1.1

Plane Curves

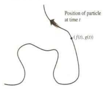
Learning Objectives:

- To define parametric equations of a plane curve.
- To discuss the parametrization of certain plane curves.

AND

- To practice related problems.

When the path of a particle moving in the plane looks like the curve in the figure below, it may not be possible to describe it with a Cartesian formula that expresses y directly in terms of x or x directly in terms of y.



Instead, we express each of the particle's coordinates as a function of time t and describe the path with a pair of equations, $x = f(t)$ and $y = g(t)$. For studying motion, equations like these are preferable to a Cartesian formula because they tell us the particle's position at any time t.

Definitions
If x and y are given as continuous functions

$$x = f(t) \text{ and } y = g(t)$$

over an interval of t values, then the set of points $(x, y) = \big(f(t), g(t)\big)$ defined by these equations is a **curve** in the coordinate plane. The equations are **parametric equations** for the curve.

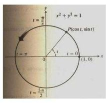
The variable t is a **parameter** for the curve and its domain I is the **parameter interval**. If I is a closed interval, $a \le t \le b$, the point $(f(a), g(a))$ is the **initial point** of the curve and $(f(b), g(b))$ is the **terminal point** of the curve. When we give parametric equations and a parameter interval for a curve in the plane, we say that we have **parametrized** the curve. The equations and interval constitute a **parametrization** of the curve.

In many applications t denotes time, but it might instead denote an angle or the distance a particle has traveled along its path from its starting point.

Example 1
The equations and parameter interval

$$x = \cos t \quad , \quad y = \sin t \qquad 0 \le t \le 2\pi$$

describe the position $P(x, y)$ of a particle that moves counterclockwise around the circle $x^2 + y^2 = 1$ as t increases.



We know that the point lies on this circle for every value of t because

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

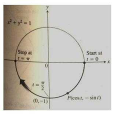
We track the motion as t runs from 0 to 2π. The parameter t is in the *radian measure* of the angle that radius OP makes with positive x-axis. The particle starts at $(1,0)$, moves up and to the left as t approaches $\pi/2$, and continues around the circle to stop again at $(1,0)$ when $t = 2\pi$. The particle traces the circle exactly once.

Example 2
The equations and parameter interval

$$x = \cos t , \qquad y = -\sin t \qquad\qquad 0 \le t \le \pi$$

describe the position $P(x, y)$ of a particle that moves clockwise around the circle $x^2 + y^2 = 1$ as t increases from 0 to π.

We know that the point P lies on this circle for all t because its coordinates satisfy the circle's equation. We track the motion as t runs from 0 to π. The particle starts at $(1,0)$. But now as t increases, y becomes negative, decreasing to -1 when $t = \pi/2$ and then increasing back to 0 as t approaches π. The motion stops at $t = \pi$ with only the lower half of the circle covered.



Example 3
The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = \sqrt{t} \quad , \quad y = t \qquad\qquad t \ge 0$$

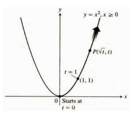
Identify the path traced by the particle and describe the motion.

Solution
We eliminate t between the equations $x = \sqrt{t}$ and $y = t$. This gives

$$y = t = \left(\sqrt{t}\right)^2 = x^2$$

This means that the particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along the parabola $y = x^2$.

The particle's x-coordinate is never negative. The particle starts at $(0,0)$ when $t = 0$ and rises into the first quadrant as t increases.



Example 4
The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = t \quad , \quad y = t^2 \qquad\qquad -\infty < t < \infty$$

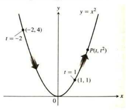
Identify the particle's path and describe the motion.

Solution
We eliminate t between the equations $x = t$ and $y = t^2$ to obtain

$$y = t^2 = x^2$$

The particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along this curve.

In contrast to the previous example, the particle now traverses the entire parabola. As t increases from $-\infty$ to ∞, the particle comes down the left-hand side, passes through the origin, and moves up the right-hand side.



As example 4 illustrates, any curve $y = f(x)$ has the parametrization $x = t, y = f(t)$. In general, such a parametrization may not be useful all the time.

Parametrizing a Line Segment:

Example5
Find a parametrization for the line segment with endpoints (-2,1) and (3, 5).
Solution:
Using (-2,1) we create the parametric equations

$$x = -2 + at \ , \ y = 1 + bt \, .$$

These represent a line, as we can see by solving each equation for t and equating to obtain

$$\frac{x+2}{a} = \frac{y-1}{b} \, .$$

This line goes through the point (-2,1) when $t = 0$. We determine a and b so that the line goes through (3, 5) when $t = 1$.

$$3 = -2 + a \Rightarrow a = 5 \qquad [x = 3 \textit{ when } t = 1]$$
$$5 = 1 + b \Rightarrow b = 4 \qquad [y = 5 \textit{ when } t = 1]$$

Therefore,

$$x = -2 + 5t \ , \ y = 1 + 4t \ , \ 0 \le t \le 1$$

is a parametrization of the line segment with initial point (-2,1) and terminal point (3, 5).

IP1:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = 2t - 5 \quad , \qquad y = 4t - 7 \qquad -\infty < t < \infty$$

Solution:

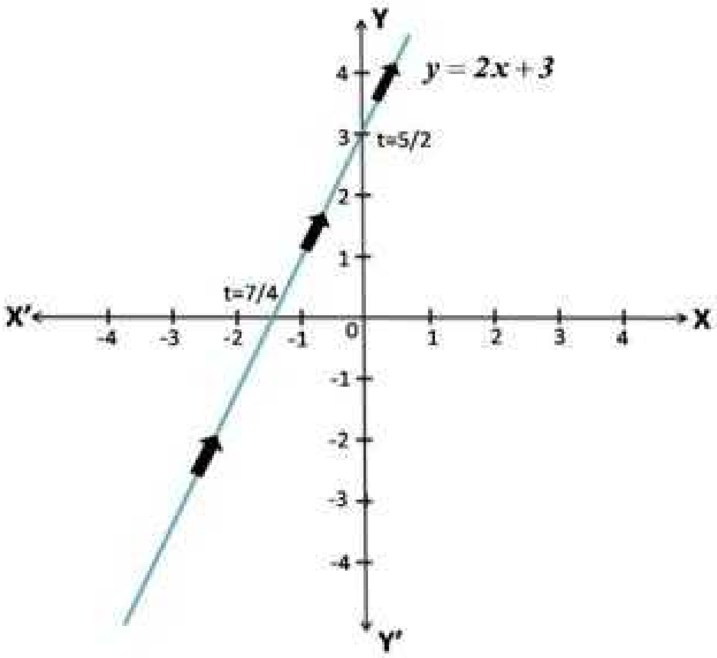
We have, $x = 2t - 5$, $\quad y = 4t - 7 \qquad -\infty < t < \infty$

$$\Rightarrow y = 4t - 7 = 2(2t - 5) + 3$$

$$\Rightarrow y = 2x + 3$$

The particle's position coordinates satisfy the equation $y = 2x + 3$, so the particle moves along this straight line.

The particle traverses the entire straight line as t increases from $-\infty$ to ∞ .



IP2:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = \cos 2t \quad , \quad y = \sin 2t \qquad 0 \le t \le \pi$$

Solution:

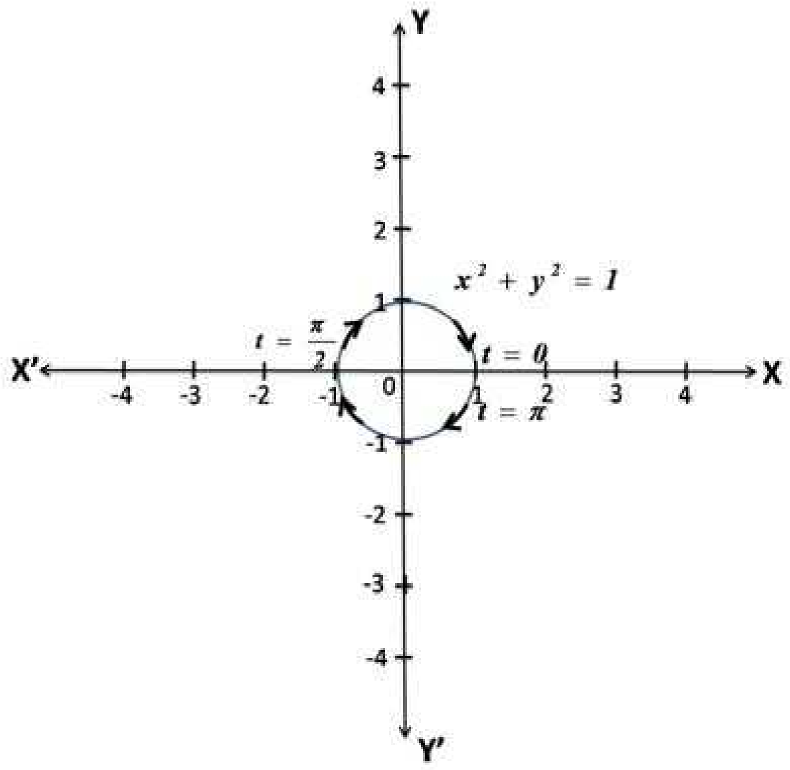
We have, $x = \cos 2t \quad , \quad y = \sin 2t \qquad 0 \le t \le \pi$

$$\Rightarrow x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1$$

$$\Rightarrow x^2 + y^2 = 1$$

The position $P(x, y)$ of a particle moves counterclockwise around the circle $x^2 + y^2 = 1$ as t increases from 0 to π.

We know that the point P lies on this circle for all t because its coordinates satisfy the circle's equation. The particle starts at $(1,0)$, moves up and to the left as t approaches $\pi/2$, and continues around the circle to stop again at $(1,0)$ when $t = \pi$. The particle traces the circle exactly once.



IP3:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = 3t \quad , \quad y = 9t^2 \quad , \quad -\infty < t < \infty$$

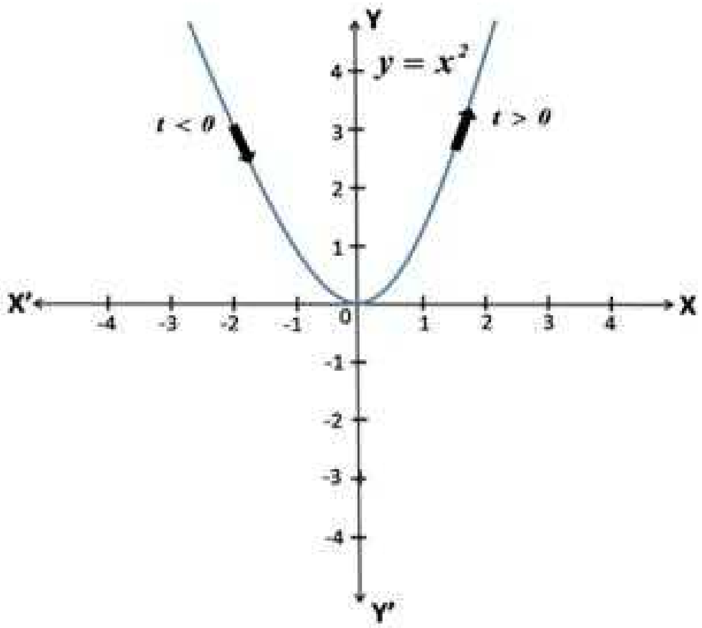
Solution:

We have, $\quad x = 3t \quad , \quad y = 9t^2 \quad , \quad -\infty < t < \infty$

$$\Rightarrow y = 9t^2 = (3t)^2 = x^2$$

The particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along this curve.

The particle traverses the entire parabola. As t increases from $-\infty$ to ∞, the particle comes down the left-hand side, passes through the origin, and moves up the right-hand side.



IP4:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = \sqrt{1-t^2} \quad , \qquad y = t \quad , \qquad -1 \le t \le 0$$

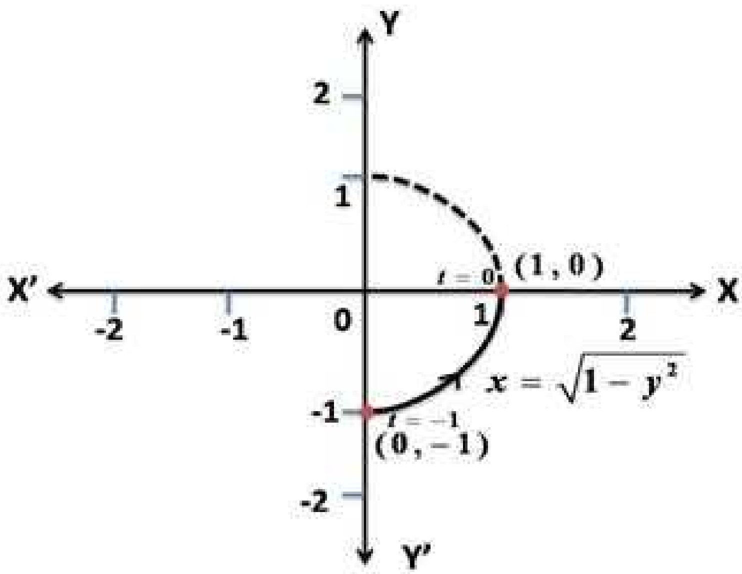
Solution:

We have, $x = \sqrt{1-t^2}\,, \qquad y = t \quad , \qquad -1 \le t \le 0$

$$\Rightarrow x = \sqrt{1-t^2} \Rightarrow x = \sqrt{1-y^2}\,; y \le 0$$

The particle's position coordinates satisfy the equation $x = \sqrt{1-y^2}$, so the particle moves along this curve.

The particle's y-coordinate is never positive. The particle starts at $(0,-1)$ when $t = -1$ and rises up as t increases. The particle stops at $(1,0)$ when $t = 0$.



P1:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = 3 - 3t \quad , \quad y = 2t \qquad 0 \le t \le 1$$

Solution:

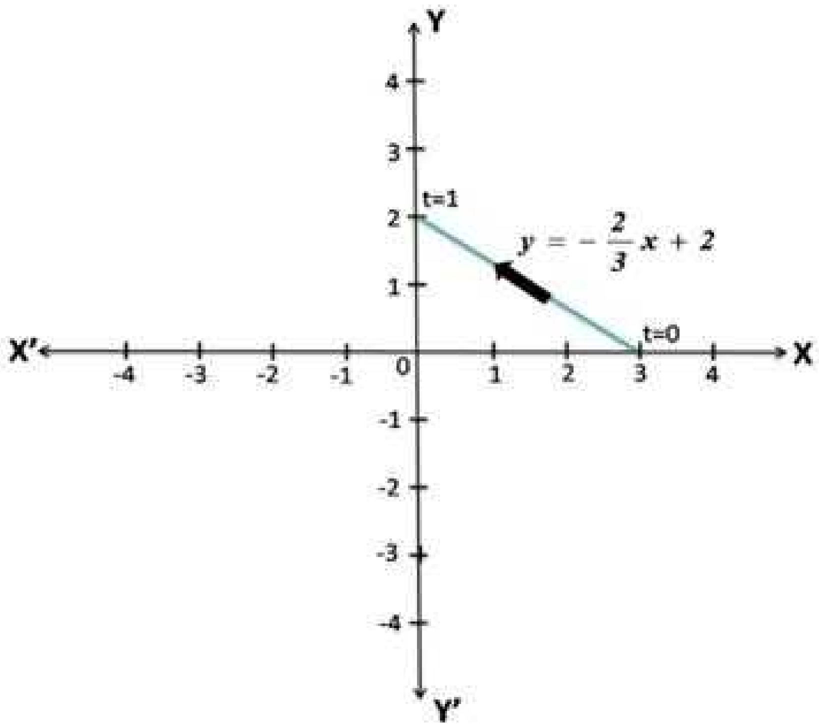
We have, $x = 3 - 3t$, $y = 2t$ $0 \le t \le 1$

$$\Rightarrow y = 2t = 2\left(\frac{3-x}{3}\right) = 2 - \frac{2}{3}x$$

$$\Rightarrow y = 2 - \tfrac{2}{3}x \,;\, 0 \le x \le 3$$

The particle's position coordinates satisfy the equation $y = 2 - \frac{2}{3}x$, so the particle moves along this line segment.

We track the motion as t runs from 0 to 1. The particle starts at $(3,0)$ when $t = 0$. As t approaches 1, the particle stops at $(0,2)$.



P2:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = \cos(\pi - t), \qquad y = \sin(\pi - t) \qquad\qquad 0 \le t \le \pi$$

Solution:

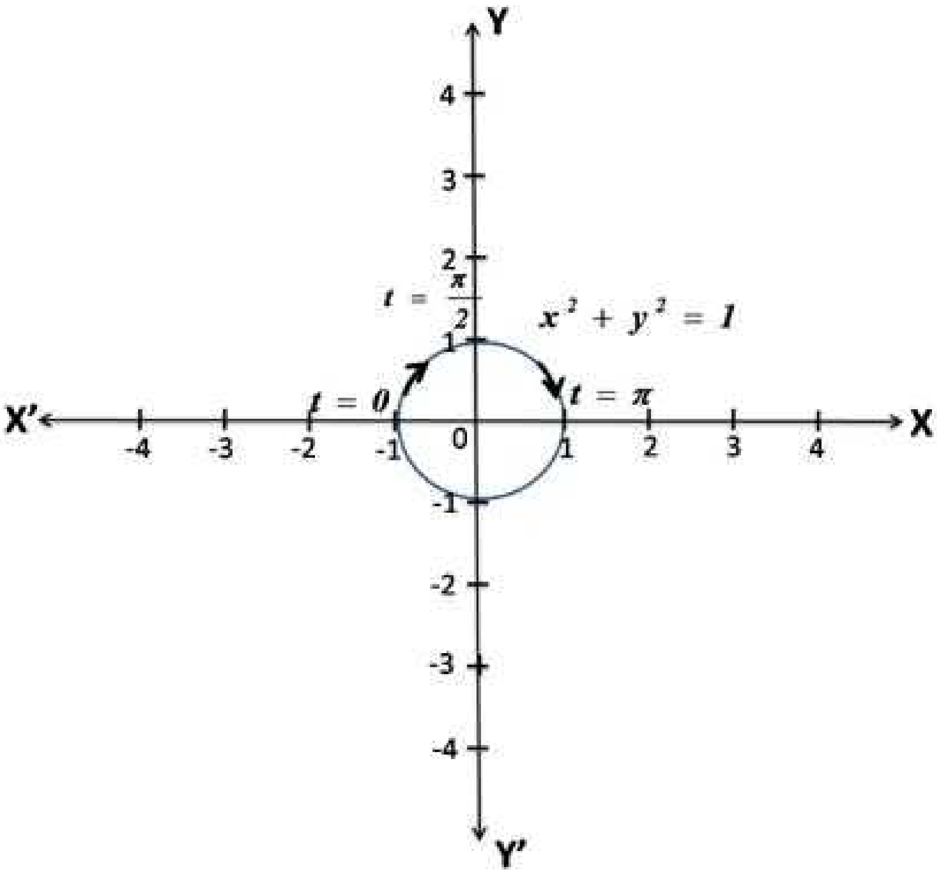
We have, $x = \cos(\pi - t), \quad y = \sin(\pi - t) \quad 0 \le t \le \pi$

$$\Rightarrow x^2 + y^2 = \cos^2(\pi - t) + \sin^2(\pi - t) = 1$$

$$\Rightarrow x^2 + y^2 = 1 ; y \ge 0$$

The position $P(x, y)$ of a particle moves clockwise around the circle $x^2 + y^2 = 1$ as t increases from 0 to π.

We know that the point P lies on this circle for all t because its coordinates satisfy the circle's equation. We track the motion as t runs from 0 to π. The particle starts at $(-1,0)$. But now as t increases, y becomes positive, increasing to 1 when $t = \pi/2$ and then decreasing back to 0 as t approaches π. The motion stops at $t = \pi$ with only the upper half of the circle covered.



P3:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = -\sqrt{t}\ , \qquad y = t \qquad\qquad t \geq 0$$

Solution:

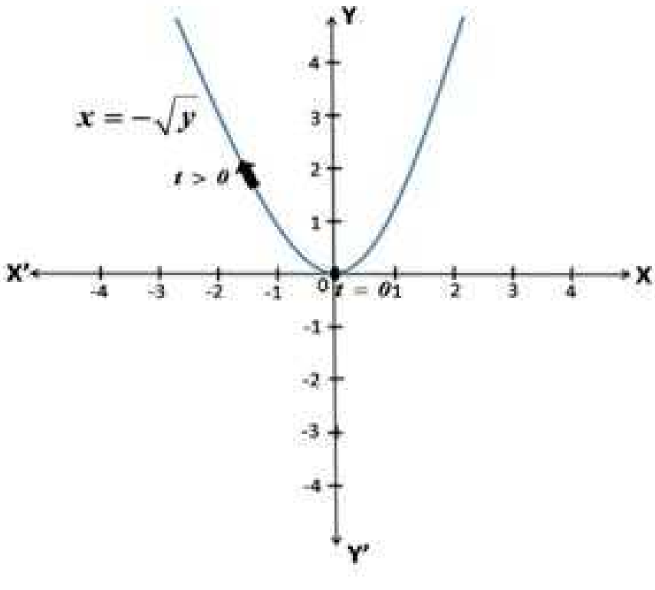
We have, $x = -\sqrt{t}$, $y = t$ $t \geq 0$

$$\Rightarrow x = -\sqrt{t} = -\sqrt{y}$$

$$\Rightarrow y = x^2 \,;\, x \leq 0$$

The particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along this curve.

The particle's x-coordinate is never positive. The particle starts at $(0,0)$ when $t = 0$ and rises into the second quadrant as t increases.



P4:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = t \quad , \quad y = \sqrt{1 - t^2} \; , \quad -1 \le t \le 0$$

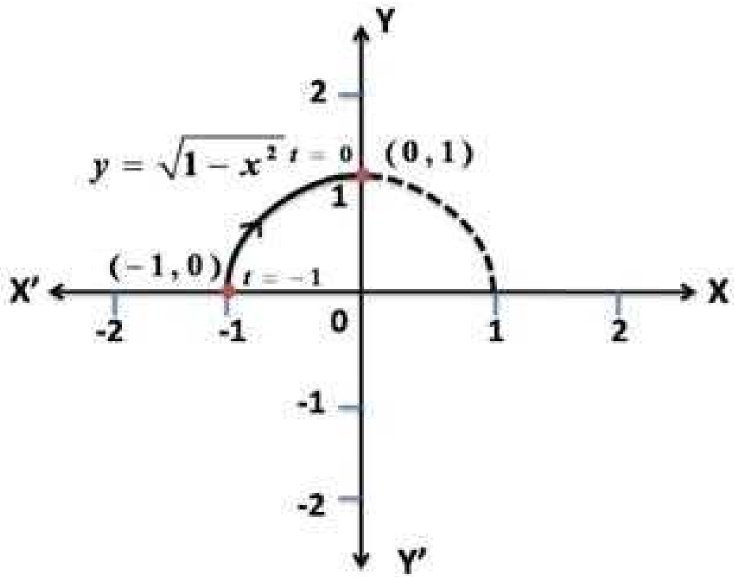
Solution:

We have, $x = t$, $y = \sqrt{1 - t^2}$, $-1 \leq t \leq 0$

$$\Rightarrow y = \sqrt{1 - t^2} \Rightarrow y = \sqrt{1 - x^2}\,; x \leq 0$$

The particle's position coordinates satisfy the equation $y = \sqrt{1 - x^2}$, so the particle moves along this curve.

The particle's x-coordinate is never positive. The particle starts at $(-1,0)$ when $t = -1$ and rises up as t increases. The particle stops at $(0,1)$ when $t = 0$.



1. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$

   a. Once clockwise.

   b. Once counterclockwise.

   c. Twice clockwise.

   d. Twice counterclockwise.

2. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

a. Once clockwise.

b. Once counterclockwise.

c. Twice clockwise.

d. Twice counterclockwise.

3. Find a parametrization for the curve.

a. The line segment with endpoints $(-1,-3)$ and $(4,1)$

b. The line segment with endpoints $(-1,3)$ and $(3,-2)$

c. The lower half of the parabola $x-1=y^2$

d. The left half of the parabola $y=x^2+2x$

e. The ray (half line) with initial point $(2,3)$ that passes through the point

1.2

Parametrization of Plane Curves

Learning Objectives:

- To present parametric equations of an ellipse, circle and hyperbola.
- To derive parametric equations of a cycloid.

AND

- To practice related problems.

In this module we give additional examples of parametrizing conic sections and cycloids.

Example 1
Describe the motion of a particle whose position $P(x, y)$ at time t is given by

$$x = a\cos t \quad , \quad y = b\sin t \quad , \quad 0 \le t \le 2\pi$$

Solution
We eliminate t from the given equations. We write

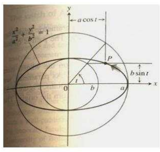
$$\cos t = \frac{x}{a} \quad , \quad \sin t = \frac{y}{b}$$

Therefore,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t = 1$$

The particle's coordinates (x, y) satisfy the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so the particle moves along the ellipse.

When $t = 0$, the particle's coordinates are $(a\cos 0, b\sin 0) = (a, 0)$ and so the motion starts at $(a, 0)$. As t increases, the particle rises and moves toward the left, moving counterclockwise. It traverses the ellipse once, returning to its starting position $(a, 0)$ at time $t = 2\pi$.



Example 2
The equations and parameter interval

$$x = a\cos t \quad , \quad y = a\sin t \quad , \quad 0 \le t \le 2\pi \, ,$$

obtained by taking $b = a$ in example 1, describe the circle

$$x^2 + y^2 = a^2$$

Example 3
Describe the motion of the particle whose position $P(x, y)$ at time t is given by

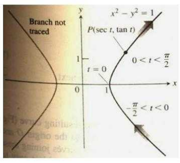
$$x = \sec t \quad , \quad y = \tan t \quad , \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Solution
We find a Cartesian equation for the coordinates of P by eliminating t between the equations

$$\sec t = x \quad , \quad \tan t = y$$

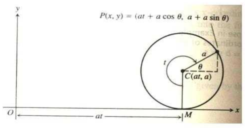
$$x^2 - y^2 = \sec^2 t - \tan^2 t = 1$$

Since the particle's coordinates (x, y) satisfy the equation $x^2 - y^2 = 1$, the motion takes place somewhere on this hyperbola. As t runs between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, $x = \sec t$ remains positive and $y = \tan t$ runs between $-\infty$ and ∞, so P traverses the hyperbola's *right-hand branch*. It comes in along the branch's lower half as $t \longrightarrow 0^-$, reaches $(1,0)$ at $t = 0$, and moves out into the first quadrant as t increases toward $\pi/2$.



Example 4
A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid.**

Solution
We take the line to be the x-axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel to the right. As parameter, we use the angle t through which the wheel turns, measured in *radians*. Figure below shows the wheel a short while later, when its base lies at units from the origin.



The wheel's center C lies at (at, a) and the coordinates of P are

$$x = at + a\cos\theta \qquad y = a + a\sin\theta$$

We observe that $t + \theta = 3\pi/2$, so that $\theta = \frac{3\pi}{2} - t$

This makes

$$\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t$$

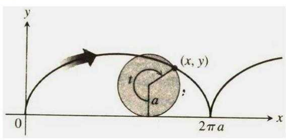
$$\sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t$$

$$x = at - a\sin t \qquad y = a - a\cos t$$

With the a factored out, they are

$$x = a(t - \sin t) \qquad y = a(1 - \cos t)$$

Figure below shows the first arch of the cycloid and part of the next.



The parametric equations of one arch of the cycloid are
$\boldsymbol{x = a(t - \sin t) \quad , \quad y = a(1 - \cos t) \quad , \quad 0 \le t \le 2\pi}$

IP1:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = 4\sin t \quad , \quad y = 5\cos t \quad , \quad 0 \le t \le 2\pi$$

Solution:

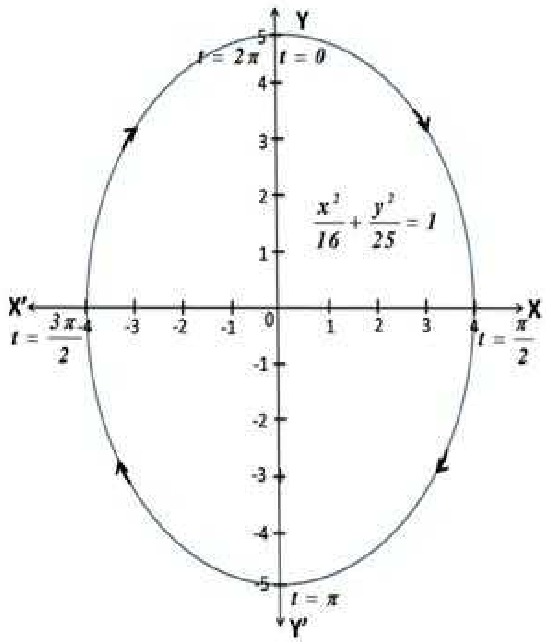
We have, $x = 4\sin t \quad , \quad y = 5\cos t \quad , \quad 0 \le t \le 2\pi$

$$\therefore \left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \sin^2 t + \cos^2 t = 1$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{25} = 1$$

The position $P(x, y)$ of a particle moves clockwise around the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ as t increases from 0 to 2π.

We know that the point P lies on this ellipse for all t because its coordinates satisfy the ellipse's equation. The particle starts at $(0,5)$, moves down and to the right as t approaches $\pi/2$ when the particle is at $(4,0)$, and continues down as t approaches π when the particle is at $(0,-5)$ and continues around the ellipse to stop again at $(0,5)$ when $t = 2\pi$. The particle traces the ellipse exactly once.



IP2:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = t \quad , \quad y = \sqrt{1+t^2} \quad , \quad t \leq 0$$

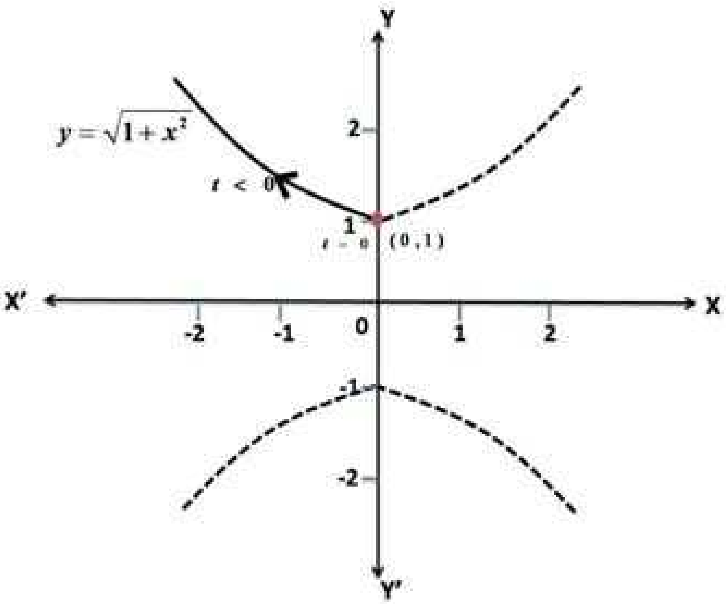
Solution:

We have, $x = t \quad , \quad y = \sqrt{1+t^2} \quad , \quad t \leq 0$

$$\Rightarrow y = \sqrt{1+t^2} \Rightarrow y = \sqrt{1+x^2}\,;\, x \leq 0$$

The particle's position coordinates satisfy the equation $y = \sqrt{1+x^2}$, so the particle moves along this curve and this is a hyperbola.

The particle's x-coordinate is never positive. The particle starts at $(0,1)$ when $t = 0$ and rises into the second quadrant as t decreases.



IP3:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = -\sec t \quad , \quad y = \tan t \quad , \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

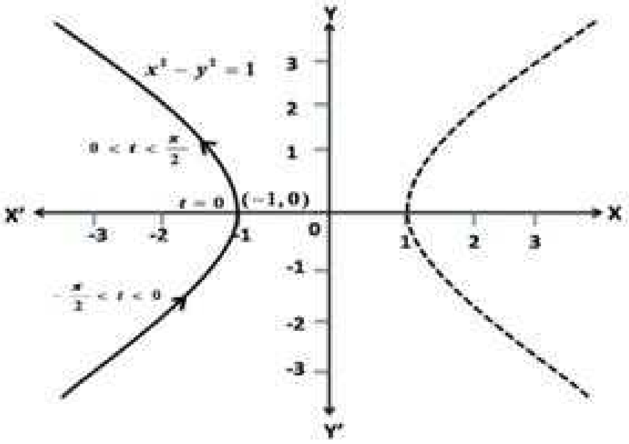
Solution:

We have, $x = -\sec t \quad , \quad y = \tan t \quad , \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$$\because \sec^2 t - \tan^2 t = 1 \Rightarrow x^2 - y^2 = 1; x < 0$$

The particle's position coordinates satisfy the equation $x^2 - y^2 = 1$, so the particle moves along this curve and the curve is a rectangular hyperbola.

The particle's x-coordinate is never positive. The particle traverses the left branch of hyperbola as t increases between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. When the t is in between $-\frac{\pi}{2}$ to 0, then the particle traverses up in the quadrant-III and when t is in between 0 to $\frac{\pi}{2}$, the particle traverses up in the quadrant-II.

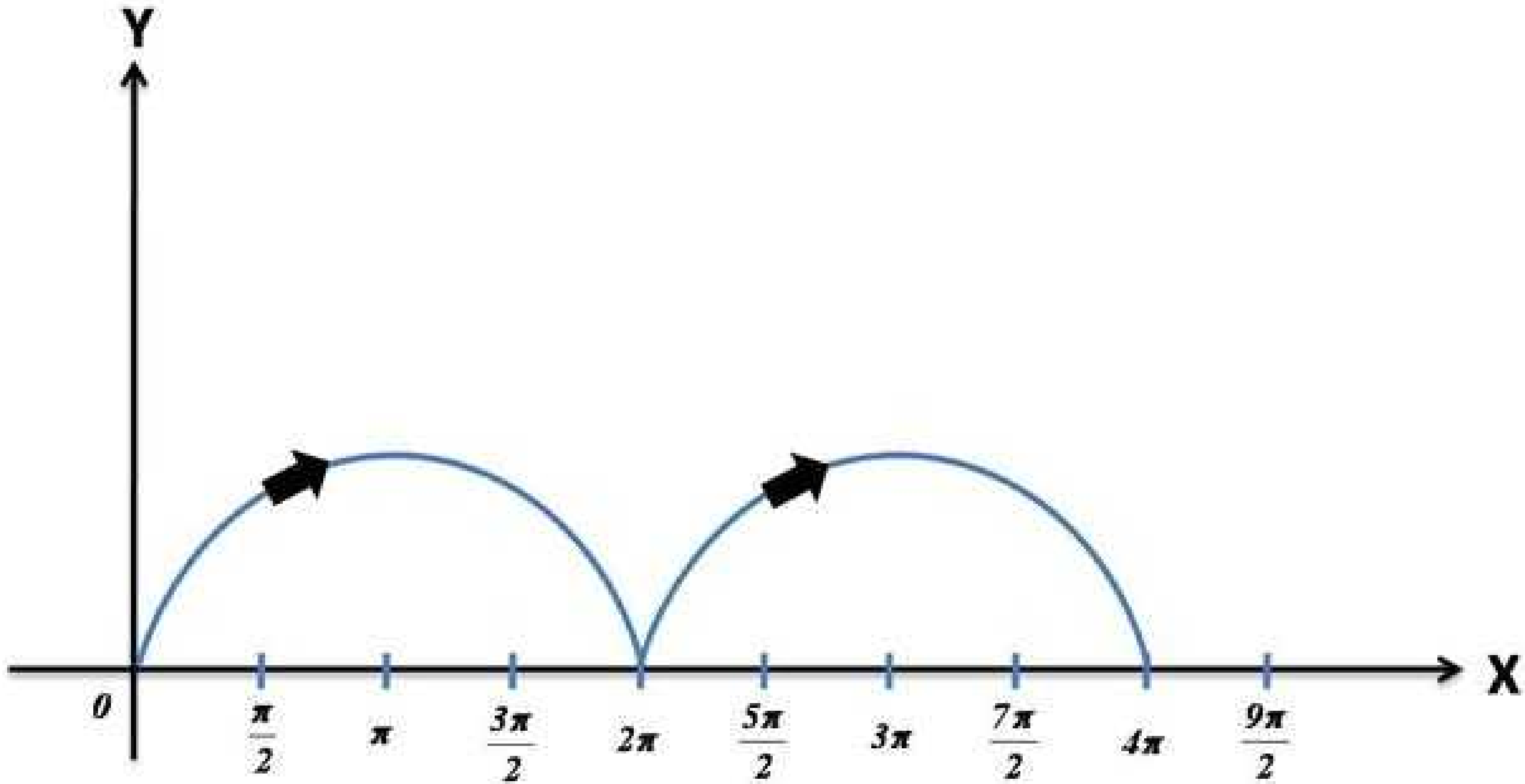


IP4:

Graph the part of the cycloid using the parametric equations $x = t - \sin t$, $y = 1 - \cos t$ and in the interval $0 \le t \le 4\pi$.

Solution:

Graph:



P1:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = 4\cos t \quad , \quad y = 2\sin t \quad , \quad 0 \le t \le 2\pi$$

Solution:

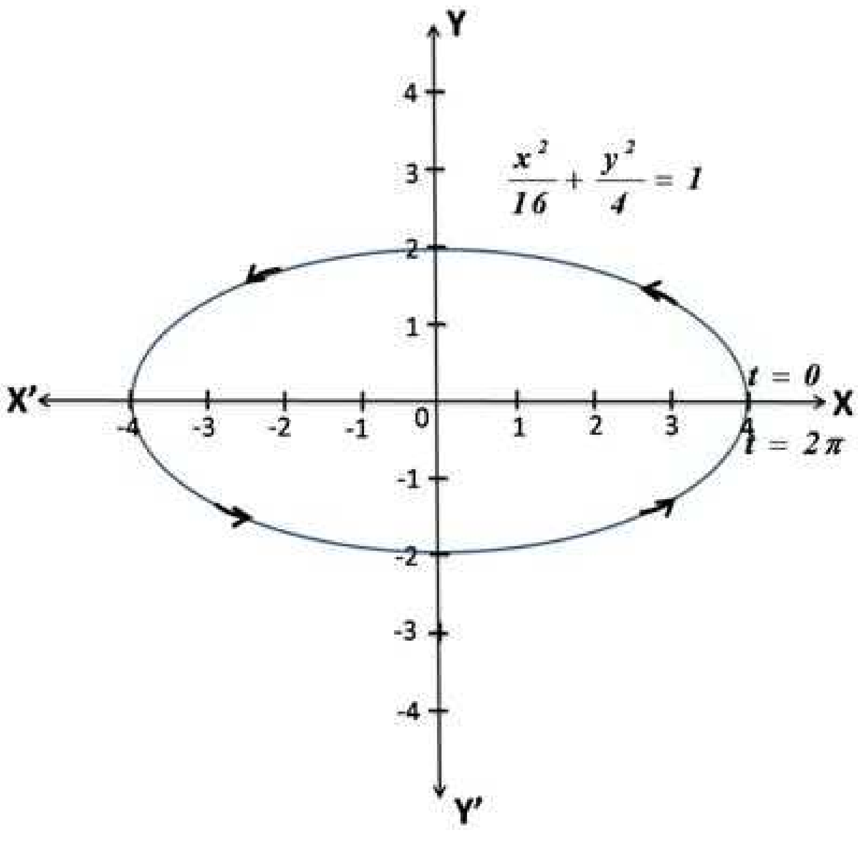
We have, $x = 4\cos t \quad , \quad y = 2\sin t \quad , \quad 0 \leq t \leq 2\pi$

$$\therefore \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$

The position $P(x, y)$ of a particle moves counterclockwise around the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ as t increases from 0 to 2π.

We know that the point P lies on this ellipse for all t because its coordinates satisfy the ellipse's equation. The particle starts at $(4,0)$, moves up and to the left as t approaches $\pi/2$ when the particle is at $(0,2)$, and continues down as t approaches π when the particle is at $(-4,0)$ and continues around the ellipse to stop again at $(4,0)$ when $t = 2\pi$. The particle traces the ellipse exactly once.



P2:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = \sqrt{t+1} \quad , \quad y = \sqrt{t} \quad , \quad t \geq 0$$

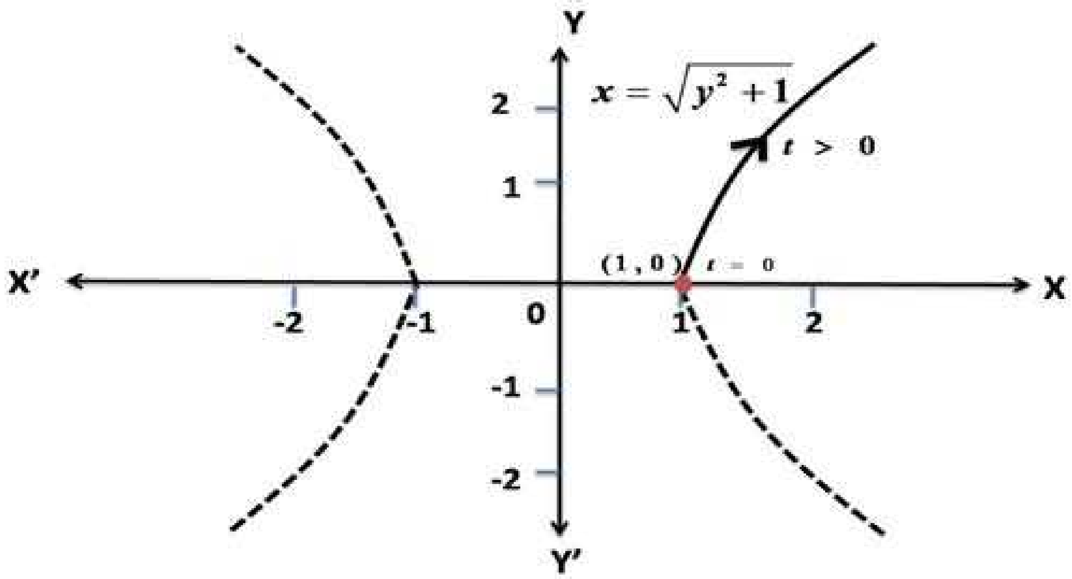
Solution:

We have, $x = \sqrt{t+1}$, $y = \sqrt{t}$, $t \geq 0$

$$\therefore y^2 = t, \text{ so } x = \sqrt{y^2+1}\,; y \geq 0$$

The particle's position coordinates satisfy the equation $x = \sqrt{y^2+1}$, so the particle moves along this curve and this is a hyperbola.

The particle's x-coordinate is never negative. The particle starts at $(1,0)$ when $t = 0$ and rises into the first quadrant as t increases.



P3:

Given parametric equations and parameter interval for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it.

$$x = \tan t \quad , \quad y = -\sec t \quad , \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

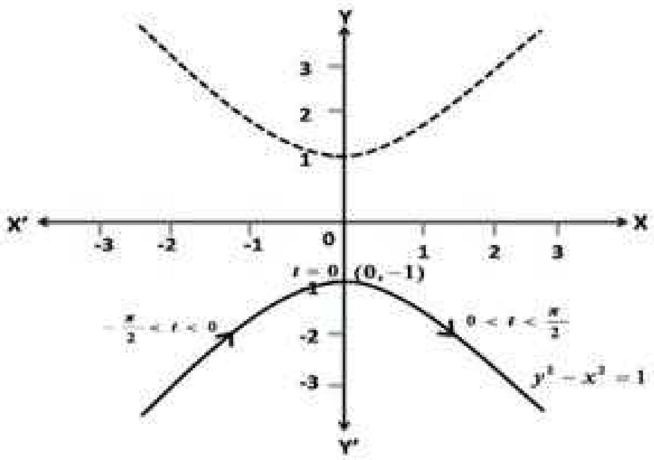
Solution:

We have, $x = \tan t \quad , \qquad y = -\sec t \quad , \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$$\because \sec^2 t - \tan^2 t = 1 \Rightarrow y^2 - x^2 = 1;\ y < 0$$

The particle's position coordinates satisfy the equation $y^2 - x^2 = 1$, so the particle moves along this curve and this curve is a rectangular hyperbola.

The particle's y-coordinate is never positive. The particle traverses the down branch of hyperbola as t increases between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. When t is in between $-\frac{\pi}{2}$ to 0, then the particle traverses up in the quadrant-III and when t is in between 0 to $\frac{\pi}{2}$ the particle traverses down in the quadrant-IV.

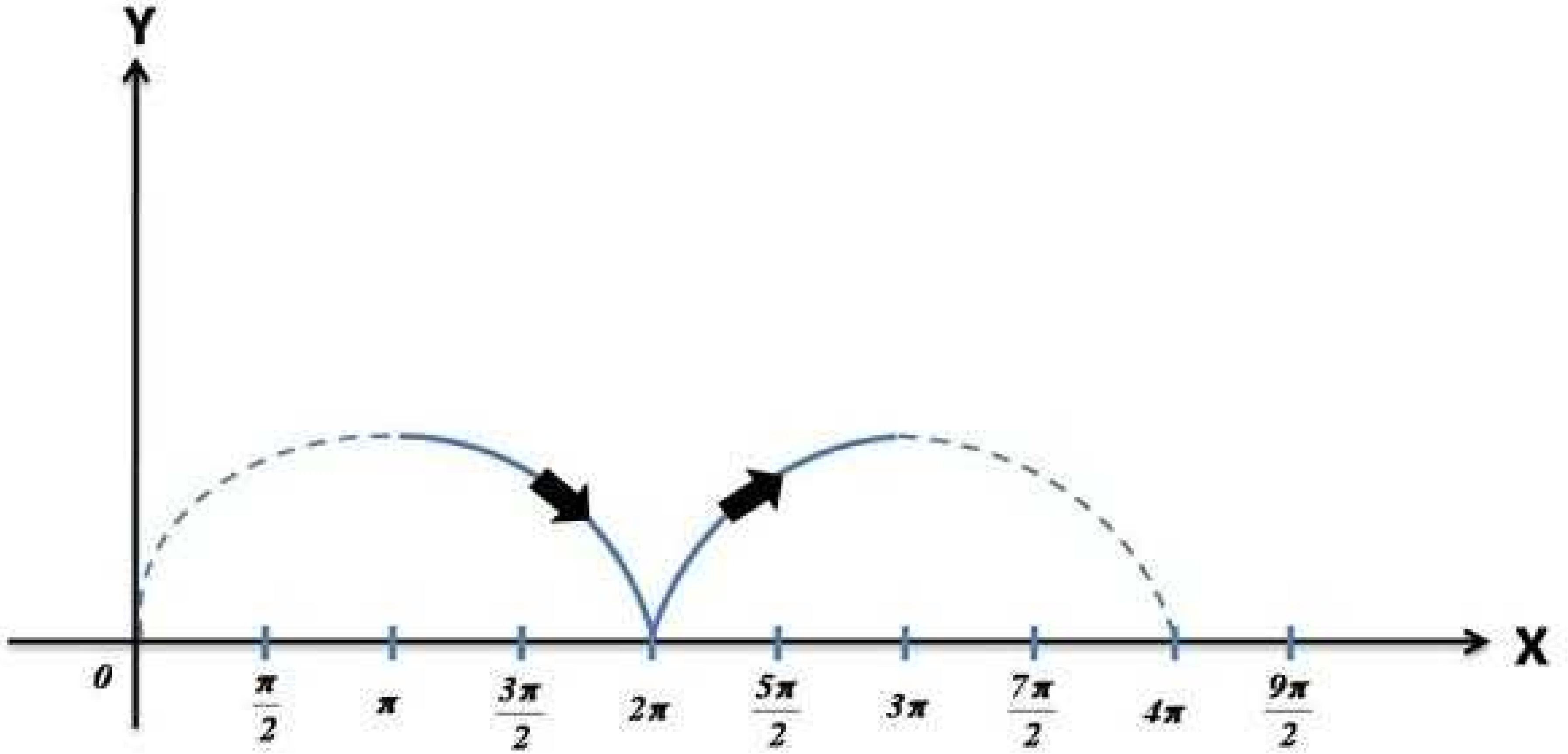


P4:

Graph the part of the cycloid using the parametric equations $x = t - \sin t$, $y = 1 - \cos t$ and in the interval $\pi \le t \le 3\pi$.

Solution:

Graph:



1.3

Differentiation with Parametrized Curves

Learning Objectives:

- To find slopes associated with parametrized curves.
- To find second derivative of parametrized curves.

In this module, we learn how to find slopes associated with parametrized curves.

Slopes of Parametrized Curves

Definitions

A parametrized curve $x = f(t), y = g(t)$ is **differentiable** at $t = t_0$ if f and g are differentiable at $t = t_0$. The curve is **differentiable** if it is differentiable at every parameter value. The curve is **smooth** if f' and g' are continuous and not simultaneously zero.

At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives $\frac{dx}{dt}, \frac{dy}{dt}$, and $\frac{dy}{dx}$ are related by the Chain Rule equation

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, we may divide both sides of this equation by $\frac{dx}{dt}$ to solve for $\frac{dy}{dx}$.

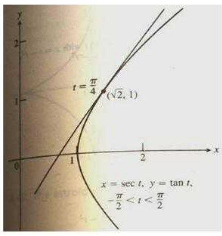
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{-- -- -- --} \quad (1)$$

Example1

Find the tangent to the right-hand branch of the hyperbola

$$x = \sec t \quad , \quad y = \tan t \quad , \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$.



Solution

The slope of the curve at t is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}$$

Setting t equal to $\pi/4$ gives

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{\sec(\pi/4)}{\tan(\pi/4)} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

The point-slope equation of the tangent is $y - y_0 = m(x - x_0)$
$\Rightarrow y - 1 = \sqrt{2}(x - \sqrt{2}) \Rightarrow y = \sqrt{2}x - 2 + 1 \Rightarrow y = \sqrt{2}x - 1$

The Parametric Formula for d^2y/dx^2

If the parametric equations for a curve define y as a twice-differentiable function of x, we may calculate $\frac{d^2y}{dx^2}$ as a function of t in the following way:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Thus the formula for finding $\frac{d^2y}{dx^2}$ from $y' = \frac{dy}{dx}$ and $\frac{dx}{dt}\left(\frac{dx}{dt} \neq 0\right)$ is

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} \quad \text{--- --- ---} \quad (2)$$

Example 2

Find $\frac{d^2y}{dx^2}$ if $x = t - t^2$ and $y = t - t^3$.

Solution

Step 1: Express y' in terms of t:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{1-2t}$$

Step 2: Differentiate y' with respect to t:

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{2-6t+6t^2}{(1-2t)^2}$$

Step 3: Divide $\frac{dy'}{dt}$ by $\frac{dx}{dt}$.

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{2-6t+6t^2}{(1-2t)^2} \cdot \frac{1}{1-2t} = \frac{2-6t+6t^2}{(1-2t)^3}$.

IP1:

Find an equation of the tangent to the curve $x = 2\cos t$, $y = 2\sin t$ at the point defined by the value of $t = \frac{\pi}{4}$ and also find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

Solution:

Finding tangent to the curve $x = 2\cos t$, $y = 2\sin t$:

The point from where the tangent of the curve is passing through at $t = \frac{\pi}{4}$ is $(x, y) = \left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right) = (\sqrt{2}, \sqrt{2})$.

Now, $x = 2\cos t \Rightarrow \frac{dx}{dt} = -2\sin t$,

$$y = 2\sin t \Rightarrow \frac{dy}{dt} = 2\cos t\ .$$

The slope of the tangent at $t = \frac{\pi}{4}$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-2\sin t} = -\cot t = -\cot\frac{\pi}{4} = -1\ .$$

The equation of the tangent which has slope -1 and is passing through the point $(\sqrt{2}, \sqrt{2})$ is

$$y - \sqrt{2} = -1(x - \sqrt{2}) \Rightarrow x + y = 2\sqrt{2}$$

Finding $\frac{d^2y}{dx^2}$:

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$, where $y' = \frac{dy}{dx}$.

We have, $y' = \frac{dy}{dx} = -\cot t \Rightarrow \frac{dy'}{dt} = \csc^2 t$

Therefore, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\csc^2 t}{-2\sin t} = -\frac{1}{2\sin^3 t}$.

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\frac{1}{2\sin^3\frac{\pi}{4}} = -\frac{2\sqrt{2}}{2} = -\sqrt{2}\ .$$

IP2:

Find an equation of the tangent to the curve $\boldsymbol{x = t,\ y = \sqrt{t}}$ at the point defined by the value of $\boldsymbol{t = \frac{1}{4}}$ and also find the value of $\boldsymbol{\frac{d^2y}{dx^2}}$ at $\boldsymbol{t = \frac{1}{4}}$.

Solution:

Finding tangent to the curve $\boldsymbol{x = t,\ y = \sqrt{t}}$:

The point from where the tangent of the curve is passing through at $t = \frac{1}{4}$ is $(x, y) = \left(\frac{1}{4}, \sqrt{\frac{1}{4}}\right) = \left(\frac{1}{4}, \frac{1}{2}\right)$.

Now, $x = t \Rightarrow \frac{dx}{dt} = 1$,

$$y = \sqrt{t} \Rightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{t}}.$$

The slope of the tangent at $t = \frac{1}{4}$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{2\sqrt{t}}}{1} = \frac{1}{2\sqrt{\frac{1}{4}}} = 1.$$

The equation of the tangent which has slope 1 and is passing through the point $\left(\frac{1}{4}, \frac{1}{2}\right)$ is

$$y - \frac{1}{2} = 1\left(x - \frac{1}{4}\right) \Rightarrow x - y + \frac{1}{4} = 0$$

Finding $\boldsymbol{\frac{d^2y}{dx^2}}$:

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$, where $y' = \frac{dy}{dx}$.

We have, $y' = \frac{dy}{dx} = \frac{1}{2\sqrt{t}} = \frac{1}{2}t^{-\frac{1}{2}} \Rightarrow \frac{dy'}{dt} = -\frac{1}{4}t^{-\frac{3}{2}}$

Therefore, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-\frac{1}{4}t^{-\frac{3}{2}}}{1} = -\frac{1}{4t^{\frac{3}{2}}}$.

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = -\frac{1}{4\left(\frac{1}{4}\right)^{\frac{3}{2}}} = -\frac{1}{4\left(\frac{1}{8}\right)} = -2.$$

IP3:

Find an equation of the tangent to the curve $\boldsymbol{x = 2t^2 + 3,\ y = t^4}$ at the point defined by the value of $\boldsymbol{t = -1}$ and also find the value of $\boldsymbol{\frac{d^2y}{dx^2}}$ at $\boldsymbol{t = -1}$.

Solution:

Finding tangent to the curve $\boldsymbol{x = 2t^2 + 3,\ y = t^4}$:

The point from where the tangent of the curve is passing through at $t = -1$ is $(x, y) = (2(-1)^2 + 3, (-1)^4) = (5,1)$.

Now, $x = 2t^2 + 3 \Rightarrow \frac{dx}{dt} = 4t$,

$$y = t^4 \Rightarrow \frac{dy}{dt} = 4t^3.$$

The slope of the tangent at $t = -1$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3}{4t} = t^2 = (-1)^2 = 1.$$

The equation of the tangent which has slope 1 and is passing through the point $(5,1)$ is

$$y - 1 = 1(x - 5) \Rightarrow x - y = 4$$

Finding $\boldsymbol{\frac{d^2y}{dx^2}}$:

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$, where $y' = \frac{dy}{dx}$.

We have, $y' = \frac{dy}{dx} = t^2 \Rightarrow \frac{dy'}{dt} = 2t$

Therefore, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{2t}{4t} = \frac{1}{2}$.

IP4:

Find an equation of the tangent to the curve $x = t - \sin t,\ y = 1 - \cos t$ at the point defined by the value of $t = \frac{\pi}{3}$ and also find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$.

Solution:

Finding tangent to the curve $x = t - \sin t,\ y = 1 - \cos t$:

The point from where the tangent of the curve is passing through at $t = \frac{\pi}{3}$ is $(x, y) = \left(\frac{\pi}{3} - \sin\frac{\pi}{3}, 1 - \cos\frac{\pi}{3}\right)$

$$= \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Now, $x = t - \sin t \Rightarrow \frac{dx}{dt} = 1 - \cos t$,

$$y = 1 - \cos t \Rightarrow \frac{dy}{dt} = \sin t\ .$$

The slope of the tangent at $t = \frac{\pi}{3}$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1-\cos t} = \frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}\ .$$

The equation of the tangent which has slope $\sqrt{3}$ and is passing through the point $\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is

$$y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) \Rightarrow \sqrt{3}x - y = \frac{\pi}{\sqrt{3}} - 2$$

Finding $\frac{d^2y}{dx^2}$:

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$, where $y' = \frac{dy}{dx}$.

We have, $y' = \frac{dy}{dx} = \frac{\sin t}{1-\cos t}$

$$\Rightarrow \frac{dy'}{dt} = \frac{\frac{d}{dt}(\sin t)(1-\cos t) - \sin t\frac{d}{dt}(1-\cos t)}{[1-\cos t]^2}$$

$$\Rightarrow \frac{dy'}{dt} = \frac{\cos t(1-\cos t) - \sin t(\sin t)}{[1-\cos t]^2} = \frac{1}{\cos t - 1}$$

Therefore, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{\cos t - 1}}{1-\cos t} = -\frac{1}{(1-\cos t)^2}$.

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = -\frac{1}{\left(1-\cos\frac{\pi}{3}\right)^2} = -\frac{1}{\left(\frac{1}{2}\right)^2} = -4\ .$$

P1:

Find an equation of the tangent to the curve

$x = \cos t$, $y = \sqrt{3}\cos t$ at the point defined by the value of

$t = \frac{2\pi}{3}$ and also find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{2\pi}{3}$.

Solution:

Finding tangent to the curve $x = \cos t,\ y = \sqrt{3}\cos t$:

The point from where the tangent of the curve is passing through at $t = \frac{2\pi}{3}$ is

$$(x, y) = \left(\cos\frac{2\pi}{3}, \sqrt{3}\cos\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Now, $x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$,

$$y = \sqrt{3}\cos t \Rightarrow \frac{dy}{dt} = -\sqrt{3}\sin t\,.$$

The slope of the tangent at $t = \frac{2\pi}{3}$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}\,.$$

The equation of the tangent which has slope $\sqrt{3}$ and is passing through the point $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is

$$y + \frac{\sqrt{3}}{2} = \sqrt{3}\left(x + \frac{1}{2}\right) \Rightarrow y = \sqrt{3}x$$

Finding $\frac{d^2y}{dx^2}$:

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$, where $y' = \frac{dy}{dx}$.

We have, $y' = \frac{dy}{dx} = \sqrt{3} \Rightarrow \frac{dy'}{dt} = 0.$

Therefore, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{0}{-\sin t} = 0\,.$

P2:

Find an equation of the tangent to the curve $x = -\sqrt{t+1},\ y = \sqrt{3t}$ at the point defined by the value of $t = 3$ and also find the value of $\frac{d^2y}{dx^2}$ at $t = 3$.

Solution:

Finding tangent to the curve $x = -\sqrt{t+1}$, $y = \sqrt{3t}$:

The point from where the tangent of the curve is passing through at $t = 3$ is $(x, y) = \left(-\sqrt{4}, \sqrt{9}\right) = (-2,3)$.

Now, $x = -\sqrt{t+1} \Rightarrow \frac{dx}{dt} = -\frac{1}{2\sqrt{t+1}}$,

$$y = \sqrt{3t} \Rightarrow \frac{dy}{dt} = \frac{\sqrt{3}}{2\sqrt{t}}.$$

The slope of the tangent at $t = 3$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{\sqrt{3}}{2\sqrt{t}}}{-\frac{1}{2\sqrt{t+1}}} = -\sqrt{3\left(1+\frac{1}{t}\right)} = -\sqrt{3\left(1+\frac{1}{3}\right)} = -2.$$

The equation of the tangent which has slope -2 and is passing through the point $(-2,3)$ is

$$y - 3 = -2(x+2) \Rightarrow 2x + y + 1 = 0$$

Finding $\frac{d^2y}{dx^2}$:

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$, where $y' = \frac{dy}{dx}$.

We have, $y' = \frac{dy}{dx} = -\sqrt{3\left(1+\frac{1}{t}\right)} \Rightarrow \frac{dy'}{dt} = -\frac{\sqrt{3}}{2\sqrt{\left(1+\frac{1}{t}\right)}}\frac{d}{dt}\left(1+\frac{1}{t}\right)$

$$\Rightarrow \frac{dy'}{dt} = -\frac{\sqrt{3}}{2\sqrt{\left(1+\frac{1}{t}\right)}}\left(-\frac{1}{t^2}\right) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{t}}{\sqrt{t+1}}\cdot\frac{1}{t^2} = \frac{\sqrt{3}}{2}\cdot\frac{t^{-\frac{3}{2}}}{\sqrt{t+1}}$$

Therefore, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{\sqrt{3}}{2}\cdot\frac{t^{-\frac{3}{2}}}{\sqrt{t+1}}}{-\frac{1}{2\sqrt{t+1}}} = -\sqrt{3}t^{-\frac{3}{2}}$.

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=3} = -3^{\frac{1}{2}}\cdot 3^{-\frac{3}{2}} = -3^{-1} = -\frac{1}{3}.$$

P3:

Find an equation of the tangent to the curve
$x = 2t^3,\ y = t^2 - t + 1$ at the point defined by the value of
$t = 1$ and also find the value of $\frac{d^2y}{dx^2}$ at $t = 1$.

Solution:

Finding tangent to the curve $x = 2t^3,\ y = t^2 - t + 1$:

The point from where the tangent of the curve is passing through at $t = 1$ is $(x, y) = (2(1)^3, (1)^2 - 1 + 1) = (2,1)$.

Now, $x = 2t^3 \Rightarrow \frac{dx}{dt} = 6t^2$,

$$y = t^2 - t + 1 \Rightarrow \frac{dy}{dt} = 2t - 1\,.$$

The slope of the tangent at $t = 1$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-1}{6t^2} = \frac{1}{6}.$$

The equation of the tangent which has slope $\frac{1}{6}$ and is passing through the point $(2,1)$ is

$$y - 1 = \frac{1}{6}(x - 2) \Rightarrow x - 6y + 4 = 0$$

Finding $\frac{d^2y}{dx^2}$:

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$, where $y' = \frac{dy}{dx}$.

We have, $y' = \frac{dy}{dx} = \frac{2t-1}{6t^2} \Rightarrow \frac{dy'}{dt} = \frac{\frac{d}{dt}(2t-1)6t^2 - (2t-1)\frac{d}{dt}(6t^2)}{[6t^2]^2}$

$$\Rightarrow \frac{dy'}{dt} = \frac{2\times 6t^2 - (2t-1)(12t)}{36t^4} = \frac{-12t^2 + 12t}{36t^4} = \frac{-t+1}{3t^3}$$

Therefore, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{-t+1}{3t^3}}{6t^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1} = 0\,.$

P4:

Find an equation of the tangent to the curve $x = \sec^2 t - 1,\ y = \tan t$ at the point defined by the value of $t = -\frac{\pi}{4}$ and also find the value of $\frac{d^2y}{dx^2}$ at $t = -\frac{\pi}{4}$.

Solution:

Finding tangent to the curve $\boldsymbol{x = \sec^2 t - 1,\ y = \tan t}$:

The point from where the tangent of the curve is passing through at $t = -\frac{\pi}{4}$ is $(x, y) = \left(\sec^2\left(-\frac{\pi}{4}\right) - 1, \tan\left(-\frac{\pi}{4}\right)\right)$

$$= (1, -1).$$

Now, $x = \sec^2 t - 1 \Rightarrow \frac{dx}{dt} = 2\sec^2 t \tan t$,

$$y = \tan t \Rightarrow \frac{dy}{dt} = \sec^2 t\ .$$

The slope of the tangent at $t = -\frac{\pi}{4}$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{2\sec^2 t \tan t} = \frac{1}{2}\cot t = \frac{1}{2}\cot\left(-\frac{\pi}{4}\right) = -\frac{1}{2}\,.$$

The equation of the tangent which has slope $-\frac{1}{2}$ and is passing through the point $(1, -1)$ is

$$y + 1 = -\frac{1}{2}(x - 1) \Rightarrow x + 2y + 1 = 0$$

Finding $\boldsymbol{\frac{d^2y}{dx^2}}$:

We have, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$, where $y' = \frac{dy}{dx}$.

We have, $y' = \frac{dy}{dx} = \frac{1}{2}\cot t \Rightarrow \frac{dy'}{dt} = \frac{1}{2}(-\csc^2 t) = -\frac{1}{2}\csc^2 t$

Therefore, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-\frac{1}{2}\csc^2 t}{2\sec^2 t \tan t} = -\frac{1}{4}\cot^3 t$.

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}} = -\frac{1}{4}\cot^3\left(-\frac{\pi}{4}\right) = \frac{1}{4}\cot^3\left(\frac{\pi}{4}\right) = \frac{1}{4}\,.$$

1. Find the equation for the tangents to the curves at the point defined by the given value of t. Also, find the value of $\frac{d^2y}{dx^2}$ at this point.

a. $x = \cos t \quad , \quad y = 1 + \sin t \quad , \quad t = \frac{\pi}{2}$

b. $x = \cos 2t \quad , \quad y = \sin 2t \quad , \quad t = \frac{3\pi}{4}$

c. $x = 4\cos t \quad , \quad y = 2\sin t \quad , \quad t = \frac{3\pi}{2}$

d. $x = -\sqrt{t} \quad , \quad y = t \quad , \quad t = 2$

e. $x = t \quad , \quad y = \sqrt{1 - t^2} \quad , \quad t = -1$

f. $x = \sqrt{t + 1} \quad , \quad y = \sqrt{t} \quad , \quad t = 0$

g. $x = \sec^2 t - 1 \quad , \quad y = \tan t \quad , \quad t = \frac{\pi}{3}$

h. $x = -\sec t \quad , \quad y = \tan t \quad , \quad t = \frac{\pi}{6}$

1.4

Integration with Parametrized Curves

Learning Objectives:

- To find lengths and areas of surface revolution of parametrized curves.

AND

- To practice related problems.

In this module, we learn how to find lengths and surface areas associated with parametrized curves.

Lengths of Parametrized Curves

We find an integral for the length of a smooth curve $x = f(t), y = g(t), a \le t \le b$, by rewriting the integral $L = \int ds$ in the following way:

$$L = \int_{t=a}^{t=b} ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\left(\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2}\right) dt^2}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The only requirement besides the continuity of the integrand is that the point $P(x, y) = P(f(t), g(t))$ not trace any portion of the curve more than once as t moves from a to b.

Length

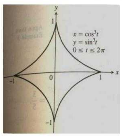
If a smooth curve $x = f(t), y = g(t), a \le t \le b$, is traversed exactly once as t increases from a to b, the curve's length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{--- --- -- (1)}$$

Example 1

Find the length of the asteroid

$$x = \cos^3 t \quad , \quad y = \sin^3 t \quad , \quad 0 \le t \le 2\pi$$



Solution

Because of symmetry, the length is four times the length of the first-quadrant portion. We have

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t\,(-\sin t)]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t\,(\cos t)]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)}$$

$$= \sqrt{9\cos^2 t \sin^2 t}$$

$$= 3|\cos t \sin t| = 3\cos t \sin t$$

$$[\because \cos t \sin t \ge 0 \; for \; 0 \le t \le \tfrac{\pi}{2}]$$

Therefore,

$$\text{Length of the first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t\, dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t\, dt = -\frac{3}{4}\cos 2t\Big]_0^{\pi/2} = \frac{3}{2}$$

The length of the asteroid is four times this: $4\left(\frac{3}{2}\right) = 6$.

Centroid

Example 2

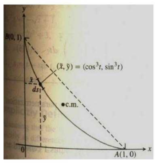
Find the centroid of the first-quadrant arc of the asteroid in Example 1.

Solution

We take the curve's density to be $\delta = 1$ and calculate the curve's mass and moments about the coordinate axes.

The distribution of mass is symmetric about the line $y = x$, so $\bar{x} = \bar{y}$.

A typical segment of the curve



has mass

$$dm = 1.\,ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 3\cos t \sin t\, dt$$

The curve's mass is

$$M = \int_0^{\pi/2} dm = \int_0^{\pi/2} 3\cos t \sin t\, dt = \frac{3}{2}$$

The curve's moment about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} \sin^3 t \cdot 3\cos t \sin t\, dt$$

$$= 3\int_0^{\pi/2} \sin^4 t \cos t\, dt = 3 \cdot \frac{\sin^5 t}{5}\Big]_0^{\pi/2} = \frac{3}{5}$$

Hence $\quad \bar{y} = \frac{M_x}{M} = \frac{3/5}{3/2} = \frac{2}{5}$

The centroid is the point $\left(\frac{2}{5}, \frac{2}{5}\right)$.

Area of a Surface of Revolution

For smooth parametrized curves, the length formula in equation (1) leads to the following formulas for surfaces of revolution. The derivations are similar to the derivations of the Cartesian formulas.

Surface Area

If a smooth curve $x = f(t), y = g(t), a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \ge 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

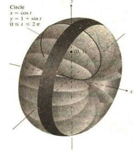
As with length, we can calculate surface area from any convenient parametrization that meets the stated criteria.

Example 3

The standard parametrization of the circle of radius 1 centered at the point $(0,1)$ in the xy-plane is

$$x = \cos t \quad , \quad y = 1 + \sin t \quad , \quad 0 \le t \le 2\pi$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis.



We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt = 2\pi[t - \cos t]_0^{2\pi} = 4\pi^2$$

IP1:

Find the length of the curve

$$x = t^3 \quad , \; y = \frac{3t^2}{2} \quad , \; 0 \le t \le \sqrt{3}\,.$$

Solution:

We have, $x = t^3 \quad , \; y = \frac{3t^2}{2} \quad , \; 0 \le t \le \sqrt{3}$

$$\Rightarrow \frac{dx}{dt} = 3t^2 \;\; , \;\; \frac{dy}{dt} = 3t$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2)^2 + (3t)^2} = 3t\sqrt{t^2+1}$$

Therefore, the length of the curve, $L = \int_0^{\sqrt{3}} ds\,.\,dt$

$$\Rightarrow L = \int_0^{\sqrt{3}} \left(3t\sqrt{t^2+1}\right).\,dt = \int_0^{\sqrt{3}} \sqrt{t^2+1}\,3t.\,dt$$

Let $u = t^2 + 1 \Rightarrow du = 2t.\,dt \Rightarrow \frac{3}{2}du = 3t.\,dt$

When $t = 0 \Rightarrow u = 1$ and $t = \sqrt{3} \Rightarrow u = 4.$

$$\therefore L = \int_1^4 \sqrt{u}\,.\frac{3}{2}\,du = \left[u^{\frac{3}{2}}\right]_1^4 = 8 - 1 = 7$$

IP2:

Find the length of the curve

$$x = \cos t \quad , \; y = t + \sin t \quad , \; 0 \le t \le \pi \,.$$

Solution:

We have, $x = \cos t \quad , \; y = t + \sin t \quad , \; 0 \le t \le \pi$

$$\Rightarrow \frac{dx}{dt} = -\sin t \;, \;\; \frac{dy}{dt} = 1 + \cos t$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1+\cos t)^2}$$

$$= \sqrt{\sin^2 t + 1 + \cos^2 t + 2\cos t}$$

$$= \sqrt{2(1+\cos t)}$$

Therefore, the length of the curve, $L = \int_0^{\pi} \sqrt{2(1+\cos t)}\,.dt$

$$\Rightarrow L = \sqrt{2}\int_0^{\pi} \sqrt{(1+\cos t)}\,.dt = \sqrt{2}\int_0^{\pi} \sqrt{\frac{(1+\cos t)(1-\cos t)}{(1-\cos t)}}\,.dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{1-\cos^2 t}{(1-\cos t)}}\,.dt = \sqrt{2}\int_0^{\pi} \sqrt{\frac{\sin^2 t}{(1-\cos t)}}\,.dt$$

$$= \sqrt{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}}\,.dt$$

Let, $1 - \cos t = u \Rightarrow \sin t \,.\, dt = du$

When $t = 0 \Rightarrow u = 0$ and $t = \pi \Rightarrow u = 2$

$$\therefore L = \sqrt{2}\int_0^{2} \frac{du}{\sqrt{u}} = \sqrt{2}\int_0^{2} u^{-\frac{1}{2}}\,.du$$

$$= 2\sqrt{2}\left[u^{\frac{1}{2}}\right]_0^2 = 2\sqrt{2}(\sqrt{2}) = 4$$

IP3:

Find the centroid of the arc of the curve in the given interval.

$$x = \cos t \qquad , \qquad y = \sin t \qquad , \qquad 0 \le t \le \frac{\pi}{2}$$

Solution:

We have, $x = \cos t \quad , \quad y = \sin t \quad , \quad 0 \le t \le \frac{\pi}{2}$

$$\Rightarrow \frac{dx}{dt} = -\sin t \qquad , \qquad \frac{dy}{dt} = \cos t$$

We take the curve's density to be $\delta = 1$ and calculate the curve's mass and moments about the coordinate axes.

The distribution of mass is symmetric about the line $y = x$, so $\bar{x} = \bar{y}$.

A typical segment of the curve has mass

$$dm = 1.ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = dt$$

The curve's mass is

$$M = \int_0^{\pi/2} dm = \int_0^{\pi/2} dt = [t]_0^{\pi/2} = \frac{\pi}{2}$$

The curve's moment about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} \sin t\, dt$$

$$= [-\cos t]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1$$

Hence, $\bar{y} = \frac{M_x}{M} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

Therefore, the centroid is $\left(\frac{2}{\pi}, \frac{2}{\pi}\right)$.

IP4:

Find the area of the surface generated by revolving the curve $x = at^2 \quad , \quad y = 2at \quad , \; 0 \le t \le 1$ about the x-axis.

Solution:

We have, $x = at^2 \quad , \quad y = 2at \quad , \; 0 \le t \le 1$

$$\Rightarrow \frac{dx}{dt} = 2at \quad , \quad \frac{dy}{dt} = 2a$$

The area of the surface generated by revolving the curve about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 2\pi(2at)\sqrt{(2at)^2 + (2a)^2}\,dt$$

$$= 4\pi a^2 \int_0^1 2t\sqrt{t^2+1}\,dt$$

Let $t^2 + 1 = u \Rightarrow 2t.dt = du$

When $t = 0 \Rightarrow u = 1$ and $t = 1 \Rightarrow u = 2$

$$\Rightarrow S = 4\pi a^2 \int_1^2 \sqrt{u}\,du = 4\pi a^2 \cdot \frac{2}{3} \cdot \left[u^{\frac{3}{2}}\right]_1^2$$

$$= \frac{8}{3}\pi a^2 [2\sqrt{2} - 1]$$

P1:

Find the length of the curve

$$x = \frac{t^2}{2} \quad , \; y = \frac{(2t+1)^{3/2}}{3} \quad , \; 0 \leq t \leq 4 .$$

Solution:

We have, $x = \frac{t^2}{2}$, $y = \frac{(2t+1)^{3/2}}{3}$, $0 \le t \le 4$

$$\Rightarrow \frac{dx}{dt} = t \ , \quad \frac{dy}{dt} = \frac{1}{3} \cdot \frac{3}{2} \cdot (2t+1)^{1/2} \cdot 2 = (2t+1)^{1/2}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2 + ((2t+1)^{1/2})^2}$$

$$= \sqrt{t^2 + 2t + 1} = \sqrt{(t+1)^2} = t + 1$$

Therefore, the length of the curve, $L = \int_0^4 ds \, . \, dt$

$$\Rightarrow L = \int_0^4 (t+1) \, . \, dt = \left[\frac{t^2}{2} + t\right]_0^4 = 12 - 0 = 12$$

P2:

Find the length of the curve

$$x = 8\cos t + 8t\sin t \ , \ y = 8\sin t - 8t\cos t \quad , \ 0 \le t \le \frac{\pi}{2}.$$

Solution:

We have,

$$x = 8\cos t + 8t\sin t\,, y = 8\sin t - 8t\cos t\,, 0 \le t \le \frac{\pi}{2}$$

$$\Rightarrow \frac{dx}{dt} = -8\sin t + 8(\sin t + t\cos t) = 8t\cos t$$

and $\dfrac{dy}{dt} = 8\cos t - 8(\cos t - t\sin t) = 8t\sin t$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2 + (8t\sin t)^2}$$

$$= 8t\sqrt{\cos^2 t + \sin^2 t} = 8t$$

Therefore, the length of the curve, $L = \int_0^{\frac{\pi}{2}} ds\,.\,dt = \int_0^{\frac{\pi}{2}} 8t\,.\,dt$

$$\Rightarrow L = 8\int_0^{\frac{\pi}{2}} t\,.\,dt = 8\left[\frac{t^2}{2}\right]_0^{\frac{\pi}{2}} = 4\left(\frac{\pi^2}{4}\right) = \pi^2$$

P3:

Find the centroid of the arc of the curve in the given interval.

$$x = t \quad , \quad y = \sqrt{1 - t^2} \quad , \quad -1 \leq t \leq 0$$

Solution:

We have, $x = t \quad , \quad y = \sqrt{1-t^2} \quad , \quad -1 \le t \le 0$

$$\Rightarrow \frac{dx}{dt} = 1 \quad , \quad \frac{dy}{dt} = -\frac{t}{\sqrt{1-t^2}}$$

We take the curve's density to be $\delta = 1$ and calculate the curve's mass and moments about the coordinate axes.

The distribution of mass is symmetric about the line $y = -x$, so $\bar{y} = -\bar{x}$.

A typical segment of the curve has mass

$$dm = 1.ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{1 + \frac{t^2}{1-t^2}}\, dt = \frac{1}{\sqrt{1-t^2}}\, dt$$

The curve's mass is

$$M = \int_{-1}^{0} dm = \int_{-1}^{0} \frac{1}{\sqrt{1-t^2}}\, dt = [\sin^{-1} t]_{-1}^{0}$$

$$= \sin^{-1} 0 - \sin^{-1}(-1) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

The curve's moment about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_{-1}^{0} \sqrt{1-t^2}\, \frac{1}{\sqrt{1-t^2}}\, dt = \int_{-1}^{0} dt$$

$$= [t]_{-1}^{0} = 1$$

Hence, $\quad \bar{y} = \frac{M_x}{M} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

$$\therefore \bar{x} = -\bar{y} = -\frac{2}{\pi}$$

Therefore, the centroid $\left(-\frac{2}{\pi}, \frac{2}{\pi}\right)$.

P4:

Find the area of the surface generated by revolving the curve

$x = \frac{2}{3}t^{3/2}$, $y = 2\sqrt{t}$, $0 \leq t \leq \sqrt{3}$ about the y-axis.

Solution:

We have, $x = \frac{2}{3}t^{3/2}$, $y = 2\sqrt{t}$, $0 \le t \le \sqrt{3}$

$$\Rightarrow \frac{dx}{dt} = t^{1/2} \quad , \quad \frac{dy}{dt} = t^{-\frac{1}{2}}$$

The area of the surface generated by revolving about the y-axis is

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\sqrt{3}} 2\pi \left(\frac{2}{3}t^{3/2}\right)\sqrt{(t^{1/2})^2 + \left(t^{-\frac{1}{2}}\right)^2}\, dt$$

$$= \frac{4\pi}{3}\int_0^{\sqrt{3}} t^{3/2}\frac{\sqrt{t^2+1}}{\sqrt{t}}\, dt = \frac{4\pi}{3}\int_0^{\sqrt{3}} t\sqrt{t^2+1}\, dt$$

Let $t^2 + 1 = u \Rightarrow 2t.dt = du \Rightarrow t.dt = \frac{1}{2}du$

When $t = 0 \Rightarrow u = 1$ and $t = \sqrt{3} \Rightarrow u = 4$

$$\Rightarrow S = \frac{4\pi}{3}\int_1^4 \sqrt{u}\,\frac{1}{2}\, du = \frac{2\pi}{3}\int_1^4 \sqrt{u}\, du$$

$$= \frac{4\pi}{9}\left[u^{3/2}\right]_1^4 = \frac{4\pi}{9}[8 - 1] = \frac{28\pi}{9}$$

1. Find the lengths of parametrized curves.

a. $x = 1 - t$, $y = 2 + 3t$, $-\frac{2}{3} \le t \le 1$

b. $x = \frac{(2t+3)^{\frac{3}{2}}}{3}$, $y = t + \frac{t^2}{2}$, $0 \le t \le 3$

2. Find the area of the surface generated by revolving the curve $x = t + \sqrt{2}$, $y = \frac{t^2}{2} + \sqrt{2}t$, $-\sqrt{2} \leq t \leq \sqrt{2}$ about the y-axis.

3. Find the area of the surface generated by revolving one arch of the cycloid

$$x = a(t - \sin t)\ , y = a(1 - \cos t)\ , 0 \le t \le 2\pi$$

about the x-axis.

2.1

Polar Coordinates

Learning Objectives:

- To define polar coordinates of a point
- To find all polar coordinates of a given point
- To write Cartesian equivalent of a polar equation and vice versa

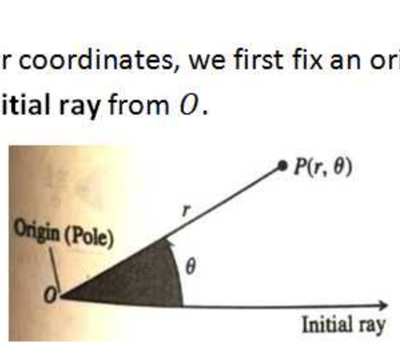
AND

- To practice related problems

In this unit, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates.

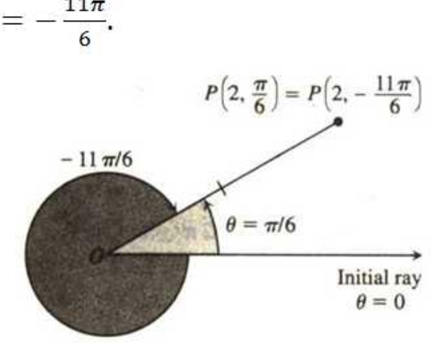
Definition

To define polar coordinates, we first fix an origin O, called the **pole** and an **initial ray** from O.



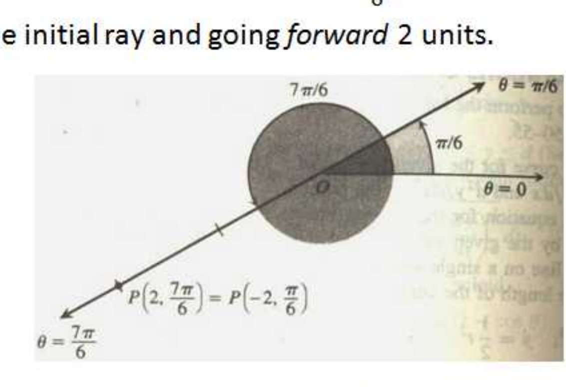
Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives *the directed distance from O to P* and θ gives the *directed angle from the initial ray to ray OP.*

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. For instance, the point 2 units from the origin along the ray $\theta = \frac{\pi}{6}$ has polar coordinates $r = 2, \theta = \frac{\pi}{6}$. It also has coordinates $r = 2, \theta = -\frac{11\pi}{6}$.



Negative Values of r

There are occasions when we wish to allow r to be negative. That is why we use directed distance in $P(r, \theta)$. The point $P(2, \frac{7\pi}{6})$ can be reached by turning $\frac{7\pi}{6}$ *rad* counterclockwise from the initial ray and going *forward* 2 units.



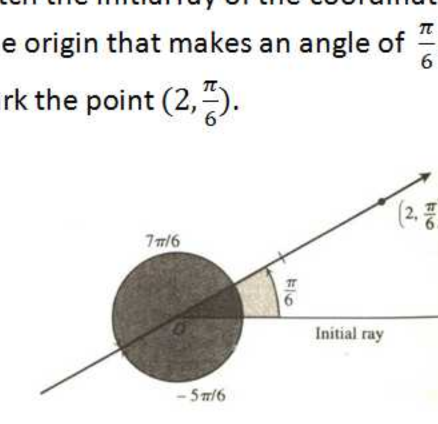
It can also be reached by turning $\frac{\pi}{6}$ *rad* counterclockwise from the initial ray and going *backward* 2 units. So, the point also has polar coordinates $r = -2, \theta = \frac{\pi}{6}$.

Example 1

Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.

Solution

We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\frac{\pi}{6}$ *rad* with the initial ray and mark the point $(2, \frac{\pi}{6})$.



We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \frac{\pi}{6} \pm 2\pi, \frac{\pi}{6} \pm 4\pi, \frac{\pi}{6} \pm 6\pi, \cdots$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, -\frac{5\pi}{6} \pm 2\pi, -\frac{5\pi}{6} \pm 4\pi, -\frac{5\pi}{6} \pm 6\pi, \cdots$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

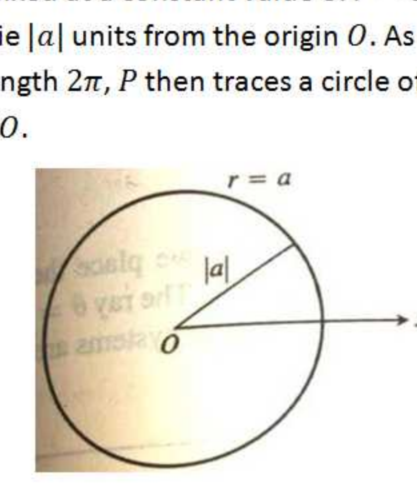
and $$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

When $n = 0$, the formulas give $\left(2, \frac{\pi}{6}\right)$ and $\left(-2, -\frac{5\pi}{6}\right)$.

When $n = 1$, they give $\left(2, \frac{13\pi}{6}\right)$ and $\left(-2, \frac{7\pi}{6}\right)$, and so on.

Elementary Coordinate Equations and Inequalities

If we hold r fixed at a constant value of $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over an interval of length 2π, P then traces a circle of radius $|a|$ centered at O.



If we hold θ fixed at a constant value of $\theta = \theta_0$, and let r vary between $-\infty$ and ∞, the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray.

Equation	Graph
$r = a$	Circle of radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray.

Example 2

a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

b) $\theta = \frac{\pi}{6}, \theta = \frac{7\pi}{6}$, and $\theta = -\frac{5\pi}{6}$ are equations for the line in example 1.

Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments, and rays.

Example 3

Graph the set of points whose polar coordinates satisfy the following conditions.

a) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$

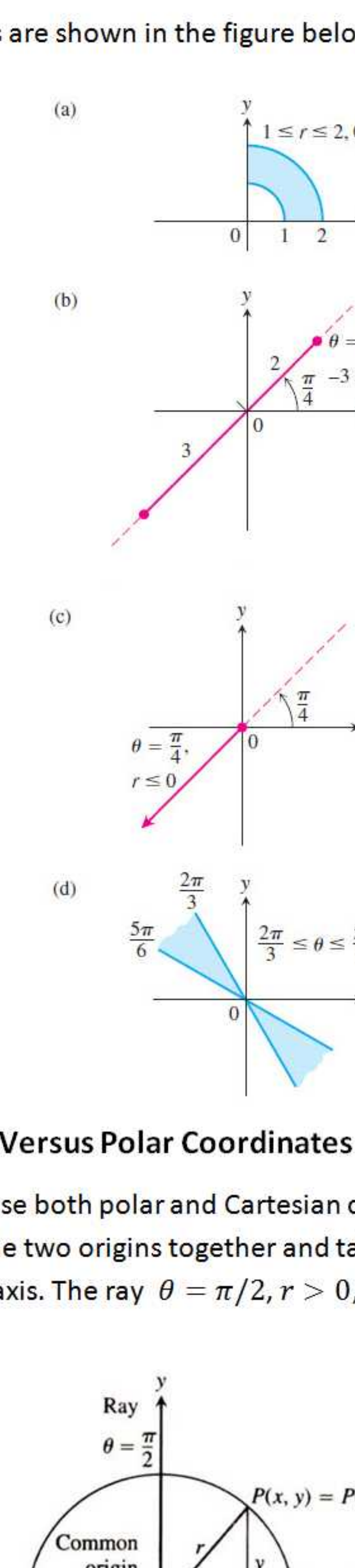
b) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$

c) $r \leq 0$ and $\theta = \frac{\pi}{4}$

d) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$ (no restriction on r)

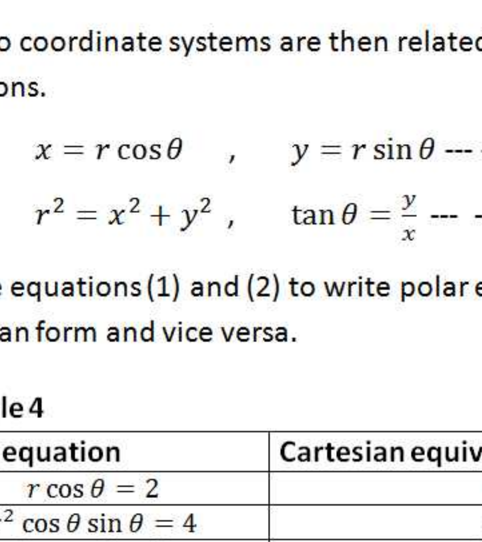
Solution

The graphs are shown in the figure below.



Cartesian Versus Polar Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial ray as the positive x-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive y-axis.



The two coordinate systems are then related by the following equations.

$$x = r\cos\theta \quad , \qquad y = r\sin\theta \text{ --- --- --- --- } \quad (1)$$

$$r^2 = x^2 + y^2 \; , \qquad \tan\theta = \frac{y}{x} \text{ --- --- --- -- } \quad (2)$$

We use equations (1) and (2) to write polar equations in Cartesian form and vice versa.

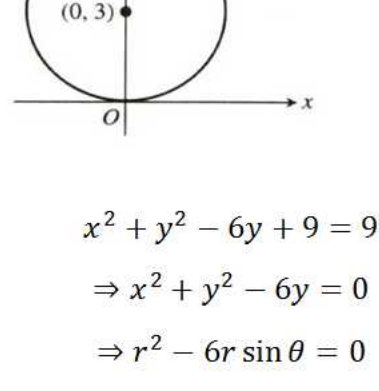
Example 4

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

From this example, we see that we are better off with polar coordinates for some curves.

Example 5

Find a polar equation for the circle $x^2 + (y-3)^2 = 9$



Solution

$$x^2 + y^2 - 6y + 9 = 9$$

$$\Rightarrow x^2 + y^2 - 6y = 0$$

$$\Rightarrow r^2 - 6r\sin\theta = 0$$

$$\Rightarrow r = 0 \qquad or \qquad r - 6\sin\theta = 0$$

Both possibilities are included in $r = 6\sin\theta$

Example 6

Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

a) $r\cos\theta = -4$

b) $r^2 = 4r\cos\theta$

c) $r = \frac{4}{2\cos\theta - \sin\theta}$

Solution

We use the substitution $r\cos\theta = x, r\sin\theta = y, r^2 = x^2 + y^2$.

a) $r\cos\theta = -4$

The Cartesian equation: $\qquad x = -4$

The graph : Vertical line through $x = -4$ on the x-axis

b) $r^2 = 4r\cos\theta$

The Cartesian equation:

$$r^2 = 4r\cos\theta$$

$$\Rightarrow x^2 + y^2 = 4x$$

$$\Rightarrow x^2 - 4x + y^2 = 0$$

$$\Rightarrow x^2 - 4x + 4 + y^2 = 4$$

$$\Rightarrow (x-2)^2 + y^2 = 4$$

The Graph: $\qquad$ Circle, radius 2, center $(h, k) = (2,0)$

c) $r = \frac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:

$$r(2\cos\theta - \sin\theta) = 4$$

$$\Rightarrow 2r\cos\theta - r\sin\theta = 4$$

$$\Rightarrow \quad 2x - y = 4$$

$$\Rightarrow \quad y = 2x - 4$$

The graph: $\qquad$ Line, slope $m = 2$, y-intercept $b = -4$

P1:

Find all polar coordinates of the point $\left(3, \frac{\pi}{4}\right)$.

Solution:

We find the angles for the other coordinate pairs of $\left(3, \frac{\pi}{4}\right)$ in which $r = 3$ and $r = -3$.

For $r = 3$, the complete list of angles is

$$\frac{\pi}{4}, \frac{\pi}{4} \pm 2\pi, \frac{\pi}{4} \pm 4\pi, \frac{\pi}{4} \pm 6\pi, \cdots$$

For $r = -3$, the angles are

$$-\frac{3\pi}{4}, -\frac{3\pi}{4} \pm 2\pi, -\frac{3\pi}{4} \pm 4\pi, -\frac{3\pi}{4} \pm 6\pi, \cdots$$

The corresponding coordinate pairs of $\left(3, \frac{\pi}{4}\right)$ are

$$\left(3, \frac{\pi}{4} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

and $$\left(-3, -\frac{3\pi}{4} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

The above are same as

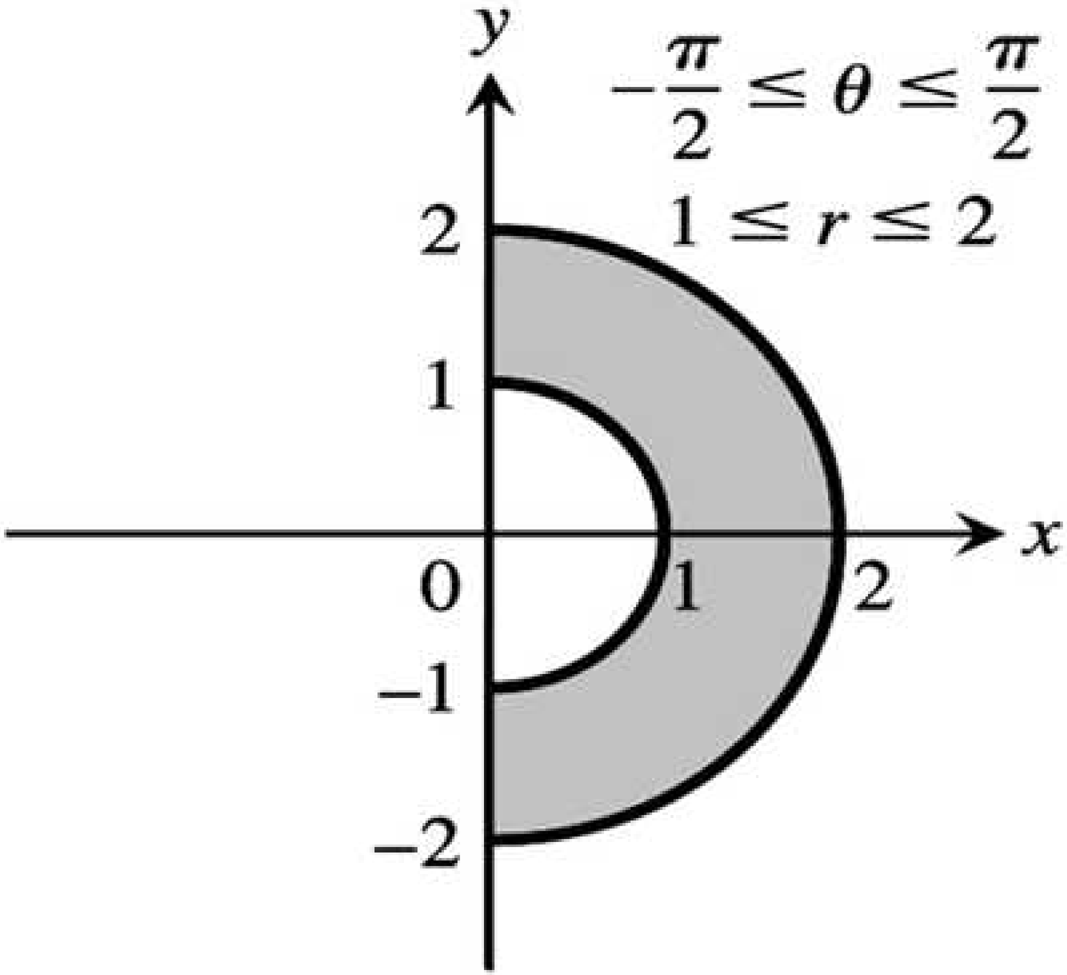
$$\left(3, \frac{\pi}{4} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

and $$\left(-3, \frac{\pi}{4} + (2n+1)\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

P2:

Graph the set of points whose polar coordinates satisfy the inequalities: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; $1 \leq r \leq 2$

Solution:



P3:

Replace the polar equation $r = 4\tan\theta\sec\theta$ by equivalent Cartesian equation. Then describe or identify the graph.

Solution:

We have, $r = 4\tan\theta\sec\theta$

$$\Rightarrow r = 4\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right)$$

$$\Rightarrow r\cos^2\theta = 4\sin\theta$$

$$\Rightarrow r^2\cos^2\theta = 4r\sin\theta$$

$$\Rightarrow x^2 = 4y$$

It is a parabola with vertex $(0,0)$ which opens upward.

P4:

Replace the Cartesian equation $(x-3)^2+(y+1)^2=4$ by equivalent polar equation.

Solution:

We have, $(x-3)^2+(y+1)^2=4$

$$\Rightarrow x^2-6x+9+y^2+2y+1=4$$

$$\Rightarrow \quad x^2+y^2-6x+2y+6=0$$

$$\Rightarrow r^2-6r\cos\theta+2r\sin\theta+6=0$$

IP1:

Find all polar coordinates of the point $\left(3, -\frac{\pi}{4}\right)$.

Solution:

We find the angles for the other coordinate pairs of $\left(3, -\frac{\pi}{4}\right)$ in which $r = 3$ and $r = -3$.

For $r = 3$, the complete list of angles are

$$-\frac{\pi}{4}, -\frac{\pi}{4} \pm 2\pi, -\frac{\pi}{4} \pm 4\pi, -\frac{\pi}{4} \pm 6\pi, \cdots$$

For $r = -3$, the angles are

$$\frac{3\pi}{4}, \frac{3\pi}{4} \pm 2\pi, \frac{3\pi}{4} \pm 4\pi, \frac{3\pi}{4} \pm 6\pi, \cdots$$

The corresponding coordinate pairs of $\left(3, -\frac{\pi}{4}\right)$ are

$$\left(3, -\frac{\pi}{4} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

and $$\left(-3, \frac{3\pi}{4} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

The above are same as

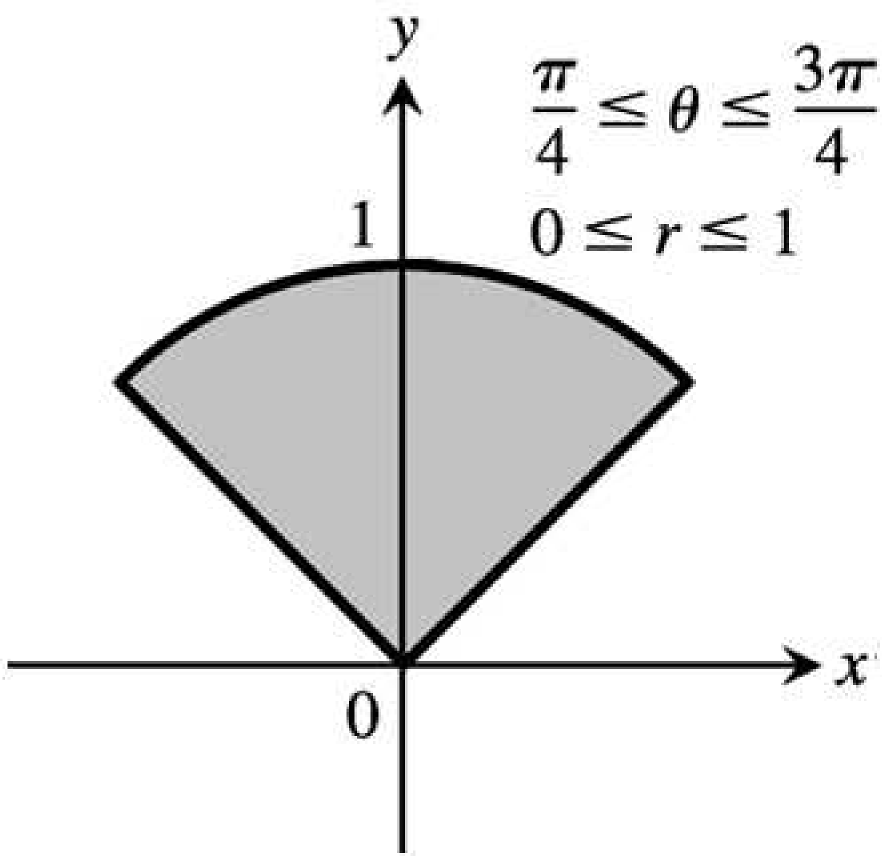
$$\left(3, -\frac{\pi}{4} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

and $$\left(-3, -\frac{\pi}{4} + (2n+1)\pi\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

IP2:

Graph the set of points whose polar coordinates satisfy the inequalities: $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$; $0 \leq r \leq 1$

Solution:



IP3:

Replace the polar equation $r^2 + 2r^2 \cos\theta \sin\theta = 1$ by equivalent Cartesian equation. Then describe or identify the graph.

Solution:

We have, $r^2 + 2r^2 \cos\theta \sin\theta = 1$

$$\Rightarrow x^2 + y^2 + 2(r\cos\theta)(r\sin\theta) = 1$$
$$\Rightarrow x^2 + y^2 + 2xy = 1$$
$$\Rightarrow (x+y)^2 = 1$$
$$\Rightarrow x + y = \pm 1$$

These are two parallel lines with slope -1 and y-intercepts ± 1.

IP4:

Replace the Cartesian equation $x^2 + xy + y^2 = 1$ by equivalent polar equation.

Solution:

We have, $x^2 + xy + y^2 = 1$

$$\Rightarrow x^2 + y^2 + xy = 1$$

$$\Rightarrow r^2 + (r\cos\theta)(r\sin\theta) = 1$$

$$\Rightarrow r^2 + r^2\cos\theta\sin\theta = 1$$

$$\Rightarrow r^2(1 + \cos\theta\sin\theta) = 1$$

I. Which polar coordinate pairs label the same point?

a. $(-3,0)$

b. $\left(2, \frac{2\pi}{3}\right)$

c. $\left(2, -\frac{2\pi}{3}\right)$

d. $(-r, \theta)$

e. $(r, \theta + \pi)$

II. Find the all polar coordinates of each point given below.

a. $(2,0)$

b. $\left(2,\frac{\pi}{2}\right)$

c. $\left(-2,\frac{\pi}{2}\right)$

d. $\left(-3,\frac{\pi}{4}\right)$

e. $\left(-3,-\frac{\pi}{4}\right)$

IV. Graph the sets of points whose polar coordinates satisfy the equations and inequalities given below

a. $r = 2$

b. $\theta = \frac{\pi}{3}$, $-1 \leq r \leq 3$

c. $\theta = \frac{\pi}{2}$, $r \geq 0$

d. $0 \leq \theta \leq \frac{\pi}{6}$, $r \geq 0$

e. $0 \leq \theta \leq \pi$, $r = 1$

V. Replace the polar equations by equivalent Cartesian equations. Then describe or identify the graph.

a. $r\cos\theta = 2$

b. $r\sin\theta = -1$

c. $r\sin\theta = 0$

d. $r\cos\theta = 0$

e. $r = 4\csc\theta$

f. $r = -3\sec\theta$

g. $r\cos\theta + r\sin\theta = 1$

h. $r\sin\theta = r\cos\theta$

i. $r^2 = 1$

j. $r^2 = 4r\sin\theta$

k. $r = \frac{5}{\sin\theta - 2\cos\theta}$

l. $r^2\sin 2\theta = 2$

m. $r = \cot\theta\csc\theta$

n. $r = \csc\theta\, e^{r\cos\theta}$

o. $r\sin\theta = \ln r + \ln\cos\theta$

p. $\cos^2\theta = \sin^2\theta$

q. $r^2 = -4r\cos\theta$

r. $r^2 = -6r\sin\theta$

s. $r = 8\sin\theta$

t. $r = 3\cos\theta$

u. $r = 2\cos\theta + 2\sin\theta$

v. $r = 2\cos\theta - \sin\theta$

w. $r\sin\left(\theta + \frac{\pi}{6}\right) = 2$

x. $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$

VI. Replace the Cartesian equations by equivalent polar equations.

a. $x - y = 3$

b. $x^2 + y^2 = 4$

c. $x^2 - y^2 = 1$

d. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

e. $xy = 2$

f. $y^2 = 4x$

g. $(x - 5)^2 + y^2 = 25$

h. $(x + 2)^2 + (y - 5)^2 = 16$

2.2

Graphing in Polar Coordinates

Learning Objectives:

- To learn techniques for graphing equations in polar coordinates.
- To find slope of the polar curve.
- To sketch the graphs of some polar curves.

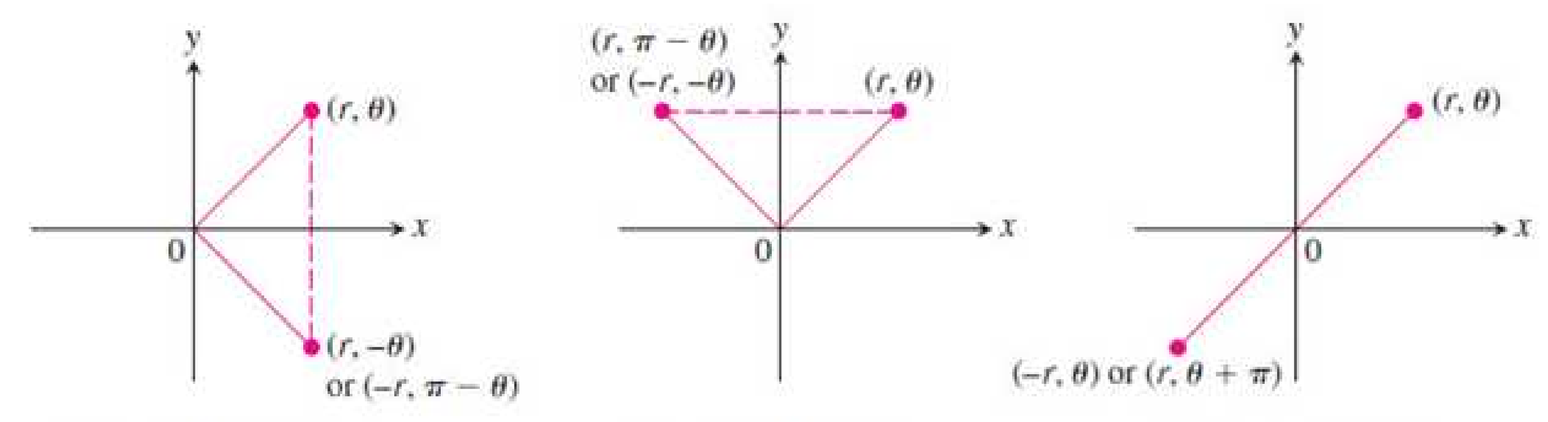
AND

- To practice related problems.

In this module, we learn techniques for graphing equations in polar coordinates.

Symmetry

Figure below illustrates the standard polar coordinate tests for symmetry.



(a) About the x-axis (b) About the y-axis (c) About the origin

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis*: If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (figure a).
2. *Symmetry about the y-axis*: If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (figure b).
3. *Symmetry about the origin*: If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \pi + \theta)$ lies on the graph (figure c)

Slope

The slope of a polar curve $r = f(\theta)$ is given by $\frac{dy}{dx}$, not by $r' = \frac{df}{d\theta}$. We think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \quad , \quad y = r\sin\theta = f(\theta)\sin\theta$$

If f is a differentiable function of θ, then so are x and y and, when $\frac{dx}{d\theta} \neq 0$, we can calculate $\frac{dy}{dx}$ from the parametric formula

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)} = \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}$$

Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \quad \text{--- ----- -- (1)}$$

provided $\frac{dx}{d\theta} \neq 0$ at (r, θ).

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and equation (1) gives

$$\left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0$$

If the graph of $r = f(\theta)$ passes through the origin at the value $\theta = \theta_0$, the slope of the curve there is $\tan\theta_0$. *A polar curve may pass through the origin more than once, with different slopes at different θ-values.*

Example 1

Graph the curve $r = 1 - \cos\theta$. ---- ----- ---- (2)

Solution

The curve is symmetric about the x-axis because

(r, θ) lies on the curve (2) $\Rightarrow r = 1 - \cos\theta$

$\Rightarrow r = 1 - \cos(-\theta)$

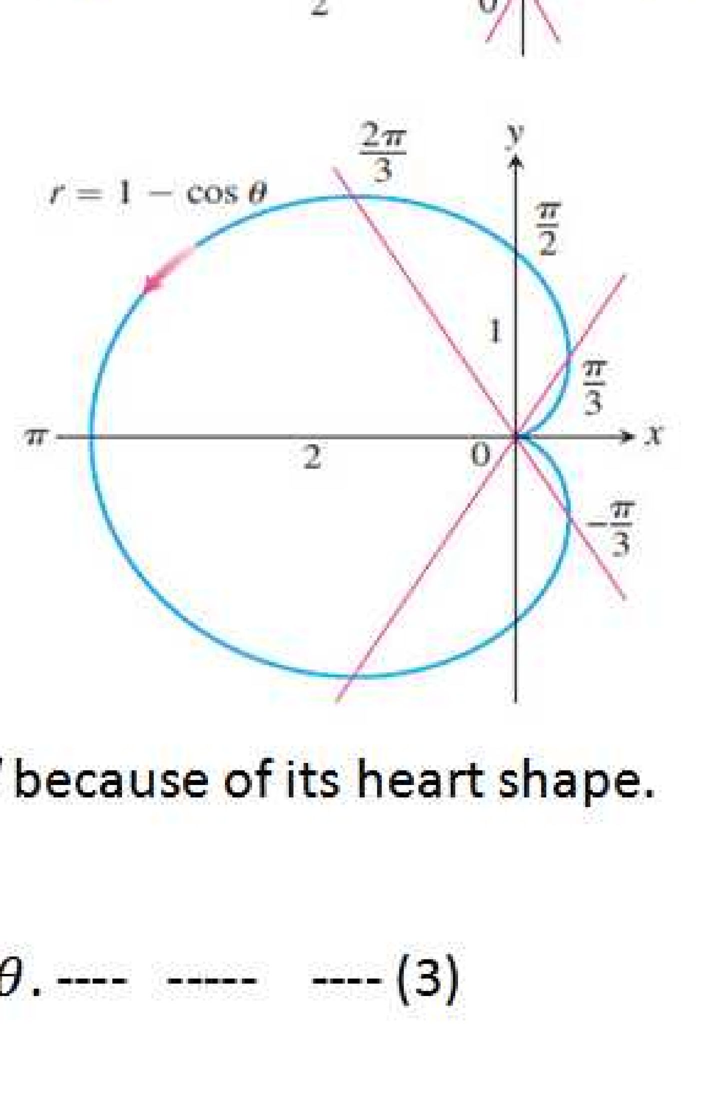
$\Rightarrow (r, -\theta)$ lies on the curve (2)

As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to 2π, $\cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope $\tan(0) = 0$ and returns to the origin with slope $\tan(2\pi) = 0$.

We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the x-axis to complete the graph.

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2



The curve is called a ***cardioid*** because of its heart shape.

Example 2

Graph the curve $r^2 = 4\cos\theta$. ---- ----- ---- (3)

Solution

The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The curve is symmetric about the x-axis because

(r, θ) on the curve (3) $\Rightarrow r^2 = 4\cos\theta$

$\Rightarrow r^2 = 4\cos(-\theta)$

$\Rightarrow (r, -\theta)$ on the curve (3)

The curve is also symmetric about the origin because (r, θ) on the graph $\Rightarrow r^2 = 4\cos\theta$

$\Rightarrow (-r)^2 = 4\cos(\theta)$

$\Rightarrow (-r, \theta)$ on the graph

Together, these two imply symmetry about the y-axis.

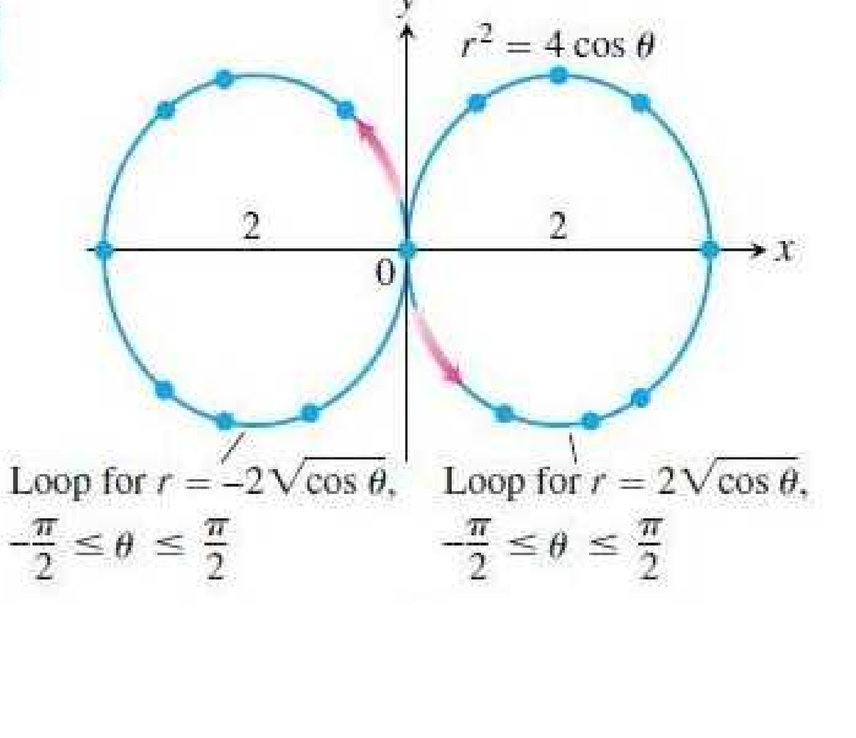
The curve passes through the origin when $\theta = -\frac{\pi}{2}$ and $= \frac{\pi}{2}$. It has a vertical tangent both times because $\tan\theta$ is infinite.

For each value of θ in the interval between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}$$

We make a table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve.

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0



P1:

Identify the symmetry of the curve $r = 1 - \sin\theta$ and sketch the graph.

Solution:

We have, $r = 1 - \sin\theta$ ---- ---- ----- (1)

(r, θ) lies on the curve (1) $\Rightarrow r = 1 - \sin\theta$

$\Rightarrow r = 1 - \sin(\pi - \theta)$

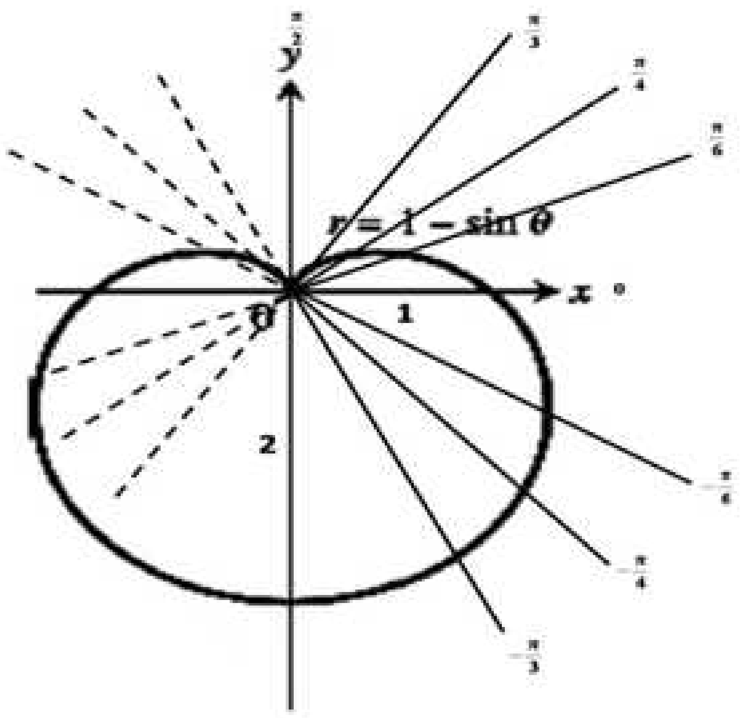
$\Rightarrow (r, \pi - \theta)$ lies on the curve (1)

Therefore, the curve $r = 1 - \sin\theta$ is symmetric about the y-axis.

As θ increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $\sin\theta$ increases from -1 to 1 and $r = 1 - \sin\theta$ decreases from a maximum value of 2 to a minimum value 0.

The curve reaches the origin with the slope $\tan\frac{\pi}{2} = \infty$. We prepare a table of values from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$, plot the points, draw a smooth curve through them with a vertical tangent at the origin and reflect the curve across the y-axis to complete the curve.

$\boldsymbol{\theta}$	$\boldsymbol{\sin\theta}$	$\boldsymbol{r = 1 - \sin\theta}$
$-\frac{\pi}{2}$	-1	2
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$	1.86
$-\frac{\pi}{4}$	$-\frac{1}{\sqrt{2}}$	1.70
$-\frac{\pi}{6}$	$-\frac{1}{2}$	1.5
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	0.5
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	0.29
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	0.13
$\frac{\pi}{2}$	1	0



P2:

Sketch the curve $r = \sin 2\theta$ (four-leaved rose)

Solution:

We have, $r = \sin 2\theta$ -------------------- (1)

The curve (1) is symmetric about the x-axis because

(r, θ) lies on the curve (1) $\Rightarrow r = \sin 2\theta$

$\Rightarrow -r = \sin 2(\pi - \theta)$

$\Rightarrow (-r, \pi - \theta)$ lies on the curve (1)

The curve (1) is symmetric about the y-axis because

(r, θ) lies on the curve (1) $\Rightarrow r = \sin 2\theta$

$\Rightarrow -r = \sin 2(-\theta)$

$\Rightarrow (-r, -\theta)$ lies on the curve (1)

And the curve (1) is symmetric about the origin because

(r, θ) lies on the curve (1) $\Rightarrow r = \sin 2\theta$

$\Rightarrow r = \sin 2(\pi + \theta)$

$\Rightarrow (r, \pi + \theta)$ lies on the curve (1)

As θ increases from 0 to $\frac{\pi}{4}$, $r = \sin 2\theta$ increases from a minimum value of 0 to a maximum value 1.

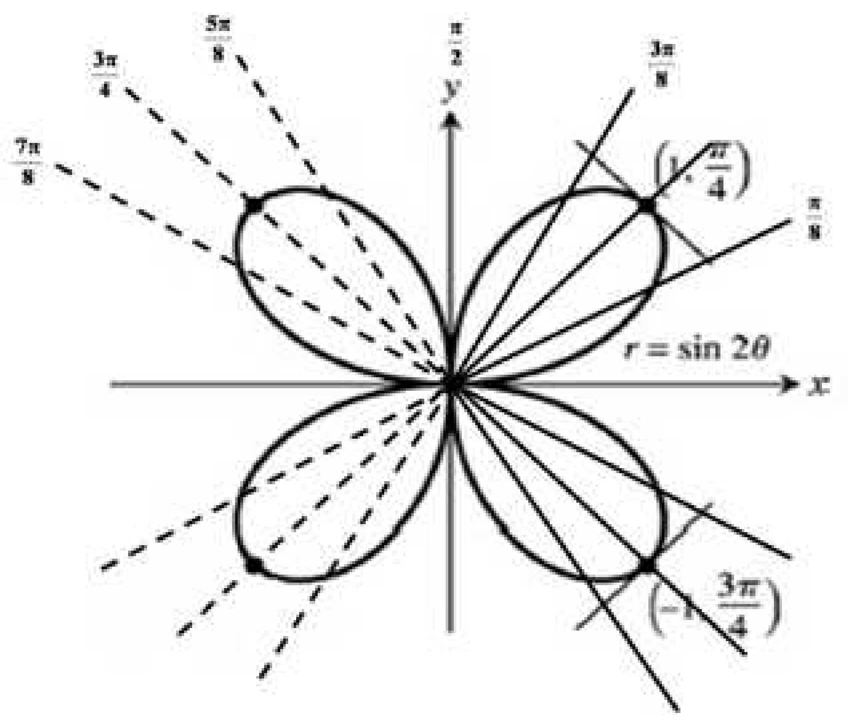
As θ increases from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, $r = \sin 2\theta$ decreases from a maximum value of 1 to a minimum value 0.

As θ increases from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, $r = \sin 2\theta$ decreases from a maximum value of 0 to a minimum value -1.

As θ increases from $\frac{3\pi}{4}$ to π, $r = \sin 2\theta$ increases from a minimum value of -1 to a maximum value 0.

The curve reaches the origin with the slope $\tan 0 = 0$, $\tan\frac{\pi}{2} = \infty, \tan \pi = 0$. We prepare a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve and reflect the curve across the x-axis to complete the curve.

$\boldsymbol{\theta}$	$\boldsymbol{2\theta}$	$\boldsymbol{r = \sin 2\theta}$
0	0	0
$\frac{\pi}{8}$	$\frac{\pi}{4}$	0.70
$\frac{\pi}{4}$	$\frac{\pi}{2}$	1
$\frac{3\pi}{8}$	$\frac{3\pi}{4}$	0.70
$\frac{\pi}{2}$	π	0
$\frac{5\pi}{8}$	$\frac{5\pi}{4}$	-0.70
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	-1
$\frac{7\pi}{8}$	$\frac{7\pi}{4}$	-0.70
π	2π	0



P3:

Find the slope of the curve $r = -1 + \sin\theta$ at the point $\theta = \pi$.

Solution:

We have, $r = -1 + \sin\theta \quad ; \theta = 0$

At $\theta = \pi$, $r = -1 + \sin\pi = -1 + 0 = -1$

The point is $(r, \theta) = (-1, \pi)$.

We have $r = -1 + \sin\theta \Rightarrow \frac{dr}{d\theta} = \cos\theta$

The slope of the curve $= \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \dfrac{(\cos\theta)\sin\theta + r\cos\theta}{(\cos\theta)\cos\theta - r\sin\theta}$

$$= \frac{\sin\theta\cos\theta + r\cos\theta}{\cos^2\theta - r\sin\theta}$$

At $(-1, \pi)$, slope $= \dfrac{\sin\pi\cos\pi + (-1)\cos\pi}{\cos^2\pi - (-1)\sin\pi} = \dfrac{0.(-1) + (-1).(-1)}{1 - (-1).0} = 1$

P4:

Find the slope of the curve $r = \cos 2\theta$ at the point $\theta = 0$.

Solution:

We have, $r = \cos 2\theta \quad ; \theta = 0$

At $\theta = 0$, $r = \cos 2(0) = \cos 0 = 1$

The point is $(r, \theta) = (1{,}0)$.

We have $r = \cos 2\theta \Rightarrow \frac{dr}{d\theta} = -2\sin 2\theta$

The slope of the curve $= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{(-2\sin 2\theta)\sin\theta + r\cos\theta}{(-2\sin 2\theta)\cos\theta - r\sin\theta}$

$$= \frac{-2\sin\theta\sin 2\theta + r\cos\theta}{-2\cos\theta\sin 2\theta - r\sin\theta}$$

At $(1{,}0)$, slope $= \frac{-2\sin 0\sin 0 + 1.\cos 0}{-2\cos 0\sin 0 - 1.\sin 0} = \frac{1}{0}$

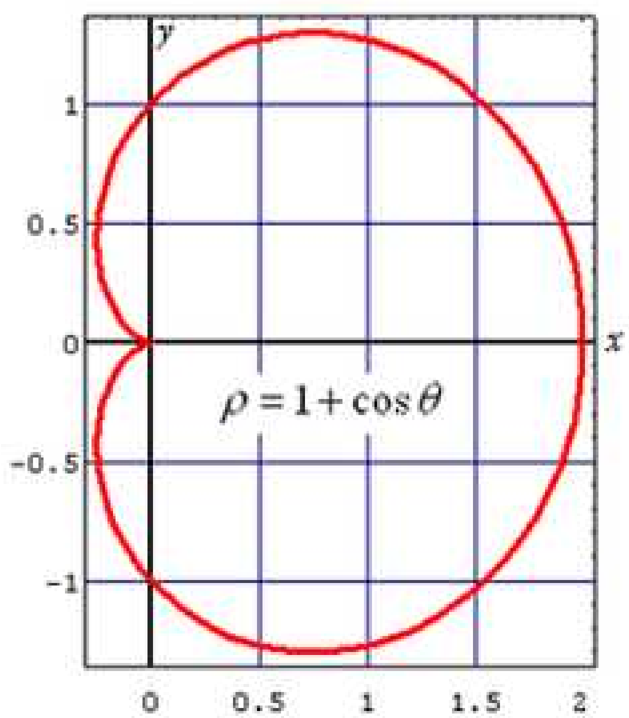
Therefore slope is not defined at $\theta = 0$.

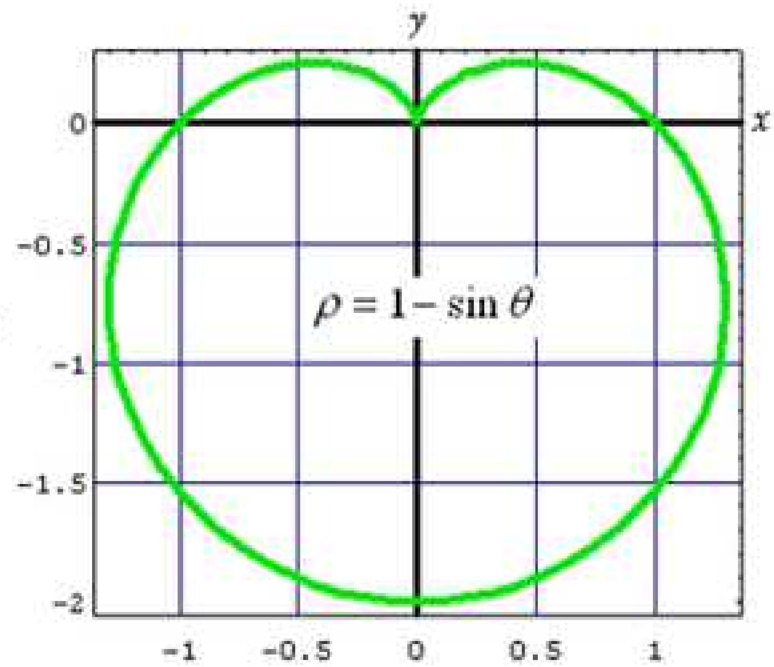
IP1:

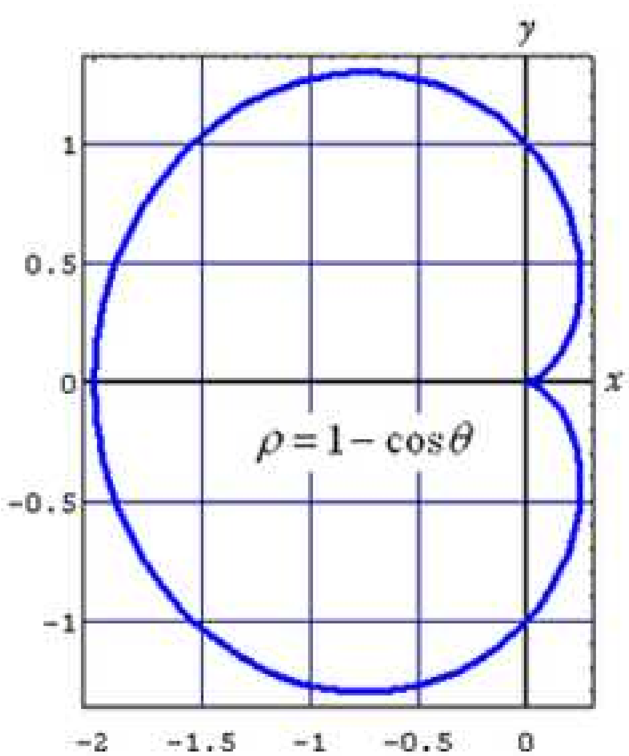
Sketch the graphs of

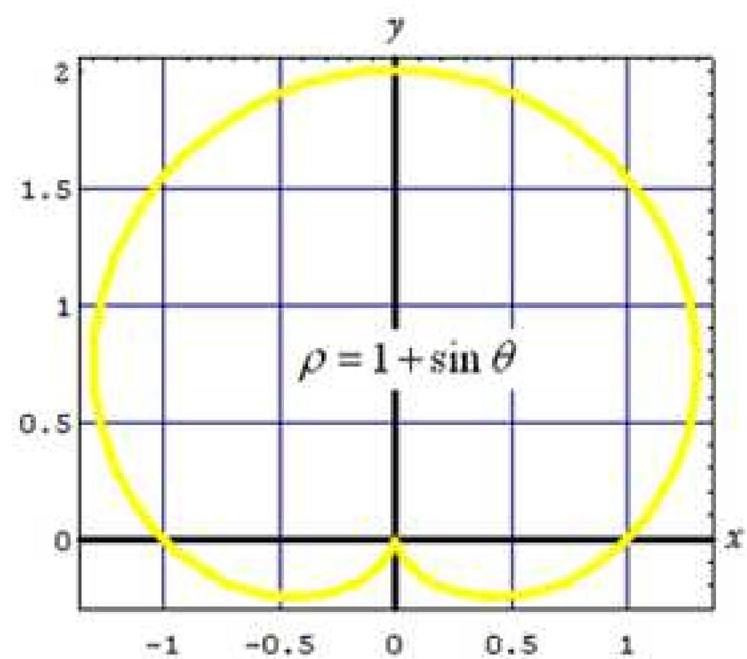
$$\rho = 1+\cos\theta \ , \rho = 1-\cos\theta \ , \ \rho = 1-\sin\theta \ , \rho = 1+\sin\theta$$

Solution:









IP2:

Sketch the curve $r = \cos 2\theta$ (four-leaved rose)

Solution:

We have, $r = \cos 2\theta$ -------------------- (1)

The curve (1) is symmetric about the x-axis because

(r, θ) lies on the curve (1) $\Rightarrow r = \cos 2\theta$

$\Rightarrow r = \cos 2(-\theta)$

$\Rightarrow (r, -\theta)$ lies on the curve (1)

The curve (1) is symmetric about the y-axis because

(r, θ) lies on the curve (1) $\Rightarrow r = \cos 2\theta$

$\Rightarrow r = \cos 2(\pi - \theta)$

$\Rightarrow (r, \pi - \theta)$ lies on the curve (1)

And the curve (1) is symmetric about the origin because

(r, θ) lies on the curve (1) $\Rightarrow r = \cos 2\theta$

$\Rightarrow r = \cos 2(\pi + \theta)$

$\Rightarrow (r, \pi + \theta)$ lies on the curve (1)

As θ increases from 0 to $\frac{\pi}{4}$, $r = \cos 2\theta$ decreases from a maximum value of 1 to a minimum value 0.

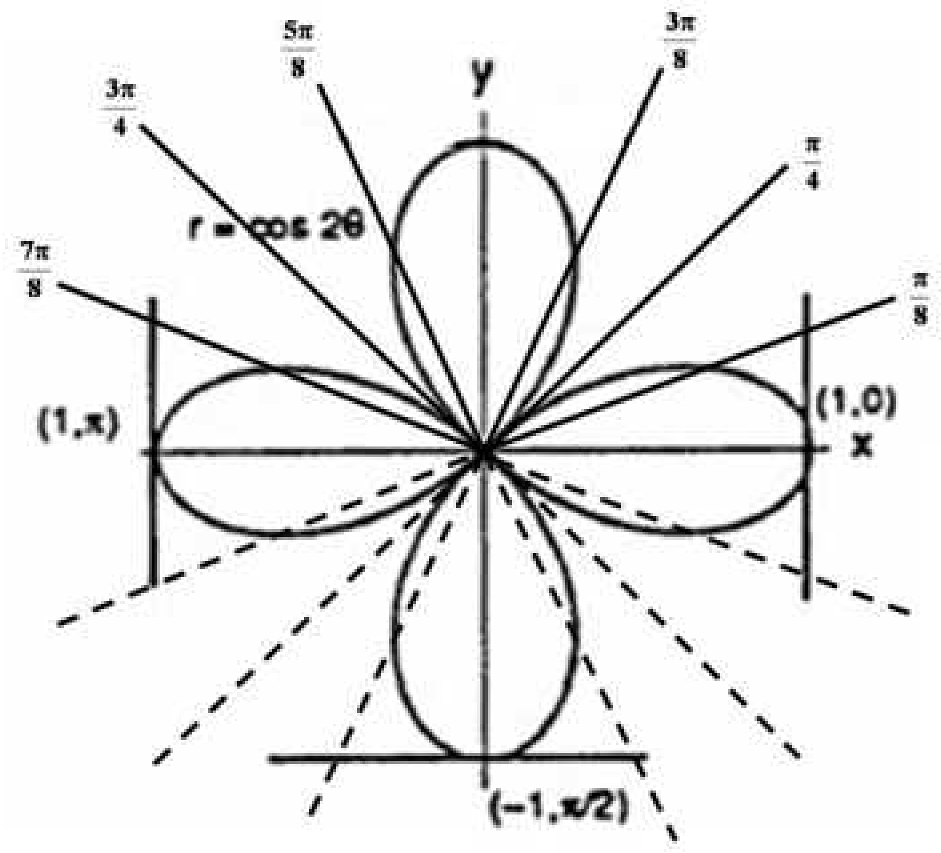
As θ increases from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, $r = \cos 2\theta$ increases from a maximum value of 0 to a minimum value -1.

As θ increases from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, $r = \cos 2\theta$ increases from a minimum value of -1 to a maximum value 0.

As θ increases from $\frac{3\pi}{4}$ to π, $r = \cos 2\theta$ increases from a minimum value of 0 to a maximum value 1.

The curve reaches the origin with the slope $\tan\frac{\pi}{4} = 1$, $\tan\frac{3\pi}{4} = -1, \tan\pi = 0$. We prepare a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve and reflect the curve across the x-axis and y-axis to complete the curve.

θ	2θ	$r = \cos 2\theta$
0	0	1
$\frac{\pi}{8}$	$\frac{\pi}{4}$	0.70
$\frac{\pi}{4}$	$\frac{\pi}{2}$	0
$\frac{3\pi}{8}$	$\frac{3\pi}{4}$	-0.70
$\frac{\pi}{2}$	π	-1
$\frac{5\pi}{8}$	$\frac{5\pi}{4}$	-0.70
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	0
$\frac{7\pi}{8}$	$\frac{7\pi}{4}$	0.70
π	2π	1



IP3:

Find the slope of the curve $r = -1 + \cos\theta$ at the point $\theta = \frac{\pi}{2}$.

Solution:

We have, $r = -1 + \cos\theta \quad ; \theta = \frac{\pi}{2}$

At $\theta = \frac{\pi}{2}$, $r = -1 + \cos\frac{\pi}{2} = -1 + 0 = -1$

The point is $(r, \theta) = \left(-1, \frac{\pi}{2}\right)$.

We have $r = -1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$

The slope of the curve $= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{(-\sin\theta)\sin\theta + r\cos\theta}{(-\sin\theta)\cos\theta - r\sin\theta}$

$$= \frac{-\sin^2\theta + r\cos\theta}{(-\sin\theta)\cos\theta - r\sin\theta}$$

At $\left(-1, \frac{\pi}{2}\right)$, slope $= \frac{-\sin^2\frac{\pi}{2} + (-1)\cos\frac{\pi}{2}}{\left(-\sin\frac{\pi}{2}\right)\cos\frac{\pi}{2} - (-1)\sin\frac{\pi}{2}} = \frac{-1 + (-1).0}{(-1).0 - (-1).1} = -1$

IP4:

Find the slope of the curve $r = \sin 2\theta$ at the point $\theta = -\frac{\pi}{4}$.

Solution:

We have, $r = \sin 2\theta \quad ; \theta = -\frac{\pi}{4}$

At $\theta = -\frac{\pi}{4}, r = \sin 2\left(-\frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

The point is $(r, \theta) = \left(-1, -\frac{\pi}{4}\right)$.

We have $r = \sin 2\theta \Rightarrow \frac{dr}{d\theta} = 2\cos 2\theta$

The slope of the curve $= \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \dfrac{(2\cos 2\theta)\sin\theta + r\cos\theta}{(2\cos 2\theta)\cos\theta - r\sin\theta}$

$$= \frac{2\sin\theta\cos 2\theta + r\cos\theta}{2\cos\theta\cos 2\theta - r\sin\theta}$$

At $\left(-1, -\frac{\pi}{4}\right)$, slope $= \dfrac{2\sin\left(-\frac{\pi}{4}\right)\cos 2\left(-\frac{\pi}{4}\right) + (-1)\cos\left(-\frac{\pi}{4}\right)}{2\cos\left(-\frac{\pi}{4}\right)\cos 2\left(-\frac{\pi}{4}\right) - (-1)\sin\left(-\frac{\pi}{4}\right)}$

$$= \frac{-2\sin\frac{\pi}{4}\cos\frac{\pi}{2} - \cos\frac{\pi}{4}}{2\cos\frac{\pi}{4}\cos\frac{\pi}{2} - \sin\frac{\pi}{4}} = 1$$

1. Identify the symmetries of the curves given below. Then sketch the curves.

a. $r = 1 + \cos\theta$

b. $r = 1 + \sin\theta$

c. $r^2 = \cos\theta$

d. $r^2 = \sin\theta$

e. $r^2 = -\cos\theta$

f. $r^2 = -\sin\theta$

2. Find the slopes of the curves given below at the given points.

a. $r = -1 + \cos\theta \quad ; \ \theta = -\frac{\pi}{2}$

b. $r = -1 + \sin\theta \quad ; \ \theta = 0$

c. $r = \sin 2\theta \quad ; \ \theta = \frac{\pi}{4}, \pm\frac{3\pi}{4}$

d. $r = \cos 2\theta \quad ; \ \theta = \pm\frac{\pi}{2}, \pi$

2.6

Length and Surface Area in Polar Coordinates

Learning Objectives:

- To compute the length of a polar plane curve
- To compute the area of surface of revolution of polar plane curve

AND

- To practice the related problems

In this module, we learn how to calculate lengths of plane curves and areas of surfaces of revolution of plane curves in polar coordinates.

The Length of a Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$\left.\begin{array}{l} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{array}\right\} \quad \alpha \le \theta \le \beta \quad \text{----- -----} \quad (1)$$

The parametric length formula then gives the length as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \quad \text{--- --- (2)}$$

When equations (1) are substituted for x and y in equation (2), it becomes $\quad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

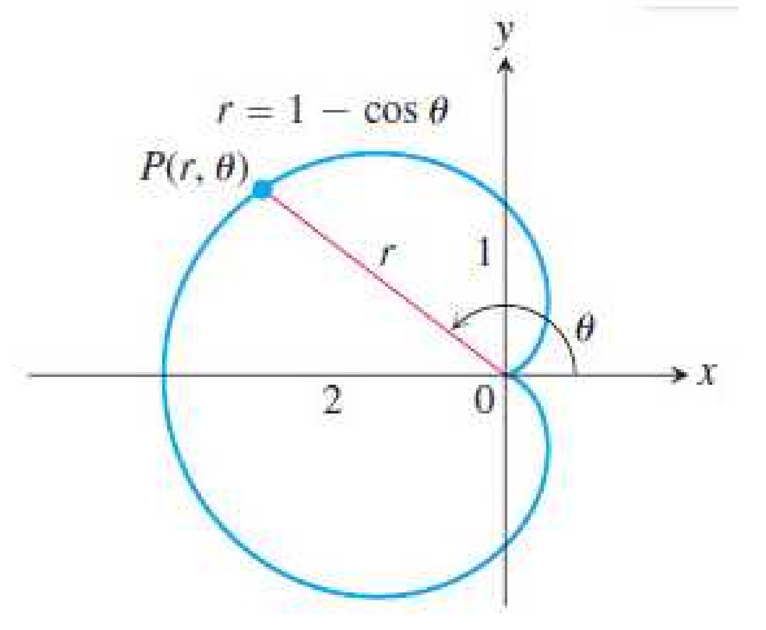
Length

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{---- ---- ----} \quad (3)$$

Example 1

Find the length of the cardioid $r = 1 - \cos\theta$.



The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.

With $r = 1 - \cos\theta$, $\frac{dr}{d\theta} = \sin\theta$

we have $r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + (\sin\theta)^2$

$$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta$$

and

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \quad \left[\sin\frac{\theta}{2} > 0 \ \ for\ 0 \le \theta \le 2\pi\right]$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8$$

The Area of a Surface of Revolution

To derive polar coordinate formula for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, with equations (1) and apply the surface area equations.

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

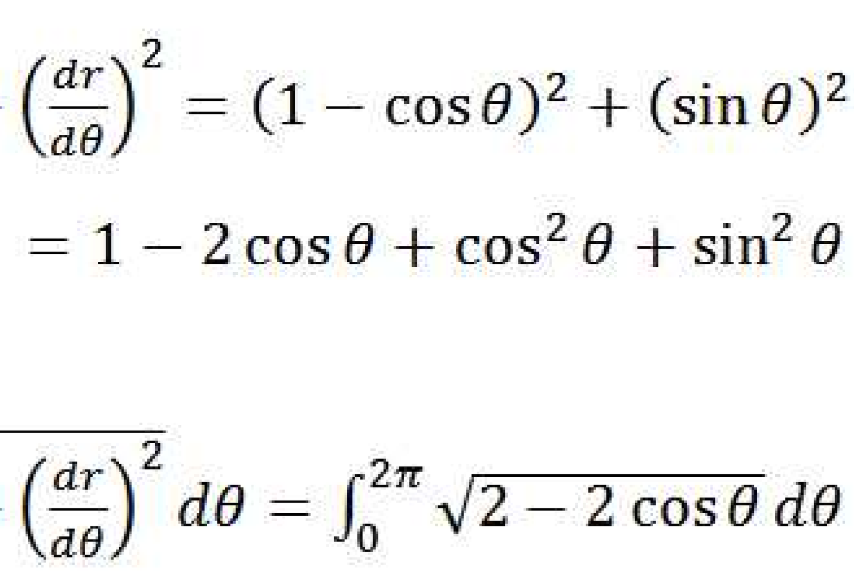
1.	Revolution about the x-axis $(y \ge 0)$	$S = \int_\alpha^\beta 2\pi r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
2.	Revolution about the y-axis $(x \ge 0)$	$S = \int_\alpha^\beta 2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Example 2

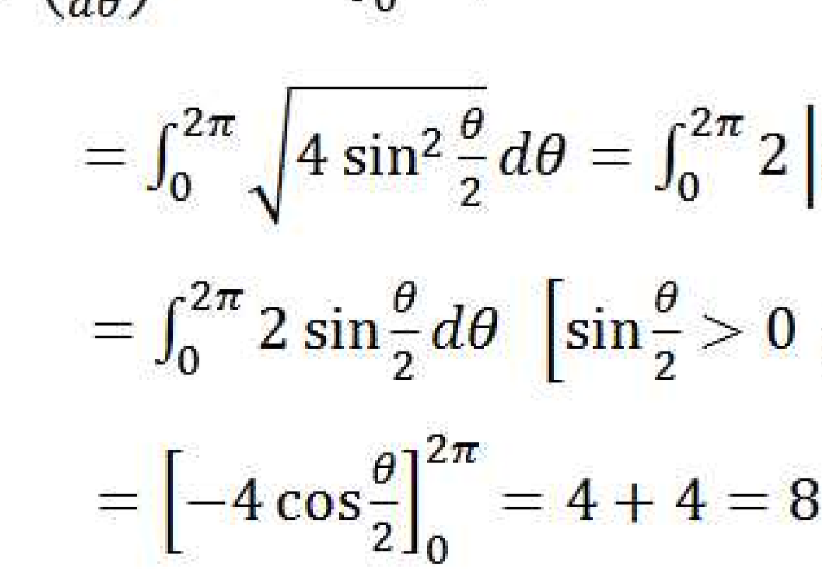
Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

Solution

We sketch the loop to determine the limits of integration.



(a)



(b)

The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

First we calculate

$$2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi\cos\theta \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}$$

Since $r^2 = \cos 2\theta$,

$\Rightarrow 2r\frac{dr}{d\theta} = -2\sin 2\theta$

$\Rightarrow r\frac{dr}{d\theta} = -\sin 2\theta$

$\Rightarrow \left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta$

We write $r^4 = (r^2)^2 = \cos^2 2\theta$. The square root simplifies to

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1$$

All together, we have

$$S = \int_\alpha^\beta 2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi\cos\theta\,.(1)d\theta$$

$$= 2\pi[\sin\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}$$

P1:

Find the length of the spiral $r = \dfrac{e^{\theta}}{\sqrt{2}}$, $0 \le \theta \le \pi$.

Solution:

We have, $r = \frac{e^{\theta}}{\sqrt{2}}$, $0 \le \theta \le \pi$

$$\Rightarrow \frac{dr}{d\theta} = \frac{e^{\theta}}{\sqrt{2}}$$

Therefore, the length of the spiral,

$$L = \int_0^{\pi} \sqrt{\left(\frac{e^{\theta}}{\sqrt{2}}\right)^2 + \left(\frac{e^{\theta}}{\sqrt{2}}\right)^2}\, d\theta = \int_0^{\pi} e^{\theta}\, d\theta = \left[e^{\theta}\right]_0^{\pi} = e^{\pi} - 1$$

P2:

Find the length of the curve $r = \sqrt{1 + \cos 2\theta}$, $0 \leq \theta \leq \pi\sqrt{2}$.

Solution:

We have, $r = \sqrt{1 + \cos 2\theta}$, $0 \leq \theta \leq \pi\sqrt{2}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{1}{2\sqrt{1+\cos 2\theta}} \times (-\sin 2\theta) \times 2 = -\frac{\sin 2\theta}{\sqrt{1+\cos 2\theta}}$$

Therefore, the length of the curve,

$$L = \int_0^{\pi\sqrt{2}} \sqrt{\left(\sqrt{1+\cos 2\theta}\right)^2 + \left(-\frac{\sin 2\theta}{\sqrt{1+\cos 2\theta}}\right)^2}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{(1+\cos 2\theta) + \frac{\sin^2 2\theta}{(1+\cos 2\theta)}}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{\frac{1+2\cos 2\theta+\cos^2 2\theta+\sin^2 2\theta}{1+\cos 2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{\pi\sqrt{2}} \sqrt{\frac{1+\cos 2\theta}{1+\cos 2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{\pi\sqrt{2}} d\theta = \sqrt{2}[\theta]_0^{\pi\sqrt{2}} = 2\pi$$

P3:

Find the areas of the surfaces generated by revolving the curve $r = \sqrt{\cos 2\theta}$, $0 \leq \theta \leq \frac{\pi}{4}$ about the x-axis.

Solution:

We have, $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \frac{\pi}{4}$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

Therefore, the surface area generated by revolving the curve $r = \sqrt{\cos 2\theta}$ about the x-axis is

$$S = \int_0^{\frac{\pi}{4}} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\pi\left(\sqrt{\cos 2\theta}\right)\sin\theta \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \left(\sqrt{\cos 2\theta}\right)\sin\theta \frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta$$

$$= 2\pi[-\cos\theta]_0^{\frac{\pi}{4}} = 2\pi\left(-\frac{1}{\sqrt{2}} + 1\right) = \pi\left(2 - \sqrt{2}\right)$$

P4:

Find the areas of the surfaces generated by revolving the curve

$r = \sqrt{2}e^{\frac{\theta}{2}}$, $0 \leq \theta \leq \frac{\pi}{2}$ about the x-axis.

Solution:

We have, $r = \sqrt{2}e^{\frac{\theta}{2}}$, $0 \leq \theta \leq \frac{\pi}{2}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{\sqrt{2}e^{\frac{\theta}{2}}}{2} = \frac{e^{\frac{\theta}{2}}}{\sqrt{2}}$$

Therefore, the surface area generated by revolving the curve $r = \sqrt{2}e^{\frac{\theta}{2}}$ about the x-axis is

$$S = \int_0^{\frac{\pi}{2}} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\pi\left(\sqrt{2}e^{\frac{\theta}{2}}\right)\sin\theta \sqrt{2e^{\theta} + \frac{e^{\theta}}{2}}\, d\theta$$

$$= 2\sqrt{5}\pi \int_0^{\frac{\pi}{2}} e^{\frac{\theta}{2}} \sin\theta\, e^{\frac{\theta}{2}}\, d\theta$$

$$= 2\sqrt{5}\pi \int_0^{\frac{\pi}{2}} e^{\theta} \sin\theta\, d\theta$$

$$= 2\sqrt{5}\pi \left[\frac{e^{\theta}}{2}(\sin\theta - \cos\theta)\right]_0^{\frac{\pi}{2}} \quad [\because \text{Integrated by parts}]$$

$$= \pi\sqrt{5}\left(e^{\frac{\pi}{2}} + 1\right)$$

IP1:

Find the length of the spiral $r = \theta^2$, $0 \leq \theta \leq \sqrt{5}$.

Solution:

We have, $r = \theta^2$, $0 \leq \theta \leq \sqrt{5}$

$$\Rightarrow \frac{dr}{d\theta} = 2\theta$$

Therefore, the length of the spiral,

$$L = \int_0^{\sqrt{5}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{\sqrt{5}} |\theta| \sqrt{\theta^2 + 4}\, d\theta$$

$$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta \qquad [\because \theta \geq 0]$$

$$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta \qquad [\because \theta \geq 0]$$

Let $\theta^2 + 4 = u \Rightarrow 2\theta\, d\theta = du \Rightarrow \theta\, d\theta = \frac{1}{2} du$

When $\theta = 0 \Rightarrow u = 4$ and $\theta = \sqrt{5} \Rightarrow u = 9$

$$\therefore L = \frac{1}{2}\int_4^9 \sqrt{u}\, du = \frac{1}{2} \times \frac{2}{3}\left[u^{\frac{3}{2}}\right]_4^9 = \frac{1}{3}[27 - 8] = \frac{19}{3}$$

IP2:

Find the length of the curve $r = \sqrt{1+\sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$.

Solution:

We have, $r = \sqrt{1+\sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{1}{2\sqrt{1+\sin 2\theta}} \times \cos 2\theta \times 2 = \frac{\cos 2\theta}{\sqrt{1+\sin 2\theta}}$$

Therefore, the length of the curve,

$$L = \int_0^{\pi\sqrt{2}} \sqrt{\left(\sqrt{1+\sin 2\theta}\right)^2 + \left(\frac{\cos 2\theta}{\sqrt{1+\sin 2\theta}}\right)^2}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{(1+\sin 2\theta) + \frac{\cos^2 2\theta}{(1+\sin 2\theta)}}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{\frac{1+2\sin 2\theta+\sin^2 2\theta+\cos^2 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi\sqrt{2}} \sqrt{\frac{1+\sin 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi\sqrt{2}} d\theta = \sqrt{2}[\theta]_0^{\pi\sqrt{2}} = 2\pi$$

IP3:

Find the areas of the surfaces generated by revolving the curve $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \frac{\pi}{4}$ about the y-axis.

Solution:

We have, $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \frac{\pi}{4}$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

Therefore, the surface area generated by revolving the curve $r = \sqrt{\cos 2\theta}$ about the y-axis is

$$S = \int_0^{\frac{\pi}{4}} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\pi\left(\sqrt{\cos 2\theta}\right) \cos\theta \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \left(\sqrt{\cos 2\theta}\right) \cos\theta \frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \cos\theta\, d\theta = 2\pi[\sin\theta]_0^{\frac{\pi}{4}} = 2\pi\left(\frac{1}{\sqrt{2}}\right) = \pi\sqrt{2}$$

IP4:

Find the areas of the surfaces generated by revolving the curve $r = 2a\cos\theta\,, a > 0\,,\ 0 \le \theta \le \pi$ about the y-axis.

Solution:

We have, $r = 2a\cos\theta\,, a > 0\,,\ 0 \le \theta \le \pi$

$$\Rightarrow \frac{dr}{d\theta} = -2a\sin\theta$$

Therefore, the surface area generated by revolving the curve $r = 2a\cos\theta$ about the y-axis is

$$S = \int_0^{\pi} 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\pi} 2\pi(2a\cos\theta)\cos\theta\sqrt{4a^2\cos^2 2\theta + 4a^2\sin^2 2\theta}\,d\theta$$

$$= 4\pi a\int_0^{\pi}\cos^2\theta\,(2a)\,d\theta \qquad [\because a > 0]$$

$$= 8\pi a^2\int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta = 4\pi a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= 2\pi a^2[2\theta + \sin 2\theta]_0^{\pi} = 2\pi a^2[2\pi] = 4a^2\pi^2$$

1. Find the lengths of the curves.

a. The cardioid, $r = 1 + \cos\theta$

b. The curve, $r = a\sin^2\frac{\theta}{2}$, $0 \le \theta \le \pi$, $a > 0$

c. The parabolic segment, $r = \dfrac{6}{1+\cos\theta}$, $0 \le \theta \le \frac{\pi}{2}$

d. The parabolic segment, $r = \dfrac{2}{1-\cos\theta}$, $\frac{\pi}{2} \le \theta \le \pi$

e. The curve, $r = \cos^3\frac{\theta}{3}$, $0 \le \theta \le \frac{\pi}{4}$

2. Find the area of the surface generated by the revolving the curve $r^2 = \cos 2\theta$ about the x-axis.

3.4. Polar Equations for Conic Sections

Learning objectives:

- To derive Polar equations for parabola, ellipse and hyperbola

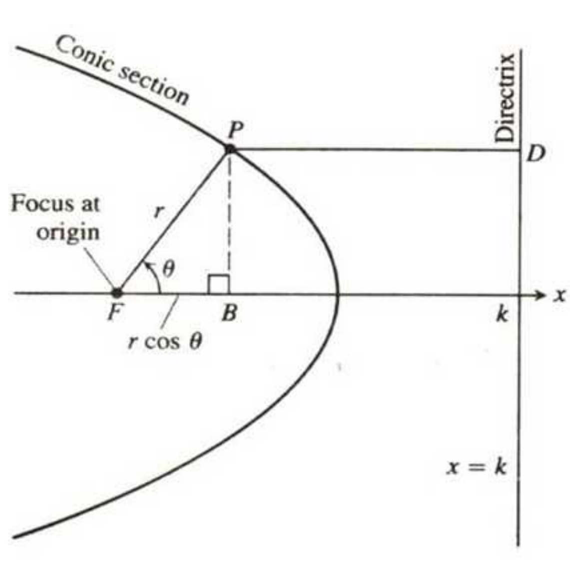
  AND

- To practice related problems

Polar coordinates are important in astronomy and astronautical engineering because the ellipses, parabolas, and hyperbolas along which satellites, moons, planets, and comets approximately move can all be described with a single relatively simple coordinate equation. We learn that equation here.

Polar Equation for a Conic:

To find polar equations for ellipses, parabolas, and hyperbolas, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$.



This makes

$$PF = r$$

and

$$PD = k - FB = k - r\cos\theta$$

The conic's focus-directrix equation $PF = e.PD$ then becomes

$$r = e(k - r\cos\theta)$$

which can be solved for r to obtain

$$r = \frac{ke}{1+e\cos\theta} \quad \text{--- ---- --- --- (1)}$$

The equation (1) represents an ellipse if $0 < e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$. Now, we have ellipses, parabolas, and hyperbolas all with the same basic equation in terms of eccentricity and location of the directrix.

Example 1 : Polar Equations of some conics:

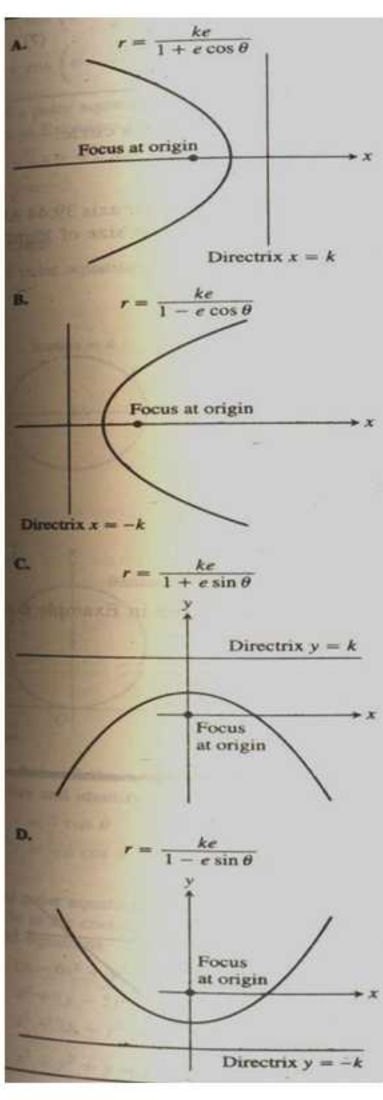
$e = \frac{1}{2}$:	**ellipse**	$r = \frac{k}{2+\cos\theta}$
$e = 1$:	**parabola**	$r = \frac{k}{1+\cos\theta}$
$e = 2$:	**hyperbola**	$r = \frac{2k}{1+2\cos\theta}$

Variations of equation (1) will occur from time to time, depending on the location of the directrix.

If the directrix is the line $x = -k$ to the left of the origin (the origin is still a focus), we replace equation (1) by

$$r = \frac{ke}{1 - e\cos\theta}$$

The denominator now has a $(-)$ instead of a $(+)$. If the directrix is either of the lines $y = k$ or $y = -k$, the equations we get have *sines* in them instead of *cosines*, as shown in the table below.



Example 2
Find an equation for the hyperbola with eccentricity 3/2 and directrix $x = 2$.

Solution
We use equation (A) in the table with $k = 2$ and $e = 3/2$ to get

$$r = \frac{2\left(\frac{3}{2}\right)}{1+\left(\frac{3}{2}\right)\cos\theta} = \frac{6}{2+3\cos\theta}$$

Example 3
Find the directrix of the parabola

$$r = \frac{25}{10+10\cos\theta}$$

Solution
We divide the numerator and denominator by 10 to put the equation in standard form:

$$r = \frac{\frac{5}{2}}{1+\cos\theta}$$

This is the equation

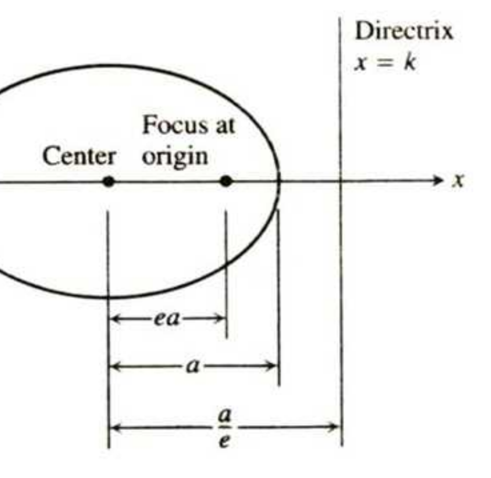
$$r = \frac{ke}{1+e\cos\theta}$$

with $k = 5/2$ and $e = 1$. The equation of the directrix is $x = 5/2$.

Polar equation for the ellipse with eccentricity e and semi-major axis a:

From the ellipse diagram in figure below, we see that k is related to the eccentricity e and the semimajor axis a by the equation

$$k = \frac{a}{e} - ea \quad \text{---- ---- ---- ---- (2)}$$



From this, we find that $ke = a(1 - e^2)$. Replacing ke in equation (1) by $a(1 - e^2)$ gives the standard polar equation for an ellipse.

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad \text{--- ---- --- --- (3)}$$

When $e = 0$, equation (3) becomes $r = a$, which represents a circle.

Equation (3) is the starting point for calculating planetary orbits.

Example 4
Find a polar equation for an ellipse with semimajor axis 39.44 AU (astronomical units) and eccentricity 0.25. This is the approximate size of Pluto's orbit around the sun.

Solution
We use equation (3) with $a = 39.44$ and $e = 0.25$ to find

$$r = \frac{39.44(1-0.25^2)}{1+0.25\cos\theta} = \frac{147.9}{4+\cos\theta}$$

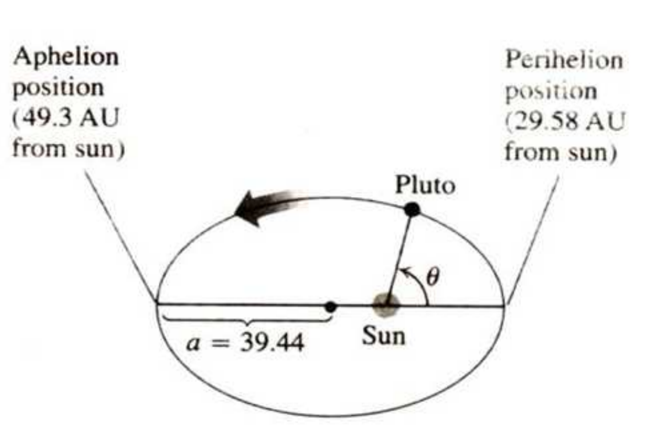
At its point of closest approach (perihelion)$(i.e., \theta = 0)$, Pluto is

$$r = \frac{147.9}{4+1} = 29.58\ AU$$

from the sun. At its most distant point (aphelion)$(i.e., \theta = \pi)$, Pluto is

$$r = \frac{147.9}{4-1} = 49.3\ AU$$

from the sun.



Example 5
Find the distance from one focus of the ellipse in example 4 to the associated directrix.

Solution
We use equation (2) with $a = 39.44$ and $e = 0.25$ to find

$$k = a\left(\frac{1}{e} - e\right)$$

$$= 39.44\left(\frac{1}{0.25} - 0.25\right) = 147.9\ AU$$

P1:

Find an equation for the conic section with eccentricity 1 and directrix $y = 2$.

Solution:

We have, eccentricity $e = 1$

and directrix, $y = 2\ (y = k) \Rightarrow k = 2$.

So, the conic section is a parabola whose axis is y-axis.

Therefore, its polar equation of the given conic section is:

$$r = \frac{ke}{1+e\sin\theta} \Rightarrow r = \frac{2\times 1}{1+1.\sin\theta}$$

$$\Rightarrow r = \frac{2}{1+\sin\theta}$$

P2:

Find an equation for the conic section with eccentricity $\frac{1}{4}$ and directrix $x = -2$.

Solution:

We have, eccentricity $e = \frac{1}{4}$

and directrix, $x = -2 \ (x = -k) \Rightarrow k = 2$.

So the conic section is an ellipse whose axis is the x-axis.

Therefore, the polar equation of the given conic section is:

$$r = \frac{ke}{1 - e\cos\theta} \Rightarrow r = \frac{2 \times \frac{1}{4}}{1 - \frac{1}{4}.\cos\theta}$$

$$\Rightarrow r = \frac{2}{4 - \cos\theta}$$

P3:

Sketch the parabola. Include the directrix that corresponds to the focus at the origin. Label the vertex with appropriate polar coordinates.

$$r = \frac{4}{2 - 2\cos\theta}$$

Solution:

We have, $r=\frac{4}{2-2\cos\theta} \Rightarrow r=\frac{2}{1-\cos\theta}$ -------------------------- (1)

We have the basic equation for conic sections

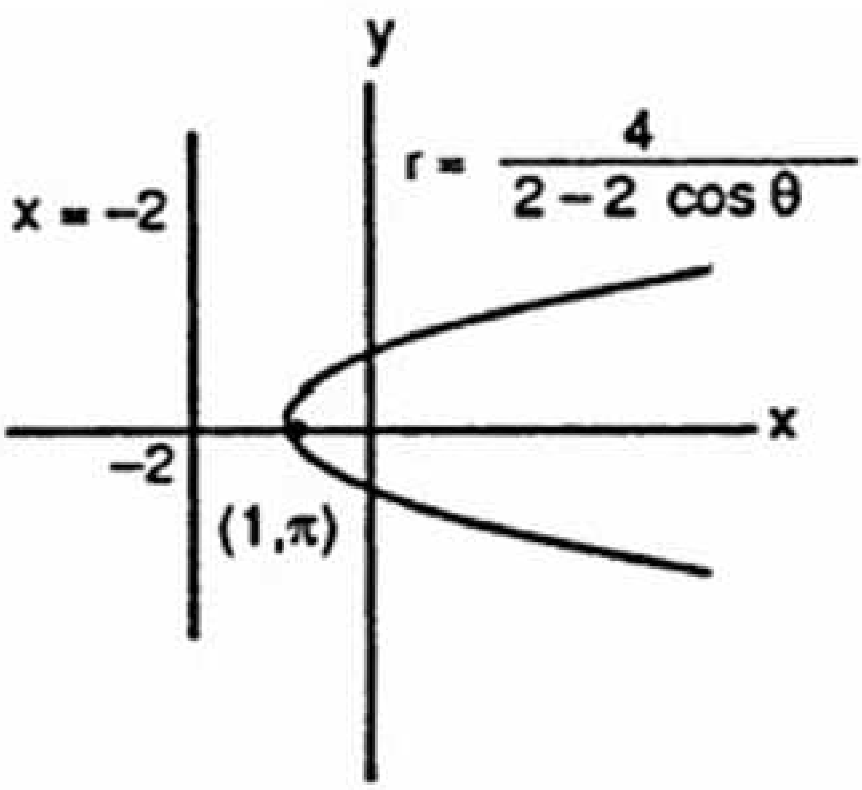
$$r=\frac{ke}{1+e\cos\theta} \text{ ------------ (2)}$$

By comparing (1) and (2), we have eccentricity $e=1$ and $ke=2 \Rightarrow k=2$

We have the eccentricity 1 and the denominator of (1) is $1-\cos\theta$ (having negative sign with cosine function), so the graph of the given equation is a parabola which opens right. So the directrix is $x=-k \Rightarrow x=-2$.

The vertex of the parabola is on negative x-axis, so $\theta=\pi$.

Now, $r=\frac{2}{1-\cos\pi}=\frac{2}{2}=1$. Therefore, the polar coordinates of vertex $=(1,\pi)$.



P4:

Sketch the ellipse. Include the directrix that corresponds to the focus at the origin. Label the vertex with appropriate polar coordinates. Find the center of the ellipse.

$$r = \frac{25}{10 - 5\cos\theta}$$

Solution:

We have, $r = \frac{25}{10-5\cos\theta} \Rightarrow r = \frac{\frac{5}{2}}{1-\frac{1}{2}\cos\theta}$ ---------------- (1)

We have the basic equation for conic sections

$$r = \frac{ke}{1+e\cos\theta} \text{ ------------ (2)}$$

By comparing (1) and (2), we have eccentricity $e = \frac{1}{2}$ and

$ke = \frac{5}{2} \Rightarrow k = 5.$

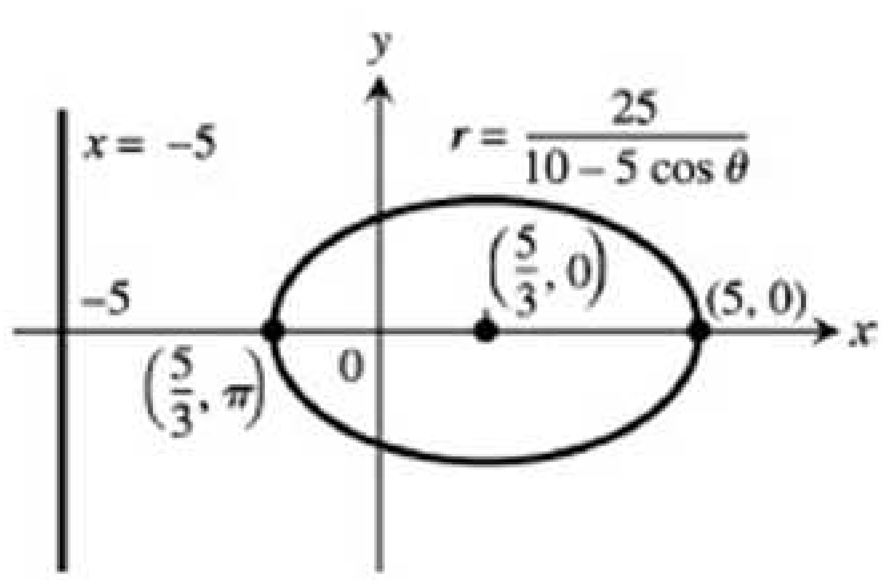
We have the eccentricity $\frac{1}{2}$ and the denominator of (1) is $1 - \frac{1}{2}\cos\theta$ (having negative sign with cosine function), so the graph of the given equation is an ellipse whose center lies on the positive x-axis $(\theta = 0)$. So the directrix is

$$x = -k \Rightarrow x = -5.$$

We have, $ke = a(1-e^2) \Rightarrow a = \frac{10}{3}$

The vertices of the ellipse in polar coordinates are $(r_1, 0)$ and (r_2, π) i.e., .$(5,0)$ and $\left(\frac{5}{3}, \pi\right)$.

The center of the ellipse in polar coordinates is $(ea, 0) = \left(\frac{5}{3}, 0\right)$



IP1:

Find an equation for the conic section with eccentricity 1 and directrix $x = 2$.

Solution:

We have, eccentricity $e = 1$

and directrix, $x = 2\ (x = k) \Rightarrow k = 2$.

So the conic section is a parabola whose axis is x-axis.

Therefore, the polar equation of the given conic section is:

$$r = \frac{ke}{1+e\cos\theta} \Rightarrow r = \frac{2\times 1}{1+1.\cos\theta}$$

$$\Rightarrow r = \frac{2}{1+\cos\theta}$$

IP2:

Find an equation for the conic section with eccentricity 5 and directrix $y = -6$.

Solution:

We have, eccentricity $e = 5$

and directrix, $y = -6 \ (y = -k) \Rightarrow k = 6.$

So the conic section is a hyperbola whose axis is the y-axis.

Therefore, the polar equation of the given conic section is:

$$r = \frac{ke}{1 - e\sin\theta} \Rightarrow r = \frac{6 \times 5}{1 - 5.\sin\theta}$$

$$\Rightarrow r = \frac{30}{1 - 5\sin\theta}$$

IP3:

Sketch the parabola. Include the directrix that corresponds to the focus at the origin. Label the vertex with appropriate polar coordinates.

$$r = \frac{1}{1 + \cos\theta}$$

Solution:

We have, $r = \frac{1}{1+\cos\theta}$ --------------------------- (1)

We have the basic equation for conic sections

$$r = \frac{ke}{1+e\cos\theta} \text{ ------------ (2)}$$

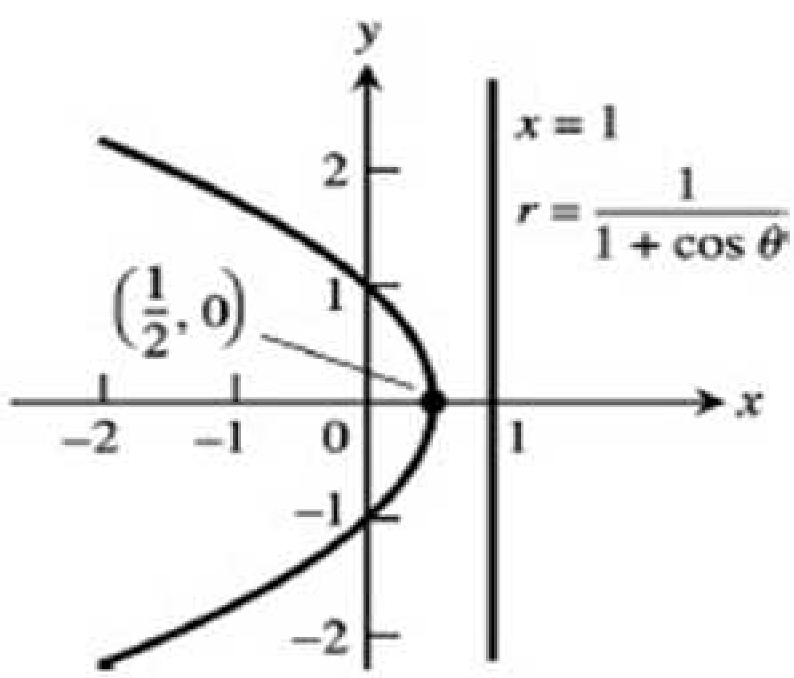
By comparing (1) and (2), we have

eccentricity $e = 1$ and $ke = 1 \Rightarrow k = 1$

We have the eccentricity 1 and the denominator of (1) is $1 + \cos\theta$ (having positive sign with cosine function), so the graph of the given equation is a parabola which opens left. So the directrix is $x = k \Rightarrow x = 1$.

The vertex of the parabola is on positive x-axis, so $\theta = 0$.

Now, $r = \frac{1}{1+\cos\theta} = \frac{1}{1+\cos 0} = \frac{1}{2}$. Therefore, the polar coordinates of vertex $= \left(\frac{1}{2}, 0\right)$.



IP4:

Sketch the ellipse. Include the directrix that corresponds to the focus at the origin. Label the vertex with appropriate polar coordinates. Find the center of the ellipse.

$$r = \frac{6}{2 + \cos\theta}$$

Solution:

We have, $r = \frac{6}{2+\cos\theta} \Rightarrow r = \frac{3}{1+\frac{1}{2}\cos\theta}$ ---------------- (1)

We have the basic equation for conic sections

$$r = \frac{ke}{1+e\cos\theta} \text{ ------------ (2)}$$

By comparing (1) and (2), we have eccentricity $e = \frac{1}{2}$ and

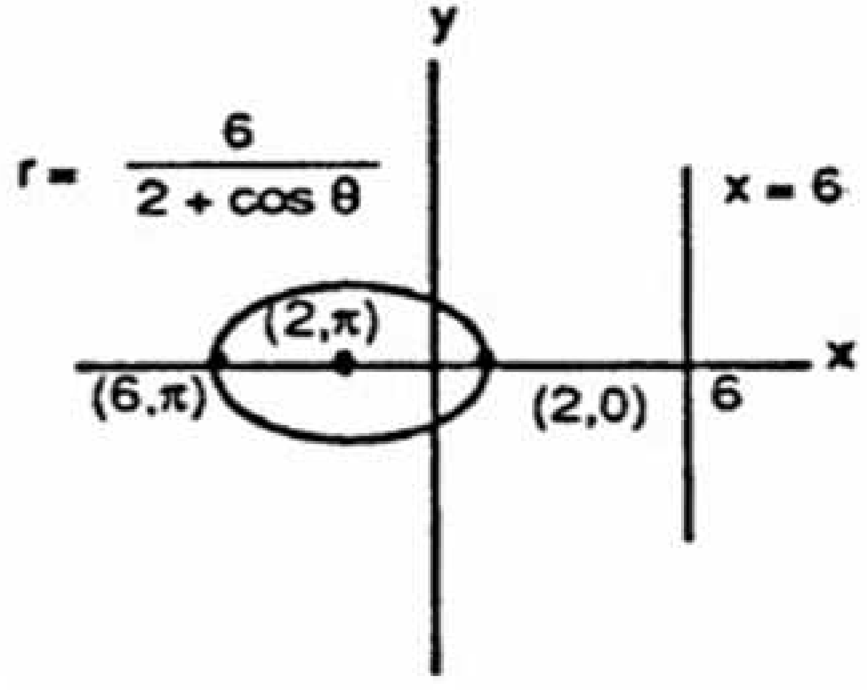
$ke = 3 \Rightarrow k = 6$

We have the eccentricity $\frac{1}{2}$ and the denominator of (1) is $1 + \frac{1}{2}\cos\theta$ (having positive sign with cosine function), so the graph of the given equation is an ellipse whose center lies on the negative x-axis $(\theta = \pi)$. So the directrix is $x = k \Rightarrow x = 6.$

We have, $ke = a(1 - e^2) \Rightarrow a = 4$

The vertices of the ellipse in polar coordinates are $(r_1, 0)$ and (r_2, π) i.e., .$(2,0)$ and $(6, \pi)$.

The center of the ellipse in polar coordinates is $(ea, \pi) = (2, \pi)$



1. Given the eccentricities of conic sections with one focus at the origin, along with the directrix corresponding to that focus. Find a polar equation for each conic section.

a. $e = 2, x = 4$

b. $e = \frac{1}{2}, x = 1$

c. $e = \frac{1}{5}, y = -10$

d. $e = \frac{1}{3}, y = 6$

2. Sketch the parabolas and ellipses given below. Include the directrix that corresponds to the focus at the origin. Label the vertices with appropriate polar coordinates. Label the centers of the ellipses as well.

a. $r = \dfrac{400}{16+8\sin\theta}$

b. $r = \dfrac{12}{3+3\sin\theta}$

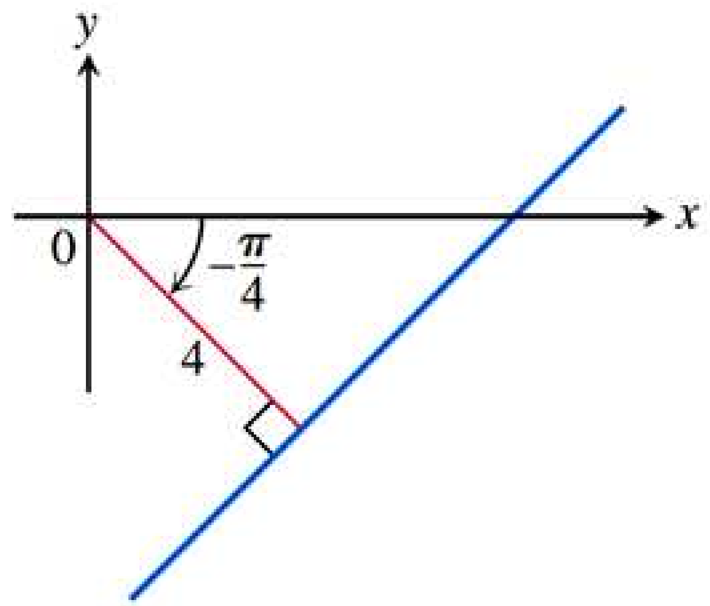
c. $r = \dfrac{8}{2-2\sin\theta}$

d. $r = \dfrac{4}{2-\sin\theta}$

HWA-7 (3.3, 3.4)

Answer all Questions and Submit.

1. Find the polar and Cartesian equations for the line in the given figure



2. Find the Cartesian equation for $r = cos\left(\theta + \frac{\pi}{3}\right)$ and sketch the graph.

3. Find a polar equation of the form $rcos(\theta - \theta_0) = r_0$ for the line $\sqrt{3}x - y = 1$

4. Find an equation for the conic section with eccentricity $\frac{1}{5}$ and directrix $y = -10$.

5. Find an equation for the conic section with eccentricity $\frac{1}{2}$ and directrix $x = 3$.

6. Sketch the ellipse. Include the directrix that corresponds to the focus at the origin. Label the vertex with appropriate polar coordinates. Find the center of the ellipse.

$$r = \frac{4}{2 - \sin\theta}$$

7. Sketch the parabola. Include the directrix that corresponds to the focus at the origin. Label the vertex with appropriate polar coordinates.

$$r = \frac{8}{2 - 2\sin\theta}$$

2.5

Area in Polar Coordinates

Learning Objectives:

- To derive a formula for the area of a plane region in polar coordinates
- To compute the areas of plane regions of some polar curves
- To compute the area between polar curves

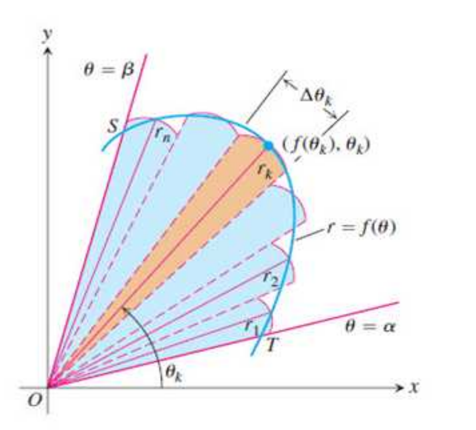
And

- To practice related problems

Area in Polar Coordinates

In this module, we learn how to calculate areas of plane regions in polar coordinates.

Area in the plane:



The region OTS is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is

$$A_k = \frac{1}{2} r_k{}^2 \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$

The area of region OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$

If f is continuous, we expect the approximations to improve as $\|P\| \to 0$, and we are led to the following formula for the region's area:

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$

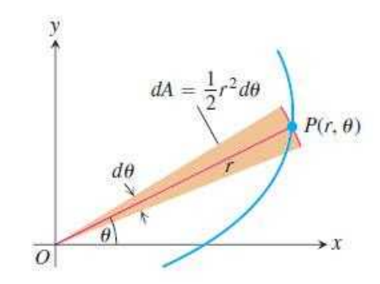
$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta$$

Area of the Fan-shaped Region between the Origin and the Curve $\boldsymbol{r = f(\theta), \alpha \le \theta \le \beta}$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

This is the integral of the **area differential**

$$dA = \frac{1}{2} r^2 d\theta$$

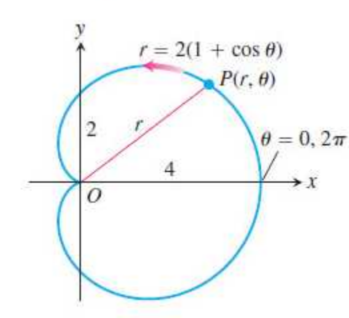


Example 1

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution

We graph the cardioid and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π.



The area is therefore

$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 d\theta = \int_{0}^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 d\theta$$

$$= \int_{0}^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) d\theta$$

$$= \int_{0}^{2\pi} \left(2 + 4\cos\theta + 2 \cdot \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_{0}^{2\pi} (3 + 4\cos\theta + \cos 2\theta) d\theta$$

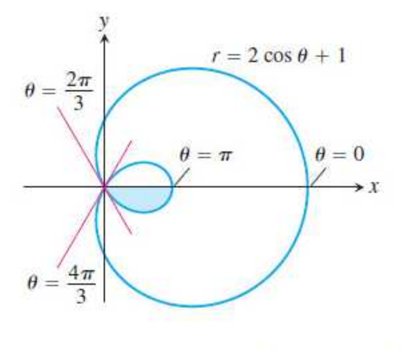
$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 6\pi - 0 = 6\pi$$

Example 2

Find the area inside the smaller loop of the limacon $r = 2\cos\theta + 1$.

Solution



The polar curve passes through the pole when

$$r = 0 \Rightarrow 2\cos\theta + 1 = 0$$

$$\Rightarrow \cos\theta = -\frac{1}{2}$$

$$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

We see that the smaller loop is traced out by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the x-axis, we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2 d\theta = \int_{2\pi/3}^{\pi} r^2 d\theta$$

Since $r^2 = (2\cos\theta + 1)^2 = 4\cos^2\theta + 4\cos\theta + 1$

$$= 4 \cdot \frac{1 + \cos 2\theta}{2} + 4\cos\theta + 1$$

$$= 2 + 2\cos 2\theta + 4\cos\theta + 1$$

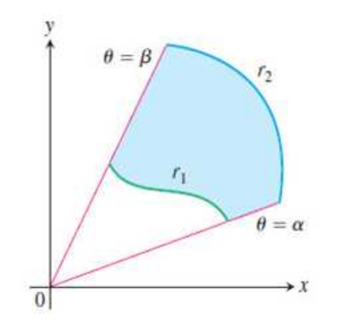
$$= 3 + 2\cos 2\theta + 4\cos\theta$$

we have $A = \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta) d\theta$

$$= [3\theta + \sin 2\theta + 4\sin\theta]_{2\pi/3}^{\pi}$$

$$= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4 \cdot \frac{\sqrt{3}}{2}\right) = \pi - \frac{3\sqrt{3}}{2}$$

To find the area of a region like the one in the figure below, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(\frac{1}{2})r_1^2 d\theta$ from the integral of $(\frac{1}{2})r_2^2 d\theta$.



This leads to the following formula:

Area of the Region $\boldsymbol{0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta}$

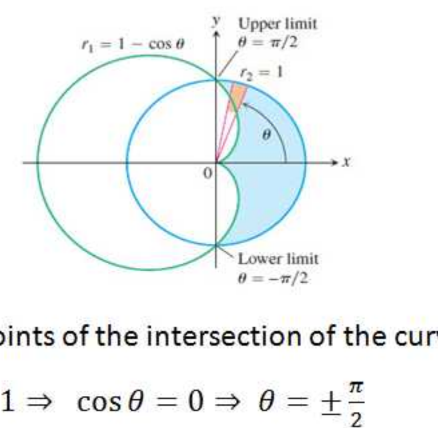
$$\boldsymbol{A = \int_{\alpha}^{\beta} \frac{1}{2} r_2{}^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1{}^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2{}^2 - r_1{}^2\right) d\theta}$$

Example 3

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution

We sketch the region to determine its boundaries and find the limits of integration.



To find the points of the intersection of the curves,

$$1 - \cos\theta = 1 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{2}$$

The points of intersection of the curves are $\left(1, \frac{\pi}{2}\right), \left(1, -\frac{\pi}{2}\right)$.

The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area is

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(r_2{}^2 - r_1{}^2\right) d\theta = 2\int_{0}^{\frac{\pi}{2}} \frac{1}{2}\left(r_2{}^2 - r_1{}^2\right) d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \left(1 - (1 - 2\cos\theta + \cos^2\theta)\right) d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} (2\cos\theta - \cos^2\theta) d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} = 2 - \frac{\pi}{4}$$

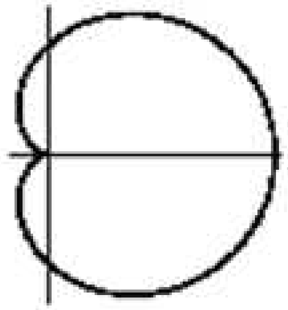
P1:

Find the area of the region inside the cardioid

$r = a(1 + \cos\theta)\ , a > 0.$

Solution:

Cardioid:



The area of the cardioid $r = a(1 + \cos\theta)$ is

$$A = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 d\theta = \frac{a^2}{2} \int_0^{2\pi} (1 + \cos\theta)^2 d\theta$$

$$= \frac{a^2}{2} \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta$$

$$= \frac{a^2}{2} \int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$

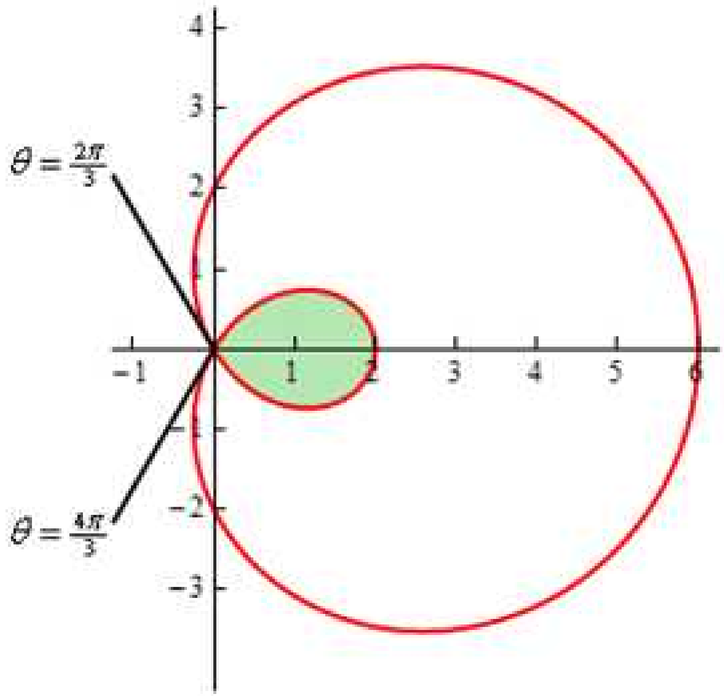
$$= \frac{a^2}{4} \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) d\theta$$

$$= \frac{a^2}{4} \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{4} [6\pi - 0] = \frac{3}{2} \pi a^2$$

P2:

Determine the area of the inner loop of $r = 2 + 4\cos\theta$



Solution:

The given polar curve is a limacon with an inner loop. The limacon passes through the pole when

$$2 + 4\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

The area of the inner loop of $r = 2 + 4\cos\theta$ is

$$A = \int_{\theta=\frac{2\pi}{3}}^{\theta=\frac{4\pi}{3}} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2 + 4\cos\theta)^2 d\theta$$

$$= 2\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 4\cos\theta + 4\cos^2\theta) d\theta$$

$$= 2\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left(1 + 4\cos\theta + 4.\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 2[3\theta + 4\sin\theta + \sin 2\theta]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

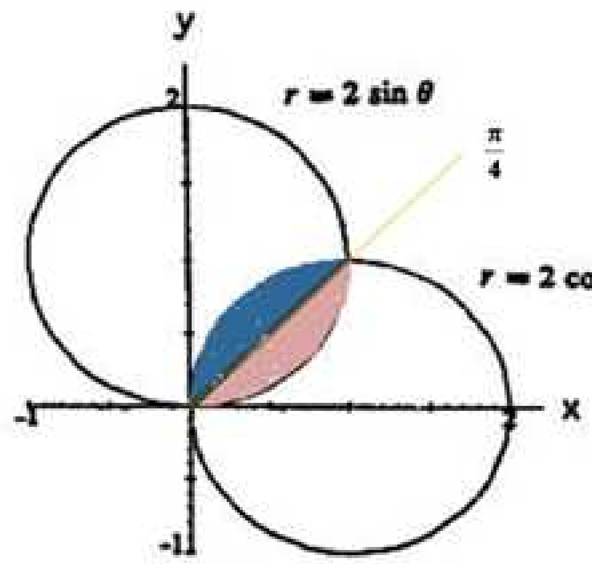
$$= 2\left[\left(4\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) - \left(2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)\right] = 4\pi - 6\sqrt{3}$$

P3:

Find the area of the region shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$.

Solution:

The area of the region shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$ is shaded in the figure given below.



We have $r = 2\cos\theta$ and $r = 2\sin\theta$

$$\Rightarrow 2\cos\theta = 2\sin\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$$

The point of intersection is $\left(\sqrt{2}, \frac{\pi}{4}\right)$. The point of intersection $(0,0)$ is found by graphing.

Therefore, the area of the shaded region is

$$A = \int_0^{\frac{\pi}{4}} \frac{1}{2}(2\sin\theta)^2 d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\sin^2\theta \, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1 - \cos 2\theta) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} + \left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

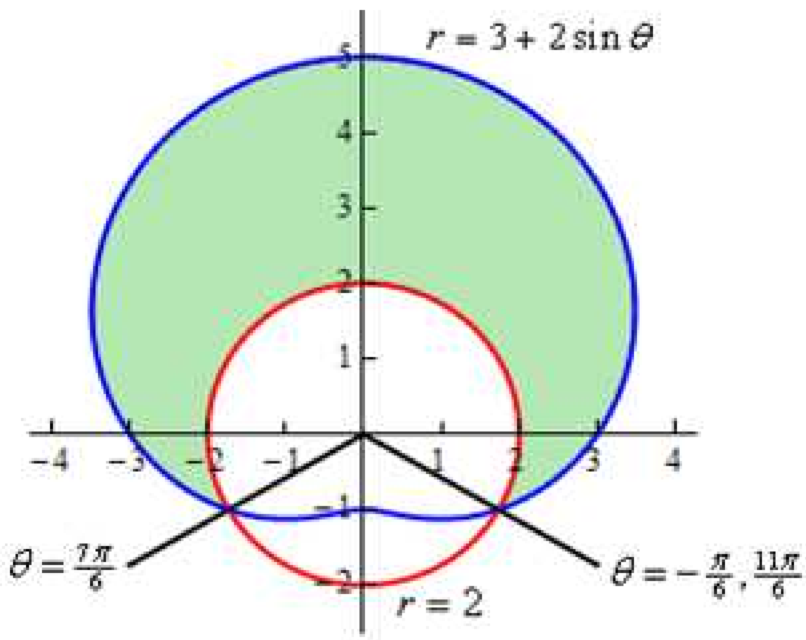
$$= \left[\frac{\pi}{4} - \frac{1}{2}\right] + \left[\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right] = \frac{\pi}{2} - 1$$

P4:

Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution:

The area of the region shared by $r = 3 + 2\sin\theta$ and $r = 2$ is shaded in the figure given below.



We have $r = 3 + 2\sin\theta$ and $r = 2$

$$\Rightarrow 3 + 2\sin\theta = 2 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6} \ or -\frac{\pi}{6} \ or \ \frac{11\pi}{6}$$

Therefore, the area of the shaded region is

$$A = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}[(3 + 2\sin\theta)^2 - (2)^2]d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}(5 + 12\sin\theta + 4\sin^2\theta)d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\big(5 + 12\sin\theta + 2(1 - \cos 2\theta)\big)d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}(7 + 12\sin\theta - 2\cos 2\theta)d\theta$$

$$= \frac{1}{2}[7\theta - 12\cos\theta - \sin 2\theta]_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \frac{1}{2}\left[\left(\frac{49\pi}{6} + 6\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \left(-\frac{7\pi}{6} - 6\sqrt{3} + \frac{\sqrt{3}}{2}\right)\right]$$

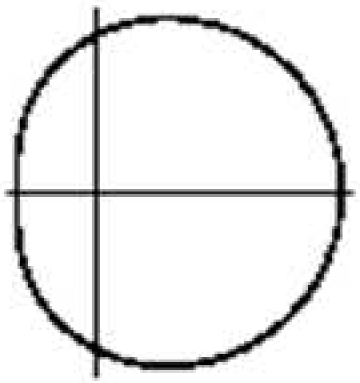
$$= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}$$

IP1:

Find the area of the region inside the oval limacon $r = 4 + 2\cos\theta$.

Solution:

Limacon:



The area of the limacon $r = 4 + 2\cos\theta$ is

$$A = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} \cdot (4 + 2\cos\theta)^2 d\theta$$

$$= \int_0^{2\pi} 2(4 + 4\cos\theta + \cos^2\theta) d\theta$$

$$= \int_0^{2\pi} \left(8 + 8\cos\theta + 2 \cdot \frac{1 + \cos 2\theta}{2}\right) d\theta$$

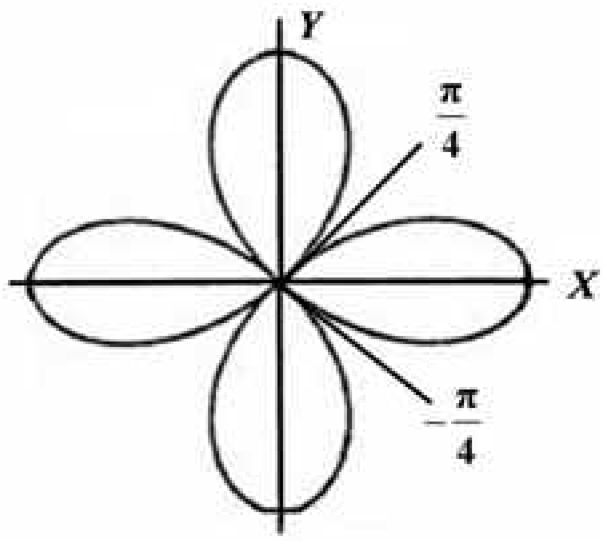
$$= \int_0^{2\pi} (9 + 8\cos\theta + \cos 2\theta) d\theta$$

$$= \left[9\theta + 8\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 18\pi - 0 = 18\pi$$

IP2:

Determine the area of the region inside one leaf of the four-leaved rose $r = \cos 2\theta$.



Solution:

The given polar curve is a four leaved rose. This curve passes through the pole when

$$\cos 2\theta = 0 \Rightarrow 2\theta = (2n+1)\frac{\pi}{2} \Rightarrow \theta = (2n+1)\frac{\pi}{4}$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\left(= -\frac{\pi}{4}\right)$$

The area of one leaf of the four-leaved rose $r = \cos 2\theta$ is

$$A = \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

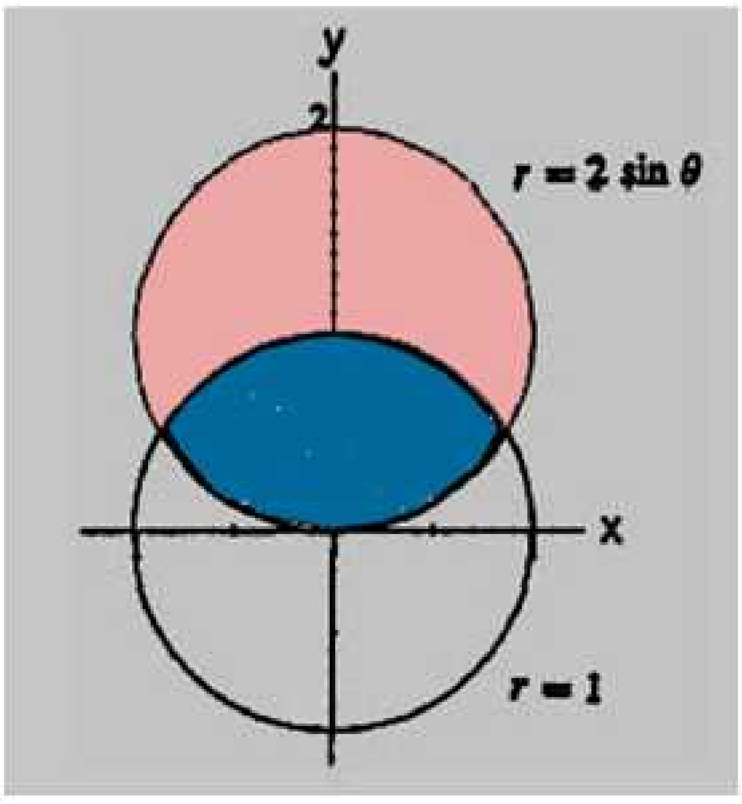
$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{2} \, d\theta = \frac{1}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{4}\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{\pi}{8}$$

IP3:

Find the area of the region shared by the circles $r = 1$ and $r = 2\sin\theta$.

Solution:

The area of the region shared by the circles $r = 1$ and $r = 2\sin\theta$ is shaded in the figure given below.



We have $r = 1$ and $r = 2\sin\theta$

The points of intersection of the curves is given by

$$1 = 2\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \ or \ \frac{5\pi}{6}$$

Therefore, the points of intersection are $\left(1, \frac{\pi}{6}\right)$ and $\left(1, \frac{5\pi}{6}\right)$.

The area of the region shaded in orange is given by

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}[(2\sin\theta)^2 - (1)^2]d\theta$$

The area of the shared region colored in blue is

$$A = \pi(1)^2 - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}[(2\sin\theta)^2 - (1)^2]d\theta$$

$$= \pi - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (4\sin^2\theta - 1)d\theta$$

$$= \pi - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2(1 - \cos 2\theta) - 1)d\theta$$

$$= \pi - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 - 2\cos 2\theta)d\theta$$

$$= \pi - \frac{1}{2}[\theta - \sin 2\theta]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \pi - \frac{1}{2}\left[\left(\frac{5\pi}{6} + \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)\right]$$

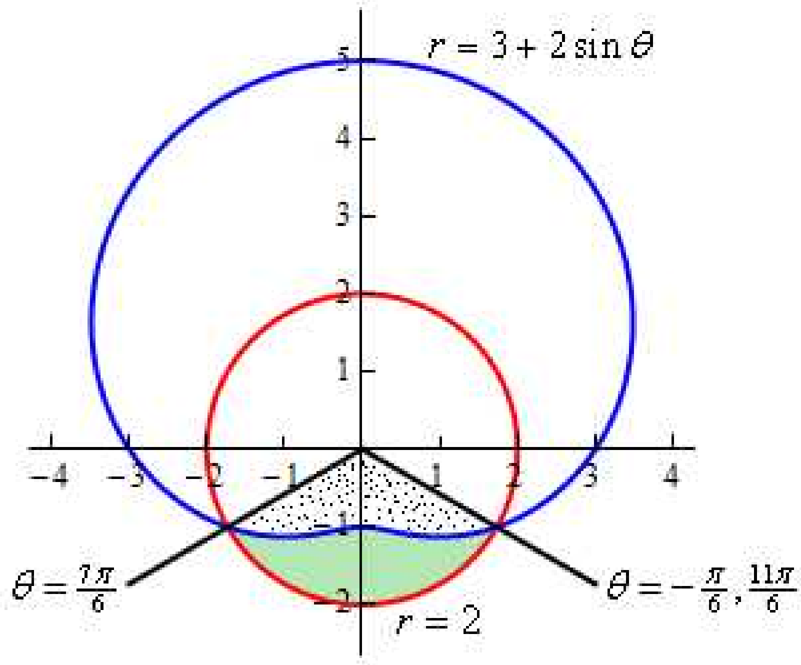
$$= \pi - \frac{1}{2}\left[\frac{2\pi}{3} + \sqrt{3}\right] = \frac{4\pi - 3\sqrt{3}}{6}$$

IP4:

Determine the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$.

Solution:

The area of the region shared by $r = 3 + 2\sin\theta$ and $r = 2$ is shaded in the figure given below.



We have $r = 3 + 2\sin\theta$ and $r = 2$

$$\Rightarrow 3 + 2\sin\theta = 2 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6} \ or -\frac{\pi}{6} \ or \ \frac{11\pi}{6}$$

Therefore, the area of the shaded region is

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}[(2)^2 - (3 + 2\sin\theta)^2]d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}(-5 - 12\sin\theta - 4\sin^2\theta)d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\big(-5 - 12\sin\theta - 2(1 - \cos 2\theta)\big)d\theta$$

$$= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (-7 - 12\sin\theta + 2\cos 2\theta)d\theta$$

$$= \frac{1}{2}[-7\theta + 12\cos\theta + \sin 2\theta]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

$$= \frac{1}{2}\left[\left(-\frac{77\pi}{6} + 6\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \left(-\frac{49\pi}{6} - 6\sqrt{3} + \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{11\sqrt{3}}{2} - \frac{7\pi}{3}$$

1. Find the areas of the regions.

a. Shared by the circle $r = 2$ and the cardioids $r = 2(1 - \cos\theta)$.

b. Shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$.

c. Inside the circle $r = 3\,a\cos\theta$ and outside the cardioids $r = a(1 + \cos\theta)\,, a > 0$.

d. Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$.

e. Inside the circle $r = 6$ above the line $r = 3\csc\theta$.

2.6

Length and Surface Area in Polar Coordinates

Learning Objectives:

- To compute the length of a polar plane curve
- To compute the area of surface of revolution of polar plane curve

AND

- To practice the related problems

In this module, we learn how to calculate lengths of plane curves and areas of surfaces of revolution of plane curves in polar coordinates.

The Length of a Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$\left.\begin{array}{l} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{array}\right\} \quad \alpha \le \theta \le \beta \quad \text{----- -----} \quad (1)$$

The parametric length formula then gives the length as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \quad \text{--- --- (2)}$$

When equations (1) are substituted for x and y in equation (2),

it becomes $\quad L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

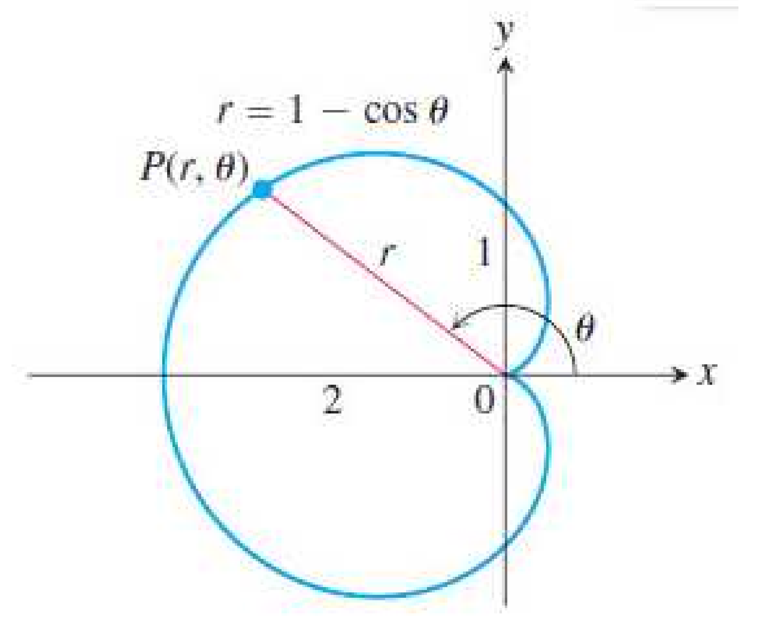
Length

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r,\theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{---- ---- ----} \quad (3)$$

Example 1

Find the length of the cardioid $r = 1 - \cos\theta$.



The point $P(r,\theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.

With $r = 1 - \cos\theta$, $\frac{dr}{d\theta} = \sin\theta$

we have $r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + (\sin\theta)^2$

$$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta$$

and

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \quad \left[\sin\frac{\theta}{2} > 0 \ \ for\ 0 \le \theta \le 2\pi\right]$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8$$

The Area of a Surface of Revolution

To derive polar coordinate formula for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, with equations (1) and apply the surface area equations.

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r,\theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

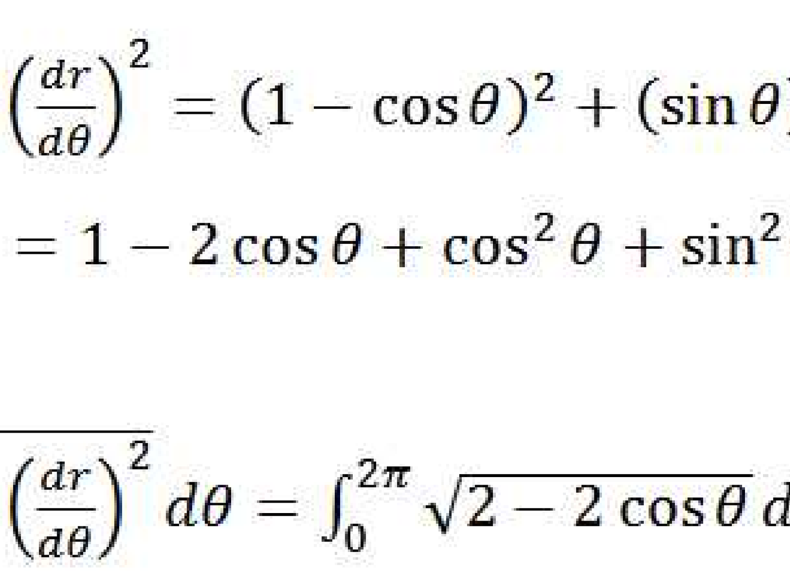
1.	Revolution about the x-axis $(y \ge 0)$	$S = \int_\alpha^\beta 2\pi r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
2.	Revolution about the y-axis $(x \ge 0)$	$S = \int_\alpha^\beta 2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Example 2

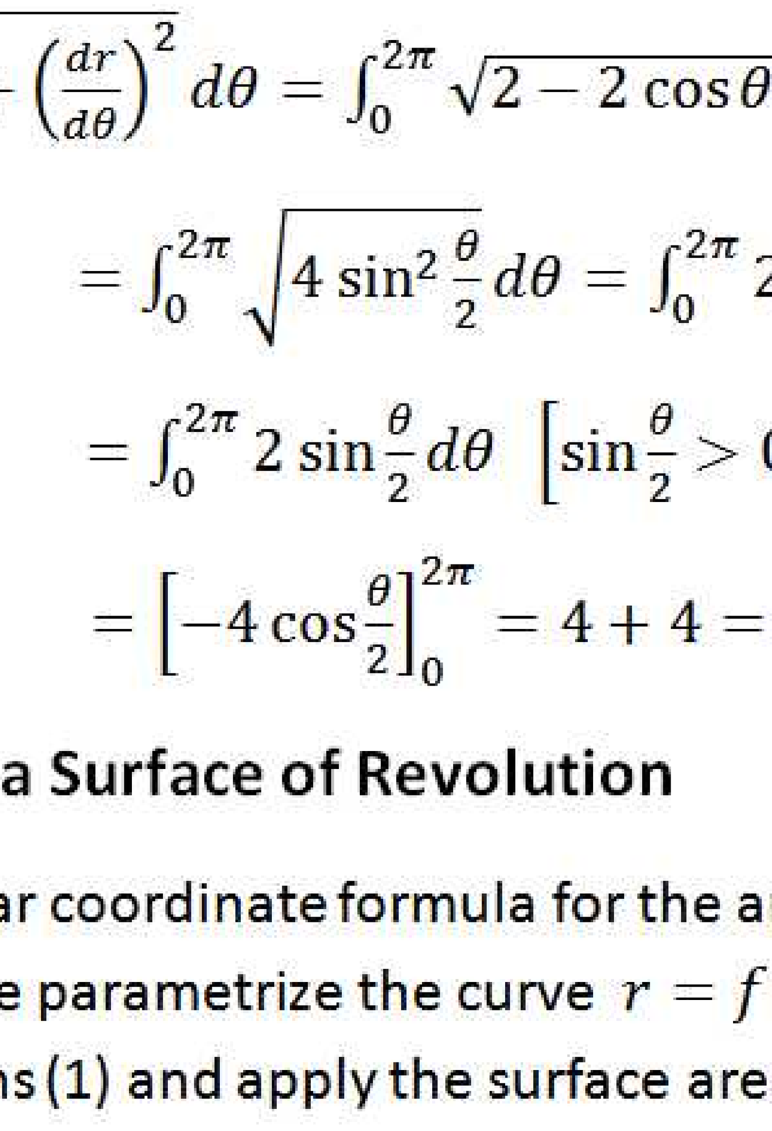
Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

Solution

We sketch the loop to determine the limits of integration.



(a)



(b)

The point $P(r,\theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

First we calculate

$$2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi\cos\theta \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}$$

Since $r^2 = \cos 2\theta$,

$\Rightarrow 2r\frac{dr}{d\theta} = -2\sin 2\theta$

$\Rightarrow r\frac{dr}{d\theta} = -\sin 2\theta$

$\Rightarrow \left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta$

We write $r^4 = (r^2)^2 = \cos^2 2\theta$. The square root simplifies to

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1$$

All together, we have

$$S = \int_\alpha^\beta 2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi\cos\theta\,.\,(1)d\theta$$

$$= 2\pi[\sin\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}$$

P1:

Find the length of the spiral $r = \frac{e^{\theta}}{\sqrt{2}}$, $0 \leq \theta \leq \pi$.

Solution:

We have, $r = \frac{e^{\theta}}{\sqrt{2}}$, $0 \leq \theta \leq \pi$

$$\Rightarrow \frac{dr}{d\theta} = \frac{e^{\theta}}{\sqrt{2}}$$

Therefore, the length of the spiral,

$$L = \int_0^{\pi} \sqrt{\left(\frac{e^{\theta}}{\sqrt{2}}\right)^2 + \left(\frac{e^{\theta}}{\sqrt{2}}\right)^2}\, d\theta = \int_0^{\pi} e^{\theta}\, d\theta = \left[e^{\theta}\right]_0^{\pi} = e^{\pi} - 1$$

P2:

Find the length of the curve $r = \sqrt{1 + \cos 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$.

Solution:

We have, $r = \sqrt{1 + \cos 2\theta}$, $0 \leq \theta \leq \pi\sqrt{2}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{1}{2\sqrt{1+\cos 2\theta}} \times (-\sin 2\theta) \times 2 = -\frac{\sin 2\theta}{\sqrt{1+\cos 2\theta}}$$

Therefore, the length of the curve,

$$L = \int_0^{\pi\sqrt{2}} \sqrt{\left(\sqrt{1 + \cos 2\theta}\right)^2 + \left(-\frac{\sin 2\theta}{\sqrt{1+\cos 2\theta}}\right)^2}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{(1 + \cos 2\theta) + \frac{\sin^2 2\theta}{(1+\cos 2\theta)}}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{\frac{1+2\cos 2\theta+\cos^2 2\theta+\sin^2 2\theta}{1+\cos 2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{\pi\sqrt{2}} \sqrt{\frac{1+\cos 2\theta}{1+\cos 2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{\pi\sqrt{2}} d\theta = \sqrt{2}[\theta]_0^{\pi\sqrt{2}} = 2\pi$$

P3:

Find the areas of the surfaces generated by revolving the curve $r = \sqrt{\cos 2\theta}$, $0 \leq \theta \leq \frac{\pi}{4}$ about the x-axis.

Solution:

We have, $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \frac{\pi}{4}$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

Therefore, the surface area generated by revolving the curve $r = \sqrt{\cos 2\theta}$ about the x-axis is

$$S = \int_0^{\frac{\pi}{4}} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\pi\left(\sqrt{\cos 2\theta}\right) \sin\theta \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \left(\sqrt{\cos 2\theta}\right) \sin\theta \frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta$$

$$= 2\pi[-\cos\theta]_0^{\frac{\pi}{4}} = 2\pi\left(-\frac{1}{\sqrt{2}} + 1\right) = \pi\left(2 - \sqrt{2}\right)$$

P4:

Find the areas of the surfaces generated by revolving the curve

$r = \sqrt{2}e^{\frac{\theta}{2}}$, $0 \leq \theta \leq \frac{\pi}{2}$ about the x-axis.

Solution:

We have, $r = \sqrt{2}e^{\frac{\theta}{2}}$, $0 \leq \theta \leq \frac{\pi}{2}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{\sqrt{2}e^{\frac{\theta}{2}}}{2} = \frac{e^{\frac{\theta}{2}}}{\sqrt{2}}$$

Therefore, the surface area generated by revolving the curve $r = \sqrt{2}e^{\frac{\theta}{2}}$ about the x-axis is

$$S = \int_0^{\frac{\pi}{2}} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\pi \left(\sqrt{2}e^{\frac{\theta}{2}}\right) \sin\theta \sqrt{2e^{\theta} + \frac{e^{\theta}}{2}}\, d\theta$$

$$= 2\sqrt{5}\pi \int_0^{\frac{\pi}{2}} e^{\frac{\theta}{2}} \sin\theta\, e^{\frac{\theta}{2}}\, d\theta$$

$$= 2\sqrt{5}\pi \int_0^{\frac{\pi}{2}} e^{\theta} \sin\theta\, d\theta$$

$$= 2\sqrt{5}\pi \left[\frac{e^{\theta}}{2}(\sin\theta - \cos\theta)\right]_0^{\frac{\pi}{2}} \quad [\because \text{Integrated by parts}]$$

$$= \pi\sqrt{5}\left(e^{\frac{\pi}{2}} + 1\right)$$

IP1:

Find the length of the spiral $r = \theta^2$, $0 \le \theta \le \sqrt{5}$.

Solution:

We have, $r = \theta^2$, $0 \le \theta \le \sqrt{5}$

$$\Rightarrow \frac{dr}{d\theta} = 2\theta$$

Therefore, the length of the spiral,

$$L = \int_0^{\sqrt{5}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{\sqrt{5}} |\theta| \sqrt{\theta^2 + 4}\, d\theta$$

$$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta \qquad [\because \theta \ge 0]$$

$$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta \qquad [\because \theta \ge 0]$$

Let $\theta^2 + 4 = u \Rightarrow 2\theta\, d\theta = du \Rightarrow \theta\, d\theta = \frac{1}{2} du$

When $\theta = 0 \Rightarrow u = 4$ and $\theta = \sqrt{5} \Rightarrow u = 9$

$$\therefore L = \frac{1}{2}\int_4^9 \sqrt{u}\, du = \frac{1}{2} \times \frac{2}{3}\left[u^{\frac{3}{2}}\right]_4^9 = \frac{1}{3}[27 - 8] = \frac{19}{3}$$

IP2:

Find the length of the curve $r = \sqrt{1 + \sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$.

Solution:

We have, $r = \sqrt{1 + \sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{1}{2\sqrt{1+\sin 2\theta}} \times \cos 2\theta \times 2 = \frac{\cos 2\theta}{\sqrt{1+\sin 2\theta}}$$

Therefore, the length of the curve,

$$L = \int_0^{\pi\sqrt{2}} \sqrt{\left(\sqrt{1 + \sin 2\theta}\right)^2 + \left(\frac{\cos 2\theta}{\sqrt{1+\sin 2\theta}}\right)^2}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{(1 + \sin 2\theta) + \frac{\cos^2 2\theta}{(1+\sin 2\theta)}}\, d\theta$$

$$= \int_0^{\pi\sqrt{2}} \sqrt{\frac{1+2\sin 2\theta+\sin^2 2\theta+\cos^2 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{\pi\sqrt{2}} \sqrt{\frac{1+\sin 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{\pi\sqrt{2}} d\theta = \sqrt{2}[\theta]_0^{\pi\sqrt{2}} = 2\pi$$

IP3:

Find the areas of the surfaces generated by revolving the curve $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \frac{\pi}{4}$ about the y-axis.

Solution:

We have, $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \frac{\pi}{4}$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

Therefore, the surface area generated by revolving the curve $r = \sqrt{\cos 2\theta}$ about the y-axis is

$$S = \int_0^{\frac{\pi}{4}} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\pi\left(\sqrt{\cos 2\theta}\right) \cos\theta \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \left(\sqrt{\cos 2\theta}\right) \cos\theta \frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \cos\theta\, d\theta = 2\pi[\sin\theta]_0^{\frac{\pi}{4}} = 2\pi\left(\frac{1}{\sqrt{2}}\right) = \pi\sqrt{2}$$

IP4:

Find the areas of the surfaces generated by revolving the curve $r = 2a\cos\theta, a > 0,\ 0 \le \theta \le \pi$ about the y-axis.

Solution:

We have, $r = 2a\cos\theta, a > 0,\ 0 \le \theta \le \pi$

$$\Rightarrow \frac{dr}{d\theta} = -2a\sin\theta$$

Therefore, the surface area generated by revolving the curve $r = 2a\cos\theta$ about the y-axis is

$$S = \int_0^{\pi} 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi} 2\pi(2a\cos\theta)\cos\theta\sqrt{4a^2\cos^2 2\theta + 4a^2\sin^2 2\theta}\, d\theta$$

$$= 4\pi a\int_0^{\pi}\cos^2\theta\,(2a)\,d\theta \qquad [\because a > 0]$$

$$= 8\pi a^2\int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta = 4\pi a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= 2\pi a^2[2\theta + \sin 2\theta]_0^{\pi} = 2\pi a^2[2\pi] = 4a^2\pi^2$$

1. Find the lengths of the curves.

a. The cardioid, $r = 1 + \cos\theta$

b. The curve, $r = a\sin^2\frac{\theta}{2}$, $0 \le \theta \le \pi$, $a > 0$

c. The parabolic segment, $r = \frac{6}{1+\cos\theta}$, $0 \le \theta \le \frac{\pi}{2}$

d. The parabolic segment, $r = \frac{2}{1-\cos\theta}$, $\frac{\pi}{2} \le \theta \le \pi$

e. The curve, $r = \cos^3\frac{\theta}{3}$, $0 \le \theta \le \frac{\pi}{4}$

2. Find the area of the surface generated by the revolving the curve $r^2 = \cos 2\theta$ about the x-axis.