4.2

Derivatives of Inverse Functions

Learning objectives:

- To study the derivative rule for inverses.

AND

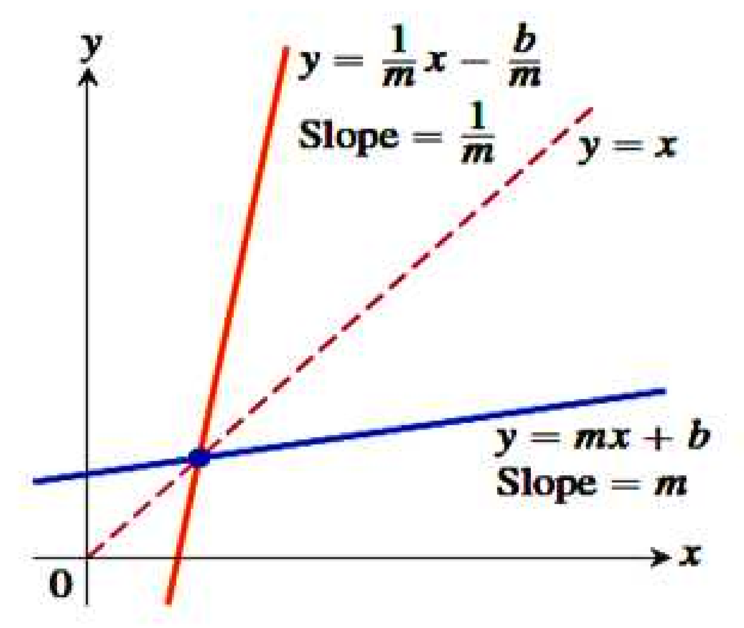
- To practice the related problems.

If we calculate the derivatives of $f(x)=\frac{1}{2}x+1$ and its inverse $f^{-1}(x)=2x-2$, we see that

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{1}{2}x+1\right)=\frac{1}{2}$$
$$\frac{d}{dx}f^{-1}(x)=\frac{d}{dx}(2x-2)=2$$

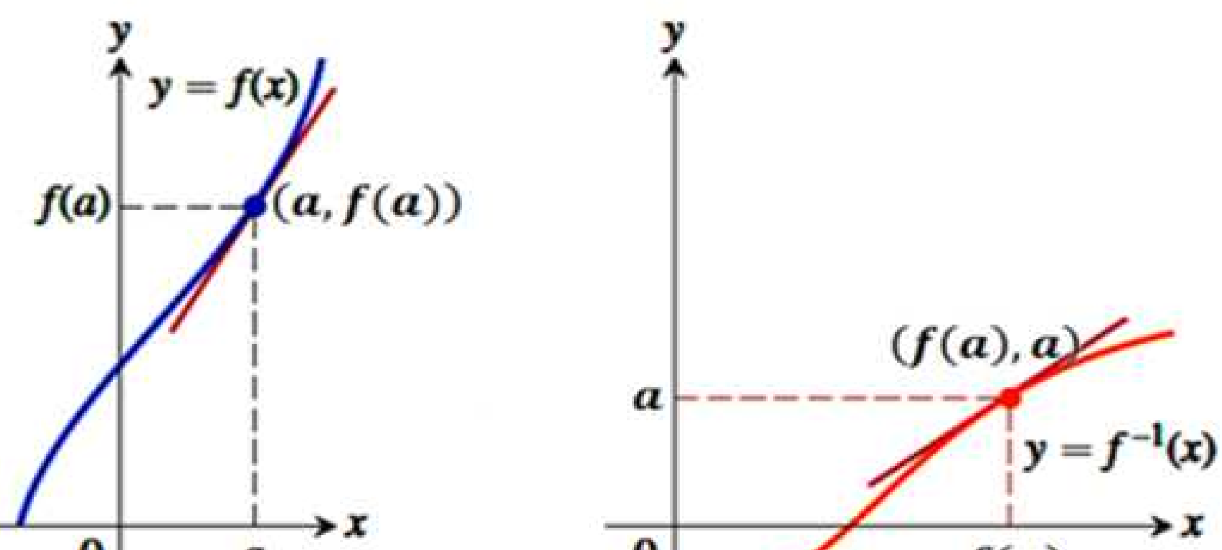
The derivatives are reciprocals of one another. The graph of f is the line $y=\frac{1}{2}x+1$, and the graph of f^{-1} is the line $y=2x-2$. Their slopes are reciprocals of one another.

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y=x$ always inverts the line's slope. If the original line has slope $m\neq 0$, the reflected line has slope $\frac{1}{m}$.



The reciprocal relation between the slopes of graphs of inverses holds for other functions as well.

If the slope of $y=f(x)$ at the point $(a,f(a))$ is $f'(a)\neq 0$, then the slope of $y=f^{-1}(x)$ at the corresponding point $(f(a),a)$ is $1/f'(a)$.



The slopes are reciprocal: $\left.\frac{df^{-1}}{dx}\right|_{f(a)}=\frac{1}{\left.\frac{df}{dx}\right|_{a}}$

Thus, the derivative of f^{-1} at $f(a)$ equals the reciprocal of the derivative of f at a.

If $y=f(x)$ has a horizontal tangent line at $(a,\ f(a))$ then the inverse function f^{-1} has a vertical tangent line at $(f(a),a)$, and this infinite slope implies that f^{-1} is not differentiable at $f(a)$. The following theorem gives the conditions under which f^{-1} is differentiable in its domain which is the same as the range of f.

Theorem 1:

The Derivative Rule for Inverses

If f is differentiable at every point of an interval I and $\frac{df}{dx}$ is never zero on I, then f^{-1} is differentiable at every point of the interval $f(I)$. The value of $\frac{df^{-1}}{dx}$ at any particular point $f(a)$ is the reciprocal of the value of $\frac{df}{dx}$ at a.

$$\left(\frac{df^{-1}}{dx}\right)_{x=f(a)}=\frac{1}{\left(\frac{df}{dx}\right)_{x=a}} \qquad \text{......... (1)}$$

In short notation, $\left(f^{-1}\right)'=\frac{1}{f'}$ .... (2)

Example 1:

For $f(x)=x^2, x\geq 0$, and its inverse $f^{-1}(x)=\sqrt{x}$,

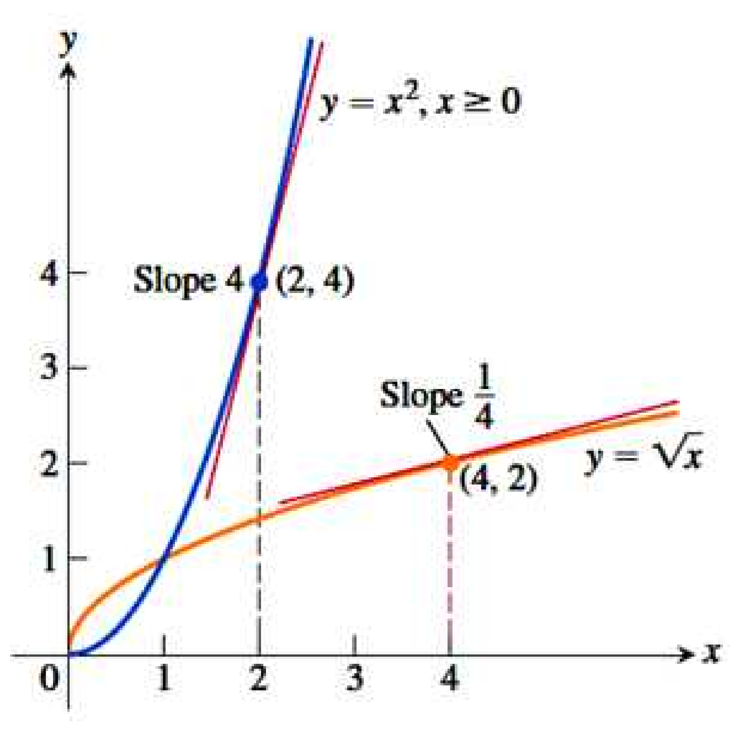
We have

$$\frac{df}{dx}=\frac{d}{dx}\left(x^2\right)=2x \qquad \frac{df^{-1}}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}},\ x>0$$

The point $(4, 2)$ is the mirror image of the point $(2, 4)$ across the line $y=x$.

At the point $(2, 4)$: $\frac{df}{dx}=2x=2(2)=4$

At the point $(4, 2)$: $\frac{df^{-1}}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{4}}=\frac{1}{4}=\frac{1}{\frac{df}{dx}}$



Equation (1) sometimes enables us to find specific values of $\frac{df^{-1}}{dx}$ without knowing a formula for f^{-1}.

Example 2:

Let $f(x)=x^3-2$. Find the value of $\frac{df^{-1}}{dx}$ at $x=6=f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

$$\left(\frac{df}{dx}\right)_{x=2}=\left[3x^2\right]_{x=2}=12$$
$$\left(\frac{df^{-1}}{dx}\right)_{x=f(2)}=\frac{1}{\left(\frac{df}{dx}\right)_{x=2}}=\frac{1}{12}$$

There is another way to look at theorem 1.

If $y=f(x)$ is differentiable at $x=a$ and we change x by a small amount dx, the corresponding change in y is approximately

$$dy=f'(a)dx$$

This means that y changes about $f'(a)$ times as fast as x and that x changes about $\frac{1}{f'(a)}$ times as fast as y.

IP1:

If $f(x) = 2x^2$, $x \geq 0$; $a = 5$ then

a. Find $f^{-1}(x)$

b. Evaluate $\frac{df}{dx}$ at $x = a$ and $\frac{df^{-1}}{dx}$ at $x = f(a)$ to show that at these points $\frac{df^{-1}}{dx} = \frac{1}{\frac{df}{dx}}$

Solution:

Given $f(x) = 2x^2$, $x \geq 0$

Clearly, f is one-to-one and so its inverse exists.

a. Now, solve for x in terms of y

$$y = 2x^2 \Rightarrow \frac{y}{2} = x^2 \Rightarrow x = \sqrt{\frac{y}{2}}$$

Interchange x and y: $y = \sqrt{\frac{x}{2}}$

The inverse of the function $f(x) = 2x^2$ is the function

$$y = f^{-1}(x) = \sqrt{\frac{x}{2}}$$

b. Now, $\frac{df}{dx} = 4x$ and $\left(\frac{df}{dx}\right)_{x=a} = \left(\frac{df}{dx}\right)_{x=5} = 4(5) = 20$

$$f(x) = 2x^2 \Rightarrow f(5) = 2(5)^2 = 50$$

$$\frac{df^{-1}}{dx} = \frac{d}{dx}\left(\sqrt{\frac{x}{2}}\right) = \frac{1}{2\sqrt{2x}} \ ;$$

$$\left(\frac{df^{-1}}{dx}\right)_{x=f(a)} = \left(\frac{df^{-1}}{dx}\right)_{x=f(5)} = \frac{1}{2\sqrt{2(50)}} = \frac{1}{20}$$

$$\therefore \left(\frac{df^{-1}}{dx}\right)_{x=f(5)} = \frac{1}{20} = \frac{1}{\left(\frac{df}{dx}\right)_{x=5}}$$

IP2:

Let $f(x) = x^2 - 4x - 5, x > 2$ then find $\frac{df^{-1}}{dx}$ at the point $x = 0 = f(5)$

Solution:

Given $f(x) = x^2 - 4x - 5, \quad x > 2$

We have $\left(\frac{df^{-1}}{dx}\right)_{x=f(a)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=a}}$

Now, $\frac{df}{dx} = 2x - 4$

$$\left(\frac{df}{dx}\right)_{x=5} = 2(5) - 4 = 6$$

$$\therefore \quad \left(\frac{df^{-1}}{dx}\right)_{x=0=f(5)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=5}} = \frac{1}{6}$$

IP3:

Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Then find the slope of the graph of g^{-1} at the origin

Solution:

Given that $y = g(x)$ is a differentiable function and has an inverse.

Since the graph of g passes through origin $(0,0)$ and has a slope 2 there,

$$\left(\frac{dg}{dx}\right)_{x=0} = 2$$

$\therefore$ The slope of the graph of g^{-1} at the origin is

$$\therefore \quad \left(\frac{dg^{-1}}{dx}\right)_{x=0} = \left(\frac{dg^{-1}}{dx}\right)_{x=f(0)} = \frac{1}{\left(\frac{dg}{dx}\right)_{x=0}} = \frac{1}{2}$$

IP4:

a. Show that $f(x) = \frac{x^3}{4}$ and $g(x) = \sqrt[3]{4x}$ are inverses of one another.

b. Find the slopes of the tangents to the graphs of f and g at $(2, 2)$ and $(-2, -2)$ (four tangents in all).

c. What lines are tangents to the curves at the origin?

Solution:

a. Given $f(x) = \frac{x^3}{4}$ and $g(x) = (4x)^{1/3}$

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{4x}\right) = \frac{\left(\sqrt[3]{4x}\right)^3}{4} = x$$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x^3}{4}\right) = \left(4\left(\frac{x^3}{4}\right)\right)^{1/3} = x$$

$$\therefore (f \circ g)(x) = (g \circ f)(x) = x$$

$\Rightarrow f(x)$ and $g(x)$ are inverses of one another.

Note:

The points of intersection of $y = \frac{x^3}{4}$ and $y = \sqrt[3]{4x}$ are given by

$$\frac{x^3}{4} = \sqrt[3]{4x} \Rightarrow x^9 = 256\,x$$

$$\Rightarrow x^9 - 256x = 0 \Rightarrow x(x^8 - 256) = 0$$

$$\Rightarrow x(x^4 - 16)(x^4 + 16) = 0$$

$$\Rightarrow x(x^2 - 2)(x^2 + 2)(x^4 + 16) = 0 \Rightarrow x = 0, \pm 2$$

The curves intersect at $(0, 0)(2, 2)(-2, -2)$

b. The slopes of the tangents to the graphs of f and g at $(2, 2)$ and $(-2, -2)$ are:

Given $f(x) = \frac{x^3}{4}$

Differentiating f w.r.t x, we get

$$f'(x) = \frac{3x^2}{4}$$

$$\left(\frac{df}{dx}\right)_{x=2} = \frac{3(2)^2}{4} = 3\,;\ \left(\frac{df}{dx}\right)_{x=-2} = \frac{3(-2)^2}{4} = 3$$

We have $g(x) = \sqrt[3]{4x}$ and $g = f^{-1}$

Now, $\left(\frac{dg}{dx}\right)_{x=2} = \left(\frac{df^{-1}}{dx}\right)_{x=2=f(2)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=2}} = \frac{1}{3}$ and

$$\left(\frac{dg}{dx}\right)_{x=-2} = \left(\frac{df^{-1}}{dx}\right)_{x=-2=f(-2)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=-2}} = \frac{1}{3}$$

c. We have $\left(\frac{df}{dx}\right) = \frac{3x^2}{4}$

$$\Rightarrow \left(\frac{df}{dx}\right)_{x=0} = 0$$

$\Rightarrow y = f(x)$ has a horizontal tangent line at $(0, 0)$

$\Rightarrow y = 0$ is a tangent to the curve $y = \frac{x^3}{4}$ at the origin.

Since $g = f^{-1}$, g has a vertical tangent line at $(0, 0)$

$\Rightarrow x = 0$ is a tangent to the curve $y = \sqrt[3]{4x}$ at the origin.

P1:

If $f(x)=\frac{x}{5}+7$; $a=-1$ then

a. Find $f^{-1}(x)$

b. Evaluate $\frac{df}{dx}$ at $x=a$ and $\frac{df^{-1}}{dx}$ at $x=f(a)$ to show that at these points $\frac{df^{-1}}{dx}=\frac{1}{\frac{df}{dx}}$

Solution:

Given $f(x)=\frac{x}{5}+7$

Clearly, f is one-to-one and so its inverse exists.

a. Now, solve for x in terms of y

$$y=\frac{x}{5}+7 \Rightarrow y-7=\frac{x}{5} \Rightarrow x=5y-35$$

Interchange x and y: $y=5x-35$

The inverse of the function $f(x)=\frac{x}{5}+7$ is the function

$$y=f^{-1}(x)=5x-35$$

b. Now, $\frac{df}{dx}=\frac{1}{5}$ and $\left(\frac{df}{dx}\right)_{x=a}=\left(\frac{df}{dx}\right)_{x=-1}=\frac{1}{5}$

$$f(x)=\frac{x}{5}+7 \Rightarrow f(-1)=-\frac{1}{5}+7=\frac{34}{5}$$

$$\frac{df^{-1}}{dx}=5 \ ; \left(\frac{df^{-1}}{dx}\right)_{x=f(a)}=\left(\frac{df^{-1}}{dx}\right)_{x=f(-1)}=5$$

$$\therefore \left(\frac{df^{-1}}{dx}\right)_{x=f(-1)}=5=\frac{1}{\left(\frac{df}{dx}\right)_{x=-1}}$$

P2:

Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$ then the value of $\dfrac{df^{-1}}{dx}$ at the point $x = -1 = f(3)$ is.

Solution:

Given $f(x) = x^3 - 3x^2 - 1,\ x \geq 2$

We have $\left(\dfrac{df^{-1}}{dx}\right)_{x=f(a)} = \dfrac{1}{\left(\dfrac{df}{dx}\right)_{x=a}}$

Now, $\dfrac{df}{dx} = 3x^2 - 6x,\ \ x \geq 2$

$$\left(\frac{df}{dx}\right)_{x=3} = 3(3)^2 - 6(3) = 9$$

$$\therefore \left(\frac{df^{-1}}{dx}\right)_{x=f(3)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=3}} = \frac{1}{9}$$

P3:

Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $\frac{1}{3}$ there. Find the value of $\frac{df^{-1}}{dx}$ at $x = 4$.

Solution:

Given that $y = f(x)$ is a differentiable function and has an inverse.

Since the graph of f passes through $(2, 4)$ and has a slope $\frac{1}{3}$ there,

$$\left(\frac{df}{dx}\right)_{x=2} = \frac{1}{3}$$

$$\therefore \quad \left(\frac{df^{-1}}{dx}\right)_{x=4} = \left(\frac{df^{-1}}{dx}\right)_{x=f(2)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=2}} = \frac{1}{\frac{1}{3}} = 3$$

P4:

a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

b. Find the slopes of the tangents to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ (four tangents in all).

c. What lines are tangents to the curves at the origin?

Solution:

a. Given $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$$

$$(g \circ f)(x) = g(f(x)) = g\left(x^3\right) = \left(x^3\right)^{1/3} = x$$

$$\therefore (f \circ g)(x) = (g \circ f)(x) = x$$

$\Rightarrow f(x)$ and $g(x)$ are inverses of one another.

Note:

The points of intersection of $y = x^3$ and $y = \sqrt[3]{x}$ are given by

$$x^3 = \sqrt[3]{x} \Rightarrow x^9 = x \Rightarrow x^9 - x = 0 \Rightarrow x(x^8 - 1) = 0$$

$$\Rightarrow x(x^4 - 1)(x^4 + 1) = 0$$

$$\Rightarrow x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0 \Rightarrow x = 0, \pm 1$$

The curves intersect at $(0, 0)(1, 1)(-1, -1)$

b. The slopes of the tangents to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ are:

Given $f(x) = x^3$

Differentiating f w.r.t x, we get

$$f'(x) = 3x^2$$

$$\left(\frac{df}{dx}\right)_{x=1} = 3(1)^2 = 3\,; \left(\frac{df}{dx}\right)_{x=-1} = 3(-1)^2 = 3$$

We have $g(x) = \sqrt[3]{x}$ and $g = f^{-1}$

Now, $\left(\frac{dg}{dx}\right)_{x=1} = \left(\frac{df^{-1}}{dx}\right)_{x=1=f(1)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=1}} = \frac{1}{3}$ and

$$\left(\frac{dg}{dx}\right)_{x=-1} = \left(\frac{df^{-1}}{dx}\right)_{x=-1=f(-1)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=-1}} = \frac{1}{3}$$

c. We have $\left(\frac{df}{dx}\right) = 3x^2$

$$\Rightarrow \left(\frac{df}{dx}\right)_{x=0} = 0$$

$\Rightarrow y = f(x)$ has a horizontal tangent line at $(0, 0)$

$\Rightarrow y = 0$ is a tangent to the curve $y = x^3$ at the origin.

Since $g = f^{-1}$, g has a vertical tangent line at $(0, 0)$

$\Rightarrow x = 0$ is a tangent to the curve $y = \sqrt[3]{x}$ at the origin.

4.2. Derivatives of Inverse Functions

Exercise:

1. The formula for $f(x)$ is given below:

   (a) Find $f^{-1}(x)$

   (b) Evaluate $\frac{df}{dx}$ at $x=a$ and $\frac{df^{-1}}{dx}$ at $x=f(a)$ to

   show that at these points $\frac{df^{-1}}{dx}=\frac{1}{\frac{df}{dx}}$.

   a. $f(x)=2x+3,\ a=-1$

   b. $f(x)=5-4x,\ a=1/2$

4.3

Natural Logarithms

Learning objectives:

- To define the Natural Logarithm Function.
- To derive the derivative of $ln\,x$.
- To prove the properties of Natural Logarithms.
- To study the graph and range of the $ln\,x$.
  And
- To practice the related problems.

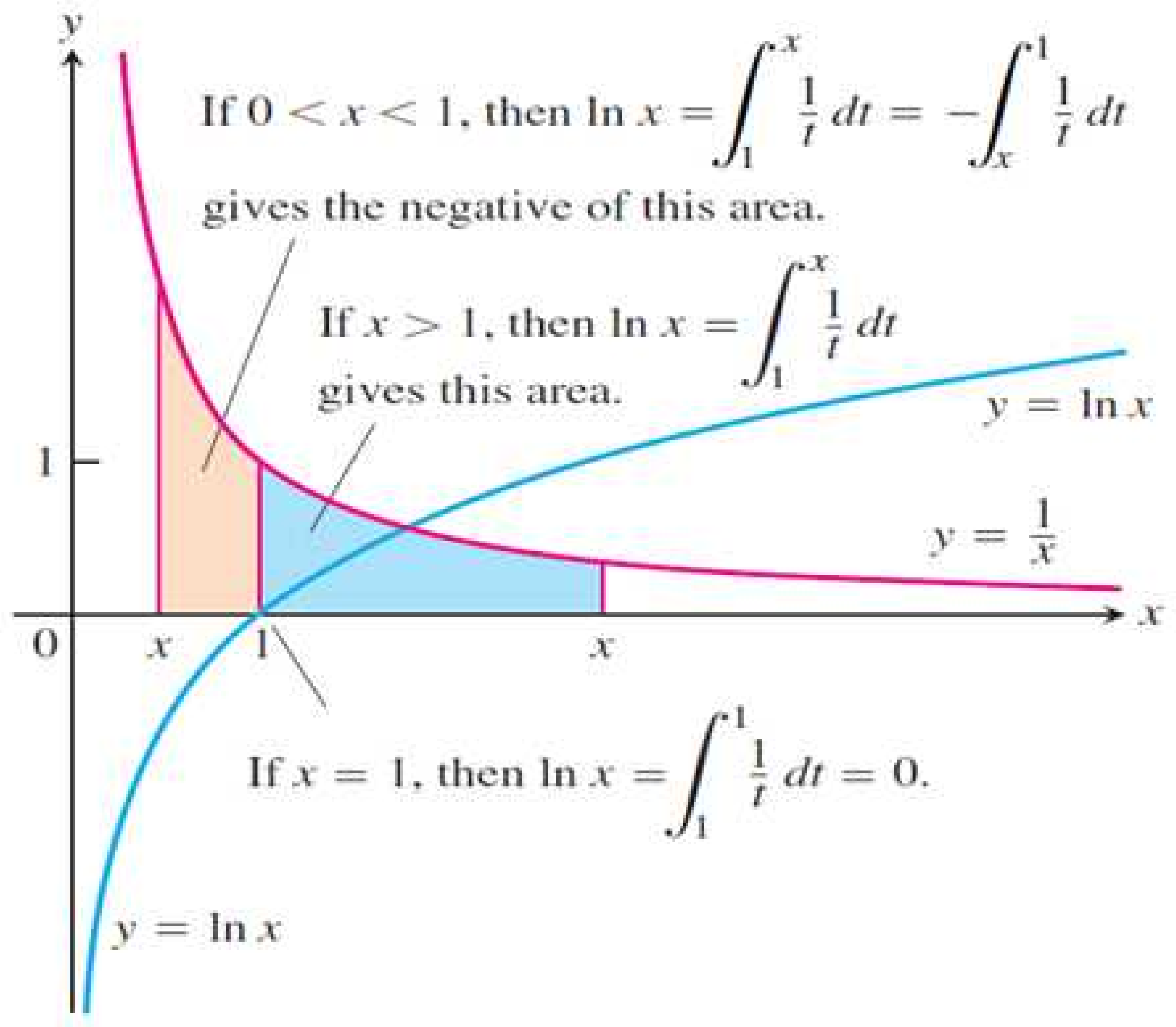
The function-inverse pair consisting of the natural logarithm function $ln\,x$ and the exponential function e^x is an important pair.

The Natural Logarithm Function

The natural logarithm of a positive number x, written as $ln\,x$, is the value of an integral.

$$\boldsymbol{ln\,x = \int_1^x \frac{1}{t}\,dt} \qquad \boldsymbol{x} > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = \frac{1}{t}$ from $t = 1$ to $t = x$.



For $0 < x < 1$, $ln\,x$ gives the negative of the area under the curve from x to 1. The function is not defined for $x \leq 0$. We also have

$$ln\,1 = \int_1^1 \frac{1}{t}\,dt = 0$$

(∵upper and lower limits are equal)

We notice that we show the graph of $y = \frac{1}{x}$ in the figure but use $y = \frac{1}{t}$ in the integral. Using x for everything would have us writing

$$ln\,x = \int_1^x \frac{1}{x}\,dx$$

with x meaning two different things. So we change the variable of integration to t.

The Derivative of $\boldsymbol{ln\,x}$:

By the first part of the Fundamental Theorem of Calculus,

$$\frac{d}{dx} ln\,x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

For every positive value of x, therefore,

$$\boldsymbol{\frac{d}{dx} ln\,x = \frac{1}{x}}$$

If u is a differentiable function of x whose values are positive, so that $ln\,u$ is defined, then applying the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ to the function $y = ln\,u$ gives

$$\frac{d}{dx} ln\,u = \frac{d}{du} ln\,u . \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$\boldsymbol{\frac{d}{dx} ln\,u = \frac{1}{u}\frac{du}{dx}}, \quad \boldsymbol{u} > 0 \qquad \text{------------ (1)}$$

Example1:

$$\frac{d}{dx} ln\,2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{x}$$

The function $y = ln\,2x$ has the same derivative as the function $y = ln\,x$. This is true for $y = ln\,ax$ for any number a.

$$\frac{d}{dx} ln\,ax = \frac{1}{ax}\frac{d}{dx}(ax) = \frac{1}{ax}(a) = \frac{1}{x} \qquad \text{-------(2)}$$

Example 2:

Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx} ln(x^2+3) = \frac{1}{x^2+3}.\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}.2x = \frac{2x}{x^2+3}$$

Properties of Logarithms

The properties of logarithms are listed below.
For any numbers $a > 0$ and $x > 0$,

Product Rule:	$ln\, ax = ln\, a + ln\, x$
Reciprocal Rule:	$ln \frac{1}{x} = -\, ln\, x$
Quotient Rule:	$ln \frac{a}{x} = ln\, a - ln\, x$
Power Rule:	$ln\, x^n = n\, ln\, x$

The properties made it possible to replace multiplication of positive numbers by addition and division of positive numbers by subtraction. They also made it possible to replace exponentiation by multiplication.

We prove the properties as follows.

(i) $\boldsymbol{ln\, ax = ln\, a + ln\, x}$
We noted that $ln\, ax$ and $ln\, x$ have the same derivative. By Corollary 1 of the Mean Value Theorem, the functions must differ by a constant,

$$\therefore\ ln\, ax = ln\, x + C \qquad \text{-------}(3)$$

for some C. This equation holds for all positive values of x, so it must hold for $x = 1$. Hence,

$$ln(a.1) = ln\, 1 + C \implies ln\, a = 0 + C \implies C = \ln a$$

Substituting $C = ln\, a$ in equation (3) we get

$$ln\, ax = ln\, a + ln\, x \qquad \text{--------}(4)$$

(ii) $\boldsymbol{ln \frac{1}{x} = -\, ln\, x}$
In Equation (4) replace a by $\frac{1}{x}$ gives

$$ln \frac{1}{x} + ln\, x = ln\left(\frac{1}{x}.x\right) = ln\, 1 = 0$$

so that $ln \frac{1}{x} = -\, ln\, x$ giving the Reciprocal Rule

(iii) $\boldsymbol{ln \frac{a}{x} = ln\, a - ln\, x}$
Equation (4) with x replaced by $\frac{1}{x}$ gives

$$ln \frac{a}{x} = ln\left(a.\frac{1}{x}\right) = ln\, a + ln \frac{1}{x} \qquad \text{Product rule}$$
$$= ln\, a - ln\, x \qquad \text{Reciprocal rule}$$

(iv) $\boldsymbol{ln\, x^n = n\, ln\, x}$
We assume n rational

$$\frac{d}{dx} ln\, x^n = \frac{1}{x^n} \frac{d}{dx}(x^n) = \frac{1}{x^n} n x^{n-1}$$
$$= n.\frac{1}{x} = \frac{d}{dx}(n\, ln\, x)$$

Since $ln\, x^n$ and $n\, ln\, x$ have the same derivative,

$$ln\, x^n = n\, ln\, x + C$$

for some constant C. Taking $x = 1$ we get $C = 0$.
This completes proof.
The rule holds for all n, rational and irrational.

Example 3:

a) $ln\, 6 = ln(2.3) = ln\, 2 + ln\, 3$ — Product rule
b) $ln\, 4 - ln\, 5 = ln \frac{4}{5} = ln\, 0.8$ — Quotient rule
c) $ln \frac{1}{8} = -\, ln\, 8$ — Reciprocal rule
$= -\, ln\, 2^3 = -3\, ln\, 2$ — Power rule

Example 4:

a) $ln\, 4 + ln \sin x = ln(4 \sin x)$ — Product rule
b) $ln \frac{x+1}{2x-3} = ln(x+1) - ln(2x-3)$ — Quotient rule
c) $ln \sec x = ln \frac{1}{\cos x} = -\, ln \cos x$ — Reciprocal rule
d) $ln \sqrt[3]{x+1} = ln(x+1)^{\frac{1}{3}} = \frac{1}{3} ln(x+1)$ — Power rule

The Graph and Range of $\ln x$

The derivative $\frac{d(ln\, x)}{dx} = \frac{1}{x}$ is positive for $x > 0$. Thus $ln\, x$ is an increasing function of x, hence it is one-to-one and invertible. The second derivative, $-\frac{1}{x^2}$, is negative, so the graph of $ln\, x$ is concave down.

We can estimate $\ln 2$ by numerical integration to be about 0.69. We therefore know that

$$ln\, 2^n = n\, ln\, 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}$$

and

$$ln\, 2^{-n} = -n\, ln\, 2 < -n\left(\frac{1}{2}\right) = -\frac{n}{2}$$

It follows that

$$\lim_{x \to \infty} ln\, x = \infty \qquad \text{and} \qquad \lim_{x \to 0^+} ln\, x = -\infty$$

The domain of $\ln x$ is the set of positive real numbers; the range is the entire real line.

IP1:

$$\frac{2\ ln\ 6+6\ ln\ 2}{4\ ln\ 2+ln\ 27-ln\ 9}=$$

Solution:

$$\frac{2\ ln\ 6+6\ ln\ 2}{4\ ln\ 2+ln\ 27-ln\ 9}$$

$$=\frac{2\ ln\ (3\times2)+6\ ln\ 2}{4\ ln\ 2+ln\ \frac{27}{9}}=\frac{2\ ln\ 3+2\ ln\ 2+6\ ln\ 2}{4\ ln\ 2+ln\ 3}$$

$$=\frac{2\ ln\ 3+8\ ln\ 2}{4\ ln\ 2+ln\ 3}=\frac{2(4\ ln\ 2+ln\ 3)}{4\ ln\ 2+ln\ 3}=2$$

IP2:

If $\boldsymbol{y = ln\left[\frac{(x^2+1)^2}{\sqrt{1-4x^3}}\right], x \neq 1}$ **then find** $\boldsymbol{y'}$ **at** $\boldsymbol{x = -1.}$

Solution:

Given $y = ln\left[\frac{(x^2+1)^2}{\sqrt{1-4x^3}}\right], x \neq 1$

$$\Rightarrow \quad y = ln\,[(x^2+1)^2] - ln\sqrt{1-4x^3}$$

$$\Rightarrow \quad y = 2\,ln(x^2+1) - \frac{1}{2}\,ln(1-4x^3)$$

Differentiating on both sides w.r.t x, we get

$$\frac{dy}{dx} = 2\,\frac{1}{x^2+1}\,\frac{d}{dx}(x^2+1) - \frac{1}{2}.\frac{1}{1-4x^3}\frac{d}{dx}(1-4x^3)$$

$$= \frac{2}{(x^2+1)}(2x) - \frac{1}{2(1-4x^3)}(-12x^2)$$

Now,

$$y'|_{x=-1} = \frac{2}{[(-1)^2+1]}.2(-1) - \frac{1}{2[1-4(-1)^3]}(-12(-1)^2)$$

$$= \frac{-4}{2} + \frac{1}{2(5)}(12) = -2 + \frac{6}{5} = -\frac{4}{5}$$

IP3:

If $\boldsymbol{y = \ln\left(sec(ln\,\theta)\right)}$, then find $\boldsymbol{\frac{dy}{d\theta}}$.

Solution:

Given $y = \ln\left(sec(ln\,\theta)\right)$

Differentiating on both sides w.r.t θ, we get

$$\frac{dy}{d\theta} = \frac{1}{sec(ln\,\theta)} \cdot \frac{d}{d\theta}\left(sec(ln\,\theta)\right)$$

$$= \frac{sec(ln\,\theta)\tan(ln\,\theta)}{sec(ln\,\theta)} \cdot \frac{d}{d\theta}(ln\,\theta)$$

$$= \frac{\tan(ln\,\theta)}{\theta}$$

IP4:

If $\boldsymbol{y = \sqrt{ln\sqrt{t}}}$**, then find** $\boldsymbol{\frac{dy}{dt}}$**.**

Solution:

Given $y = \sqrt{ln\sqrt{t}} = \left(\ln t^{\frac{1}{2}}\right)^{\frac{1}{2}}$

Differentiating on both sides w.r.t t, we get

$$\frac{dy}{dt} = \frac{1}{2}\left(ln\, t^{\frac{1}{2}}\right)^{-\frac{1}{2}} \frac{d}{dt}\left(ln\, t^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(ln\, t^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \frac{1}{t^{\frac{1}{2}}} \cdot \frac{d}{dt}\left(t^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(ln\, t^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \frac{1}{t^{\frac{1}{2}}} \cdot \frac{1}{2} t^{-\frac{1}{2}}$$

$$= \frac{1}{4t}\left(ln\, t^{\frac{1}{2}}\right)^{-\frac{1}{2}} = \frac{1}{4t\sqrt{ln\sqrt{t}}}$$

P1:

$$\boldsymbol{7\,ln\frac{16}{15} + 5\,ln\frac{25}{24} + 3\,ln\frac{81}{80} =}$$

Solution:

$$7\;ln\frac{16}{15} + 5\,ln\frac{25}{24} + 3\,ln\frac{81}{80}$$

$$=\;7\;ln\left(\frac{2^4}{3\times5}\right) + 5\,ln\left(\frac{5^2}{3\times2^3}\right) + 3\,ln\left(\frac{3^4}{5\times2^4}\right)$$

$$= 7[ln\,2^4 - ln(3\times5)] + 5[ln\,5^2 - ln(3\times2^3)]$$

$$+3[ln\,3^4 - ln(5\times2^4)]$$

$$= 7[4\,ln\,2 - ln\,3 - ln\,5] + 5[2\,ln\,5 - ln\,3 - 3\,ln\,2]$$

$$+3[4\,ln\,3 - ln\,5 - 4\,ln\,2]$$

$$=\;[28-15-12]\,ln\,2 + [-7+10-3]\,ln\,5$$

$$+[-7-5+12]\,ln\,3$$

$$= 1.\,ln\,2 + 0 + 0 = ln\,2$$

P2:

Find the derivative of $\boldsymbol{f(x) = ln\left(x + \sqrt{x^2 - 1}\right)}$.

Solution:

Given $f(x) = ln\left(x + \sqrt{x^2 - 1}\right)$

Differentiating on both sides w.r.t x, we get

$$f'(x) = \frac{1}{x+\sqrt{x^2-1}}\ \frac{d}{dx}\left(x + \sqrt{x^2 - 1}\right)$$

$$= \frac{1}{x+\sqrt{x^2-1}}.\left[1 + \frac{1}{2\sqrt{x^2-1}}\frac{d}{dx}(x^2 - 1)\right]$$

$$= \frac{1}{x+\sqrt{x^2-1}}\left[1 + \frac{1}{2\sqrt{x^2-1}}(2x)\right]$$

$$= \frac{1}{x+\sqrt{x^2-1}}\left[\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right]$$

$$= \frac{1}{\sqrt{x^2-1}}$$

P3:

If $\boldsymbol{y = \theta[\sin(\ln\theta) + \cos(\ln\theta)]}$, then find $\boldsymbol{\frac{dy}{d\theta}}$.

Solution:

Given $y = \theta[\sin(ln\,\theta) + \cos(ln\,\theta)]$

Differentiating on both sides w.r.t θ, we get

$$\frac{dy}{d\theta} = \theta.\left[\cos(ln\,\theta).\frac{d}{d\theta}(ln\,\theta) + \left(-\sin(ln\,\theta)\right)\frac{d}{dx}(ln\,\theta)\right] + [\sin(ln\,\theta) + \cos\,(ln\,\theta)].(1)$$

$$= \theta\left[\frac{1}{\theta}\cos(ln\,\theta) - \frac{1}{\theta}\sin(ln\,\theta)\right] + \sin(ln\,\theta) + \cos(ln\,\theta)$$

$$= \cos(ln\,\theta) - \sin(ln\,\theta) + \sin(ln\,\theta) + \cos(ln\,\theta)$$

$$= 2\cos(ln\,\theta)$$

P4:

If $y = ln\frac{\sqrt{\sin\theta\cos\theta}}{1+2\,ln\,\theta}$, then find $\frac{dy}{d\theta}$.

Solution:

Given $y = ln\frac{\sqrt{\sin\theta\cos\theta}}{1+2\,ln\,\theta}$

$$y = \frac{1}{2}ln(\sin\theta\cos\theta) - ln(1+2\ln\theta)$$

$$= \frac{1}{2}(ln(\sin\theta) + ln(\cos\theta)) - ln(1+2\ln\theta)$$

Differentiating on both sides w.r.t θ, we get

$$\frac{dy}{d\theta} = \frac{1}{2}\left(\frac{1}{\sin\theta}\cdot\cos\theta - \frac{1}{\cos\theta}\cdot\sin\theta\right) - \frac{1}{1+2\,ln\,\theta}\cdot\frac{d}{d\theta}(1+2\,ln\,\theta)$$

$$= \frac{1}{2}(\cot\theta - \tan\theta) - \frac{1}{1+2\,ln\,\theta}\left(\frac{2}{\theta}\right)$$

$$= \frac{1}{2}\left(\cot\theta - \tan\theta - \frac{4}{\theta(1+2\,ln\,\theta)}\right)$$

4.3. Natural Logarithms

Exercise:

1. Express the following logarithms in terms of $ln\,2$ and $ln\,3$.

(a) $ln\,0.75$ (b) $ln\left(\frac{4}{9}\right)$ (c) $ln\left(\frac{4}{9}\right)$

(d) $ln\sqrt[3]{9}$ (e) $ln\,3\sqrt{2}$ (f) $ln\sqrt{13.5}$

2. Express the following logarithms in terms of $ln\,5$ and $ln\,7$

(a) $ln\frac{1}{125}$ (b) $ln\,9.8$ (c) $ln\,7\sqrt{7}$

(d) $ln\,1225$ (e) $ln\,0.056$ (f) $\frac{ln\,35+ln\left(\frac{1}{7}\right)}{ln\,25}$

3. Use the properties of logarithms to simplify the expressions.

a. $ln\sin\theta - ln\left(\frac{\sin\theta}{5}\right)$

b. $ln(3x^2-9x)+ln\left(\frac{1}{3x}\right)$

c. $\frac{1}{2}ln(4t^4)-ln\,2$

d. $ln\sec\theta + ln\cos\theta$

e. $ln(8x+4)-2\,ln\,2$

f. $3\,ln\sqrt[3]{t^2-1}-ln(t+1)$

4. Find the derivative of y with respect to x, t, or θ, as appropriate.

a. $y = ln\,3x$

b. $y = ln\,kx$

c. $y = ln(t^2)$

d. $y = ln\left(t^{\frac{3}{2}}\right)$

e. $y = ln\frac{3}{x}$

f. $y = ln\frac{10}{x}$

g. $y = ln(\theta+1)$

h. $y = ln(2\theta+2)$

i. $y = ln\,x^3$

j. $y = (ln\,x)^3$

k. $y = t(ln\,t)^2$

l. $y = t\sqrt{ln\,t}$

m. $y = \frac{x^4}{4}ln\,x - \frac{x^4}{16}$

n. $y = \frac{x^3}{3}ln\,x - \frac{x^3}{9}$

o. $y = \frac{ln\,t}{t}$

p. $y = \frac{1+ln\,t}{t}$

q. $y = \frac{ln\,x}{1+ln\,x}$

r. $y = \frac{x\,ln\,x}{1+ln\,x}$

s. $y = ln(ln\,x)$

t. $y = ln(ln(ln\,x))$

u. $y = ln\frac{1}{x\sqrt{x+1}}$

v. $y = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)$

w. $y = \frac{1+ln\,t}{1-ln\,t}$

x. $y = ln(sec(ln\,\theta))$

y. $y = ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right)$

z. $y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

4.4

Logarithmic Differentiation and $\int \frac{du}{u}$

Learning Objectives:

- To learn logarithmic differentiation
- To evaluate $\int \frac{du}{u}$, where u is a nonzero differentiable function
  AND
- To practice the related problems

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm on both sides before differentiating. This enables us to use the properties of logarithms to simplify the formulas before differentiating. This process is called ***logarithmic differentiation***.

Example 1:

Find $\frac{dy}{dx}$ if $y = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$, $x > 1$.

Solution:

We take the natural logarithm on both sides and simplify using the properties of logarithms.

$$\ln y = \ln \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}$$

$$= \ln(x^2+1)(x+3)^{\frac{1}{2}} - \ln(x-1)$$

$$= \ln(x^2+1) + \ln(x+3)^{\frac{1}{2}} - \ln(x-1)$$

$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

We then take derivatives of both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}$$

Next we solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

The Integral $\int \left(\frac{1}{u}\right) du$

We have, $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}, u > 0$ --- (1)

Equation (1) leads to the integral formula

$$\int \frac{1}{u} du = \ln u + C \quad \text{----- (2)}$$

where u is a positive differentiable function. If u is negative, then $-u$ is positive and

$$\int \frac{1}{u} du = \int \frac{1}{(-u)} d(-u) = \ln(-u) + C \quad \text{---- (3)}$$

(equation (2) with u replaced by $-u$)

We can combine equations (2) and (3) into a single formula by noticing that in each case the expression on the right is $\ln|u| + C$. In equation (2), $\ln u = \ln|u|$ because $u > 0$; in equation (3), $\ln(-u) = \ln|u|$ because $u < 0$. Whether u is positive or negative, the integral of $\int \left(\frac{1}{u}\right) du$ is $\ln|u| + C$.

If u is a nonzero differentiable function,

$$\int \frac{1}{u} du = \ln|u| + C \quad \text{---- (4)}$$

We know that

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, n \neq -1$$

Equation (4) handles the case when n equals −1.

Equation (4) says that integrals of a certain *form* lead to logarithms. That is,

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that maintains a constant sign on the domain given for it.

Example 2:

$$\int_0^2 \frac{2x}{x^2-5} dx = \int_{-5}^{-1} \frac{du}{u} \quad \begin{pmatrix} u = x^2 - 5 \Rightarrow du = 2x\, dx \\ u(0) = -5, u(2) = -1 \end{pmatrix}$$

$$= \ln|u|]_{-5}^{-1} = \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

Example 3:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3+2\sin\theta} d\theta = \int_1^5 \frac{2}{u} du$$

$$\begin{pmatrix} u = 3 + 2\sin\theta \Rightarrow du = 2\cos\theta\, d\theta \\ u\left(-\frac{\pi}{2}\right) = 1, u\left(\frac{\pi}{2}\right) = 5 \end{pmatrix}$$

$$= 2\ln|u|]_1^5 = 2\ln|5| - 2\ln|1| = 2\ln|5|$$

The Integrals of $\tan x$ and $\cot x$

Equation (4) helps us in the evaluation of integrals of the tangent and cotangent functions.

For the tangent,

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u}$$

$$(u = \cos x \Rightarrow du = -\sin x\, dx)$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C \quad \text{(Reciprocal rule)}$$

$$= \ln|\sec x| + C$$

For the cotangent,

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{du}{u}$$

$$(u = \sin x \Rightarrow du = \cos x\, dx)$$

$$= \ln|u| + C$$

$$= \ln|\sin x| + C$$

$$= -\ln\frac{1}{|\sin x|} + C \quad \text{(Reciprocal rule)}$$

$$= -\ln|\csc x| + C$$

$$\boldsymbol{\int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C}$$

$$\boldsymbol{\int \cot u\, du = \ln|\sin u| + C = -\ln|\csc u| + C}$$

Example 4:

$$\int_0^{\frac{\pi}{6}} \tan 2x\, dx = \int_0^{\frac{\pi}{3}} \tan u \frac{du}{2} \quad \begin{pmatrix} u = 2x \Rightarrow du = 2dx \\ u(0) = 0, u\left(\frac{\pi}{6}\right) = \frac{\pi}{3} \end{pmatrix}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \tan u\, du = \frac{1}{2}\ln|\sec u|]_0^{\frac{\pi}{3}} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

IP1:

Find the derivative of $\boldsymbol{y = (ln\,x)^{\frac{1}{ln\,x}}}$ with respect to $\boldsymbol{x}$.

Solution:

Given $y = (ln\,x)^{\frac{1}{ln\,x}}$

Taking logarithm on both sides, we get

$$ln\,y = \frac{1}{ln\,x}(ln(ln\,x))$$

Differentiating on both sides w.r.t x, we get

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{ln\,x\frac{d}{dx}(ln(ln\,x)) - ln(ln\,x)\frac{d}{dx}ln\,x}{(ln\,x)^2}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{ln\,x\frac{1}{ln\,x}\frac{1}{x} - ln(ln\,x)\frac{1}{x}}{(ln\,x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = (ln\,x)^{\frac{1}{ln\,x}}\left[\frac{1 - ln(ln\,x)}{x(lnx)^2}\right]$$

IP2:

If $x^m y^n = (x+y)^{m+n}$, then find $\frac{dy}{dx}$.

Solution:

Given, $x^m y^n = (x+y)^{m+n}$

Taking logarithm on both sides, we get

$$ln(x^m y^n) = ln(x+y)^{m+n}$$

$$\Longrightarrow m\ lnx + n\ lny = (m+n)ln(x+y)$$

Differentiating on both sides w.r.t x, we get

$$\Longrightarrow \frac{m}{x} + \frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{x+y}.\frac{d}{dy}(x+y)$$

$$\Longrightarrow \frac{m}{x} + \frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}.\frac{dy}{dx}$$

$$\Longrightarrow \frac{m}{x} - \frac{m+n}{x+y} = \frac{dy}{dx}\left[\frac{m+n}{x+y} - \frac{n}{y}\right]$$

$$\Longrightarrow \frac{mx+my-mx-nx}{x(x+y)} = \frac{dy}{dx}\left[\frac{my+ny-nx-ny}{y(x+y)}\right]$$

$$\Longrightarrow \left[\frac{my-nx}{x}\right] = \frac{dy}{dx}\left[\frac{my-nx}{y}\right]$$

$$\Longrightarrow \frac{dy}{dx} = \frac{y}{x}$$

IP3:

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\boldsymbol{sin2x}}{\boldsymbol{acos^2x+bsin^2x}}\, \boldsymbol{dx}$.

Solution:

Given, $\int_0^{\frac{\pi}{2}} \frac{sin2x}{acos^2x+bsin^2x}\, dx$

$$\frac{d}{dx}(acos^2x + bsin^2x) = 2acosx(-sinx) + 2bsinx\ cosx$$

$$= -a(2sinxcosx) + b(2sinxcosx)$$

$$= (b-a)sin2x$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{sin2x}{acos^2x+bsin^2x}\, dx = \frac{1}{b-a}\int_0^{\frac{\pi}{2}} \frac{(b-a)sin2x}{acos^2x+bsin^2x}\, dx$$

$$= \frac{1}{b-a}\int_0^{\frac{\pi}{2}} \frac{\frac{d}{dx}(acos^2x+bsin^2x)}{(acos^2x+bsin^2x)}\, dx$$

$$= \frac{1}{b-a}[ln|acos^2x + bsin^2x|]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{b-a}[ln|b| - ln|a|]$$

$$= \frac{1}{b-a} ln\left|\frac{b}{a}\right|$$

IP4:

Evaluate $\int \frac{dx}{x(1+lnx)^3}$.

Solution:

Given, $\int \frac{dx}{x(1+lnx)^3}$

Put $1 + lnx = t \Rightarrow \frac{dx}{x} = dt$

$$\therefore \int \frac{\left(\frac{1}{x}\right) dx}{(1 + lnx)^3} = \int \frac{1}{t^3} dt = \frac{-2}{t^2} + C = \frac{-2}{(1 + lnx)^3} + C$$

P1:

If $\boldsymbol{y}=\dfrac{(1-2x)^{\frac{2}{3}}(1+3x)^{\frac{-3}{4}}}{(1-6x)^{\frac{5}{6}}(1+7x)^{\frac{-6}{7}}}$ **, then find** $\dfrac{dy}{dx}$**.**

Solution:

Given, $y=\dfrac{(1-2x)^{\frac{2}{3}}(1+3x)^{\frac{-3}{4}}}{(1-6x)^{\frac{5}{6}}(1+7x)^{\frac{-6}{7}}}$

Taking logarithm on both sides, we get

$$ln\,y = ln\frac{(1-2x)^{\frac{2}{3}}(1+3x)^{\frac{-3}{4}}}{(1-6x)^{\frac{5}{6}}(1+7x)^{\frac{-6}{7}}}$$

$$ln\,y = ln\left[(1-2x)^{\frac{2}{3}}(1+3x)^{\frac{-3}{4}}\right] - ln\left[(1-6x)^{\frac{5}{6}}(1+7x)^{\frac{-6}{7}}\right]$$

$$\Rightarrow lny = \frac{2}{3}ln(1-2x) + \left(-\frac{3}{4}\right)ln(1+3x) - \frac{5}{6}ln(1-6x) + \frac{6}{7}ln\,(1+7x)$$

Differentiating on both sides w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{3}\frac{1}{1-2x}(-2) - \frac{3}{4}\frac{1}{1+3x}(3) - \frac{5}{6}\left(\frac{1}{1-6x}\right)(-6) + \frac{6}{7}\frac{1}{1+7x}(7)$$

$$\frac{dy}{dx} = y\left[\frac{-4}{3(1-2x)} - \frac{9}{4(1+3x)} + \frac{5}{1-6x} + \frac{6}{1+7x}\right]$$

$$= \frac{(1-2x)^{\frac{2}{3}}(1+3x)^{\frac{-3}{4}}}{(1-6x)^{\frac{5}{6}}(1+7x)^{\frac{-6}{7}}}\left[\frac{5}{1-6x} + \frac{6}{1+7x} - \frac{4}{3(1-2x)} - \frac{9}{4(1+3x)}\right]$$

P2:

If $xy = (x+y)^n$ and $\frac{dy}{dx} = \frac{y}{x}$, then find the value of n.

Solution:

Given, $xy = (x+y)^n$

Taking logarithm on both sides, we get

$$ln\,x + ln\,y = n\ ln(x+y)$$

Differentiating on both sides w.r.t x, we get

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = n.\frac{1}{x+y}\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \frac{1}{x} - \frac{n}{x+y} = \left[\frac{n}{x+y} - \frac{1}{y}\right]\frac{dy}{dx}$$

$$\Rightarrow \frac{x+y-nx}{x(x+y)} = \frac{ny-x-y}{y(x+y)}.\frac{y}{x} \quad \left(\text{Given}, \frac{dy}{dx} = \frac{y}{x}\right)$$

$$\Rightarrow x + y - nx = ny - x - y$$

$$\Rightarrow 2(x+y) = n(x+y)$$

$$\Rightarrow n = 2$$

P3:

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{\cos t}{1-\sin t}\, dt$ **.**

Solution:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{\cos t}{1-\sin t}\, dt = -\int_{2}^{\frac{1}{2}} \frac{1}{u}\, du \quad \begin{pmatrix} u = 1 - \sin t \Rightarrow du = -\cos t\, dt \\ t = -\frac{\pi}{2} \Rightarrow u = 2, t = \frac{\pi}{6} \Rightarrow u = \frac{1}{2} \end{pmatrix}$$

$$= -[ln|u|]_{2}^{\frac{1}{2}} = -\left[ln\left|\frac{1}{2}\right| - ln|2|\right] = ln\, 4$$

P4:

Evaluate $\int \frac{\sec^2 x \tan x}{\sec^2 x+\tan^2 x} dx$

Solution:

Given, $\int \frac{\sec^2 x \tan x}{\sec^2 x+\tan^2 x} dx$

$$\frac{d}{dx}(\sec^2 x + \tan^2 x) = 2\sec^2 x \tan x + 2\tan x \sec^2 x$$

$$= 4\sec^2 x \tan x$$

$$\therefore \int \frac{\sec^2 x \tan x}{\sec^2 x+\tan^2 x} dx = \frac{1}{4}\int \frac{4\sec^2 x \tan x}{\sec^2 x+\tan^2 x} dx$$

$$= \frac{1}{4}\int \frac{\frac{d}{dx}(\sec^2 x+\tan^2 x)}{\sec^2 x+\tan^2 x} dx$$

$$= \frac{1}{4} ln|\sec^2 x + \tan^2 x| + C$$

4.4. Logarithmic Differentiation and $\int \frac{du}{u}$

Exercises:

I. Use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

a. $y = \sqrt{x(x+1)}$
b. $y = \sqrt{(x^2+1)(x-1)^2}$
c. $y = \sqrt{\frac{t}{t+1}}$
d. $y = \sqrt{\frac{1}{t(t+1)}}$
e. $y = \sqrt{\theta + 3}\sin\theta$
f. $y = \tan\theta\sqrt{2\theta+1}$
g. $y = t(t+1)(t+2)$
h. $y = \frac{1}{t(t+1)(t+2)}$
i. $y = \frac{\theta+5}{\theta\cos\theta}$
j. $y = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}$
k. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{\frac{2}{3}}}$
l. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$
m. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$
n. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

II. Evaluate the integrals.

a. $\int_{-3}^{-2} \frac{dx}{x}$
b. $\int_{-1}^{0} \frac{3dx}{3x-2}$
c. $\int \frac{2y\,dy}{y^2-25}$
d. $\int \frac{8r\,dr}{4r^2-5}$
e. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$
f. $\int_0^{\frac{\pi}{3}} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$
g. $\int_1^2 \frac{2\ln x}{x}\,dx$
h. $\int_2^4 \frac{dx}{x\ln x}$
i. $\int_2^4 \frac{dx}{x(\ln x)^2}$
j. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$
k. $\int \frac{3\sec^2 t}{6+3\tan t}\,dt$
l. $\int \frac{\sec y\tan y}{2+\sec y}\,dy$
m. $\int_0^{\frac{\pi}{2}} \tan\frac{x}{2}\,dx$
n. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot t\,dt$
o. $\int_{\frac{\pi}{2}}^{\pi} 2\cot\frac{\theta}{3}\,d\theta$
p. $\int_0^{\frac{\pi}{12}} 6\tan 3x\,dx$
q. $\int \frac{dx}{2\sqrt{x}+2x}$
r. $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x+\tan x)}}$

4.5

The Exponential Function

Learning objectives:

- To discuss the inverse of $\ln x$
- To define the number e and the exponential function e^x
- To state and prove the laws of exponents.
- To find the derivative and integral of e^x w.r.t x

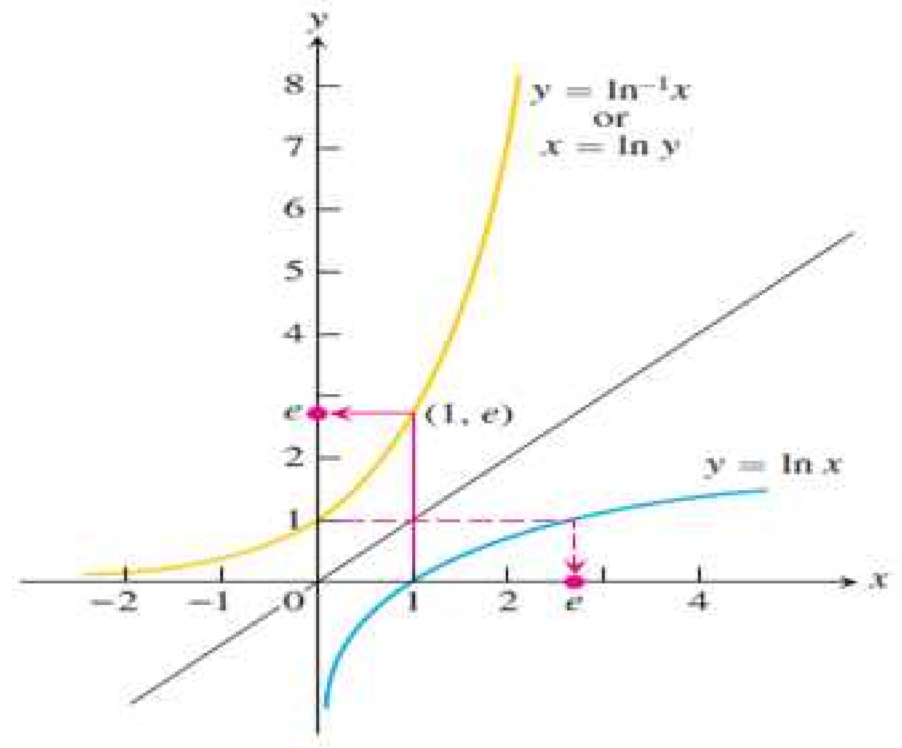
  And
- To practice the related problems.

The Inverse of $\ln x$ and the Number e

The function $\ln x$, being an increasing functions of x with domain $(0,\infty)$ and range $(-\infty,\infty)$, has an inverse $\ln^{-1}x$ with domain $(-\infty,\infty)$, and range $(0,\infty)$. The graph of $\ln^{-1}x$ is the graph of $\ln x$ reflected across the line $y=x$. As seen from the graph,

$$\lim_{x\to\infty} \ln^{-1}x=\infty \qquad \text{and} \qquad \lim_{x\to-\infty} \ln^{-1}x=0$$

The number $\ln^{-1}1$ is denoted by the letter e (figure below).



Definition:

$$e=\ln^{-1}1$$

Although e is not a rational number, we will see that it is possible to find its value with a computer to as many places as we want with the formula

$$e=\lim_{n\to\infty}\left(1+1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n!}\right)$$

To 3 places, e=2.718

The Function $y=e^x$

We can raise the number e to a rational power x in the usual way: $e^2=e\cdot e,\ e^{-2}=\frac{1}{e^2}, e^{1/2}=\sqrt{e}$

and so on. Since e is positive, e^x is positive too. This means that e^x has a logarithm. When we take the logarithm we find that

$$\ln e^x=x\ln e=x\cdot 1=x \qquad \text{... (1)}$$

Since $\ln x$ is one-to-one and $\ln\left(\ln^{-1}x\right)=x$, equation (1) tells us that

$$e^x=\ln^{-1}x \qquad \text{for } x \text{ rational} \qquad \text{.... (2)}$$

Equation (2) provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1}x$ is defined for all x, so we can use it to assign a value to e^x at every point where e^x had no previous value.

Definition: The natural exponential function

For every real number x, $e^x=\ln^{-1}x$

Inverse equations for e^x and $\ln x$

Since $\ln x$ and e^x are inverses of one another, we have

$$e^{\ln x}=x \quad (\text{for all } x>0) \qquad \text{... (3)}$$

$$\ln e^x=x \quad (\text{for all } x) \qquad \text{... (4)}$$

The domain of $\ln x$ is $(0,\infty)$ and its range is $(-\infty,\infty)$. Therefore the domain of e^x is $(-\infty,\infty)$ and its range is $(0,\infty)$

Example 1:

a) $\ln e^2 = 2$

b) $\ln e^{-1} = -1$

c) $\ln \sqrt{e} = \frac{1}{2}$

d) $\ln e^{\sin x} = \sin x$

e) $e^{\ln 2} = 2$

f) $e^{\ln\left(x^2+1\right)} = \left(x^2+1\right)$

g) $e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$

h) $e^{3\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$

The following operational rules are useful.

1. To remove logarithms from an equation, exponentiate both sides.
2. To remove exponentials, take the logarithm of both sides.

Example 2:

Find y if $\ln y = 3t+5$

Solution

Exponentiate both sides:

$$e^{\ln y} = e^{3t+5}$$

$$y = e^{3t+5}$$

Example 3:

Find k if $e^{2k} = 10$

Solution

Take the natural logarithm of both sides:

$$e^{2k} = 10 \Rightarrow \ln e^{2k} = \ln 10 \Rightarrow 2k = \ln 10 \Rightarrow k = \frac{1}{2}\ln 10$$

Laws of Exponents:

Even though e^x is defined as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra.

For all numbers x, x_1, and x_2 we have,

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $\left(e^{x_1}\right)^{x_2} = e^{x_1 x_2} = \left(e^{x_2}\right)^{x_1}$

These laws can be proved using the following pattern.

Let

$$y_1 = e^{x_1} \quad \text{and} \quad y_2 = e^{x_2}$$

$\Rightarrow x_1 = \ln y_1 \quad , x_2 = \ln y_2$ *Take logs of both sides*

$\Rightarrow x_1 + x_2 = \ln y_1 + \ln y_2 = \ln y_1 y_2$ *Product Rule*

$\Rightarrow e^{x_1+x_2} = e^{\ln y_1 y_2}$ *Exponentiate*

$$= y_1 y_2 \; (\because e^{\ln u} = u) = e^{x_1} e^{x_2}$$

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1.

Example 4:

a) $e^{x+\ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ *Law 1*

b) $e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$ *Law 2*

c) $\frac{e^{2x}}{e} = e^{2x-1}$ *Law 3*

d) $\left(e^3\right)^x = e^{3x} = \left(e^x\right)^3$ *Law 4*

The Derivative and Integral of e^x:

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. We calculate its derivative using theorem 1 and our knowledge of the derivative of $ln\ x$

Let $f(x) = ln\ x$ and $y = e^x = ln^{-1} x = f^{-1}(x)$

Then $\frac{df}{dx} = \frac{1}{x}$ and

$$\frac{dy}{dx} = \frac{d\left(e^x\right)}{dx} = \frac{d\left(ln^{-1} x\right)}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right)$$

$$= \frac{1}{\left(\frac{df}{dx}\right)_{x=f^{-1}(x)}} = \frac{1}{\left(\frac{1}{x}\right)_{x=f^{-1}(x)}} = f^{-1}(x) = e^x$$

Thus, for $y = e^x$, we find that $\frac{dy}{dx} = e^x$

$i.e.,$ e^x is its own derivative. We will see later that the only functions that behave this way are constant multiples of e^x. In summary

$$\frac{d}{dx} e^x = e^x \qquad \dots (5)$$

Example 5:

$$\frac{d}{dx}\left(5e^x\right) = 5\frac{d}{dx}\left(e^x\right) = 5e^x$$

The Chain Rule extends equation (5) in the usual way to a more general form.

If u is any differentiable function of x, then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx} \qquad \dots (6)$$

The integral equivalent of equation 6 is

$$\int e^u\, du = e^u + C$$

Example 6:

a)

$$\frac{d}{dx} e^{-x} = e^{-x} \frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$$

(Equation 6 with $u = -x$)

b)

$$\frac{d}{dx} e^{sin x} = e^{sin x} \frac{d}{dx}(sin x) = e^{sin x}(cos\, x)$$ The integral

(Equation 6 with $u = sinx$)

Example 7:

$$\int_0^{ln2} e^{3x}\, dx = \int_0^{ln8} e^u \cdot \frac{1}{3} du \quad (u = 3x,\ du = 3dx,\ u(0) = 0,\ u(ln2) = 3ln2 = ln2^3)$$

$$= \frac{1}{3}\int_0^{ln8} e^u\, du \quad = \frac{1}{3} e^u \Big]_0^{ln8} = \frac{1}{3}[8-1] = \frac{7}{3}$$

Example 8:

$$\int_0^{\pi/2} e^{sinx} cos\, x\, dx \quad (\text{Put } sinx = t \Rightarrow cosx\ dx = dt$$

$$\text{Limits: } x = 0 \Longrightarrow t = 0; x = \frac{\pi}{4} \Longrightarrow t = 1)$$

$$= \int_0^1 e^t\, dt = e^1 - e^0 = e - 1$$

IP1:

Solve for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$ if

$$\boldsymbol{ln(y^2-1)-ln(y+1)=ln\,(sinx)}$$

Solution:

Given $ln(y^2-1)-ln(y+1)=ln\,(sinx)$

$$\Rightarrow ln\left(\frac{y^2-1}{y+1}\right)=ln\,(sinx)$$

$$\Rightarrow ln(y-1)=ln(sinx)$$

$$\Rightarrow y-1=sinx$$ (By exponentiation)

$$\Rightarrow y=sinx+1$$

IP2:

If $\boldsymbol{x + y = ln\ xy}$ then $\boldsymbol{\frac{dy}{dx}}$ =

Solution:

Given $x + y = ln\,xy = ln\,x + ln\,y$

$$\Longrightarrow 1 + \frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx}$$

$$\Longrightarrow 1 - \frac{1}{x} = \frac{dy}{dx}\left(\frac{1}{y} - 1\right)$$

$$\Longrightarrow \frac{x-1}{x} = \left(\frac{1-y}{y}\right)\frac{dy}{dx}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{y(x-1)}{x(1-y)}$$

IP3:

$$\int_{0}^{\sqrt{ln\pi}} \boldsymbol{2x\, e^{x^2}\, cos(e^{x^2})dx} =$$

Solution:

Given

$$\int_{0}^{\sqrt{ln\pi}} 2x\, e^{x^2}\, cos(e^{x^2})dx$$

Put $e^{x^2} = t \Rightarrow 2x\, e^{x^2}\, dx = dt$

Limits: $x = 0 \Rightarrow t = 1;\ x = \sqrt{ln\pi} \Rightarrow t = \pi$

$$\int_{0}^{\sqrt{ln\pi}} 2x\, e^{x^2}\, cos(e^{x^2})dx = \int_{1}^{\pi} cos\, t\, dt$$

$$= [sin\, t\,]_1^\pi = \sin(\pi) - \sin(1) = -\sin(1)$$

IP4:

$$\int_{0}^{\frac{\pi}{4}} \left(1 + e^{tan\theta}\right) sec^2\theta \, d\theta =$$

Solution:

Given

$$\int_{0}^{\frac{\pi}{4}} \left(1 + e^{tan\theta}\right) sec^2\theta \, d\theta =$$

Put $tan\theta = t \Rightarrow sec^2\theta \, d\theta = dt$

Limits: $\theta = 0 \Rightarrow t = 0$ and $\theta = \frac{\pi}{4} \Rightarrow t = 1$

$$\int_{0}^{\frac{\pi}{4}} \left(1 + e^{tan\theta}\right) sec^2\theta \, d\theta = \int_{0}^{1} (1 + e^{t}) \, dt$$

$$= \left[t + e^{t}\right]_0^1$$

$$= (1 + e - 0 - 1) = e$$

P1:

Find the simpler expressions for the quantity $\boldsymbol{ln\ (lne^e)}$

Solution:

Given $y = ln\ (lne^e)$

$$= ln\ (e\ lne)$$

$$= ln(e) = 1 \qquad (\ since\ ln\ e = 1)$$

P2:

If $\boldsymbol{y = ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right)}$ then $\boldsymbol{\frac{dy}{dx} =}$

Solution:

Given $y = ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right)$

$$y = ln\left(\sqrt{1+e^x}-1\right) - ln\left(\sqrt{1+e^x}+1\right)$$

Differentiating on both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+e^x}-1}\frac{d}{dx}\left(\sqrt{1+e^x}-1\right) - \frac{1}{\sqrt{1+e^x}+1}\frac{d}{dx}\left(\sqrt{1+e^x}+1\right)$$

$$= \frac{1}{\sqrt{1+e^x}-1}\cdot\frac{1}{2\sqrt{1+e^x}}\frac{d}{dx}(1+e^x) - \frac{1}{\sqrt{1+e^x}+1}\cdot\frac{1}{2\sqrt{1+e^x}}\frac{d}{dx}(1+e^x)$$

$$= \frac{e^x}{2\sqrt{1+e^x}[\sqrt{1+e^x}-1]} - \frac{e^x}{2\sqrt{1+e^x}[\sqrt{1+e^x}+1]}$$

$$= \frac{e^x}{2\sqrt{1+e^x}}\left[\frac{\sqrt{1+e^x}+1-\sqrt{1+e^x}+1}{(\sqrt{1+e^x}-1)(\sqrt{1+e^x}+1)}\right]$$

$$= \frac{e^x}{2\sqrt{1+e^x}}\left[\frac{2}{1+e^x-1}\right] = \frac{1}{\sqrt{1+e^x}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1+e^x}}$$

P3:

$$\int_{ln\left(\frac{\pi}{6}\right)}^{ln\left(\frac{\pi}{2}\right)} 2e^{y}\, cose^{y}\, dy =$$

Solution:

Given $\int_{ln\left(\frac{\pi}{6}\right)}^{ln\left(\frac{\pi}{2}\right)} 2e^{y}\, cose^{y}\, dy$

Put $e^{y} = t \Rightarrow e^{y}dy = dt$

Limits: $y = \ln\left(\frac{\pi}{6}\right) \Rightarrow t = \frac{\pi}{6}$ and $y = \ln\left(\frac{\pi}{2}\right) \Rightarrow t = \frac{\pi}{2}$

$$\therefore \int_{ln\left(\frac{\pi}{6}\right)}^{ln\left(\frac{\pi}{2}\right)} 2e^{y}\, cose^{y}\, dy = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos t\, dt$$

$$= 2[sint]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left[sin\frac{\pi}{2} - sin\frac{\pi}{6}\right]$$

$$= 2\left[1 - \frac{1}{2}\right] = 2\left(\frac{1}{2}\right) = 1$$

P4:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + e^{cot\theta}) cosec^2\theta \, d\theta =$$

Solution:

Given

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + e^{cot\theta}) cosec^2\theta \, d\theta$$

Put $cot\theta = t \Rightarrow -cosec^2\theta \, d\theta = dt \Rightarrow cosec^2\theta \, d\theta = -dt$

Limits $\theta = \frac{\pi}{4} \Rightarrow t = cot\frac{\pi}{4} = 1$ and $\theta = \frac{\pi}{2} \Rightarrow t = cot\frac{\pi}{2} = 0$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + e^{cot\theta}) cosec^2\theta \, d\theta = -\int_1^0 (1 + e^t) dt = \int_0^1 (1 + e^t) dt$$

$$= [t + e^t]_0^1 = [1 + e - 1] = e$$

4.5. The Exponential Function

Exercise:

1. **Find simpler expressions for the quantities**

a) $e^{\ln 7.2}$

b) $e^{-\ln x^2}$

c) $e^{\ln x - \ln y}$

d) $2\ln\sqrt{e}$

e) $\ln\left(e^{-x^2-y^2}\right)$

II. solve for y in terms of t

a) $\ln y = 2t + 4$

b) $\ln(y-40) = 5t$

c) $\ln(y-1) - \ln 2 = t + \ln t$

III. Solve for y in terms of k

a) $e^{2k} = 4$

b) $100e^{10k} = 200$

c) $e^{k/1000} = a$

IV. Solve for t.

a) $e^{-0.3t} = 27$

b) $e^{kt} = \frac{1}{2}$

c) $e^{(\ln 0.2)t} = 0.4$

d) $e^{\sqrt{t}} = x^2$

V. Find the derivative of y with respect to x, t, or θ, as appropriate.

a) $y = e^{-5x}$

b) $y = e^{5-7x}$

c) $y = xe^x - e^x$

d) $y = \left(x^2 - 2x + 2\right)e^x$

e) $y = e^{\theta}(\sin\theta + \cos\theta)$

f) $y = \cos\left(e^{-\theta^2}\right)$

g) $y = \ln\left(3te^{-t}\right)$

h) $y = \ln\left(\frac{e^{\theta}}{1+e^{\theta}}\right)$

i) $y = e^{(\cos t + \ln t)}$

j) $y = \int_0^{\ln x} \sin e^t \, dt$

VI. Find $\frac{dy}{dx}$ of the following functions:

a) $\ln y = e^y \sin x$

b) $e^{2x} = \sin(x + 3y)$

VII. Evaluate the integrals.

a) $\int\left(e^{3x} + 5e^{-x}\right)dx$

b) $\int_{\ln 2}^{\ln 3} e^x \, dx$

c) $\int 8e^{(x+1)} dx$

d) $\int_{\ln 4}^{\ln 9} e^{x/2} dx$

e) $\int \frac{e^{\sqrt{r}}}{\sqrt{r}} dr$

f) $\int 2te^{-t^2} dt$

g) $\int \frac{e^{1/x}}{x^2} dx$

h) $\int e^{\sec \pi t} \sec \pi t \tan \pi t \, dt$

i) $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v \, dv$

j) $\int \frac{e^r}{1+e^r} dr$

1.6. General Exponential Functions

Learning objectives:

- To define the general exponential functions.
- To derive the power rule of differentiation for any real number.
- To compute the derivative and integration of the general exponential functions.

  AND

- To practice the related problems.

The definition $e^x = ln^{-1} x$ defines e^x for every real value of x, irrational as well as rational. This enables us to raise any other positive number to an arbitrary power and thus to define an exponential function $y = a^x$ for any positive number a. We will also define functions like x^x and $(\sin x)^{\tan x}$ that involve raising the values of one function to powers given by another.

The General Exponential Function a^x

Since $a = e^{ln a}$ for any positive number a, we can think of a^x as $\left(e^{ln a}\right)^x = e^{x ln a}$. We therefore make the following definition.

Definition

For any numbers $a > 0$ and x, the exponential function with base a is

$$\boxed{a^x = e^{x ln a}} \quad \text{..............} \; (1)$$

If $a = e$ the definition gives $a^x = e^{x ln a} = e^{x ln e} = e^{x.1} = e^x$

Example 1

a) $2^{\sqrt{3}} = e^{\sqrt{3} ln 2}$

b) $2^{\pi} = e^{\pi ln 2}$

The function a^x obeys the usual laws of exponents.

For $a > 0$ and any x and y:

1. $a^x \cdot a^y = a^{x+y}$
2. $a^{-x} = \dfrac{1}{a^x}$
3. $\dfrac{a^x}{a^y} = a^{x-y}$
4. $\left(a^x\right)^y = a^{xy} = \left(a^y\right)^x$

The Power Rule

We can now define x^n for any $x > 0$ and any real number n as $x^n = e^{n ln x}$. Therefore, the n in the equation $ln x^n = n ln x$ no longer needs to be rational – it can be any number as long as $x > 0$.

$$ln x^n = ln\left(e^{n ln x}\right) = n ln x \cdot ln e = n ln x$$

Together, the law $a^x / a^y = a^{x-y}$ and the definition $x^n = e^{n ln x}$ enable us to establish the Power Rule for differentiation. Differentiating x^n with respect to x gives

$$\frac{d}{dx} x^n = \frac{d}{dx} e^{n ln x} = e^{n ln x} \cdot \frac{d}{dx}(n ln x) = x^n \cdot \frac{n}{x} = n x^{n-1}$$

In short, as long as $x > 0$,

$$\frac{d}{dx} x^n = n x^{n-1}$$

The Chain Rule extends this equation to the Power Rule.

Power Rule (General form):

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\boxed{\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}}$$

Example 2

a) $\dfrac{d}{dx} x^{\sqrt{2}} = \sqrt{2} x^{\sqrt{2}-1} \quad (x > 0)$

b) $\dfrac{d}{dx} (\sin x)^{\pi} = \pi (\sin x)^{\pi-1} \cos x \qquad (\sin x > 0)$

The Derivative of a^x

We use the definition $a^x = e^{x ln a}$,

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x ln a} = e^{x ln a} \cdot \frac{d}{dx}(x ln a) = a^x ln a$$

$$\therefore \; \boxed{\frac{d}{dx}\left(a^x\right) = a^x ln a}$$

With the Chain Rule, we get a more general form.

➢ If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u ln a \frac{du}{dx} \quad \text{.........} \; (2)$$

If $a = e$, then $ln a = 1$ and equation (2) simplifies to

$$\frac{d}{dx} e^x = e^x ln e = e^x$$

This shows why e^x, is the exponential function preferred in calculus.

Example 3

a) $\dfrac{d}{dx} 3^x = 3^x ln 3$

b) $\dfrac{d}{dx} 3^{-x} = 3^{-x} ln 3 \dfrac{d}{dx}(-x) = -3^{-x} ln 3$

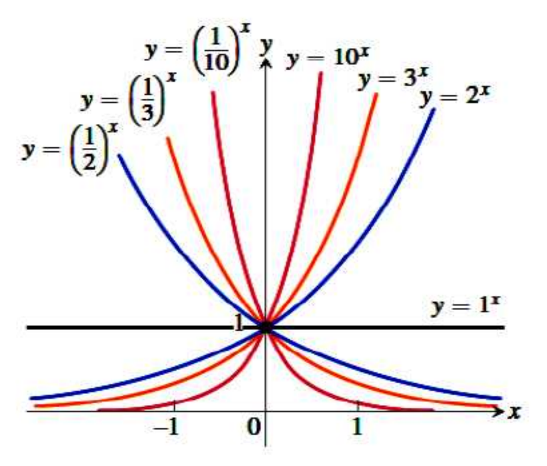
c) $\dfrac{d}{dx} 3^{\sin x} = 3^{\sin x} ln 3 \dfrac{d}{dx}(\sin x) = 3^{\sin x} (ln 3) \cos x$

From equation (2), we see that the derivative of a^x is positive if $ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus,

a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}\left(a^x\right) = \frac{d}{dx}\left(a^x ln a\right) = (ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line.



Note:

As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and

$a^x \to \infty$ if $a > 1$

As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and

$a^x \to 0$ if $a > 1$

Other Power Functions

The ability to raise positive numbers to arbitrary real powers makes it possible to define functions like x^x and $x^{ln x}$ for $x > 0$. We find the derivatives of such functions by rewriting the functions as powers of e.

Example 4

Find $\dfrac{dy}{dx}$ if $y = x^x$, $x > 0$.

Solution

Write x^x as power of e.

$$y = x^x = e^{x ln x} \qquad \text{equation (1) with } a = x$$

Then differentiate as usual:

$$\frac{dy}{dx} = \frac{d}{dx} e^{x ln x} = e^{x ln x} \frac{d}{dx}(x ln x)$$

$$= x^x \left(x \cdot \frac{1}{x} + ln x\right) = x^x (1 + ln x)$$

The Integral of a^u

If $a \neq 1$, so that $ln a \neq 0$, we can divide both sides of equation (2) by $ln a$ to obtain

$$a^u \frac{du}{dx} = \frac{1}{ln a} \frac{d}{dx}\left(a^u\right)$$

Integrating with respect to x then gives

$$\int a^u \frac{du}{dx} dx = \int \frac{1}{ln a} \frac{d}{dx}\left(a^u\right) dx = \frac{1}{ln a} \int \frac{d}{dx}\left(a^u\right) dx = \frac{1}{ln a} a^u + C$$

Writing the first integral in differential form gives

$$\int a^u \, du = \frac{a^u}{ln a} + C \quad \text{.........} \; (3)$$

Example 5

a. $\displaystyle\int 2^x \, dx = \frac{2^x}{ln 2} + C$

b. $\displaystyle\int 2^{\sin x} \cos x \, dx$

$$= \int 2^u \, du = \frac{2^u}{ln 2} + C$$

$$= \frac{2^{\sin x}}{ln 2} + C$$

IP1.

Solve the equation $\boldsymbol{a^{3-x} . b^{5x} = a^{3x} . b^{x+5}}$ for $\boldsymbol{x}$

Solution:

Given $a^{3-x} . b^{5x} = a^{3x} . b^{x+5}$

$$\Longrightarrow \frac{b^{5x}}{b^{x+5}} = \frac{a^{3x}}{a^{3-x}}$$

$$\Longrightarrow lnb^{4x-5} = lna^{4x-3}$$

$$\Longrightarrow (4x-5)lnb = (4x-3)loga$$

$$\Longrightarrow 4x\ lnb - 4x\ lna = 5\ lnb - 3lna$$

$$\Longrightarrow x[4\ lnb - 4lna] = 5\ lnb - 3\ lna$$

$$\Longrightarrow x = \frac{5\ lnb - 3\ lna}{4(lnb - lna)}$$

IP2.

If $x^y = e^{x-y}$ then show that $\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2}$

Solution:

Given $x^y = e^{x-y}$

Taking logarithms on both sides, we get

$$y \ln x = x - y$$

That is, $y = \frac{x}{1+\ln x}$

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \frac{(1+\ln x).1 - x.\frac{1}{x}}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2}$$

IP3.

If $y = (tanx)^{(tanx)^{(tanx)}}$ then find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$

Solution:

Given $y = (tanx)^{(tanx)^{(tanx)}}$

Taking logarithms on both sides, we get

$ln\, y = (tanx)^{(tanx)}\, ln\, tanx$ .... (1)

Again taking logarithms on both sides, we get

$ln(ln\, y) = tanx(ln\, tanx) + ln(ln\, tanx)$

Differentiating both sides w.r.t x, we get

$$\frac{1}{ln\, y}\cdot\frac{1}{y}\frac{dy}{dx} = sec^2x\, ln\, tanx + tanx.\frac{sec^2x}{tanx} + \frac{1}{ln\, tanx}\cdot\frac{1}{tanx}.\, sec^2x$$

$$\frac{dy}{dx} = y\, ln\, y\, .\, sec^2x\left[ln\, tanx + 1 + \frac{1}{(tanx.ln\, tanx)}\right]$$

$ln(ln\, y) = tanx(ln\, tanx) + ln(ln\, tanx)$

Differentiating both sides w.r.t x, we get

$$\frac{1}{ln\, y}\cdot\frac{1}{y}\frac{dy}{dx} = sec^2x\, ln\, tanx + tanx.\frac{sec^2x}{tanx} + \frac{1}{ln\, tanx}\cdot\frac{1}{tanx}.\, sec^2x$$

$$\frac{dy}{dx} = y\, ln\, y\, .\, sec^2x\left[ln\, tanx + 1 + \frac{1}{(tanx.ln\, tanx)}\right]$$

$$\frac{dy}{dx} =$$

$$y(tanx)^{(tanx)}\, ln\, tanx\, .\, sec^2x[ln\, tanx\, (ln\, tanx + 1) + cotx]$$

Now, at $x = \frac{\pi}{4} \Longrightarrow y = 1$ and $ln\, tan\left(\frac{\pi}{4}\right) = ln\, 1 = 0$

$$\therefore \frac{dy}{dx} = 1.1.2[0 + 1] = 2$$

IP4.

Evaluate $\int_{1}^{2} x^{2x}(1+\ln x)\, dx$

Solution:

To evaluate $\int_{1}^{2} x^{2x}(1+\ln x)\, dx$

Put $u = x^{2x}$

Taking logarithms on both sides, we get

$$\ln u = 2x \ln x$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = 2 \ln x + 2x \cdot \frac{1}{x}$$

$$\Rightarrow \frac{du}{2} = x^{2x}(\ln x + 1)\, dx$$

Limits: $= 1 \Rightarrow u = 1\,; \quad x = 2 \Rightarrow u = 2^4 = 16$

$$\therefore \int_{1}^{2} x^{2x}(1+\ln x)\, dx = \frac{1}{2}\int_{1}^{16} du$$

$$= \frac{1}{2}[u]_1^{16} = \frac{15}{2}$$

P1.

Solve for x: $\left(\frac{1}{2}\right)^x . \mathbf{4^{x+1} = 8^{-2x+1}}$

Solution:

Given $\left(\frac{1}{2}\right)^{x}.4^{x+1} = 8^{-2x+1}$

$$\Rightarrow 2^{-x}.2^{2(x+1)} = 2^{3(-2x+1)}$$

$$\Rightarrow 2^{-x}.2^{(2x+2)} = 2^{(-6x+3)}$$

$$\Rightarrow 2^{-x+2x+2} = 2^{-6x+3}$$

$$\Rightarrow 2^{x+2} = 2^{-6x+3}$$

$$\Rightarrow x + 2 = -6x + 3$$

$$\Rightarrow 7x = 1 \Rightarrow x = \frac{1}{7}$$

P2.

If $a^x + a^y = a^{x+y}$ then find $\frac{dy}{dx}$

Solution:

Given $a^x + a^y = a^{x+y}$

Differentiating both sides w.r.t x, we get

$$a^x \ln a + a^y \ln a \, .\frac{dy}{dx} = a^{x+y} \ln a \, \frac{d}{dx}(x+y)$$

$$a^x + a^y \, .\frac{dy}{dx} = a^{x+y} \left[1 + \frac{dy}{dx}\right]$$

$$a^x - a^{x+y} = [a^{x+y} - a^y\,]\frac{dy}{dx}$$

$$a^x - a^x . a^y = [a^x . a^y - a^y\,]\frac{dy}{dx}$$

$$a^x(1 - a^y) = a^y[a^x - 1\,]\frac{dy}{dx}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{a^x(a^y - 1)}{a^y(1 - a^x)}$$

P3.

If $y = lnx^{lnx}$ then find $\frac{dy}{dx}$

Solution:

Given $y = (lnx)^{lnx}$

Taking logarithms on both sides, we get

$$lny = lnx.\,ln\,(lnx)$$

Differentiating both sides w.r.t x, we get

$$\frac{1}{y}\,\frac{dy}{dx} = \frac{1}{x}ln(lnx) + lnx.\frac{1}{lnx}\frac{d}{dx}(lnx)$$

$$\frac{dy}{dx} = y\left[\frac{ln(lnx)}{x} + \frac{1}{x}\right] = (lnx)^{lnx}\left[\frac{ln(lnx)+1}{x}\right]$$

P4.

Evaluate the $\displaystyle\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t \, dt$

Solution:

To evaluate $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t \, dt$

Put $u = tant \Longrightarrow du = sec^2 t \;\; dt$

Limits: $t = 0 \Longrightarrow u = 0 \quad ; \quad t = \frac{\pi}{4} \Longrightarrow u = 1$

Now,

$$\int_0^{\pi/4} \left(\frac{1}{3}\right)^{tant} sec^2 t \, dt = \int_0^1 \left(\frac{1}{3}\right)^u du$$

$$= \left[\frac{(1/3)^u}{ln(1/3)}\right]_0^1 = -\frac{1}{ln(1/3)}\left[\left(\frac{1}{3}\right)^1 - \left(\frac{1}{3}\right)^0\right]$$

$$= \frac{2}{3ln3}$$

Now,

$$\int_0^{\pi/4} \left(\frac{1}{3}\right)^{tant} sec^2 t \, dt = \int_0^1 \left(\frac{1}{3}\right)^u du$$

$$= \left[\frac{(1/3)^u}{ln(1/3)}\right]_0^1 = -\frac{1}{ln(1/3)}\left[\left(\frac{1}{3}\right)^1 - \left(\frac{1}{3}\right)^0\right]$$

$$= \frac{2}{3ln3}$$

$$\therefore \int_0^{\pi/4} \left(\frac{1}{3}\right)^{tant} sec^2 t \, dt = \frac{2}{3ln3}$$

1. Find the derivative of y with respect to the given independent variable.

a. $y = 2^{x}$

b. $y = 5^{\sqrt{s}}$

c. $y = x^{\pi}$

d. $y = (\cos\theta)^{\sqrt{2}}$

e. $y = 7^{sec\theta}\, ln 7$

f. $y = 2^{\sin 3t}$

2. Use logarithmic differentiation to find the derivative of y w.r.t the given independent variable.

a) $y = (x+1)^x$
b) $y = x^{(x+1)}$
c) $y = \left(\sqrt{t}\right)^t$
d) $y = t^{\sqrt{t}}$
e) $y = (sinx)^x$
f) $y = x^{sinx}$
g) $y = x^{lnx}$

3. Evaluate the integrals given below:

a. $\int 5^x \, dx$

b. $\int (1.3)^x \, dx$

c. $\int_0^1 2^{-\theta} \, d\theta$

d. $\int_1^{\sqrt{2}} x2^{\left(x^2\right)} \, d\theta$

e. $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}} \, dx$

f. $\int_{-2}^0 5^{-\theta} \, d\theta$

g. $\int_1^{\sqrt{2}} x.2^{\left(x^2\right)} \, dx$

h. $\int_0^{\pi/2} 7^{\cos t} \sin t \, dt$

i. $\int_1^2 \frac{2^{\ln x}}{x} \, dx$

1.7
General Logarithmic Functions

Learning objectives

- To define the general logarthmic function.
- To derive the Derivative of $\log_a u$.
- To evaluate integrals involving $\log_a x$.
- To study the uses of common logarithm$(\log_{10} x)$.
  And
- To practice the related problems.

Logarithms with Base a:

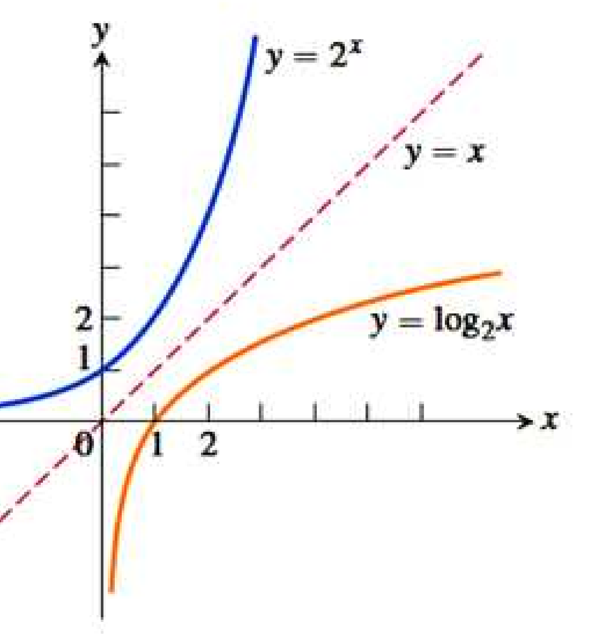
As we saw earlier, if a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the ***logarithm of x with base a*** and denote it by $\log_a x$.

Definition

For any positive number $a \neq 1$,

$$\log_a x = \text{inverse of } a^x$$

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$.



Since $\log_a x$ and a^x are inverses of one another, composing them in either order gives the identity function.

$$a^{\log_a x} = x \qquad (x > 0) \qquad \text{------}(1)$$

$$\log_a a^x = x \qquad (all\, x) \qquad \text{-------}(2)$$

Example 1:

a) $\log_2 2^5 = 5$
b) $\log_{10}(10^{-7}) = -7$
c) $2^{\log_2 3} = 3$
d) $10^{\log_{10} 4} = 4$

The Evaluation of $\log_a x$

The evaluation of $\log_a x$ is simplified by the observation that $\log_a x$ is a numerical multiple of $ln\, x$.

$$\log_a x = \frac{1}{ln\, a} \cdot ln\, x = \frac{ln\, x}{ln\, a} \qquad \text{------}(3)$$

We can derive equation (3) from equation (1):

$$a^{\log_a x} = x \qquad \Rightarrow \ln a^{\log_a x} = ln\, x$$

$$\Rightarrow \log_a x \cdot ln\, a = ln\, x \quad \Rightarrow \log_a x = \frac{ln\, x}{ln\, a}$$

Example 2:

$$\log_2 10 = \frac{ln\, 2}{ln\, 10} \approx \frac{0.69315}{2.30259} \approx 0.30103$$

The arithmetic properties of $\log_a x$ are the same as the ones for $ln\, x$.

For any numbers $x > 0$ and $y > 0$,

1. Product Rule : $\log_a xy = \log_a x + \log_a y$
2. Reciprocal Rule: $\log_a \frac{1}{y} = -\log_a y$
3. Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$
4. Power Rule: $\log_a x^y = y \log_a x$

These rules can be proved by dividing the corresponding rules for the natural logarithm function by $ln\, a$.

The Derivative of $\log_a u$

To find the derivative of a base a logarithm, we first convert it to a natural logarithm. If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{ln\, u}{ln\, a}\right) = \frac{1}{ln\, a}\frac{d}{dx}(ln\, u) = \frac{1}{ln\, a} \cdot \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{ln\, a} \cdot \frac{1}{u}\frac{du}{dx} \qquad \text{---------}(4)$$

Example 3:

$$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{ln\, 10} \cdot \frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(ln\, 10)(3x+1)}$$

Integrals involving $\log_a x$

To evaluate integrals involving base a logarithms, we convert them to natural logarithms.

Example 4:

$$\int \frac{\log_2 x}{x}\, dx = \frac{1}{ln\, 2}\int \frac{ln\, x}{x}\, dx$$
$$= \frac{1}{ln\, 2}\int u\, du \text{ , where } u = ln\, x$$
$$= \frac{1}{ln\, 2}\frac{u^2}{2} + C = \frac{1}{ln\, 2}\frac{(ln\, x)^2}{2} + C = \frac{(ln\, x)^2}{2\, ln\, 2} + C$$

Base 10 Logarithms

Base 10 logarithms, often called *common logarithms*, appear in many scientific formulas. For example, earthquake intensity if often reported on the logarithmic ***Richter scale***. Here the formula for magnitude R is

$$R = \log_{10}\left(\frac{a}{T}\right) + B$$

where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that allows for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

Example 5:

For an earthquake 10,000 km from the receiving station, $B = 6.8$. If the recorded vertical ground motion is $a = 10$ microns and the period is $T = 1\ s$, the earthquake's magnitude is

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 1 + 1.68 = 7.8$$

An earthquake of this magnitude does great damage near its epicenter.

The **pH scale** for measuring the acidity of a solution is a base 10 logarithmic scale. The pH value (hydrogen potential) of the solution is the common logarithm of the reciprocal of the solution's hydronium ion concentration, $[H_3O^+]$.

$$\text{pH} = \log_{10}\frac{1}{[H_3O^+]} = -\log_{10}[H_3O^+]$$

The hydronium ion concentration is measured in mole per liter. The total scale ranges from about 0.1 for normal hydrochloric acid to 14 for a normal $(1\ N)$ solution of sodium hydroxide.

Another example of the use of common logarithm is the ***decibel*** or db scale for measuring loudness. If I is the ***intensity*** of sound in watts per square meter, the decibel level of the sound is

$$sound\ level = 10\log_{10}(I \times 10^{12})\ db \qquad \text{-------}(5)$$

Example 6:

Doubling I in equation (5) adds about 3 db. Writing log for $\log_{10}$ (a common practice), we have

$$\text{sound level with } I \text{ doubled} = 10\log(2I \times 10^{12})$$
$$= 10\log(2 \cdot I \times 10^{12})$$
$$= 10\log 2 + 10\log(I \times 10^{12})$$
$$= \text{original sound level} + 10\log 2$$
$$\approx \text{original sound level} + 3 \quad (\log_{10} 2 \approx 0.30)$$

IP1:

If $\boldsymbol{y} = \ \log_3\left(\left(\frac{(x+1)}{(x-1)}\right)^{\ln 3}\right)$,then $\frac{dy}{dx} =$

Solution:

Given $y = \ \log_3\left(\left(\frac{(x+1)}{(x-1)}\right)^{\ln 3}\right)$

$$\Rightarrow y = \frac{\ln\left[\left(\frac{x+1}{x-1}\right)^{\ln 3}\right]}{\ln 3} = \frac{\ln 3 \cdot \ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln(x+1) - \ln(x-1)$$

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)} = \frac{-2}{x^2-1}$$

IP2:

If $\boldsymbol{y = \log_2\left(\frac{x^2\, e^2}{2\sqrt{x+1}}\right)}$, then $\boldsymbol{\frac{dy}{dx} =}$

Solution:

Given $y = \log_2\left(\frac{x^2\, e^2}{2\sqrt{x+1}}\right)$

$$= \frac{ln\left(\frac{(x^2\, e^2)}{2\sqrt{x+1}}\right)}{ln 2} = \frac{ln\, x^2 + ln\, e^2 - ln\, 2 - ln\sqrt{x+1}}{ln\, 2}$$

$$= \frac{2\, ln\, x + 2 - ln\, 2 - \frac{1}{2} ln(x+1)}{ln\, 2}$$

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{ln\, 2}\left[\frac{2}{x} - \frac{1}{2} \cdot \frac{1}{x+1}\right]$$

$$= \frac{1}{ln\, 2}\left[\frac{4x+4-x}{2x(x+1)}\right]$$

$$= \frac{1}{ln\, 2}\left[\frac{3x+4}{2x(x+1)}\right] = \frac{3x+4}{2x(x+1)\, ln\, 2}$$

IP3:

$$\int_{1}^{4} \frac{\log_2 x}{x}\, dx =$$

Solution:

$$\int_{1}^{4} \frac{\log_2 x}{x}\, dx = \int_{1}^{4} \left(\frac{ln\, x}{\ln 2}\right)\left(\frac{1}{x}\right) dx$$

Let $u = ln\, x \implies du = \frac{1}{x}\, dx$

$$x = 1 \implies u = 0; \; x = 2 \implies u = ln\, 4$$

$$\int_{1}^{4} \left(\frac{ln\, x}{\ln 2}\right)\left(\frac{1}{x}\right) dx = \int_{0}^{ln\, 4} \left(\frac{1}{ln\, 2}\right) u\, du$$

$$= \left(\frac{1}{ln\, 2}\right)\left[\frac{u^2}{2}\right]_0^{ln\, 4} = \frac{(ln\, 4)^2}{2\, ln\, 2} = \frac{(ln\, 4)^2}{ln\, 4} = ln\, 4$$

$$\therefore \int_{1}^{4} \frac{\log_2 x}{x}\, dx = ln\, 4$$

IP4:

$$\int_{2}^{3} \frac{2\log_2(x-1)}{x-1}\,dx =$$

Solution:

$$\int_{2}^{3} \frac{2\log_2(x-1)}{x-1}\,dx = \frac{2}{ln\,2}\int_{2}^{3} \frac{ln(x-1)}{x-1}\,dx$$

$$= \frac{2}{ln\,2}\left[\frac{(ln(x-1))^2}{2}\right]_2^3$$

$$= \frac{1}{ln\,2}\left[(ln\,2)^2 - (ln\,1)^2\right] = ln\,2$$

$$\therefore \int_{2}^{3} \frac{2\log_2(x-1)}{x-1}\,dx = ln\,2$$

P1:

If $y = 3\log_8(\log_2 t)$, then find $\frac{dy}{dt}$.

Solution:

Given $y = 3\log_8(\log_2 t) = \frac{3\,ln\,(\log_2 t)}{ln\,8} = \frac{3\,ln\left(\frac{ln\,t}{ln\,2}\right)}{ln\,8}$

Differentiating both sides w.r.t t, we get

$$\frac{dy}{dt} = \frac{3}{\ln 8} \cdot \left(\frac{1}{\left(\frac{ln\,t}{ln\,2}\right)}\right) \cdot \frac{d}{dt}\left(\frac{ln\,t}{ln\,2}\right) = \frac{3\ln 2}{(3\ln 2)(\ln t)}\left(\frac{1}{t\,ln\,2}\right)$$

$$\frac{dy}{dt} = \frac{1}{t(\ln t)(\ln 2)}$$

P2:

If $y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}$ then $\frac{dy}{dx} =$

Solution:

$$y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \frac{\ln\left(\frac{7x}{3x+2}\right)^{\frac{\ln 5}{2}}}{\ln 5}$$

$$= \frac{\ln 5 \ \ln\left(\frac{7x}{3x+2}\right)}{2 \ \ln 5} = \frac{1}{2}\left[\ln 7x - \ln(3x+2)\right]$$

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{7x}\ \frac{d}{dx}(7x) - \frac{1}{3x+2} \cdot \frac{d}{dx}(3x+2)\right]$$

$$= \frac{1}{2}\left[\frac{1}{7x}(7) - \frac{1}{3x+2}(3)\right]$$

$$= \frac{1}{2}\left[\frac{1}{x} - \frac{3}{3x+2}\right]$$

$$= \frac{1}{2}\left[\frac{3x+2-3x}{x(3x+2)}\right] = \frac{1}{2}\left[\frac{2}{x(3x+2)}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x(3x+2)}$$

P3:

$$\int \frac{dx}{x \log_{10} x} =$$

Solution:

$$\int \frac{dx}{x\log_{10}x} = \int \frac{1}{x\left(\frac{ln\,x}{ln\,10}\right)}\,dx$$

$$= \int \left(\frac{ln\,10}{ln\,x}\right)\left(\frac{1}{x}\right)dx$$

let $u = ln\,x \implies du = \frac{1}{x}\,dx$

$$\therefore \int \left(\frac{ln\,10}{ln\,x}\right)\left(\frac{1}{x}\right)dx = ln\,10\int \frac{1}{u}\,du$$

$$= ln\,10\;ln|u| + C$$

$$= ln\,10\;ln|ln\,x| + C$$

P4:

$$\int_{1}^{e} \frac{2\, ln\, 10\, \log_{10} x}{x}\, dx =$$

Solution:

$$\int_1^e \frac{2\ ln\, 10\ \ \log_{10} x}{x}\, dx = \int_1^e \frac{2\ ln\, 10\ ln\, x}{ln\, 10}\left(\frac{1}{x}\right) dx$$

$$= 2\int_1^e \frac{ln\, x}{x}\, dx = 2\left[\frac{(ln\, x)^2}{2}\right]_1^e$$

$$= (ln\, e)^2 - (ln\, 1)^2 = 1$$

Exercises:

1. Find the derivative of y with respect to the given independent variable.

a. $y = \log_2 5\theta$

b. $y = \log_3(1 + \theta\, ln\, 3)$

c. $y = \log_{25} e^{x} - \log_5 \sqrt{x}$

d. $y = \log_4 x + \log_4 x^2$

e. $y = \log_3 r \cdot \log_9 r$

f. $y = \log_2 r \cdot \log_4 r$

g. $y = \theta \sin(\log_7 \theta)$

h. $y = \log_5 e^{x}$

i. $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right)$

j. $y = 3^{\log_2 t}$

k. $y = \log_2\left(8t^{\ln 2}\right)$

l. $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

2. Evaluate the integrals.

a. $\int \frac{\log_{10} x}{x} dx$

b. $\int_1^4 \frac{\ln 2 \log_2 x}{x} dx$

c. $\int_0^2 \frac{\log_2 (x+2)}{x+2} dx$

d. $\int_{\frac{1}{10}}^{10} \frac{\log_{10} (10x)}{x} dx$

e. $\int_0^9 \frac{2 \log_{10} (x+1)}{x+1} dx$

f. $\int \frac{dx}{x(\log_8 x)^2}$

g. $\int_1^8 \frac{\log_4 \theta}{\theta} d\theta$

h. $\int_1^e \frac{8 \ln 3 \log_3 \theta}{\theta} d\theta$

1.8
L'Hôpital's Rule

Learning Objectives:

- To discuss the indeterminate form $\frac{0}{0}$ and its evaluation by L'Hôpital's Rule
- To state and prove Cauchy's Mean Value Theorem and apply it to prove the stronger form of L'Hôpital's Rule
- To discuss the other indeterminate forms and their evaluation by L'Hôpital's Rule

AND

- To practice the related problems

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ cannot be found by substituting $x = a$. The substitution produces $\frac{0}{0}$, an expression known as an ***indeterminate form***. L'Hôpital's rule enables us to evaluate the limits that otherwise lead to indeterminate forms.

Theorem: L'Hôpital's Rule (First form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \quad \ldots\ldots\ldots\ldots (1)$$

Proof

$$\frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$
$$= \lim_{x\to a}\frac{f(x)-0}{g(x)-0} = \lim_{x\to a}\frac{f(x)}{g(x)}$$

Example 1

a) $$\lim_{x\to 0}\frac{3x-\sin x}{x} = \left.\frac{3-\cos x}{1}\right|_{x=0} = 2$$

b) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \left.\frac{\frac{1}{2\sqrt{1+x}}}{1}\right|_{x=0} = \frac{1}{2}$$

Some times after differentiation, the new numerator and denominator both equal to zero at $x = a$. In this case, we apply a stronger form of L'Hôpital's Rule.

The proof of the stronger form of L'Hôpital's Rule is based on Cauchy's Mean Value theorem, a Mean Value Theorem that involves two functions instead of one.

Theorem: Cauchy's Mean Value Theorem

If the functions f and g are continuous on $[a, b]$, differentiable throughout (a, b) and $g'(x) \neq 0$ throughout (a, b), then there exists a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof: We apply Lagrange's Mean Value Theorem twice

(i) First we use it to show that $g(a) \neq g(b)$. If $g(a) = g(b)$, then by Lagrange's Mean Value Theorem
$$g'(c) = \frac{g(b)-g(a)}{b-a} = 0$$
for some $c \in (a, b)$, which is not possible, since $g'(x) \neq 0$ in (a, b).

(ii) We next apply Mean Value Theorem to the function
$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$$
Notice that F is continuous on $[a, b]$, differentiable on (a, b) and $F(b) = 0 = F(a)$. Therefore, by Rolle's theorem, there exists a $c \in (a, b)$ such that $F'(c) = 0$, i.e., $F'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(c) = 0$
$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Hence the theorem.

Note: Lagrange's Mean Value Theorem is a special case of Cauchy's Mean Value Theorem.

If $g(x) = x$, then Cauchy's Mean Value Theorem reduces to

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Theorem: L'Hôpital's Rule (Stronger form)

If f and g are differentiable on an open interval I containing a , $f(a) = g(a) = 0$ and $g'(x) \neq 0$ on I when $x \neq a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

assuming that the limit on the right hand side exists.

Proof: We first establish the limit equation for the case $x \to a^+$.

Suppose that x lies to the right of a . Then $g'(x) \neq 0$ and we can apply Cauchy Mean Value Theorem to the closed interval $[a, x]$. Thus there exists $c \in (a, x)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)}{g(x)} \quad \text{(Since } f(a) = g(a) = 0\text{)}$$

As x approaches a, c approaches a, since it lies between a and x. Therefore

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{c\to a^+}\frac{f'(c)}{g'(c)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

Thus, L'Hôpital's Rule is proved for the case when x approaches a from above.

The case when x approaches a from below can be proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a]$, $x < a$. The combination of these two cases now establishes the result.

Hence the theorem.

Example:

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \left.\frac{1-\cos x}{3x^2}\right|_{x=0} = ? \qquad \text{Still } \frac{0}{0}$$

This example can be solved by using a stronger form of L'Hôpital's Rule which says that whenever the rule gives $\frac{0}{0}$ we can apply it again, repeating the process until we get a different result. With this stronger rule we get

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} \qquad \text{Still } \frac{0}{0}\text{; apply the rule again}$$
$$= \lim_{x\to 0}\frac{\sin x}{6x} \qquad \text{Still } \frac{0}{0}\text{; apply the rule again}$$
$$= \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \frac{0}{0}\text{; limit is found}$$

Example 2

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-(x/2)}{x^2} \qquad \frac{0}{0}$$
$$= \lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-(1/2)}{2x} \qquad \frac{0}{0}$$
$$= \lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2} \qquad \text{Not } \frac{0}{0}$$
$$= -\frac{1}{8}$$

When you apply L'Hôpital's Rule, look for a change from $\frac{0}{0}$ to something else. This is where the limit is revealed.

Example 3

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} \qquad \frac{0}{0}$$
$$= \lim_{x\to 0}\frac{\sin x}{1+2x} \qquad \text{Not } \frac{0}{0}$$
$$= \frac{0}{1} = 0$$

If we continue to differentiate in an attempt to apply L'Hôpital's Rule once more, we get

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} = \lim_{x\to 0}\frac{\sin x}{1+2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$$

which is wrong.

L'Hospital's rule applies to one-sided limits also.

Example 4

$$\lim_{x\to 0^+}\frac{\sin x}{x^2} \qquad \frac{0}{0}$$
$$= \lim_{x\to 0^+}\frac{\cos x}{2x} = \infty \qquad \text{Not } \frac{0}{0}$$

L'Hôpital's Rule also applies to quotients that lead to the indeterminate form $\frac{\infty}{\infty}$. If $f(x)$ and $g(x)$ both approach infinity as $x \to a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. The a here may itself be either finite or infinite.

Example 5

a) $$\lim_{x\to(\pi/2)^-}\frac{\sec x}{1+\tan x} \qquad \frac{\infty}{\infty}$$
$$= \lim_{x\to(\pi/2)^-}\frac{\sec x\tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-}\sin x = 1$$

b) $$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$$

Indeterminate Products and Differences

We can sometimes handle the indeterminate forms $0\cdot\infty$ and $\infty-\infty$ by using algebra to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ instead. It must be remembered that all these forms are not some numbers but only notations for functional behaviors when considering limits.

Example 6

$$\lim_{x\to 0^+} x\cot x \qquad 0\cdot\infty$$
$$= \lim_{x\to 0^+} x\cdot\frac{1}{\tan x}$$
$$= \lim_{x\to 0^+}\frac{x}{\tan x} \qquad \text{Now } \frac{0}{0}$$
$$= \lim_{x\to 0^+}\frac{1}{\sec^2 x} = \frac{1}{1} = 1$$

Example 7

Find $\lim\limits_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

Solution

If $x\to 0^+$, then $\sin x\to 0^+$ and $\frac{1}{\sin x}-\frac{1}{x}\to\infty-\infty$

If $x\to 0^-$, then $\sin x\to 0^-$ and $\frac{1}{\sin x}-\frac{1}{x}\to-\infty-(-\infty) = -\infty+\infty$

Neither form reveals what happens in the limit. To find out, we first combine the fractions.

$$\frac{1}{\sin x}-\frac{1}{x} = \frac{x-\sin x}{x\sin x}$$

and then apply L'Hôpital's Rule to the result:

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} \qquad \frac{0}{0}$$
$$= \lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x} \qquad \frac{0}{0}$$
$$= \lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x} = \frac{0}{2} = 0$$

Indeterminate Powers

Limits that lead to the indeterminate forms 1^∞, 0^0 and ∞^0 can sometimes be handled by taking logarithms first. We use L'Hôpital's Rule to find the limit of the logarithm and then exponentiate to find the original function behavior.

If $\lim\limits_{x\to a}\ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L$$

Here a may be either finite or infinite.

Example 8

Show that $\lim\limits_{x\to 0^+}(1+x)^{1/x} = e$

Solution

The limit leads to the indeterminate form 1^∞. We let $f(x) = (1+x)^{1/x}$ and find $\lim\limits_{x\to 0^+}\ln f(x)$. Since

$$\ln f(x) = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x),$$

L'Hôpital's Rule gives

$$\lim_{x\to 0^+}\ln f(x) = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} \qquad \frac{0}{0}$$
$$= \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1$$

Therefore,

$$\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e$$

Example 9

Find $\lim\limits_{x\to\infty} x^{1/x}$.

Solution

The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim\limits_{x\to 0^+}\ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

L'Hôpital's Rule gives

$$\lim_{x\to\infty}\ln f(x) = \lim_{x\to\infty}\frac{\ln x}{x} \qquad \frac{\infty}{\infty}$$
$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \frac{0}{1} = 0$$

Therefore,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

IP1:

Find $\lim_{t \to 0} \frac{10(\sin t - t)}{t^3}$ by using L'Hôpital's Rule.

Solution:

$$\lim_{t \to 0} \frac{10(\sin t - t)}{t^3} \qquad \frac{0}{0}\text{ form}$$

$$= \lim_{t \to 0} \frac{10(\cos t - 1)}{3t^2} \qquad \frac{0}{0}\text{ form}$$

$$= \lim_{t \to 0} \frac{-10 \sin t}{6t} \qquad \frac{0}{0}\text{ form}$$

$$= \lim_{t \to 0} \frac{-10 \cos t}{6} = -\frac{10}{6} = -\frac{5}{3}$$

IP2:

Find $\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\tan x$ by using L'Hôpital's Rule.

Solution:

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\tan x \qquad (0.\infty \text{ form})$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x}{\cos x} \qquad \frac{0}{0}\text{ form}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\cos x+\sin x\,(-1)}{-\sin x}=\frac{-1}{-1}=1$$

IP3:

Find $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$ by using L'Hôpital's Rule.

Solution:

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x \qquad 1^\infty \text{ form}$$

The limit leads to the indeterminate form 1^∞.

Let $f(x)=\left(1+\frac{3}{x}\right)^x$ and find $\lim_{x\to\infty}\ln f(x)$.

$$\because \ln f(x)=\ln\left(1+\frac{3}{x}\right)^x = x\ln\left(1+\frac{3}{x}\right)=\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}$$

L'Hopital's rule gives

$$\lim_{x\to\infty}\ln f(x)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} \qquad \frac{0}{0}\text{ form}$$

$$=\lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{3}{x}}\right)\left(-\frac{3}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{3x}{x+3} \qquad \frac{\infty}{\infty}\text{ form}$$

$$=\lim_{x\to\infty}\frac{3}{1}=3$$

$$\therefore \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x=\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^3$$

IP4:

Find $\lim_{x\to\infty}(1+2x)^{\frac{1}{2\ln x}}$ by using L'Hôpital's Rule.

Solution:

$\lim_{x\to\infty}(1+2x)^{\frac{1}{2\ln x}}$ $\qquad \infty^0$ form

The limit leads to the indeterminate form ∞^0.

Let $f(x)=(1+2x)^{\frac{1}{2\ln x}}$ and find $\lim_{x\to\infty}\ln f(x)$.

$$\because \ln f(x)=\ln(1+2x)^{\frac{1}{2\ln x}}=\frac{\ln(1+2x)}{2\ln x}$$

L'Hopital's rule gives

$$\lim_{x\to\infty}\ln f(x)=\lim_{x\to\infty}\frac{\ln(1+2x)}{2\ln x} \qquad \frac{\infty}{\infty}\text{ form}$$

$$=\lim_{x\to\infty}\frac{\frac{2}{1+2x}}{\frac{2}{x}}=\lim_{x\to\infty}\frac{x}{1+2x} \qquad \frac{\infty}{\infty}\text{ form}$$

$$=\lim_{x\to\infty}\frac{1}{2}=\frac{1}{2}$$

$$\therefore \lim_{x\to\infty}(1+2x)^{\frac{1}{2\ln x}}=\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^{\frac{1}{2}}=\sqrt{e}$$

P1:

Find $\lim_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1}$ by using L'Hôpital's Rule.

Solution:

$$\lim_{x\to\infty}\frac{5x^2-3x}{7x^2+1} \qquad \frac{\infty}{\infty}\text{ form}$$

$$=\lim_{x\to\infty}\frac{10x-3}{14x} \qquad \frac{\infty}{\infty}\text{ form}$$

$$=\lim_{x\to\infty}\frac{10}{14}=\frac{5}{7}$$

P2:

Find $\lim_{x \to 0} \left(\csc x - \cot x \right)$ by using L'Hôpital's Rule.

Solution:

$$\lim_{x\to 0}(\csc x - \cot x) \qquad \infty - \infty \text{ form}$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right) \qquad \frac{0}{0}\text{ form}$$

$$= \lim_{x\to 0}\frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

P3:

Find $\lim_{x \to 0^+} x^{-\frac{1}{\ln x}}$ by using L'Hôpital's Rule.

Solution:

$$\lim_{x\to 0^+} x^{-\frac{1}{\ln x}} \qquad 0^0 \text{ form}$$

The limit leads to the indeterminate form 0^0.

Let $f(x) = x^{-\frac{1}{\ln x}}$ and find $\lim_{x\to 0^+} \ln f(x)$.

$$\because \ln f(x) = \ln x^{-\frac{1}{\ln x}} = -\frac{\ln x}{\ln x} = -1$$

L'Hopital's rule gives

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} (-1) = -1$$

$$\therefore \lim_{x\to 0^+} x^{-\frac{1}{\ln x}} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

P4:

Find all values of c, that satisfy the conclusion of Cauchy's Mean Value Theorem for the functions $f(x) = x$, $g(x) = x^2$, and interval (a, b).

Solution:

We have $f(x) = x$, $g(x) = x^2$ and interval (a, b). Now, $f'(x) = 1$ and $g'(x) = 2x$.

By Cauchy's Mean Value Theorem, we seek c in the interval (a, b) so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{b-a}{b^2-a^2} = \frac{b-a}{(b-a)(b+a)} = \frac{1}{b+a}$$

$$\Rightarrow \frac{1}{2c} = \frac{1}{b+a}$$

$$\Rightarrow c = \frac{b+a}{2}$$

1. Use L'Hôpital's Rule to find the limits.

a. $\lim_{x\to 2} \frac{x-2}{x^2-4}$

b. $\lim_{t\to -3} \frac{t^3-4t+15}{t^2-t-12}$

c. $\lim_{x\to\infty} \frac{5x^2-3x}{7x^2+1}$

d. $\lim_{t\to 0} \frac{\sin t^2}{t}$

e. $\lim_{x\to 0} \frac{8x^2}{\cos x-1}$

f. $\lim_{\theta\to\pi/2} \frac{2\theta-\pi}{\cos(2\pi-\theta)}$

g. $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{1+\cos 2\theta}$

h. $\lim_{x\to 0} \frac{x^2}{\ln(\sec x)}$

i. $\lim_{t\to 0} \frac{t(1-\cos t)}{t-\sin t}$

j. $\lim_{x\to(\pi/2)^-} \left(x-\frac{\pi}{2}\right)\sec x$

k. $\lim_{\theta\to 0} \frac{3^{\sin\theta}-1}{\theta}$

l. $\lim_{x\to 0} \frac{x2^x}{2^x-1}$

m. $\lim_{x\to\infty} \frac{\ln(x+1)}{\log_2 x}$

n. $\lim_{x\to 0^+} \frac{\ln(x^2+2x)}{\ln x}$

o. $\lim_{y\to 0} \frac{\sqrt{5y+25}-5}{y}$

p. $\lim_{x\to\infty} (\ln 2x-\ln(x+1))$

q. $\lim_{x\to 0^+} \left(\frac{1}{x}-\frac{1}{\sin x}\right)$

r. $\lim_{x\to 1^+} \left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$

s. $\lim_{x\to\infty} \int_x^{2x} \frac{1}{t}dt$

t. $\lim_{\theta\to 0} \frac{\cos\theta-1}{e^\theta-\theta-1}$

u. $\lim_{t\to\infty} \frac{e^t+t^2}{e^t-t}$

2. Find the limits.

a. $\lim_{x \to 1^+} x^{1/(1-x)}$

b. $\lim_{x \to \infty} (\ln x)^{1/x}$

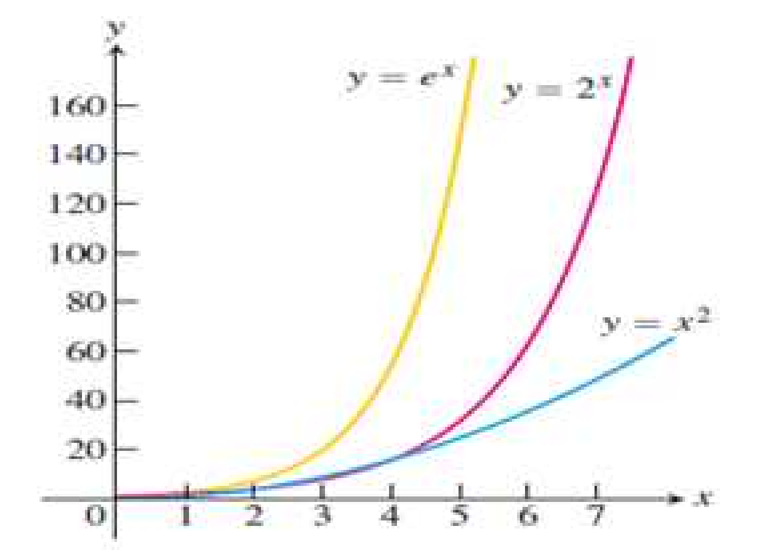
c. $\lim_{x \to 0^+} x^x$

1.9
Relative Rates of Growth

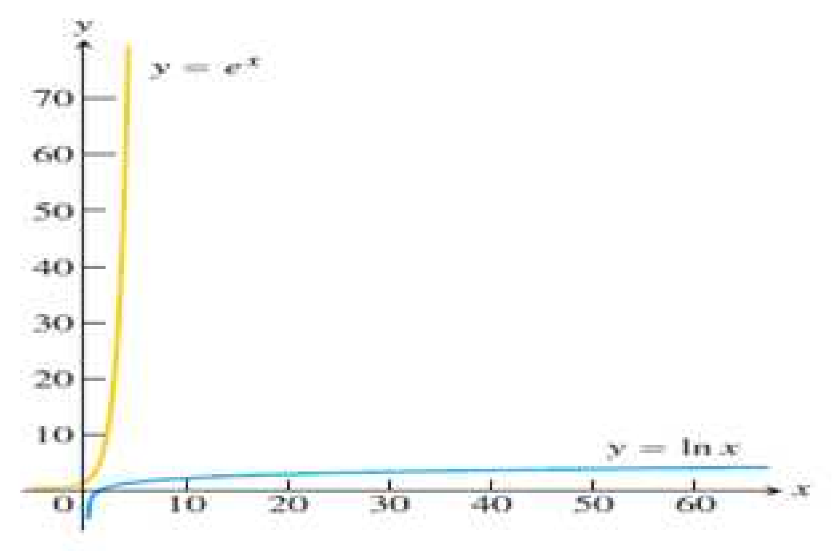
Learning Objectives:

1) To discuss the comparison of functions as $x \to \infty$.
2) To define order, little - o and big – O of functions. And
3) To practice the related problems.

If we look at the graphs, we notice that exponential functions like 2^x and e^x grow more rapidly as x gets large than the polynomials and rational functions. The graphs of x^2, 2^x, and e^x are shown below. In fact, as $x \to \infty$, the functions 2^x and e^x grow faster than any power of x.



In contrast, logarithmic functions like $y = \log_2 x$ and $y = \ln x$ grow more slowly as $x \to \infty$ than any positive power of x.



These important comparisons of exponential, polynomial, and logarithmic functions can be made precise by defining what it means for a function $f(x)$ to grow faster than a function $g(x)$ as $x \to \infty$.

Definition

Rates of Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f *grows faster than* g as $x \to \infty$ if
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$$
or, equivalently, if
$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0.$$
We also say that g *grows slower than* f as $x \to \infty$.

2. f *and* g *grow at the same rate* as $x \to \infty$ if
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \text{ where } L \textit{ is finite and not zero}$$

According to these definitions, $y = 2x$ does not grow faster than $y = x$. The two functions grow at the same rate because
$$\lim_{x\to\infty} \frac{2x}{x} = \lim_{x\to\infty} 2 = 2$$
which is a finite, nonzero limit. This is sensible if we want "f grows faster than g" to mean that for large x-values g is negligible when compared with f.

Example 1

e^x grows faster than x^2 as $x \to \infty$ because
$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$
using L'Hôpital's Rule twice

Example 2

a) 3^x grows faster than 2^x as $x \to \infty$ because
$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty \qquad \left(\because \frac{3}{2} > 1\right)$$

b) As part (a) suggests, exponential functions with different bases never grow at the same rate as $x \to \infty$. If $a > b > 0$, then a^x grows faster than b^x. Since $\left(\frac{a}{b} > 1\right)$,
$$\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty$$

Example 3

x^2 grows faster than $\ln x$ as $x \to \infty$ because
$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty \text{ (L'Hôpital's Rule)}$$

Example 4

$\ln x$ grows slower than x as $x \to \infty$ because
$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0 \qquad \text{(L'Hôpital's Rule)}$$

Example 5

In contrast to exponential functions, logarithmic functions with different bases a and b always grow at the same rate as $x \to \infty$.
$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x / \ln a}{\ln x / \ln b} = \frac{\ln b}{\ln a}$$
The limiting ratio is always finite and never zero.

If f grows at the same rate as g as $x \to \infty$, and g grows at the same rate as h as $x \to \infty$, then f grows at the same rate as h as $x \to \infty$. The reason is that
$$\lim_{x\to\infty} \frac{f}{g} = L_1 \text{ and } \lim_{x\to\infty} \frac{g}{h} = L_2$$
Together imply $\lim_{x\to\infty} \frac{f}{h} = \lim_{x\to\infty} \frac{f}{g} \cdot \frac{g}{h} = L_1 L_2$

If L_1 and L_2 are finite and nonzero, then so is $L_1 L_2$.

Example 6

Show that $\sqrt{x^2+5}$ and $\left(2\sqrt{x}-1\right)^2$ grow at the same rate as $x \to \infty$.

Solution

We show that the functions grow at the same rate by showing that they both grow at the same rate as the function $g(x) = x$.
$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = \lim_{x\to\infty} \sqrt{1 + \frac{5}{x^2}} = 1$$
$$\lim_{x\to\infty} \frac{\left(2\sqrt{x}-1\right)^2}{x} = \lim_{x\to\infty} \left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty} \left(2 - \frac{1}{\sqrt{x}}\right)^2 = 4$$

Order and O-Notation

We introduce the "little-o" and "big-O" notation invented by number theorists a hundred years ago and now commonplace in mathematical analysis and computer science.

Definition

A function f is of ***smaller order than*** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing

$$\boldsymbol{f = o(g)} \text{ ("} f \text{ is little-o of } g\text{").}$$

We note that saying $f = o(g)$ as $x \to \infty$ is another way to say that f grows slower than g as $x \to \infty$.

Example 7

$$ln\, x = o(x) \ \ as\ x \to \infty \ \text{ because } \lim_{x\to\infty} \frac{ln\, x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$x^2 = o(x^3 + 1)$ as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{x^2}{x^3+1} = \lim_{x\to\infty} \frac{2x}{3x^2} = \lim_{x\to\infty} \frac{2}{6x} = 0$$

Definition

Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is ***of at most the order of*** g as $x \to \infty$ if there is a positive integer M for which $\frac{f(x)}{g(x)} \le M$ for x sufficiently large. We indicate this by writing

$$\boldsymbol{f = O(g)} \text{ ("} f \text{ is big-O of } g\text{").}$$

Example 8

$x + sinx = O(x)$ as $x \to \infty$ because $\frac{x+\sin x}{x} \le 2$ for x sufficiently large.

Example 9

$e^x + x^2 = O(e^x)$ as $x \to \infty$ because $\frac{e^x+x^2}{e^x} \to 1$ as $x \to \infty$

$x = O(e^x)$ as $x \to \infty$ because $\frac{x}{e^x} \to 0$ as $x \to \infty$

From the definitions, we see that $f = o(g)$ implies $f = O(g)$ for functions that are positive for x sufficiently large. Also, if f and g grow at the same rate, then $f = O(g)$ and $= O(f)$.

IP1.

Which of the following statement is true

I. If $f(x) = \log_a x$ and $(x) = \log_b x$, $a \neq b$ then f and g grow at the same rate as $x \to \infty$.

II. If $f(x) = \sqrt{x^2+5}$ and $g(x) = \left(2\sqrt{x}-1\right)^2$ grow at the same rate as $x \to \infty$.

Solution:

I.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\log_a x}{\log_b x}$$

$$= \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a}$$

Since limiting ratio is finite and never zero, f and g grow at the same rate as $x \to \infty$.

II.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sqrt{x^2+5}}{\left(2\sqrt{x}-1\right)^2} = \lim_{x\to\infty} \frac{x\sqrt{1+\frac{5}{x^2}}}{x\left(2-\frac{1}{\sqrt{x}}\right)^2}$$

$$= \frac{1}{4} \quad \text{(finite and nonzero)}$$

$\Longrightarrow f$ and g grow at the same rate as $x \to \infty$.

Thus, I and II are true.

IP2.

Which of the following functions grow slower than $\ln x$ as

$x \longrightarrow \infty$

a) $\log_2 x^2$ b) $\log_{10} 10x$ c) $\ln(\ln x)$ d) $\ln(2x+5)$

Solution:

a)

$$\lim_{x\to\infty} \frac{\log_2 x^2}{\ln x} = \lim_{x\to\infty} \frac{\left(\frac{\ln x^2}{\ln 2}\right)}{\ln x} = \frac{1}{\ln 2} \lim_{x\to\infty} \frac{2\ln x}{\ln x} = \frac{2}{\ln 2}$$

Thus $\log_2 x^2$ grows at the same rate as $\ln x$ as $x \longrightarrow \infty$

b)

$$\lim_{x\to\infty} \frac{\log_{10} 10x}{\ln x} = \lim_{x\to\infty} \frac{\left(\frac{\ln 10x}{\ln 10}\right)}{\ln x}$$

$$= \frac{1}{\ln 10} \lim_{x\to\infty} \frac{\ln 10x}{\ln x} = \frac{1}{\ln 10} \lim_{x\to\infty} \frac{\left(\frac{10}{10x}\right)}{\left(\frac{1}{x}\right)} = \frac{1}{\ln 10}$$

Thus $\log_{10} 10x$ grows at the same rate as $\ln x$ as $x \longrightarrow \infty$.

c)

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x\to\infty} \frac{\left(\frac{\frac{1}{x}}{\ln x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\ln x} = 0$$

Thus $\ln(\ln x)$ grows slower than $\ln x$ as $x \longrightarrow \infty$.

d)

$$\lim_{x\to\infty} \frac{\ln(2x+5)}{\ln x} = \lim_{x\to\infty} \frac{\left(\frac{2}{2x+5}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \left(\frac{2x}{2x+5}\right) = \lim_{x\to\infty} \frac{2}{2} = 1$$

Thus $\ln(2x+5)$ grows at the same rate as $\ln x$ as $x \longrightarrow \infty$.

Therefore, $\boldsymbol{\ln(\ln x)}$ grows slower than $\boldsymbol{\ln x}$ as $x \longrightarrow \infty$.

IP3.

Order the following from functions from slowest growing to fastes growing as $x \longrightarrow \infty$

a) 2^x b) x^2 c) $(ln\ 2)^x$ d) e^x

Solution:

I.

$$\lim_{x\to\infty} \frac{(ln\ 2)^x}{x^2} = \lim_{x\to\infty} \frac{ln(ln\ 2)\,(ln\ 2)^x}{2x} = \lim_{x\to\infty} \frac{(ln(ln\ 2))^2 (ln\ 2)^x}{2}$$

$$= \frac{(ln(ln\,2))^2}{2} \lim_{x\to\infty} (ln\ 2)^x = 0 \ (\because\ ln\ 2 < 1)$$

Thus $(ln\ 2)^x$ grows slower than x^2 as $x \longrightarrow \infty$.

II. $\lim_{x\to\infty} \frac{x^2}{2^x} = \lim_{x\to\infty} \frac{2x}{(ln\ 2)2^x} = \lim_{x\to\infty} \frac{2}{(ln\ 2)^2 2^x} = 0;$

Thus x^2 grows slower than 2^x as $x \longrightarrow \infty$.

III. $\lim_{x\to\infty} \frac{2^x}{e^x} = \lim_{x\to\infty} \left(\frac{2}{e}\right)^x = 0;$

Thus 2^x grows slower than e^x as $x \longrightarrow \infty$.

Therefore, the order of the functions from slowest growing to fastest growing as $x \longrightarrow \infty$ is: $(ln\ 2)^x$, x^2, 2^x and e^x.

IP4.

Which of the following statements are true.

I. $x + sinx = O(x)$
II. $e^x + x^2 = O(e^x)$

Solution:

I.

$$\lim_{x\to\infty}\left(\frac{x+\sin x}{x}\right)=\lim_{x\to\infty}\left(1+\frac{\sin x}{x}\right)$$

$$=\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{\sin x}{x}\leq 2$$

$$\Rightarrow x + sinx = O(x)$$

II.

$$\lim_{x\to\infty}\left(\frac{e^x+x^2}{e^x}\right)=\lim_{x\to\infty}\left(1+\frac{x^2}{e^x}\right)$$

$$=1+\lim_{x\to\infty}\frac{x^2}{e^x}$$

$$=1+\lim_{x\to\infty}\frac{2x}{e^x} \qquad (by\ L'Hopital's\ rule)$$

$$=1+\lim_{x\to\infty}\frac{2}{e^x} \qquad (by\ L'Hopital's\ rule)$$

$$=1$$

$$\because\ e^x+x^2=O(e^x)$$

So, both I and II are true

P1.

Which of the following statements is false?

A. e^x grows faster than x^2 as $x \to \infty$

B. 2^x grows faster than 7^x as $x \to \infty$

C. 5^x grows faster than 3^x as $x \to \infty$

D. x^2 grows faster than $\ln x$ as $x \to \infty$

Solution:

A.

$$\lim_{x \to \infty} \frac{e^x}{x^2} \qquad \left[Form \frac{\infty}{\infty} \right]$$

$$= \lim_{x \to \infty} \frac{e^x}{2x} \begin{pmatrix} by\ L'Hopital's\ rule \\ and\ still\ \frac{\infty}{\infty}\ form \end{pmatrix}$$

$$= \lim_{x \to \infty} \frac{e^x}{2} (by\ L'Hopital's\ rule)$$

$$= \frac{\infty}{2} = \infty$$

$\Rightarrow$ e^x grows faster than x^2 as $x \to \infty$

P2.

Which of the following functions grow faster than e^x as $x \longrightarrow \infty$

a) e^{-x} b) $x\, e^x$ c) e^{cosx} d) e^{x-1}

Solution:

a)

$$\lim_{x\to\infty}\frac{e^{-x}}{e^{x}}=\lim_{x\to\infty}\frac{1}{e^{2x}}=0$$

$\Rightarrow e^{-x}$ grows slower than e^{x} as $x \longrightarrow \infty$.

b)

$$\lim_{x\to\infty}\frac{xe^{x}}{e^{x}}=\lim_{x\to\infty}x=\infty$$

$\Rightarrow xe^{x}$ grows faster than e^{x} as $x \longrightarrow \infty$.

c)

For all real values of x, we have

$-1 \le cosx \le 1 \Rightarrow e^{-1} \le e^{cosx} \le e^{1}$

$$\Rightarrow \frac{e^{-1}}{e^{x}} \le \frac{e^{cosx}}{e^{x}} \le \frac{e^{1}}{e^{x}}$$

Further, $\lim\limits_{x\to\infty}\frac{e^{-1}}{e^{x}}=0$ and $\lim\limits_{x\to\infty}\frac{e^{1}}{e^{x}}=0$

By Sandwich theorem we conclude that $\lim\limits_{x\to\infty}\frac{e^{cosx}}{e^{x}}=0.$

Thus e^{cosx} grows slower than e^{x} as $x \longrightarrow \infty$.

d)

$$\lim_{x\to\infty}\frac{e^{x-1}}{e^{x}}=\lim_{x\to\infty}\frac{1}{e}=\frac{1}{e}$$

Thus, e^{x-1} grows at the same rate as e^{x} as $x \longrightarrow \infty$.

Therefore, only $\boldsymbol{x\,e^{x}}$ ***grows faster than*** $\boldsymbol{e^{x}}$.

P3.

Order the following functions from fastest growing to slowest growing as $x \longrightarrow \infty$.

a) e^{x} b) x^{x} c) $(\ln x)^{x}$ d) $e^{\frac{x}{2}}$

Solution:

I. $\lim\limits_{x \to \infty} \frac{e^x}{e^{\frac{x}{2}}} = \lim\limits_{x \to \infty} e^{\frac{x}{2}} = \infty$

$\Longrightarrow e^x$ grows faster than $e^{\frac{x}{2}}$ as $x \longrightarrow \infty$.

II. For $x > e^e$, we have $\ln x > e$ and $\frac{\ln x}{e} > 1$

Now, $\lim\limits_{x \to \infty} \frac{(\ln x)^x}{e^x} = \lim\limits_{x \to \infty} \left(\frac{\ln x}{e}\right)^x = \infty$

$\Longrightarrow (\ln x)^x$ grows faster than e^x as $x \longrightarrow \infty$.

III. We have $x > \ln x$ for $x > 0$. Now,

$\lim\limits_{x \to \infty} \frac{x^x}{(\ln x)^x} = \lim\limits_{x \to \infty} \left(\frac{x}{\ln x}\right)^x = \infty \quad (\because \frac{x}{\ln x} > 1 \; for \; x > 0)$

$\Longrightarrow x^x$ grows faster than $(\ln x)^x$;

Therefore, the order of the functions from fastest growing to slowest growing as $x \longrightarrow \infty$ is x^x, $(\ln x)^x$, e^x and $e^{\frac{x}{2}}$.

P4.

Which of the following statement is false

I. $x = o(x+5)$

II. $ln x = o(ln 2x)$

Solution:

We recall,

$f = o(g)$ if $\lim\limits_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = 0$

I. We have,

$$\lim_{x\to\infty} \frac{x}{x+5} \qquad \left[Form\ \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{1}{1} = 1 \qquad (by\ L'Hopital's\ rule)$$

Therefore, $x = o(x+5)$ is false.

II. We have,

$$\lim_{x\to\infty} \frac{ln x}{ln 2x} \qquad \left[Form\ \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2x}.2} \qquad (by\ L'Hopital's\ rule)$$

$$= \lim_{x\to\infty} 1 = 1$$

Therefore, $lnx = o(ln2x\ )$ is false.

Hence, both I and II are false.

1. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?
   a. $x + 3$
   b. $x^3 + sin^2 x$
   c. $\sqrt{x}$
   d. 4^x
   e. $(3/2)^x$
   f. $e^{x/2}$
   g. $e^x/2$
   h. $\log_{10} x$

2. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

a. $x^2 + 4x$
b. $x^5 - x^2$
c. $\sqrt{x^4 + x^3}$
d. $(x+3)^2$
e. $x \ln x$
f. 2^x
g. $x^3 e^{-x}$
h. $8x^2$

3. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

a. $\log_3 x$
b. $\ln 2x$
c. $\ln \sqrt{x}$
d. $\sqrt{x}$
e. x
f. $5 \ln x$
g. $1/x$
h. e^x

4. True, or false? As $x \to \infty$

a. $x = o(x)$

b. $x = o(x+5)$

c. $x = O(x+5)$

d. $x = O(2x)$

e. $e^x = o(e^{2x})$

f. $x + \ln x = O(x)$

g. $\ln x = o(\ln 2x)$

h. $\sqrt{x^2+5} = O(x)$

i. $\frac{1}{x+3} = O(x)$

j. $\frac{1}{x} + \frac{1}{x^2} = O(\frac{1}{x})$

k. $\frac{1}{x} - \frac{1}{x^2} = o(\frac{1}{x})$

l. $2 + cosx = O(2)$

m. $e^x + x = O(e^x)$

n. $xlnx = o(x^2)$

o. $ln(lnx) = O(lnx)$

p. $ln(x) = o(ln\,(x^2+1))$

1.11

Derivatives of Inverse Trigonometric Functions

Learning objectives:

- To find the deriatives of inverse trigonometric functions. And
- To solve the related problems.

Inverse trigonometric functions provide anti derivatives for a variety of functions that arise in engineering. The derivatives of the inverse trigonometric functions are given below.

1. $\frac{d(\sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1$

2. $\frac{d(\cos^{-1}u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1$

3. $\frac{d(\tan^{-1}u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

4. $\frac{d(\cot^{-1}u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$

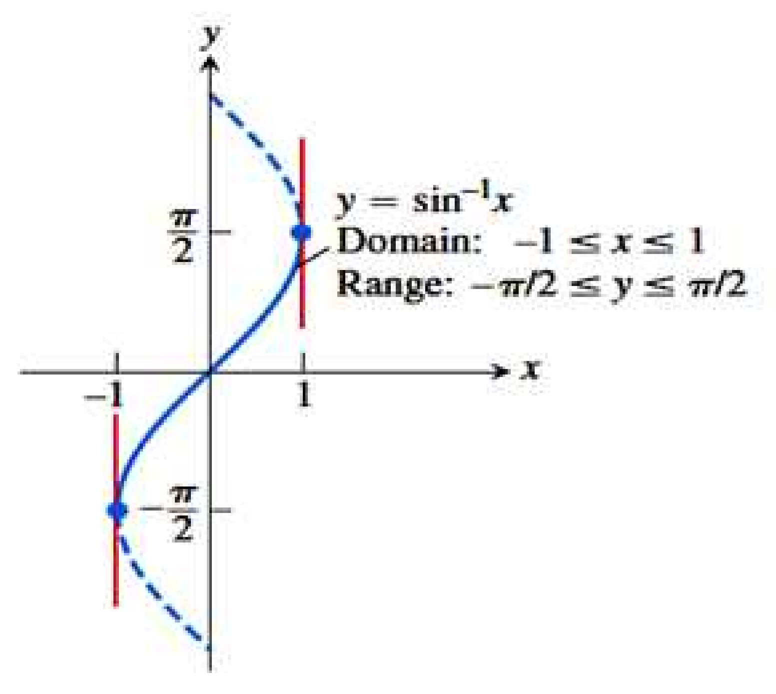
5. $\frac{d(\sec^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1$

6. $\frac{d(\csc^{-1}u)}{dx} = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1$

We derive formulas 1 and 5 below.

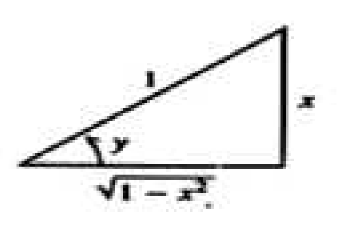
The Derivative of $y = \sin^{-1} u$:

The function $x = \sin y$ is differentiable in the interval $\frac{-\pi}{2} < y < \frac{\pi}{2}$ and that its derivative, the cosine, is positive there. Therefore, the inverse function $y = \sin^{-1} x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points.



We find the derivative of $y = \sin^{-1} x$ as follows.

$\sin y = x \implies \frac{d}{dx}(\sin y) = 1 \implies \cos y \frac{dy}{dx} = 1$



$$\implies \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Therefore, the derivative of $y = \sin^{-1} x$ with respect to x is

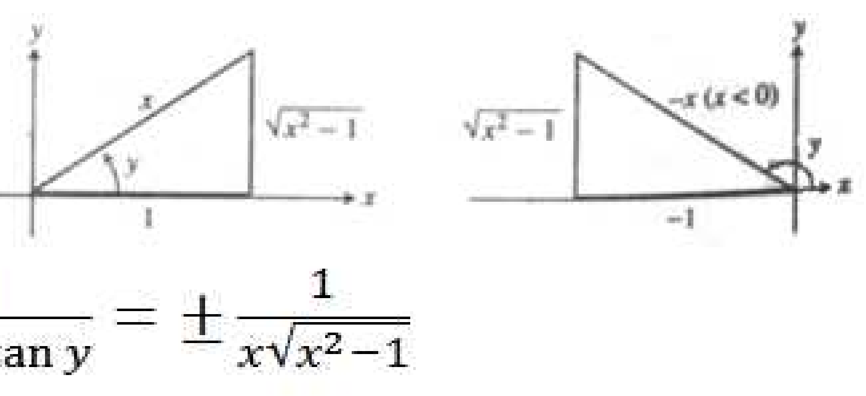
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

- If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
to $y = \sin^{-1} u$ to obtain
$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1$$

The Derivative of $y = \sec^{-1} u$:

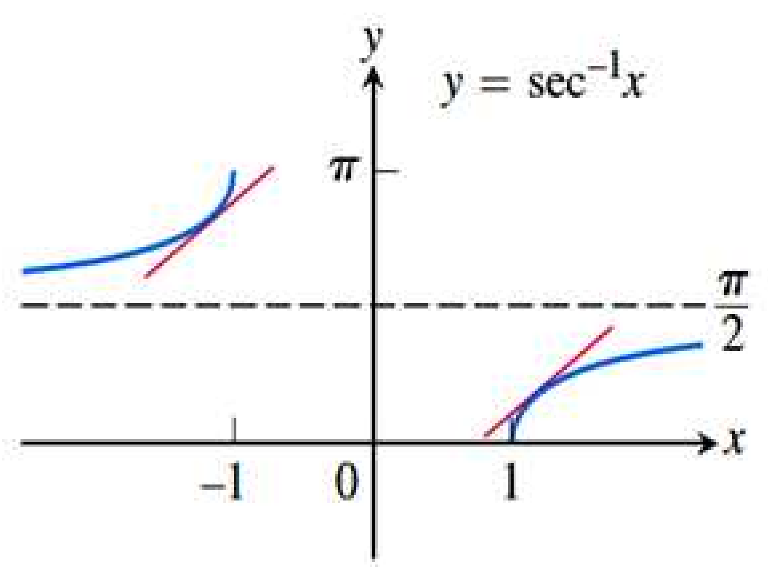
We find the derivative of $y = \sec^{-1} x$, $|x| > 1$, in a similar way.

$\sec y = x \implies \frac{d}{dx}(\sec y) = 1 \implies \sec y \tan y \frac{dy}{dx}$



$$\implies \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \pm\frac{1}{x\sqrt{x^2-1}}$$

From the figure below, we see that for $|x| > 1$ the slope of the graph of $y = \sec^{-1} x$ is always positive.



Therefore, $\frac{d}{dx}(\sec^{-1} x) = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\frac{1}{x\sqrt{x^2-1}} & \text{if } x < -1 \end{cases}$

With absolute values, we can write this equation as a single formula.

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad |x| > 1$$

- If u is a differentiable function of x with $|u| > 1$, we apply the Chain Rule to obtain
$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1$$

The derivation of formula 3 is similar to the above derivations.. Formulas 2, 4, and 6 can be derived from formulas 1, 3, and 5 by differentiating appropriate identities.

Example 1:

(a) $\frac{d}{dx}\sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$

(b) $\frac{d}{dx}\tan^{-1}\sqrt{x+1} = \frac{1}{1+(\sqrt{x+1})^2} \cdot \frac{d}{dx}\left(\sqrt{x+1}\right)$

$$= \frac{1}{x+2} \cdot \frac{1}{2\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}(x+2)}$$

(c) $\frac{d}{dx}\sec^{-1}(-3x) = \frac{1}{|-3x|\sqrt{(-3x)^2-1}} \cdot \frac{d}{dx}(-3x)$

$$= \frac{-3}{|3x|\sqrt{9x^2-1}} = \frac{-1}{|x|\sqrt{9x^2-1}}$$

Example 2:

$$\int_0^1 \frac{e^{\tan^{-1} x}}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} e^u\,du, \text{ where } u = \tan^{-1} x \text{ and } du = \frac{dx}{1+x^2}$$

$$= \left[e^u\right]_0^{\pi/4} = e^{\pi/4} - 1$$

IP1:

If $\boldsymbol{y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1}x, x > 1}$, then find $\boldsymbol{\frac{dy}{dx}}$.

Solution:

Given $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1}x, x > 1$

$$\frac{dy}{dx} = \frac{1}{1+\left(\sqrt{x^2-1}\right)^2}\frac{d}{dx}\left(\sqrt{x^2-1}\right) - \frac{1}{|x|\sqrt{x^2-1}}$$

$$= \frac{1}{x^2}\cdot\frac{1}{2\sqrt{x^2-1}}(2x) - \frac{1}{|x|\sqrt{x^2-1}}$$

$$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0 \qquad (\because x > 1)$$

IP2:

If $f(x) = \left(a^2 - b^2\right)^{-1/2} \cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$, then show that $f'(x) = (a + b\cos x)^{-1}$.

Solution:

Given $f(x) = (a^2 - b^2)^{-1/2} \cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$

$$f'(x) = \frac{1}{\sqrt{a^2-b^2}} \cdot \frac{-1}{\sqrt{1-\left(\frac{a\cos x+b}{a+b\cos x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

$$= \frac{-(a+b\cos x)}{\sqrt{a^2-b^2}\sqrt{(a+b\cos x)^2-(a\cos x+b)^2}} \cdot \frac{(a+b\cos x)(-a\sin x)-(a\cos x+b)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{-\left[-a^2\sin x - ab\sin x\cos x + ab\sin x\cos x + b^2\sin x\right]}{\sqrt{a^2-b^2}\sqrt{a^2(1-\cos^2 x)-b^2(1-\cos^2 x)}(a+b\cos x)}$$

$$= \frac{\left(a^2-b^2\right)\sin x}{\sqrt{a^2-b^2}\sqrt{a^2-b^2}\sin x} \cdot \frac{1}{(a+b\cos x)}$$

$$= (a + b\cos x)^{-1}$$

IP3:

If $\boldsymbol{f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)}$, then find $\boldsymbol{f'(1)}$.

Solution:

Given $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$.

Let $u = x^x$

$\Longrightarrow ln\, u = x\, ln\, x$

Differentiate both sides w.r.t x, we get

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x} + ln\, x \quad \Longrightarrow \frac{d}{dx}(x^x) = x^x(1 + ln\, x)$$

Let $v = x^{-x}$

$\Longrightarrow ln\, v = -x\, ln\, x$

Differentiate both sides w.r.t x, we get

$$\frac{1}{v}\frac{dv}{dx} = -x \cdot \frac{1}{x} - ln\, x \quad \Longrightarrow \frac{d}{dx}(x^{-x}) = -x^{-x}(1 + ln\, x)$$

$$\therefore\ f'(x) = -\frac{1}{1+\left(\frac{x^x - x^{-x}}{2}\right)^2}\frac{d}{dx}\left(\frac{x^x - x^{-x}}{2}\right)$$

$$= -\frac{4}{x^{2x} + x^{-2x} + 2}\left(\frac{1}{2}(x^x + x^{-x})(1 + ln\, x)\right)$$

$$f'(1) = -\frac{4}{1+1+2}\left(\frac{1}{2}(1+1)(1 + ln\, 1)\right) = -1$$

IP4:

$$\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1} x)}{x\sqrt{x^2-1}}\,dx =$$

Solution:

Let $u = \sec^{-1} x \;\Rightarrow du = \frac{1}{x\sqrt{x^2-1}}dx \qquad (\because x \in [\sqrt{2}, 2])$

$$x = \sqrt{2} \Rightarrow u = \frac{\pi}{4} \;;\; x = 2 \Rightarrow u = \frac{\pi}{3}$$

$$\therefore \int_{\sqrt{2}}^{2} \frac{sec^2(\sec^{-1} x)}{x\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 u \; du$$

$$= [\tan u]_{\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4}$$

$$= \sqrt{3} - 1$$

P1:

If $y = x \sin^{-1} x + \sqrt{1 - x^2}$, then find $\frac{dy}{dx}$.

Solution:

Given $y = x\sin^{-1}x + \sqrt{1 - x^2}$

$$\frac{dy}{dx} = \sin^{-1}x + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}(-2x)$$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x$$

P2:

$$\frac{d}{dx}\left[\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\right] =$$

Solution:

$$\frac{d}{dx}\left[\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\right] = \frac{\sqrt{1-x^2}\frac{d}{dx}[x\sin^{-1}x]-x\sin^{-1}x\frac{d}{dx}(\sqrt{1-x^2})}{\left[\sqrt{1-x^2}\right]^2}$$

Now, $\frac{d}{dx}[x\sin^{-1}x] = x\frac{d}{dx}[\sin^{-1}x] + \sin^{-1}x\frac{d}{dx}(x)$

$$= x.\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x.1$$

And $\frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}(1-x^2)$

$$= \frac{1}{2\sqrt{1-x^2}}.(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}\left[\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\right] = \frac{\sqrt{1-x^2}\left[\frac{x}{\sqrt{1-x^2}}+\sin^{-1}x\right]-x.\sin^{-1}x\left(\frac{-x}{\sqrt{1-x^2}}\right)}{1-x^2}$$

$$= \frac{x+\sin^{-1}x\left[\sqrt{1-x^2}+\frac{x^2}{\sqrt{1-x^2}}\right]}{1-x^2}$$

$$= \frac{x\sqrt{1-x^2}+(1-x^2+x^2)\sin^{-1}x}{(1-x^2)\sqrt{1-x^2}}$$

$$= \frac{x\sqrt{1-x^2}+\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$$

P3:

Find the derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Solution:

Derivative of a function w.r.t another function

Suppose $f(x)$ and $g(x)$ are two functions and we have to find the derivative of $f(x)$ w.r.t $g(x)$. Now let $y = f(x), u = g(x)$.

Then $\frac{dy}{dx} = f'(x), \frac{du}{dx} = g'(x)$.

$$\therefore \frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}} = \frac{f'(x)}{g'(x)}.$$

Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Put $x = \tan\theta \implies \theta = \tan^{-1} x$

$$u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \text{ and } v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$\implies u = \tan^{-1}(\tan 2\theta)$ and $v = \sin^{-1}(\sin 2\theta)$

$\implies u = 2\theta$ and $v = 2\theta$

$\implies u = 2\tan^{-1} x$ and $v = 2\tan^{-1} x$

$$\implies \frac{du}{dx} = \frac{2}{1+x^2} \text{ and } \frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\implies \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}} = 1.$$

P4:

$$\int \frac{1}{\tan^{-1} y\,(1+y^2)}\,dy =$$

Solution:

$$\int \frac{1}{\tan^{-1} y\,(1+y^2)}\,dy = \int \frac{\left(\frac{1}{1+y^2}\right)}{\tan^{-1} y}\,dy$$

Let $u = \tan^{-1} y \implies du = \frac{1}{1+y^2}\,dy$

$$\int \frac{\left(\frac{1}{1+y^2}\right)}{\tan^{-1} y}\,dy = \int \frac{1}{u}\,du = ln|u| + C = ln|\tan^{-1} y| + C$$

$$\therefore \int \frac{1}{\tan^{-1} y\,(1+y^2)}\,dy = ln|\tan^{-1} y| + C$$

1. Find the derivative of y w.r.t appropriate variable.

a. $y = \cos^{-1}(x^2)$

b. $y = \cos^{-1}\left(\frac{1}{x}\right)$

c. $y = \sin^{-1}\sqrt{2}t$

d. $y = \sin^{-1}(1-t)$

e. $y = \sec^{-1}(2s+1)$

f. $y = \sec^{-1} 5s$

g. $y = \csc^{-1}(x^2+1)\,, x > 0$

h. $y = \csc^{-1}\frac{x}{2}$

i. $y = \sec^{-1}\left(\frac{1}{t}\right)\,, 0 < t < 1$

j. $y = \sin^{-1}\left(\frac{3}{t^2}\right)$

k. $y = \cot^{-1}\sqrt{t}$

l. $y = \cot^{-1}\sqrt{t-1}$

m. $y = \ln(\tan^{-1} x)$

n. $y = \tan^{-1}(\ln x)$

o. $y = \csc^{-1}(e^t)$

p. $y = \cos^{-1}(e^{-t})$

q. $y = s\sqrt{1-s^2}\cos^{-1} s$

r. $y = \sqrt{s^2-1} - \sec^{-1} s$

s. $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1} x \quad , x > 1$

t. $y = \cot^{-1}\frac{1}{x} - \tan^{-1} x$

u. $y = ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right)$

2. Find the derivative of the following w.r.t x.

a. $\sin^{-1}(3x - 4x^3)$

b. $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

c. $\cos^{-1}(4x^3 - 3x)$

d. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

e. $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

f. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

g. $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

h. $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

3. Find the derivative of

a. $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t $\sqrt{1-x^2}$

b. $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t $\tan^{-1} x$

c. $x^{\sin^{-1} x}$ w.r.t $\sin^{-1} x$.

d. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t $\tan^{-1} \frac{2x}{1-x^2}$.

4. Evaluate the integrals.

a. $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\,dx$

b. $\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}}\,dx$

c. $\int \frac{\left(\sin^{-1} x\right)^2}{\sqrt{1-x^2}}\,dx$

d. $\int \frac{\sqrt{\tan^{-1} x}}{1+x^2}\,dx$

e. $\int \frac{1}{(\sin^{-1} y)\sqrt{1-y^2}}\,dx$

f. $\int \frac{\sec^2\left(2\tan^{-1} x\right)}{1+x^2}\,dx$

g. $\int_{2\sqrt{3}}^{2} \frac{\cos\left(\sec^{-1} x\right)}{x\sqrt{x^2-1}}\,dx$

1.13
Hyperbolic Functions

Learning objectives:

1. To find derivatives and integrals of hyperbolic functions.
   And
2. To practice the related problems.

Every function f that is defined on an interval centered at the origin can be written in a unique way as the sum of one even function and one odd function. The decomposition is

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{Even\ function} + \underbrace{\frac{f(x)-f(-x)}{2}}_{Odd\ function}$$

If we write e^x this way, we get

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

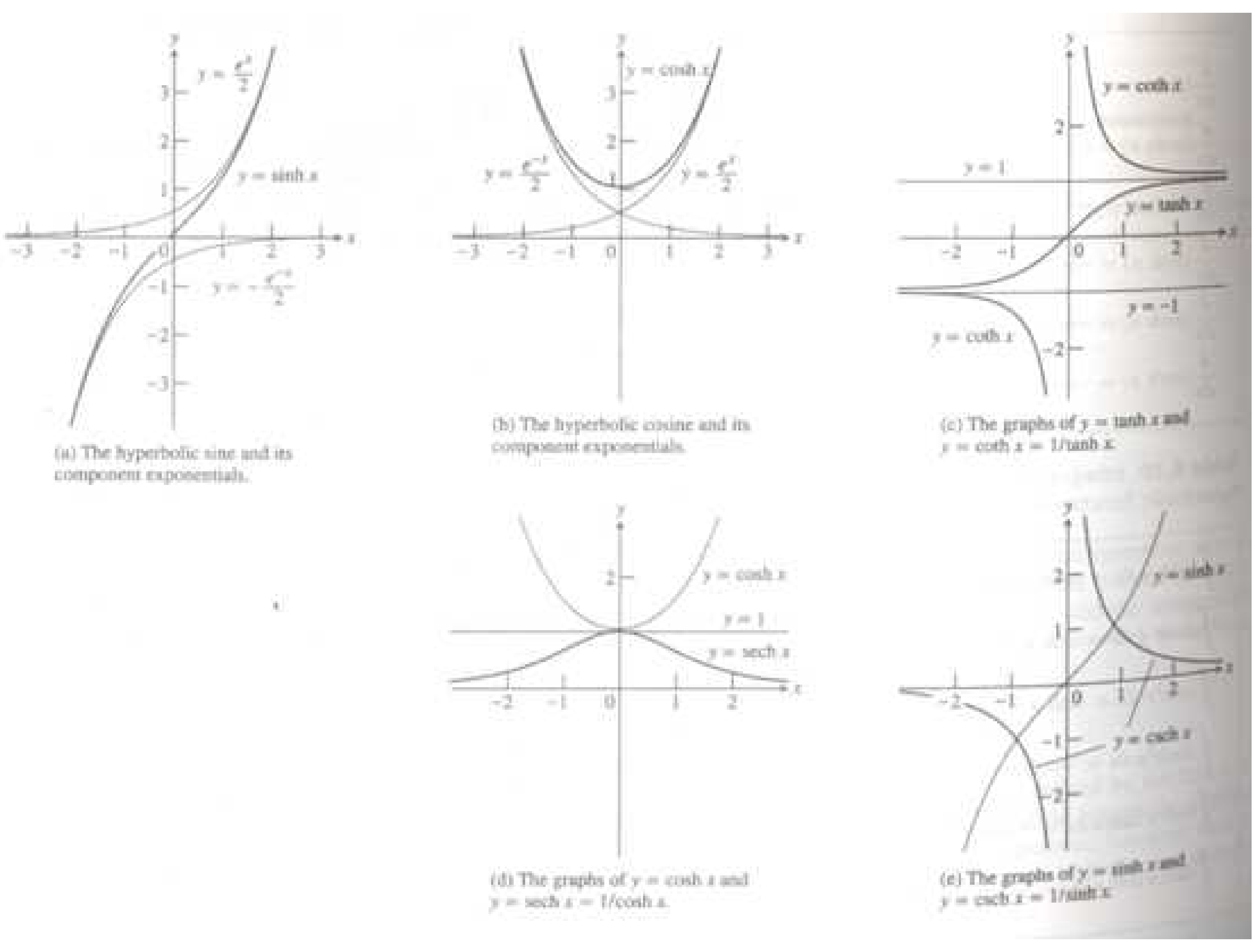
The even and odd parts of e^x, called the ***hyperbolic cosine*** and ***hyperbolic sine*** of x, respectively, are useful in their own right. They describe the motion of waves in elastic solids, the shapes of hanging electric power lines, and the temperature distributions in metal cooling fins.

Definitions

The hyperbolic cosine and hyperbolic sine functions are defined by the first two equations in the table below. The table also lists the definitions of the hyperbolic tangent, cotangent, secant, and cosecant. We will see the hyperbolic functions bear a number of similarities to the trigonometric functions after which they are named.

1. $\cosh x = \dfrac{e^x + e^{-x}}{2}$
2. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
6. $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

The graphs of the six hyperbolic functions are shown below.



Identities

Hyperbolic functions satisfy the identities in the table below. Except for differences in the sign, these are identities already known for trigonometric functions.

1. $\sinh 2x = 2\sinh x \cosh x$
2. $\cosh 2x = \cosh^2 x + \sinh^2 x$
3. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
4. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
5. $\cosh^2 x - \sinh^2 x = 1$
6. $\tanh^2 x = 1 - \operatorname{sech}^2 x$
7. $\coth^2 x = 1 + \operatorname{csch}^2 x$

Derivatives and Integrals

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined. Again, there are similarities with trigonometric functions.

1. $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$
2. $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$
3. $\frac{d}{dx}(\tanh u) = \mathrm{sech}^2 u \frac{du}{dx}$
4. $\frac{d}{dx}(\coth u) = -\mathrm{csch}^2 u \frac{du}{dx}$
5. $\frac{d}{dx}(\mathrm{sech} u) = -\mathrm{sech} u \tanh u \frac{du}{dx}$
6. $\frac{d}{dx}(\mathrm{csch} u) = -\mathrm{csch} u \coth u \frac{du}{dx}$

The derivative formulas in the table above lead to the integral formulas in the table below.

1. $\int \sinh u\, du = \cosh u + C$
2. $\int \cosh u\, du = \sinh u + C$
3. $\int \mathrm{sech}^2 u\, du = \tanh u + C$
4. $\int \mathrm{csch}^2 u\, du = -\coth u + C$
5. $\int \mathrm{sech}\, u \tanh u\, du = -\mathrm{sech}\, u + C$
6. $\int \mathrm{csch}\, u \coth u\, du = -\mathrm{csch}\, u + C$

Example 1

$$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \mathrm{sech}^2\sqrt{1+t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$$
$$= \frac{t}{\sqrt{1+t^2}}\mathrm{sech}^2\sqrt{1+t^2}$$

Example 2

$$\int \coth 5x\, dx = \int \frac{\cosh 5x}{\sinh 5x}dx \qquad u = \sinh 5x$$
$$= \frac{1}{5}\int\frac{du}{u}$$
$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

Example 3

$$\int_0^1 \sinh^2 x\, dx = \int_0^1 \frac{\cosh 2x - 1}{2}dx$$
$$= \frac{1}{2}\int_0^1(\cosh 2x - 1)dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$
$$= \frac{\sinh 2}{4} - \frac{1}{2}$$

Example 4

$$\int_0^{\ln 2} 4e^x \sinh x\, dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}dx = \int_0^{\ln 2}\left(2e^{2x} - 2\right)dx$$
$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = \left(e^{2\ln 2} - 2\ln 2\right) - (1 - 0)$$
$$= 4 - 2\ln 2 - 1 = 3 - 2\ln 2$$

IP1.

If $y = sech^2(e^t)$ then $\frac{dy}{dt} =$

Solution:

Given, $y = sech^2(e^t)$

$$\frac{dy}{dt} = 2\,sech(e^t)\,(-sech(e^t)\,tanh(e^t))e^t$$

$$= -2e^t sech^2(e^t) tanh(e^t)$$

$$= -2e^t(tanh(e^t) - tanh^3(e^t))$$

IP2.

If $y = ln\, sinhv - \frac{1}{2} coth^2 v$ then $\frac{dy}{dv} =$

Solution:

Given $y = ln\, sinhv - \frac{1}{2} coth^2 v$

Differentiating on both sides, w.r.t v, we get

$$\frac{dy}{dv} = \frac{coshv}{sinhv} - \frac{1}{2}(2cothv)(-\mathrm{cosech}^2 v)$$

$$= cothv + cothv(cosech^2 v)$$

$$= cothv(1 + cosech^2 v)$$

$$= cothv(coth^2 v)$$

$$= coth^3 v$$

IP3.

$$\int \frac{\operatorname{sech}(\sqrt{t})\tanh(\sqrt{t})}{\sqrt{t}}\, dt =$$

Solution:

$$\int \frac{\operatorname{sech}(\sqrt{t})\tanh(\sqrt{t})}{\sqrt{t}}\, dt =$$

Put, $\sqrt{t} = x \Longrightarrow \frac{1}{2\sqrt{t}}\, dt = dx \Longrightarrow \frac{dt}{\sqrt{t}} = 2dx$

$$\int \frac{\operatorname{sech}(\sqrt{t})\tanh(\sqrt{t})}{\sqrt{t}}\, dt = 2\int \operatorname{sech}x \tanh x \; dx$$

$$= -2\operatorname{sech}x + c$$

$$= -2\operatorname{sech}(\sqrt{t}) + c$$

IP4.

$$\int_0^{ln\,10} 4\,sinh^2\left(\frac{x}{2}\right)dx =$$

Solution:

$$\int_0^{ln\,10} 4\,sinh^2\left(\frac{x}{2}\right)dx =$$

We have, $sinh^2\left(\frac{x}{2}\right) = \frac{coshx-1}{2} \Longrightarrow 4sinh^2\left(\frac{x}{2}\right) = 2(coshx-1)$

$$\int_0^{ln\,10} 4\,sinh^2\left(\frac{x}{2}\right)dx = 2\int_0^{ln\,10}(coshx-1)dx$$

$$= 2(sinhx - x)_0^{\;ln\,10}$$

$$= 2(sinhx - x)_0^{\;ln\,10}$$

$$= 2(sinh\,(ln\,10) - ln\,10)$$

$$= e^{ln\,10} - e^{-ln\,10} - 2\,ln\,10$$

$$= 10 - \frac{1}{10} - 2\,ln\,10 = 9.9 - 2\,ln\,10$$

P1.

If $y = cscht(1 - ln\, cscht)$, then $\frac{dy}{dt} =$

Solution:

Given, $y = cscht(1 - ln\, cscht)$

$$\frac{dy}{dt} = cscht\left(-\frac{1}{cscht} \times -cscht\ cotht\right) + (1 - ln\, cscht)(-cscht\ cotht)$$

$$= cscht\ cotht - cscht\ cotht + cscht\ cotht\ ln\, cscht$$

$$= cscht\ cotht\ ln\, cscht$$

P2.

If $y = (4x^2 - 1)cosech(\ln 2x)\ then\ \frac{dy}{dx} =$

Solution:

Given $y \quad = (4x^2 - 1)cosech(ln\, 2x)$

$$= (4x^2 - 1).\frac{1}{\sinh(ln\, 2x)}$$

$$= (4x^2 - 1).\frac{2}{e^{ln\, 2x} + e^{-ln\, 2x}}$$

$$= (4x^2 - 1).\frac{2}{2x - \frac{1}{2x}}$$

$$= (4x^2 - 1).\frac{4x}{(4x^2 - 1)} = 4x$$

$$\therefore y = 4x$$

Differentiating on both sides w.r.t x, we get $\frac{dy}{dx} = 4.$

P3.

$$\int \frac{csch(\ln t)coth(\ln t)}{t}\,dt =$$

Solution:

$$\int \frac{csch(\ln t)coth(\ln t)}{t}\,dt =$$

Put, $\ln t = u \Longrightarrow \frac{1}{t}dt = du$

$$= \int cschu.\,cothu\,du$$

$$= \int cschu.\,cothu\,du = -cschu + c$$

$$= -csch(\ln t) + c$$

P4.

$$\int_{-\ln 4}^{-\ln 2} 2e^{\theta} \cosh\theta \, d\theta =$$

Solution:

$$\int_{-ln\,4}^{-ln\,2} 2e^{\theta} cosh\theta \, d\theta = \int_{-ln\,4}^{-ln\,2} 2.e^{\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) d\theta$$

$$= \int_{-ln\,4}^{-ln\,2} (e^{2\theta}+1) d\theta$$

$$= \int_{-ln\,4}^{-ln\,2} (e^{2\theta}+1) d\theta = \left[\frac{e^{2\theta}}{2}+\theta\right]_{-ln\,4}^{-ln\,2}$$

$$= \left[\frac{e^{-2ln\,2}}{2} - ln\,2\right] - \left[\frac{e^{-2ln\,4}}{2} - ln\,4\right]$$

$$= \frac{2^{-2}}{2} - ln\,2 - \frac{4^{-2}}{2} + ln\,4$$

$$= \frac{1}{8} - \frac{1}{32} + ln\,\frac{4}{2}$$

$$= \frac{3}{32} + ln\,2$$

I. Rewrite the expression in terms of exponentials and simplify the results.

a) $2cosh\ (ln\, x)$
b) $sinh(2\ ln\, x)$
c) $\cosh 5x + \sinh 5x$
d) $\cosh 3x - \sinh 3x$
e) $(\sinh x + \cosh x)^4$
f) $ln(coshx + sinhx) + ln(coshx - sinhx)$

II. Find the derivative of y with respect to appropriate variable.

a) $y = 6 sinh \frac{x}{3}$

b) $y = \frac{1}{2} sinh(2x + 1)$

c) $y = 2\sqrt{t}\ tanh\sqrt{t}$

d) $y = t^2 tanh \frac{1}{t}$

e) $y = ln(sinhz)$

f) $y = ln(coshz)$

g) $y = sech\theta(1 - ln\ sech\theta)$

h) $y = ln\ cosh\ v - \frac{1}{2} tanh^2 v$

i) $y = (x^2 + 1) sech(lnx)$

III. Evaluate the following indefinite integrals

a) $\int sinh\, 2x \;\; dx$

b) $\int sinh\, \frac{x}{5}\; dx$

c) $\int 6\, cosh\left(\frac{x}{2} - ln3\right) dx$

d) $\int 4\, cosh(3x - ln2) dx$

e) $\int tanh\, \frac{x}{7}\; dx$

f) $\int coth\, \frac{\theta}{\sqrt{3}}\; d\theta$

g) $\int sech^2\left(x - \frac{1}{2}\right) dx$

h) $\int csch^2\, (5 - x)\, dx$

IV. Evaluate the following definite integrals

a) $\int_{ln2}^{ln4} coth\, x\, dx$

b) $\int_{0}^{ln2} tanh\, 2x\, dx$

c) $\int_{0}^{ln2} 4e^{-\theta} sinh\, \theta\, d\theta$

d) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cosh(tan\, \theta)\, sec^2\theta\, d\theta$

e) $\int_{0}^{\frac{\pi}{2}} 2\, sinh(sin\theta)\, cos\theta\, d\theta$

f) $\int_{1}^{2} \frac{cosh\,(ln\, t)}{t}\, dt$

g) $\int_{1}^{4} \frac{8\, cosh\sqrt{x}}{\sqrt{x}}\, dx$

h) $\int_{-ln2}^{0} cosh^2\left(\frac{x}{2}\right) dx$

1.14

Inverse Hyperbolic Functions

Learning objectives:

- To define six basic Inverse hyperbolic functions
- To find the derivatives and integrals of inverse hyperbolic functions
  AND
- To practice the related problems.

We use the inverses of the six basic hyperbolic functions in integration. Since $\frac{d(sinhx)}{dx} = coshx > 0$, the *hyperbolic sine* is an increasing function of x. We denote its inverse by

$$y = sinh^{-1}x$$

For every value of x in the interval $-\infty < x < \infty$, the value of $y = sinh^{-1}x$ is the number whose *hyperbolic sine* is x. The graphs of $y = sinhx$ and $y = sinh^{-1}x$ are shown in figure (a) below.

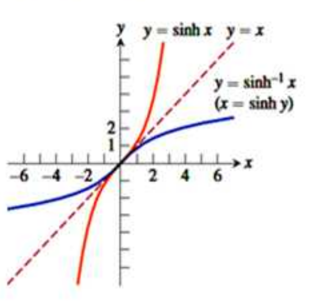


Fig (a)

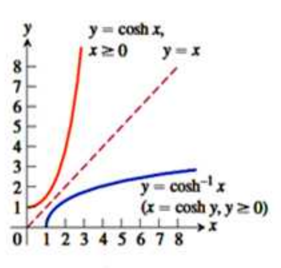


Fig (b)

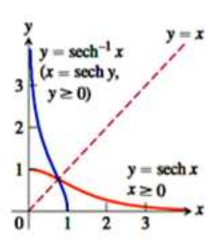


Fig (c)

The function $y = coshx$ is not one-to-one as we can see from its graph in the previous module. But the restricted function $y = coshx, x \geq 0$, is one-to-one and therefore has an inverse, denoted by

$$y = cosh^{-1}x$$

For every value of $x \geq 1$, $y = cosh^{-1}x$ is the number in the interval $0 \leq y < \infty$ whose *hyperbolic cosine* is x. The graphs of $y = coshx$, $x \geq 0$, and $y = cosh^{-1}x$ are shown in figure (b) above.

Like $y = coshx$, the function $y = \text{sech}\,x = 1/coshx$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

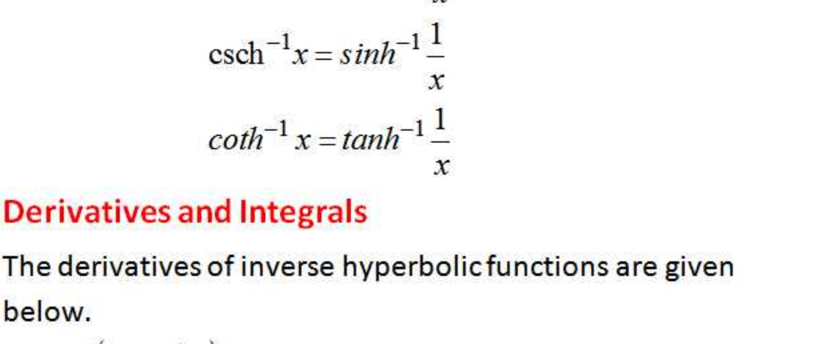
$$y = \text{sech}^{-1}x$$

For every value of x in the interval $(0, 1)$, $y = \text{sech}^{-1}x$ is the nonnegative number whose *hyperbolic secant* is x. The graphs of $y = \text{sech}\,x$, $x \geq 0$, and $y = \text{sech}^{-1}x$ are shown in figure (c) above.

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = tanh^{-1}x \quad , \quad y = coth^{-1}x \quad , \quad y = \text{csch}^{-1}x$$

The functions are graphed below.



The following identities for inverse hyperbolic functions are useful.

$$\text{sech}^{-1}x = cosh^{-1}\frac{1}{x}$$

$$\text{csch}^{-1}x = sinh^{-1}\frac{1}{x}$$

$$coth^{-1}x = tanh^{-1}\frac{1}{x}$$

Derivatives and Integrals

The derivatives of inverse hyperbolic functions are given below.

1. $\frac{d\left(sinh^{-1}u\right)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
2. $\frac{d\left(cosh^{-1}u\right)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \quad , \quad u > 1$
3. $\frac{d\left(tanh^{-1}u\right)}{dx} = \frac{1}{1-u^2}\frac{du}{dx} \quad |u| < 1$
4. $\frac{d\left(coth^{-1}u\right)}{dx} = \frac{1}{1-u^2}\frac{du}{dx} \quad |u| > 1$
5. $\frac{d\left(\text{sech}^{-1}u\right)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx} \quad 0 < u < 1$
6. $\frac{d\left(\text{csch}^{-1}u\right)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx} \quad u \neq 0$

The chief use of inverse hyperbolic functions lies in integrations that reverse the derivative formulas above.

The restrictions $|u| < 1$ and $|u| > 1$ on the derivative formulas for $tanh^{-1}u$ and $coth^{-1}u$ come from the natural restrictions on the values of these functions. The distinction between $|u| < 1$ and $|u| > 1$ becomes important when we convert the derivative formulas into integral formulas. If $|u| < 1$, the integral of $\frac{1}{\left(1-u^2\right)}$ is $tanh^{-1}u + C$. If $|u| > 1$, the integral is $coth^{-1}u + C$.

Example 1

Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}\left(cosh^{-1}u\right) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

Solution

First we find the derivative of $y = cosh^{-1}x$ for $x > 1$:

$y = cosh^{-1}x \Rightarrow x = cosh\,y$

Differentiating both sides w.r.t x, we get

$$1 = sinh\,y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{sinh\,y} = \frac{1}{\sqrt{cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}$$

In short, $\frac{d}{dx}\left(cosh^{-1}x\right) = \frac{1}{\sqrt{x^2-1}}$. The Chain Rule gives the final result:

$$\frac{d}{dx}\left(cosh^{-1}u\right) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

With appropriate substitutions, the derivative formulas, given earlier, lead to the integration formulas given below:

1. $\int\frac{du}{\sqrt{a^2+u^2}} = sinh^{-1}\left(\frac{u}{a}\right) + C \quad , \quad a > 0$
2. $\int\frac{du}{\sqrt{u^2-a^2}} = cosh^{-1}\left(\frac{u}{a}\right) + C \quad u > a > 0$
3. $\int\frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if} \quad u^2 < a^2 \\ \frac{1}{a}coth^{-1}\left(\frac{u}{a}\right) + C & \text{if} \quad u^2 > a^2 \end{cases}$
4. $\int\frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\text{sech}^{-1}\left(\frac{u}{a}\right) + C \quad 0 < u < a$
5. $\int\frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\text{csch}^{-1}\left|\frac{u}{a}\right| + C \quad u \neq 0$

Example 2

Evaluate $\int_0^1\frac{2dx}{\sqrt{3+4x^2}}$

Solution

The indefinite integral is

$$\int\frac{2dx}{\sqrt{3+4x^2}} = \int\frac{du}{\sqrt{a^2+u^2}} \quad , \quad u = 2x,\ a = \sqrt{3}$$

$$= sinh^{-1}\left(\frac{u}{a}\right) + C = sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

Therefore,

$$\int_0^1\frac{2dx}{\sqrt{3+4x^2}} = \left[sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right]_0^1 = sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - sinh^{-1}(0)$$

$$= sinh^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

Domain and Range of Inverse Hyperbolic Functions:

Inverse Hyperbolic function $y = f(x)$	Domain (x)	Range (y)
$y = sinh^{-1}x$	$\mathbb{R}$	$\mathbb{R}$
$y = cosh^{-1}x$	$[1, \infty)$	$[0, \infty)$
$y = tanh^{-1}x$	$(-1, 1)$	$\mathbb{R}$
$y = coth^{-1}x$	$\mathbb{R} - [-1, 1]$	$\mathbb{R} - \{0\}$
$y = sech^{-1}x$	$(0, 1]$	$[0, \infty)$
$y = csch^{-1}x$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$

Formulas for Inverse Hyperbolic Functions in terms of Natural logarithms.

1. $sinh^{-1}x = ln\left(x + \sqrt{x^2+1}\right), -\infty < x < \infty$
2. $cosh^{-1}x = ln\left(x + \sqrt{x^2-1}\right),\ x \geq 1$
3. $tanh^{-1}x = \frac{1}{2}\,ln\left(\frac{1+x}{1-x}\right),\ |x| < 1$
4. $sech^{-1}x = ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), 0 < x \leq 1$
5. $csch^{-1}x = ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right), x \neq 0$
6. $coth^{-1}x = \frac{1}{2}ln\left(\frac{x+1}{x-1}\right), |x| > 1$

Proof of 1:

Let $x \in \mathbb{R}$ and $y = sinh^{-1}x$. Then

$x = sinhy = \frac{e^y - e^{-y}}{2} = \frac{e^{2y}-1}{2e^y} \Rightarrow e^{2y} - e^y(2x) - 1 = 0$

This is a quadratic equation in e^y. Therefore,

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $e^y > 0$ for all $y \in \mathbb{R}$ and $x < \sqrt{x^2+1}$, we get

$$e^y = x + \sqrt{x^2+1}$$

Thus, $y = ln\left(x + \sqrt{x^2+1}\right)$

Hence,

$$\boxed{\boldsymbol{sinh^{-1}x = ln\left(x + \sqrt{x^2+1}\right)}, -\infty < x < \infty}$$

Proof of 2:

Let $x \in [1, \infty)$ and $y = cosh^{-1}x$. Then

$x = coshy = \frac{e^y + e^{-y}}{2} = \frac{e^{2y}+1}{2e^y} \Rightarrow e^{2y} - e^y(2x) + 1 = 0$

This is a quadratic equation in e^y. Therefore,

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

But $x - \sqrt{x^2-1} = \frac{1}{x+\sqrt{x^2-1}} < 1$, since $x > 1$

Further $e^y \geq 1$ for all $y \geq 0$

$$e^y = x + \sqrt{x^2-1}$$

Thus, $y = ln\left(x + \sqrt{x^2-1}\right)$

Hence,

$$\boxed{\boldsymbol{cosh^{-1}x = ln\left(x + \sqrt{x^2+1}\right), x \geq 1}}$$

Proof of 3:

Let $x \in (-1, 1)$ and $y = tanh^{-1}x$. Then

$$x = tanhy = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{(e^y)^2-1}{(e^y)^2+1}$$

$\Rightarrow x(e^y)^2 + x = (e^y)^2 - 1$

$\Rightarrow e^{2y} = \frac{1+x}{1-x} \Rightarrow 2y = ln\left(\frac{1+x}{1-x}\right)$

(Note that $\frac{1+x}{1-x} > 0$ for $x \in (-1, 1)$)

$\Rightarrow y = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right)$

Hence,

$$\boxed{\boldsymbol{tanh^{-1}x = \frac{1}{2}\,ln\left(\frac{1+x}{1-x}\right)},\ |x| < 1}$$

Similarly, we can prove the other formulas.

Example

Evaluate the integral $\int_0^{1/3}\frac{6\,dx}{\sqrt{1+9x^2}}$ in terms of Natural logarithms.

Solution:

To evaluate $\int_0^{1/3}\frac{6\,dx}{\sqrt{1+9x^2}}$

Put $u = 3x \Rightarrow du = 3dx$

Limits: $x = 0 \Rightarrow u = 0$; $x = \frac{1}{3} \Rightarrow u = 1$

$$\therefore \int_0^{1/3}\frac{6\,dx}{\sqrt{1+9x^2}} = 2\int_0^1\frac{du}{\sqrt{1+u^2}}$$

$$= \left[2sinh^{-1}u\right]_0^1 = \left[2sinh^{-1}1 - 2sinh^{-1}0\right]$$

$$= 2sinh^{-1}1 = 2ln\left(1 + \sqrt{1+1^2}\right) = 2ln\left(1+\sqrt{2}\right)$$

Example

Evaluate the integral $\int_{5/4}^{2}\frac{dx}{1-x^2}$ in terms of Natural logarithms.

Solution:

$$\int_{5/4}^{2}\frac{dx}{1-x^2} = \left[coth^{-1}x\right]_{5/4}^{2}$$

$$= coth^{-1}2 - coth^{-1}\frac{5}{4}$$

$$= \frac{1}{2}ln\left(\frac{2+1}{2-1}\right) - \frac{1}{2}ln\left(\frac{5/4+1}{5/4-1}\right)$$

$$= \frac{1}{2}[ln3 - ln9] = \frac{1}{2}ln\frac{1}{3}$$

IP1.

If $y = cosh^{-1}2\sqrt{x+1}$ then find $\frac{dy}{dx}$

Solution:

Given $y = cosh^{-1}2\sqrt{x+1}$

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{\left(2\sqrt{x+1}\right)^2 - 1}} \frac{d}{dx}\left[2(x+1)^{1/2}\right]$$

$$= \frac{1}{\sqrt{4(x+1)-1}}\left[2\,.\frac{1}{2}(x+1)^{\frac{1}{2}-1}\right]$$

$$= \frac{1}{\sqrt{4x+3}}\cdot\frac{1}{\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{(4x+3)(x+1)}}$$

$$= \frac{1}{\sqrt{4x^2+7x+3}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{4x^2+7x+3}}$$

IP2.

If $\boldsymbol{y = csch^{-1}(sinhx)}$ then find $\boldsymbol{\frac{dy}{dx}}$

Solution:

Given $y = csch^{-1}(sinhx)$

Differentiating w.r.t x on both sides, we get

$$\frac{dy}{dx} = \frac{d}{dx}[csch^{-1}(sinhx)]$$

$$= -\frac{1}{|sinhx|\sqrt{sinh^2x+1}} \frac{d}{dx}[sinhx]$$

$$= -\frac{coshx}{sinhx\sqrt{sinh^2x+1}}$$

$$= -\frac{coshx}{sinhx\sqrt{cosh^2x}} \qquad (\because \ cosh^2x - sinh^2x = 1)$$

$$= -\frac{coshx}{sinhx\ coshx}$$

$$= -\frac{1}{sinhx} = -cschx$$

$$\therefore \frac{dy}{dx} = -cschx$$

IP3.

Evaluate $\displaystyle\int_{1/5}^{3/13} \frac{1}{x\sqrt{1-16x^2}}\,dx$

Solution:

To evaluate $\displaystyle\int_{1/5}^{3/13} \frac{1}{x\sqrt{1-16x^2}}\,dx$

Put $u = 4x \Rightarrow du = 4\,dx \Rightarrow dx = \frac{du}{4}$

Limits: $= \frac{1}{5} \Rightarrow u = \frac{4}{5}$; $x = \frac{3}{13} \Rightarrow u = \frac{12}{13}$

Now,

$$\int_{1/5}^{3/13} \frac{1}{x\sqrt{1-16x^2}}\,dx = \int_{4/5}^{12/13} \frac{1}{\frac{u}{4}\sqrt{1-u^2}}\,\frac{du}{4}$$

$$= \left[-sech^{-1}u\right]_{4/5}^{12/13}$$

$$= -\left[sech^{-1}\frac{12}{13} - sech^{-1}\frac{4}{5}\right]$$

$$\therefore \int_{1/5}^{3/13} \frac{1}{x\sqrt{1-16x^2}}\,dx = -sech^{-1}\frac{12}{13} + sech^{-1}\frac{4}{5}$$

IP4.

Evaluate $\int \frac{dx}{\sqrt{2x^2+5x+6}}$

Solution:

$$\int \frac{dx}{\sqrt{2x^2+5x+6}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+\frac{5}{2}x+3}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{5}{4}\right)^2+3-\frac{25}{16}}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{5}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} sinh^{-1}\left(\frac{x+5/4}{\sqrt{23}/4}\right)+C = \frac{1}{\sqrt{2}} sinh^{-1}\left(\frac{4x+5}{\sqrt{23}}\right)+C$$

where C is an arbitrary constant

P1.

Find the derivative of y w.r.t the appropriate variable

a. $y = \cos^{-1} x - x\, \mathrm{sech}^{-1} x$

b. $y = (1 - t^2) \coth^{-1} t$

Solution:

a. Given $y = \cos^{-1} x - x\, \mathrm{sech}^{-1} x$

Differentiating w.r.t x on both sides, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x) - \frac{d}{dx}(x\, \mathrm{sech}^{-1} x)$$

$$= -\frac{1}{\sqrt{1-x^2}} - \left[x\left(-\frac{1}{x\sqrt{1-x^2}}\right) + (1)\, \mathrm{sech}^{-1} x\right], 0 < x < 1$$

$$= -\frac{1}{\sqrt{1-x^2}} - \left(-\frac{1}{\sqrt{1-x^2}} + \mathrm{sech}^{-1} x\right),\ 0 < x < 1$$

$$= -\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} - \mathrm{sech}^{-1} x, 0 < x < 1$$

$$= -\mathrm{sech}^{-1} x\ ,\ 0 < x < 1$$

$$\therefore\ \frac{dy}{dx} = -\mathrm{sech}^{-1} x\ ,\ 0 < x < 1$$

P2.

If $y = (\theta^2 + 2\theta)tanh^{-1}(\theta + 1)$ then find $\frac{dy}{d\theta}$

Solution:

Given $y = (\theta^2 + 2\theta)tanh^{-1}(\theta + 1)$

Differentiating both sides w.r.t θ, we get

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[(\theta^2 + 2\theta)tanh^{-1}(\theta + 1)]$$

$$= (\theta^2 + 2\theta)\frac{1}{1-(\theta+1)^2}\frac{d}{d\theta}(\theta + 1) + (2\theta + 2)tanh^{-1}(\theta + 1)$$

$$= (\theta^2 + 2\theta)\frac{1}{1-\theta^2-2\theta-1}(1) + 2(\theta + 1)tanh^{-1}(\theta + 1)$$

$$= \frac{(\theta^2+2\theta)}{-(\theta^2+2\theta)} + 2(\theta + 1)tanh^{-1}(\theta + 1)$$

$$= -1 + 2(\theta + 1)tanh^{-1}(\theta + 1)$$

$$\therefore \quad \frac{dy}{d\theta} = 2(\theta + 1)tanh^{-1}(\theta + 1) - 1$$

P3.

Evaluate $\displaystyle\int \frac{1}{\sqrt{9x^2-25}}\,dx$

Solution:

$$\int \frac{1}{\sqrt{9x^2-25}}\,dx = \int \frac{1}{5\sqrt{\frac{9x^2}{25}-1}}\,dx$$

$$= \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3x}{5}\right)^2-1}}\,dx$$

Put $u = \frac{3x}{5} \Longrightarrow du = \frac{3}{5}\,dx$

$$\int \frac{1}{\sqrt{9x^2-25}}\,dx = \frac{1}{5}\int \frac{5}{3}\frac{du}{\sqrt{u^2-1}}$$

$$= \frac{1}{3}cosh^{-1}u + C = \frac{1}{3}cosh^{-1}\left(\frac{3x}{5}\right) + C$$

$$\therefore \int \frac{1}{\sqrt{9x^2-25}}\,dx = \frac{1}{3}cosh^{-1}\left(\frac{3x}{5}\right) + C$$

where C is an arbitrary constant

P4.

Evaluate $\displaystyle\int \frac{x+1}{\sqrt{x^2-x+1}}\,dx$

Solution:

To evaluate $\int \frac{x+1}{\sqrt{x^2-x+1}}\ dx$

Put $u = x^2 - x + 1 \Rightarrow du = (2x-1)\,dx$

$$\therefore\ \ x+1 = \tfrac{1}{2}(2x-1) + \tfrac{3}{2}$$

Now,

$$\int \frac{x+1}{\sqrt{x^2-x+1}}\ dx = \int \frac{\frac{1}{2}(2x-1)+\frac{3}{2}}{\sqrt{x^2-x+1}}\ dx$$

$$= \frac{1}{2}\int \frac{2x-1}{\sqrt{x^2-x+1}}\ dx + \frac{3}{2}\int \frac{1}{\sqrt{x^2-x+1}}\ dx$$

$$= \frac{1}{2}\int \frac{\frac{d}{dx}\left(x^2-x+1\right)}{\sqrt{x^2-x+1}}\ dx + \frac{3}{2}\int \frac{1}{\sqrt{(x-1/2)^2+\frac{3}{4}}}\ dx$$

$$= \frac{1}{2}.\,2\sqrt{x^2-x+1} + \frac{3}{2}\int \frac{1}{\sqrt{(x-1/2)^2+\left(\sqrt{3}/2\right)^2}}\ dx$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}sinh^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right) + C$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$\therefore\ \ \int \frac{x+1}{\sqrt{x^2-x+1}}\ dx = \sqrt{x^2-x+1} + \frac{3}{2}sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

where C is an arbitrary constant

1. Find the derivatives of y with respect to the appropriate variable:

a. $y = sinh^{-1}\sqrt{x}$

b. $y = (1-\theta)tanh^{-1}\theta$

c. $y = (1-t)coth^{-1}\sqrt{t}$

d. $y = lnx + \sqrt{1-x^2}sech^{-1}x$

e. $y = csch^{-1}\left(\frac{1}{2}\right)^{\theta}$

f. $y = csch^{-1}(2)^{\theta}$

g. $y = sinh^{-1}(tanx)$

h. $y = cosh^{-1}(secx)$

2. Evaluate the following integrals:

a. $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$

b. $\displaystyle\int_0^{1/3} \frac{6\,dx}{\sqrt{1+9x^2}}$

c. $\displaystyle\int_{5/4}^{2} \frac{dx}{1-x^2}$

d. $\displaystyle\int_0^{1/2} \frac{dx}{1-x^2}$

e. $\displaystyle\int_1^{2} \frac{dx}{x\sqrt{4+x^2}}$

f. $\displaystyle\int_0^{\pi} \frac{\cos x\,dx}{x\sqrt{1+\sin^2 x}}$

g. $\displaystyle\int_1^{e} \frac{dx}{x\sqrt{1+(\ln x)^2}}$

h. $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+10}}$

i. $\displaystyle\int \frac{2x+5}{\sqrt{x^2-2x+10}}dx$

j. $\displaystyle\int \frac{dx}{\sqrt{2x^2-x+1}}$

k. $\displaystyle\int \frac{dx}{\sqrt{2x^2+5x+6}}$

l. $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+13}}$

m. $\displaystyle\int \frac{dt}{\sqrt{t^2+t+1}}$