there are just 3 positions to be filled and 7 letters are at our disposal.

By extended product rule, the number of 5 letter words that begin with E and end with S is

$$= 7 \times 6 \times 5 = 210$$

P4.

How many numbers are there between 100 and 1000 which have exactly one of their digits as 7?

Solution:

The numbers between 100 and 1000 having 7 as exactly one of their digits can be classified into three types.

(i) When the unit's place has 7:

Here the ten's place can have any one of the digits except 7. It can be filled in 9 different ways. The hundred's place can have any one of the digits except 0 and 7. So hundred's place can be filled in 8 different ways. Therefore, there are $9 \times 8 = 72$ such numbers.

(ii) When the ten's place has 7:

The unit's place can be filled in 9 different ways. It can have any one of the digits except 7. The hundred's place can have any one of the digits except 0 and 7. So hundred's place can be filled in 8 different ways. So, there are $9 \times 8 = 72$ such numbers.

(iii) When the hundred's place has 7:

Here the unit's place can be filled by 9 different ways (except 7) and ten's place can be filled by 9 different ways (except by 7). So there are $9 \times 9 = 81$ such numbers.

Hence the number of desired numbers is

$$= 72 + 72 + 81 = 225$$

EXERCISES

1. Given 4 flags of different colors, how many different signals can be generated, if the signal requires the use of 2 flags one below the other?

2. How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

3. Using the letters of the word MARKING and calling any arrangement a word,

   a. How many different 7-letter words can be formed,

   b. How many different 3-letter words can be formed?

4. If repetitions are allowed:

a. How many three-digit numbers can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?

b. How many of these are odd numbers?

c. How many are even numbers?

d. How many are divided by 5?

5. Find the number of 4 letter words that can be formed using the letters of the word EQUATION. How many of these words begin with E? How many end with N? How many begin with E and end with N?

6. Find the number of 4-digit numbers that can be formed using the digits 2, 3, 5, 6, 8 (without repetition).

   a. How many of them are divisible by 2?
   b. How many of them are divisible by 5?
   c. How many of them are divisible by 25?

7.2

Permutations

Learning objectives:

- To define a permutation and a circular permutation on a finite set.
- To determine the number of permutations and circular permutations on a set with n elements.
  AND
- To practice the related problems.

We wish to know the possible ordered arrangements of the letters a, b and c. By direct enumeration we see that there are 6: namely

$$abc, acb, bac, bca, cab, cba$$

Each arrangement is known as a ***permutation***. Thus there are 6 possible permutations of a set of 3 objects. The result could also have been obtained from the product rule, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then chosen from the remaining 1. Thus there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

Suppose that we have n objects. Reasoning similar to that we have just used for the three letters shows that there are

$$n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

different permutations of the n objects.

The notation $n!$ is read "n factorial" and is defined by

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

It is the product of all the consecutive integers from n down to 1.

For example,

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Note:
The expression $0!$ is defined to be 1.

Example 1:
How many different batting orders are possible for a baseball team consisting of 9 players?
Solution:
There are $9! = 3{,}62{,}880$ possible batting orders.

Example 2:
A class in probability theory consists of 6 boys and 4 girls. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

(a) How many different rankings are possible?
(b) If the boys are ranked just among themselves and the girls among themselves, how many different rankings are possible?

Solution:
(a) As each ranking corresponds to a particular ordered arrangement of the 10 people, we see that the answer to this part is $10! = 3{,}628{,}800$.

(b) As there are $6!$ possible rankings of the boys among themselves and $4!$ possible rankings of the girls among themselves, it follows from the product rule that there are $(6!)(4!) = (720)(24) = 17{,}280$ possible rankings in this case.

Example 3:
Mr. Jones has 10 books that he is going to put on his bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Jones wants to arrange his books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?
Solution:
There are $4!\,3!\,2!\,1!$ arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are $4!\,3!\,2!\,1!$ possible arrangements. Hence, as there are $4!$ possible orderings of the subjects, the desired answer is $4!\;4!\,3!\,2!\,1! = 6912$.

Circular permutation of n objects:
An arrangement of n distinct objects in definite order in a circle is called a **circular permutation.**

Example 4:
In how many ways can 10 boys be arranged (a) in a straight line, (b) in a circle?
Solution:
(a) The boys may be arranged in a straight line in $10!$ Ways.
(b) We first place a boy at any point on the circle. The other 9 boys may then be arranged in $9!$ Ways.

This is an example of a ***circular permutation***.

In general n objects may be arranged in a circle in $(n-1)!$ ways.

IP1.

Find the value of $\boldsymbol{n}$ if $\dfrac{(2n)!}{3!(2n-3)!} : \dfrac{n!}{2!(n-2)!} = 44:3$

Solution:

Given that $\dfrac{(2n)!}{3!(2n-3)!} : \dfrac{n!}{2!(n-2)!} = 44:3$

We have $\dfrac{\dfrac{(2n)!}{3!(2n-3)!}}{\dfrac{n!}{2!(n-2)!}} = \dfrac{44}{3}$

$$\Rightarrow \frac{2n(2n-1)(2n-2)(2n-3)!}{3!(2n-3)!} \times \frac{2!(n-2)!}{n(n-1)(n-2)!} = \frac{44}{3}$$

$$\Rightarrow \frac{2n(2n-1)2(n-1)}{3n(n-1)} = \frac{44}{3}$$

$$\Rightarrow \frac{4(2n-1)}{3} = \frac{44}{3}$$

$$\Rightarrow 2n-1 = 11 \Rightarrow n = 6$$

IP2.

Find the number of ways of arranging 5 History, 4 Economics and 4 Civics books in a shelf (in a row) such that the books on the same subject are together.

Solution:

Consider 5 History books as one unit, 4 Economics books as a second unit and 4 Civics books as a third unit.

The three units can be arranged in $3!$ ways. Moreover,

5 History books are arranged themselves in $5!$ ways

4 Economics books are arranged themselves in $4!$ ways

4 Civics books are arranged themselves in $4!$ ways

The number of the required arrangements

$$= 3! \times 5! \times 4! \times 4! = 6 \times 120 \times 24 \times 24 = 4{,}14{,}720$$

IP3.

Ten guests are to be seated in a row. Three of them are to be seated together. Of the remaining two guests do not wish to sit side by side. Find the number of possible arrangements.

Solution:

Treat the 3 guests who are to be seated together as one unit. Then we have 7 guests $+$ 1 unit of 3 guests. They can be arranged in $8!$ ways and the one unit of 3 guests can be permuted in $3!$ ways. Therefore,

Number of permutations of 10 guests in which 3 guests are always sit together is $= 8! \times 3!$

Now, if the two guests who donot wish to sit side by side, are considered to be sitting side by side, then the number of permutations is $= 7! \times 3! \times 2!$

Therefore, the number of permutations of 10 guests so that 3 particular guests are seated together and two particular guests do not sit side by side is

$$= 8! \times 3! - 7! \times 3! \times 2!$$

$$= 7! \times 3!\,(8 - 2) = 1{,}81{,}440$$

IP4.

The countries Japan, China and Russia sent 15, 14 and 13 representatives for a round table conference to discuss the international cooperation among them. Find the number of ways of these representatives sits at a round table so that

a. All Japanese are together

b. Representatives of same nationality together

Solution:

a. Treat 15 Japanese as one unit. Then we have

14 Chinese $+$ 13 Russians $+$ 1 unit of Japanese $= 28$ entities.

They can be arranged at a round table conference in

$$(28-1)! = 27! \text{ ways}$$

Now, the 15 Japanese among themselves can be arranged in $15!$ ways.

Hence the required arrangement is $27! \times 15!$

b. Treating 3 nationalities as three units, can arrange at a round table in $(3-1)! = 2!$ ways.

Now, 15 Japanese among themselves can be permuted in $15!$ ways.

Similarly, 14 Chinese in $14!$ ways

13 Russians in $13!$ ways

Hence the required number of ways arrangements is

$$2! \times 15! \times 14! \times 13!$$

P1.

I. If $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$ then find x

II. If $(n+2)! = 2550(n!)$ then find n

Solution:

I. $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$

$$\Rightarrow x = 11!\left(\frac{1}{9!} + \frac{1}{10!}\right) = \frac{11!}{9!} + \frac{11!}{10!}$$

$$= \frac{11 \times 10 \times 9!}{9!} + \frac{11 \times 10!}{10!} = 110 + 11 = 121$$

$$\Rightarrow x = 121$$

II. $(n+2)! = 2550(n!)$

$$\Rightarrow (n+2)(n+1)n! = 2550(n!)$$

$$\Rightarrow (n+2)(n+1) = 2550$$

$$\Rightarrow n^2 + 3n - 2548 = 0$$

$$\Rightarrow (n-49)(n+52) = 0$$

$\Rightarrow n = 49$, since $n = -52 \notin \boldsymbol{N}$

P2.

Find the number of ways of arranging the letters of the word KRISHNA in which all the vowels come together

Solution:

The word KRISHNA has 7 letters in which there are two vowels namely I and A. Treat the vowels as one unit.

Thus, we have 5 consonants + 1 unit vowels = 6 things, which can be arranged in 6! Ways. Now the vowels can be arranged among themselves in 2! Ways. Therefore, by product rule, the number of arrangements in which the 2 vowels come together is $6! \times 2! = 720 \times 2 = 1440$

P3.

Find the number of ways of arranging 6 boys and 6 girls in a row. In how many of these arrangements

a. All the girls are together
b. Boys and girls come alternately

Solution:

The number of ways of arranging 6 boys and 6 girls in a row is $12!$ ways.

a. Treat 6 girls as one unit. Then we have 6 boys $+ 1$ unit girls
They can be arranged in $7!$ ways and the one unit of girls can be permuted in $6!$ ways. Hence the number of arrangements in which all 6 girls are together is $7! \times 6!$

b. Let us take 12 places. The row may begin with either a boy or a girl which can be arranged in 2 ways.
If it begins with a boy, then all the odd places $(1, 3, 5, 7, 9, 11)$ will be occupied by boys and the even places $(2, 4, 6, 8, 10, 12)$ occupied by girls.
The 6 boys can be arranged in the 6 odd places in 6! Ways and the 6 girls can be arranged in the 6 even places in $6!$ Ways. Thus the number of arrangements in which boys and girls come alternately is $2 \times 6! \times 6!$

P4.

Find the number of ways of arranging 8 persons around a circular table if two particular persons were to sit together.

Solution:

Treat the two particular persons as one unit. Then we have $6+1=7$ entites. They can be arranged around a circular table in $(7-1)!=6!$ ways.

Now, the two particular persons can be permuted among themselves in 2!

Therefore, the number of arrangements is $6! \times 2! = 1{,}440$

EXERCISES

1. Evaluate (a) 5! (b) 7! (c) 7! – 5!

2. Compute (a) $\frac{7!}{5!}$ (b) $\frac{12!}{(10!)(2!)}$

3. Evaluate $\frac{n!}{r!(n-r)!}$ when n = 5, r = 2.

4. Simplify

   a. $\frac{n!}{(n-1)!}$

   b. $\frac{(n+2)!}{n!}$

   c. $\frac{(r+1)!}{(r-1)!}$

5. If $\frac{1}{8!}+\frac{1}{9!}=\frac{x}{10!}$ then the value of x

6. Find the number of ways that 4 people can sit in a row of 4 seats.

7. A family has 3 boys and 2 girls.
   a. Find the number of ways they can sit in a row.
   b. Find the number of ways they can sit in a circle.
   c. Find the number of ways the boys and girls sit in a row alternately.

8. A debating team consists of 3 boys and 3 girls. Find the number n of ways they can sit in row where:
   a. there are no restrictions,
   b. the boys and girls sit alternately,
   c. just the girls are to sit together.

9. Find the number n of ways 5 large books, 4 medium-size books, and 3 small books can be placed on a shelf so that all books of the same size are together.

10. Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that
    a. All vowels occur together
    b. All vowels do not occur together.

7.3

Permutations with Repetitions

Learning objectives:

- To find the number of permutations of a set with n elements when certain elements are indistinguishable from each other.

  AND

- To practice the related problems.

We shall now determine the number of permutations of a set of n objects when certain of the objects are *indistinguishable* from each other.

Example 1:

How many different letter arrangements can be formed using the letters P E P P E R?

Solution:

We first note that there are 6! permutations of the letters $P_1E_1P_2P_3E_2R$ when the 3P's and the 2E's are distinguished from each other. However, consider any one of these permutations – for instance, $P_1P_2E_1P_3E_2R$. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangements would still be of the form P P E P E R. That is, all 3! 2! permutations

$$
\begin{array}{ll}
P_1P_2E_1P_3E_2R & P_1P_2E_2P_3E_1R \\
P_1P_3E_1P_2E_2R & P_1P_3E_2P_2E_1R \\
P_2P_1E_1P_3E_2R & P_2P_1E_2P_3E_1R \\
P_2P_3E_1P_1E_2R & P_2P_3E_2P_1E_1R \\
P_3P_1E_1P_2E_2R & P_3P_1E_2P_2E_1R \\
P_3P_2E_1P_1E_2R & P_3P_2E_2P_1E_1R
\end{array}
$$

are of the form P P E P E R. Hence there are $6!/(3!\,2!) = 60$ possible letter arrangements of the letters P E P P E R.

In general, the same reasoning as that used in example 1 shows that there are

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, …, n_r are alike.

Example 2:

A chess tournament has 10 competitors of which 4 are from Russia, 3 are from the United States, 2 from Great Britain, and 1 from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Solution:

There are

$$\frac{10!}{4!\,3!\,2!\,1!} = 12{,}600$$

possible outcomes.

Example 3:

How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Solution:

There are $\dfrac{9!}{4!\,3!\,2!} = 1{,}260$ different signals.

Example 4:

Find the number m of seven-letter words that can be formed using the letters of the word "BENZENE".

Solution:

We seek the number of permutations of seven objects of which three are alike, the three E's, and two are alike, the two N's. Therefore,

$$m = \frac{7!}{3!\;2!} = 420$$

Example 5:

A set of snooker balls consists of a white, a yellow, a green, a brown, a blue, a pink, a black and 15 reds. How many distinguishable permutations of the balls are there?

Solution:

In total there are 22 balls, the 15 reds being indistinguishable. Thus, the number of distinguishable permutations is

$$\frac{22!}{(1!)(1!)(1!)(1!)(1!)(1!)(1!)(15!)} = \frac{22!}{15!} = 85{,}95{,}41{,}760$$

IP1.

Find the number of 5 digit even numbers that can be formed using the digits $1, 1, 2, 2, 3$.

Solution:

To find 5 digit even numbers, fill the units place by 2 and the remaining 4 places can be arranged using the remaining digits $1, 1, 2, 3$ in $\frac{4!}{2!} = 12$ ways.

Thus, the number of 5 digit even numbers that can be formed using the digits $1, 1, 2, 2, 3$ is 12.

IP2.

Find the number of ways of arranging the letters of the word SPECIFIC. In how many ways can be arranged that the two C's come together.

Solution:

a. The given word has 8 letters in which there are 2 I's and 2 C's. Hence, they can be arranged in

$$=\frac{8!}{2!2!}=10{,}080 \text{ ways}$$

b. Teat the 2 C's as one unit, then we have, 6 letters$+1$ unit $=7$ objects in which two letters (I's) are alike.

Hence, they can be arranged in $=\frac{7!}{2!}=2520$ ways

Now, the 2 C's among themselves can be arranged in $\frac{2!}{2!}=1$way.

Thus, the number of required arrangements is 2520.

IP3.

How many numbers greater than $10,00,000$ can be formed by using the digits $1,2,0,2,4,2,4$?

Solution:

Notice that $10{,}00{,}000$ is a 7-digit number and the number of given digits is 7.

Therefore, the numbers to be counted will be 7-digit only.

Also, the numbers have to be greater than $10{,}00{,}000$, so they have to begin with $1,2$ or 4:

The numbers begins with 1:

1	0	2	2	2	4	4

1 is fixed at the extreme left position and the remaining digits to be rearranged will be $0,2,2,2,4,4$

Therefore, the number of numbers begins with 1 is $= \frac{6!}{2!3!} = 60$

The numbers begins with 2:

2	1	0	2	2	4	4

2 is fixed at the extreme left position, the remaining digits to be rearranged will be $1,0,2,2,4,4$

Therefore, the number of numbers begins with 2 is $= \frac{6!}{2!2!} = 180$

The numbers begin with 4:

4	1	0	2	2	2	4

4 is fixed at the extreme left position, the remaining digits to be rearranged will be $1,0,2,2,2,4$

Therefore, the total number of numbers begins with 4 is $= \frac{6!}{3!} = 120$

Thus, the required number of numbers is

$$= 60 + 180 + 120 = 360$$

IP4.

How many arrangements van be made with the letters of the word MATHEMATICS? In how many of them vowels are together?

Solution:

There are 11 letters in the given word MATHEMATICS of which we have 2 M's, 2 A's, 2 T's and all other are distinct.

Therefore, the required number of arrangements is $= \frac{11!}{2!\times 2!\times 2!}$

There are 4 vowels i.e., A, E, A, I.

Treat 4 vowels as one unit and in the remaining 7 letters (i.e., M,T,H,M,T,C,S) we have (1 unit + 7 letters) 2 M's, 2 T's and the rest are different. These 8 objects can be arranged in $\frac{8!}{2!\times 2!}$ ways.

But the 4 vowels (A, E, A, I) can be put together in $\frac{4!}{2!}$ ways.

Hence, the total number of arrangements in which vowels are always together is

$$= \frac{8!}{2!\times 2!} \times \frac{4!}{2!} = 1{,}20{,}960$$

P1.

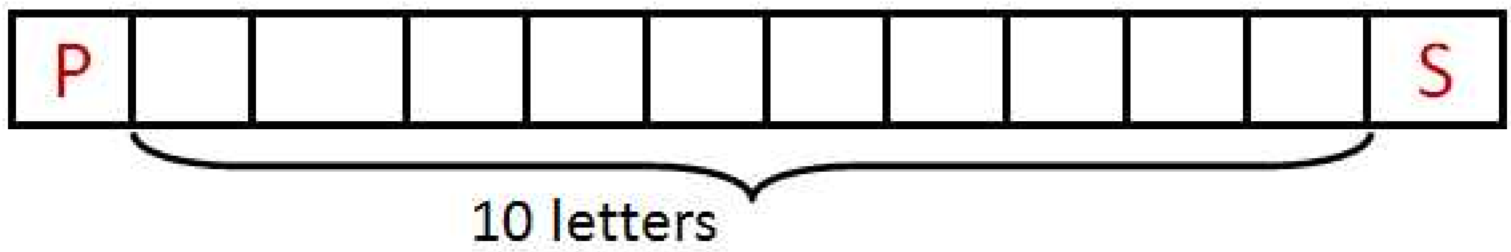
In how many ways can the letters of the word PERMUTATIONS be arranged if the

(i) words start with P and end with S
(ii) All the vowels are occur together

Solution:

Notice that the given word PERMUTATIONS has 12 letters in which there are 2 T’s and all the other letters occur only once.

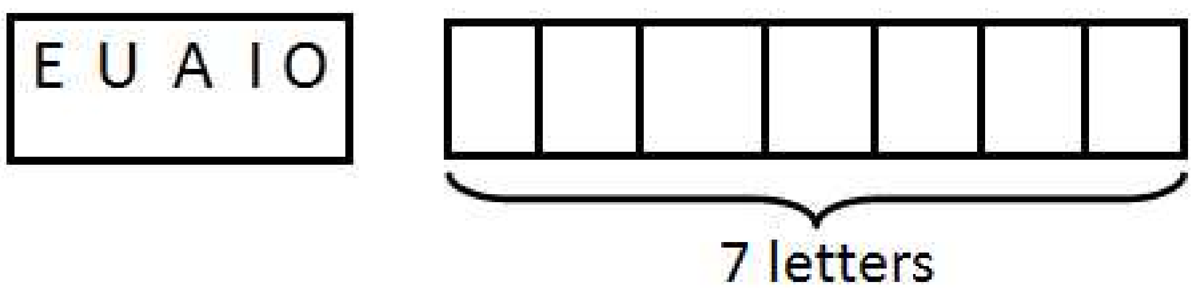
(i) If P and S are fixed at the extreme ends (P at the left end and S at the right end), then 10 letters are left in which we have 2 T’s



Hence, the required number of arrangements is $= \frac{10!}{2!}$

(ii) There are 5 vowels in the given word namely E, U, A, I, O each appearing only once.
Treat 5 vowels as one unit. Thus, we have 8 objects (i.e., 7 letters + 1 unit).

i.e.,



In these 8 objects there are 2 T’s, which can be arranged in $\frac{8!}{2!}$ ways.

Corresponding to each of these arrangements, the 5 different vowels can be arranged in $5!$ ways.

Therefore, the required number of arrangements is $\frac{8!}{2!} \times 5!$

P2.

Find the number of ways of arranging the letters of the word SINGING so that

a. They begin and end with I
b. The two G’s come together

Solution:

a. First we fill first and last places with I's in $\frac{2!}{2!} = 1$ way as shown below

I						I

Now, we fill the remaining 5 places with the remaining 5 letters S, N, G, N, G in $\frac{5!}{2!2!} = 30$ ways.

Hence, the number of required permutations is 30.

b. Treat the two G's as one unit. Then we have, 5 letters $+ 1$ unit $= 6$ objects, in which we have 2 I's and 2N's.

Hence, they can be arranged in $\frac{6!}{2!2!} = 180$ ways

Now, the G's among themselves can be arranged in $\frac{2!}{2!} = 1$ way. Hence the number of required permutations is 180.

P3.

In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same color are indistinguishable.

Solution:

Total number of discs is $4 + 3 + 2 = 9$

Out of 9 discs, 4 are red, 3 are yellow and 2 are green

$\therefore$ The number of arrangements is $\dfrac{9!}{4!\,3!\,2!} = 1260$

P4.

How many different words can be formed by using all the letters of the word ALLAHABAD?

a. In how many of them vowels occupy the even positions?
b. In how many of them both L do not come together?

Solution:

There are 9 letters in the given word ALLAHABAD, in which 4 A's, 2 L's and the rest are all distinct.

So, the required number of words is $= \frac{9!}{4!2!} = 7560$

a. There are 4 vowels i.e., 4 A's and all are alike. Also, there are 4 even places i.e., 2, 4, 6, 8.

   These 4 even places can be occupied by 4 vowels in

   $$= \frac{4!}{4!} = 1 \text{way}$$

   Now, we are left with 5 places in which 5 letters, of which 2 L's are alike and other are distinct, can be arranged in $\frac{5!}{2!}$ ways.

   Hence, the total number of words in which vowels occupy the even places is

   $$= \frac{5!}{2!} \times \frac{4!}{4!} = \frac{5!}{2!} = 60$$

b. Treat 2 L's as one unit and we have 8 objects (7 letters + 1 unit), out of which A repeats 4 times and others are distinct. These 8 letters can be arranged in $\frac{8!}{4!}$ ways.

   So, the number of words in which both L's come together is $\frac{8!}{4!} = 1680$

   Hence, the number of words in which both L's do not come together

   $= \text{Total no. of words} - \begin{cases} \text{no. of words in which both L's} \\ \text{come together} \end{cases}$

   $= 7560 - 1680 = 5880$

EXERCISES

1. Find the number n of distinct permutations that can be formed from all the letters of each word
    a. THOSE
    b. UNUSUAL
    c. SOCIOLOGICAL
    d. QUEUE
    e. COMMITTEE
    f. PROPOSITION
    g. BASEBALL

2. Find the number n of different signals, each consisting of 6 flags hung in a vertical line, which can be formed from 4 identical red flags and 2 identical blue flags.

3. Find the number n of different signals, each consisting of 8 flags hung in a vertical line, which can be formed from 4 identical red flags, 2 identical blue flags, and 2 identical green flags.

4. Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

a. do the words start with P
b. do all the vowels always occur together
c. do the vowels never occur together
d. do the words begin with I and end in P?

5. How many permutations can be made of the letters, taken all together, of the “word” MASSESS?
   a. In how many ways will the four S’s be together?
   b. How many will end in SS?

6. Find the number of 5-digit numbers that can be formed using the digits 1, 1, 2, 2, 3. (Ans: 230)

7. Garlands are formed using 4 red roses and 4 yellow roses of different sizes. In how many of them

   a. All 4 red roses come together.

   b. Red roses and yellow roses come alternately.

   (Ans: 288, 72)

8. How many ways can the letters of the word ENGINEERING be arranged so that the 3 N's come together but the 3 E's do not come together?(Ans: 13,860)

9. How many permutations can be made of the letters of the word ARRANGEMENT? In how many of these the vowels occur together? (Ans: 2491800, 10800)

10. Find the number of 5-digit numbers that can be formed using the digits 2, 2, 3, 3, 4. How many of them are greater than 30,000? (Ans: 30, 18)

7.4. Ordered Samples

Learning objectives:

- To define a $r-$ permutation of a set S with n elements and to derive a formula for the number of $r-$permutations of S

- To introduce the concept of sampling and to derive the formulae for the number of ordered samples of size r with and without replacement from a set S with n elements.

  AND

- To practice the related problems.

The permutations of a set of letters a, b and c, taken all at a time, were discussed in a previous module and they are $abc, acb, bac, bca, cab, cba$.
Any arrangement of a set of objects in a definite order is called a ***permutation** of the set taken all at a time*.
If a set contains n objects, any ordered arrangement of any $r \le n$ of the objects is called *permutation of the n objects taken r at a time*. For example,

$$ab, ac, ad, bc, bd, cd,$$
$$ba, ca, da, cb, db, dc$$

are permutations of n = 4 letters a, b, c, d taken r = 2 at a time

The number of permutations of n objects taken r at a time is denoted by $\boldsymbol{{}^nP_r}$, where $r \le n$. We now derive the general formula for ${}^nP_r$.

There are n different ways to choose the first element; following this, the second element can be chosen in n − 1 ways; and, following this, the third element can be chosen n − 2 ways. Continuing in this manner, we have that the r^{th} element can be chosen in $n-(r-1)=n-r+1$ ways. Thus, by the fundamental principle of counting, we have

$$\begin{aligned} {}^n\boldsymbol{P}_r &= \boldsymbol{n(n-1)(n-2)\cdots(n-r+1)} \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)!}{(n-r)!} \end{aligned}$$

$${}^n\boldsymbol{P}_r = \frac{\boldsymbol{n!}}{\boldsymbol{(n-r)!}}$$

If $r = n$, then

$${}^nP_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

There are $n!$ permutations of n objects taken all at a time and the same result we obtained in the previous module.

If $r = 0$,

$${}^nP_0 = \frac{n!}{(n-0)!} = 1$$

The symbol ${}^nP_0$ is the number of arrangements which have no objects at all in the arrangement. It means all the

The symbol ${}^nP_0$ is the number of arrangements which have no objects at all in the arrangement. It means all the objects are left behind, and there is only one way of doing so. Therefore, ${}^nP_0$ = 1.

Example 1:

Solve for n, given (a) ${}^nP_2 = 110$, (b) ${}^nP_4 = 30\ {}^nP_2$

Solution:

(a)

$$\begin{aligned} &{}^nP_2 = n(n-1) = 110 \\ &n^2 - n - 110 = 0 \\ &(n-11)(n+10) = 0 \end{aligned}$$

Since n is positive, n = 11.

(b) We have

$$\begin{aligned} &n(n-1)(n-2)(n-3) = 30n(n-1) \\ &n(n-1)(n-2)(n-3) - 30n(n-1) = 0 \\ &n(n-1)\left[(n-2)(n-3) - 30\right] = 0 \\ &n(n-1)\left(n^2 - 5n - 24\right) = 0 \\ &n(n-1)(n-8)(n+3) = 0 \end{aligned}$$

Since $n \ge 4$, the required solution is n = 8.

Example 2:

How many different numbers can be formed by using six out of nine digits 1, 2, 3, … , 9?

Solution:

Here we have 9 different things and we have to find the number of permutations of them taken 6 at a time. Therefore,

$$\text{The desired answer } = {}^9P_6 = 9\times 8\times 7\times 6\times 5\times 4 = 60480$$

Many problems in combinatorial analysis are concerned with choosing an element from a set S consisting of n elements. A card from a deck or a person from a population is an example. When we choose one element after another from the set S, say r times, we call the choice as an ordered sample of size r. The following two cases are of common occurrence.

(i) Sampling with replacement
(ii) Sampling without replacement

In sampling with replacement, the element is replaced in the set S before the next element is chosen. Since there are n different ways to choose each element (repetitions are allowed), the product rule principle tells us that there are

$$\overbrace{n\cdot n\cdot n\cdots n}^{r \text{ times}} = n^r$$

different ordered samples with replacement of size r.

In sampling without replacement, the element is not replaced in the set S before the next element is chosen. Thus, there are no repetitions in the ordered sample. Accordingly, an ordered sample of size r without replacement is simply a r-permutation of the elements in the set S with n elements. Thus, there are

$${}^nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

different ordered samples without replacement of size r from a population (set) with n elements. In other words, by the product rule, the first element can be chosen in n ways, the second in n − 1 ways, and so on.

Example 3:

Three cards are chosen in succession from a deck of 52 cards. Find the number of ways this can be done: (a) with replacement (b) without replacement.

Solution:

(a) Since each card is replaced before the next card is chosen, each card can be chosen in 52 ways. Thus,

$$(52)(52)(52) = 52^3 = 140{,}608$$

is the number of different ordered samples of size $r = 3$ with replacement.

(b) Since there is no replacement, the first card can be chosen in 52 ways, the second card in 51 ways, and the last card in 50 ways. Thus,

$${}^{52}P_3 = (52)(51)(50) = 132{,}600$$

is the number of different ordered samples of size $r = 3$ without replacement.

Example 4:

Suppose repetitions are not allowed. (a) Find the number n of three-digit numbers that can be formed from the six digits: 2, 4, 5, 8, 7, 9. (b) How many of them are even? (c) How many of them exceed 400?

Solution:

There are 6 digits, and we have to form three-digit numbers.

(a) There are 6 ways to fill the first position, 5 ways for the second position, and 3 ways for the third position. Thus,

$$n = 6\cdot 5\cdot 4 = 120$$

Alternatively, n is the number of permutations of 6 things taken 3 at a time, and so

$$n = {}^6P_3 = 6\cdot 5\cdot 4 = 120$$

(b) Since the numbers must be even, the last digit must be either 2 or 4. Thus, the third position is filled first and it can be done in 2 ways. Then there are now 5 ways to fill the middle position and 4 ways to fill the first position.
Thus, $4\cdot 5\cdot 2 = 40$ of the numbers are even.

(c) Since the numbers must exceed 400, they must begin with $5, 8, 7, or\ 9$. Thus, we first fill the first position

and it can be done in 4 ways. Then there are 5 ways to fill the second position and 4 ways to fill the third position. Thus, $4\cdot 5\cdot 4 = 80$ of the numbers exceed 400.

Example 5:

A class contains 8 students. Find the number of ordered samples of size 3: (a) with replacement, (b) without replacement.

Solution:

(a) Each student in the ordered sample can be chosen in 8 ways; hence there are $8\cdot 8\cdot 8 = 8^3 = 512$ samples of size 3 with replacement.
(b) The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there are $8\cdot 7\cdot 6 = 336$ samples of size 3 without replacement.

IP1.

If $^{5}P_r = 2\cdot {}^{6}P_{r-1}$ **then find the value of** r**.**

Solution:

Given $^{5}P_r = 2\cdot {}^{6}P_{r-1}$

$$\Rightarrow \frac{5!}{(5-r)!} = 2\,.\frac{6!}{(6-(r-1))!} \qquad \textit{(By definition)}$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{2\times6\times5!}{(7-r)!}$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{12\times5!}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow (7-r)(6-r) = 12$$

$$\Rightarrow (7-r)(6-r) = 12$$

$$\Rightarrow r^2 - 13r + 30 = 0$$

$$\Rightarrow (r-10)(r-3) = 0$$

$$\Rightarrow r = 3 \;,\quad (\because r \le n = 5)$$

IP2.

18 guests have to be seated half on each side of a long table. 4 particular guests desire to sit on one particular side and 3 other on the other side. Determine the number of ways in which the sitting arrangements can be made?

Solution:

Assume that two sides of a long table be P and Q.

Now, 4 particular guests to be sit on the side P of the table in 9 chairs in $9P_4$ ways.

After this arrangement, again 3 particular guests to be sit on the side Q of the table in 9 chairs in $9P_3$ ways.

Now, the remaining 11 guests can be permuted on 11 chairs on both sides of the table, in $11!$ ways.

Hence, by the product rule, the total number of ways in which 18 guests can be seated $= 9P_4 \times 9P_3 \times 11!$

IP3:

Find the number of 5- letter words that can be formed using the letters of the word *EXPLAIN* that begin and end with a vowel when repetitions are allowed.

Solution:

We can fill the first and last places with vowels each in 3 ways (A or E or I).

X				X

Now, each of the remaining 3 places can be filled in 7 ways (using any letter of the given 7 letters).

Hence the number of 5 letter words which begin and end with vowels when the repetitions are allowed is

$$3^2 \times 7^3 = 9 \times 343 = 3087$$

IP4:

Find the number of 4- digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 that are divisible by

(i) 2 (ii) 3 when repetition is allowed

Solution:

(i) **Numbers divisible by 2**

For a number to be divisible by 2, the unit's place should be filled with an even digit. This can be done in 3 ways (2 or 4 or 6)

			X

Now, each of the remaining 3 places can be filled in 6 ways.

Hence the number of 4- digit numbers that are divisible by 2 is

$$3 \times 6^3 = 3 \times 216 = 648$$

(ii) **Numbers divisible by 3**

Fill the first 3 places with the given 6 digits in 6^3 ways.

X	X	X	

Now, after filling up the first 3 places with three digits, if we fill up the units place in 6 ways, we get 6 consecutive positive integers. ***Out of any six consecutive integers exactly two are divisible by 3***. Therefore, the units place can be filled in 2 ways.

Hence the number of $4-$ digit numbers divisible by 3 is

$$= 2 \times 216 = 432.$$

P1.

If ${}^{(2n+1)}P_{n-1} : {}^{(2n-1)}P_{n} = 3:5$ then find the value of n

Solution:

Given ${}^{(2n+1)}P_{n-1} : {}^{(2n-1)}P_{n} = 3:5$

$$\Rightarrow \frac{{}^{(2n+1)}P_{n-1}}{{}^{(2n-1)}P_{n}} = \frac{3}{5}$$

$$\Rightarrow \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$ *(By definition)*

$$\Rightarrow \frac{(2n+1)(2n)(2n-1)!}{(n+2)\,(n+1)\,n\,(n-1)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \ 5\,(4n+2) = 3[(n+2)(n+1)]$$

$$\Rightarrow 3n^2 - 11n - 4 = 0$$

$$\Rightarrow (3n+1)(n-4) = 0$$

$$\Rightarrow n = 4\,, \qquad (\because n \in \boldsymbol{N})$$

P2.

There are 8 students appearing in an examination of which 3 have to appear in mathematics paper and the remaining 5 in different subjects. In how many ways can they be made to sit in a row if the candidates in mathematics cannot sit next to each other?

Solution:

The total number of candidates is $= 8$.Now, the 5 different subjects candidates can be seated in $^5P_5 = 5!$ ways.

In between 5 candidates there are 6 places for 3 mathematics candidates. Therefore, the mathematics candidates can be seated in $6P_3$ ways.

Hence, by product rule, the required number of ways they can sit in a row such that no two candidates in mathematics sit side by side is

$$= 5! \times 6P_3 = 14,400$$

P3.

Find the number of 4 letter words that can be formed using the letters of the word PISTON in which at least one letter is repeated.

Solution:

The given word has 6 letters. The number of 4 letter words that can be formed using these 6 letters

i. when repetition is allowed is $6^4 = 1296$

ii. when repetition is not allowed is $6_{P_4} = 6 \times 5 \times 4 \times 3 = 360$

Hence, the number of 4 letter words in which at least one letter repeated is $6^4 - 6_{P_4} = 1296 - 360 = 936$

P4.

Find the sum of all $4-$ digit numbers that can be formed using the digits $1, 3, 5, 7, 9$. (if the repetition of digits not allowed).

Solution:

We know that the number of $4-$ digits numbers that can be formed using the given 5 digits is ${}^5P_4 = 120$. Now we find their sum.

We first find the sum of the digits in the unit place of all these 120 numbers. If we fill the units place with 1 as shown below, then the

			1

remaining 3 places can be filled with the remaining 4 digits in ${}^4P_3 = 24$ ways. This means, the number of 4 digit numbers having 1 in units place is ${}^4P_3$. Similarly, each of the digits 3, 5, 7, 9 appears 24 times in units place. By adding all these digits we get the sum of the digits in unit's place of all 120 numbers is

$${}^4P_3 \times 1 + {}^4P_3 \times 3 + {}^4P_3 \times 5 + {}^4P_3 \times 7 + {}^4P_3 \times 9 = {}^4P_3 \times 25$$

Similarly, we get the sum of the digits in Ten's place is ${}^4P_3 \times 25$. Since it is in 10's place, its value is ${}^4P_3 \times 25 \times 10$.

Similarly, the values of the sum of the digits in 100's place and 1000's place are

$${}^4P_3 \times 25 \times 100 \text{ and } {}^4P_3 \times 25 \times 1000$$

respectively.

Hence the sum of all the 4-digit numbers formed by using the digits 1, 3, 5, 7, 9

$${}^4P_3 \times 25 \times 1 + {}^4P_3 \times 25 \times 10 + {}^4P_3 \times 25 \times 100 + {}^4P_3 \times 25 \times 1000$$

$$= {}^4P_3 \times 25 \times 1111 \ldots (1)$$

$$= 24 \times 25 \times 1111 = 6{,}66{,}600.$$

Note:

1. From (1) in the above example, we can derive that the sum of all r- digit numbers that can be formed using the given 'n' non – zero digits $(1 \le r \le n \le 9)$ is

   $${}^{n-1}P_{r-1} \times \text{ Sum of the given digits} \times 111\ldots1 \ (r \text{ times})$$

2. In the above, if '0' is one digit among the given n digits, then we get that the sum of the $r-$ digit numbers that can be formed using the given n digits (including '0')

   $$= \{ {}^{n-1}P_{r-1} \times \text{ sum of the given digits} \times 111\ldots1 \ (r \text{ times})\}$$

   $$- \{ {}^{n-2}P_{r-2} \times \text{ sum of the given digits} \times 111\ldots1 \ ((r\text{-}1) \text{ times})\}$$

1. Find n if : (a) ${}^{n}P_{2} = 72$ (b) $2 \cdot {}^{n}P_{2} + 50 = {}^{2n}P_{2}$

2. Find the value of n such that

(a) $^{n}P_5 = 42\, ^{n}P_3, \quad n > 4$

(b) $\dfrac{^{n}P_4}{^{n-1}P_4} = \dfrac{5}{3}\ n > 4$

3. Find r, if $5\,{}^{4}P_{r} = 6\,{}^{5}P_{r-1},\ n > 4$

4. Find the number n of ways a judge can award first, second, and third places in a contest with 18 contestants.

5. A box contains 12 light bulbs. Find the number n of ordered samples of size 3:

(a) with replacement (b) without replacement.

6. A class contains 10 students. Find the number n of ordered samples of size 4:

(a) with replacement (b) without replacement.

7. How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

8. How many numbers lying between 100 and 1000 can be formed with the digits $0, 1, 2, 3, 4, 5$, if the repetition of the digits is not allowed?

9. In how many ways can 5 prizes be given away to 4 boys, when each boy is eligible for all the prizes?

10. How many different words can be formed of the letters of the word MALENKOV so that

a. no two vowels are together

b. The relative position of the vowels and consonants remains unaltered?

7.5. Combinations

Learning objectives:

* To introduce the concept of combinations.
* To derive a formula for the number of possible combinations of n elements taken r at a time AND
* To practice the related problems.

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of 3 could be selected from the 5 items A, B, C, D, E? We reason as follows.

Since there are 5 ways to select the initial item, 4 ways to then select the next item, and 3 ways to select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant. However, since every group of 3, say, the group consisting of items A, B, C, will be counted 6 times (that is, all of the permutations ABC, ACB, BAC, BCA, CAB, CBA will be counted when the order of selection is relevant), it follows that the total number of groups that can be formed is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

In general, as $n(n-1)\cdots(n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!(n-r!)}{r!}$$

Notation and terminology

We define $\binom{n}{r}$, for $r \le n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time. The notation ${}^{n}C_{r}$ is also used for the number of possible combinations of n objects taken r at a time. The two notations $\binom{n}{r}$ and ${}^{n}C_{r}$ are equivalent.

By convention, $0!$ is defined to be 1. Thus

$$\binom{n}{0} = \binom{n}{n} = 1$$

We also take

$$\binom{n}{i} = 0$$

when either $i < 0$ or $i > n$.

Thus $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the *order of selection is not considered relevant*.

The number of combinations of n things taken r at a time is equal to the number of n things taken $n-r$ at a time.

This is seen as follows. To each group of r things we select, there is left a corresponding group of $n-r$ things; that is, the number of combinations of n things r at a time is the same as the number of combinations of things $n-r$ at a time. Therefore,

$$\binom{n}{r} = \binom{n}{n-r}$$

If we put $r = n$, then $\binom{n}{0} = \binom{n}{n} = 1$

Example 1:

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution:

There are ${}^{20}C_3 = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$ possible committees.

Example 2:

From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Solution:

As there are $\binom{5}{2}$ possible groups of 2 women, and $\binom{7}{3}$ possible groups of 3 men, it follows from the basic principle that there are $\binom{5}{2}\binom{7}{3} = \left(\frac{5 \cdot 4}{2 \cdot 1}\right)\left(\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}\right) = 350$ possible committees consisting of 2 women and 3 men.

On the other hand, if 2 of the men refuse to serve on the committee together, then there are $\binom{2}{0}\binom{5}{3}$ possible groups of 3 men not containing either of the 2 feuding men and $\binom{2}{1}\binom{5}{2}$ groups of 3 men containing exactly 1 of the feuding men, it follows that there are $\binom{2}{0}\binom{5}{3} + \binom{2}{1}\binom{5}{2} = 30$ groups of 3 men not containing both of the feuding men. Since there are $\binom{5}{2}$ ways to choose the 2 women, it follows that in this case there are $30\binom{5}{2} = 300$ possible committees.

Example 3:

Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution:

Imagine that the $n-m$ functional antennas are lined up among themselves. Now, if no two defectives are to be consecutive, then the spaces between the functional antennas must each contain at most one defective antenna. That is, in the $n-m+1$ possible positions, represented by $\times$, we must select m of these in which to put the defective antennas.

$$\times 1 \times 1 \times \cdots \times 1 \times 1 \times \qquad \begin{aligned} 1 &= \text{functional} \\ \times &= \text{place for atmost one defective} \end{aligned}$$

Hence there are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones.

Example 4:

A hand of 13 playing cards is dealt from a well-shuffled pack of 52 (a) what is the number of distinct hands? (b) What is the number of hands containing two aces?

Solution:

The order of the cards in the hand is immaterial.

(a) The total number of distinct hands is equal to the number of combination of 13 objects drawn from 52:

The desired answer is $\binom{52}{13}$.

(b) The number of hands containing two aces is equal to the number of ways, $\binom{4}{2}$, in which the two aces can be drawn from the four available, multiplied by the number of ways, $\binom{48}{11}$, in which the remaining 11 cards in the hand can be drawn from the 48 cards that are not aces:

The desired answer is $\binom{4}{2} \times \binom{48}{11}$.

Example 5:

A class contains 10 students with 6 men and 4 women. Find the number n of ways:

a) A 4-member committee can be selected from the students.
b) A 4-member committee with 2 men and 2 women.
c) The class can elect a president, vice president, treasurer and secretary.

Solution:

(a) This concerns combinations, not permutations, since order does not count in a committee.

$$n = \binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

(b) The 2 men can be chosen from the 6 men in $\binom{6}{2}$ ways and the 2 women can be chosen from the 4 women in $\binom{4}{2}$ ways. Thus, by the product rule,

$$n = \binom{6}{2} \times \binom{4}{2} = \frac{6 \cdot 5}{2 \cdot 1} \times \frac{4 \cdot 3}{2 \cdot 1} = 90$$

(c) This concerns permutations, not combinations, since order does count. Thus,

$$n = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

Example 6:

A box contains 7 blue socks and 5 red socks. Find the number n of ways two socks can be drawn from the box if: (a) They can be any color; (b) They must be the same color.

Solution:

(a) There are "12 choose 2" ways to select 2 of the 12 socks.

$$n = \binom{12}{2} = \frac{12 \cdot 11}{2 \cdot 1} = 66$$

(b) There are $\binom{7}{2} = 21$ ways to choose 2 of the 7 blue socks and $\binom{5}{2} = 10$ ways to choose 2 of the 5 red socks. By the sum rule,

$$n = 21 + 10 = 31$$

IP1.

If ${}^{n}C_3 : {}^{(n-1)}C_4 = 8:5$ **then the value of** $\boldsymbol{n} =$

Solution:

We have ${}^{n}C_3 : {}^{(n-1)}C_4 = 8:5 \Rightarrow {}^{n}C_3 = 8\,{}^{(n-1)}C_4$

$$\Rightarrow 5\frac{n!}{(n-3)!\,3!} = 8\frac{(n-1)!}{(n-1-4)!\,4!}$$

$$\Rightarrow \frac{5n(n-1)!}{(n-3)(n-4)(n-5)!\,6} = \frac{8(n-1)!}{(n-5)!\,24}$$

$$\Rightarrow 5n = 2[(n-3)(n-4)]$$

$$\Rightarrow 2n^2 - 19n + 24 = 0$$

$$\Rightarrow (2n-3)(n-8) = 0$$

$$\Rightarrow 2n-3 = 0,\ n-8 = 0 \Rightarrow n = 8,\ \frac{3}{2}$$

$$\Rightarrow n = 8,\ \ (\because n \in \boldsymbol{N})$$

IP2.

Find the number of ways of selecting 11 member cricket team from 7 bats men, 6 bowlers and 2 wicket keepers so that the team contains atleast 4 bowlers and one wicket keeper?

Solution:

Bowlers	Wicket Keepers	Batsmen	Number of ways of selecting team
4	1	6	$6_{C_4} \times 2_{C_1} \times 7_{C_6} = 15 \times 2 \times 7 = 210$
5	1	5	$6_{C_5} \times 2_{C_1} \times 7_{C_5} = 6 \times 2 \times 21 = 252$
6	1	4	$6_{C_6} \times 2_{C_1} \times 7_{C_4} = 1 \times 2 \times 35 = 70$
4	2	5	$6_{C_4} \times 2_{C_2} \times 7_{C_5} = 15 \times 1 \times 21 = 315$
5	2	4	$6_{C_5} \times 2_{C_2} \times 7_{C_4} = 6 \times 1 \times 35 = 210$
6	2	3	$6_{C_6} \times 2_{C_2} \times 7_{C_3} = 1 \times 1 \times 35 = 35$

Therefore, the number of ways of selecting the required cricket team is

$$= 210 + 252 + 70 + 315 + 210 + 35 = 1092$$

IP3:

How many diagonals are there in a polygon with n sides?

Solution:

A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of a $n -$ sided polygon taken two at a time is

$= $ Number of ways of selecting 2 out of n

$$= {}^{n}C_{2} = \frac{n(n-1)}{2}$$

Out of these lines, n lines are the sides of the polygon.

$\therefore$ Number of diagonals of the polygon $= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

Application:

If a polygon has 44 diagonals then find the number sides of the polygon.

Solution:

We know that the number of diagonals of n sided polygon is

$\frac{n(n-3)}{2}$

By the hypothesis, we have $\frac{n(n-3)}{2} = 44$

$$\Rightarrow n^2 - 3n - 88 = 0$$

$$\Rightarrow (n - 11)(n + 8) = 0$$

$$\Rightarrow n = 11 \quad (\because n \in \boldsymbol{N})$$

Hence, there are 11 sides for the polygon with 44 diagonals.

IP4.

What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

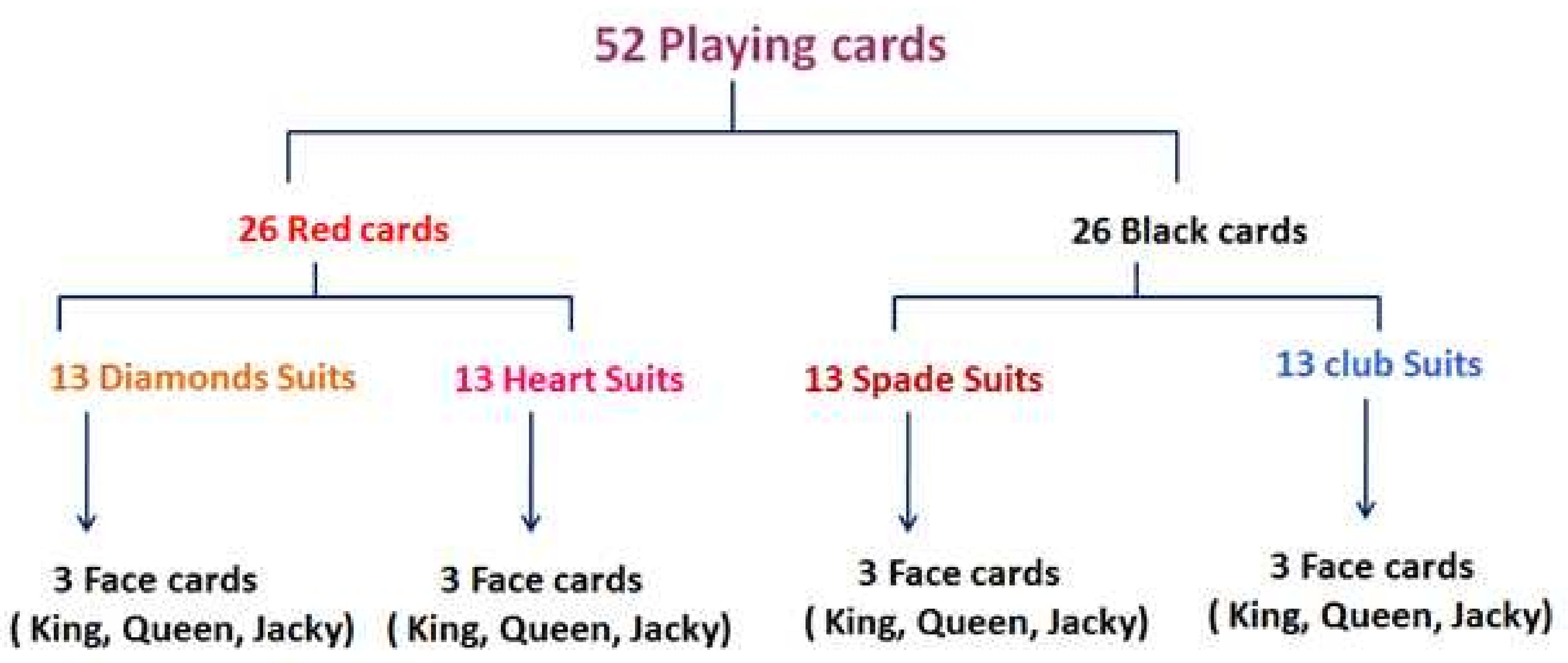
a) **four cards are of the same suit,**
b) **four cards belong to four different suits,**
c) **are face cards,**
d) **two are red cards and two are black cards,**
e) **Cards are of the same color?**

Solution:

The number of ways of choosing 4 cards from a pack of 52 playing cards is

$$= {}^{52}C_4 = \frac{52!}{4!(52-4)!} = \frac{52!}{4!\,48!} = \frac{52\times 51\times 50\times 49}{4\times 3\times 2\times 1} = 2,70,725$$

Observe the following diagram:



a) There are four types of suits namely diamond, club, spade, heart and there are 13 cards in each suit. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are ${}^{13}C_4$ ways of choosing 4 clubs, ${}^{13}C_4$ ways of choosing 4 spades and ${}^{13}C_4$ ways of choosing 4 hearts. Therefore, the required number of ways

$$= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 \qquad (\text{by sum rule})$$

$$= 4\times \frac{13!}{4!9!} = 2,860$$

b) There are13 cards in each suit.

Therefore, there are ${}^{13}C_1$ ways of choosing 1 card from 13 cards of diamond, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of spades.

Hence, by product rule, the required number of ways

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4 \qquad (\text{By product rule})$$

c) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in ${}^{12}C_4$ ways. Therefore, the required number of ways $= {}^{12}C_4 = \frac{12!}{4!\ 8!} = 495$

d) There are 26 red cards and 26 black cards. Therefore, the required number of ways

$$= {}^{26}C_2 \times {}^{26}C_2 = \left(\frac{26!}{2!\ 24!}\right)^2 = (325)^2 = 1,05,625$$

e) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ ways.

Therefore, the required number of ways

$$= {}^{26}C_4 + {}^{26}C_4 = 2\times \frac{26!}{4!\ 22!} = 29,900$$

P1.

If ${}^{n}P_{r} = 840$ **and** ${}^{n}C_{r} = 35$ **then the value of** $n =$

Solution:

We have $\dfrac{{}^nC_r}{{}^nP_r} = \dfrac{35}{840}$

$$\Rightarrow \frac{\dfrac{n!}{(n-r)!\,r!}}{\dfrac{n!}{(n-r)!}} = \frac{1}{24} \Rightarrow r! = 24 \Rightarrow r! = 4! \Rightarrow r = 4$$

$$\Rightarrow {}^nP_4 = 840 = 7 \times 6 \times 5 \times 4$$

$$\Rightarrow {}^nP_4 = {}^7P_4 \Rightarrow n = 7$$

P2.

A committee of 5 members is to be formed out of 6 gentlemen and 4 ladies. In how many ways that can be done when at least two ladies are included?

Solution:

a. Given that there are 6 gentlemen and 4 ladies. To select a committee of 5 members there are 3 possibilities included.

a) 3 gentlemen and 2 ladies

b) 2 gentlemen and 3 ladies

c) 1 gentleman and 4 ladies

Hence, the required number of ways (by product and sum rules) is

$$= {}^6C_3 \times {}^4C_2 + {}^6C_2 \times {}^4C_3 + {}^6C_1 \times {}^4C_4$$

$$= \left(\frac{6!}{3!\,3!} \times \frac{4!}{2!\,2!}\right) + \left(\frac{6!}{4!\,2!} \times \frac{4!}{1!\,3!}\right) + \left(\frac{6!}{5!\,1!} \times \frac{4!}{0!\,4!}\right)$$

$$= 120 + 60 + 6 = 186$$

b. A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways:

I. Selecting 5 gentlemen out of 6 (no ladies). This can be done in ${}^6C_5$ ways.

II. Selecting 4 gentlemen out of 6 and one lady out of 4. This can be done in ${}^6C_3 \times {}^4C_1$ ways.

III. Selecting 3 gentlemen out of 6 and two ladies out of 4. This can be done is ${}^6C_3 \times {}^4C_2$ ways.

Therefore, the total number of ways of forming the committee is

$${}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186$$

P3:

A box contains 5 red balls and 6 white balls. In how many ways can 6 balls be selected so that there are at least two balls of each color?

Solution:

The selection of 6 balls, consisting of at least two balls of each colour from 5 red and 6 white balls, can be made in the following ways:

I. By selecting 2 red balls out of 5 and 4 white balls out of 6. This can be done in ${}^5C_2 \times {}^6C_4$ ways.

II. By selecting 3 red balls out of 5 and 3 white balls out of 6. This can be done in ${}^5C_3 \times {}^6C_3$ ways.

III. By selecting 4 red balls out of 5 and 2 white balls out of 6. This can be done in ${}^5C_4 \times {}^6C_2$ ways.

By the sum rule, the total number of ways to select the 6 balls with at least two balls of each color is

$$= {}^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2$$

$$= 10 \times 15 + 10 \times 20 + 5 \times 15 = 425$$

P4.

A person invites 13 guests to a dinner party and arranging 8 of them at one circular table and the remaining 5 at the other. In how many ways can he arrange the guests?

Solution:

13 guests in the party, can be divided into two groups of 8 and 5 in ${}^{13}C_8$ *or* ${}^{13}C_5$, *i.e.*, $\frac{13!}{5!8!}$.

The first group of 8 guests can be arranged around at one table in $(8-1)! = 7!$ ways.

The second group of 5 guests can be arranged around at the other table in $(5-1)! = 4!$ ways.

Therefore, the number of arrangements is

$$= \frac{13!}{5!8!} \times 4! \times 7!$$

$$= \frac{13!}{5.4!8.7!} \times 4! \times 7! = \frac{13!}{40} = 15,56,75,520$$

1.

a. If ${}^{n}C_{9} = {}^{n}C_{8}$ *then* ${}^{n}C_{17} =$

b. Find the value of $\sum_{r=1}^{5} {}^{5}C_{r}$ (Ans: 31)

c. If ${}^{n}C_{r} : {}^{n}C_{r+1} = 1:2$ *and* ${}^{n}C_{r+1} : {}^{n}C_{r+2} = 2+3$

then find the values of n and r (Ans: $n = 14, r = 4$)

2. A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

3. A student is to answer 8 out of 10 questions on an exam.

   a. Find the number n of ways the student can choose the eight questions.

   b. Find n if the student must answer the first three questions.

4. Find the number n of committees of 5 with a given chairperson that can be selected from 12 persons.

5. From 12 books in how many ways can a selection of 5 be made,

   a. when one specified book is always included,

   b. when one specified book is always excluded?

6. From 7 Englishmen and 4 Americans a committee of 6 is to be formed; in how many ways can this be done,

   a. when the committee contains exactly 2 Americans,

   b. at least 2 Americans?

7. Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels?

8. A man has 7 friends. In how many ways can he invite one or more friends to a party? (Ans: 127)

9. A man has 7 relatives, 4 of them are ladies and 3 gentlemen, his wife has 7 relatives and 3 of them are ladies and 4 gentlemen. In how many ways can they invite them to a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives? (Ans: 485)

10. A question paper contains 12 questions, divided into three parts A, B and C. Part A contains 6 questions while B and C contain 3 questions each. A candidate is required to attempt 6 questions selecting atleast two from part A and atleast one from each of part B and part C. In how many ways can the candidate select 6 questions? (Ans: 720)