3.2

Volume of Solids by Slicing

Learning objectives:

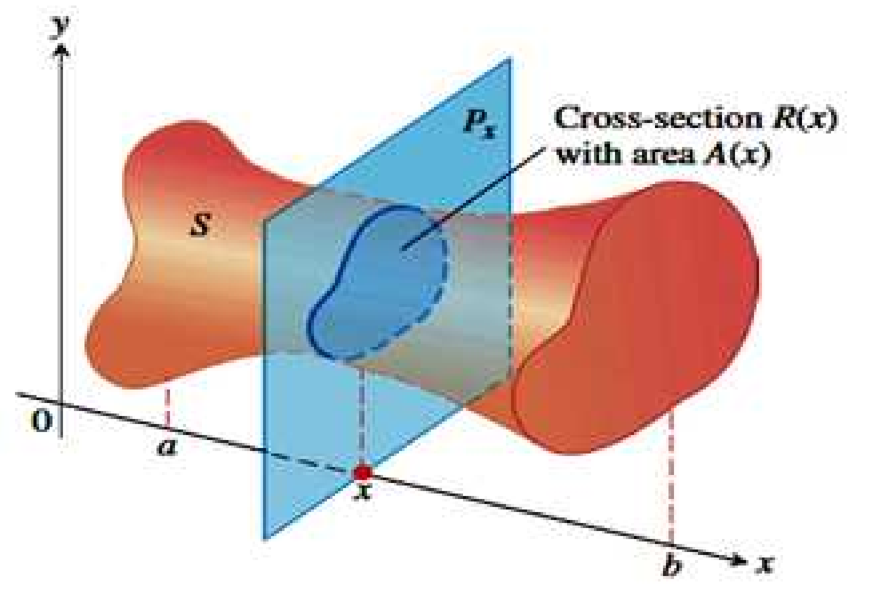
- To find the volume of a solid by the method of slicing.

AND

- To practice the related problems.

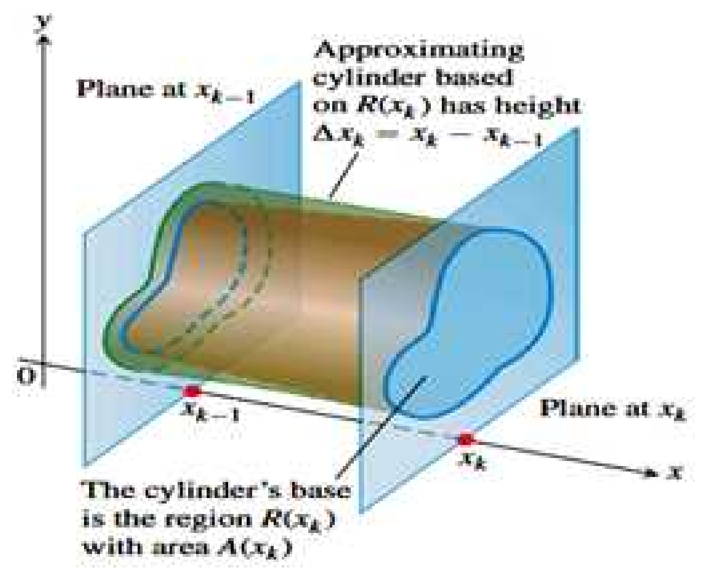
Volume of Solids by Slicing

A *cross section* of a solid *S* is the plane region formed by intersecting *S* with a plane (see the figure below).
Suppose we want to find the volume of a solid like the one shown below.



At each point x in the closed interval $[a,b]$ the cross section of the solid is a region $R(x)$ whose area is $A(x)$. Then A is a real-valued function of x. If it is also a continuous function of x, we can calculate the volume of the solid as an integral in the following way.

We partition the interval $[a,b]$ along the x-axis in the usual manner and slice the solid by planes perpendicular to the x-axis at the partition points. The k^{th} slice, the one between the planes at x_{k-1} and x_k, has approximately the same volume as the cylinder between these two planes based on the region $R(x_k)$.



The volume of the solid is therefore approximated by the cylinder volume sum

$$\sum_{k=1}^{n} A(x_k)\Delta x_k$$

This is a Riemann sum for the function $A(x)$ on $[a,b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a,b]$ goes to zero, so we define their limiting integral to be the volume of the solid.

Volume of a solid

The volume of a solid of known integrable cross-section area $A(x)$ from $x=a$ to $x=b$ is the integral of A from a to b:

$$V=\int_a^b A(x)\,dx \qquad \text{... (1)}$$

Steps involved to calculate the volume of a solid

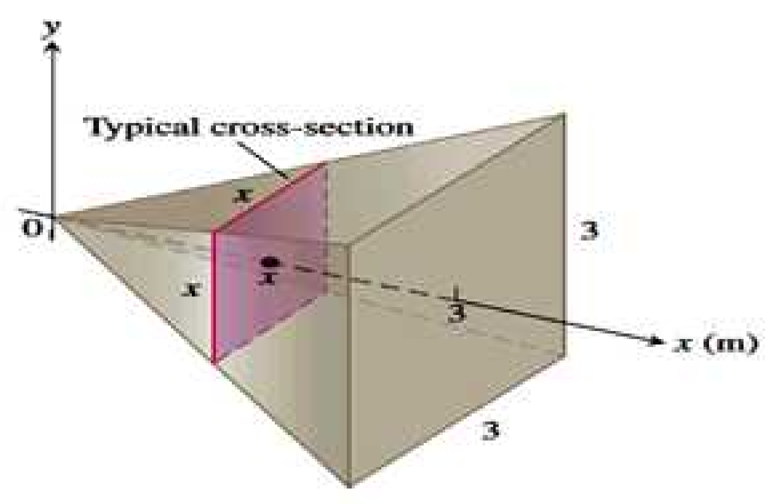
1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$, the area of a typical cross-section
3. Find the limits of integration.
4. Integrate $A(x)$ using the fundamental theorem to find the volume of the solid.

Example 1:

A pyramid 3 m high has a square base that is 3 m on a side. The cross section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

We draw the pyramid with its altitude along the x-axis and its vertex at the origin.



The area of cross section is $A(x)=x^2$

The squares go from $x=0$ to $x=3$. The volume is

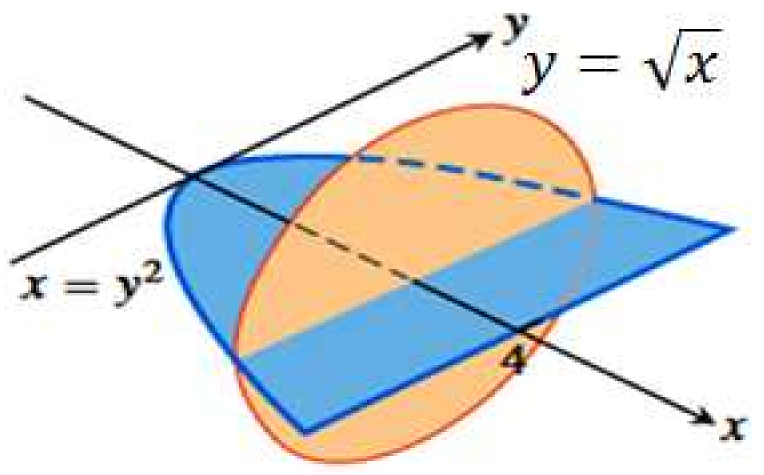
$$V=\int_a^b A(x)\,dx=\int_0^3 x^2\,dx=\left.\frac{x^3}{3}\right]_0^3=9\text{ m}^3$$

IP1:

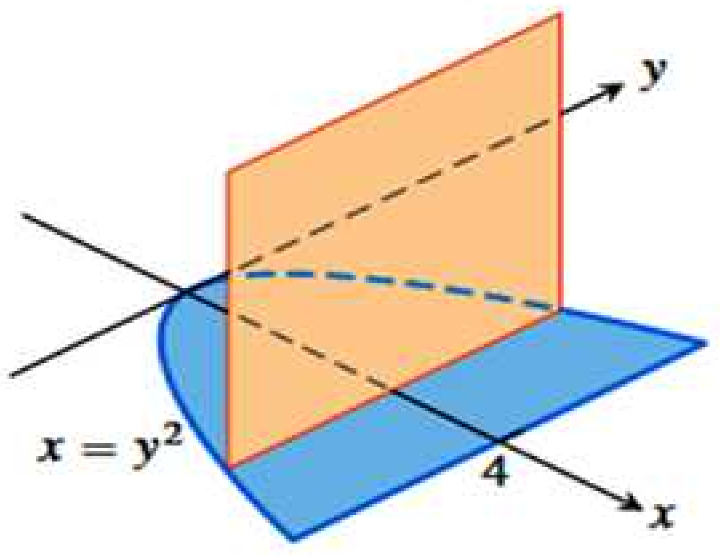
Find a formula for the area $A(x)$ of the cross sections of the solid perpendicular to the x –axis.

The solid lies between planes at $x = 0$ and $x = 4$. In each case, the cross sections are perpendicular to the x – axis between these planes run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$

a. The cross sections are circular disks with diameters in the xy –plane.



b. The cross sections are squares with bases in the xy –plane.



c. The cross sections are squares with diagonals in the xy –plane.

d. The cross sections are equilateral triangles with bases in the xy –plane.

Solution:

a. Since the cross sections are circular disks with diameters in the xy – plane,

Area of the cross section $A(x) = \pi(radius)^2$

From the figure, $radius = \sqrt{x}$

$\therefore\ A(x) = \pi\left(\sqrt{x}\,\right)^2 = \pi x$

b. Since the cross sections are squares with bases in the xy –plane ,

Area of the cross section $A(x) = width \times height$

From the figure, $width = height = 2\sqrt{x}$

$\therefore\ A(x) = 2\sqrt{x} \times 2\sqrt{x} = 4x$

c. Since The cross sections are squares with diagonals in the xy –plane,

Area of the cross section $\mathrm{A(x)} = (\mathrm{side})^2$ and

$\mathrm{diagonal} = \sqrt{2}\,\mathrm{side}$. Thus, $side = \frac{diagonal}{\sqrt{2}}$

From the figure, $diagonal = 2\sqrt{x}$

$\therefore\ A(x) = \pi\left(\frac{2\sqrt{x}}{\sqrt{2}}\right)^2 = 2x$

d. Since the cross sections are equilateral triangles with bases in the xy –plane,

Area of the cross section $A(x) = \frac{\sqrt{3}}{4}(side)^2$ and

From the figure, $side = 2\sqrt{x}$

$\therefore\ A(x) = \frac{\sqrt{3}}{4}\left(2\sqrt{x}\right)^2 = \sqrt{3}x$

IP2:

The solid lies between planes perpendicular to the x −axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x − axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$. Find the volume of the solid.

Solution:

Given the solid lies between planes perpendicular to the x −axis at $x = -1$ and $x = 1$.

The cross sections perpendicular to the x − axis between these planes, are squares whose diagonals run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$

Area of the cross section is $A(x) = (side)^2$ and

$$diagonal = \sqrt{2}\ side$$

Now, diagonal $= \left(\sqrt{1-x^2}\right) - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$

$$\therefore\ side = \frac{diagonal}{\sqrt{2}} = \frac{2\sqrt{1-x^2}}{\sqrt{2}}$$

$$\Rightarrow A(x) = \left(\frac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2 = 2(1-x^2)$$

Limits of integration: $a = -1$ and $b = 1$

$\therefore$ Volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_{-1}^{1} 2\left(1-x^2\right)dx$$

$$= 2\int_{-1}^{1} dx - 2\int_{-1}^{1} x^2\,dx$$

$$= 2[x]_{-1}^{1} - 2\left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= 2[1-(-1)] - 2\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] = \frac{8}{3}$$

IP3:

The solid lies between the planes perpendicular to the x −axis at $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$. Find the volume of the solid, if the cross sections perpendicular to the x −axis are

a. **Circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.**
b. **Squares with bases running from the curve $y = \tan x$ to the curve $y = \sec x$.**

Solution:

Given the solid lies between the planes perpendicular to the x −axis at $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$.

a. Since the cross sections perpendicular to the x −axis are **circular disks** with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$

∴ Area of the cross section

$$A(x) = \pi(\text{radius})^2 = \pi\frac{(\text{diameter})^2}{4} \text{ and}$$

$$\text{diameter} = \sec x - \tan x$$

$$\Rightarrow A(x) = \frac{\pi}{4}(\sec x - \tan x)^2$$

$$= \frac{\pi}{4}[\sec^2 x + \tan^2 x - 2\sec x . \tan x]$$

$$= \frac{\pi}{4}[\sec^2 x + \sec^2 x - 1 - 2\sec x . \tan x]$$

$$= \frac{\pi}{4}[2\sec^2 x - 1 - 2\sec x . \tan x]$$

Limits of integration: $a = -\frac{\pi}{3}$ and $b = \frac{\pi}{3}$

∴ Volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}\left[2\sec^2 x - 1 - 2\sec x . \tan x\right] dx$$

$$= \frac{\pi}{2}\int_{-\pi/3}^{\pi/3} \sec^2 x\,dx - \frac{\pi}{4}\int_{-\pi/3}^{\pi/3} dx - \frac{\pi}{2}\int_{-\pi/3}^{\pi/3} \sec x . \tan x\,dx$$

$$= \frac{\pi}{2}\left[\tan x\right]_{-\pi/3}^{\pi/3} - \frac{\pi}{4}\left[x\right]_{-\pi/3}^{\pi/3} - \frac{\pi}{2}\left[\sec x\right]_{-\pi/3}^{\pi/3}$$

$$= \frac{\pi}{2}\left[\sqrt{3} + \sqrt{3}\right] - \frac{\pi}{4}\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] - \frac{\pi}{2}\left(\sqrt{2} - \sqrt{2}\right)$$

$$= \pi\sqrt{3} - \frac{\pi^2}{6}$$

b. Since the cross sections perpendicular to the x −axis are ***squares*** with bases running from the curve $y = \tan x$ to the curve $y = \sec x$

∴ Area of the cross section

$$A(x) = (\text{side})^2 = (\sec x - \tan x)^2$$
$$= \sec^2 x + \tan^2 x - 2\sec x . \tan x$$
$$= \sec^2 x + \sec^2 x - 1 - 2\sec x . \tan x$$
$$= 2\sec^2 x - 1 - 2\sec x . \tan x$$

Limits of integration: $a = -\frac{\pi}{3}$ and $b = \frac{\pi}{3}$

∴ Volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \left[2\sec^2 x - 1 - 2\sec x . \tan x\right] dx$$

$$= 2\int_{-\pi/3}^{\pi/3} \sec^2 x\,dx - \int_{-\pi/3}^{\pi/3} dx - 2\int_{-\pi/3}^{\pi/3} \sec x . \tan x\,dx$$

$$= 2\left[\tan x\right]_{-\pi/3}^{\pi/3} - \left[x\right]_{-\pi/3}^{\pi/3} - 2\left[\sec x\right]_{-\pi/3}^{\pi/3}$$

$$= 2\left[\sqrt{3} + \sqrt{3}\right] - \left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] - 2\left(\sqrt{2} - \sqrt{2}\right)$$

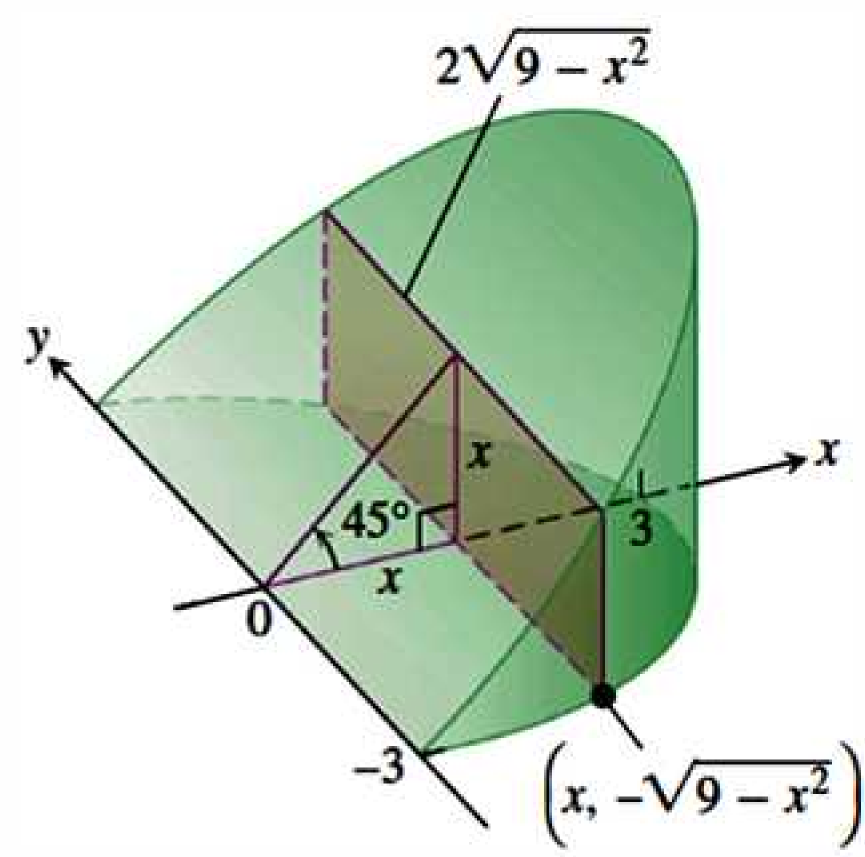
$$= 4\sqrt{3} - \frac{2\pi}{3}$$

IP4:

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution:

We draw the wedge and sketch a typical cross section perpendicular to the x-axis.



The base of the cylinder is the circle $x^2 + y^2 = 9$.

The cross section at x is a rectangle of area

$$A(x) = (x)\left(2\sqrt{9-x^2}\right) = 2x\sqrt{9-x^2}$$

The rectangles runs from $x = 0$ to $x = 3$.

The volume of the wedge is

$$V = \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx$$

$$\text{put } 9 - x^2 = u \Rightarrow -2x\,dx = du \Rightarrow 2x\,dx = -du$$

Limits :

$$x = 0 \Rightarrow u = 9 \quad \textit{and} \quad x = 3 \Rightarrow u = 0$$

$$\therefore V = \int_0^3 2x\sqrt{9-x^2}\,dx = \int_9^0 u^{1/2}(-du) = \int_0^9 u^{1/2}\,du$$

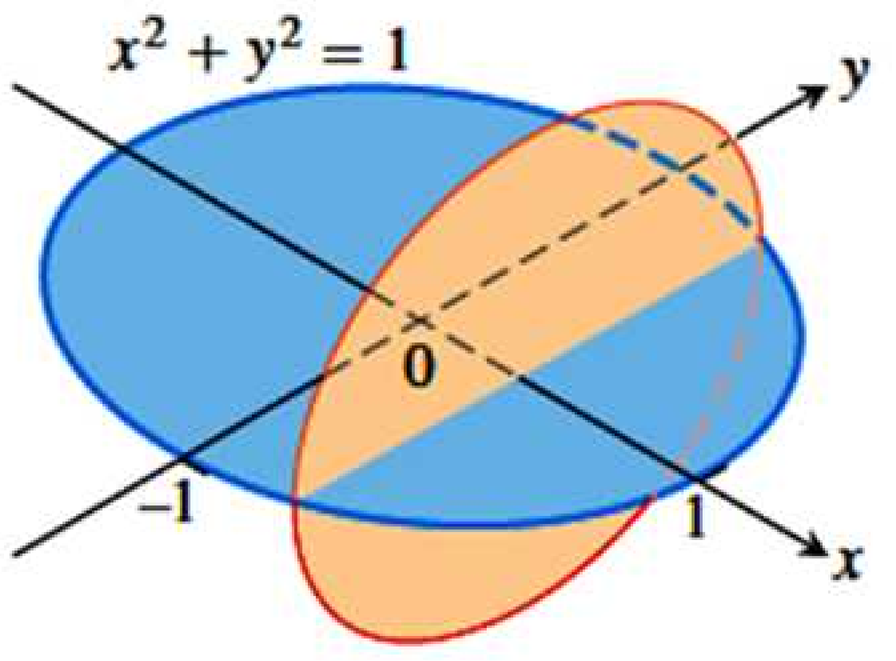
$$= \left[\frac{u^{3/2}}{3/2}\right]_0^9 = \frac{2}{3}\left[9\sqrt{9} - 0\right] = 18$$

P1:

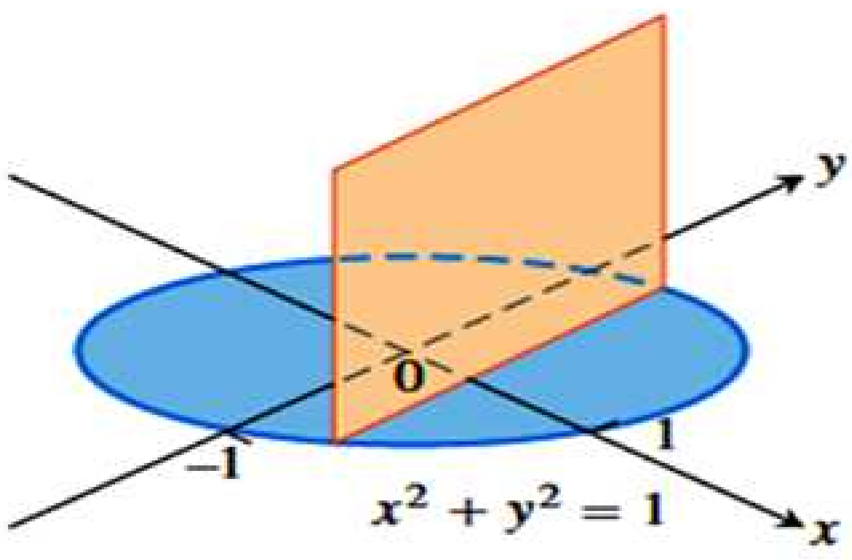
Find a formula for the area $A(x)$ of the cross sections of the solid perpendicular to the x −axis.

The solid lies between planes perpendicular to the x −axis at $x = -1$ and $x = 1$. In each case, the cross sections are perpendicular to the x − axis between these planes run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$

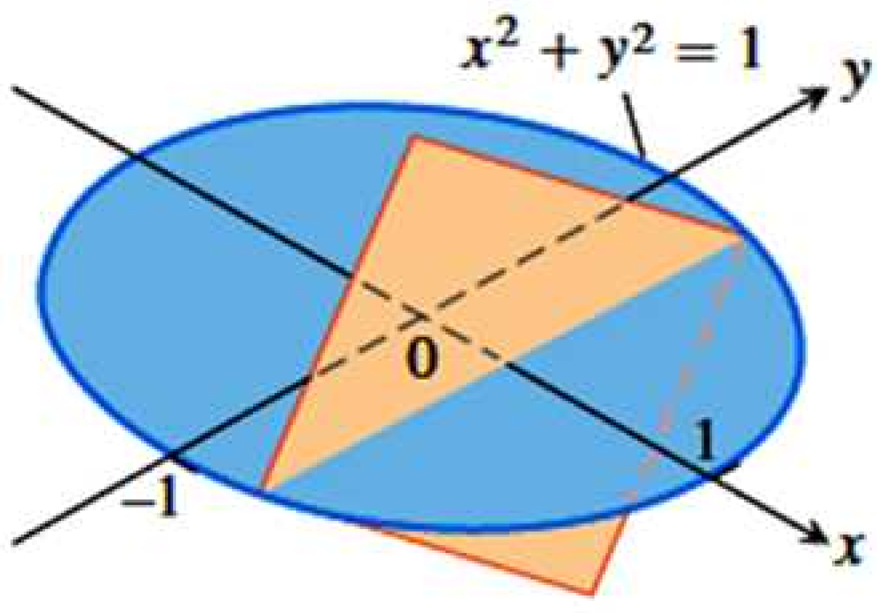
a. The cross sections are circular disks with diameters in the xy −plane.



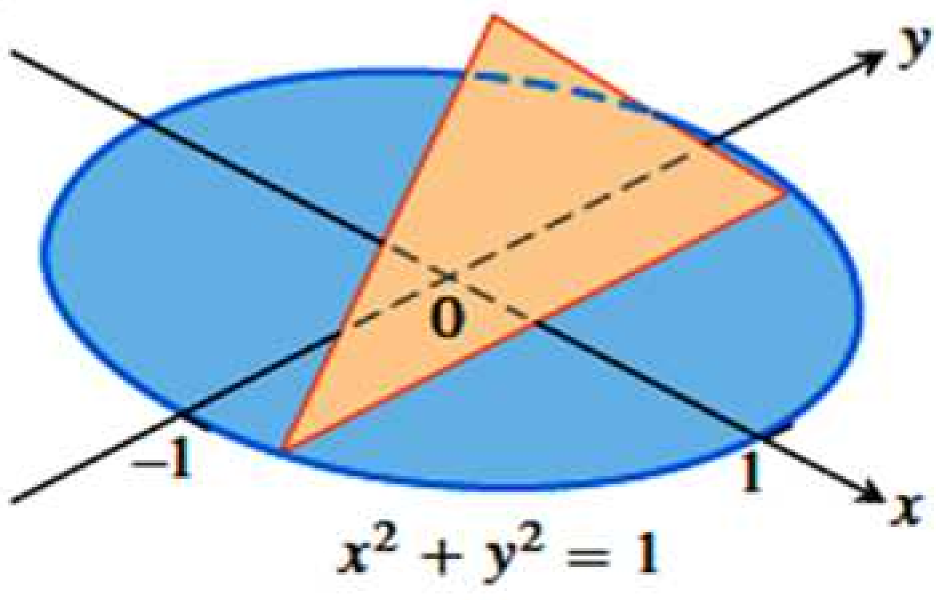
b. The cross sections are squares with bases in the xy −plane.



c. The cross sections are squares with diagonals in the xy −plane.



d. The cross sections are equilateral triangles with bases in the xy −plane.



Solution:

a. Since the cross sections are circular disks with diameters in the xy − plane,

Area of the cross section $A(x) = \pi(radius)^2$

From the figure, $radius = \sqrt{1-x^2}$

$\therefore\ A(x) = \pi\left(\sqrt{1-x^2}\right)^2 = \pi(1-x^2)$

b. Since the cross sections are squares with bases in the xy −plane ,

Area of the cross section $A(x) = width \times height$

From the figure, $width = height = 2\sqrt{1-x^2}$

$\therefore\ A(x) = 2\sqrt{1-x^2} \times 2\sqrt{1-x^2} = 4(1-x^2)$

c. Since The cross sections are squares with diagonals in the xy −plane,

Area of the cross section $A(x) = (side)^2$ and

$diagonal = \sqrt{2}\, side$. Thus, $side = \frac{diagonal}{\sqrt{2}}$

From the figure, $diagonal = 2\sqrt{1-x^2}$

$\therefore\ A(x) = \pi\left(\frac{2\sqrt{1-x^2}}{\sqrt{2}}\right)^2 = 2(1-x^2)$

d. Since the cross sections are equilateral triangles with bases in the xy −plane,

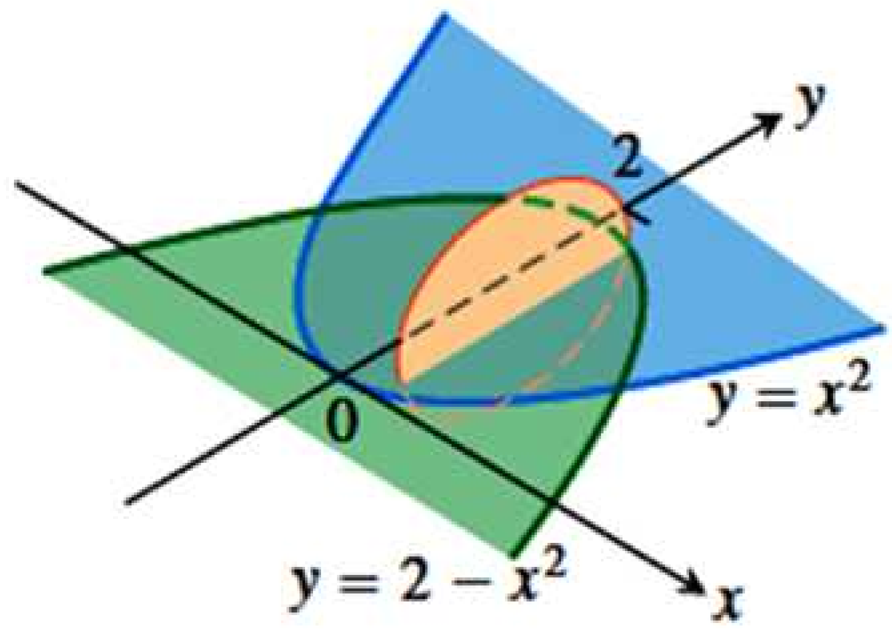
Area of the cross section $A(x) = \frac{\sqrt{3}}{4}(side)^2$ and

From the figure, $side = 2\sqrt{1-x^2}$

$\therefore\ A(x) = \frac{\sqrt{3}}{4}\left(2\sqrt{1-x^2}\right)^2 = \sqrt{3}(1-x^2)$

P2:

The solid lies between the planes perpendicular to the x −axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x −axis, are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.



Solution:

Given the solid lies between the planes perpendicular to the x −axis at $x = -1$ and $x = 1$.

The cross sections perpendicular to the x −axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$

∴ Area of the cross section $A(x) = \pi(\text{radius})^2 = \pi \frac{(\text{diameter})^2}{4}$

where diameter $= (2 - x^2) - x^2 = 2(1 - x^2)$

$$\Rightarrow A(x) = \pi \frac{\left(2(1-x^2)\right)^2}{4} = \pi(1 - 2x^2 + x^4)$$

Limits of integration: $a = -1$ and $b = 1$

∴ Volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_{-1}^{1} \pi\left(1 - 2x^2 + x^4\right)dx$$

$$= \pi\left[\int_{-1}^{1} dx - 2\int_{-1}^{1} x^2 dx + \int_{-1}^{1} x^4 dx\right]$$

$$= \pi\left\{[x]_{-1}^{1} - 2\left[\frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^5}{5}\right]_{-1}^{1}\right\}$$

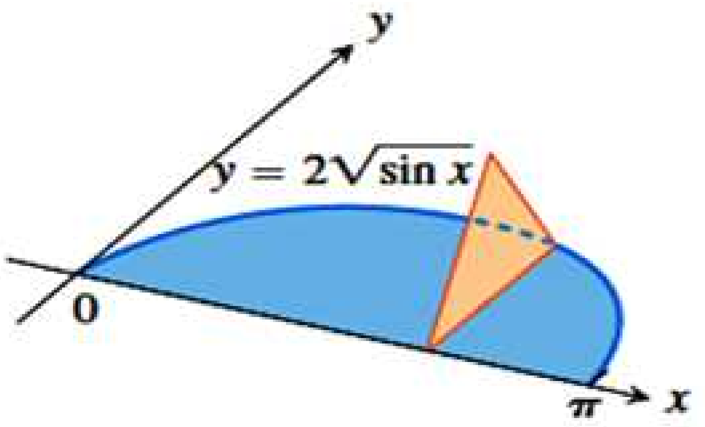
$$= \pi\left[(1-(-1)) - \frac{2}{3}(1-(-1)) + \frac{1}{5}(1-(-1))\right]$$

$$= 2\pi\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{16\pi}{15}$$

P3:

The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x –axis. Find the volume of the solid, if the cross sections perpendicular to the x –axis, are

a. Equilateral triangles with bases running from the x –axis to the curve as shown in figure



b. Squares with bases running from the x –axis to the curve

Solution:

Given the base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x –axis.

a. Since the cross sections perpendicular to the x –axis are ***equilateral triangles*** with bases running from the x –axis to the curve as shown in figure.

∴ Area of the cross section

$$A(x) = \frac{1}{2} \times (side) \times (side) \times sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2} \times \left(2\sqrt{sin\,x}\right) \times \left(2\sqrt{sin\,x}\right) \times \frac{\sqrt{3}}{2} = \sqrt{3}\,sin\,x$$

Limits of integration: $a = 0$ and $b = \pi$

∴ Volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_0^\pi \sqrt{3}\sin x\,dx$$

$$= \sqrt{3}\left[-\cos x\right]_0^\pi = -\sqrt{3}\left[-1-1\right] = 2\sqrt{3}$$

b. Since the cross sections perpendicular to the x –axis are ***squares*** with bases running from the x –axis to the curve.

∴ Area of the cross section

$$A(x) = (\text{side})^2 = \left(2\sqrt{\sin x}\right)^2 = 4\sin x$$

Limits of integration: $a = 0$ and $b = \pi$

∴ Volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_0^\pi 4\sin x\,dx$$

$$= 4\left[-\cos x\right]_0^\pi = -4\left[-1-1\right] = 8$$

P4:

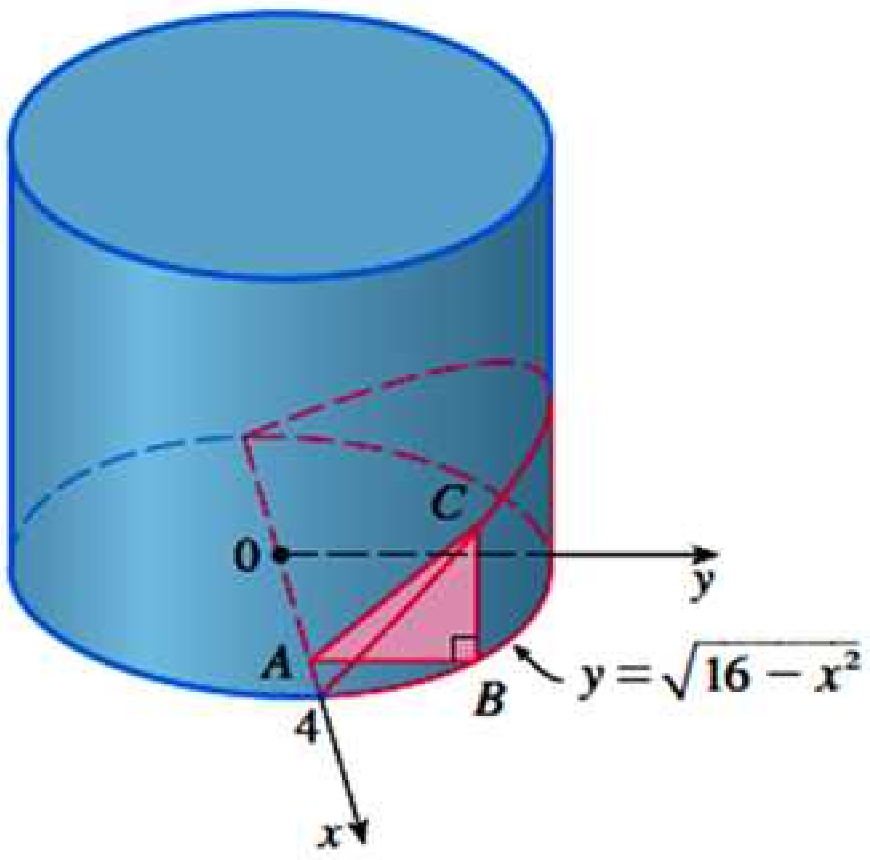
A wedge is cut out of a circular cylinder of radius 4 by two planes; one of the planes is perpendicular to the axis of the cylinder and the other plane intersects the first at an angle of $30°$along a diameter of the cylinders. Find the volume of the wedge?

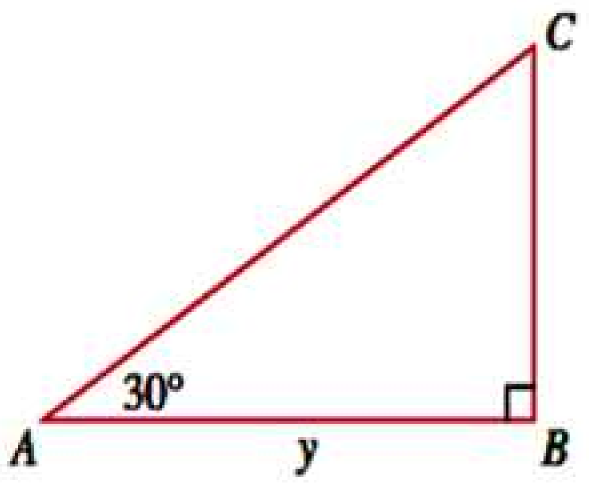
Solution:

If we place the x −axis along the diameter where the planes meet, then the base of the solid is semicircle with equation $= \sqrt{16 - x^2}$, $-4 \le x \le 4$.

A cross section perpendicular to the x −axis at a distance x from the origin is triangle ABC, as shown in figure, whose base is $y = \sqrt{16 - x^2}$ and whose height is

$$|BC| = y \tan 30° = \sqrt{16 - x^2}/\sqrt{3}$$





Thus, the cross section area is

$$A(x) = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times \sqrt{16 - x^2} \times \sqrt{16 - x^2}/\sqrt{3}$$

$$= \frac{16 - x^2}{2\sqrt{3}}$$

∴ Volume of the wedge is

$$V = \int_a^b A(x)\,dx = \int_{-4}^{4} \frac{16 - x^2}{2\sqrt{3}}\,dx$$

$$= \frac{1}{\sqrt{3}} \int_0^4 \left(16 - x^2\right) dx = \frac{16}{\sqrt{3}} \int_0^4 dx - \frac{1}{\sqrt{3}} \int_0^4 x^2\,dx$$

$$= \frac{16}{\sqrt{3}} [x]_0^4 - \frac{1}{\sqrt{3}} \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{\sqrt{3}} - \frac{64}{3\sqrt{3}} = \frac{128}{3\sqrt{3}}$$

3.2. Volume of Solids by Slicing

Exercise:

1. The solid lies between planes perpendicular to the x-axis at x = 0 and x = 4. The cross sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonal run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

2. The solid lies between planes perpendicular to the x-axis at x = −1 and x = 1. The cross sections perpendicular to the axis between these planes are vertical squares whose base edges run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$. Find the volume of the solid.

3. The solid lies between the planes perpendicular to the y-axis at y = 0 and y = 2. The cross sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$. Find the volume of the solid.

3.3

Volume of Solids of Revolution- Disks

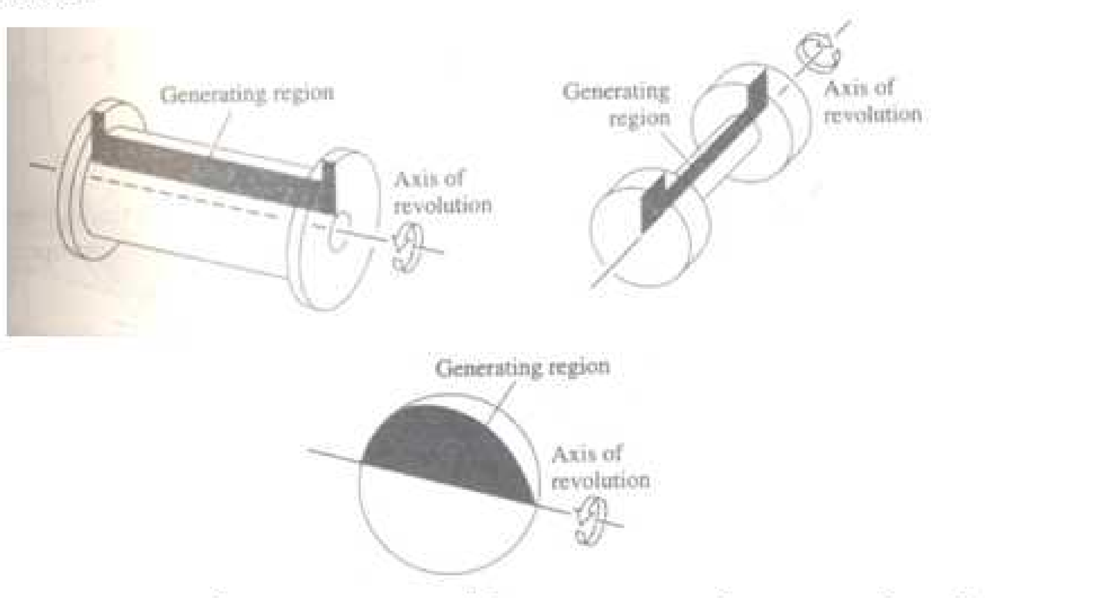
Learning objectives:

- To find the volume of solids of revolution by disk method.

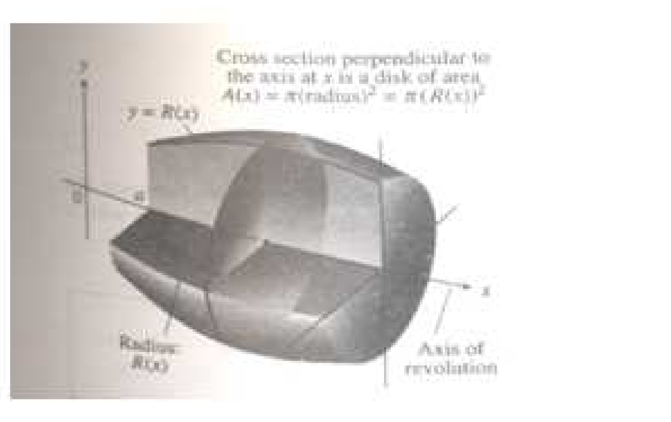
AND

- To practice the related problems.

The most common application of the method of slicing is to solids of revolution. ***Solids of revolution*** are solids whose shapes can be generated by revolving plane regions about axes.



If the region can be arranged between the graph of a continuous function $y = R(x)$, $a \le x \le b$, and the x-axis, we can find the solid's volume in the following way.



The typical cross section of the solid perpendicular to the axis of revolution is a disk of radius $R(x)$ and area

$$A(x) = \pi\left[R(x)\right]^2$$

The solid's volume, being the integral of A from $x = a$ to $x = b$, is the integral of $\pi\left[R(x)\right]^2$ from a to b.

The volume of the solid generated by revolving about the x-axis the region between the x-axis and the graph of the continuous function $y = R(x)$, $a \le x \le b$, is

$$V = \int_a^b \pi\left[R(x)\right]^2 dx \qquad \text{.... (1)}$$

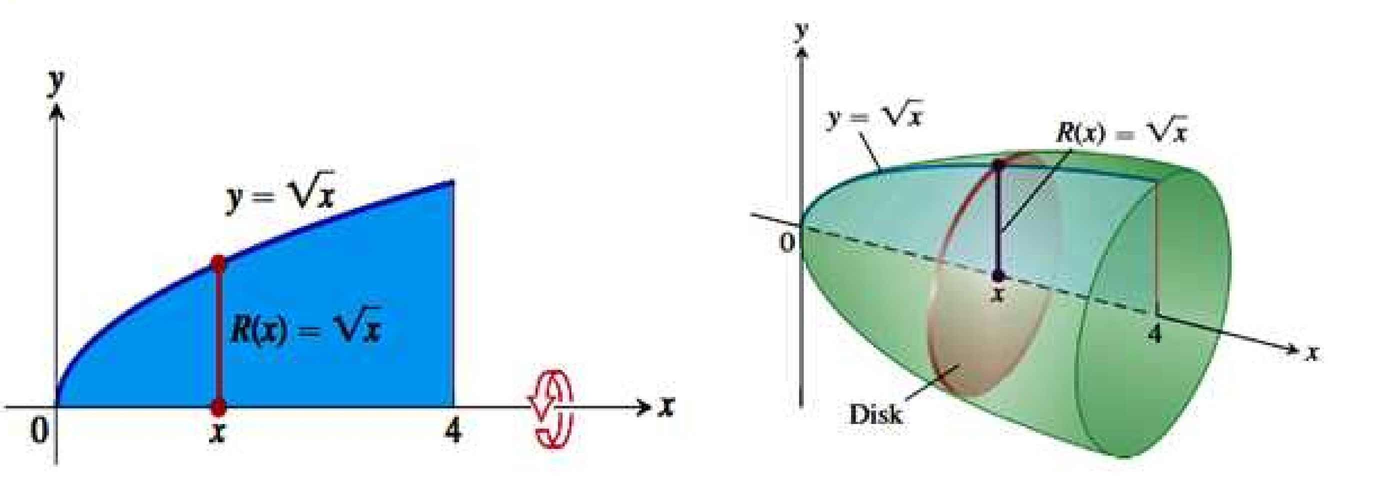
Example 1:

A solid of revolution (Rotation about x −axis)

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution

We draw figures showing the region, a typical radius, and the generated solid.



The volume is

$$V = \int_a^b \pi\left[R(x)\right]^2 dx = \int_0^4 \pi\left[\sqrt{x}\right]^2 dx = \pi\int_0^4 x\,dx$$

$$= \pi\left.\frac{x^2}{2}\right]_0^4 = \pi\frac{(4)^2}{2} = 8\pi$$

The axis of revolution need not necessarily be the x-axis, but the rule for calculating volume is the same.

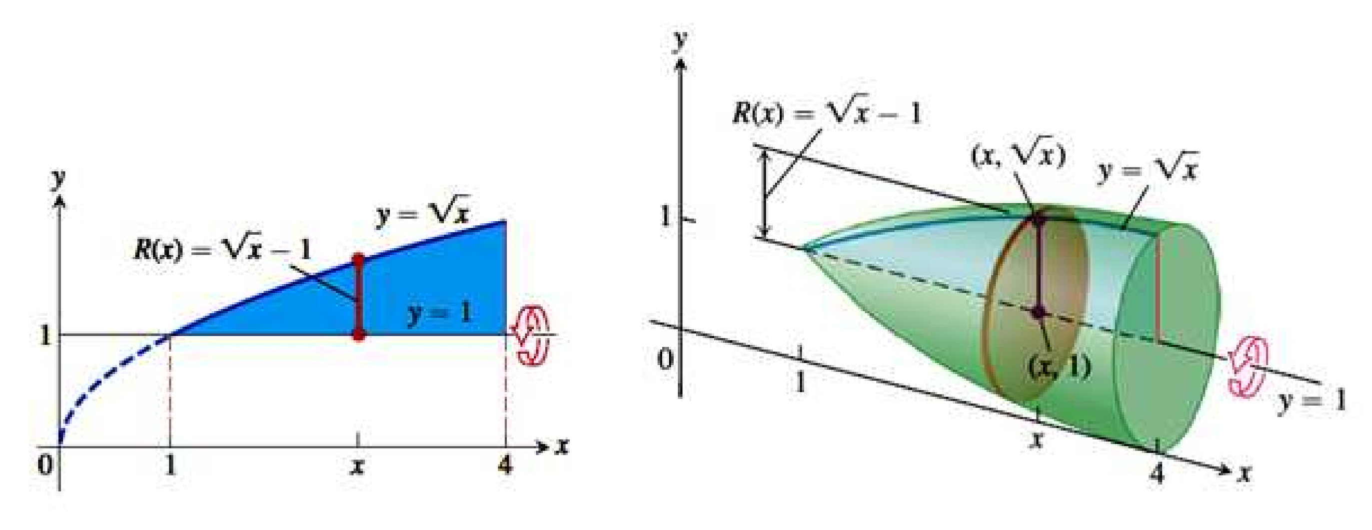
Example 2:

A solid of revolution (Rotation about the line $y = 1$)

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution

We draw figures showing the region, a typical radius, and the generated solid.



The volume is

$$V = \int_1^4 \pi\left[R(x)\right]^2 dx = \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx$$

$$= \pi\int_1^4 [x - 2\sqrt{x} + 1]\,dx$$

$$= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}$$

- To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \le y \le d$ about the y-axis, we use equation (1) with x replaced by y.

$$V = \int_c^d \pi\left[R(y)\right]^2 dy$$

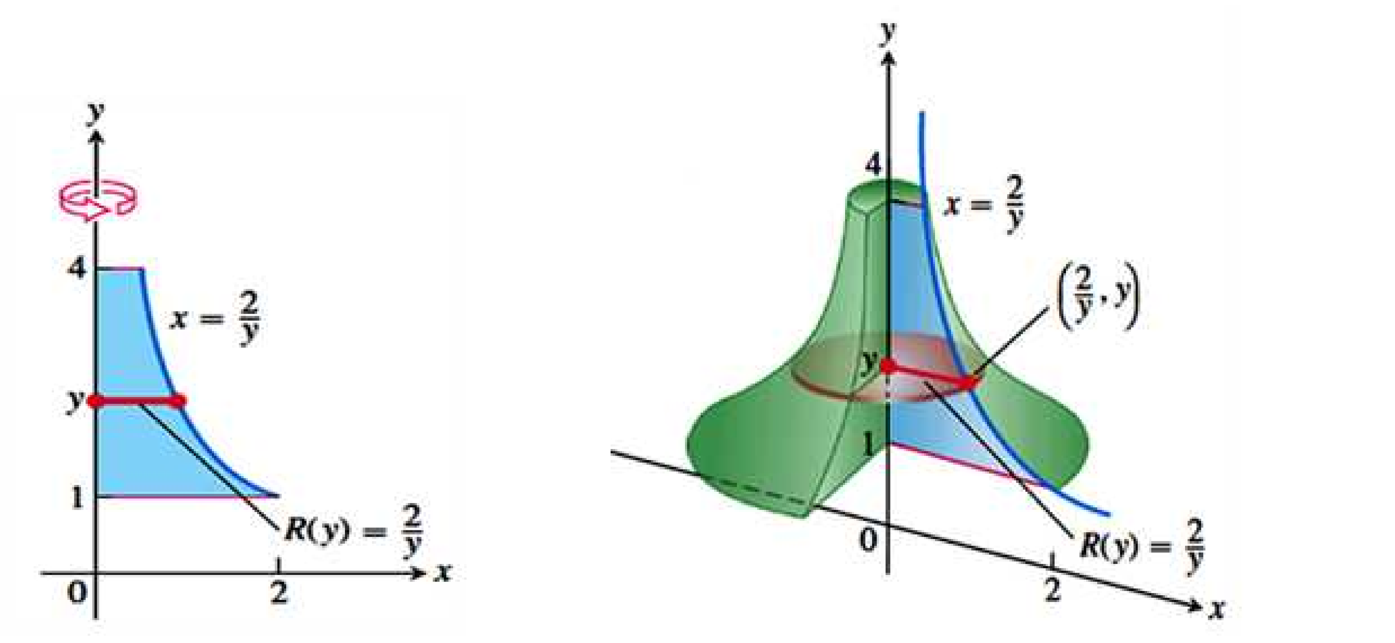
Example 3:

A solid of revolution (Rotation about y −axis)

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Solution

We draw figures showing the region, a typical radius, and the generated solid.



The volume is

$$V = \int_1^4 \pi\left[R(y)\right]^2 dy = \int_1^4 \pi\left[\frac{2}{y}\right]^2 dy = \pi\int_1^4 \frac{4}{y^2}dy$$

$$= 4\pi\left[-\frac{1}{y}\right]_1^4 4\pi\cdot\frac{3}{4} = 3\pi$$

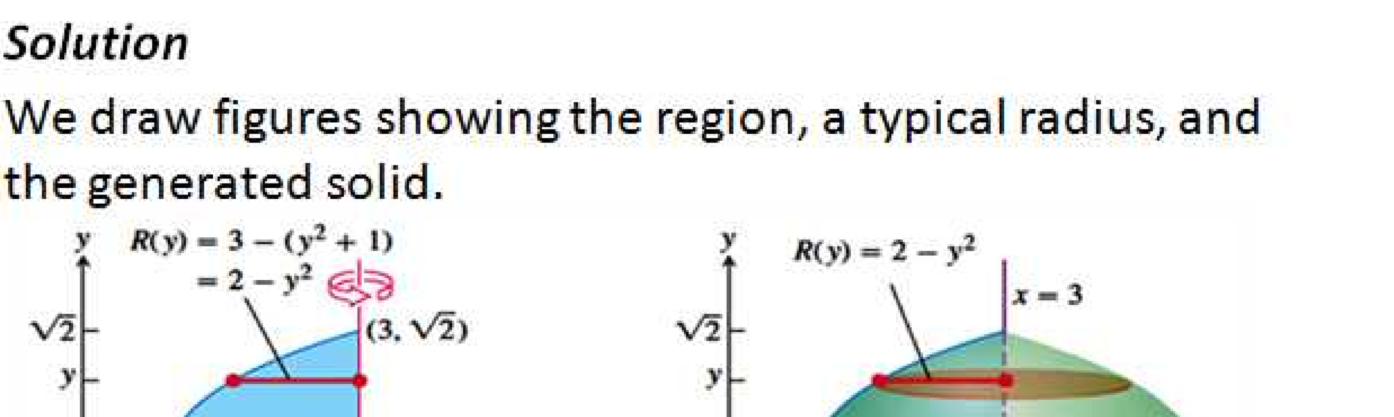
Example 4:

A solid of revolution (Rotation about the line $x = 3$)

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution

We draw figures showing the region, a typical radius, and the generated solid.



The volume is

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left[R(y)\right]^2 dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left[2 - y^2\right]^2 dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left[4 - 4y^2 + y^4\right] dy = \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{64\pi\sqrt{2}}{15}$$

P1:

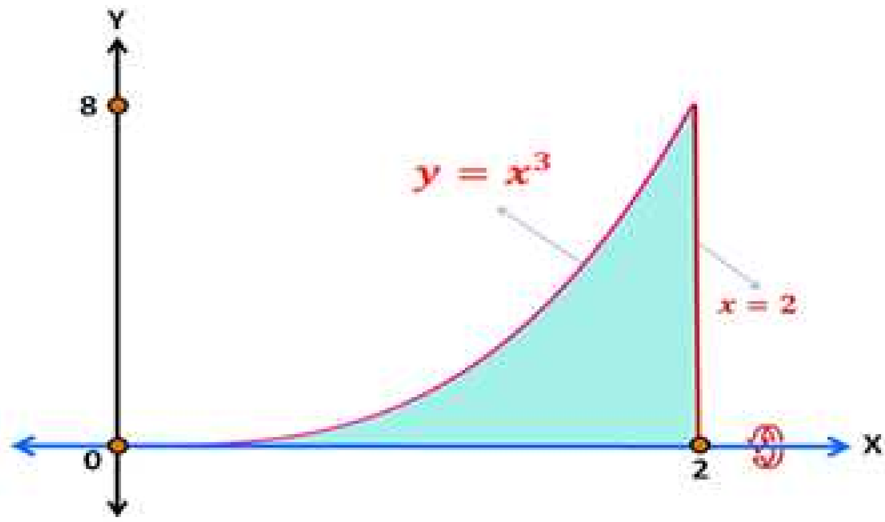
Find the volume of the solid generated by revolving the regions bounded by the curve $y = x^3$ and lines $x = 2, y = 0$ about the x −axis.

P1:

Find the volume of the solid generated by revolving the regions bounded by the curve $y = x^3$ and lines $x = 2, y = 0$ about the x -axis.

Solution: Given curve is $y = x^3$ and lines are $x = 2, y = 0$.

A typical cross section of the solid perpendicular to the axis of revolution is shown in figure, which is a disk of radius $R(x) = x^3$



Area of the region $A(x) = \pi[R(x)]^2 = \pi[x^3]^2 = \pi x^6$

Limits of integration: $x^3 = 0 \Rightarrow x = 0$ and $x = 2$

$\Rightarrow a = 0$ and $b = 2$

$\therefore$ Volume of the solid generated by revolving the regions bounded by the curve $y = x^3$ and lines $x = 2, y = 0$ about the x -axis is

$$V = \int_a^b \pi[R(x)]^2\, dx = \pi\int_0^2 x^6\, dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \pi\left[\frac{2^7}{7} - 0\right] = \frac{128\pi}{7}$$

P2:

Find the volume of the solid generated by revolving the region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$ and on the left by the $y-$axis about the line $y = \sqrt{2}$?

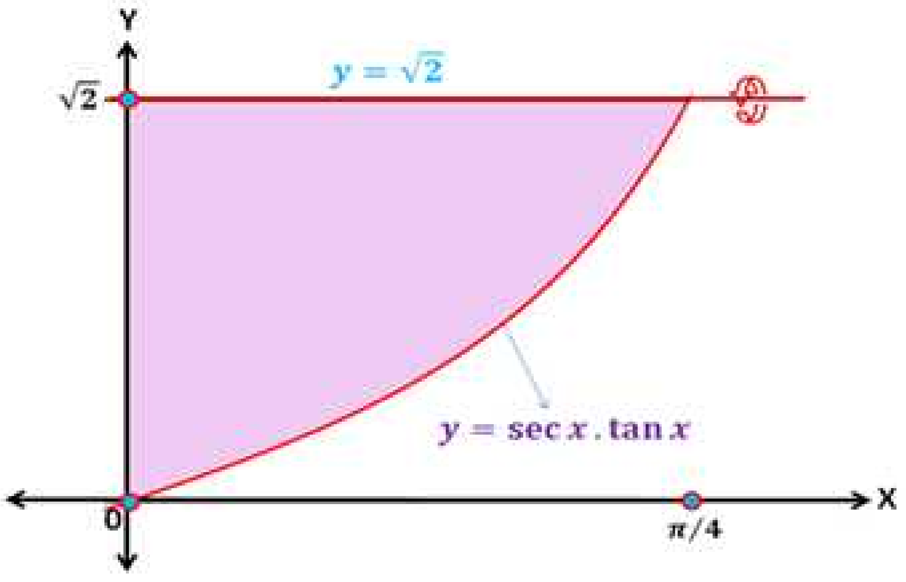
P2:

Find the volume of the solid generated by revolving the region in the first quadrant bounded above by the line $\boldsymbol{y} = \sqrt{\mathbf{2}}$, below by the curve $\boldsymbol{y} = \boldsymbol{\sec x}\ \boldsymbol{\tan x}$ and on the left by the $\boldsymbol{y}$ −axis about the line $\boldsymbol{y} = \sqrt{\mathbf{2}}$?

Solution: Given curve is $y = \sec x \,.\tan x$ and line is $y = \sqrt{2}$

A typical cross section of the solid perpendicular to the axis of revolution is shown in figure, which is a disk of radius

$$R(x) = \left(\sqrt{2} - \sec x \,.\tan x\right)$$



Area of the region

$$A(x) = \pi[R(x)]^2 = \pi\left[\sqrt{2} - \sec x \,.\tan x\right]^2$$

$$= \pi\left[2 + \sec^2 x \,.\tan^2 x - 2\sqrt{2}\sec x \,.\tan x\right]$$

Limits of integration: $x = 0$ and $\sec x \,.\tan x = \sqrt{2} \Rightarrow x = \frac{\pi}{4}$

$a = 0$ and $b = \frac{\pi}{4}$

$\therefore$ Volume of the solid generated by revolving the region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \,.\tan x$ and on the left by the y −axis about the line $y = \sqrt{2}$ is

$$V = \int_a^b \pi\left[R(x)\right]^2\, dx$$

$$= \pi\int_0^{\pi/4}\left[2 + (\tan x)^2 \sec^2 x - 2\sqrt{2}\sec x \tan x\right] dx$$

$$= 2\pi\int_0^{\pi/4} dx + \pi\int_0^{\pi/4}(\tan x)^2\sec^2 x\, dx - 2\sqrt{2}\pi\int_0^{\pi/4}\sec x \tan x\, dx$$

$$= 2\pi\left[x\right]_0^{\pi/4} + \pi\left[\frac{(\tan x)^3}{3}\right]_0^{\pi/4} - 2\sqrt{2}\pi\left[\sec x\right]_0^{\pi/4}$$

$$= \pi\left[\frac{\pi}{2} + \left(\frac{1}{3} - 0\right) - 2\sqrt{2}\left(\sqrt{2} - 1\right)\right]$$

$$= \pi\left[\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right]$$

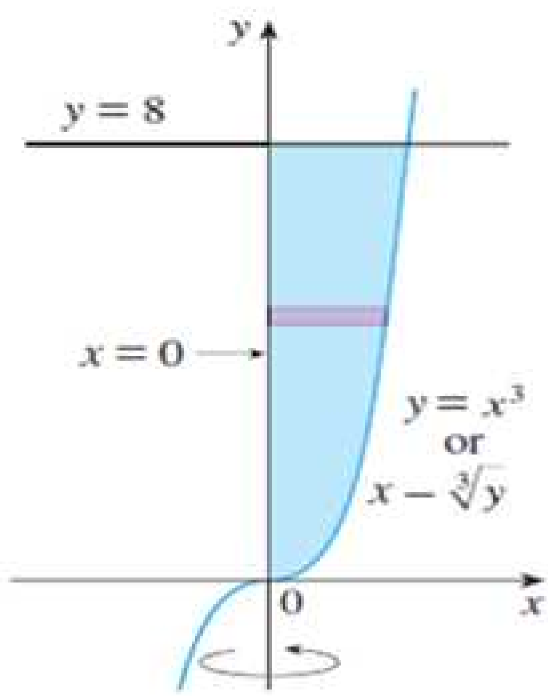
P3:

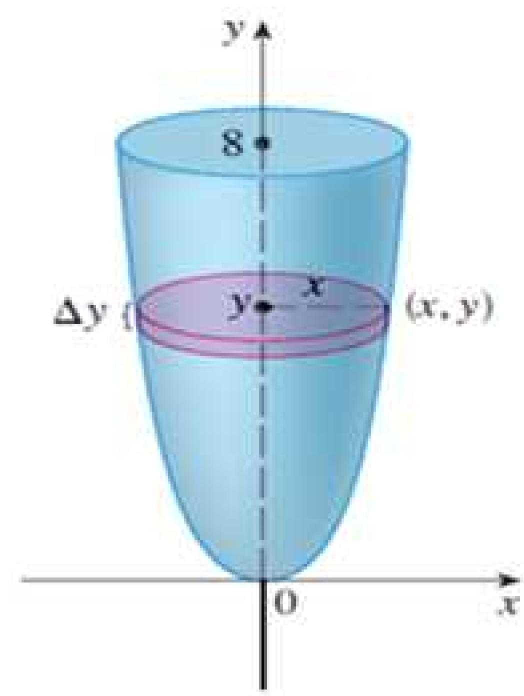
Find the volume of the solid obtained by rotating the region bounded by the curve $y = x^3$, $y = 8$ and $x = 0$ about the y −axis.

P3:

Find the volume of the solid obtained by rotating the region bounded by the curve $y = x^3$, $y = 8$ and $x = 0$ about the y −axis.

Solution: The region between the curve $y = x^3$ and lines $y = 8$ and $x = 0$ is shown in figure and the resulting solid is shown in figure.





The region is rotated about the y −axis, it makes sense to slice the solid perpendicular to the y −axis and therefore to integrate with respect to y.

If we slice at height y, we get a circular disk with radius $R(x) = x$, where $x = \sqrt[3]{y}$. So the area of a cross section through y is

$$A(y) = \pi[R(x)]^2 = \pi\left(\sqrt[3]{y}\right)^2 = \pi y^{2/3}$$

Limits of integration: $c = 0$ and $d = 8$

∴ Volume of the solid is

$$V = \int_c^d \pi\left[R(x)\right]^2 dy = \int_0^8 \pi y^{2/3}\, dy$$

$$= \pi\left[\frac{y^{2/3+1}}{2/3+1}\right]_0^8 = \frac{3\pi}{5}\left[y^{5/3}\right]_0^8$$

$$= \frac{3\pi}{5}[32 - 0] = \frac{96\pi}{5}$$

P4:

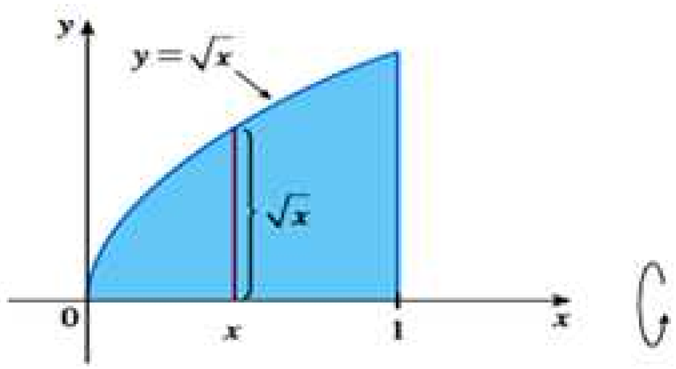
Find the volume of the solid obtained by rotating about the x $-$axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

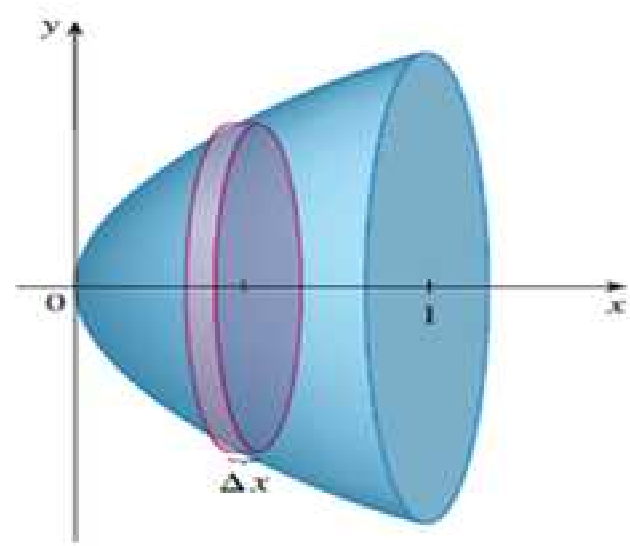
P4:

Find the volume of the solid obtained by rotating about the x −axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Solution:

The region under the curve $y = \sqrt{x}$ from 0 to 1 is shown in figure. If we rotate about the x −axis, we get the solid which is shown in figure.





When we slice through the point x, we get a disk with radius $R(x) = \sqrt{x}$. The area of this cross section is

$$A(x) = \pi[R(x)]^2 = \pi\left[\sqrt{x}\right]^2 = \pi x$$

Limits of integration: $a = 0$ and $b = 1$

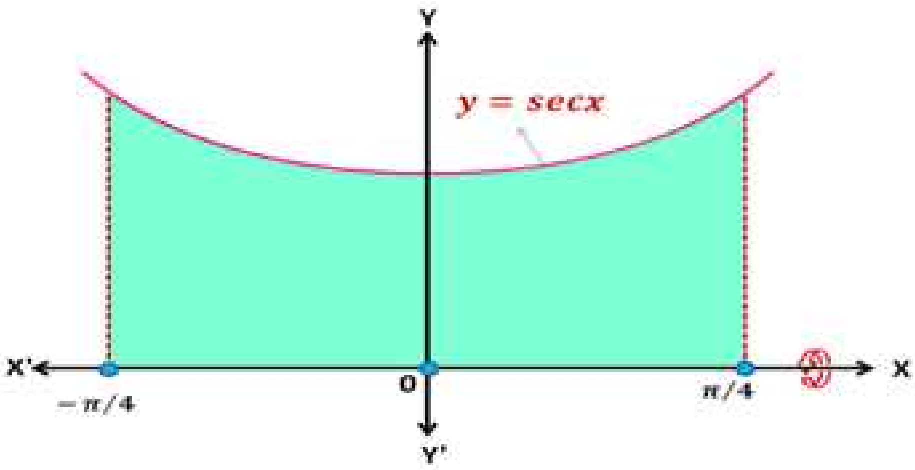
∴ Volume of the solid is

$$V = \int_a^b \pi\left[R(x)\right]^2\, dx = \int_0^1 \pi x\, dx = \pi\left[\frac{x^2}{2}\right]_0^1 = \pi\left[\frac{1}{2} - 0\right] = \frac{\pi}{2}$$

IP1:

Find the volume of the solid generated by revolving the regions bounded by the curve $y = \sec x$ and lines $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ about the x −axis.

Solution: Given curve is $y = \sec x$ and lines are $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$. A typical cross section of the solid perpendicular to the axis of revolution is shown in figure, which is a disk of radius $R(x) = \sec x$.



Area of the region $A(x) = \pi[R(x)]^2 = \pi[\sec x]^2 = \pi \sec^2 x$

Limits of integration: $a = -\frac{\pi}{4}$ and $b = \frac{\pi}{4}$

∴ Volume of the solid generated by revolving the regions bounded by the curve $y = \sec x$ and lines $y = 0, x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ about the x −axis is

$$V = \int_a^b \pi\left[R(x)\right]^2\, dx = \pi\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \pi\left[\tan x\right]_{-\pi/4}^{\pi/4}$$

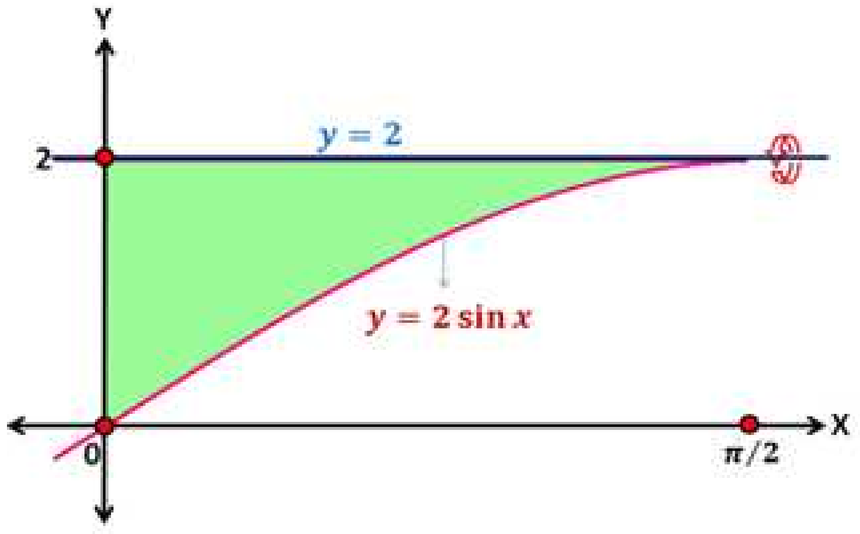
$$= \pi\left[\tan(\pi/4) - \tan(-\pi/4)\right] = 2\pi$$

IP2:

Find the volume of the solid generated by revolving the region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2\sin x$, $0 \le x \le \pi/2$ and on the left by the y −axis about the line $y = 2$?

Solution: Given curve is $y = 2\sin x$, $0 \le x \le \pi/2$ and line is $y = 2$. A typical cross section of the solid perpendicular to the axis of revolution is shown in figure, which is a disk of radius

$$R(x) = (2 - 2\sin x) = 2(1 - \sin x)$$



Area of the region

$$A(x) = \pi[R(x)]^2 = \pi[2(1 - \sin x)]^2$$

$$= \pi[4(1 + \sin^2 x - 2\sin x)]$$

$$= 4\pi\left[\left(1 + \frac{1-\cos 2x}{2} - 2\sin x\right)\right] = 4\pi\left[\left(\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right)\right]$$

Limits of integration: $a = 0$ and $b = \frac{\pi}{2}$

∴ Volume of the solid generated by revolving the region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2\sin x$, $0 \le x \le \pi/2$ and on the left by the y −axis about the line $y = 2$ is

$$V = \int_a^b \pi[R(x)]^2\, dx$$

$$= 4\pi\int_0^{\pi/2}\left[\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right]dx$$

$$= 4\pi\left[\frac{3}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2}\cos 2x\, dx - 2\int_0^{\pi/2}\sin x\, dx\right]$$

$$= 4\pi\left\{\frac{3}{2}[x]_0^{\pi/2} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2} - 2[-\cos x]_0^{\pi/2}\right\}$$

$$= 4\pi\left[\frac{3}{2}\left[\frac{\pi}{2} - 0\right] - \frac{1}{4}[0 - 0] + 2[0 - 1]\right]$$

$$= 4\pi\left[\frac{3\pi}{4} - 2\right] = \pi(3\pi - 8)$$

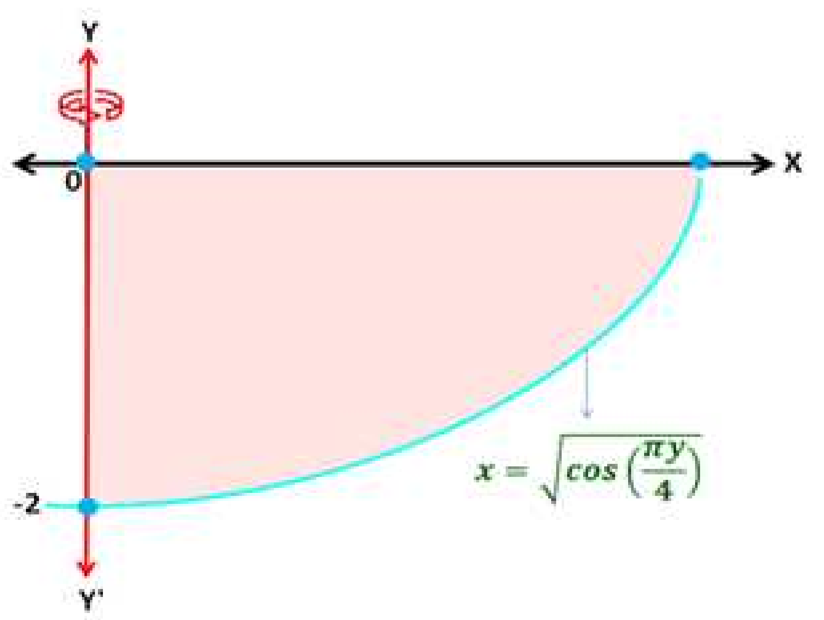
IP3:

Find the volume of the solid generated by revolving the regions enclosed by the curve $\boldsymbol{x = \sqrt{\cos\left(\frac{\pi y}{4}\right)}, -2 \le y \le 0}$ and lines $\boldsymbol{x = 0}$ about the $\boldsymbol{y}$ −axis.

Solution: Given curve is $x = \sqrt{\cos\left(\frac{\pi y}{4}\right)}, -2 \le y \le 0$ and line $x = 0$.

A typical cross section of the solid perpendicular to the axis of revolution is shown in figure, which is a disk of radius

$$R(y) = \sqrt{\cos\left(\frac{\pi y}{4}\right)}$$



Area of the region $A(y) = \pi[R(y)]^2$

$$= \pi\left[\sqrt{\cos\left(\frac{\pi y}{4}\right)}\right]^2 = \pi\cos\left(\frac{\pi y}{4}\right)$$

Limits of integration: $c = -2$ and $d = 0$

∴ Volume of the solid generated by revolving the regions enclosed by the curve $x = \sqrt{\cos\left(\frac{\pi y}{4}\right)}, -2 \le y \le 0$ and lines $x = 0$ about the y −axis is

$$V = \int_c^d \pi[R(y)]^2\, dy = \pi\int_0^2 \cos\left(\frac{\pi y}{4}\right)dy$$

$$= \pi.\frac{4}{\pi}\left[\sin\left(\frac{\pi y}{4}\right)\right]_{-2}^{0} = 4\left[0 + \sin\left(\frac{\pi}{2}\right)\right] = 4$$

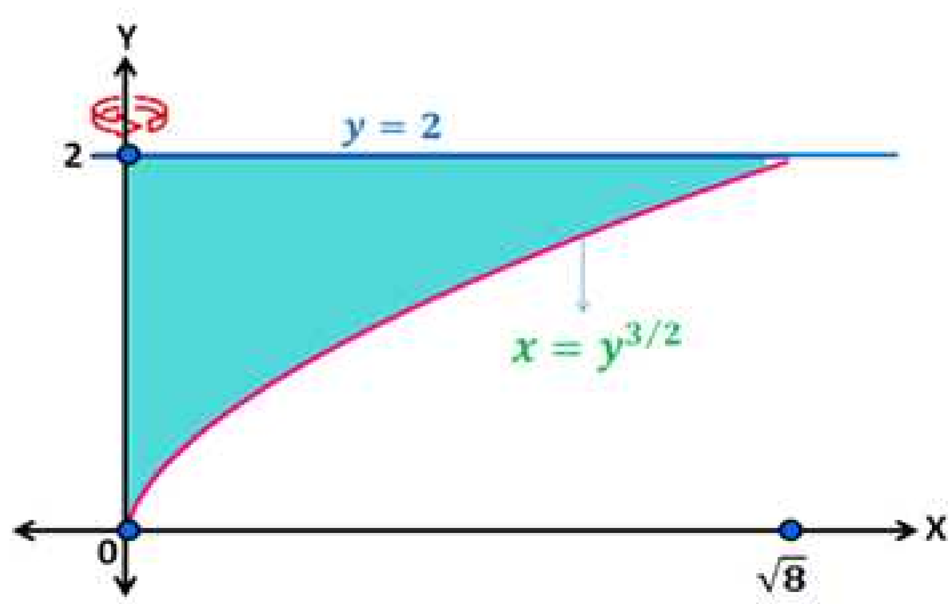
IP4:

Find the volume of the solid generated by revolving the regions enclosed by the curve $x = y^{3/2}$ and bounded by the lines $x = 0$, $y = 2$ about the y −axis.

Solution: Given curve is $x = y^{3/2}$, and lines are $x = 0\,, y = 2$

A typical cross section of the solid perpendicular to the axis of revolution is shown in figure, which is a disk of radius

$$R(y) = y^{3/2}$$



Area of the region $A(y) = \pi[R(y)]^2 = \pi\left[y^{3/2}\right]^2 = \pi y^3$

Limits of integration:

$$y^{3/2} = 0 \Rightarrow y = 0 \text{ and } y = 2$$

$$c = 0 \text{ and } d = 2$$

∴ Volume of the solid generated by revolving the regions enclosed by the curve $x = y^{3/2}$ and bounded by the lines $x = 0$, $y = 2$ about the y −axis is

$$V = \int_c^d \pi\left[R(y)\right]^2 dy$$

$$= \pi\int_0^2 y^3\, dy$$

$$= \pi\left[\frac{y^4}{4}\right]_0^2 = \pi\left[\frac{2^4}{4} - 0\right] = 4\pi$$

3.3. Volume of Solids of Revolution- Disks

Exercise:

1. Find the volumes of solids generated by revolving the regions bounded by the lines and curves in problems about the x-axis.

   a. $y = x^2, \quad y = 0, \quad x = 2$

   b. $y = \sqrt{9 - x^2}, \quad y = 0$

   c. $y = \sqrt{\cos x}, \quad 0 \le x \le \pi / 2, \ y = 0, x = 0$

2. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in problems about the y-axis.

   a) $x = \sqrt{5} y^2, \quad x = 0, \quad y = -1, \quad y = 1$

   b) $x = \sqrt{2 \sin 2y}, \quad 0 \le y \le \pi / 2, \ x = 0$

   c) $x = 2 / (y + 1), \quad x = 0, \quad y = 0, \quad y = 3$

3.4

Volumes of solids of Revolution-Washers

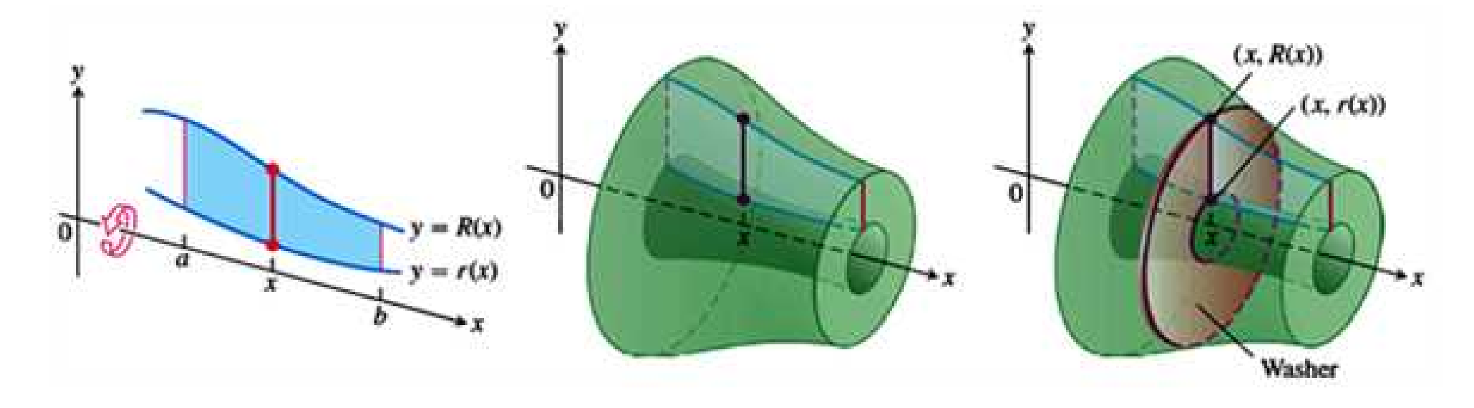
Learning objectives:

- To find the volume of a solid of revolution by Washer method.

  AND

- To practice the related problems.

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it.



The cross sections which are perpendicular to the axis of revolution are washers (rings) instead of disks. We denote the outer radius by $R(x)$ and the inner radius by $r(x)$. The washer's (ring's) area is

$$A(x)=\pi\left[R(x)\right]^2-\pi\left[r(x)\right]^2=\pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)$$

The volume is

$$V=\int_a^b \pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$$

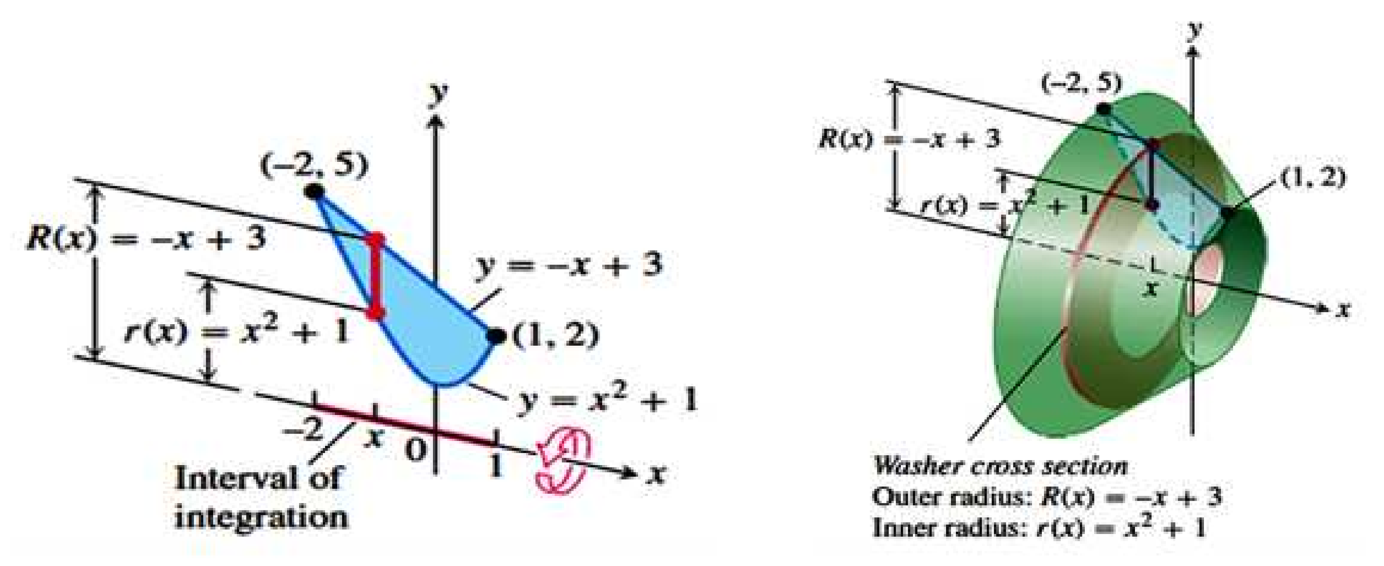
This method for calculating volume of a solid of revolution is called the *Washer method,* because a slab is a circular washer of outer radius *R(x)* and inner radius *r(x).*

Example 1:

The region bounded by the curve $y=x^2+1$ and the line $y=-x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution:

Draw the region and sketch a line segment across it perpendicular to the axis of revolution – x –axis.



When the region is revolved, this segment will generate a typical ring cross section of the generated solid.

We find the limits of integration by finding the x-coordinates of the intersection points.

$$x^2+1=-x+3 \Rightarrow x^2+x-2=0 \Rightarrow (x+2)(x-1)=0$$
$$\Rightarrow x=-2, x=1$$

We find the outer and inner radii of the ring that would be swept out by the line segment if it were revolved about the x-axis along with the region. These radii are the distances of the ends of the line segment from the axis of revolution.

$$R(x)=-x+3\ ,\quad r(x)=x^2+1$$

The volume is

$$V=\int_a^b \pi\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$$
$$=\int_{-2}^{1}\pi\left(\left(-x+3\right)^2-\left(x^2+1\right)^2\right)dx$$
$$=\int_{-2}^{1}\pi\left(8-6x-x^2-x^4\right)dx=\pi\left[8x-3x^2-\frac{x^3}{3}-\frac{x^5}{5}\right]_{-2}^{1}=\frac{117\pi}{5}$$

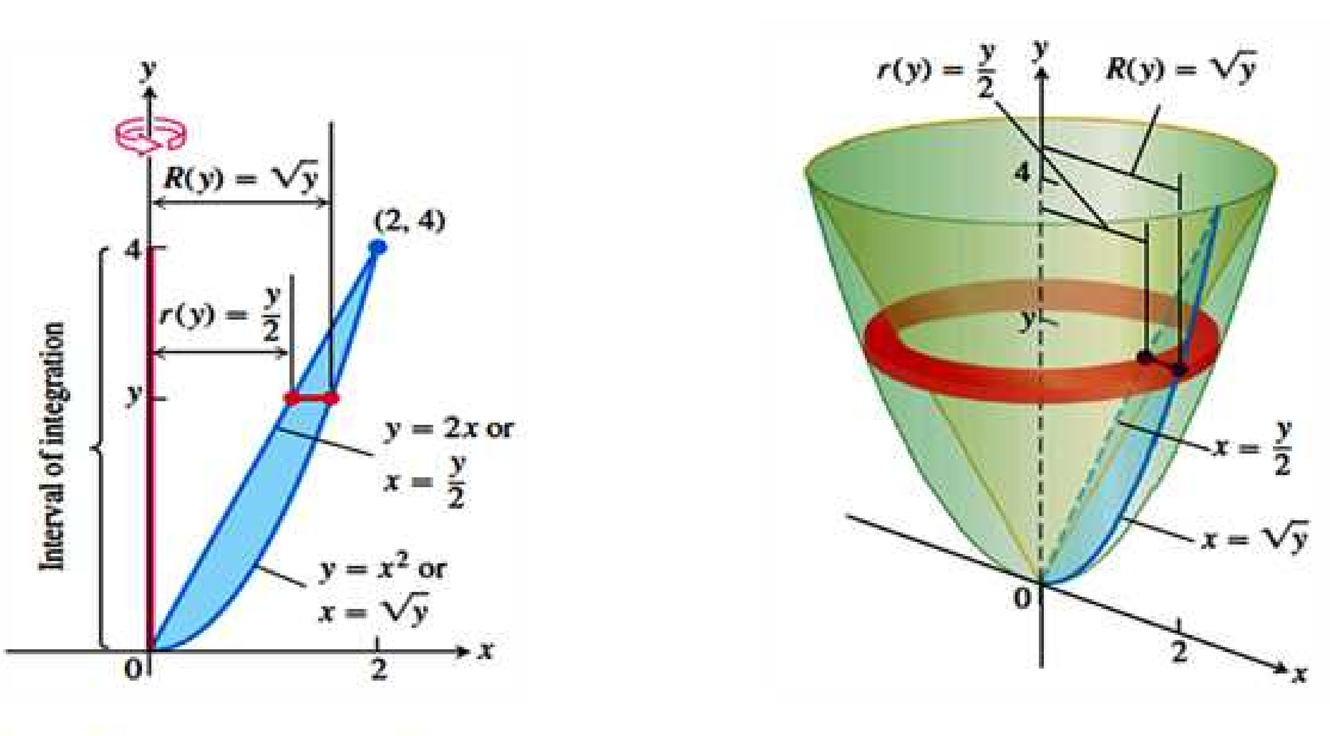
To find the volume of a solid generated by revolving a region about the y-axis, we use the same procedure but integrate with respect to y instead of x.

Example 2:

The region bounded by the parabola $y=x^2$ and the line $y=2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution

We draw the region and sketch a line segment across it perpendicular to the axis of revolution - y-axis.



The line and parabola intersect at $y=0$ and $y=4$, so the limits of integration are $c=0$ and $d=4$.

The radii of the ring swept out by the line segment are $R(y)=\sqrt{y}, r(y)=y/2$.

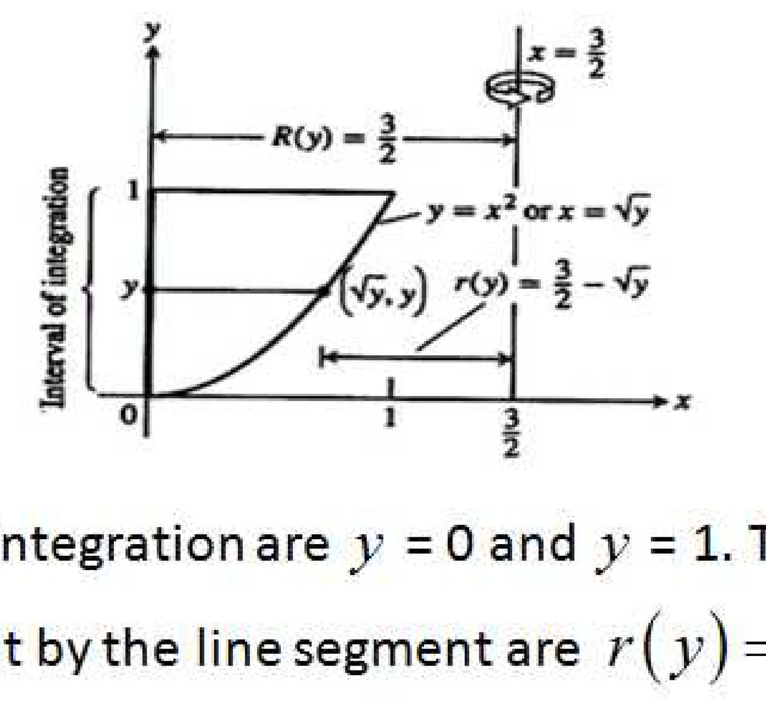
The volume is

$$V=\int_c^d \pi\left(\left[R(y)\right]^2-\left[r(y)\right]^2\right)dy=\int_0^4 \pi\left(\left[\sqrt{y}\right]^2-\left[\frac{y}{2}\right]^2\right)dy$$
$$=\int_0^4 \pi\left(y-\frac{y^2}{4}\right)dy=\pi\left[\frac{y^2}{2}-\frac{y^3}{12}\right]_0^4=\frac{8}{3}\pi$$

Example 3:

The region in the first quadrant enclosed by the parabola $y=x^2$, the y-axis, and the line $y=1$ is revolved about the line $x=3/2$ to generate a solid. Find the volume of the solid.

Solution: We draw the region and sketch a line segment across it perpendicular to the axis of revolution, in this case the line $x=\frac{3}{2}$



The limits of integration are $y=0$ and $y=1$. The radii of the ring swept out by the line segment are $r(y)=(3/2)-\sqrt{y}$, $R(y)=3/2$.

The volume is

$$V=\int_c^d \pi\left(\left[R(y)\right]^2-\left[r(y)\right]^2\right)dy$$
$$=\int_0^1 \pi\left(\left[\frac{3}{2}\right]^2-\left[\frac{3}{2}-\sqrt{y}\right]^2\right)dy$$
$$=\int_0^1 \pi\left(3\sqrt{y}-y\right)dy=\pi\left[2y^{3/2}-\frac{y^2}{2}\right]_0^4=\frac{3}{2}\pi$$

P1:

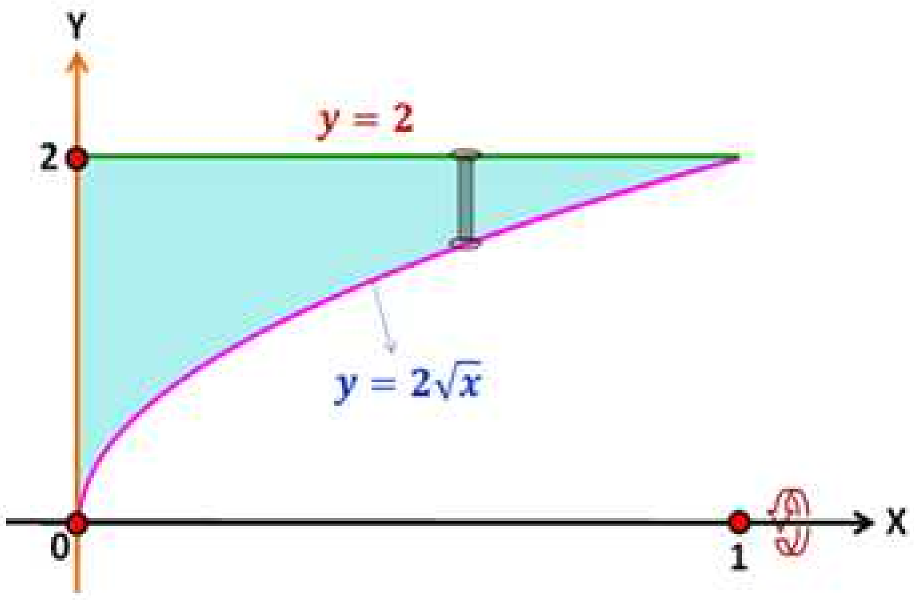
Find the volume of the solid generated by revolving the region bounded by the curve $y = 2\sqrt{x}$ and lines $y = 2, x = 0$ about the x −axis.

P1:

Find the volume of the solid generated by revolving the region bounded by the curve $y = 2\sqrt{x}$ and lines $y = 2$, $x = 0$ about the x –axis.

Solution: Given curve $y = 2\sqrt{x}$ and lines $y = 2, x = 0$

The region between the curve $y = 2\sqrt{x}$ and lines $y = 2, x = 0$ is shown in figure and sketch a line segment across it perpendicular to the axis of revolution about $x -$ axis.



When the region is revolved about x –axis, it will generate a typical ring cross section of the generated solid.

Limits on integration:

$$2\sqrt{x} = 2 \Rightarrow x = 1 \text{ and } x = 0 \Rightarrow a = 0 \text{ and } b = 1$$

Outer radius $R(x) = 2$ and inner radius $r(x) = 2\sqrt{x}$

Therefore, the ring's area of cross section is

$$A(x) = \pi\{[R(x)]^2 - [r(x)]^2\}$$

$$= \pi\left\{[2]^2 - \left[2\sqrt{x}\right]^2\right\} = 4\pi(1 - x)$$

$\therefore$ Volume of the solid is

$$V = \int_a^b \pi\left\{\left[R(x)\right]^2 - \left[r(x)\right]^2\right\} dx$$

$$= \int_0^1 4\pi\left[1 - x\right] dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 4\pi\left[\left(1 - \frac{1}{2}\right) - 0\right] = 2\pi$$

P2:

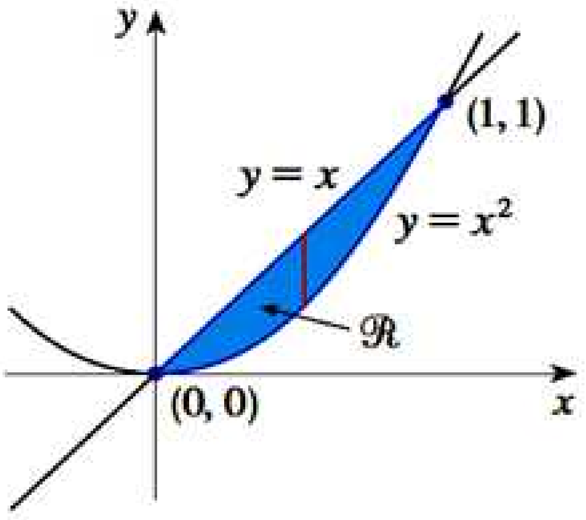
The region $\Re$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the $x-$axis. Find the volume of the resulting solid.

P2:

The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x −axis. Find the volume of the resulting solid.

Solution: The curves $y = x$ and $y = x^2$ intersect at the points $(0, 0)$ and $(1, 1)$. The region between them, the solid of rotation, and a cross section perpendicular to x −axis are shown in figure.

Sketch a line segment across it perpendicular to the axis of revolution about the x −axis.



Limits of integration:

$$x^2 = x \Rightarrow (x^2 - x) = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, x = 1$$

$$\Rightarrow a = 0 \text{ and } b = 1$$

A cross section in the plane has the shape of a washer (an annular ring) with inner radius $r(x) = x^2$ and outer radius $R(x) = x$.

Now, we find the cross sectional area by subtracting the area of the inner circle from the area of the outer circle

$$A(x) = \pi\{[R(x)]^2 - [r(x)]^2\}$$

$$A(x) = \pi\{[x]^2 - [x^2]^2\} = \pi\{x^2 - x^4\}$$

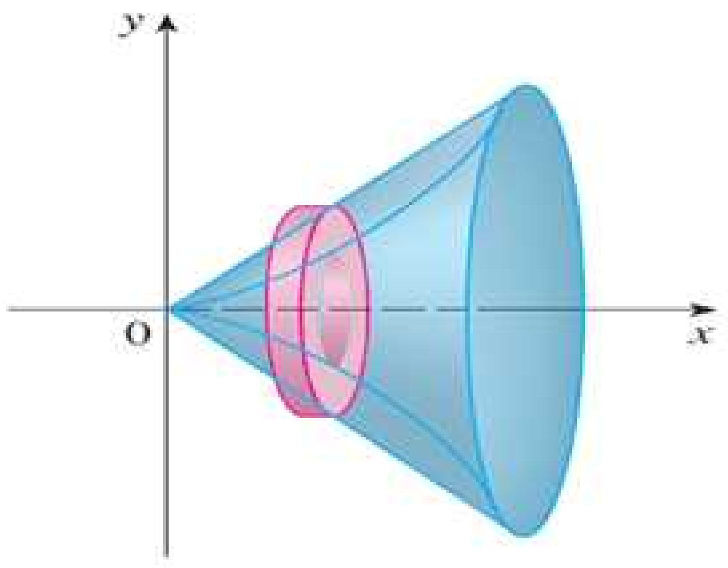
∴ Volume of the solid is

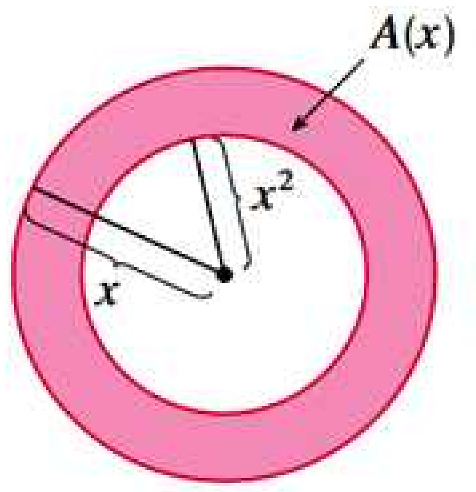
$$V = \int_a^b \pi\left\{[R(x)]^2 - [r(x)]^2\right\} dx$$

$$= \int_0^1 \pi\left[x^2 - x^4\right] dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \pi\left[\left(\frac{1}{3} - \frac{1}{5}\right) - 0\right] = \frac{2\pi}{15}$$

The solid of revolution and a washer is shown below.





P3:

Find the volume of the solid generated by revolving the region enclosed by the triangle with vertices $(1, 0)$, $(2, 1)$ and $(1, 1)$ about $y-$ axis.

P3:

Find the volume of the solid generated by revolving the region enclosed by the triangle with vertices $(\mathbf{1},\mathbf{0}),(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{1})$ about $\boldsymbol{y}-$ axis.

Solution: Given vertices of a triangle is $A(1,0), B(2,1)$ and $C(1,1)$. The equation of the line $\overline{AB}$ is

$$(y-0)(2-1)=(1-0)(x-1) \Rightarrow x=y+1$$

The equation of the line $\overline{BC}$ is

$$(y-1)(1-2)=(1-1)(x-2) \Rightarrow y=1$$

The equation of the line $\overline{CA}$ is

$$(y-1)(1-1)=(0-1)(x-1) \Rightarrow x=1$$

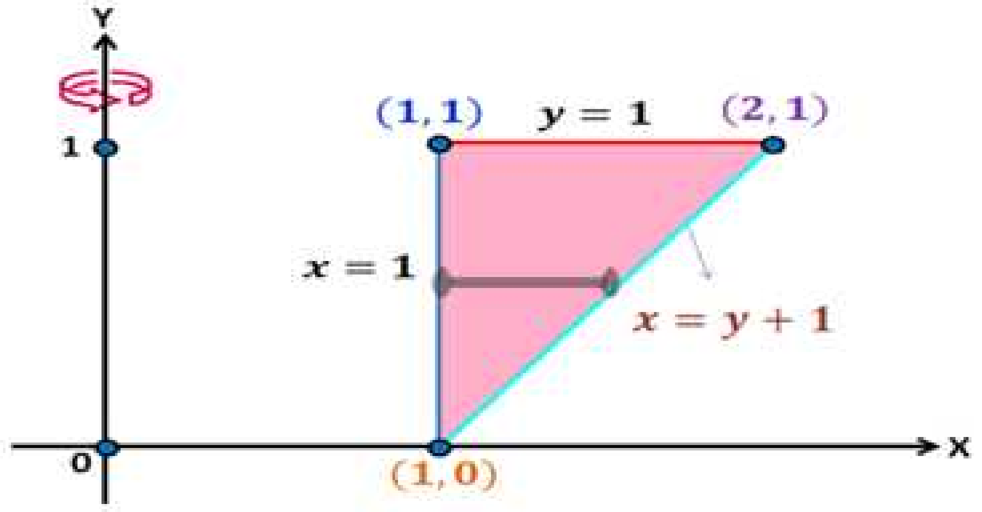
The triangular region of the solid is plotted between the lines $x=y+1,\ y=1,\ x=1$ is shown below.

Sketch a line segment across it perpendicular to the axis of revolution – y –axis.

When the region is revolved about y –axis, it will generate a typical ring cross section of the generated solid.

Limits of integration:

$y+1=1 \Rightarrow y=0$, and $y=1 \Rightarrow c=0$ and $d=1$



From figure, outer radius is $R(y)=1+y$ and inner radius is $r(y)=1$. Therefore, the ring's cross sectional area is

$$A(x)=\pi\{[R(y)]^2-[r(y)]^2\}$$
$$=\pi\{[1+y]^2-[1]^2\}=\pi(y^2+2y)$$

$\therefore$ Volume of the solid is

$$V=\int_c^d \pi\left\{[R(y)]^2-[r(y)]^2\right\}dy=\pi\int_0^1\left[y^2+2y\right]dx$$

$$=4\pi\left[\frac{y^3}{3}+y^2\right]_0^1=4\pi\left[\left(\frac{1}{3}+1\right)-0\right]=\frac{4\pi}{3}$$

P4:

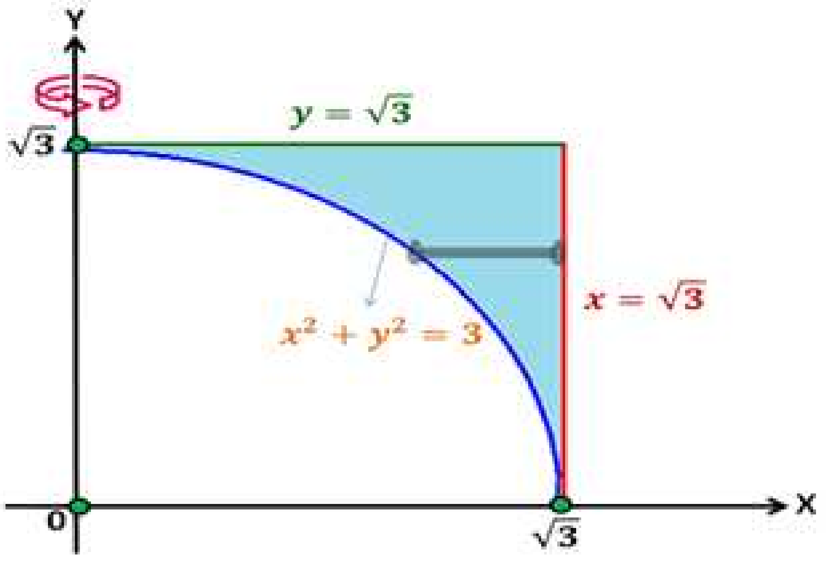
Find the volume of the solid generated by revolving the region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$ and above by the line $y = \sqrt{3}$ about the y −axis.

P4:

Find the volume of the solid generated by revolving the region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$ and above by the line $y = \sqrt{3}$ about the y −axis.

Solution: The region is plotted in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$ and above by the line $y = \sqrt{3}$ is shown in figure.

Sketch a line segment across it perpendicular to the axis of revolution about $y -$ axis.



When the region is revolved about y −axis, it will generate a typical ring cross section of the generated solid.

Limits on integration:

$$\sqrt{3 - y^2} = \sqrt{3} \Rightarrow y = 0 \text{ and } y = \sqrt{3} \Rightarrow c = 0 \text{ and } d = \sqrt{3}$$

From figure, outer radius is $R(y) = \sqrt{3}$ and inner radius is $r(y) = \sqrt{3 - y^2}$

Therefore, the ring's area of cross section is

$$A(y) = \pi\{[R(y)]^2 - [r(y)]^2\}$$

$$= \pi\left\{\left[\sqrt{3}\right]^2 - \left[\sqrt{3 - y^2}\right]^2\right\} = \pi y^2$$

∴ Volume of the solid is

$$V = \int_c^d \pi\left\{\left[R(y)\right]^2 - \left[r(y)\right]^2\right\} dy = \int_0^{\sqrt{3}} \pi\left[y^2\right] dy$$

$$= \pi\left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\left[\left(\frac{\left(\sqrt{3}\right)^3}{3}\right) - 0\right] = \pi\sqrt{3}$$

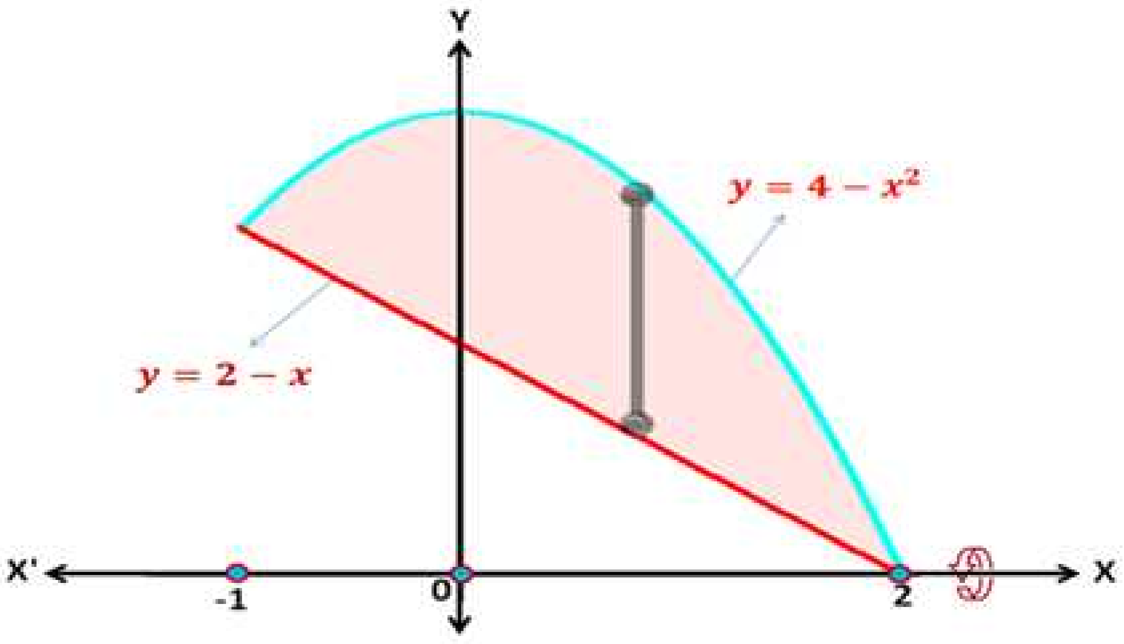
IP1:

Find the volume of the solid generated by revolving the regions bounded by the curve $y = 4 - x^2$ and line $y = 2 - x$ about the x −axis.

Solution: Given curve $y = 4 - x^2$ and line $y = 2 - x$

The region between the curve $y = 4 - x^2$ and line $y = 2 - x$

is shown in figure and sketch a line segment across it perpendicular to the axis of revolution about $x -$ axis.



When the region is revolved about x −axis, it will generate a typical ring cross section of the generated solid.

Limits on integration:

$$4 - x^2 = 2 - x \Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, x = 2$$

$$\Rightarrow a = -1 \text{ and } b = 2$$

Outer radius $R(x) = 4 - x^2$ and inner radius $r(x) = 2 - x$

Therefore, the ring's area of cross section is

$$A(x) = \pi\{[R(x)]^2 - [r(x)]^2\} = \pi\{[4 - x^2]^2 - [2 - x]^2\}$$

$$= 4\pi(x^4 - 9x^2 + 4x + 12)$$

∴ Volume of the solid is

$$V = \int_a^b \pi\left\{[R(x)]^2 - [r(x)]^2\right\} dx$$

$$= \int_0^1 4\pi\left[x^4 - 9x^2 + 4x + 12\right] dx$$

$$= 4\pi\left[\frac{x^5}{5} - 3x^3 + 2x^2 + 12x\right]_0^1$$

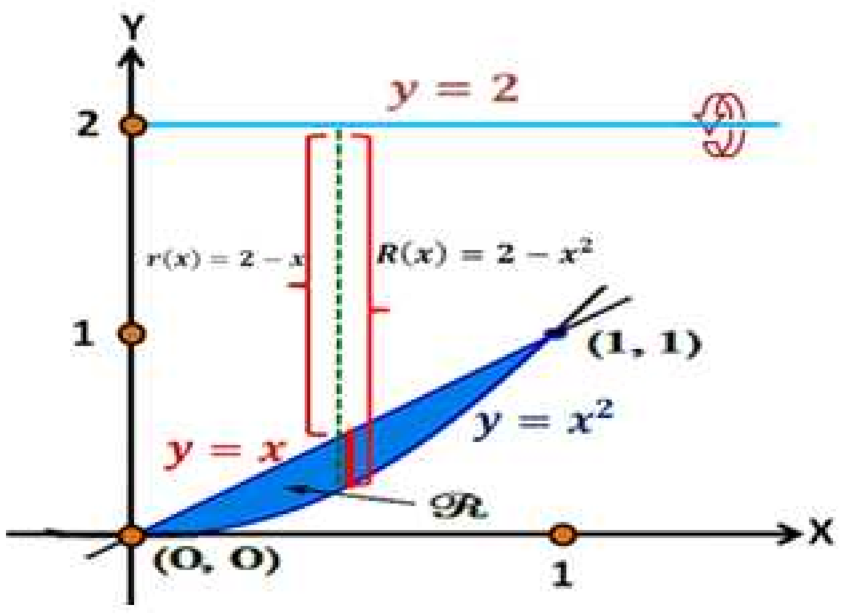
$$= 4\pi\left[\left(\frac{1}{5} - 3 + 2 + 12\right) - 0\right] = \frac{108\pi}{5}$$

IP2:

The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume of the solid.

Solution: The curves $y = x$ and $y = x^2$ intersect at the points $(0,0)$ and $(1,1)$. The region between them, the solid of revolution, and a cross section perpendicular to x −axis are shown in figure.

Sketch a line segment perpendicular to the axis of revolution i.e., the line $y = 2$.



Limits of integration:

$$x^2 = x \Rightarrow (x^2 - x) = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, x = 1$$

$$\Rightarrow a = 0 \text{ and } b = 1$$

A cross section in the plane has the shape of a washer (an annular ring) with inner radius $r(x) = 2 - x$ and outer radius $R(x) = 2 - x^2$.

Now, we find the cross sectional area by subtracting the area of the inner circle from the area of the outer circle

$$A(x) = \pi\{[R(x)]^2 - [r(x)]^2\}$$

$$= \pi\{[2 - x^2]^2 - [2 - x]^2\} = \pi\{x^4 - 5x^2 + 4x\}$$

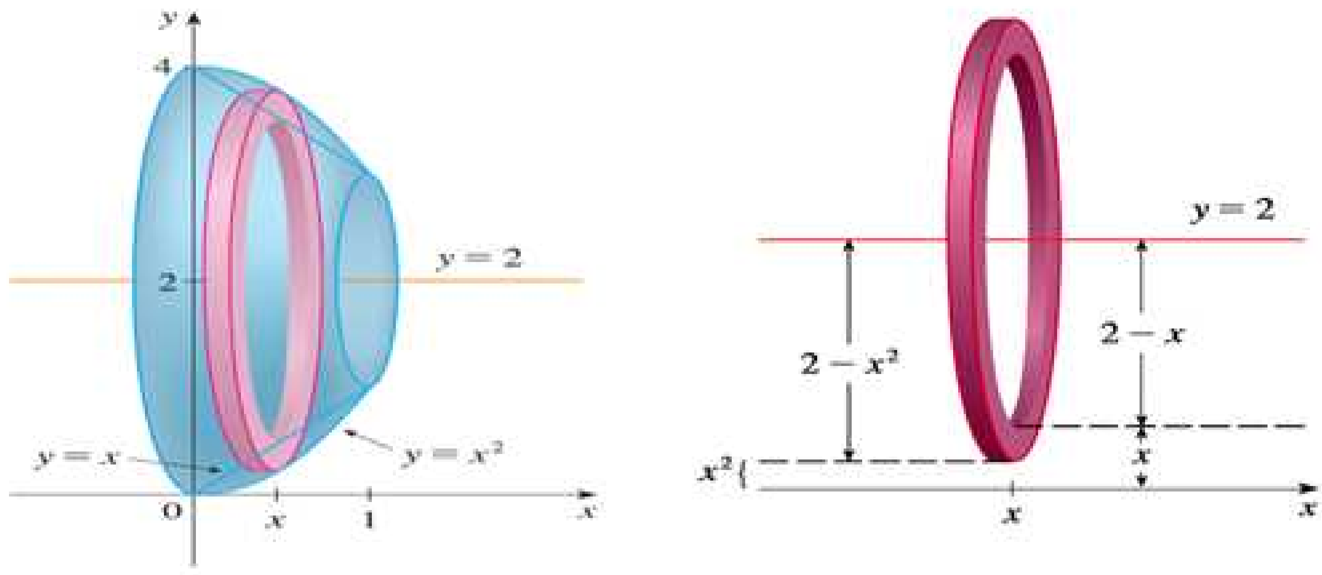
∴ Volume of the solid is

$$V = \int_a^b \pi\left\{[R(x)]^2 - [r(x)]^2\right\} dx$$

$$= \int_0^1 \pi\left[x^4 - 5x^2 + 4x\right] dx = \pi\left[\frac{x^5}{5} - \frac{5x^3}{3} + \frac{4x^2}{2}\right]_0^1$$

$$= \pi\left[\left(\frac{1}{5} - \frac{5}{3} + 2\right) - 0\right] = \frac{8\pi}{15}$$

The solid of revolution and a typical washer is shown below.



IP3:

Find the volume of the solid generated by revolving the region enclosed by the triangle with vertices $(0,1)$, $(1,0)$ and $(1,1)$ about $y-$ axis.

Solution: Given vertices of a triangle is $A(0,1), B(1,0)$ and $C(1,1)$. The equation of the line $\overline{AB}$ is

$$(y-1)(1-0) = (0-1)(x-0) \Rightarrow x = 1-y$$

The equation of the line $\overline{BC}$ is

$$(y-0)(1-1) = (1-0)(x-1) \Rightarrow x = 1$$

The equation of the line $\overline{CA}$ is

$$(y-1)(0-1) = (1-1)(x-1) \Rightarrow y = 1$$

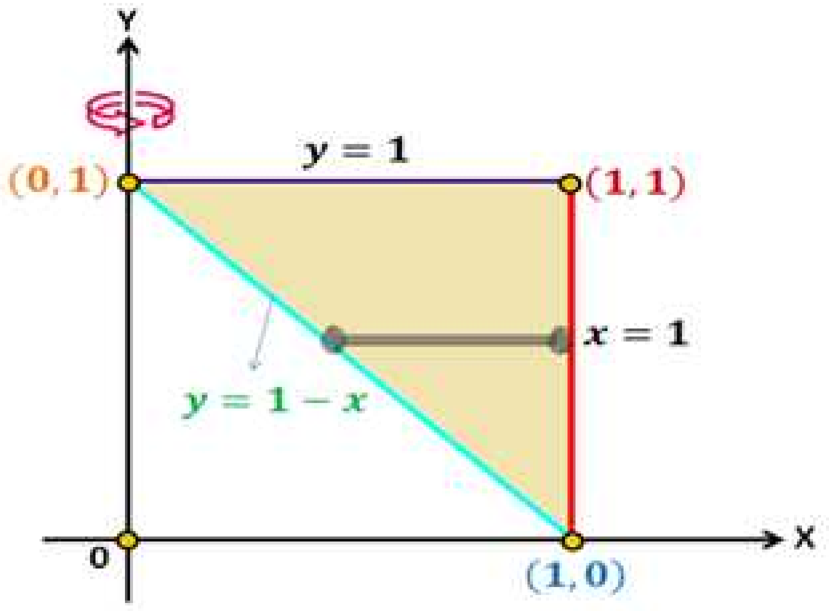
The triangular region of the solid is plotted between the lines $x = 1-y,\ y = 1,\ x = 1$ is shown below.

Sketch a line segment across it perpendicular to the axis of revolution – y –axis.

When the region is revolved about y –axis, it will generate a typical ring cross section of the generated solid.

Limits of integration:

$1-y = 1 \Rightarrow y = 0$, and $y = 1 \Rightarrow c = 0$ and $d = 1$



From figure, outer radius is $R(y) = 1$ and inner radius is $r(y) = 1-y$. Therefore, the ring's cross sectional area is

$$A(x) = \pi\{[R(y)]^2 - [r(y)]^2\}$$
$$= \pi\{[1]^2 - [1-y]^2\} = \pi(2y - y^2)$$

$\therefore$ Volume of the solid is

$$V = \int_c^d \pi\left\{[R(y)]^2 - [r(y)]^2\right\} dy = \pi\int_0^1 \left[2y - y^2\right] dx$$

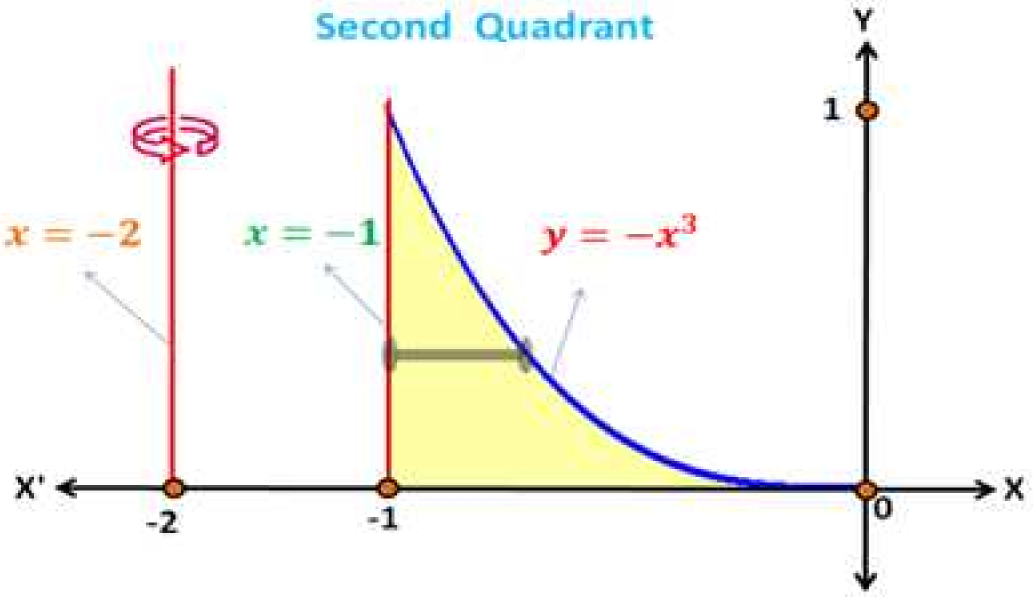
$$= 4\pi\left[y^2 - \frac{y^3}{3}\right]_0^1 = 4\pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{2\pi}{3}$$

IP4:

Find the volume of the solid generated by revolving the region in the second quadrant bounded above by the curve $y = -x^3$, below by the x –axis and on the right by the line $x = -1$ about the line $x = -2$.

Solution: The region is plotted in the second quadrant bounded above by the curve $y = -x^3$, below by the x –axis and on the right by the line $x = -1$ is shown in figure.

Sketch a line segment across it perpendicular to the axis of revolution about y – axis.



When the region is revolved about y –axis, it will generate a typical ring cross section of the generated solid.

Limits on integration:

$-y^{1/3} = -1 \Rightarrow y = 1$ and $y = 0 \Rightarrow c = 0$ and $d = 1$

From figure, outer radius is $R(y) = -y^{1/3} - (-2) = 2 - y^{1/3}$ and inner radius is $r(y) = 1$

Therefore, the ring's area of cross section is

$$A(y) = \pi\{[R(y)]^2 - [r(y)]^2\}$$

$$= \pi\left\{\left[2 - y^{1/3}\right]^2 - [1]^2\right\} = \pi\left[3 + y^{2/3} - 4y^{1/3}\right]$$

∴ Volume of the solid is

$$V = \int_c^d \pi\left\{[R(y)]^2 - [r(y)]^2\right\} dy$$

$$= \int_0^1 \pi\left[3 + y^{2/3} - 4y^{1/3}\right] dy = \pi\left[3y + \frac{3}{5}y^{5/3} - 3y^{4/3}\right]_0^{\sqrt{3}}$$

$$= \pi\left[\left(3 + \frac{3}{5} - 3\right) - 0\right] = \frac{3\pi}{5}$$

3.4. Volumes of solids of Revolution-Washers

Exercise:

1. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in problems 1-4 about the x-axis.

   a) $y = x, \quad y = 1, \quad x = 0$

   b) $y = x^2 + 1, \quad y = x + 3$

   c) $y = \sec x, \quad y = \sqrt{2}, \ -\pi/4 \le x \le \pi/4$

2. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line x = 2. Find the volume of the solid generated by revolving the region about the y-axis.

3. In the first quadrant, a region is bounded above by the curve $y = x^2$, below by the x-axis, and on the right by the line $x = 1$. Find the volume of the solid generated by revolving the region about the line $x = -1$.

4. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines y = 2 and x = 0 about

   a. the x-axis;

   b. the y-axis;

   c. the line y = 2;

   d. the line x = 4.

5. Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the line $y = 1$ about

   a. the line y = 1;

   b. the line y = 2;

   c. the line $y = -1$.

3.5

Shell Method of Finding Volumes

Learning objectives:

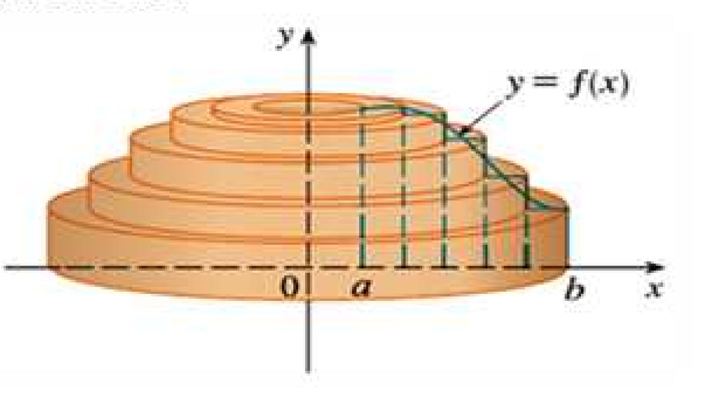
- To find the volume of the solid generated by shell method.

  AND

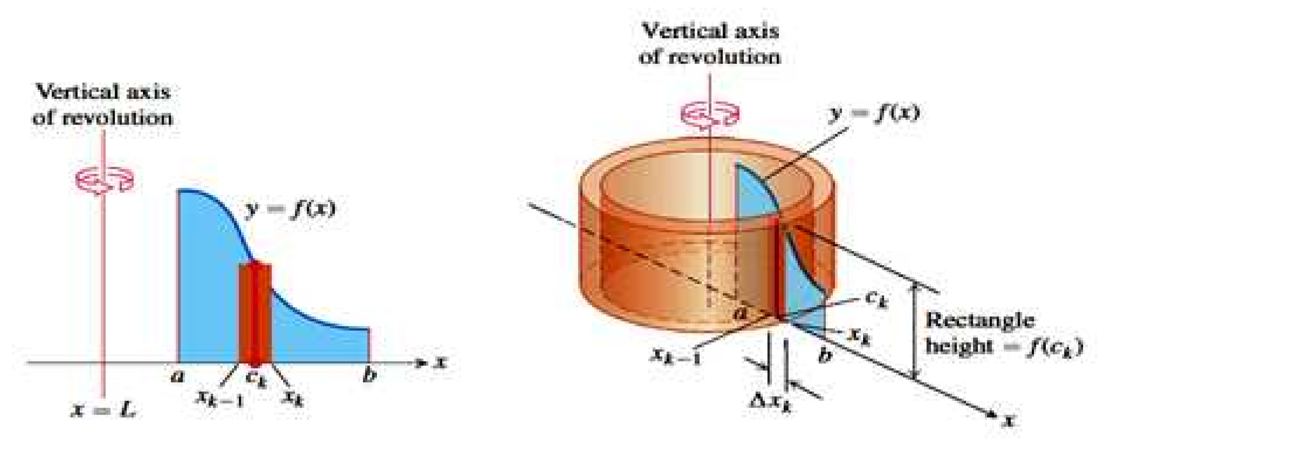
- To practice the related problems.

In finding the volume of a solid of revolution, cylindrical shell method sometimes works better than rings. We derive the shell formula below.

In the figure below, a solid of revolution is approximated by the cylindrical shells.



To estimate the volume of the solid, we can approximate the region with rectangles based on a partition P of the interval $[a,b]$ over which the region stands. The typical approximating rectangle is Δx_k units wide by $f(c_k)$ units high, where c_k is the midpoint of the rectangle's base.



A formula from geometry tells us that the volume of the shell swept out by the rectangle is

$$\Delta V_k = 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness}$$
$$= 2\pi \times c_k \times f(c_k) \times \Delta x_k$$

We approximate the volume of the solid by adding the volumes of the shells swept out by the n rectangles based on P: $\quad V = \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi c_k f(c_k)\Delta x_k$

The limit of this sum as $\|P\| \to 0$ gives the volume of the solid: $\quad V = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi c_k f(c_k)\Delta x_k = \int_a^b 2\pi\, x\, f(x)dx$

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0, 0 \leq a \leq x \leq b$, about the y-axis is

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx$$

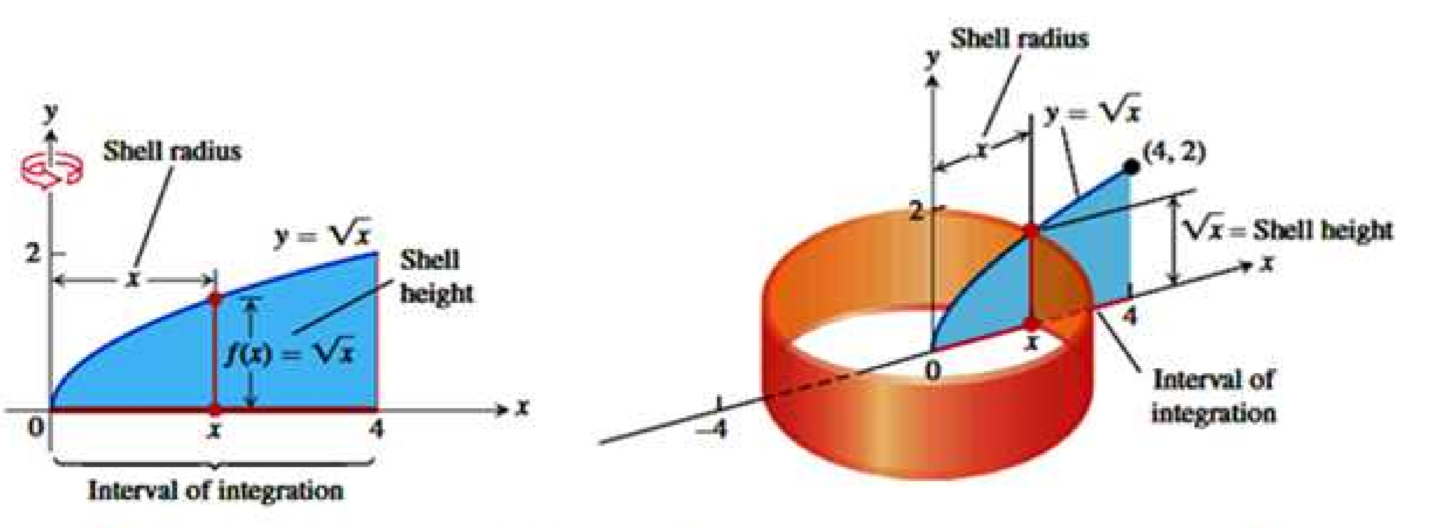
$$V = \int_a^b 2\pi\, x\, f(x)dx \qquad \text{...... (1)}$$

Example 1:

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution:

Sketch the region and draw a line segment across it parallel to the axis of revolution.



Label the segment's height, and distance from the axis of revolution. The width of the segment is the shell thickness dx. The limits of integration are $a = 0$ and $b = 4$.

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx$$
$$= \int_a^b 2\pi\, x\, f(x)\, dx = \int_0^4 2\pi\, x\sqrt{x}\, dx$$
$$= 2\pi\int_0^4 x^{3/2}\, dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

❖ Equation (1) is for the vertical axes of revolution. For horizontal axes, we replace x's with y's.

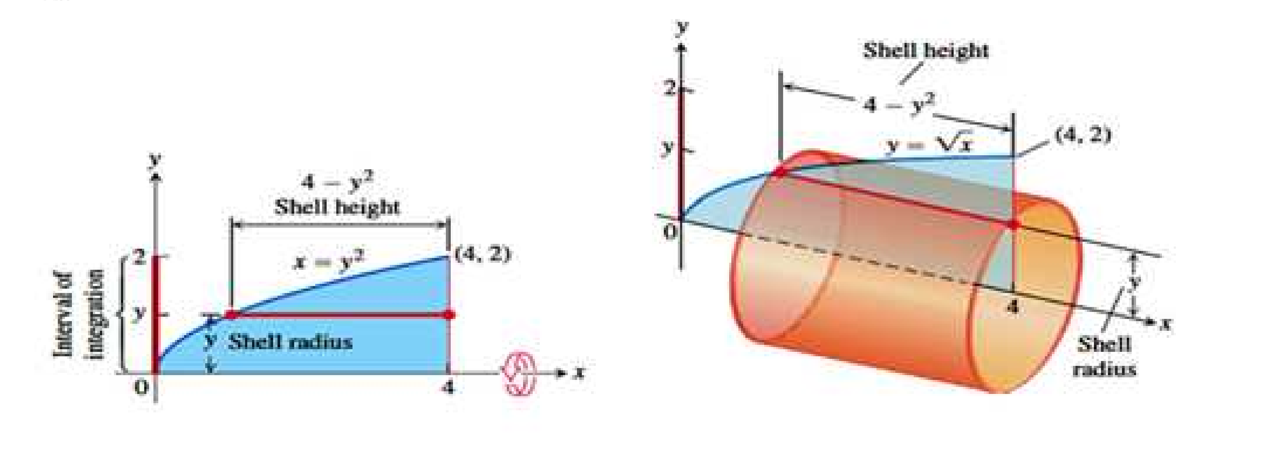
$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\, dy$$
$$= \int_c^d 2\pi\, y\, f(y)dy$$

Example 2:

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line x = 4 is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution:

Sketch the region and draw a line segment across it parallel to the axis of revolution.



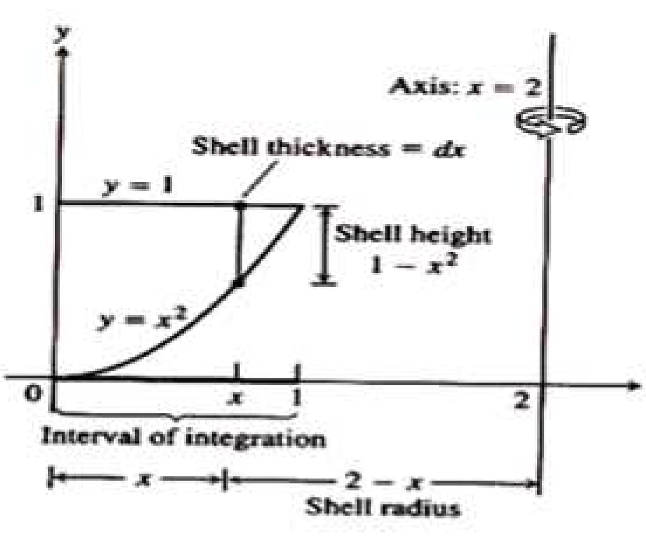
Label the segment's length (shell height), and distance from the axis of revolution (shell radius). The width of the segment is the shell thickness dy. The limits of integration are c = 0 and d = 2.

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\, dy$$
$$= \int_c^d 2\pi\, y\, f(y)dy = \int_0^2 2\pi\, y\left(4 - y^2\right)dy$$
$$= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi$$

Example 3:

The region in the first quadrant bounded by the parabola $y = x^2$, the y-axis, and the line y = 1 is revolved about the line x=2 to generate a solid. Find the volume of the solid.

Solution: Draw a line segment across it parallel to the axis of revolution (the line x = 2).



Label the segment's height, and distance from the axis of revolution. The width of the segment is the shell thickness dx.

The limits of integration are a = 0 and b = 1.

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx$$
$$= \int_0^1 2\pi(2 - x)\left(1 - x^2\right)dx$$
$$= 2\pi\int_0^1 \left(2 - x - 2x^2 + x^3\right)dx = \frac{13\pi}{6}$$

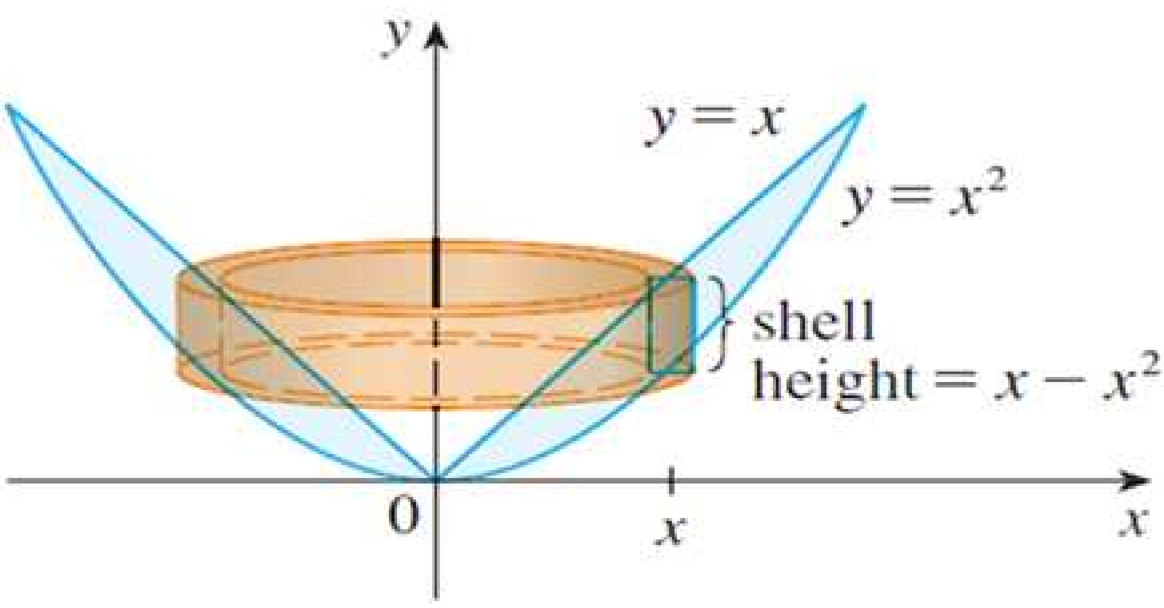
IP1:

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve $y = x^2$ and the line $y = x$ about the y –axis.

Solution:

Given curve $y = x^2$ and the line is $y = x$.

The region between the curves $y = x^2$ and the line $y = x$ is shown in figure and draw a line segment across it parallel to the axis of revolution: $y -$ axis.



Limits of integration:

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0$ and $x = 1$

$\Rightarrow a = 0$ and $b = 1$

From the figure, $\text{shell radius} = x, \ \text{shell height} = x - x^2$

$$\text{shell thickness} = dx$$

Therefore, the volume of the solid is

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx$$

$$= \int_0^1 2\pi x\left[x - x^2\right] dx$$

$$= 2\pi \int_0^2 \left[x^2 - x^3\right] dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right] = \frac{\pi}{6}$$

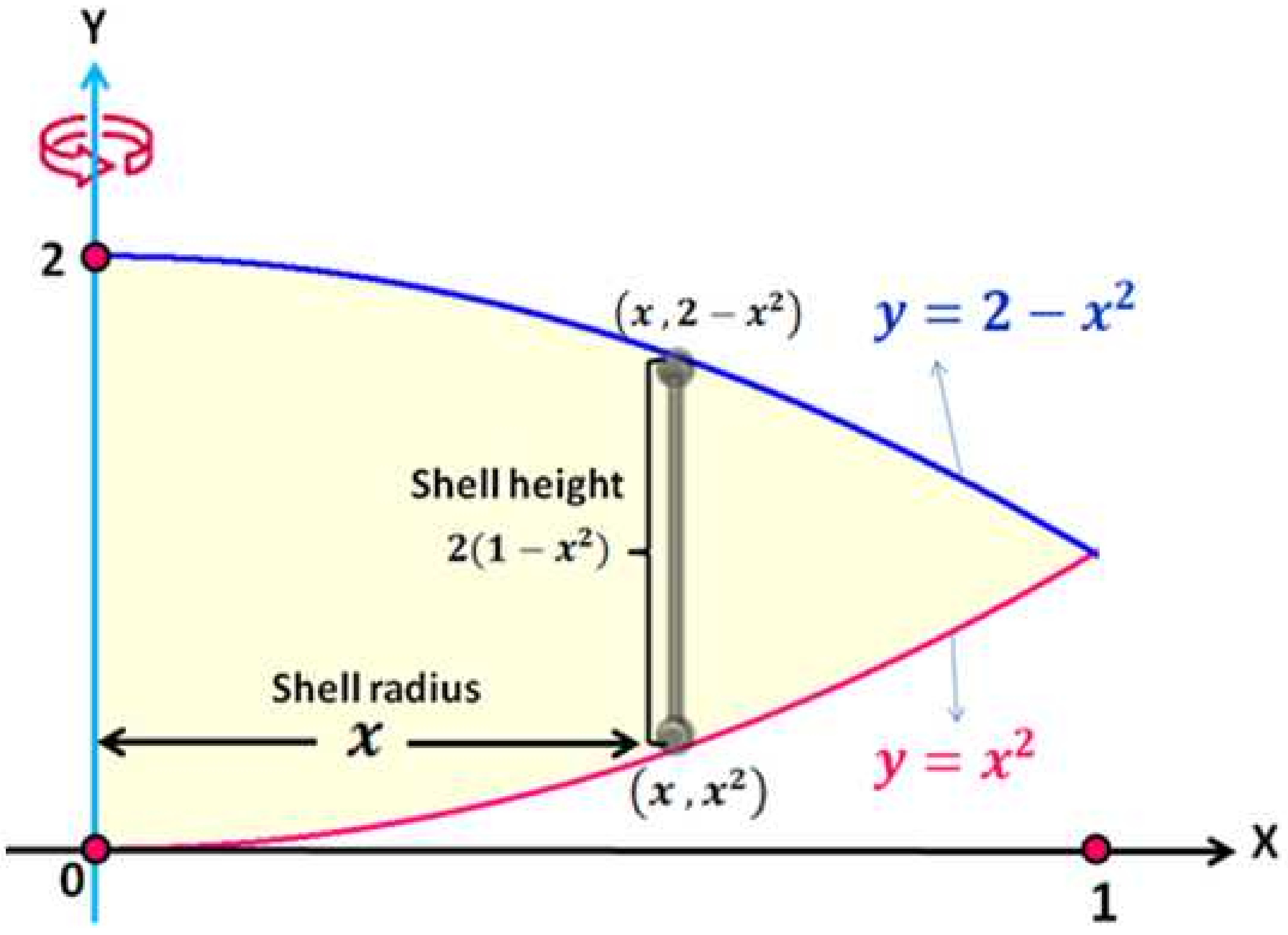
IP2:

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curves $y = 2 - x^2$, $y = x^2$ and the line $x = 0$ about the y –axis.

Solution:

Given curves are $y = 2 - x^2$, $y = x^2$ and the line is $x = 0$.

The region between the curves $y = 2 - x^2$, $y = x^2$ and the line $x = 0$ is shown in figure and draw a line segment across it parallel to the axis of revolution: y – axis.



Limits of integration:

$2 - x^2 = x^2 \Longrightarrow 2x^2 = 2 \Longrightarrow x = 1$ and $x = 0$

$\Longrightarrow a = 0$ and $b = 1$

From the figure, shell radius $= x$

$$\text{shell height} = (2 - x^2) - x^2 = 2(1 - x^2)$$

$$\text{shell thickness} = dx$$

Therefore, the volume of the solid is

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx$$

$$= \int_0^1 2\pi x\left[2\left(1 - x^2\right)\right] dx$$

$$= 4\pi \int_0^1 \left(x - x^3\right) dx$$

$$= 4\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 4\pi\left[\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right] = \pi$$

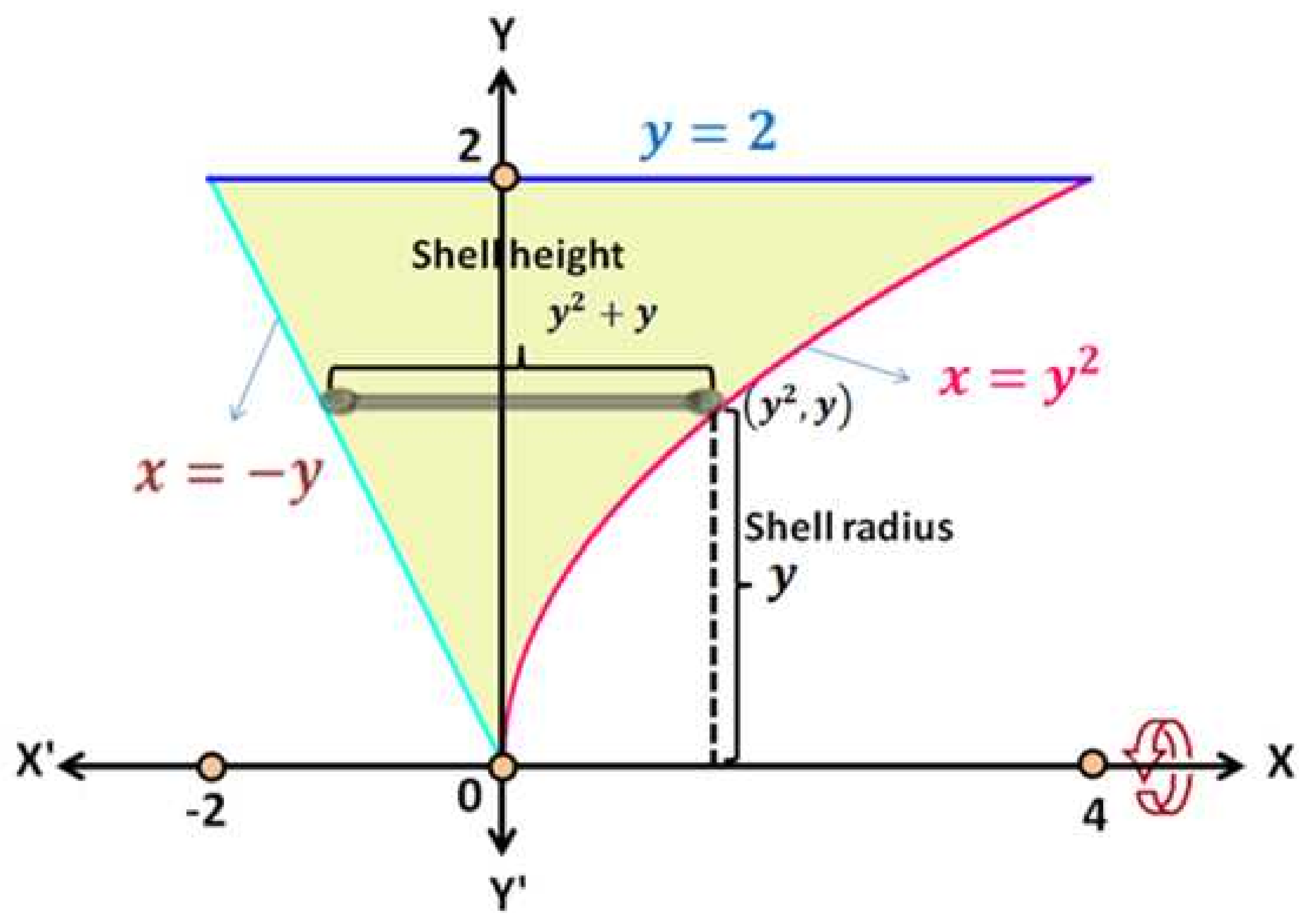
IP3:

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve $x = y^2$ and the lines $x = -y,\ y = 2,\ y \geq 0$ about the $x-$ axis.

Solution:

Given curve is $x = y^2$ and the lines are $x = -y,\ y = 2, y \geq 0.$

The region between the curve $x = y^2$ and the lines $x = -y$, $y = 2$ is shown in figure for $y \geq 0$ and draw a line segment across it parallel to the axis of revolution: x −axis.



Limits of integration:

$y^2 = -y \Rightarrow y(y+1) = 0 \Rightarrow y = 0\,, y = 1$(not a solution because $y \geq 0$) and $y = 2$

$\Rightarrow a = 0$ and $b = 2$

From the figure, shell radius $= y$

$$\text{shell height} = y^2 - (-y) = y^2 + y,$$

$$\text{shell thickness} = dy$$

Therefore, the volume of the solid is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\, dy$$

$$= \int_0^2 2\pi\, y\left[y^2 + y\right]dy$$

$$= 2\pi \int_0^2 \left[y^3 + y^2\right]dy$$

$$= 2\pi\left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2$$

$$= 2\pi\left[\left(4 + \frac{8}{3}\right) - 0\right] = \frac{40\pi}{3}$$

IP4:

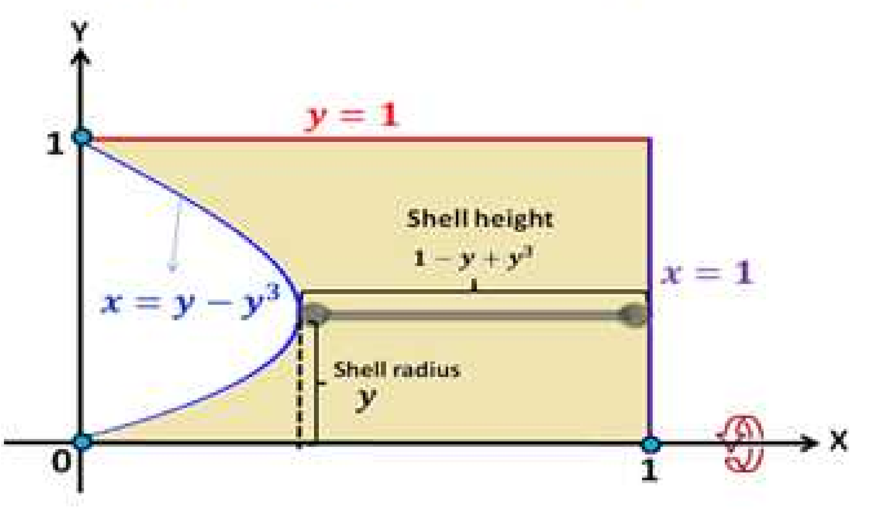
Use the shell method or washer method to find the volume of the solid generated by revolving the region in the first quadrant bounded by the curve $x = y - y^3$ and the lines $x = 1,\ y = 1$ about

A. The $x-$axis **C. The $y-$axis**

B. The line $x = 1$ **D. The line $y = 1$**

Solution: Given curve $x = y - y^3$ and the lines $x = 1,\ y = 1$.

A. The region between the curve $x = y - y^3$ and the lines $x = 1,\ y = 1$ is as shown in figure and draw a line segment across it parallel to the axis of revolution: $x-$axis.



Limits of integration:

Since the region in the first quadrant, so $c = 0$ and $d = 1$

From the figure, shell radius $= y$, shell thickness $= dy$

$$\text{shell height} = 1 - (y - y^3) = 1 - y + y^3$$

Therefore, the volume of the solid is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy$$

$$= \int_0^1 2\pi y\left[1 - y + y^3\right]dy = 2\pi\int_0^1\left[y - y^2 + y^4\right]dy$$

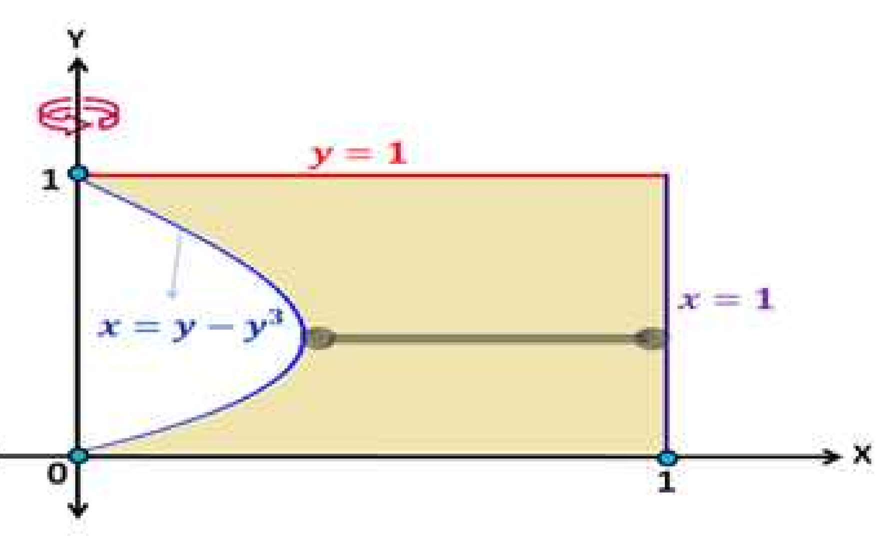
$$= 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5}\right]_0^1 = 2\pi\left[\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right) - 0\right] = \frac{11\pi}{15}$$

B. In this case, we cannot express y explicitly in terms of x. So, shell method cannot be used. Here we are using the washer method to solve the problem.

The region is shown in figure. Sketch a line segment across it perpendicular to the axis of revolution about the $y-$axis.

Limits of integration:

Since the region in the first quadrant, so $c = 0$ and $d = 1$



When the region is revolved about $y-$axis, it will generate a typical washer cross section of the generated solid.

From the figure, inner radius $r(y) = (y - y^3)$ and outer radius $R(y) = 1$

Therefore, the cross section's area is

$$A(y) = \pi\left[(R(y))^2 - (r(y))^2\right]$$

$$= \pi[1 - (y - y^3)^2] = \pi[1 - y^2 - y^6 + 2y^4]$$

Therefore, the volume of the solid is

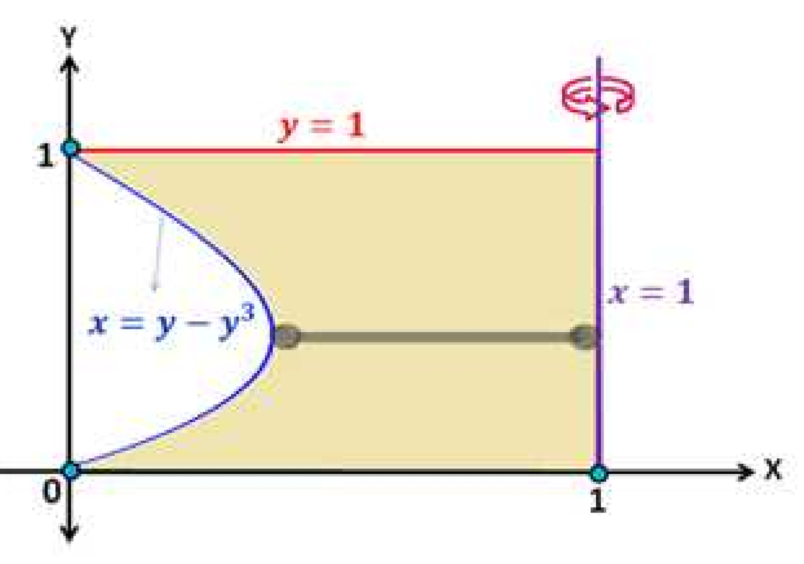
$$V = \int_c^d \pi\left[(R(y))^2 - (r(y))^2\right]dy = \int_0^1 \pi\left[1 - y^2 - y^6 + 2y^4\right]dy$$

$$= \pi\left[y - \frac{y^3}{3} - \frac{y^7}{7} + \frac{2y^5}{5}\right]_0^1 = \pi\left[\left(1 - \frac{1}{3} - \frac{1}{7} + \frac{2}{5}\right) - 0\right] = \frac{97\pi}{105}$$

C. In this case, we cannot express y explicitly in terms of x. So, shell method cannot be used. Here we are using the washer method to solve the problem.

The region is shown in figure. Sketch a line segment across it perpendicular to the axis of revolution about the line $x = 1$.

Limits of integration: Since the region in the first quadrant, so $c = 0$ and $d = 1$



When the region is revolved about $y-$axis, it will generate a typical washer cross section of the generated solid.

From the figure, inner radius $r(y) = 0$ and outer radius $R(y) = [1 - (y - y^3)]$

Therefore, the cross section's area is

$$A(y) = \pi\left[(R(y))^2 - (r(y))^2\right] = \pi\left[\left(1 - (y - y^3)\right)^2 - 0\right]$$

$$= \pi[1 + y^2 + y^6 - 2y + 2y^3 - 2y^4]$$

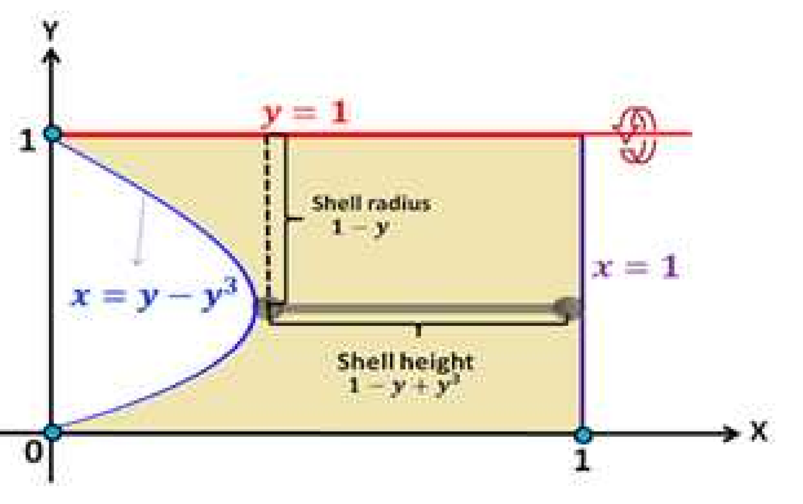
Therefore, the volume of the solid is

$$V = \int_c^d \pi\left[(R(y))^2 - (r(y))^2\right]dy = \int_0^1 \pi\left[1 + y^2 + y^6 - 2y + 2y^3 - 2y^4\right]dy$$

$$= \pi\left[y + \frac{y^3}{3} + \frac{y^7}{7} - y^2 + \frac{y^4}{2} - \frac{2y^5}{5}\right]_0^1$$

$$= \pi\left[\left(1 + \frac{1}{3} + \frac{1}{7} - 1 + \frac{1}{2} - \frac{2}{5}\right) - 0\right] = \frac{121\pi}{210}$$

D. The region is as shown in figure and draw a line segment across it parallel to the axis of revolution: $y = 1$.



Limits of integration: $c = 0$ and $d = 1$

From the figure, shell radius $= (1 - y)$, shell thickness $= dy$

$$\text{shell height} = 1 - (y - y^3) = 1 - y + y^3$$

Therefore, the volume of the solid is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy$$

$$= \int_0^1 2\pi(1 - y)\left[1 - y + y^3\right]dy = 2\pi\int_0^1\left[1 - y + y^3 - y + y^2 - y^4\right]dy$$

$$= 2\pi\int_0^1\left[1 - 2y + y^2 + y^3 - y^4\right]dy = 2\pi\left[y - y^2 + \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$$

$$= 2\pi\left[\left(1 - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right) - 0\right] = \frac{23\pi}{30}$$

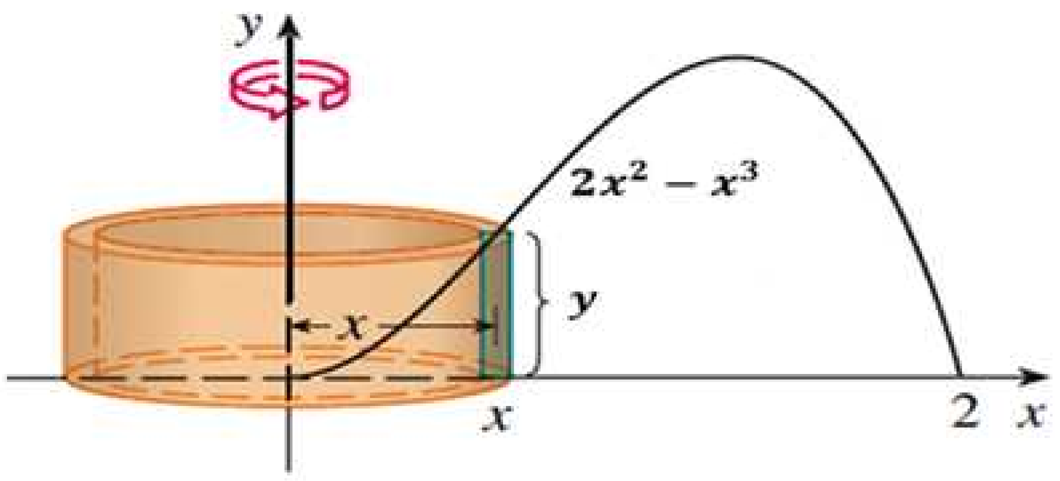
P1:

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve $y = 2x^2 - x^3$, and the line $y = 0$ about the y -axis.

Solution:

Given curves are $y = 2x^2 - x^3$ and the line is $y = 0$.

The region between the curves $y = 2x^2 - x^3$ and the line $y = 0$ is shown in figure and draw a line segment across it parallel to the axis of revolution: $y -$ axis.



Limits of integration:

$2x^2 - x^3 = 0 \Rightarrow x^2(2 - x) = 0 \Rightarrow x = 0$ and $x = 2$

$\Rightarrow a = 0$ and $b = 2$

From the figure, shell height $= 2x^2 - x^3$, shell radius $= x$

$$\text{shell thickness} = dx$$

Therefore, the volume of the solid is

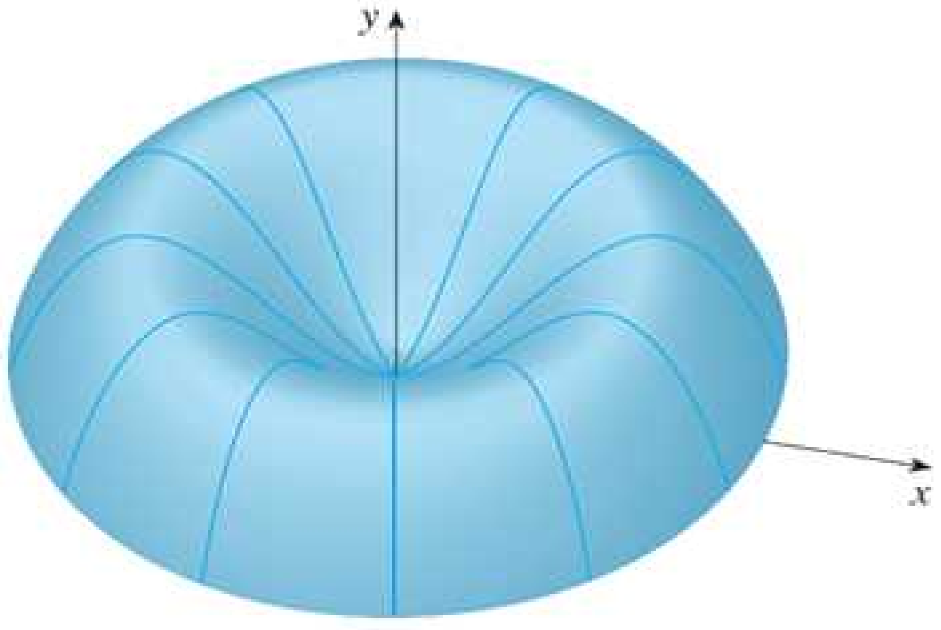
$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\,dx$$

$$= \int_0^2 2\pi x\left[2x^2 - x^3\right]dx$$

$$= 2\pi\int_0^2 \left[2x^3 - x^4\right]dx$$

$$= 2\pi\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = 2\pi\left[\left(8 - \frac{32}{5}\right) - 0\right] = \frac{16\pi}{5}$$

The solid generated by the revolution is shown below.



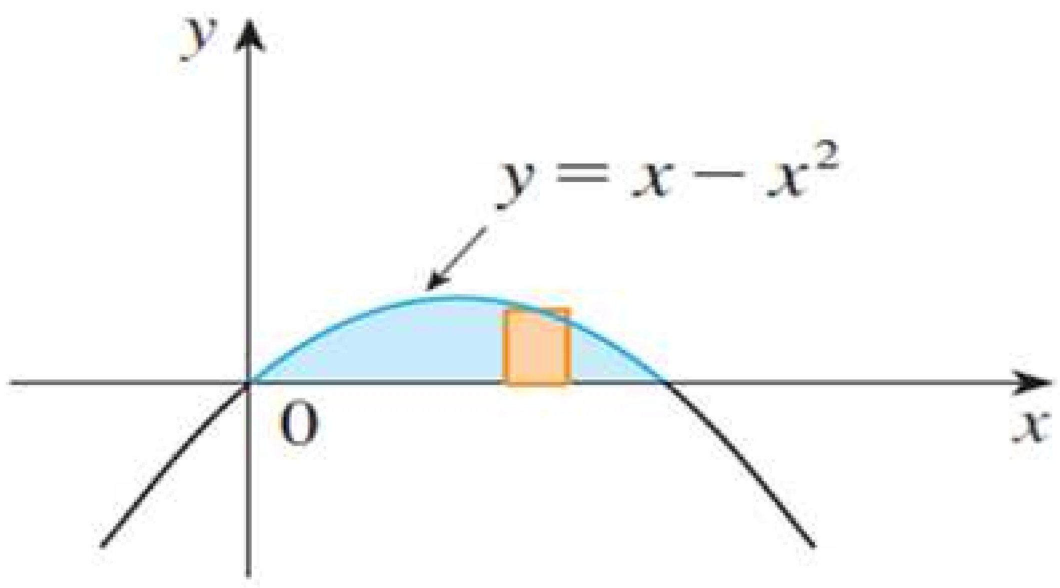
P2:

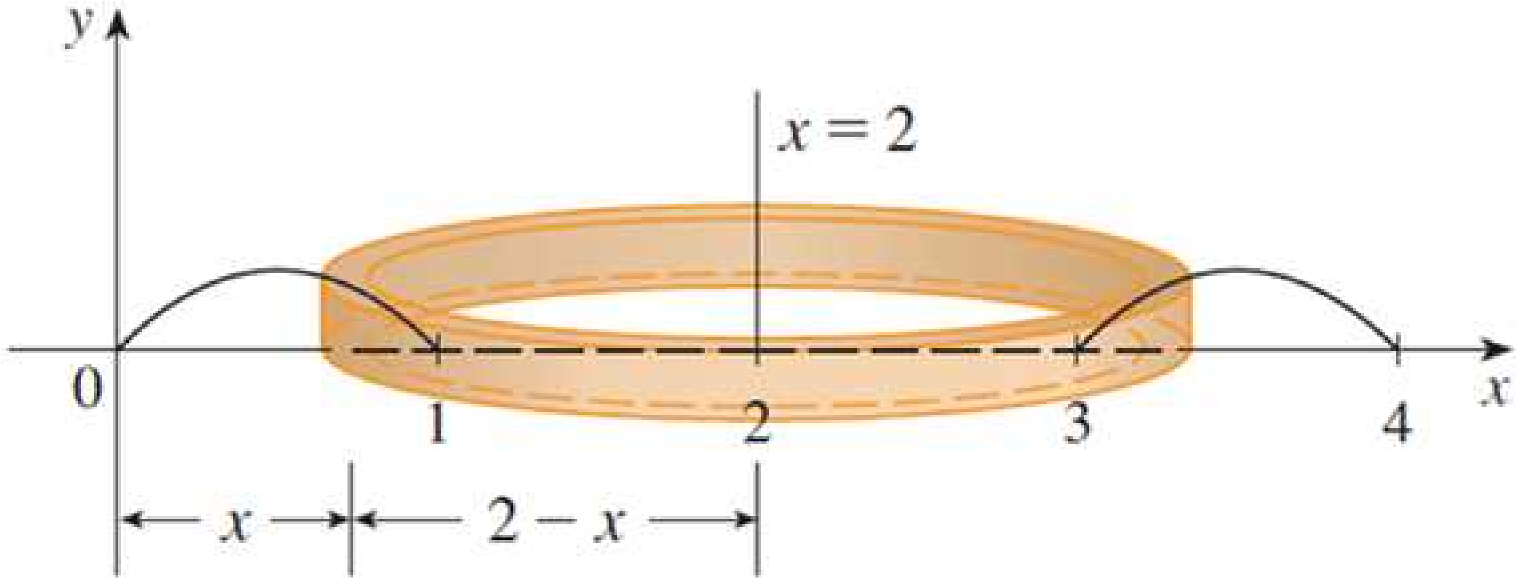
Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve $y = x - x^2$ and the line $y = 0$ about the line $x = 2$.

Solution:

Given curve $y = x - x^2$ and the line is $y = 0$.

The region between the curves $y = x - x^2$ and the line $y = 0$ is shown in figure and draw a line segment across it parallel to the axis of revolution: the line $x = 2$.





Limits of integration:

$x - x^2 = 0 \Rightarrow x(1 - x) = 0 \Rightarrow x = 0$ and $x = 1$

$\Rightarrow a = 0$ and $b = 1$

From the figure, shell radius $= 2 - x$

shell height $= x - x^2$

shell thickness $= dx$

Therefore, the volume of the solid is

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx$$

$$= \int_0^1 2\pi(2-x)\left[x - x^2\right] dx$$

$$= 2\pi \int_0^1 \left[x^3 - 3x^2 + 2x\right] dx$$

$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = 2\pi\left[\left(\frac{1}{4} - 1 + 1\right) - 0\right] = \frac{\pi}{2}$$

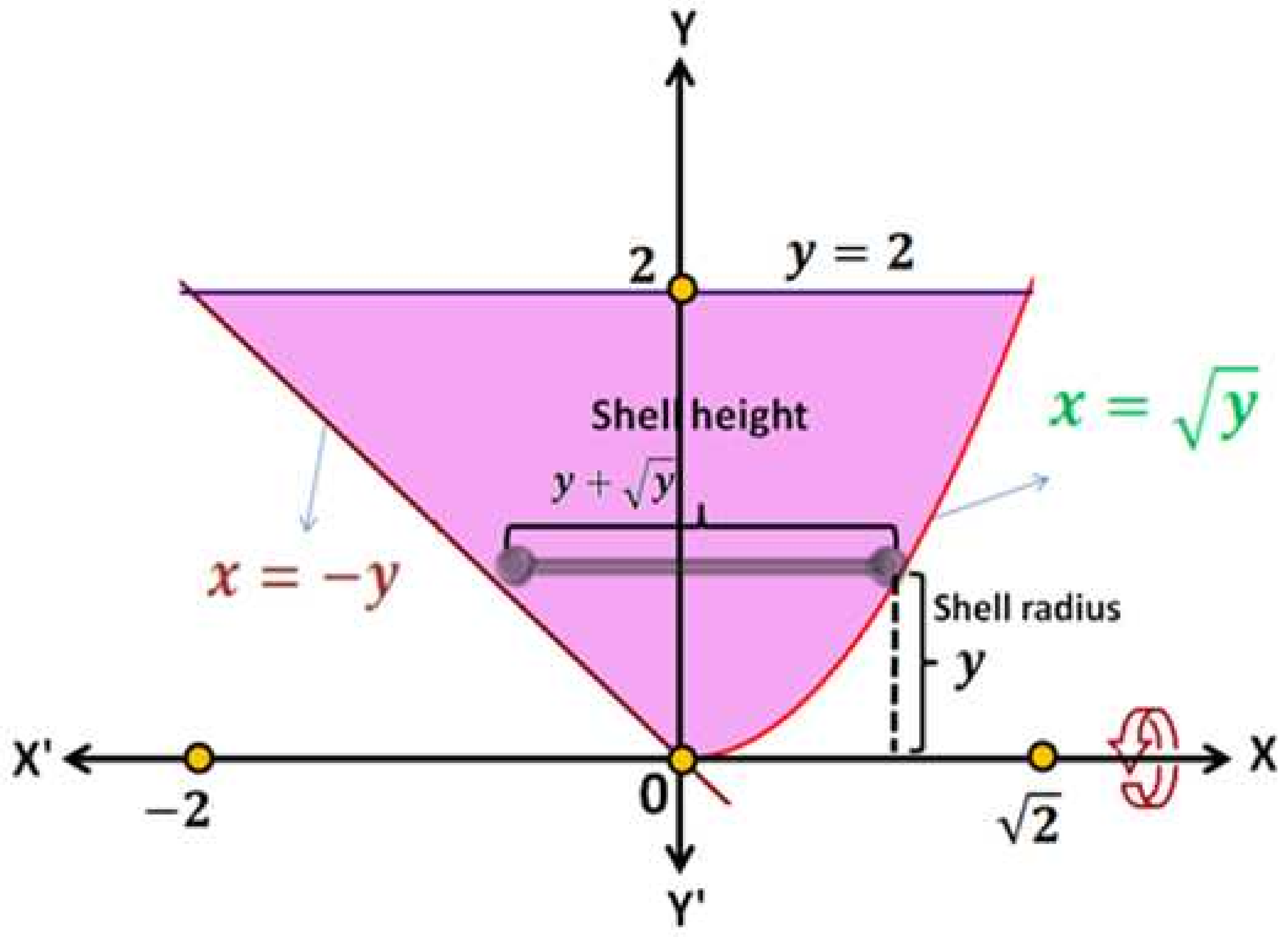
P3:

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve $x = \sqrt{y}$ and the lines $x = -y,\ y = 2$ about the $x-$axis.

Solution:

Given curve is $x = \sqrt{y}$ and the lines are $x = -y,\ y = 2$.

The region between the curve $x = \sqrt{y}$ and the lines $x = -y$, $y = 2$ is shown in figure and draw a line segment across it parallel to the axis of revolution: x $-$axis.



Limits of integration:

$\sqrt{y} = -y \Rightarrow y(y-1) = 0 \Rightarrow y = 0\,, y = 1$ and $y = 2$

$\Rightarrow a = 0$ and $b = 2$

From the figure, shell radius $= y$

$$\text{shell height} = \sqrt{y} - (-y) = \sqrt{y} + y,$$

$$\text{shell thickness} = dy$$

Therefore, the volume of the solid is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\, dy$$

$$= \int_0^2 2\pi\, y\left[\sqrt{y} + y\right] dy$$

$$= 2\pi \int_0^2 \left[y^{3/2} + y^2\right] dy$$

$$= 2\pi\left[\frac{y^{3/2+1}}{3/2+1} + \frac{y^3}{3}\right]_0^2 = 2\pi\left[\frac{2}{5}y^{5/2} + \frac{y^3}{3}\right]_0^2$$

$$= 2\pi\left[\left(\frac{2}{5}(2)^{5/2} + \frac{8}{3}\right) - 0\right] = \frac{16\pi}{15}\left(3\sqrt{2} + 5\right)$$

P4:

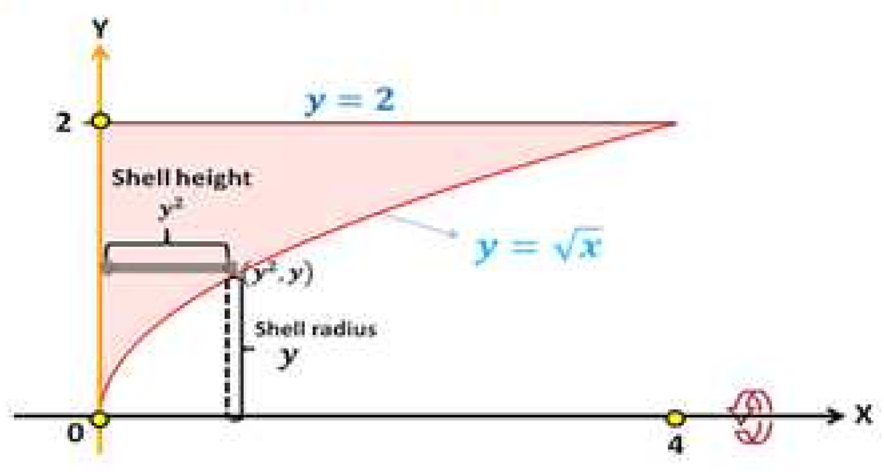
Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve $y=\sqrt{x}$ and the lines $x=0,\ y=2$ about

A. The $x-$ axis B. The $y-$axis

C. The line $x=4$ D. The line $y=2$

Solution: Given curve $y=\sqrt{x}$ and the lines $x=0,\ y=2$.

A. The region between the curve $y=\sqrt{x}$ and the lines $x=0$, $y=2$ is shown in figure and draw a line segment across it parallel to the axis of revolution: $x-$axis.



Limits of integration: $y^2=0 \Rightarrow y=0$ and $y=2$

$\Rightarrow c=0$ and $d=2$

From the figure, shell radius $=y$; shell thickness $=dy$

shell height $=y^2-0=y^2$

Therefore, the volume of the solid is

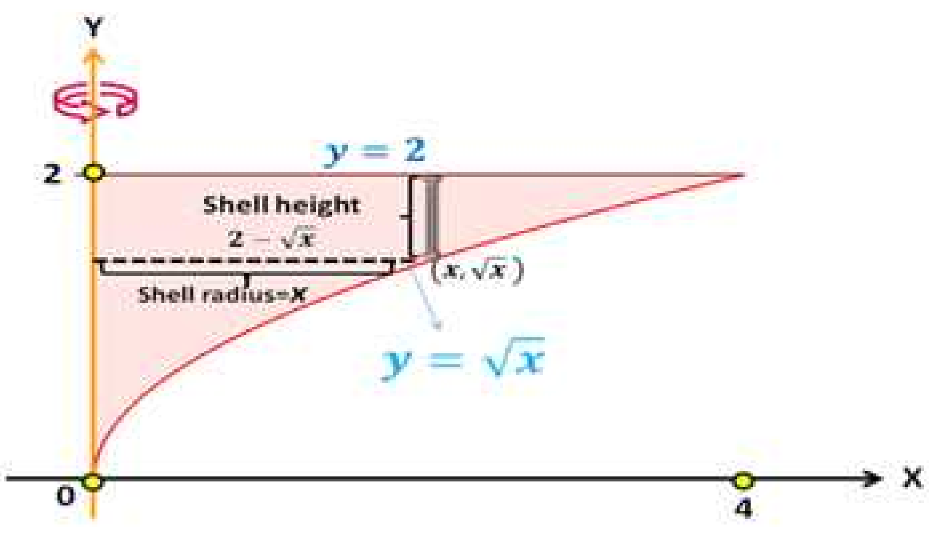
$$V=\int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy$$

$$=\int_0^2 2\pi\, y\left[y^2\right]dy=2\pi\int_0^2\left[y^3\right]dy=2\pi\left[\frac{y^4}{4}\right]_0^2=2\pi\left[(4)-0\right]=8\pi$$

B. The region is as shown in figure. Draw a line segment across it parallel to the axis of revolution: $y-$axis.

Limits of integration: $\sqrt{x}=2 \Rightarrow x=4$ and $x=0$

$\Rightarrow a=0$ and $b=2$



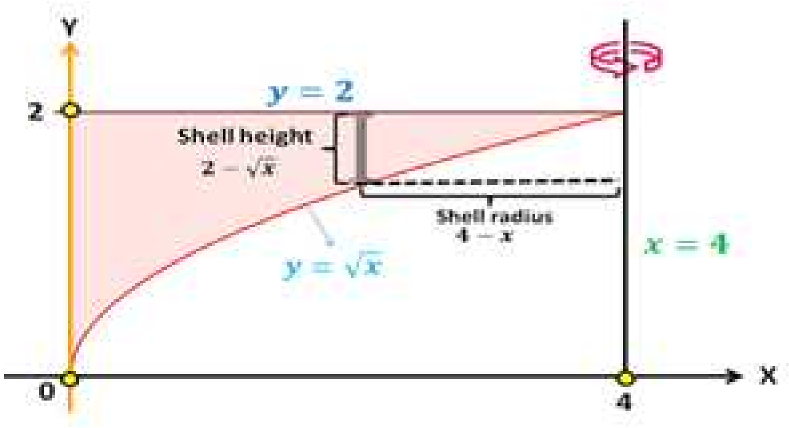
From the figure, shell radius $=x$, shell height $=2-\sqrt{x}$ and shell thickness $=dx$. Therefore, the volume of the solid is

$$V=\int_a^b 2\pi(\text{shell radius})(\text{shell height})\,dx$$

$$=\int_0^4 2\pi x\left[2-\sqrt{x}\right]dx=2\pi\int_0^4\left[2x-x^{3/2}\right]dx$$

$$=2\pi\left[x^2-\frac{x^{3/2+1}}{3/2+1}\right]_0^4=2\pi\left[\left(16-\frac{64}{5}\right)-0\right]=\frac{32\pi}{5}$$

C. The region is as shown in figure and draw a line segment across it parallel to the axis of revolution: $x=4$



Limits of integration: $a=0$ and $b=4$

From the figure, shell radius $=4-x$, shell height $=2-\sqrt{x}$,

shell thickness $=dx$. Therefore, the volume of the solid is

$$V=\int_a^b 2\pi(\text{shell radius})(\text{shell height})\,dx$$

$$=\int_0^4 2\pi(4-x)\left[2-\sqrt{x}\right]dx=2\pi\int_0^4\left[8-4x^{1/2}-2x+x^{3/2}\right]dx$$

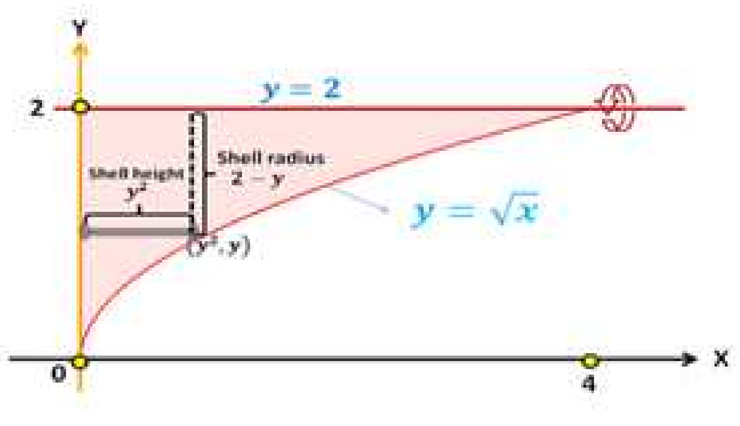
$$=2\pi\left[8x-4\frac{x^{1/2+1}}{1/2+1}-x^2+\frac{x^{3/2+1}}{3/2+1}\right]_0^4$$

$$=2\pi\left[8x-\frac{8x^{3/2}}{3}-x^2+\frac{2x^{5/2}}{5}\right]_0^4$$

$$=2\pi\left[\left(32-\frac{64}{3}-16+\frac{64}{5}\right)-0\right]=\frac{224\pi}{15}$$

D. The region is as shown in figure and draw a line segment across it parallel to the axis of revolution: $y=2$.

Limits of integration: $c=0$ and $d=2$



From the figure, shell radius $=2-y$

shell height $=y^2-0=y^2$, shell thickness $=dy$

Therefore, the volume of the solid is

$$V=\int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy$$

$$=\int_0^2 2\pi(2-y)y^2\,dy=2\pi\int_0^2\left(2y^2-y^3\right)dy$$

$$=2\pi\left[\frac{2y^3}{3}-\frac{y^4}{4}\right]_0^2=2\pi\left[\left(\frac{16}{3}-\frac{16}{4}\right)-0\right]=\frac{8\pi}{3}$$

3.5. Shell Method of Finding Volumes

Exercise:

1. Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines about the y-axis.

   a) $y = x, \quad y = -x/2, \quad x = 2$
   b) $y = x^2, \quad y = 2 - x, \quad x = 0, \quad \text{for } x \geq 0$
   c) $y = 1/x, \; y = 0, \quad x = 1/2, \; x = 2$

2. Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines about the x-axis.

   a. $x = 2y - y^2, \quad x = 0$
   b. $y = |x|, \quad y = 1$
   c. $y = \sqrt{x}, \; y = 0, \quad y = x - 2$

3. Find the volumes of the solids generated by revolving the regions about the given axes.

   A. The triangle with vertices (1, 1), (1, 2), and (2, 2) about (a) the x-axis; (b) the y-axis; (c) the line $x = \frac{10}{3}$; (d) the line $y = 1$

   B. The region in the first quadrant bounded by $x = y - y^3, x = 1$, and $y = 1$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 1$; (d) the line $y = 1$

   C. The region in the first quadrant bounded by $y = x^3$, and $y = 4x$ about (a) the x-axis; (b) the line $y = 8$

   D. The region bounded by $y = 2x - x^2$ and $y = x$ about (a) the y-axis (b) the line $x = 1$

3.6

Lengths of plane curves

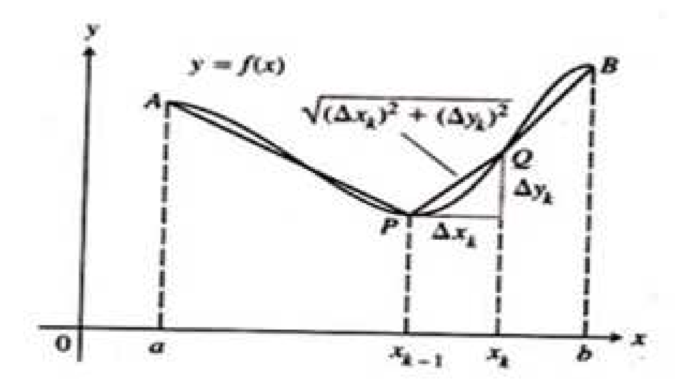
Learning objectives:

- To find the length of a smooth plane curve in Cartesian coordinates.

  AND

- To practice the related problems.

Suppose we want to find the length of the curve $y = f(x)$ from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and connect the corresponding points on the curve with line segments to form a polygonal path that approximates the curve.

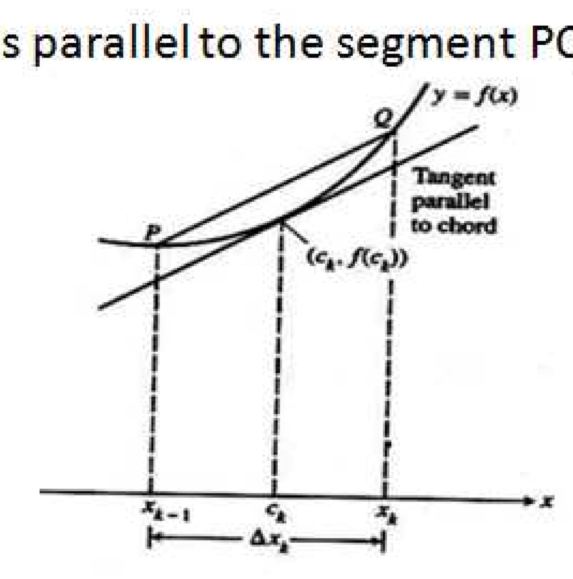


The length of a typical line segment PQ is $\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}$.
The length of the curve is therefore approximated by the sum

$$\sum_{k=1}^{n}\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}.$$

We expect the approximation to improve as the partition of $[a,b]$ becomes finer. We will show that the sums above approach a calculable limit as the norm of the partition goes to zero.

A function with a continuous first derivative is said to be ***smooth*** and its graph is called a ***smooth curve***.
If f is smooth, by the Mean Value Theorem there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ.



At this point,

$$f'(c_k)=\frac{\Delta y_k}{\Delta x_k} \quad , \quad \Delta y_k = f'(c_k)\Delta x_k$$

With this substitution, we have

$$\sum_{k=1}^{n}\sqrt{(\Delta x_k)^2+(\Delta y_k)^2} = \sum_{k=1}^{n}\sqrt{(\Delta x_k)^2+(f'(c_k)\Delta x_k)^2}$$
$$= \sum_{k=1}^{n}\sqrt{1+(f'(c_k))^2}\,\Delta x_k \quad \text{(a Riemann sum)}$$

Because $\sqrt{1+(f'(x))^2}$ is continuous on $[a,b]$, the limit of the sums on the right as the norm of the partition goes to zero is $\int_a^b\sqrt{1+(f'(x))^2}\,dx$. We define the length of the curve to be the value of this integral.
If f is smooth on $[a,b]$, the ***length*** of the curve $y = f(x)$ from a to b is

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

Example 1:

Find the length of the curve $y=\frac{4\sqrt{2}}{3}x^{3/2}-1, \quad 0\le x\le 1.$

Solution:

$$y=\frac{4\sqrt{2}}{3}x^{3/2}-1 \Rightarrow \frac{dy}{dx}=\frac{4\sqrt{2}}{3}\cdot\frac{3}{2}x^{1/2}=2\sqrt{2}x^{1/2}$$
$$\Rightarrow\left(\frac{dy}{dx}\right)^2=8x$$

The length of the curve from x = 0 to x = 1 is

$$L=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_a^b\sqrt{1+8x}\,dx$$
$$=\frac{2}{3}\cdot\frac{1}{8}(1+8x)^{3/2}\Big]_0^1=\frac{13}{6} \quad \text{....... (1)}$$

❖ At a point on a curve where $\frac{dy}{dx}$ fails to exist, $\frac{dx}{dy}$ may exist and we may be able to find the curve's length by expressing x as a function of y say $x = g(y)$ and applying the following analogue of equation (1):

$$L=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_c^d\sqrt{1+(g'(y))^2}\,dy \quad \text{.........(2)}$$

Example 2:

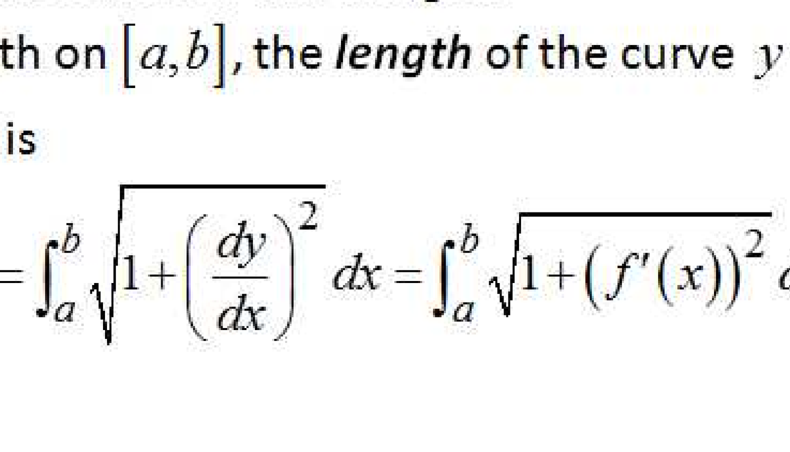
Find the length of the curve $y=(x/2)^{2/3}$ from x = 0 to x = 2.

Solution:

The derivative

$$\frac{dy}{dx}=\frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right)=\frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at x = 0, so we cannot find the curve's length with equation (1).
We therefore rewrite the equation to express x in terms of y: $\quad y=(x/2)^{2/3}\Rightarrow y^{3/2}=\frac{x}{2}\Rightarrow x=2y^{3/2}$
From this we see that the curve whose length we want is also the graph of $x=2y^{3/2}$ from y = 0 to y = 1.



The derivative $\frac{dx}{dy}=2\left(\frac{3}{2}\right)y^{1/2}=3y^{1/2}$ is continuous on [0, 1]. We may therefore use equation (2) to find the curve's length.

$$L=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_0^1\sqrt{1+9y}\,dy=\frac{1}{9}\cdot\frac{2}{3}(1+9y)^{3/2}\Big]_0^1$$
$$=\frac{2}{27}\left(10\sqrt{10}-1\right)=2.27$$

The equations

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad\text{and}\quad L=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \quad \text{..... (3)}$$

are often written with differentials instead of derivatives. This is done formally by thinking of the derivatives as quotients of differentials and bringing the dx and dy inside the radicals to cancel the denominators. In the first integral we have

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\frac{dy^2}{dx^2}}\,dx=\sqrt{dx^2+\frac{dy^2}{dx^2}dx^2}=\sqrt{dx^2+dy^2}$$

In the second integral we have

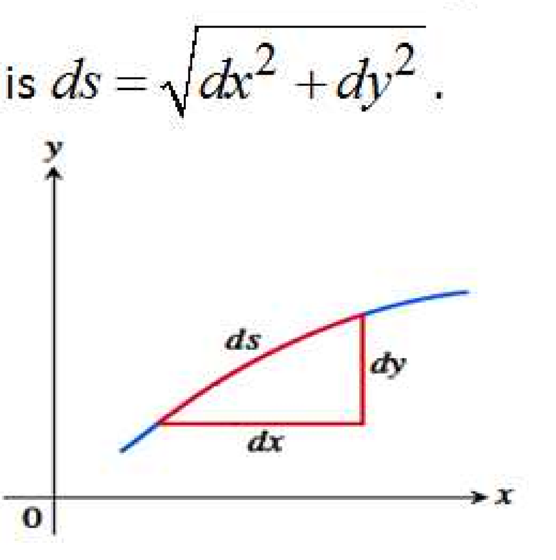
$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\sqrt{1+\frac{dx^2}{dy^2}}\,dy=\sqrt{dy^2+\frac{dx^2}{dy^2}dy^2}=\sqrt{dx^2+dy^2}$$

Thus the integrals in (3) reduce to the same differential formula:

$$L=\int_a^b\sqrt{dx^2+dy^2} \quad \text{........(4)}$$

The differentials dx and dy must be expressed in terms of a common variable, and appropriate limits of integration must be found for performing the integration.

We can also view equation (5) as follows. We think of dx and dy as two sides of a small triangle whose "hypotenuse" is $ds=\sqrt{dx^2+dy^2}$.



The differential ds is then regarded as a differential of arc length that can be integrated between appropriate limits to give the length of the curve.
The arc length differential and the differential formula for the arc length are given by

$$ds=\sqrt{dx^2+dy^2} \qquad L=\int ds$$

IP1:

Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$

Solution: Given curve $y = x^{3/2}$, $0 \le x \le 4$

Differentiating y w.r.t x, we get

$$\frac{dy}{dx} = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2}$$

Now, $1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{3}{2}x^{1/2}\right)^2 = 1+\frac{9}{4}x$

Therefore, the length of the given curve is

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx = \int_0^4 \sqrt{1+\frac{9}{4}x}\ dx$$

Put $1+\frac{9}{4}x = u \Rightarrow dx = \frac{4}{9}du$

Limits:

$x = 0 \Rightarrow u = 1$ and $x = 4 \Rightarrow u = 10$

$$= \int_1^{10} \sqrt{t}\left(\frac{4}{9}\ dt\right) = \frac{4}{9}\int_1^{10} t^{1/2}\ dt = \frac{4}{9}\left[\frac{t^{1/2+1}}{1/2+1}\right]_1^{10}$$

$$= \frac{4}{9}.\frac{2}{3}\left[t^{3/2}\right]_1^{10} = \frac{8}{27}\left[10\sqrt{10}-1\right]$$

IP2:

Find the length of the curve $\boldsymbol{x}=\frac{\boldsymbol{y}^3}{6}+\frac{1}{2\boldsymbol{y}}$ from $\boldsymbol{y}=2$ to $\boldsymbol{y}=3$.

Solution: Given curve is $x=\frac{y^3}{6}+\frac{1}{2y}$, $2 \le y \le 3$

Differentiating x w.r.t y, we get

$$\frac{dx}{dy}=\frac{1}{6}.3y^2-\frac{1}{2}y^{-2}=\frac{1}{2}\left(y^2-y^{-2}\right)$$

Now,

$$1+\left(\frac{dx}{dy}\right)^2=1+\frac{1}{4}\left(y^2-y^{-2}\right)^2=1+\frac{1}{4}\left(y^4+y^{-4}-2\right)$$

$$=\frac{1}{4}\left(y^4+y^{-4}\right)+\frac{1}{2}=\frac{1}{4}\left(y^2+y^{-2}\right)^2$$

Therefore, the length of the given curve is

$$L=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_2^3\sqrt{\frac{1}{4}\left(y^2+y^{-2}\right)^2}\,dy$$

$$=\frac{1}{2}\int_2^3\left(y^2+y^{-2}\right)dy$$

$$=\frac{1}{2}\left[\frac{y^3}{3}-y^{-1}\right]_2^3=\frac{1}{2}\left[\left(\frac{3^3}{3}-3^{-1}\right)-\left(\frac{2^3}{3}-2^{-1}\right)\right]=\frac{13}{4}$$

IP3:

Find the length of the curve $y=\int_0^x \sqrt{\sin 2t}\ dt$, $\mathbf{0 \le x \le \frac{\pi}{4}}$.

Solution: Given curve is $y=\int_0^x \sqrt{\sin 2t}\ dt\ \ 0 \le x \le \frac{\pi}{4}$

By the Fundamental Theorem of calculus part 1, we have

$$\frac{dy}{dx}=\sqrt{\sin 2x}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\sqrt{\sin 2x}\right)^2 = 1+\sin 2x$$

$$=\sin^2 x+\cos^2 x+2\sin x.\cos x$$

$$=(\sin x+\cos x)^2$$

Therefore, the length of the curve is

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx=\int_0^{\pi/4}\sqrt{(\sin x+\cos x)^2}\ dx$$

$$=\int_0^{\pi/4}(\sin x+\cos x)\ dx=\left[-\cos x+\sin x\right]_0^{\pi/4}$$

$$=\left[\left(-\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)\right)-(-\cos 0+\sin 0)\right]$$

$$=\left[\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-(-1+0)\right]=1$$

IP4:

a. Find a curve through the point $(\mathbf{0}, \mathbf{1})$ whose length integral is $L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy$

b. How many such curves are there? Give reasons for your answer.

Solution:

a) Given the length integral is $L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy$ ..... (1)

Comparing (1) with $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$, we get

$\left(\frac{dx}{dy}\right)^2 = \frac{1}{y^4} \Rightarrow \frac{dx}{dy} = \frac{1}{y^2}$ and integrating, we get

$\Longrightarrow x = -\frac{1}{y} + C$, where C arbitrary constant

Since the curve passes through the point $(0, 1)$, we get $C = 1$

Required curve $x = -\frac{1}{y} + 1 \Longrightarrow y = \frac{1}{1-x}$

b) If we take $\frac{dx}{dy} = \frac{1}{y^2}$ then there is only one curve because we know the derivative of the function and the value of the function at one value of x.

P1:

Find the length of the curve $y = \frac{x^3}{3} + x^2 + x + \frac{1}{4x+4}, 0 \le x \le 2.$

Solution: Given curve is $y = \frac{x^3}{3} + x^2 + x + \frac{1}{4x+4}, 0 \le x \le 2$

Differentiating y w.r.t x, we get

$$\frac{dy}{dx} = \frac{3x^2}{3} + 2x + 1 - \frac{1}{4(x+1)^2} = x^2 + 2x + 1 - \frac{1}{4(x+1)^2}$$

$$\frac{dy}{dx} = (x+1)^2 - \frac{1}{4(x+1)^2}$$

Squaring on both sides, we get

$$\left(\frac{dy}{dx}\right)^2 = \left[(x+1)^2 - \frac{1}{4(x+1)^2}\right]^2 = (x+1)^4 + \frac{1}{16(x+1)^4} - \frac{1}{2}$$

Now,

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + (x+1)^4 + \frac{1}{16(x+1)^4} - \frac{1}{2}$$

$$= (x+1)^4 + \frac{1}{16(x+1)^4} + \frac{1}{2} = \left[(x+1)^2 + \frac{1}{4(x+1)^2}\right]^2$$

Therefore, Length of the given curve is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 \sqrt{\left[(x+1)^2 + \frac{1}{4(x+1)^2}\right]^2}\, dx$$

$$= \int_0^2 \left[(x+1)^2 + \frac{1}{4(x+1)^2}\right] dx = \left[\frac{(x+1)^3}{3} - \frac{1}{4(x+1)}\right]_0^2$$

$$= \left[\left(\frac{(2+1)^3}{3} - \frac{1}{4(2+1)}\right) - \left(\frac{(0+1)^3}{3} - \frac{1}{4(0+1)}\right)\right] = \frac{53}{6}$$

P2:

Find the length of the curve $x = \frac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$.

Solution: Given curve is $x = \frac{y^{3/2}}{3} - y^{1/2} \quad , \quad 1 \le y \le 9$

Differentiating x w.r.t y, we get

$$\frac{dx}{dy} = \frac{1}{3}.\frac{3}{2}y^{1/2} - \frac{1}{2}y^{-1/2} = \frac{1}{2}\left(y^{1/2} - y^{-1/2}\right)$$

Now,

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}\left(y^{1/2} - y^{-1/2}\right)^2 = 1 + \frac{1}{4}\left(y + y^{-1} - 2\right)$$

$$= \frac{1}{4}\left(y + y^{-1}\right) + \frac{1}{2} = \frac{1}{4}\left(y^{1/2} + y^{-1/2}\right)^2$$

Therefore, the length of the given curve is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^9 \sqrt{\frac{1}{4}\left(y^{1/2} + y^{-1/2}\right)^2}\, dy$$

$$= \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right) dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \left[\frac{1}{3}y^{3/2} + y^{1/2}\right]_1^9$$

$$= \left[\left(\frac{1}{3}(9)^{3/2} + (9)^{1/2}\right) - \left(\frac{1}{3}(1)^{3/2} + (1)^{1/2}\right)\right] = \frac{32}{3}$$

P3:

Find the length of the curve $y=\int_{-2}^{x}\sqrt{3t^4-1}\,dt$ **,** $\mathbf{-2 \le x \le -1.}$

Solution: Given curve is $y=\int_{-2}^{x}\sqrt{3t^4-1}\,dt$, $-2\le x\le -1$

By the Fundamental theorem of calculus part 1, we have

$$\frac{dy}{dx}=\sqrt{3x^4-1}$$

Now, $1+\left(\frac{dy}{dx}\right)^2=1+\left(\sqrt{3x^4-1}\right)^2=1+3x^4-1=3x^4$

Therefore, the length of the curve is

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_{-2}^{-1}\sqrt{3x^4}\,dx$$

$$=\sqrt{3}\int_{-2}^{-1}x^2\,dx=\sqrt{3}\left[\frac{x^3}{3}\right]_{-2}^{-1}$$

$$=\sqrt{3}\left[\frac{(-1)^3}{3}-\frac{(-2)^3}{3}\right]=\frac{7\sqrt{3}}{3}$$

P4:

Find the arc length of the semi-cubical parabola $\boldsymbol{y^2 = x^3}$ between the points $\boldsymbol{(1,1)}$ and $\boldsymbol{(4,8)}$

Solution: We have to compute the arc length of the given semi-cubical parabola $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$.

$$y^2 = x^3 \Rightarrow y = x^{3/2} \qquad \dots \ (1)$$

Differentiating y w.r.t x, we get

$$2y\,dy = 3x^2 dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y} = \frac{3x^2}{2x^{3/2}} \quad (\because \ from\ (1))$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}\sqrt{x}$$

We have

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$ds = \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\ dx = \sqrt{1 + \frac{9}{4}x}\ \ dx$$

Limits of integration: From the given points $(1,1)$ and $(4,8)$, we have $a = 1$ and $b = 4$

Therefore, the length of the curve is

$$L = \int_a^b ds = \int_1^4 \sqrt{1 + \frac{9x}{4}}\ dx$$

$$\text{Put } 1 + \frac{9x}{4} = u \Rightarrow dx = \frac{4}{9}du$$

$$\text{Limits}: \ x = 1 \Rightarrow u = \frac{13}{4} \ \ and \ \ x = 4 \Rightarrow u = 10$$

$$= \int_{13/4}^{10} \sqrt{u}\left(\frac{4}{9}du\right) = \frac{4}{9}\int_{13/4}^{10} u^{1/2}\ du$$

$$= \frac{4}{9}\left[\frac{u^{1/2+1}}{1/2+1}\right]_{13/4}^{10} = \frac{4}{9}.\frac{2}{3}\left[u^{3/2}\right]_{13/4}^{10}$$

$$= \frac{8}{27}\left[(10)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] = \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right]$$

3.6. Lengths of plane curves

Exercises:

1. **Find the lengths of the curves in problems 1-5.**

a. $y=(1/3)\left(x^2+2\right)^{3/2}$ from $x=0$ to $x=3$

b. $x=\left(y^3/3\right)+1/(4y)$ from $y=1$ to $y=3$

c. $x=\left(y^4/4\right)+1/\left(8y^2\right)$ from $y=1$ to $y=2$

d. $y=(3/4)x^{4/3}-(3/8)x^{2/3}+5$ from $1\le x\le 8$

e. $x=\int_0^y \sqrt{\sec^4 t-1}\,dt$, $-\pi/4\le y\le \pi/4$

f. $y=\int_0^x \sqrt{\cos 2t}\,dt$, $x=0$ to $x=\pi/4$

2.

a) Find a curve through the point $(1,1)$ whose length integral is

$$L=\int_1^4 \sqrt{1+\frac{1}{4x}}\,dx$$

b) How many such curves are there? Give reasons for your answer.

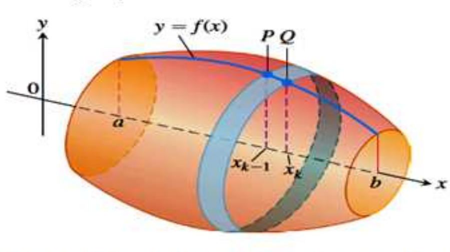
3.7

Areas of surfaces of Revolution

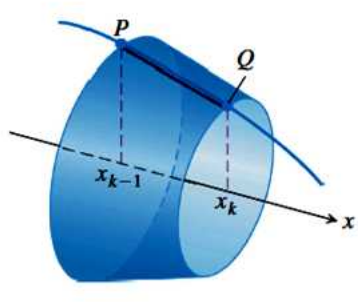
Learning objectives:

- To find the area of the surface swept out by revolving the graph of a non-negative function between given limits about axes.
  AND
- To practice the related problems.

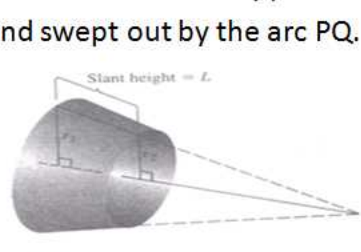
Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative function $y=f(x)$, $a \le x \le b$, about the x-axis. We partition $[a,b]$ in the usual way and use the points in the partition to partition the graph into short arcs.



As the arc PQ revolves about the x-axis, the line segment joining P and Q sweeps out part of a cone whose axis lies along the x-axis.



A piece of a cone like this is called a *frustum* of the cone. The surface area of the frustum approximates the surface area of the band swept out by the arc PQ.

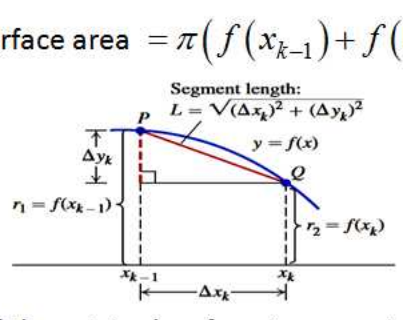


The surface area of the frustum of a cone is 2π times the average of the base radii times the slant height.

Frustum surface area = $2\pi \cdot \dfrac{r_1+r_2}{2} \cdot L = \pi(r_1+r_2)L$

For the frustum swept out by the segment PQ, this works out to be

Frustum surface area $=\pi\left(f(x_{k-1})+f(x_k)\right)\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}$

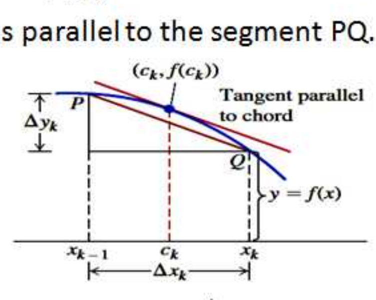


The area of the original surface is approximated by the frustum area sum

$$\sum_{k=1}^{n}\pi\left(f(x_{k-1})+f(x_k)\right)\sqrt{(\Delta x_k)^2+(\Delta y_k)^2} \quad \ldots\ldots (1)$$

We expect the approximation to improve as the partition of $[a,b]$ becomes finer. We will show that the sum in (1) approach a calculable limit as the norm of the partition goes to zero.

If f is smooth, by the Mean Value Theorem there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ.



At this point, $f'(c_k)=\dfrac{\Delta y_k}{\Delta x_k} \Rightarrow \Delta y_k = f'(c_k)\Delta x_k$

With this substitution for Δy_k, the sums in (1) take the form

$$\sum_{k=1}^{n}\pi\left(f(x_{k-1})+f(x_k)\right)\sqrt{(\Delta x_k)^2+\left(f'(c_k)\Delta x_k\right)^2}$$

$$=\sum_{k=1}^{n}\pi\left(f(x_{k-1})+f(x_k)\right)\sqrt{1+\left(f'(c_k)\right)^2}\,\Delta x_k \quad \ldots\ldots (2)$$

The sums in (2) are not the Riemann sums of any function because the points x_{k-1}, x_k, and c_k are not the same and there is no way to make them the same. However, a theorem called Bliss's theorem, from advanced calculus, assures us that as the norm of the partition of $[a,b]$ goes to zero, the sum in equation (2) converges to

$$\int_a^b 2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of f from a to b.

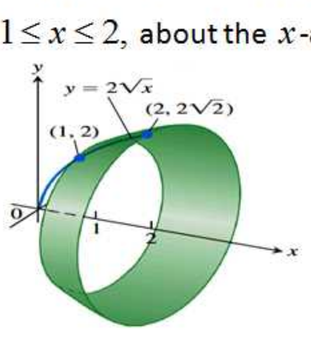
➢ If the function $f(x)\ge 0$ is smooth on $[a,b]$, the area of the surface generated by revolving the curve $y=f(x)$ about the x-axis is

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S=\int_a^b 2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx \ldots\ldots (3)$$

Example 1:

Find the area of the surface generated by revolving the curve $y=2\sqrt{x}$, $1\le x\le 2$, about the x-axis.



Solution:

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}=\sqrt{\frac{x+1}{x}}=\frac{\sqrt{x+1}}{\sqrt{x}}$$

$$S=\int_1^2 2\pi\cdot 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\,dx=4\pi\int_1^2\sqrt{x+1}\,dx$$

$$=4\pi\cdot\frac{2}{3}(x+1)^{3/2}\Big]_1^2=\frac{8\pi}{3}\left(3\sqrt{3}-2\sqrt{2}\right)$$

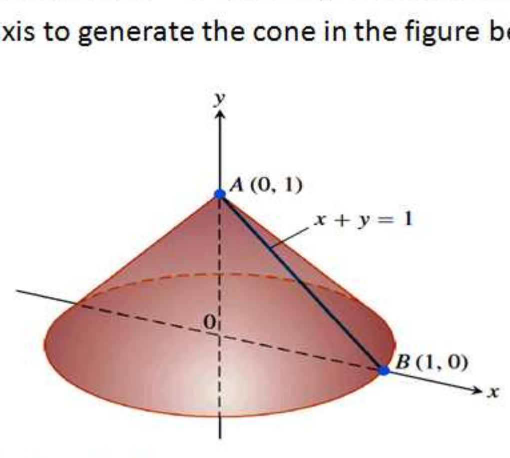
❖ For revolution about the y-axis, we interchange x and y in equation (3).

If $x=g(y)\ge 0$ is smooth on $[c,d]$, the area of the surface generated by revolving the curve $x=g(y)$ about the y-axis is

$$S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_c^d 2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\,dy \ldots. (4)$$

Example 2:

The line segment $x=1-y$, $0\le y\le 1$, is revolved about the y-axis to generate the cone in the figure below.



Find its lateral surface area.

Solution:

From geometry, we know

Lateral surface area = $\dfrac{\text{base circumference}}{2}\times\text{slant height}=\pi\sqrt{2}$

We can use equation (4) to obtain the same result.

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+(-1)^2}=\sqrt{2}$$

$$S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_0^1 2\pi(1-y)\sqrt{2}\,dy$$

$$=2\pi\sqrt{2}\left[y-\frac{y^2}{2}\right]_0^1=2\pi\sqrt{2}\left(1-\frac{1}{2}\right)=\pi\sqrt{2}$$

❖ The equations

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad\text{and}\quad S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

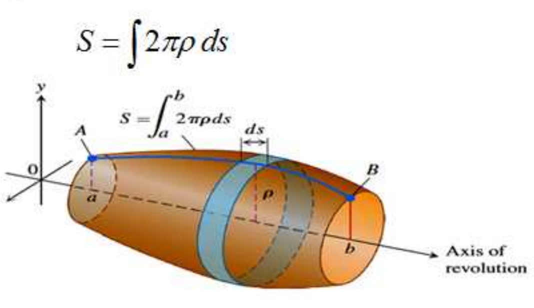
are often written in terms of the arc length differential $ds=\sqrt{dx^2+dy^2}$ as

$$S=\int_a^b 2\pi y\,ds \qquad\text{and}\qquad S=\int_c^d 2\pi x\,ds$$

In the first of these, y is the distance from the x-axis to an element of arc length ds. In the second, x is the distance from the y-axis to an element of arc length ds. Both integrals have the form

$$S=\int 2\pi(\text{radius})(\text{band width})=\int 2\pi\rho\,ds$$ where ρ is the radius from the axis of revolution to an element of arc length ds.

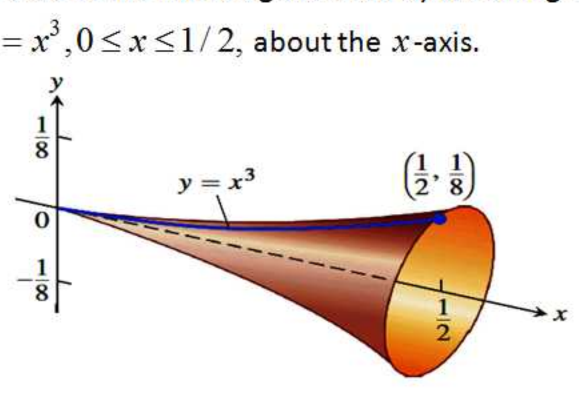
$$S=\int 2\pi\rho\,ds$$



In any particular problem, we will then express the radius function ρ and the arc length differential ds in terms of a common variable and supply limits of integration for that variable.

Example 3:

Find the area of the surface generated by revolving the curve $y=x^3$, $0\le x\le 1/2$, about the x-axis.



Solution:

$$S=\int 2\pi\rho\,ds$$

$$=\int 2\pi y\,ds \quad , \qquad \text{For revolution about the } x\text{-axis}$$

$$=\int 2\pi y\sqrt{dx^2+dy^2}$$

We then decide whether to express dy in terms of dx or dx in terms of dy.

$y=x^3, \quad dy=3x^2\,dx,$

$$\sqrt{dx^2+dy^2}=\sqrt{dx^2+\left(3x^2\,dx\right)^2}=\sqrt{1+9x^4}\,dx$$

With these substitutions, x becomes the variable of integration and

$$S=\int_{x=0}^{x=1/2}2\pi y\sqrt{dx^2+dy^2}=\int_0^{1/2}2\pi x^3\sqrt{1+9x^4}\,dx$$

$$=\frac{\pi}{27}\left[\left(1+\frac{9}{16}\right)^{3/2}-1\right]=\frac{61\pi}{1728}$$

P1:

The curve $y = \sqrt{4 - x^2}$, $-1 \leq x \leq 1$ is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x –axis.

P1:

The curve $y = \sqrt{4 - x^2}, -1 \le x \le 1$ is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x –axis.

Solution: Given curve $y = \sqrt{4 - x^2}, -1 \le x \le 1$

Differentiating y w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(4 - x^2\right)^{-1/2}(-2x) = -\frac{x}{\sqrt{4 - x^2}} \quad \text{and}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4}{4 - x^2}$$

Therefore, the surface area of the arc about the x –axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-1}^{1} 2\pi y \sqrt{\frac{4}{4 - x^2}}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} . \frac{2}{\sqrt{4 - x^2}}\, dx$$

$$= 2\pi \int_{-1}^{1} 2\, dx = 2\pi \left[2x\right]_{-1}^{1} = 2\pi\left[2(1) - 2(-1)\right] = 8\pi$$

P2:

Find the area of the surface generated by revolving the curve

$\boldsymbol{x = \sqrt{2y - 1}}$, $\frac{5}{8} \leq \boldsymbol{y} \leq \mathbf{1}$ about $\boldsymbol{y}$ −axis.

P2:

Find the area of the surface generated by revolving the curve $\boldsymbol{x = \sqrt{2y-1}}$, $\frac{5}{8} \leq \boldsymbol{y} \leq \mathbf{1}$ about $\boldsymbol{y}$ −axis.

Solution: Given curve $x = \sqrt{2y-1}$, $\frac{5}{8} \leq y \leq 1$

Differentiating x w.r.t y, we get

$$\frac{dx}{dy} = \frac{1}{2\sqrt{2y-1}}(2) = \frac{1}{\sqrt{2y-1}} \text{ and}$$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\left(\frac{1}{\sqrt{2y-1}}\right)^2 = 1+\frac{1}{2y-1} = \frac{2y}{2y-1}$$

Therefore, the area of the surface generated by revolving the curve $x = \sqrt{2y-1}$, $\frac{5}{8} \leq y \leq 1$ about y −axis is

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_{5/8}^1 2\pi\sqrt{2y-1}\sqrt{\frac{2y}{2y-1}}\, dy$$

$$= 2\sqrt{2}\pi\int_{5/8}^1 y^{1/2} dy = 2\sqrt{2}\pi\left[\frac{y^{1/2+1}}{1/2+1}\right]_{5/8}^1$$

$$= 2\sqrt{2}\pi.\frac{2}{3}\left[y^{3/2}\right]_{5/8}^1 = \frac{4\sqrt{2}\pi}{3}\left[1-\frac{5\sqrt{5}}{8\sqrt{8}}\right]$$

$$= \frac{\pi}{12}\left[16\sqrt{2}-5\sqrt{5}\right]$$

P3:

Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \le x \le 4$ about the y −axis. Check the answer with its geometry formula.

P3:

Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \le x \le 4$ about the y −axis. Check the answer with its geometry formula.

Solution: Given $y = \frac{x}{2}$, $0 \le x \le 4$

Since the lateral surface of the cone is generated by revolving about y −axis,

$$x = 2y, \quad 0 \le y \le 2$$

Differentiating x w.r.t , we get

$$\frac{dx}{dy} = 2 \qquad and \quad 1 + \left(\frac{dx}{dy}\right)^2 = 1 + (2)^2 = 5$$

Therefore, the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \le x \le 4$ about the y −axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^2 2\pi(2y)\sqrt{5}\, dy$$

$$= 4\pi\sqrt{5}\int_0^2 y\, dy = 4\pi\sqrt{5}\left[\frac{y^2}{2}\right]_0^2$$

$$= 2\pi\sqrt{5}[4 - 0] = 8\pi\sqrt{5}$$

Verification:

From the geometry, we know that

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}$$

$$\text{Base circumference} = 2\pi \times \text{radius of the base circle}$$

$$= 2\pi(4) = 8\pi$$

$$\text{Slant height} = \sqrt{(\text{radius})^2 + (\text{height})^2}$$

$$= \sqrt{(4)^2 + (2)^2} = 2\sqrt{5}$$

Therefore, the lateral surface area is

$$= \frac{1}{2} \times 8\pi \times 2\sqrt{5} = 8\pi\sqrt{5}$$

Hence verified

P4:

Find the area of the surface generated by revolving the curve

$x = \frac{y^4}{4} + \frac{1}{8y^2}$, $1 \le y \le 2$ about the x $-$axis.

P4:

Find the area of the surface generated by revolving the curve

$x=\dfrac{y^4}{4}+\dfrac{1}{8y^2}$, $1\le y\le 2$ about the x −axis.

Solution: Given curve is $x=\dfrac{y^4}{4}+\dfrac{1}{8y^2}$, $1\le y\le 2$

Differentiating x w.r.t y , we get

$$\frac{dx}{dy}=y^3-\frac{1}{4y^3}$$

$$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$=\sqrt{1+\left(y^3-\frac{1}{4y^3}\right)^2}\ dy=\sqrt{1+\left(y^6+\frac{1}{16y^6}-\frac{1}{2}\right)}\ dy$$

$$=\sqrt{\left(y^6+\frac{1}{16y^6}+\frac{1}{2}\right)}\ dy=\sqrt{\left(y^3+\frac{1}{4y^3}\right)^2}\ dy=\left(y^3+\frac{1}{4y^3}\right)dy$$

Therefore, the area of the surface generated by revolving the

curve $x=\dfrac{y^4}{4}+\dfrac{1}{8y^2}$, $1\le y\le 2$ about the x −axis is

$$S=\int_c^d 2\pi y\ ds=\int_1^2 2\pi y\left(y^3+\frac{1}{4y^3}\right)dy$$

$$=2\pi\int_1^2\left(y^4+\frac{1}{4y^2}\right)dy=2\pi\left[\frac{y^5}{5}-\frac{1}{4y}\right]_1^2$$

$$=2\pi\left[\left(\frac{2^5}{5}-\frac{1}{4(2)}\right)-\left(\frac{1}{5}-\frac{1}{4}\right)\right]=\frac{253\pi}{20}$$

IP1:

Find the area of the surface generated by revolving the curve $y = \sqrt{x+1}\,, 1 \le x \le 5$ about the x –axis.

Solution: Given curve $y = \sqrt{x+1}\,,\quad 1 \le x \le 5$

Differentiating y w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} \quad \text{and} \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1}{4(x+1)} = \frac{4x+4+1}{4(x+1)} = \frac{4x+5}{4(x+1)}$$

Therefore, the area of the surface generated by revolving the curve $y = \sqrt{x+1}\,, 1 \le x \le 5$ about the x –axis is

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx = \int_1^5 2\pi y\sqrt{\frac{4x+5}{4(x+1)}}\ dx$$

$$= 2\pi\int_1^5 \sqrt{x+1}\,.\frac{\sqrt{4x+5}}{2\sqrt{x+1}}\ dx = \pi\int_1^5 \sqrt{4x+5}\ dx$$

Put $4x+5 = u \Rightarrow dx = \frac{1}{4}du$

Limits: $x = 1 \Rightarrow u = 9$ and $x = 5 \Rightarrow u = 25$

$$= \pi\int_9^{25} \sqrt{u}\left(\frac{du}{4}\right) = \frac{\pi}{4}\int_9^{25} u^{1/2}\ du = \frac{\pi}{4}\left[\frac{u^{1/2+1}}{1/2+1}\right]_9^{25}$$

$$= \frac{\pi}{4}.\frac{2}{3}\left[u^{3/2}\right]_9^{25} = \frac{\pi}{6}[125-27] = \frac{49\pi}{3}$$

IP2:

The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y −axis. Find the area of the resulting surface.

Solution: Given curve $y = x^2$

$$\Rightarrow x = \sqrt{y} \ , 1 \le y \le 4$$

Differentiating x w.r.t y, we get

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}} \text{ and } 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{1}{2\sqrt{y}}\right)^2 = 1 + \frac{1}{4y}$$

Therefore, the area of the surface generated by revolving the curve $x = \sqrt{y}$, $1 \le y \le 4$ about y −axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_1^4 2\pi \sqrt{y} \sqrt{1 + \frac{1}{4y}} \, dy = \pi \int_1^4 \sqrt{4y+1} \, dy$$

$$\text{Put } 4y + 1 = u \Rightarrow dy = \frac{du}{4}$$

$$\text{Limits: } y = 1 \Rightarrow u = 5 \text{ and } y = 5 \Rightarrow u = 17$$

$$S = \pi \int_5^{17} \sqrt{u} \left(\frac{du}{4}\right) = \frac{\pi}{4} \int_5^{17} u^{1/2} \, du$$

$$= \frac{\pi}{4} \left[\frac{u^{1/2+1}}{1/2+1}\right]_5^{17} = \frac{\pi}{6} \left[u^{3/2}\right]_5^{17} = \frac{\pi}{6} \left[17\sqrt{17} - 5\sqrt{5}\right]$$

IP3:

Find the surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}$, $1 \le x \le 3$ about the y −axis. Check the answer with its geometry formula.

Solution: Given $y = \frac{x}{2} + \frac{1}{2}$, $1 \le x \le 3$

Since the lateral surface of the cone is generated by revolving about y −axis,

$$x = 2y - 1, \quad 1 \le y \le 2$$

Differentiating x w.r.t , we get

$$\frac{dx}{dy} = 2 \qquad and \quad 1 + \left(\frac{dx}{dy}\right)^2 = 1 + (2)^2 = 5$$

Therefore, the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}$, $1 \le x \le 3$ about the y −axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 2\pi(2y-1)\sqrt{5}\; dy$$

$$= 2\pi\sqrt{5}\int_1^2 (2y-1)\, dy = 2\pi\sqrt{5}\left[y^2 - y\right]_1^2$$

$$= 2\pi\sqrt{5}\left[(4-2)-(1-1)\right] = 4\pi\sqrt{5}$$

Verification:

From the geometry, we know that

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times Slant\ height$$

By the hypothesis, $r_1 = 1$, $r_2 = 3$

$$\text{Slant height} = \sqrt{(radius)^2 + (height)^2}$$

$$= \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5}$$

Therefore, the frustum surface area is

$$= \pi(1+3) \times \sqrt{5} = 4\pi\sqrt{5}$$

Hence verified

IP4:

Find the area of the surface generated by revolving the curve $y = \frac{1}{3}\left(x^2+2\right)^{3/2}$, $0 \le x \le \sqrt{2}$ about the $\boldsymbol{y}$ −axis.

Solution: Given curve is $y = \frac{1}{3}\left(x^2+2\right)^{3/2}$, $0 \le x \le \sqrt{2}$

Differentiating y w.r.t x , we get

$$\frac{dy}{dx} = \frac{1}{3}.\frac{3}{2}\left(x^2+2\right)^{1/2}(2x) = x\sqrt{x^2+2}$$

$$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \sqrt{1+\left(x\sqrt{x^2+2}\right)^2}\,dx = \sqrt{1+x^2\left(x^2+2\right)}\,dx$$

$$= \sqrt{\left(x^4+2x^2+1\right)}\,dx = \sqrt{\left(x^2+1\right)^2}\,dx = \left(x^2+1\right)dx$$

Therefore, the area of the surface generated by revolving the curve $y = \frac{1}{3}\left(x^2+2\right)^{3/2}$, $0 \le x \le \sqrt{2}$ about the y −axis is

$$S = \int_a^b 2\pi x\,ds = \int_0^{\sqrt{2}} 2\pi x\left(x^2+1\right)dx = 2\pi\int_0^{\sqrt{2}}\left(x^3+x\right)dx$$

$$= 2\pi\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^{\sqrt{2}} = 2\pi\left[\left(\frac{\left(\sqrt{2}\right)^4}{4} + \frac{\left(\sqrt{2}\right)^2}{2}\right) - 0\right] = 4\pi$$

3.7. Areas of surfaces of Revolution

Exercise:

1. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the x-axis.
2. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the x-axis.
3. Find the areas of the surfaces generated by revolving the curves in problems 3- about the indicated axes.
   a) $y = x^3/9, \quad 0 \le x \le 2; \quad x\text{-axis}$
   b) $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5;$ x-axis
   c) $y = \sqrt{x} \quad , 3/4 \le x \le 15/4; \quad x$ –axis
   d) $x = \frac{1}{3}y^{3/2} - y^{1/2} \quad , 1 \le y \le 3 \quad ; \; y$ –axis
   e) $x = y^3/3, \quad 0 \le y \le 1; \quad y\text{-axis}$
   f) $x = 2\sqrt{4 - y}, \; 0 \le y \le 15/4; \quad y\text{-axis}$