2.1

Indefinite Integrals

Learning objectives:

- To define the indefinite integral of a function.
- To list some standard indefinite integrals.
- To study some standard arithmetic rules for indefinite integration.
- To determine a function from one of its known values and its derivative.

AND

- To practice the related problems.

Definition

A function $F(x)$ is an ***anti-derivative*** of a function $f(x)$ if

$$\boldsymbol{F'(x) = f(x)}$$

for all x in the domain of f.

The set of all anti-derivatives of f is the ***indefinite integral*** of f with respect to x, denoted by

$$\int f(x)dx$$

The symbol $\int$ is an ***integral sign***. The function f is the ***integrand*** of the integral and x is the ***variable of integration***.

Once we have found one anti-derivative F of a function f, the other anti-derivatives of f differ from F by a constant. We indicate this in the integral notation in the following way:

$$\int f(x)dx = F(x)+C \qquad ...(1)$$

The constant *C* is the ***constant of integration*** or ***arbitrary constant***. Equation (1) is read, "The indefinite integral of f with respect to x is $F(x)+C$." When we find $F(x)+C$, we say that we have ***integrated*** f and ***evaluated*** the integral.

Example 1: Evaluate $\int 2x\,dx$

Solution

$$\int 2x\,dx = x^2 + C$$

The formula $x^2 + C$ generates all the anti-derivatives of the function $2x$. The functions x^2+1, $x^2-\pi$, and $x^2+\sqrt{2}$ are all anti-derivatives of the function $2x$, as can be verified by differentiation.

Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas. The following table lists a number of standard integral forms side by side with their derivative-formula sources.

Indefinite Integral	Reversed derivative formula
$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C,$ $n \neq -1$, n rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$
$\int dx = \int 1\,dx = x + C$ (special case)	$\frac{d}{dx}(x)=1$
$\int \sin kx\,dx = -\frac{\cos kx}{k}+C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right)=\sin kx$
$\int \cos kx\,dx = \frac{\sin kx}{k}+C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right)=\cos kx$
$\int \sec^2 x\,dx = \tan x + C$	$\frac{d}{dx}(\tan x)=\sec^2 x$
$\int \csc^2 x\,dx = -\cot x + C$	$\frac{d}{dx}(-\cot x)=\csc^2 x$
$\int \sec x \tan x\,dx = \sec x + C$	$\frac{d}{dx}(\sec x)=\sec x \tan x$
$\int \csc x \cot x\,dx = -\csc x + C$	$\frac{d}{dx}(-\csc x)=\csc x \cot x$

Example 2:

a) $\int x^5\,dx = \frac{x^6}{6}+C$

b) $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = 2x^{1/2}+C = 2\sqrt{x}+C$

c) $\int \sin 2x\,dx = -\frac{\cos 2x}{2}+C$

d) $\int \cos\frac{x}{2}\,dx = \frac{\sin(x/2)}{1/2}+C = 2\sin\frac{x}{2}+C$

Once an integral formula is identified, it can be easily checked. The derivative should be the integrand.

Example 3:

Suppose we think $\int x\cos x\,dx = x\sin x + \cos x + C$.

We can easily check whether this is correct.

$$\begin{aligned}\frac{d}{dx}&(x\sin x+\cos x+C)\\ &= x\cos x + \sin x - \sin x + 0\\ &= x\cos x\end{aligned}$$

The derivative of the right-hand side is the integrand, and so it is correct. The standard arithmetic rules for indefinite integration are

1. $\int k\,f(x)\,dx = k\int f(x)\,dx$ **Constant Multiple Rule**
2. $\int -f(x)\,dx = -\int f(x)\,dx$ **Rule for Negatives**
3. $\int [f(x) \pm g(x)]dx = \int f(x)\,dx \pm \int g(x)\,dx$

Sum and Difference Rule

Example 4:

$$\begin{aligned}\int 5\sec x\tan x\,dx &= 5\int \sec x\tan x\,dx\\ &= 5(\sec x + C) = 5\sec x + 5C\end{aligned}$$

The constant of integration 5 times *C* is an arbitrary constant and so it can be written simply as *C* itself. Therefore, we can write simply

$$\int 5\sec x\tan x\,dx = 5\sec x + C$$

The Sum and Difference Rule for integration enables us to integrate expressions term by term. When we do so, we combine the individual constants of integration into a single arbitrary constant at the end.

Example 5:

Evaluate $\int (x^2-2x+5)\,dx$

Solution:

We can generate the anti-derivative term by term with the Sum and Difference Rule.:

$$\begin{aligned}\int (x^2-2x+5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx\\ &= \frac{x^3}{3}+C_1 - x^2 + C_2 + 5x + C_3\end{aligned}$$

If we combine C_1, C_2, and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and still gives all the anti-derivatives there are. For this reason, we can go right to the final form when you integrate term by term.

$$\begin{aligned}\int (x^2-2x+5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx\\ &= \frac{x^3}{3} - x^2 + 5x + C\end{aligned}$$

We find the simplest anti-derivative for each part and add the constant at the end.

We can sometimes use trigonometric identities such as

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

to transform integrals into integrals that can be evaluated using standard integral formulas. For example:

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx\\ &= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx\\ &= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C\\ &= \frac{x}{2} - \frac{\sin 2x}{4} + C\end{aligned}$$

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

The process of determining a function from one of its known values and its derivative $f(x)$ has two steps. The first is to find a formula that gives us all the functions that could possibly have f as a derivative. As we have seen earlier, these functions are the anti-derivatives of f and the set of all anti-derivatives of f is the indefinite integral of f. The second step is to use the known function value to select the particular anti-derivative we want from the indefinite integral. This is illustrated through an example.

Example 6:

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution:

Since the derivative of $(-\cos x)$ is $\sin x$, the function $f(x)$ will be $f(x) = -\cos x + C$ for some constant C. Since

$f(0) = 2$,

$$2 = -\cos(0) + C \Rightarrow C = 3$$

The formula for f is $f(x) = -\cos x + 3$

IP1:

Evaluate $\int \frac{x^2+5x-1}{\sqrt{x}}dx$

Solution:

$$\int \frac{x^2+5x-1}{\sqrt{x}}dx$$

$$= \int \frac{x^2}{\sqrt{x}}dx + \int \frac{5x}{\sqrt{x}}dx - \int \frac{1}{\sqrt{x}}dx$$

$$= \int x^{3/2}dx + 5\int x^{1/2}\,dx - \int x^{-1/2}\,dx$$

$$= \left[\frac{x^{3/2+1}}{3/2+1}\right] + 5\left[\frac{x^{1/2+1}}{1/2+1}\right] - \left[\frac{x^{-1/2+1}}{-1/2+1}\right] + C$$

where C is an arbitrary constant

$$= \frac{2}{5}x^{5/2} + 5.\frac{2}{3}x^{3/2} - 2x^{1/2} + C$$

$$= \frac{2}{5}x^{5/2} + \frac{10}{3}x^{3/2} - 2x^{1/2} + C$$

IP2:

Evaluate $\int \sin^3 2x \; dx$

Solution:

$$\int \sin^3 2x \; dx = \int \left[\frac{3\sin 2x - \sin 6x}{4} \right] dx$$

$$= \frac{1}{4} \int \left(3\sin 2x - \sin 6x \right) \, dx$$

$$= \frac{3}{4} \int \sin 2x \, dx - \frac{1}{4} \int \sin 6x \, dx$$

$$= -\frac{3}{4} \frac{\cos 2x}{2} + \frac{1}{4} \frac{\cos 6x}{6} + C$$

where C is an arbitrary constant

$$= \frac{\cos 6x}{24} - \frac{3\cos 2x}{8} + C$$

IP3:

Evaluate $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}\, dx$

Solution:

To evaluate $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}\, dx$

The integrand can be written as

$$\frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x} = \frac{\left(\sin^2 x\right)^3 + \left(\cos^2 x\right)^3}{\sin^2 x.\cos^2 x}$$

$$= \frac{\left(\sin^2 x + \cos^2 x\right)^3 - 3\left(\sin^2 x.\cos^2 x\right)\left(\sin^2 x + \cos^2 x\right)}{\left(\sin^2 x.\cos^2 x\right)}$$

$$= \frac{\left(\sin^2 x + \cos^2 x\right)^3}{\left(\sin^2 x.\cos^2 x\right)} - 3 = \sec^2 x.\csc^2 x - 3$$

$$\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}\, dx = \int \sec^2 x.\csc^2 x\, dx - 3\int dx$$

$$= \int \frac{1}{\cos^2 x.\sin^2 x} dx - 3x$$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x.\sin^2 x}\, dx - 3x$$

$$= \int \csc^2 x\, dx + \int \sec^2 x\, dx - 3x$$

$$= -\cot x + \tan x - 3x + C$$

where C is an arbitrary constant

IP4:

Find the function $F(t)$ whose derivative is $cost + sint$ and whose graph passes through the point $(\pi, 1)$.

Solution:

Given the derivative of $F(t)$ is $f(t) = cost + sint$

$\therefore$ The function $F(t)$ is the anti-derivative of $f(t)$ and

$$F(t) = \int f(t)\,dt + C$$

$$= \int (\cos t + \sin t)\,dt + C$$

$$= \sin t - \cos t + C$$

where C is an arbitrary constant

Given $F(\pi) = 1$

$\Rightarrow sin\pi - cos\pi + C = 1$

$\Rightarrow 0 + 1 + C = 1 \Rightarrow C = 0$

$\therefore F(t) = sint - cost$

P1:

Find $\int \left(4x^{3/2} - \frac{5}{\sqrt{x}} + \sin x \right) dx$

Solution:

$$\int \left(4x^{3/2} - \frac{5}{\sqrt{x}} + \sin x \right) dx$$

$$= 4\int x^{3/2} dx - 5\int \frac{1}{\sqrt{x}}\, dx + \int \sin x\, dx$$

$$= 4\left[\frac{x^{3/2+1}}{3/2+1} \right] - 5\left[\frac{x^{-1/2+1}}{-1/2+1} \right] + (-\cos x) + C$$

where C is an arbitrary constant

$$= 4\left[\frac{2}{5} x^{2/5} \right] - 10\sqrt{x} - cosx + C$$

$$= 2\sqrt{x}\left[\frac{4}{5} x^{2} - 5 \right] - \cos x + C$$

P2:

Find $\int \sqrt{1+\sin\frac{x}{2}}\,dx$

Solution:

$$\int \sqrt{1+\sin\frac{x}{2}}\,dx = \int \sqrt{\cos^2\frac{x}{4}+\sin^2\frac{x}{4}+2\sin\frac{x}{4}.\cos\frac{x}{4}}\;dx$$

$$= \int \sqrt{\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)^2}\;dx$$

$$= \int \left(\sin\frac{x}{4}+\cos\frac{x}{4}\right) dx$$

$$= \int \sin\frac{x}{4}\,dx + \int \cos\frac{x}{4}\,dx$$

$$= -\frac{\cos(x/4)}{1/4} + \frac{\sin(x/4)}{1/4} + C$$

$$= 4\left[\sin(x/4) - \cos(x/4)\right] + C$$

P3:

$$\int \frac{2\cos^3 x+3\sin^3 x}{\cos^2 x.\sin^2 x}\,dx = f(x)+C \Rightarrow f(x)=$$

Solution:

To evaluate $\displaystyle\int \frac{2\cos^3 x+3\sin^3 x}{\cos^2 x.\sin^2 x}\,dx$

The integrand can be written as

$$\frac{2\cos^3 x+3\sin^3 x}{\cos^2 x.\sin^2 x}=\frac{2\cos^3 x}{\cos^2 x.\sin^2 x}+\frac{3\sin^3 x}{\cos^2 x.\sin^2 x}$$
$$=2\cot x.\csc x+3\tan x.\sec x$$

$$\int \frac{2\cos^3 x+3\sin^3 x}{\cos^2 x.\sin^2 x}\,dx$$
$$=2\int \csc x.\cot x\;dx+3\int \sec x.\tan x\;dx$$
$$=-2\csc x+3\sec x+C$$
$$=3\sec x-2\csc x+C=f(x)+C$$
$$\Rightarrow f(x)=3\sec x-2\csc x$$

where C is an arbitrary constant

P4:

Find the function $\boldsymbol{F(x)}$ whose derivative is $\mathbf{9}\boldsymbol{x}^{\mathbf{2}} - \mathbf{4}\boldsymbol{x} + \mathbf{5}$ and whose graph passes through the point $(\mathbf{-1}, \mathbf{0})$.

Solution:

Given the derivative of $F(x)$ is $f(x) = 9x^2 - 4x + 5$

$\therefore$ The function $F(x)$ is the anti-derivative of $f(x)$ and

$$
\begin{aligned}
F(x) &= \int f(x)\,dx + C \\
&= \int \left(9x^2 - 4x + 5\right) dx + C \\
&= 9\left[\frac{x^3}{3}\right] - 4\left[\frac{x^2}{2}\right] + 5x + C \\
&= 3x^3 - 2x^2 + 5x + C
\end{aligned}
$$

where C is an arbitrary constant

Given $F(-1) = 0$

$\Rightarrow 3(-1)^3 - 2(-1)^2 + 5(-1) + C = 0$

$\Rightarrow -3 - 2 - 5 + C = 0 \Rightarrow C = 10$

$\therefore F(x) = 3x^3 - 2x^2 + 5x + 1$

2.1. Indefinite Integrals

Exercise:

Find an anti-derivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. $2x,\ x^2,\ x^2+1, -2x+1$
2. $-3x^{-4},\ \ x^{-4},\ \ x^{-4}+2x+3$
3. $\dfrac{1}{x^2},\ \dfrac{5}{x^2},\ 2-\dfrac{5}{x^2}$
4. $\dfrac{3}{2}\sqrt{x},\quad \dfrac{1}{2\sqrt{x}},\quad \sqrt{x}+\dfrac{1}{\sqrt{x}}$
5. $\dfrac{2}{3}x^{-1/3}\quad,\quad \dfrac{1}{3}x^{-2/3}\quad,\quad -\dfrac{1}{3}x^{-4/3}$
6. $-\pi\sin\pi x,\ \ 3\sin x,\ \ \sin\pi x-3\sin x$
7. $\sec^2 x,\qquad \dfrac{2}{3}\sec^2\dfrac{x}{3},\qquad -\sec^2\dfrac{3x}{2}$
8. $\csc x\cot x,\ \ -\csc 5x\cot 5x,\ \ -\pi\csc\dfrac{\pi x}{2}\cot\dfrac{\pi x}{2}$
9. $\left(\sin x-\cos x\right)^2$

Evaluate the integrals. Check your answers by differentiation.

10. $\int\left(x+1\right)dx$
11. $\int\left(3t^2+\dfrac{t}{2}\right)dt$
12. $\int\left(2x^3-5x+7\right)dx$
13. $\int\left(\dfrac{1}{x^2}-x^2-\dfrac{1}{3}\right)dx$
14. $\int x^{-1/3}\,dx$
15. $\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx$
16. $\int\left(8y-\dfrac{2}{y^{1/4}}\right)dy$
17. $\int 2x\left(1-x^{-3}\right)dx$
18. $\int\dfrac{t\sqrt{t}+\sqrt{t}}{t^2}dt$
19. $\int\left(-2\cos t\right)dt$
20. $\int 7\sin\dfrac{\theta}{3}d\theta$
21. $\int\left(-3\csc^2 x\right)dx$
22. $\int\dfrac{\csc\theta\cot\theta}{2}d\theta$
23. $\int\left(4\sec x\tan x-2\sec^2 x\right)dx$
24. $\int\left(\sin 2x-\csc^2 x\right)dx$
25. $\int 4\sin^2 y\,dy$
26. $\int\dfrac{1+\cos 4t}{2}dt$
27. $\int\left(1+\tan^2\theta\right)d\theta$
28. $\int\cot^2 x\,dx$
29. $\int\cos\theta\left(\tan\theta+\sec\theta\right)d\theta$

In problems 30-32, find the function with the given derivative whose graph passes through the point P.

30. $f'(x)=0\qquad P(-1,3)$
31. $f'(x)=2x-1\qquad P(0,0)$
32. $r'(\theta)=8-\csc^2\theta\qquad P(\dfrac{\pi}{4},0)$

2.2

Integration by Substitution

Learning objectives:

- To study power rule in integral form.
- To evaluate indefinite integrals by substitution method.

AND

- To practice the related problems.

Integration by Substitution

A change of variable can often turn an unfamiliar integral into one we can evaluate. The method for doing this is called the *substitution method of integration*. It is one of the principal methods for evaluating the integrals.

The Power rule in integral form:

If u is a differentiable function of x and n is a rational number different from -1, the Chain Rule tells us

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)=u^n\frac{du}{dx}$$

This same equation, from another point of view, says that $u^{n+1}/(n+1)$ is one of the anti-derivatives of the function $u^n(du/dx)$. Therefore,

$$\int\left(u^n\frac{du}{dx}\right)dx=\frac{u^{n+1}}{n+1}+C$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n\,du,$$

obtained by treating the dx's as differentials that cancel. Therefore, we have the following rule.

If u is any differentiable function,

$$\int u^n\,du=\frac{u^{n+1}}{n+1}+C \qquad (n\neq-1,\ n \text{ rational})$$

In deriving the above equation we assumed u to be a differentiable function of the variable x, but the name of the variable does not matter and does not appear in the final formula.

Example 1: **Evaluate** $\int(x+2)^5\,dx$

Solution:

We put the integral in the form $\int u^n\,du$ by substituting

$$u=(x+2), \quad du=d(x+2)=\frac{d}{dx}(x+2)\cdot dx=1\cdot dx=dx$$

Then $\int(x+2)^5\,dx=\int u^5du=\dfrac{u^6}{6}+C=\dfrac{(x+2)^6}{6}+C$

Example 2:

$$\int\sqrt{1+y^2}\cdot 2y\,dy=\int u^{1/2}du\ ,\quad u=1+y^2,\ du=2y\,dy$$

$$=\frac{u^{(1/2)+1}}{(1/2)+1}+C=\frac{2}{3}u^{3/2}+C$$

$$=\frac{2}{3}\left(1+y^2\right)^{3/2}+C$$

Example 3:

$$\int\sqrt{4t-1}\,dt=\int u^{1/2}\cdot\frac{1}{4}du,$$

$$u=4t-1;\ du=4dt;\ \frac{1}{4}du=dt$$

$$=\frac{1}{4}\int u^{1/2}du=\frac{1}{4}\cdot\frac{u^{3/2}}{3/2}+C=\frac{1}{6}u^{3/2}+C$$

$$=\frac{1}{6}(4t-1)^{3/2}+C$$

If u is a differentiable function of x, then $\sin u$ is a differentiable function of x. The Chain Rule gives the derivative of $\sin u$ as

$$\frac{d}{dx}\sin u=\cos u\frac{du}{dx}$$

From another point of view, however, this same equation says that $\sin u$ is one of the anti-derivatives of the product $\cos u\cdot\dfrac{du}{dx}$. Therefore,

$$\int\left(\cos u\frac{du}{dx}\right)dx=\sin u+C$$

A formal cancellation of the dx's in the integral on the left leads to the following rule.

If u is a differentiable function, then

$$\int\cos u\,du=\sin u+C$$

The companion formula for the integral of $\sin u$ when u is a differentiable function is

$$\int\sin u\,du=-\cos u+C$$

Example 4:

$$\int\cos(7\theta+5)d\theta=\int\cos u\cdot\frac{1}{7}du\ ,\quad u=7\theta+5;\ du=7d\theta$$

$$=\frac{1}{7}\int\cos u\,du=\frac{1}{7}\sin u+C$$

$$=\frac{1}{7}\sin(7\theta+5)+C$$

Example 5:

$$\int x^2\sin\left(x^3\right)dx=\int\sin\left(x^3\right)\cdot x^2dx \qquad u=x^3;\ du=3x^2dx$$

$$=\int\sin u\cdot\frac{1}{3}du=\frac{1}{3}\int\sin u\,du$$

$$=\frac{1}{3}(-\cos u)+C=-\frac{1}{3}\cos\left(x^3\right)+C$$

The Chain Rule formulas for the derivatives of the tangent, cotangent, secant, and cosecant of a differentiable function u lead to the following integrals.

$$\int\sec^2u\,du=\tan u+C,\qquad \int\sec u\tan u\,du=\sec u+C$$

$$\int\csc^2u\,du=-\cot u+C,\qquad \int\csc u\cot u\,du=-\csc u+C$$

In each formula, **u** is a differentiable function of a real variable.

Example 6:

$$\int\frac{1}{\cos^2 2\theta}d\theta=\int\sec^2 2\theta\,d\theta \quad ,\ u=2\theta;\ du=2d\theta$$

$$=\int\sec^2u\cdot\frac{1}{2}du$$

$$=\frac{1}{2}\int\sec^2u\,du=\frac{1}{2}\tan u+C=\frac{1}{2}\tan 2\theta+C$$

Example 7:

$$\int\left(x^2+2x-3\right)^2(x+1)dx$$

put $u=x^2+2x-3;\ du=2xdx+2dx=2(x+1)dx$

$$=\int u^2\cdot\frac{1}{2}du=\frac{1}{2}\int u^2du=\frac{1}{2}\cdot\frac{u^3}{3}+C$$

$$=\frac{1}{6}u^3+C=\frac{1}{6}\left(x^2+2x-3\right)^3+C$$

Example 8:

$$\int\sin^4t\cos t\,dt=\int u^4du\quad ,u=\sin t;\ du=\cos t\,dt$$

$$=\frac{u^5}{5}+C=\frac{\sin^5t}{5}+C$$

The success of the substitution method depends on finding a substitution that will change an integral we cannot evaluate directly into one that we can.

P1:

Evaluate $\int \frac{3x+1}{\left(3x^2+2x+1\right)^3}\,dx$

Solution:

To evaluate $\int \frac{3x+1}{\left(3x^2+2x+1\right)^3}\,dx$

$$Put\ \ 3x^2+2x+1=u$$

$$\Rightarrow (6x+2)\,dx=du \Rightarrow (3x+1)\,dx=\frac{1}{2}du$$

$$\int \frac{3x+1}{\left(3x^2+2x+1\right)^3}\,dx=\frac{1}{2}\int \frac{du}{u^3}$$

$$=\frac{1}{2}\int u^{-3}\,du=\frac{1}{2}\left[\frac{u^{-3+1}}{-3+1}\right]$$

$$=-\frac{1}{4u^2}+C=-\frac{1}{4\left(3x^2+2x+1\right)^2}+C$$

where C is an arbitrary constant

P2:

Evaluate $\int \frac{x^2}{\sqrt{1+x}} dx$

Solution:

To evaluate $\int \frac{x^2}{\sqrt{1+x}} dx$

Put $1+x=u^2 \Rightarrow dx = 2u\,du$ and $x = u^2-1$

$$\int \frac{x^2}{\sqrt{1+x}} dx = \int \frac{\left(u^2-1\right)^2}{u} .2u\ du$$

$$= 2\int\left(u^4+1-2u^2\right)du$$

$$= 2\left[\int u^4 du + \int du - 2\int u^2\ du\right]$$

$$= 2\left[\frac{u^5}{5}+u-2.\frac{u^3}{3}\right]+C$$

$= \frac{2}{5}(1+x)^{5/2} - \frac{4}{3}(1+x)^{3/2} + 2\sqrt{1+x}+C$, where C is an arbitrary constant

P3:

Evaluate $\int \frac{\sin^4 x}{\cos^6 x}\, dx$

Solution:

$$\int \frac{\sin^4 x}{\cos^6 x}\, dx = \int \tan^4 x . \sec^2 x \, dx$$

Put $tanx = u \Rightarrow sec^2x \;\; dx = du$

$$\therefore \int \frac{\sin^4 x}{\cos^6 x}\, dx = \int u^4 \; du$$

$$= \frac{u^5}{5} + C$$

$$= \frac{\tan^5 x}{5} + C$$, where C is an arbitrary constant

P4:

Evaluate $\int \frac{\tan^4 \sqrt{x}.\sec^2 \sqrt{x}}{\sqrt{x}}\, dx$

Solution:

To evaluate $\int \frac{\tan^4 \sqrt{x}.\sec^2 \sqrt{x}}{\sqrt{x}}\, dx$

Put $\sqrt{x} = u \Longrightarrow \frac{1}{2\sqrt{x}}\, dx = du$

$$\int \frac{\tan^4 \sqrt{x}.\sec^2 \sqrt{x}}{\sqrt{x}}\, dx = 2\int \tan^4 u.\sec^2 u\ du$$

Again put $tanu = v \Longrightarrow sec^2u\, du = dv$

$$\int \frac{\tan^4 \sqrt{x}.\sec^2 \sqrt{x}}{\sqrt{x}}\, dx = 2\int v^4\, dv = 2\left[\frac{v^5}{5}\right] + C$$

$$= \frac{2}{5}\tan^5 u + C$$

$$= \frac{2}{5}\tan^5 \sqrt{x} + C$$

where C, is an arbitrary constant

IP1:

Evaluate $\int \frac{\left(x^4 - x\right)^{1/4}}{x^5}\, dx$

Solution:

$$\int \frac{\left(x^4 - x\right)^{1/4}}{x^5}\, dx = \int \frac{x\left(1 - \frac{1}{x^3}\right)^{1/4}}{x^5}\, dx = \int \frac{1}{x^4}\left(1 - \frac{1}{x^3}\right)^{1/4} dx$$

Put $1 - \frac{1}{x^3} = u \Rightarrow \frac{3}{x^4} dx = du \Rightarrow \frac{dx}{x^4} = \frac{du}{3}$

$$\int \frac{1}{x^4}\left(1 - \frac{1}{x^3}\right)^{1/4} dx$$

$$= \frac{1}{3}\int u^{1/4}\, du = \frac{1}{3}\left[\frac{u^{1/4+1}}{1/4+1}\right] + C$$

$$= \frac{4}{15} u^{5/4} + C = \frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{5/4} + C,$$

where C is an arbitrary constant

IP2:

Evaluate $\int \frac{x^2}{\sqrt{x+5}}\,dx,\ x \in (-5, \infty)$

Solution:

To evaluate $\int \frac{x^2}{\sqrt{x+5}}\,dx$

Put $x + 5 = u^2$ so that $u > 0$ on $(-5, \infty)$

$\Longrightarrow dx = 2u\,du$ and $x = u^2 - 5$

$$\int \frac{x^2}{\sqrt{x+5}}\,dx = \int \frac{\left(u^2-5\right)^2}{u}.2u\,du$$

$$= 2\int \left(u^4 + 25 - 10u^2\right) du$$

$$= 2\left[\int u^4 du + 25\int du - 10\int u^2 du\right]$$

$$= 2\left[\frac{u^5}{5} + 25u - 10\frac{u^3}{3}\right] + C$$

$$= \frac{2}{5}(x+5)^{5/2} + 50\sqrt{x+5} - \frac{20}{3}(x+5)^{3/2} + C$$, where

C is an arbitrary constant

IP3:

Evaluate $\int \sin^2 x.\cos^5 x\, dx$

Solution:

To evaluate $\int \sin^2 x.\cos^5 x\, dx$

Put $sinx = u \Rightarrow cosx\ \ dx = du$

$$\int \sin^2 x.\cos^5 x\, dx = \int \sin^2 x.\cos^4 x.\left(\cos x\, dx\right)$$

$$= \int \sin^2 x.\left(1-\sin^2 x\right)^2.\left(\cos x\, dx\right)$$

$$= \int u^2\left(1-u^2\right)^2\, du$$

$$= \int u^2\left(1+u^4-2u^2\right)du$$

$$= \int \left(u^2+u^6-2u^4\right)du$$

$$= \frac{u^3}{3}+\frac{u^7}{7}-\frac{2u^5}{5}+C$$

$= \dfrac{\sin^3 x}{3}+\dfrac{\sin^7 x}{7}-\dfrac{2\sin^5 x}{5}+C$, where C is an arbitrary constant

IP4:

Evaluate $\int \frac{\tan x . \sec^2 x}{\left(a+b\tan^2 x\right)^2}\, dx$

Solution:

To evaluate $\int \frac{\tan x . \sec^2 x}{\left(a+b\tan^2 x\right)^2}\, dx$

Put $a + btan^2x = u \Rightarrow 2btanx.sec^2x \;\; dx = du$

$$\Rightarrow tanx.sec^2x \;\; dx = \frac{du}{2b}$$

$$\int \frac{\tan x . \sec^2 x}{\left(a+b\tan^2 x\right)^2}\, dx = \frac{1}{2b}\int \frac{1}{u^2}\, du$$

$$= \frac{1}{2b}\left(-\frac{1}{u}\right) + C$$

$$= -\frac{1}{2b\left(a+b\tan^2 x\right)} + C$$, where C is an arbitrary constant

2.2. Integration by Substitution

Exercise:

Evaluate the integrals

1. $\int\left(3t^2+\frac{t}{2}\right)dt$
2. $\int\left(2x^3-5x+17\right)dx$
3. $\int\left(\frac{1}{x^3}-x^2-\frac{1}{3}\right)dx$
4. $\int x^{-1/3}\,dx$
5. $\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx$
6. $\int\left(8y-\frac{2}{y^{-1/4}}\right)dy$
7. $\int 2x\left(1-x^{-3}\right)dx$
8. $\int\left(\frac{t\sqrt{t}+\sqrt{t}}{t^2}\right)dt$
9. $\int(-2\cos t)\,dt$
10. $\int 7\sin\frac{\theta}{3}\,d\theta$

Evaluate the indefinite integrals by using the given substitution to reduce the integrals to standard form.

1. $\int\sin 3x\,dx,\qquad u=3x$
2. $\int\sec 2t\tan 2t\,dt,\qquad u=2t$
3. $\int 28(7x-2)^{-5}\,dx,\qquad u=7x-2$
4. $\int\frac{9r\,dr}{\sqrt{1-r}},\qquad u=1-r$
5. $\int\sqrt{x}\sin(x-1)dx\quad,\qquad u=x-1$
6. $\int\csc 2\theta\cot 2\theta\,d\theta$
   a. using $u=\cot 2\theta$
   b. using $u=\csc 2\theta$

Evaluate the integrals

7. $\int(3-2s)\,ds$
8. $\int\frac{1}{\sqrt{5s+4}}\,ds$
9. $\int\theta\sqrt[4]{1-\theta^2}\,d\theta$
10. $\int 3y\sqrt{7-3y^2}\,dy$
11. $\int\frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx$
12. $\int\cos(3z+4)dz$
13. $\int\sec(3x+2)dx$
14. $\int\sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$
15. $\int r^2\left(\frac{r^3}{18}-1\right)^5dr$
16. $\int x^{1/2}\sin\left(x^{3/2}+1\right)dx$
17. $\int\sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)dv$
18. $\int\frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$
19. $\int\sqrt{\cot y}\csc^2 y\,dy$
20. $\int\frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$
21. $\int\frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$
22. $\int\left(s^3+2s^2-5s+5\right)\left(3s^2+4s-5\right)ds$
23. $\int t^3\left(1+t^4\right)^3dt$

2.3

Approximation by Finite Sums

Learning objectives:

- To show how areas, volumes and the distances traveled by objects over time can be approximated by the finite sums.

  AND
- To practice the related problems.

One of the great achievements of classical geometry was to obtain formulas for the areas and volumes of triangles, spheres and cones. We now study a method to calculate the areas and volumes of these and other more general shapes. The method is *integration* and it is a tool for calculating much more than areas and volumes. The *integral* has many applications in sciences, engineering, economics and statistics.

The idea behind integration is that we can effectively compute many quantities by breaking them into small pieces and then summing the contributions from each small part. We begin with examples involving finite sums. These lead to question of what happens when more and more terms are summed. Passing to the limit, as the number of terms goes to infinity, gives an integral.

This module shows how areas, volumes and distance travelled by an object over time can be approximated by finite sums. Finite sums are the basis for defining the integral.

Distance Traveled

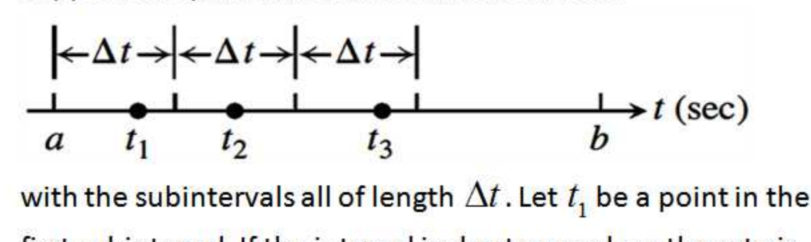
Suppose we know the velocity function $v=\frac{ds}{dt}=f(t)$ *m/s* of a car moving down a highway and want to know how far the car will travel in the time interval $a\le t\le b$.

We can approximate the distance traveled with a sum in the following way. We partition $[a,b]$ into short time intervals on each of which v is fairly constant. We approximate the distance traveled on each time interval with the formula

Distance $= f(t)\cdot\Delta t$

and add the results across $[a,b]$.

Suppose the partitioned interval looks like this



with the subintervals all of length Δt. Let t_1 be a point in the first subinterval. If the interval is short enough so the rate is almost constant, the car will move about $f(t_1)\Delta t$ *m* during that interval. If t_2 is a point in the second interval, the car will move an additional $f(t_2)\Delta t$ *m* during that interval, and so on. The sum of these products approximates the total distance D traveled from $t=a$ to $t=b$.

If we use n subintervals, then

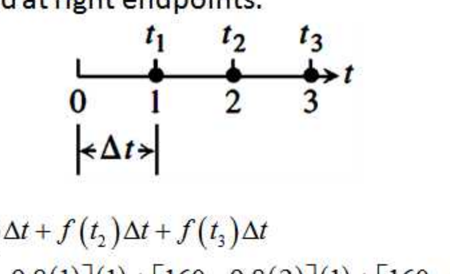
$$D\approx f(t_1)\Delta t+f(t_2)\Delta t+\cdots+f(t_n)\Delta t$$

Example 1:

The velocity function of a projectile fired straight into the air is $f(t)=160-9.8t$. Use the summation technique just described to estimate how far the projectile rises during the first *3 seconds*. How close do the sums come to the exact figure of 435.9 *m*?

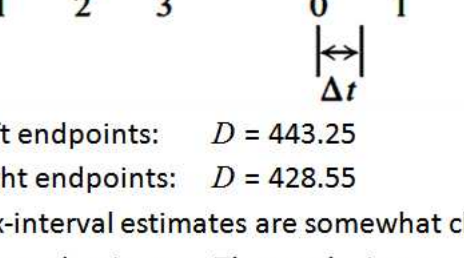
Solution

We consider 3 subintervals of length 1, with f evaluated at left endpoints.



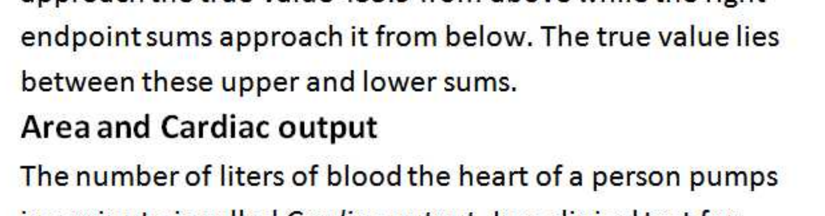
$$D\approx f(t_1)\Delta t+f(t_2)\Delta t+f(t_3)\Delta t$$
$$=[160-9.8(0)](1)+[160-9.8(1)](1)+[160-9.8(2)](1)=450.6$$

We now consider 3 subintervals of length 1, with f evaluated at right endpoints.



$$D\approx f(t_1)\Delta t+f(t_2)\Delta t+f(t_3)\Delta t$$
$$=[160-9.8(1)](1)+[160-9.8(2)](1)+[160-9.8(3)](1)=421.2$$

With 6 subintervals of length $\frac{1}{2}$, we get



Using left endpoints: $D = 443.25$

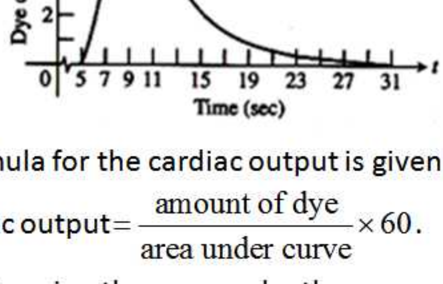
Using right endpoints: $D = 428.55$

These six-interval estimates are somewhat closer than three-interval estimates. The results improve as the subintervals get closer.

As we can see from these figures, the left endpoint sums approach the true value 435.9 from above while the right-endpoint sums approach it from below. The true value lies between these upper and lower sums.

Area and Cardiac output

The number of liters of blood the heart of a person pumps in a minute is called *Cardiac output*. In a clinical test for determining the cardiac output, 5.6 *mg* of dye is injected in a main vein near the heart. The concentration (*mg/L*) of the dye is measured every few seconds and the data is plotted as shown below.

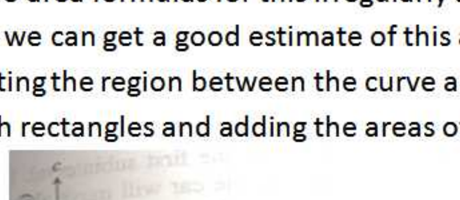


The formula for the cardiac output is given by

$$\text{Cardiac output}=\frac{\text{amount of dye}}{\text{area under curve}}\times 60.$$

If we determine the area under the curve, we can find the cardiac output of the patient.

There are no area formulas for this irregularly shaped region. But we can get a good estimate of this area by approximating the region between the curve and the t −axis with rectangles and adding the areas of rectangles.



Each rectangle omits some of the area under the curve but includes area from outside the curve, which compensates. Each rectangle has a base 2 units long and a height that is equal to the height of the curve above the ***midpoint*** of the base. The rectangle's height acts as a sort of average value of the function over the time interval on which the rectangle stands.

Area under curve ≈ sum of rectangle areas

$$=f(6)\cdot 2+f(8)\cdot 2+f(10)\cdot 2+\cdots+f(28)\cdot 2$$
$$=(1.4)(2)+(6.3)(2)+(7.5)(2)+\cdots+(0.1)(2)$$
$$=(28.8)(2)=57.6\ mg\cdot s/L$$

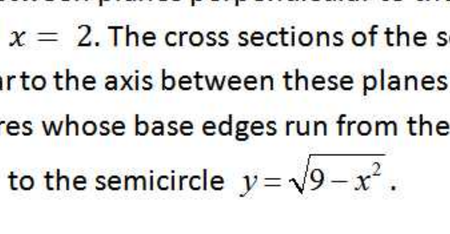
$$Cardiac\ output=\frac{5.6\ mg}{57.6\ mg.s/L}\times\frac{60\ s}{1\ min}=5.8\ L/min$$

The patient's cardiac output is about 5.8 *L/min*.

Additional Examples

Example 2:

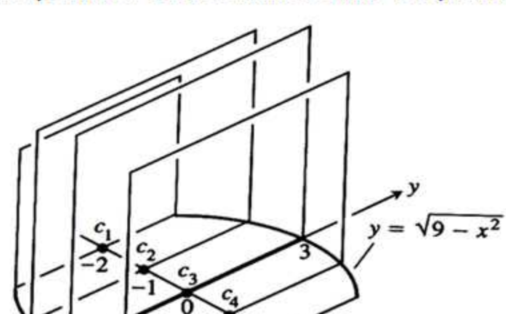
A solid lies between planes perpendicular to the x −axis at $x = -2$ and $x = 2$. The cross sections of the solid perpendicular to the axis between these planes are vertical squares whose base edges run from the semicircle $y=-\sqrt{9-x^2}$ to the semicircle $y=\sqrt{9-x^2}$.



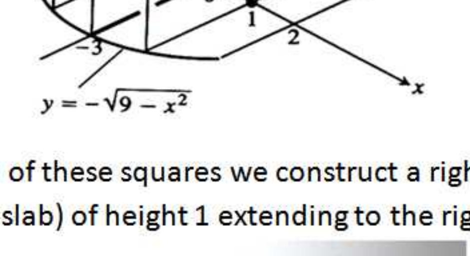
The height of the square at x is $2\sqrt{9-x^2}$. Estimate the volume of the solid.

Solution

We partition the interval [−2, 2] on the x −axis into four subintervals of length $\Delta x = 1$. The solid's cross section at the left endpoint of each subinterval is a square.



On each of these squares we construct a right cylinder (Square slab) of height 1 extending to the right.



We add the cylinders' volumes to estimate the volume of the solid.

The area of the solid's cross section at x is

The area of the solid's cross section at x is

$$A(x)=\left(2\sqrt{9-x^2}\right)^2=4\left(9-x^2\right).$$

So, the sum of the volumes of the cylinders is

$$S_4=A(c_1)\Delta x+A(c_2)\Delta x+A(c_3)\Delta x+A(c_4)\Delta x$$
$$=4\left(9-c_1^2\right)(1)+4\left(9-c_2^2\right)(1)+4\left(9-c_3^2\right)(1)+4\left(9-c_4^2\right)(1)$$
$$=4\left[\left(9-(-2)^2\right)+\left(9-(-1)^2\right)+\left(9-(0)^2\right)+\left(9-(1)^2\right)\right]$$
$$=4\left[(9-4)+(9-1)+(9-0)+(9-1)\right]$$
$$=4(30)=120$$

We will show later that the true volume of solid is $V=368/3\approx 122.67$. The difference between S and V is a small percentage of V:

Error percentage $=\frac{|V-S_4|}{V}=\frac{(368/3)-120}{(368/3)}=\frac{8}{368}\approx 2.2\%$

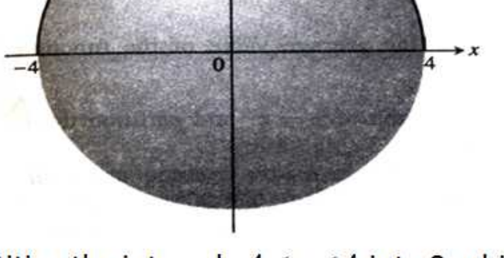
With a finer partition (more subintervals) the approximation would be even better.

Example 3:

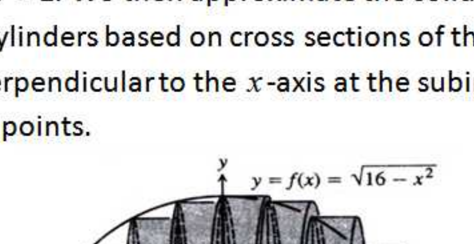
Estimate the volume of a solid sphere of radius 4.

Solution

We picture the sphere as if its surface were generated by revolving the graph of the function $f(x)=\sqrt{16-x^2}$ about the x −axis.



We partition the interval $-4\le x\le 4$ into 8 subintervals of length Δx = 1. We then approximate the solid with right circular cylinders based on cross sections of the solid by planes perpendicular to the x-axis at the subintervals' left-hand endpoints.



The cylinder at $x = -4$ is degenerate because the cross section there is just a point. We add the cylinders' volumes to estimate the volume of a sphere.

The sum of the eight cylinders' volume is

$$S_8=\pi\left[f(c_1)\right]^2\Delta x+\pi\left[f(c_2)\right]^2\Delta x+\cdots+\pi\left[f(c_8)\right]^2\Delta x$$
$$=\pi\left[\sqrt{16-c_1^{\,2}}\right]^2\Delta x+\pi\left[\sqrt{16-c_2^{\,2}}\right]^2\Delta x+\cdots+\pi\left[\sqrt{16-c_8^{\,2}}\right]^2\Delta x$$
$$=\pi\left[+\left(16-(-4)^2\right)+\left(16-(-3)^2\right)+\cdots+\left(16-(3)^2\right)\right]$$
$$=\pi\left[0+7+12+15+16+15+12+7\right]=84\pi$$

The sphere's true volume is $V=\frac{4}{3}\pi r^3=\frac{4}{3}\pi(4)^3=\frac{256\pi}{3}$

The difference between S_8 and V is a small percentage of V:

Error percentage

$$=\frac{|V-S_8|}{V}=\frac{(256/3)\pi-84\pi}{(256/3)\pi}=\frac{256-253}{256}=\frac{1}{64}\approx 1.6\%$$

All the examples above describe instances in which sums of function values multiplied by interval lengths provide approximations that are good enough to solve practical problems.

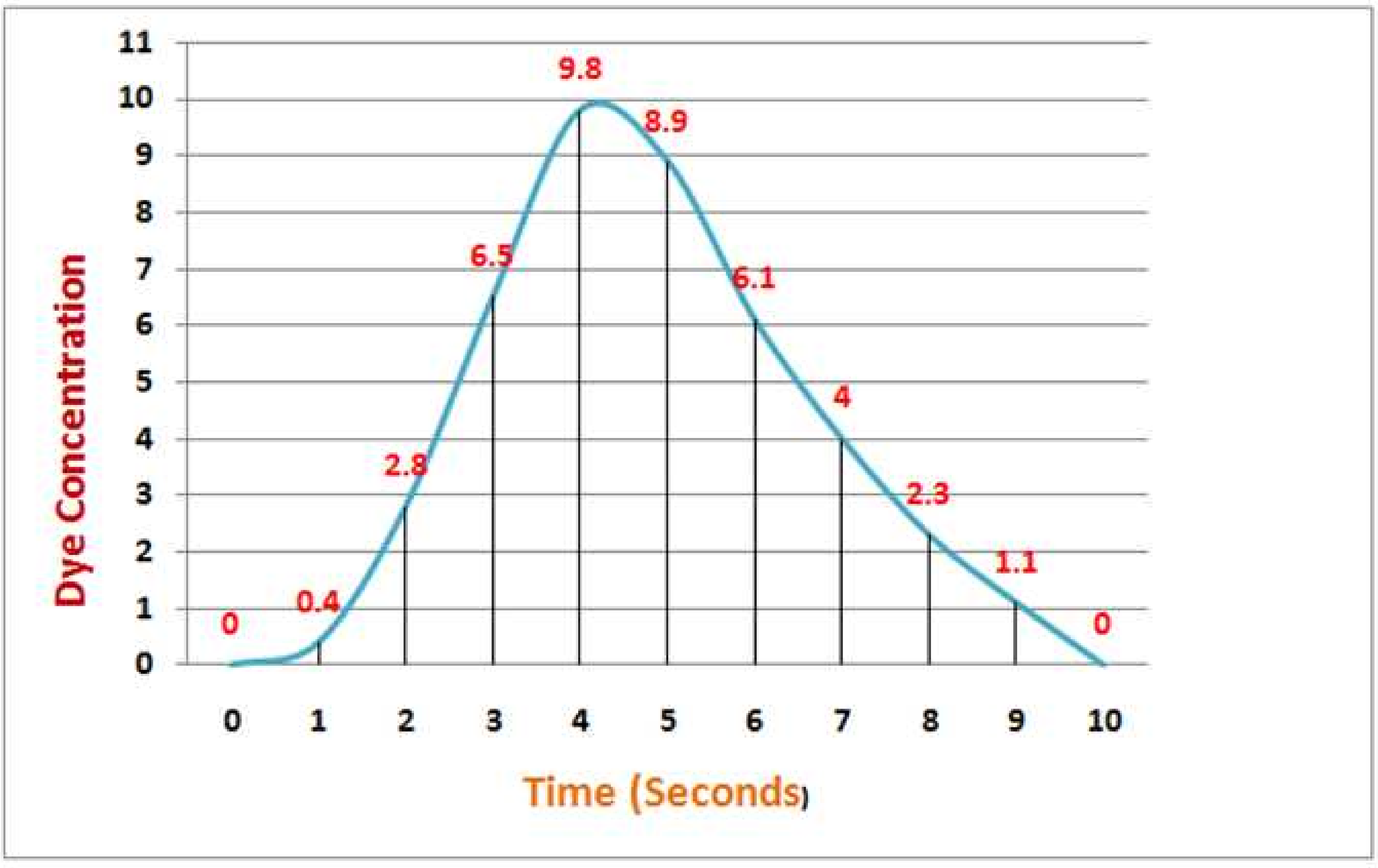
IP1:

A 5-mg dye is injected in a main vein near the heart. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the chart. Estimate the patient's cardiac output.

Seconds after injection (t)	Dye concentration (mg/L)
0	0
1	0.4
2	2.8
3	6.5
4	9.8
5	8.9
6	6.1
7	4.0
8	2.3
9	1.1
10	0

Solution:

From the figure, each rectangle has a base of 1 unit. The rectangle's height acts as average value of the function over the time interval on which the rectangle stands.



∴ Area under the curve ≈ sum of the rectangles areas

$$= f(0).1 + f(1).1 + f(2).1 + \dots\dots\dots + f(9) + f(10)$$

$$= \begin{bmatrix} 0 + 0.4 + 2.8 + 6.5 + 9.8 + 8.9 + 6.1 + 4.0 \\ +2.3 + 1.1 + 0 \end{bmatrix}$$

$= 41.9$ mg.s/L

Now, the cardiac output is $= \dfrac{5\ mg}{41.9\ mg.s/L} \times \dfrac{60s}{1\ min} = 7.2\ L/min$

Hence the patient's cardiac output is $\mathbf{7.2}$ ***L /min***

IP2:

Distance traveled upstream

You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 using

a. Left endpoints
b. Right endpoints

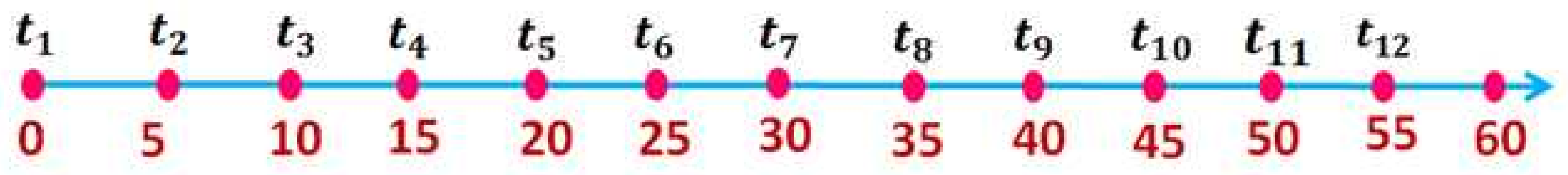
Time t (minutes)	Velocity v m/sec)
0	1
5	1.2
10	1.7
15	2.0
20	1.8
25	1.6
30	1.4
35	1.2
40	1.0
45	1.8
50	1.5
55	1.2
60	0

Solution:

In upstream the distance travelled by the bottle in an hour can be estimated as follows:

Using left endpoints

The estimation using 12 subintervals of length 5minutes (300 sec) using left endpoints is



$$D \approx v(t_1)\Delta t + v(t_2)\Delta t + v(t_3)\Delta t + \ldots\ldots + v(t_{12})\Delta t$$

$$= [v(0) + v(5) + v(10) + \ldots\ldots + v(55)](300)$$

$$(\because \Delta t = 300sec)$$

$$= \begin{bmatrix} 1 + 1.2 + 1.7 + 2.0 + 1.8 + 1.6 + 1.4 \\ +1.2 + 1.0 + 1.8 + 1.5 + 1.2 \end{bmatrix}(300)$$

$$= 5220\ m$$

Using right endpoints

The estimation using 12 subintervals of length 5minutes (300 sec) using right endpoints is

$$D \approx v(t_1)\Delta t + v(t_2)\Delta t + v(t_3)\Delta t + \ldots\ldots + v(t_{12})\Delta t$$

$$= [v(5) + v(10) + v(15) + \ldots\ldots + v(60)](300)$$

$$= \begin{bmatrix} 1.2 + 1.7 + 2.0 + 1.8 + 1.6 + 1.4 \\ +1.2 + 1.0 + 1.8 + 1.5 + 1.2 + 0 \end{bmatrix}(300)$$

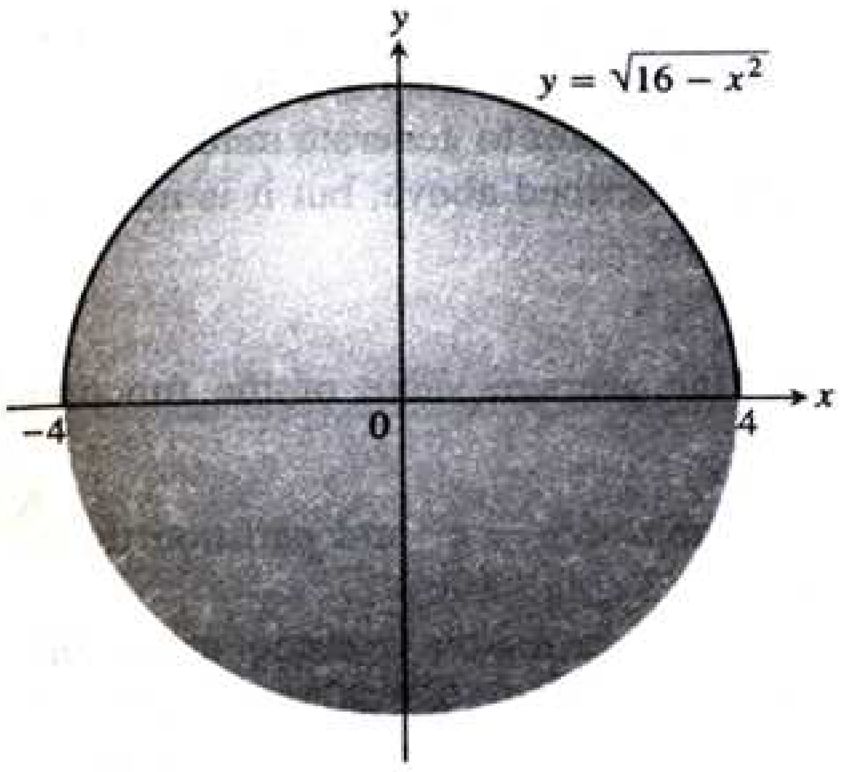
$$= 4920\ m$$

IP3:

Estimate the volume of a solid sphere of radius 4.

Solution

We picture the sphere as if its surface were generated by revolving the graph of the function $f(x)=\sqrt{16-x^2}$ about the x -axis.



We partition the interval $-4 \le x \le 4$ into 4 subintervals of length $\Delta x = 2$. We then approximate the solid with right circular cylinders based on cross sections of the solid by planes perpendicular to the x-axis at the subintervals' left-hand endpoints.

The cylinder at $x = -4$ is degenerate because the cross section there is just a point. We add the cylinders' volumes to estimate the volume of a sphere.

The sum of the four cylinders' volume is

$$S_4 = \pi\left[f(c_1)\right]^2 \Delta x + ... + \pi\left[f(c_4)\right]^2 \Delta x$$

$$= \pi\left[\sqrt{16-c_1^{\ 2}}\right]^2 \Delta x + \cdots + \pi\left[\sqrt{16-c_4^{\ 2}}\right]^2 \Delta x$$

$$= 2\pi\left[\left(16-(-4)^2\right)+\left(16-(-2)^2\right)+\left(16-(0)^2\right)+\left(16-(2)^2\right)\right]$$

$$= 2\pi\left[0+12+16+12\right] = 80\pi$$

The sphere's true volume is $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi(4)^3 = \dfrac{256\pi}{3}$

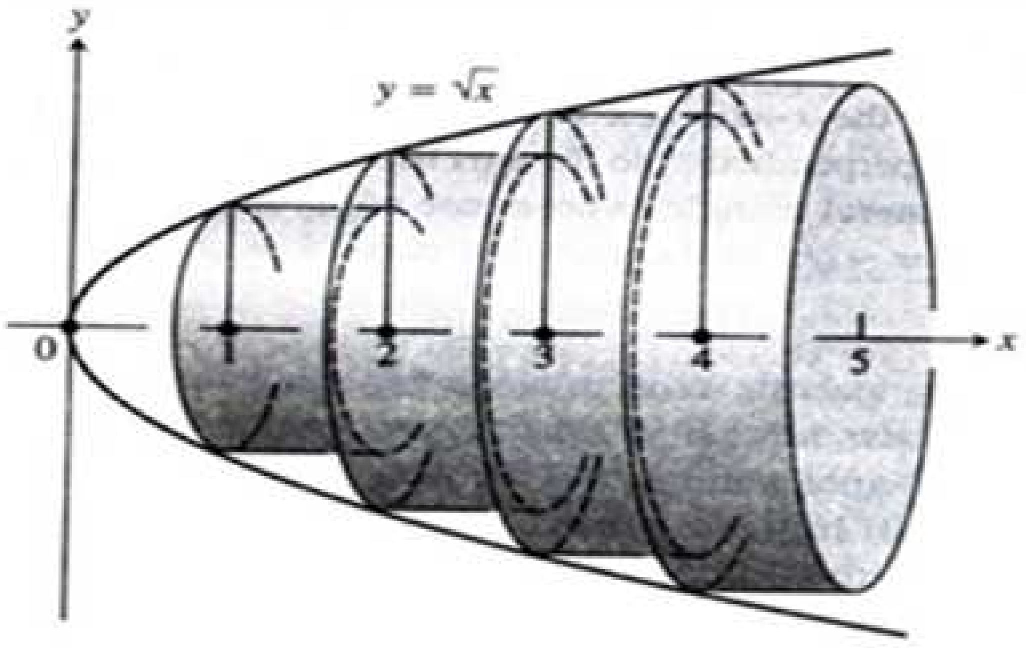
The difference between S_4 and V is a small percentage of V:

Error percentage

$$= \frac{|V - S_4|}{V} = \frac{(256/3)\pi - 80\pi}{(256/3)\pi} = \frac{256-240}{256} = \frac{1}{16} \approx 6.2\%$$

IP4:

The nose "cone" of a rocket is a paraboloid obtained by revolving the curve $y=\sqrt{x}$, $0\le x\le 5$, about the x-axis, where x is measured in meters. To estimate the volume V of the nose cone, we partition $[0,5]$ into five subintervals of equal length, slice the cone with planes perpendicular to the x-axis at the subintervals' left-hand endpoints, and construct cylinders of height 1 based on cross sections at these points.



a. Find the sum S_5 of the volumes of the cylinders. Do you expect to S_5 to overestimate V, or to underestimate V? Give reasons for your answer.

b. The true volume of the nose cone is $V = 25\pi/2\ m^3$. Express $|V-S_5|$ as a percentage of V to the nearest percent.

Solution:

a. We partition the interval $[0, 5]$ into 5 subintervals of length $\Delta x = 1$ and by slicing the cone with planes perpendicular to the x −axis at the subintervals' left hand endpoints, we construct cylinders of height 1 based on cross sections at these points.

Given that $f(x) = \sqrt{x}$

Now, we add the cylinders' volumes to estimate the volume of the nose cone

$$S_5 = \pi\left[f(c_1)\right]^2 \Delta x + \pi\left[f(c_2)\right]^2 \Delta x + \ldots\ldots + \pi\left[f(c_5)\right]^2 \Delta x$$

$$= \pi\left\{\left[\sqrt{c_1}\right]^2 + \left[\sqrt{c_2}\right]^2 + \ldots\ldots + \left[\sqrt{c_5}\right]^2\right\}\Delta x$$

$$= \pi\left[0+1+2+3+4\right](1) = 10\pi$$

Given true volume of the nose cone is

$$V = \frac{25\pi}{2} = 12.5m^3$$

By comparing the estimated volume S_5 and true volume V of the nose cone, S_5 is underestimated V, because $V > S_5$

b. Error percentage is

$$= \frac{|V-S_5|}{V} = \frac{|12.5\pi-10\pi|}{12.5\pi} = 0.2 \approx 20\%$$

P1:

The table below gives dye concentration for a dye-dilution cardiac output determination. The amount of dye injected is 5mg. use rectangles to estimate the area under the dye concentration curve and then estimate the patient's cardiac output.

Seconds after injection(t)	Dye concentration (mg/L)
2	0
4	0.6
6	1.4
8	2.7
10	3.7
12	4.1
14	3.8
16	2.9
18	1.7
20	1.0
22	0.5
24	0

P2:

The table below shows the velocity of a model train engine moving along a track for 10 seconds. Estimate the distance travelled by the engine using 10 subintervals of length 1 using

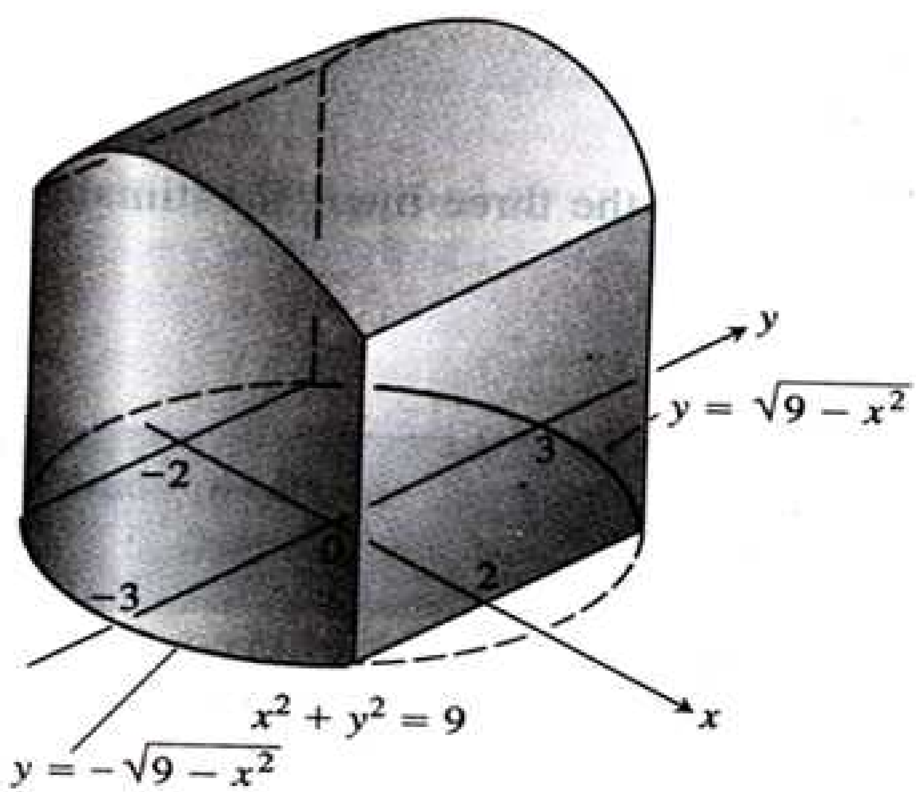
a. Left endpoints

b. Right endpoints

Time t (sec)	Velocity $v(t)$ (m/sec)
0	0
1	12
2	22
3	10
4	5
5	13
6	11
7	6
8	2
9	6
10	0

P3:

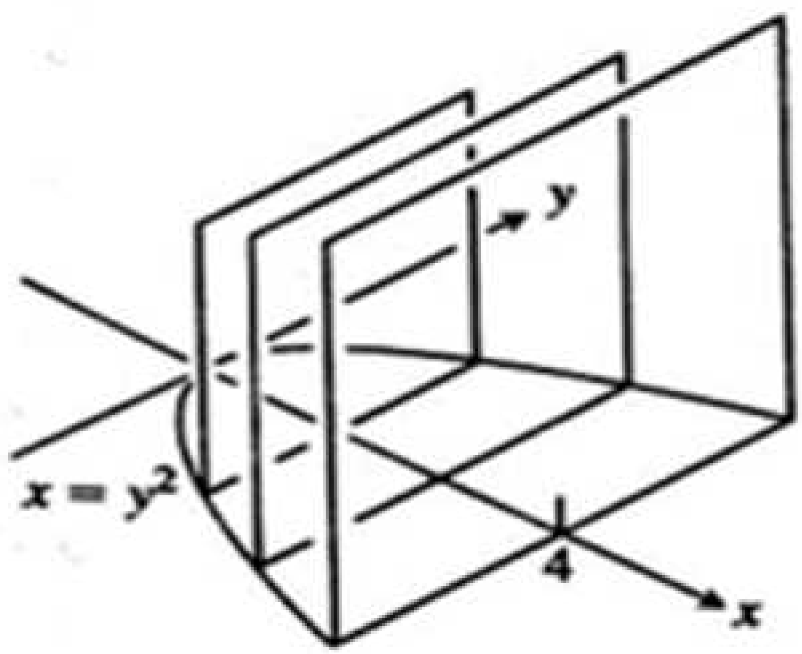
A solid lies between planes perpendicular to the x −axis at $x = -2$ and $x = 2$. The cross sections of the solid perpendicular to the axis between these planes are vertical squares whose base edges run from the semicircle $y = -\sqrt{9-x^2}$ to the semicircle $y = \sqrt{9-x^2}$.



The height of the square at x is $2\sqrt{9-x^2}$. Estimate the volume of the solid.

P4:

A solid lies between planes perpendicular to the x-axis at $x=0$ hand $x=4$. The cross sections of the solid perpendicular to the axis between these planes are vertical squares whose base edges run from the parabolic curve $y=-\sqrt{x}$ to the parabolic curve $y=\sqrt{x}$.



a. Find the sum S_4 of the volumes of the cylinders obtained by partitioning $0 \le x \le 4$ into four subintervals of length 1 based on the cross sections at the subinterval's right-hand endpoints.

b. The true volume is $V=32$. Express $|V-S_4|$ as a percentage of V to the nearest percent.

2.3. Approximation by Finite Sums

Exercise:

1. The table below gives dye concentrations for a dye-dilution cardiac-output determination. The amount of dye injected is 5 mg. Use rectangles to estimate the area under the dye concentration curve and then estimate the patient's cardiac output.

Seconds after Injection (*t*)	Dye concentration (*c*)
2	0
4	0.6
6	1.4
8	3.7
10	4.7
12	5.1
14	4.2
16	3.0
18	2.7
20	1.5
22	0.6
24	0

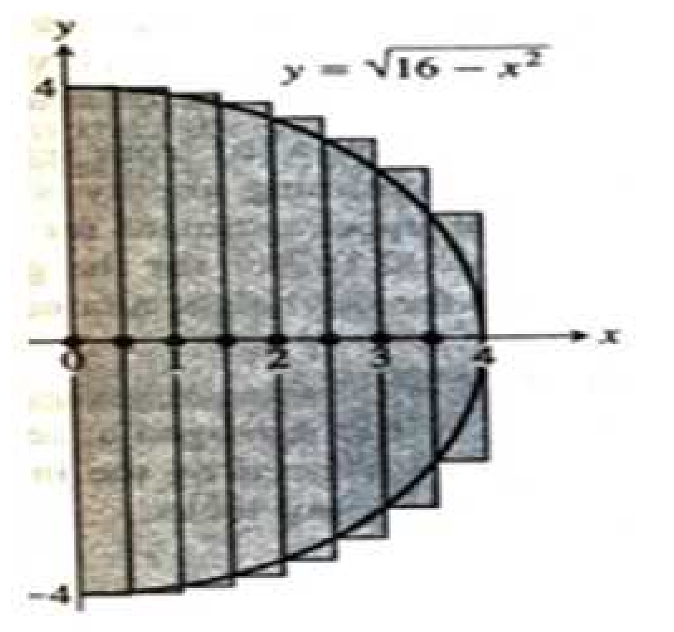
2. The table below shows the velocity of a model train engine moving along a track for 12 seconds. Estimate the distance traveled by the engine using 10 subintervals of length 2 with (a) left-endpoint values and (b) right-endpoint values.

Time (*seconds*)	Velocity (*m/sec*)
0	0
2	14
4	24
6	11
8	6
10	3
12	0

3. Read example 2. Suppose we use only two square cylinders to estimate the volume V of the solid.
   a. Find the sum S_2 of the volumes of the cylinders.
   b. Express $|V - S_2|$ as a percentage of V to the nearest percent.

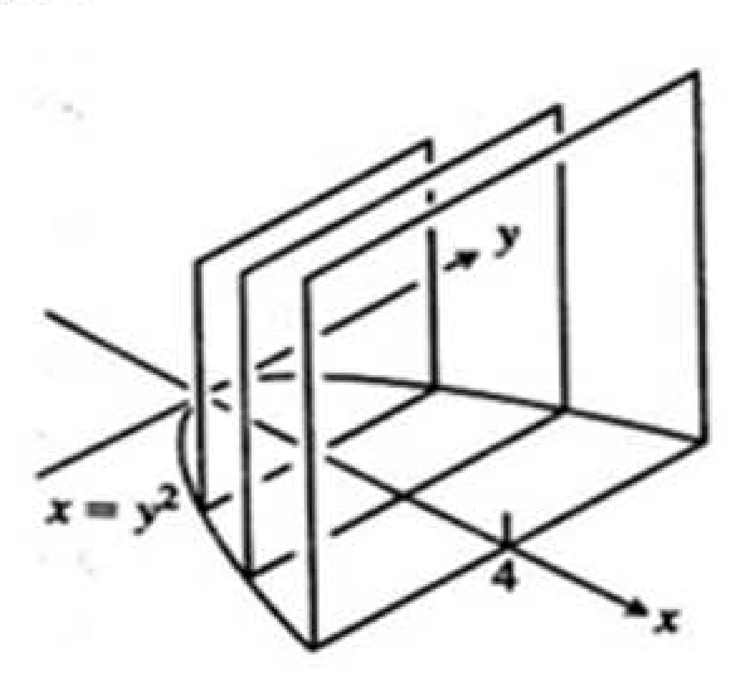
4. Read example 3. Suppose we approximate the volume V of the sphere by partitioning the interval $-4 \le x \le 4$ into four subintervals of length 2 and using cylinders based on the cross sections at the subintervals' right-hand endpoints.
   a. Find the sum S_4 of the volume of the cylinders.
   b. Express $|V - S_4|$ as a percentage of V to the nearest percent.

5. To estimate the volume V of a solid hemisphere of radius 4, imagine its axis of symmetry to be the interval [0, 4] on the x-axis. Partition [0, 4] into eight subintervals of equal length and approximate the solid with cylinders based on the circular cross sections of the hemisphere perpendicular to the x-axis at the subintervals' right-hand endpoints.



   a. Find the sum S_8 of the volumes of cylinders. Do you expect S_8 to overestimate V, or to underestimate V? Give reasons for your answer.
   b. Express $|V - S_8|$ as a percentage of V to the nearest percent.

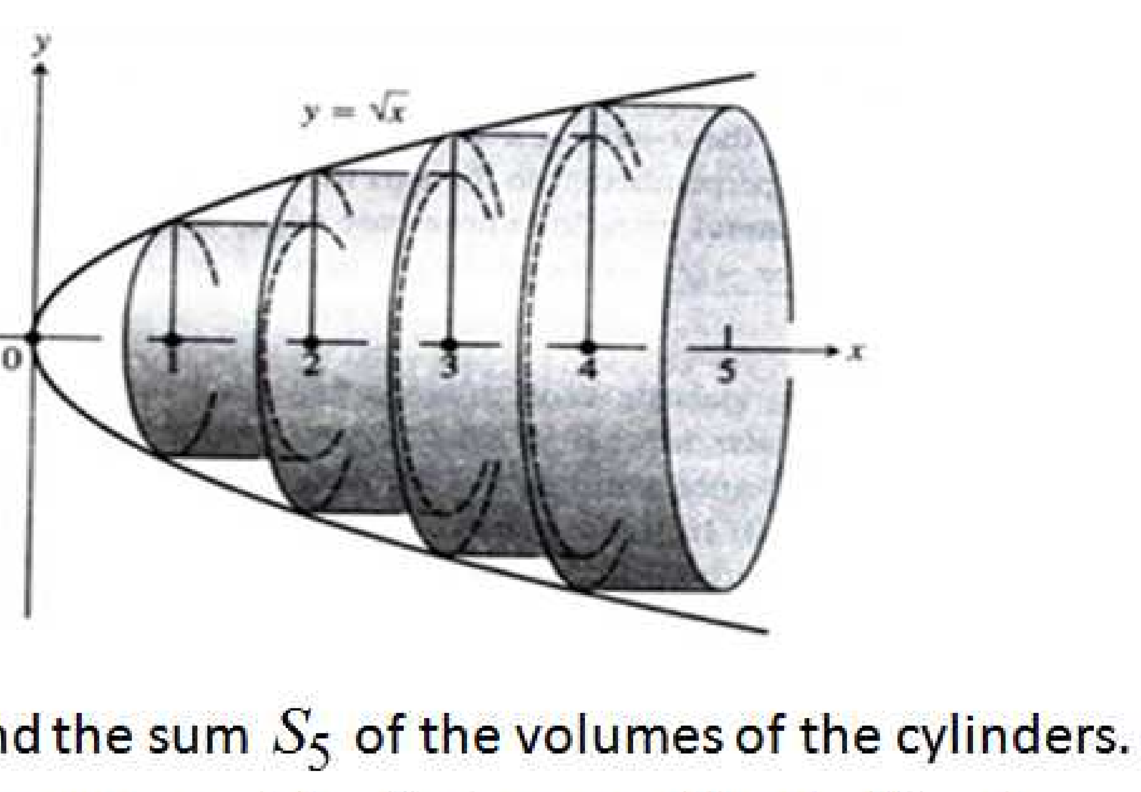
6. A solid lies between planes perpendicular to the x-axis at x = 0 and x = 4. The cross sections of the solid perpendicular to the axis between these planes are vertical squares whose base edges run from the parabolic curve $y = -\sqrt{x}$ to the parabolic curve $y = \sqrt{x}$.



   a. Find the sum S_4 of the volumes of the cylinders obtained by partitioning $0 \le x \le 4$ into four subintervals of length 1 based on the cross sections at the subinterval's left-hand endpoints.
   b. The true volume is V = 32. Express $|V - S_4|$ as a percentage of V to the nearest percent.
   c. Repeat parts (a) and (b) for the sum S_8.

7. A reservoir shaped like a hemispherical bowl of radius 8 m is filled with water to a depth of 4 m.
   a. Find an estimate S of the water's volume by approximating the water with eight circumscribed solid cylinders.
   b. The true water's volume is V = 320 π/3 m³. Find the error $|V - S|$ as a percentage of V to the nearest percent.

8. The nose "cone" of a rocket is a paraboloid obtained by revolving the curve $y = \sqrt{x}$, $0 \le x \le 5$, about the x-axis, where x is measured in meters. To estimate the volume V of the nose cone, we partition [0,5] into five subintervals of equal length, slice the cone with planes perpendicular to the x-axis at the subintervals' right-hand endpoints, and construct cylinders of height 1 based on cross sections at these points.



   a. Find the sum S_5 of the volumes of the cylinders. Do you expect to S_5 to overestimate V, or to underestimate V? Give reasons for your answer.
   b. The true volume of the nose cone is V = 25 π/2 m³. Express $|V - S_5|$ as a percentage of V to the nearest percent.

2.4

Average value of a Non-Negative function

Learning objectives:

- To estimate the average value of a non-negative continuous function in the given interval.

  AND

- To practice the related problems.

The average value of a collection of n numbers $x_1, x_2, \dots x_n$ is obtained by adding them together and dividing by n. But what is the average value of a continuous function f on an interval $[a, b]$? Such a function assumes infinitely many values.

The average value of a non-negative continuous function

The average value of a non-negative continuous function f over an interval $[a, b]$ can be approximated by finite sums.

We first subdivide the interval $[a, b]$ into n subintervals of equal width $(i.e., length)\ \Delta x = \frac{b-a}{n}$. If $f(c_k)$ is the value of f at the chosen point c_k in the k^{th} subinterval then

Average value of f over $[a, b]$

$$\approx \frac{f(c_1)+f(c_2)+\dots\dots+f(c_k)+\dots\dots+f(c_n)}{n}$$

$$= \frac{1}{b-a}\left[f(c_1)\left(\frac{b-a}{n}\right)+f(c_2)\left(\frac{b-a}{n}\right)+\dots+f(c_k)\left(\frac{b-a}{n}\right)+\dots+f(c_n)\left(\frac{b-a}{n}\right)\right]$$

$$= \frac{1}{b-a}\left[f(c_1).\Delta x+f(c_2).\Delta x+\dots+f(c_k).\Delta x+\dots+f(c_n).\Delta x\right]$$

$$= \frac{1}{length\ of\ [a,b]}\ .A$$

where A is the approximate area that lies above x −axis, below the graph of the funtion $f(x)$ and between vertical lines $x = a$ and $x = b$.

By taking more and more subintervals (i.e., n sufficiently large), we get better and better approximation for A and thereby we get better and better average values of f over $[a, b]$.

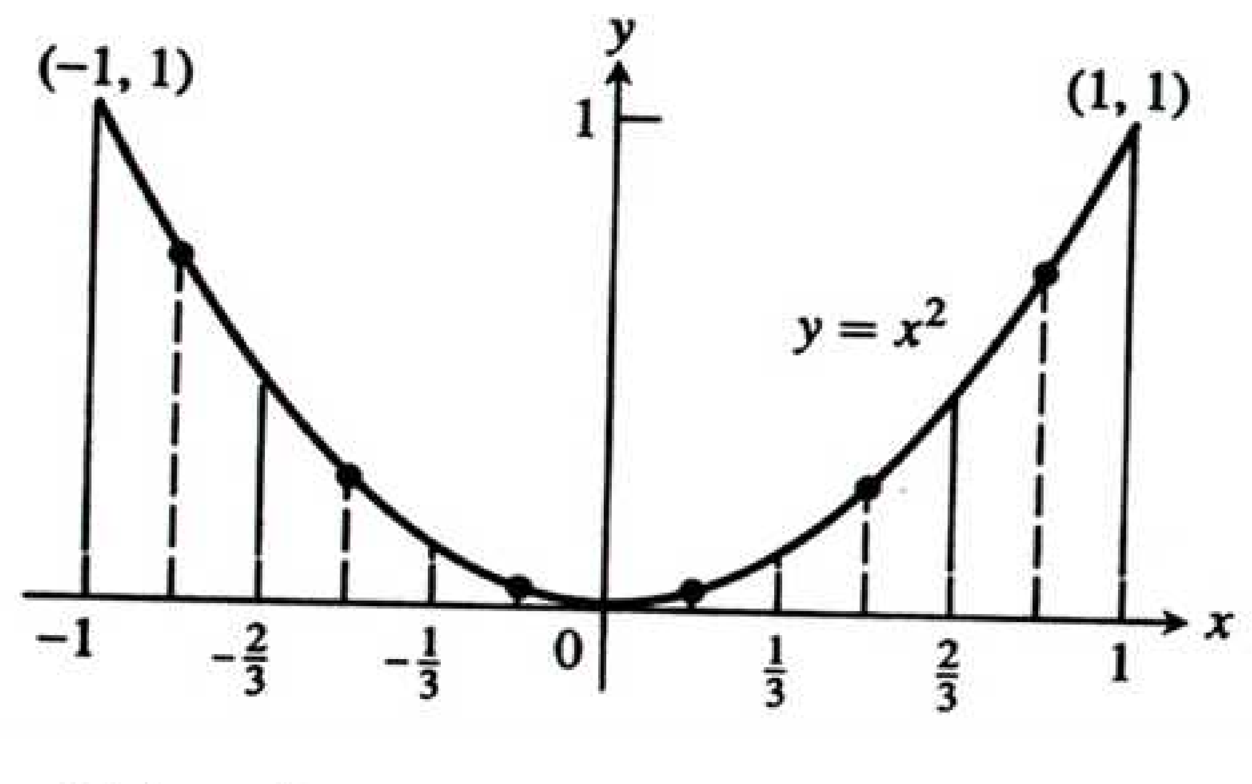
We define the average value of a non-negative function on an interval $[a, b]$ as the area under its graph divided by $b - a$.

Example 1:

Estimate the average value of the function $f(x) = x^2$ on the interval $[-1, 1]$.

Solution:

The graph of $y = x^2$ and the partition of the interval $[-1, 1]$ into 6 subintervals of length $\Delta x = 1/3$ are shown below:



The six subintervals are

$$\left[-1,-\frac{2}{3}\right],\ \left[-\frac{2}{3},-\frac{1}{3}\right],\ \left[-\frac{1}{3},0\right],\ \left[0,\frac{1}{3}\right],\ \left[\frac{1}{3},\frac{2}{3}\right],\ \left[\frac{2}{3},1\right]$$

We estimate the average value with a finite sum.

It appears that a good estimate for the average of square on each subinterval is the square of the *midpoint* of the subinterval.

We have $c_1 = -\frac{5}{6}$, $c_2 = -\frac{3}{6}$, $c_3 = -\frac{1}{6}$, $c_4 = \frac{1}{6}$, $c_5 = \frac{3}{6}$, $c_6 = \frac{5}{6}$ are the midpoints of the above six subintervals respectively.

The average value of the function f on $[-1, 1]$ is

$$\approx \frac{1}{\text{length }[-1,1]}\left[f(c_1).\Delta x+f(c_2).\Delta x+\dots+f(c_6).\Delta x\right]$$

$$= \frac{1}{1-(-1)}\left[\left(-\frac{5}{6}\right)^2+\left(-\frac{3}{6}\right)^2+\left(-\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^2+\left(\frac{3}{6}\right)^2+\left(\frac{5}{6}\right)^2\right]\frac{1}{3}$$

$$= \frac{1}{6}\left[\frac{25+9+1+1+9+25}{36}\right] = \frac{70}{216} \approx 0.324$$

Note: We can calculate the average value in the above example using anti-derivative, as we see in a module on definite integrals. Later, we will show that the true average value of the function is 1/3, and this approximation compares well with the true value.

IP1:

Use a finite sum to estimate the average value of a function $f(x) = 2x^2$ on the interval $[0, 8]$ by partitioning the interval into 4 subintervals of equal length and evaluating f at the subinterval midpoints.

Solution:

We partition the interval $[0, 8]$ into 4 subintervals of length $\Delta x = 2$ which are as follows.

$$[0, 2] \;,\quad [2, 4] \;,\quad [4, 6] \;,\quad [6, 8]$$

We estimate the average value with a finite sum by using midpoint rule.

We have $c_1 = 1,\ c_2 = 3,\ c_3 = 5,\ c_4 = 7$ are the midpoints of the above four subintervals respectively.

Average value of the function f on $[0, 8]$ is

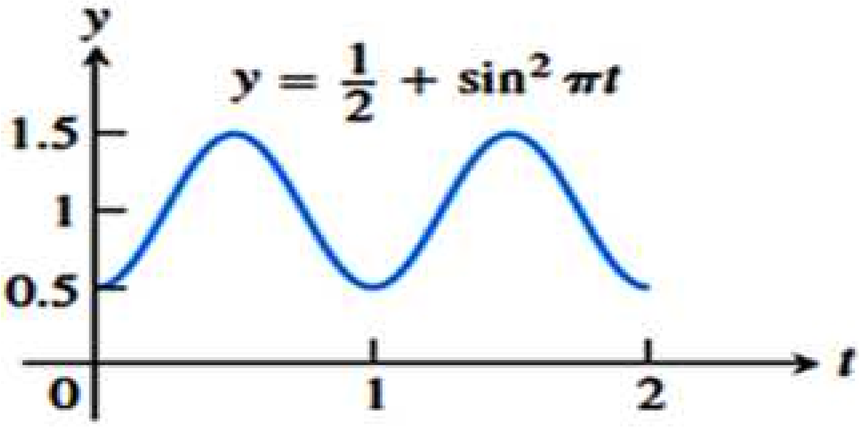
$$\approx \frac{1}{\text{length}\,[0, 8]}\left[f(c_1).\Delta x + f(c_2).\Delta x + f(c_3).\Delta x + f(c_4).\Delta x\right]$$

$$= \frac{1}{8}[f(1).2 + f(3).2 + f(5).2 + f(7).2]$$

$$= \frac{1}{4}[2(1)^2 + 2(3)^2 + 2(5)^2 + 2(7)^2] \approx 42$$

IP2:

Estimate the average value of the function $f(t) = \frac{1}{2} + sin^2\pi t$ on the interval $[0, 2]$.



Solution:

We partition the interval $[0, 2]$ into 4 subintervals of lengths $\Delta x = 0.5$ which are as follows.

$$[0, 0.5]\ ,\quad [0.5, 1]\ ,\quad [1, 1.5]\ ,\quad [1.5, 2]$$

We estimate the average value of finite sum by using midpoint rule.

We have $c_1 = \frac{1}{4},\ c_2 = \frac{3}{4},\ c_3 = \frac{5}{4},\ c_4 = \frac{7}{4}$ are the midpoints of the above four subintervals respectively.

Average value of the function f on $[0, 2]$ is

$$\approx \frac{1}{\text{length}\,[0, 2]}\left[f(c_1).\Delta x + f(c_2).\Delta x + f(c_3).\Delta x + f(c_4).\Delta x\right]$$

$$= \frac{1}{2}\left[f\left(\frac{1}{4}\right)(0.5) + f\left(\frac{3}{4}\right)(0.5) + f\left(\frac{5}{4}\right)(0.5) + f\left(\frac{7}{4}\right)(0.5)\right]$$

where, $f\left(\frac{1}{4}\right) = \frac{1}{2} + sin^2\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$

$$f\left(\frac{3}{4}\right) = \frac{1}{2} + sin^2\frac{3\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

$$f\left(\frac{5}{4}\right) = \frac{1}{2} + sin^2\frac{5\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$$

$$f\left(\frac{7}{4}\right) = \frac{1}{2} + sin^2\frac{\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$$

Average value$= \frac{1}{2}\left[1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)\right] \approx 1$

IP3:

Estimate the average value of the function $f(x) = cosx + 2$ on the interval $[0, \pi]$ using n rectangles of equal width.

Solution:

In general the average value can be approximated by using the formula

Average value$= \frac{1}{length\ of\ the\ interval}\begin{bmatrix} a\ sum\ of\ the\ function \\ values\ multiplied\ by \\ interval\ length \end{bmatrix}$

Here $\Delta x = \frac{\pi}{n}$ and $x_i = \frac{i\pi}{n}$

Using Right hand endpoint approximation, we have

Average value

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{n}\right).f\left(\frac{\pi}{n}\right) + \left(\frac{\pi}{n}\right).f\left(\frac{2\pi}{n}\right) + \cdots + \left(\frac{\pi}{n}\right).f\left(\frac{n\pi}{n}\right)\right]$$

$$= \frac{1}{n}\left[f\left(\frac{\pi}{n}\right) + f\left(\frac{2\pi}{n}\right) + \cdots + f\left(\frac{n\pi}{n}\right)\right]$$

$$= \frac{1}{n}\left[\left(cos\frac{\pi}{n} + 2\right) + \left(cos\frac{2\pi}{n} + 2\right) + \cdots + \left(cos\frac{n\pi}{n} + 2\right)\right]$$

$$= \frac{1}{n}\left[cos\frac{\pi}{n} + cos\frac{2\pi}{n} + cos\frac{3\pi}{n} + \cdots + cos\frac{n\pi}{n}\right] + 2$$

IP4:

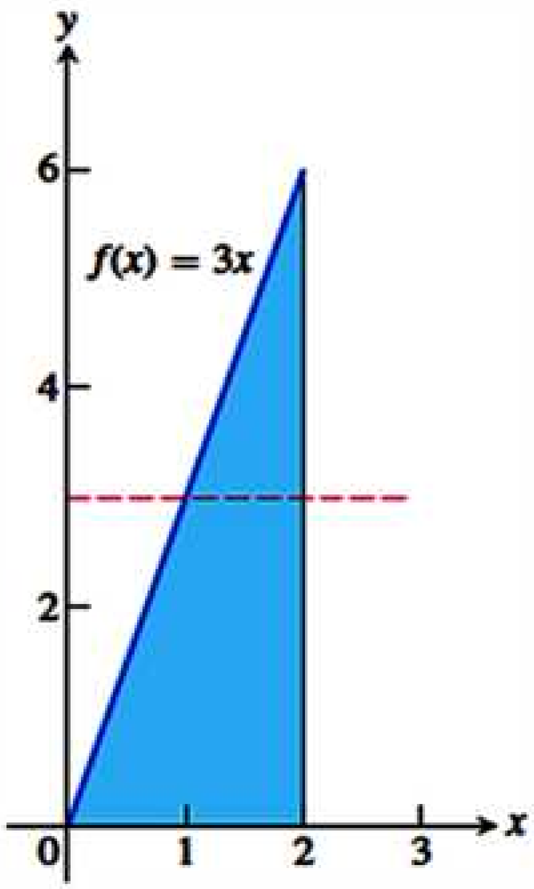
What is the average value of the of the function $f(x) = 3x$ on the interval $[0, 2]$?

Solution:

The average value equals the area under the graph divided by the width of the interval. In this case we don't need finite approximation to estimate the area of the region under the graph: a triangle of height 6 and base 2 has area 6 (from figure)

The width of the interval is $b - a = 2 - 0 = 2$

The average value of the function is $= \frac{6}{2} = 3$



Hence the average value of the function $f(x) = 3x$ over the interval $[0, 2]$ is 3.

P1:

Use a finite sum to estimate the average value of a function $f(x) = x^3$ on the interval $[0, 2]$ by partitioning the interval into 4 subintervals of equal length and evaluating f at the subinterval midpoints.

Solution:

We partition the interval $[0, 2]$ into 4 subintervals of length $\Delta x = \frac{2-0}{4} = 0.5$ which are

$$[0, 0.5]\ ,\quad [0.5, 1]\ ,\quad [1, 1.5]\ ,\quad [1.5, 2]$$

We estimate the average value with a finite sum by using midpoint rule.

We have $c_1 = 0.25,\ c_2 = 0.75,\ c_3 = 1.25,\ c_4 = 1.75$ are the midpoints of the above four subintervals respectively.

Average value of the function f on $[0, 2]$ is

$$\approx \frac{1}{\text{length}\,[0, 2]}\left[f(c_1).\Delta x + f(c_2).\Delta x + f(c_3).\Delta x + f(c_4).\Delta x\right]$$

$$= \frac{1}{2}[f(0.25)(0.5) + f(0.75)(0.5) + f(1.25)(0.5) + f(1.75)(0.5)]$$

$$= (0.5)(0.5)[(0.25)^3 + (0.75)^3 + (1.25)^3 + (1.75)^3]$$

$$= 0.25[0.01563 + 0.42187 + 1.95312 + 5.35937] \approx 1.9375$$

P2:

Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution:

We partition the interval $[0, \pi]$ into 4 subintervals of lengths $\Delta x = \frac{\pi}{4}$ which are as follows.

$$\left[0, \frac{\pi}{4}\right], \quad \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \quad \left[\frac{\pi}{2}, \frac{3\pi}{4}\right], \quad \left[\frac{3\pi}{4}, \pi\right]$$

We estimate the average value of finite sum by using midpoint rule.

We have $c_1 = \frac{\pi}{8},\ c_2 = \frac{3\pi}{8},\ c_3 = \frac{5\pi}{8},\ c_4 = \frac{7\pi}{8}$ are the midpoints of the above four subintervals respectively.

Average value of the function f on $[0, \pi]$ is

$$\approx \frac{1}{\text{length}\,[0,\,\pi]}\left[f(c_1).\Delta x + f(c_2).\Delta x + f(c_3).\Delta x + f(c_4).\Delta x\right]$$

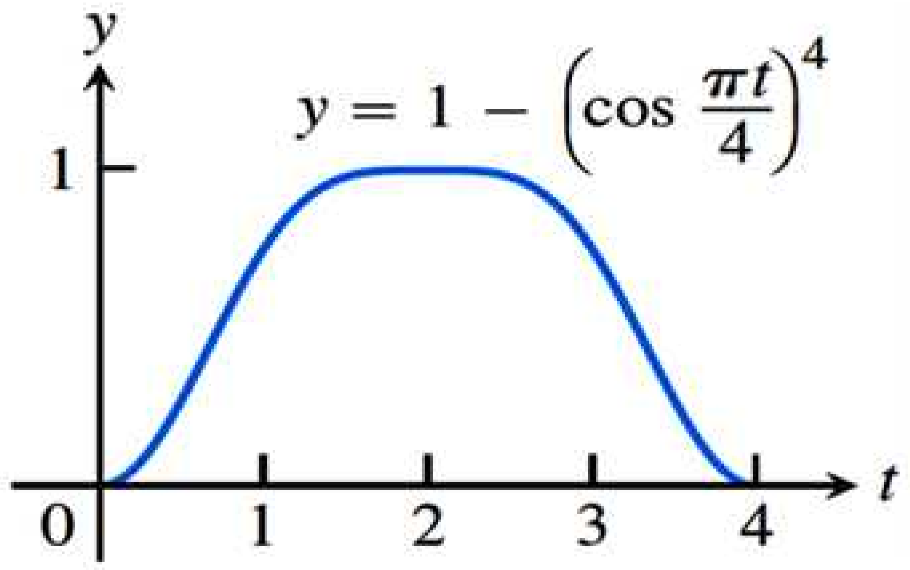
$$= \frac{1}{\pi}\left[f\left(\frac{\pi}{8}\right)\frac{\pi}{4} + f\left(\frac{3\pi}{8}\right)\frac{\pi}{4} + f\left(\frac{5\pi}{8}\right)\frac{\pi}{4} + f\left(\frac{7\pi}{8}\right)\frac{\pi}{4}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4}.\sin\frac{\pi}{8} + \frac{\pi}{4}.\sin\frac{3\pi}{8} + \frac{\pi}{4}.\sin\frac{5\pi}{8} + \frac{\pi}{4}.\sin\frac{7\pi}{8}\right]$$

$$= \frac{1}{4}[0.3825 + 0.9247 + 0.9237 + 0.3825] \approx 0.65335$$

P3:

Estimate the average value of the function $\boldsymbol{f(t) = 1 - \left(cos\frac{\pi t}{4}\right)^4}$ on the interval $[\mathbf{0}, \mathbf{4}]$.



Solution:

We partition the interval $[0, 4]$ into 4 subintervals of lengths $\Delta x = 1$ which are as follows.

$$[0, 1]\ ,\quad [1, 2]\ ,\quad [2, 3]\ ,\quad [3, 4]$$

We estimate the average value with a finite sum by using midpoint rule.

We have $c_1 = \frac{1}{2},\ c_2 = \frac{3}{2},\ c_3 = \frac{5}{2},\ c_4 = \frac{7}{2}$ are the midpoints of the above four subintervals respectively.

Average value of the function f on $[0, 4]$ is

$$\approx \frac{1}{\text{length}\,[0, 4]}\left[f(c_1).\Delta x + f(c_2).\Delta x + f(c_3).\Delta x + f(c_4).\Delta x\right]$$

$$= \frac{1}{4}\left[f\left(\frac{1}{2}\right)(1) + f\left(\frac{3}{2}\right)(1) + f\left(\frac{5}{2}\right)(1) + f\left(\frac{7}{2}\right)(1)\right]$$

$$= \frac{1}{4}\left[f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right)\right]$$

where, $f\left(\frac{1}{2}\right) = 1 - \left(cos\frac{\pi}{8}\right)^4 = 0.2712$

$$f\left(\frac{3}{2}\right) = 1 - \left(cos\frac{3\pi}{8}\right)^4 = 0.97842$$

$$f\left(\frac{5}{4}\right) = 1 - \left(cos\frac{5\pi}{8}\right)^4 = 0.97876$$

$$f\left(\frac{7}{4}\right) = 1 - \left(cos\frac{7\pi}{8}\right)^4 = 0.27313$$

Average value

$$= \frac{1}{4}[0.2712 + 0.97842 + 0.97876 + 0.27313] \approx 0.62538$$

P4:

Use a finite sum to estimate the average value of a function $f(x) = \frac{1}{x}$ on the interval $[1, 9]$ by partitioning the interval into 4 subintervals of equal length and evaluating f at the subinterval midpoints.

Solution:

We partition the interval $[1, 9]$ into 4 subintervals of lengths $\Delta x = 2$ which are as follows.

$$[1, 3] \;,\quad [3, 5] \;,\quad [5, 7] \;,\quad [7, 9]$$

We estimate the average value with a finite sum by using midpoint rule.

We have $c_1 = 2,\ c_2 = 4,\ c_3 = 6,\ c_4 = 8$ are the midpoints of the above four subintervals respectively.

Average value of the function f on $[1, 9]$ is

$$\approx \frac{1}{\text{length } [1, 9]}\left[f(c_1).\Delta x + f(c_2).\Delta x + f(c_3).\Delta x + f(c_4).\Delta x\right]$$

$$= \frac{1}{8}[f(2).2 + f(4).2 + f(6).2 + f(8).2]$$

$$= \frac{1}{8}\left[\frac{1}{2}(2) + \frac{1}{4}(2) + \frac{1}{6}(2) + \frac{1}{8}(2)\right] \approx 0.26$$

2.4. Average value of a Non-Negative function

Exercise:

Use a finite sum to estimate the average value of the function on the given interval by partitioning the interval into 4 subintervals of equal length and evaluating f at the subinterval midpoints.

1. $f(x) = 3x^3$, $[0, 4]$
2. $f(x) = \frac{2}{x}$, $[1, 9]$
3. $f(x) = sin^2 x$, $[0, \pi]$
4. $f(x) = 4 - x^2$, $[-2, 2]$
5. $f(t) = cos^2 \pi t$, $[0, 2]$

2.5

Algebra of Finite sums

Learning objectives:

- To introduce sigma notation to write a sum with many terms in the compact form.
- To study the rules for finite sums.
- To state the formula for the sum of the first n natural numbers and formulas for the sums of the squares and cubes of the first n natural numbers.

AND

- To practice the related problems.

We introduce a compact notation for sums that contain large number of terms.

Sigma Notation for Finite Sums

We use the capital Greek letter $\sum$ ("sigma") to write an abbreviation for the sum

Definitions

The symbol $\sum_{k=1}^{n} a_k$ denotes the sum $a_1 + a_2 + \cdots + a_n$. The a's are terms of the sum: a_1 is the first term, a_2 is the second term, a_k is the k^{th} term, and a_n is the n^{th} and last term. The variable k is the ***index of summation***. The values of k runs through the integers from 1 to n. The number 1 is the ***lower limit of summation***; the number n is the ***upper limit of summation***.

Example 1:

$$\sum_{k=1}^{5} k = 1+2+3+4+5=15$$

$$\sum_{k=1}^{3} (-1)^k k = +(-1)^1(1)+(-1)^2(2)+(-1)^3(3) = -1+2-3=-2$$

$$\sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{1+1}+\frac{2}{2+1} = \frac{1}{2}+\frac{2}{3} = \frac{7}{6}$$

The lower limit of summation does not have to be 1; it can be any integer.

Example 2:

Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

Solution:

Starting with $k=2$: $1 + 3 + 5 + 7 + 9 = \sum_{k=2}^{6}(2k-3)$

Starting with $k=-3$: $1 + 3 + 5 + 7 + 9 = \sum_{k=-3}^{1}(2k+7)$

The formula generating the terms changes with the lower limit of summation, but the terms generated remain the same. It is often simplest to start with $k=0$ or $k=1$.

Starting with $k=0$: $1 + 3 + 5 + 7 + 9 = \sum_{k=0}^{6}(2k+1)$

Starting with $k=1$: $1 + 3 + 5 + 7 + 9 = \sum_{k=1}^{5}(2k-1)$

Algebra with Finite Sums

We can use the following rules whenever we work with finite sums.

1. Sum Rule: $\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. Difference Rule: $\sum_{k=1}^{n}(a_k-b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. Constant Multiple Rule: $\sum_{k=1}^{n} c a_k = c \cdot \sum_{k=1}^{n} a_k$ (any number c)

4. Constant Value Rule: $\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value)

The formal proofs of these rules can be done by mathematical induction.

Example 3:

a) $\sum_{k=1}^{n}\left(3k-k^2\right) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$

b) $\sum_{k=1}^{n}(-a_k) = \sum_{k=1}^{n}(-1)\cdot a_k = -1\cdot\sum_{k=1}^{n} a_k = -\sum_{k=1}^{n} a_k$

c) $\sum_{k=1}^{3}(k+4) = \sum_{k=1}^{3} k + \sum_{k=1}^{3} 4 = (1+2+3)+(3\cdot 4) = 6+12=18$

Sum Formulas for Positive Integers

Gauss discovered the formula for the sum of first n integers. The formulas for the sums of the squares and cubes of the first n integers are also famous.

The first n integers: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

The first n squares: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

The first n cubes: $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 4:

Evaluate

$$\sum_{k=1}^{4}\left(k^2-3k\right) = \sum_{k=1}^{4} k^2 - 3\sum_{k=1}^{4} k$$

$$= \frac{4(4+1)(8+1)}{6} - 3\left(\frac{4(4+1)}{2}\right)$$

$$= 30-30=0$$

IP1:

Find the following:

1. $\sum_{k=1}^{5} \sin k\pi$

2. $\sum_{k=1}^{4} (-1)^k \cos k\pi$

Solution:

1. $\sum_{k=1}^{5} \sin k\pi = sin(1.\pi) + sin(2\pi) + sin(3\pi) + sin(4\pi) + sin(5\pi)$

$$= 0 + 0 + 0 + 0 + 0 \ = 0$$

2. $\sum_{k=1}^{4} (-1)^k \cos k\pi \ =$

$$(-1)^1 cos\pi + (-1)^2 cos2\pi + (-1)^3 cos3\pi + (-1)^4 cos4\pi$$

$$= -\ cos\pi + cos2\pi - cos3\pi + cos4\pi$$

$$= \ -(-1) + 1 - (-1) + 1 = 4$$

IP2:

Which of the following express $\mathbf{1-2+4-8+16-32}$ in sigma notation?

A. $\sum_{k=1}^{6}(-2)^{K-1}$ B. $\sum_{k=0}^{5}(-1)^{K}2^{k}$ C. $\sum_{k=-2}^{3}(-1)^{K+1}2^{k+2}$

Solution:

A. $\sum_{k=1}^{6}(-2)^{K-1}=(-2)^{1-1}+(-2)^{2-1}+(-2)^{3-1}+(-2)^{4-1}+(-2)^{5-1}+(-2)^{6-1}$

$$=1-2+4-8+16-32$$

B. $\sum_{k=0}^{5}(-1)^{K}2^{k}=(-1)^{0}(2)^{0}+(-1)^{1}(2)^{1}+(-1)^{3}(2)^{3}$

$$+(-1)^{4}(2)^{4}+(-1)^{5}(2)^{5}$$

$$=1-2+4-8+16-32$$

C. $\sum_{k=-2}^{3}(-1)^{K+1}2^{k+2}$

$$=(-1)^{-2+1}2^{-2+2}+(-1)^{-1+1}2^{-1+2}+(-1)^{0+1}2^{0+2}$$

$$+(-1)^{1+1}2^{1+2}+(-1)^{2+1}2^{2+2}+(-1)^{3+1}2^{3+2}$$

$$=\ -1+2-4+8-16+32$$

A and B represents $1-2+4-8+16-32$, but C does not represents the given pattern.

IP3:

Suppose that $\sum_{k=1}^{n} a_k = 5$ and $\sum_{k=1}^{n} b_k = -2$. Find the values of

a) $\sum_{k=1}^{n}(2a_k + 3b_k)$

b) $\sum_{k=1}^{n}(5a_k - 3b_k + 1)$

c) $\sum_{k=1}^{n}(5a_k + 13b_k - 5)$

Solution:

a) $\sum_{k=1}^{n}(2a_k + 3b_k) = \sum_{k=1}^{n} 2a_k + \sum_{k=1}^{n} 3b_k$

$$= 2\sum_{k=1}^{n} a_k + 3\sum_{k=1}^{n} b_k = 2(5) + 3(-2) = 4$$

b) $\sum_{k=1}^{n}(5a_k - 3b_k + 1) = \sum_{k=1}^{n} 5a_k - \sum_{k=1}^{n} 3b_k + \sum_{k=1}^{n} 1$

$$= 5\sum_{k=1}^{n} a_k - 3\sum_{k=1}^{n} b_k + n.1$$

$$= 5(5) - 3(-2) + n = 31 + n$$

c) $\sum_{k=1}^{n}(6a_k + 13b_k - 5) = \sum_{k=1}^{n} 6a_k + \sum_{k=1}^{n} 13b_k - \sum_{k=1}^{n} 5$

$$= 6\sum_{k=1}^{n} a_k + 13\sum_{k=1}^{n} b_k - 5n$$

$$= 6(5) + 13(-2) - 5n = 4 - 5n$$

IP4:

$$\sum_{k=1}^{10}\left\{k\left(k^2+2k+1\right)+2\right\}=$$

Solution:

$$\sum_{k=1}^{10}\left\{k\left(k^2+2k+1\right)+2\right\}$$

$$=\sum_{k=1}^{10}(k^3+2k^2+k)+\sum_{k=1}^{10}2$$

$$=\sum_{k=1}^{10}k^3+\sum_{k=1}^{10}2k^2+\sum_{k=1}^{10}k+2(10)$$

$$=\sum_{k=1}^{10}k^3+2\sum_{k=1}^{10}k^2+\sum_{k=1}^{10}k+20$$

$$=\left[\frac{10(10+1)}{2}\right]^2+\left[\frac{10(10+1)(2(10)+1)}{6}\right]+\left[\frac{10(10+1)}{2}\right]+20$$

$$=3870$$

P1:

Find the following:

1. $\sum_{k=1}^{4} \frac{3k}{k+2} =$

2. $\sum_{k=1}^{3} \frac{k-1}{k} =$

Solution:

1. $\sum_{k=1}^{4} \frac{3k}{k+2} = \frac{3(1)}{1+2} + \frac{3(2)}{2+2} + \frac{3(3)}{3+2} + \frac{3(4)}{4+2} = \frac{3}{3} + \frac{6}{4} + \frac{9}{5} + \frac{12}{6} = \frac{63}{10}$

2. $\sum_{k=1}^{3} \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

P2:

Express the following in sigma notation.

a. $\mathbf{1 + 4 + 9 + 16}$

b. $\mathbf{2 + 4 + 6 + 8 + 10}$

Solution:

a. $1 + 4 + 9 + 16 = (1)^2 + (2)^2 + (3)^2 + (4)^2$

$$= \sum_{k=1}^{4} k^2$$

b. $2 + 4 + 6 + 8 + 10 = 2(1) + 2(2) + 2(3) + 2(4) + 2(5)$

$$= \sum_{k=1}^{5} 2K$$

P3:

Suppose that $\sum_{k=1}^{n} a_k = 0$ **and** $\sum_{k=1}^{n} b_k = 1$ **then find the values of**

a) $\sum_{k=1}^{n} 8a_k$ **b)** $\sum_{k=1}^{n} 250b_k$ **c)** $\sum_{k=1}^{n} (a_k + 1)$ **d)** $\sum_{k=1}^{n} (b_k - 1)$

Solution:

a) $\sum_{k=1}^{n} 8a_k = 8 \sum_{k=1}^{n} a_k = 8\ (0) = 0$

b) $\sum_{k=1}^{n} 250b_k = 250 \sum_{k=1}^{n} b_k = 250(1) = 250$

c) $\sum_{k=1}^{n} (a_k + 1) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} 1 = 0 + n = n$

d) $\sum_{k=1}^{n} (b_k - 1) = \sum_{k=1}^{n} b_k - \sum_{k=1}^{n} 1 = 1 - n$

P4:

$$\sum_{k=1}^{7} k(2k+1) =$$

Solution:

$$\sum_{k=1}^{7} k(2k+1) = \sum_{k=1}^{7} (2k^2+k) = \sum_{k=1}^{7} 2k^2 + \sum_{k=1}^{7} k$$

$$= 2\sum_{k=1}^{7} k^2 + \sum_{k=1}^{7} k$$

$$= 2\left[\frac{7(7+1)(2(7)+1)}{6}\right] + \left(\frac{7(7+1)}{2}\right)$$

$$= \frac{7.8.15}{3} + \frac{7.8}{2} = 280 + 28 = 308$$

2.5. Algebra of Finite sums

Exercise:

1. **Write the sums without sigma notation. Then evaluate them.**

   $a.\ \sum_{k=1}^{2} \frac{6k}{k+1}$

   $b.\ \sum_{k=1}^{4} \cos k\pi$

   $c.\ \sum_{k=1}^{3} (-1)^{k+1} \sin\frac{\pi}{k}$

2. **Which of the following express** $1+2+4+8+16+32$ **in sigma notation?**

   a. $\sum_{k=1}^{6} 2^{k-1}$

   b. $\sum_{k=0}^{5} 2^{k}$

   c. $\sum_{k=-1}^{4} 2^{k+1}$

3. **Which formula is not equivalent to the other two?**

   a. $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$

   b. $\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$

   c. $\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$

4. **Express the following problems in sigma notation.**

   a. $1+2+3+4+5+6$

   $b.\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$

   $c.\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

5. **Suppose that** $\sum_{k=1}^{n} a_k = -5$ **and** $\sum_{k=1}^{n} b_k = 6$. **Find the values of**

   a. $\sum_{k=1}^{n} 3a_k$

   b. $\sum_{k=1}^{n} \frac{b_k}{6}$

   c. $\sum_{k=1}^{n} (a_k + b_k)$

   d. $\sum_{k=1}^{n} (a_k - b_k)$

   e. $\sum_{k=1}^{n} (b_k - 2a_k)$

6. **Evaluate the sums**

   a. $\sum_{k=1}^{10} k$

   b. $\sum_{k=1}^{10} k^2$

   c. $\sum_{k=1}^{10} k^3$

   $d.\ \sum_{k=1}^{7} (-2k)$

   e. $\sum_{k=1}^{6} \left(3-k^2\right)$

   $f.\ \sum_{k=1}^{5} k(3k+5)$

   $g.\ \sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

2.6

Limit of Riemann Sums

Learning objectives:

- To define a partition of an interval and its norm.
- To define Riemann sum of a given function on an interval for a partition of the interval.
- To define the definite integral of a given function on a given interval as a limit of Riemann sums.
  AND
- To practice the related problems.

Limit of Riemann Sums

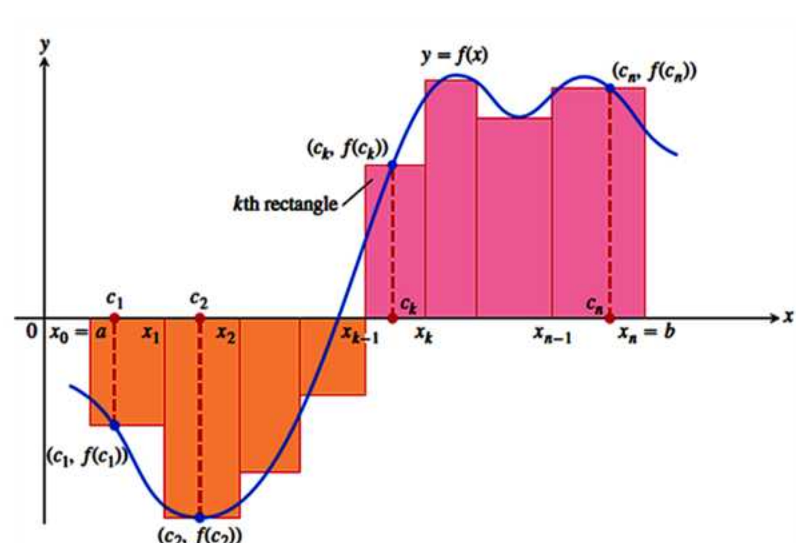
In the preceding module, we estimated distances, areas, volumes, and average values with finite sums. The terms in the sums were obtained by multiplying selected function values by the lengths of intervals. We now inquire what happens to the sums like these as the intervals involved become more numerous and shorter.

Riemann Sums

The approximating sums in the modules 3.3 and 3.4 are examples of a more general kind of sum called a ***Riemann sum***. The functions in the examples had nonnegative values, but the more general notion has no such restriction.

Given an arbitrary function $y=f(x)$ on an interval $[a,b]$, we partition the interval into n subintervals by choosing $n-1$ points, say $x_1, x_2, \cdots, x_{n-1}$, between a and b subject to the only condition that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$

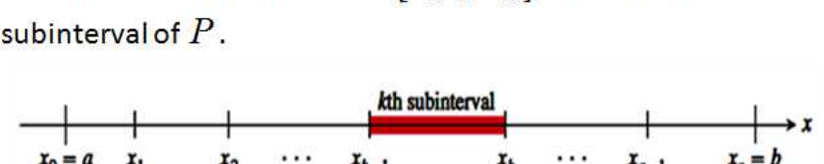


To make the notation consistent, we usually denote a by x_0 and b by x_n. The set $P=\{x_0, x_1, \cdots, x_n\}$ is called a ***partition*** of $[a,b]$.

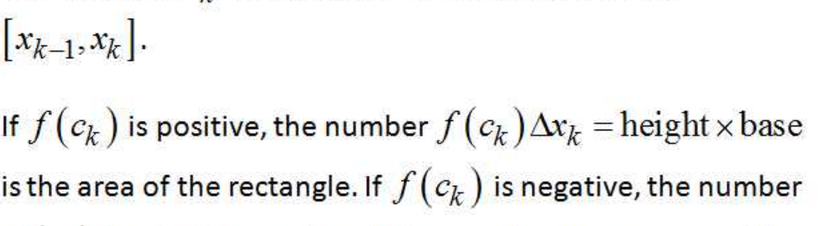
The partition P defines n closed ***subintervals***

$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n].$$

The typical closed subinterval $[x_{k-1}, x_k]$ is called the k^{th} subinterval of P.



The width (length) of the k^{th} subinterval is $\Delta x_k = x_k - x_{k-1}$.



In each subinterval $[x_{k-1}, x_k]$, we select a point c_k and construct a vertical rectangle on the subinterval $[x_{k-1}, x_k]$ to touch the curve $y=f(x)$ at the point $(c_k, f(c_k))$.

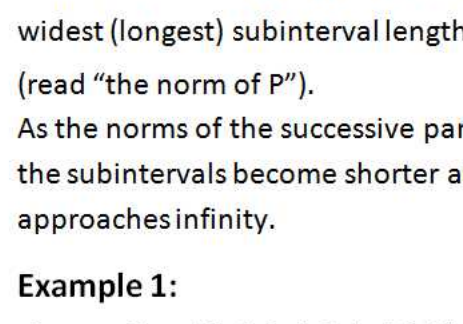
The choice of c_k does not matter as long as it lies in $[x_{k-1}, x_k]$.

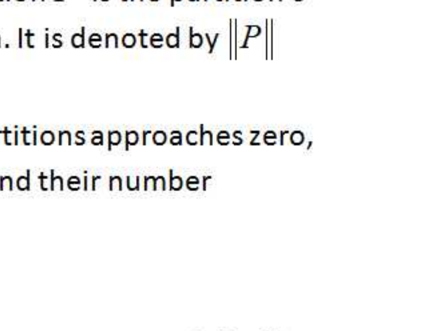
If $f(c_k)$ is positive, the number $f(c_k)\Delta x_k = \text{height} \times \text{base}$ is the area of the rectangle. If $f(c_k)$ is negative, the number $f(c_k)\Delta x_k$ is the negative of the area. In any case, we add the n products $f(c_k)\Delta x_k$ to form the sum

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

This sum, which depends on P and the choice of the numbers c_k, is called a ***Riemann sum for f on the interval*** $[a,b]$, after German mathematician Riemann, who studied the limits of such sums.

As the partitions of $[a,b]$ become finer, the rectangles defined by the partition approximate the region between the x-axis and the graph of f with increasing accuracy. Finer partitions create more rectangles with shorter bases





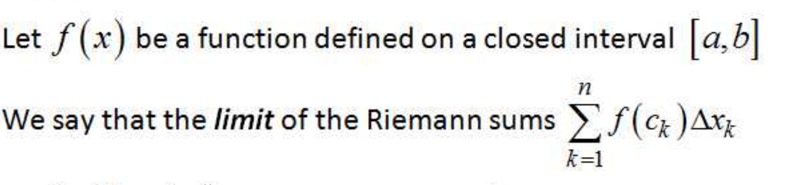
So, we expect the associated Riemann sums to have a limiting value. The norm of a partition P is the partition's widest (longest) subinterval length. It is denoted by $\|P\|$ (read "the norm of P").

As the norms of the successive partitions approaches zero, the subintervals become shorter and their number approaches infinity.

Example 1:

The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$.

There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$ and $[1.5, 2]$.



The lengths of the subintervals are

$\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$ and $\Delta x_5 = 0.5$.

The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length.

Definition

Let $f(x)$ be a function defined on a closed interval $[a,b]$

We say that the ***limit*** of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ on $[a,b]$ as $\|P\| \to 0$ is the number I if the following condition is satisfied:

Given any number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for every partition P of $[a,b]$

$$\|P\| < \delta \quad \Rightarrow \quad \left|\sum_{k=1}^{n} f(c_k)\Delta x_k - I\right| < \varepsilon$$

for any choice of the numbers c_k in the subinterval $[x_{k-1}, x_k]$.

If the limit exists, we write $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$

We call I the ***definite integral*** of f over $[a,b]$, we say that f is ***integrable*** over $[a,b]$, and we say that the Riemann sums of f on $[a,b]$ converge to the number I.

We usually write I as $\int_a^b f(x)\,dx$, which is read "integral of f from a to b." Thus if the limit exists,

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx$$

When each partition has n equal subintervals, each of width $\Delta x = \frac{b-a}{n}$, we will also write

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = I = \int_a^b f(x)\,dx$$

Despite the variety in the Riemann sums $\sum f(c_k)\Delta x_k$ as the partitions change and the arbitrary choice of c_k's in the interval of each new partition, the sums always have the same limit as $\|P\| \to 0$ as long as f is continuous.

Theorem 1:

All continuous functions are integrable. That is, if a function f is continuous on an interval $[a,b]$, then its definite integral over $[a,b]$ exists.

The theorem is proved in advanced calculus.

Example 2:

Express the limit of Riemann sums

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(3c_k^2 - 2c_k + 5\right)\Delta x_k$$

as an integral if P denotes a partition of the interval $[-1, 3]$.

Solution:

The function being evaluated at c_k in each term of the sum is $f(x) = 3x^2 - 2x + 5$ a polynomial and hence a continuous function. The interval being partitioned is $[\mathbf{-1}, \mathbf{3}]$.

The limit is therefore the integral of f from -1 to 3:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(3c_k^2 - 2c_k + 5\right)\Delta x_k = \int_{-1}^{3} \left(3x^2 - 2x + 5\right)dx$$

Constant Functions

Suppose that f has the constant value $f(x) = c$ over $[a,b]$. Then, no matter how the c_k's are chosen,

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} c \cdot \Delta x_k = c \cdot \sum_{k=1}^{n} \Delta x_k = c(b-a)$$

Since the sums all have the value $c(b-a)$, their limit, the integral, does too. We have the following result.

If $f(x)$ has the constant value c on $[a,b]$, then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a)$$

Example 3:

a) $\int_{-1}^{4} 3\,dx = 3(4-(-1)) = (3)(5) = 15$

b) $\int_{-1}^{4} -3\,dx = -3(4-(-1)) = (-3)(5) = -15$

IP1:

Let $f(x) = sinx$ be a function defined over the interval $[-\pi, \pi]$. Partition the interval into 4 subintervals of equal length. Then compute the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x_k$.

Given that c_k is the

a. Left hand endpoint
b. Right hand endpoint
c. Midpoint of the k^{th} subinterval.

Solution:

The given function is $f(x) = sinx$, $x \in [-\pi, \pi]$

By partitioning the interval $[0, 2]$ into 4 subintervals with $\Delta x = \frac{\pi-(-\pi)}{4} = \frac{\pi}{2}$.

We have $\left[-\pi, -\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2}, 0\right]$, $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$

Left hand endpoints:

The left hand endpoints of the subintervals are $c_1 = -\pi$, $c_2 = -\frac{\pi}{2}$, $c_3 = 0$, $c_4 = \frac{\pi}{2}$

We now compute the corresponding Riemann sum.

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4$$
$$= \left(\frac{\pi}{2}\right)\left[\sin(-\pi) + \sin\left(-\frac{\pi}{2}\right) + \sin 0 + \sin\left(\frac{\pi}{2}\right)\right]$$
$$= \left(\frac{\pi}{2}\right)[0 - 1 + 0 + 1] = 0$$

Right hand endpoints:

The right hand endpoints of the subintervals are $c_1 = -\frac{\pi}{2}$, $c_2 = 0$, $c_3 = \frac{\pi}{2}$, $c_4 = \pi$

We now compute the corresponding Riemann sum.

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4$$
$$= \left(\frac{\pi}{2}\right)\left[\sin\left(-\frac{\pi}{2}\right) + \sin 0 + \sin\left(\frac{\pi}{2}\right) + \sin(\pi)\right]$$
$$= \left(\frac{\pi}{2}\right)[-1 + 0 + 1 + 0] = 0$$

C. Midpoints of the subintervals:

The midpoints of the subintervals are $c_1 = -\frac{3\pi}{4}$, $c_2 = -\frac{\pi}{4}$, $c_3 = \frac{\pi}{4}$, $c_4 = \frac{3\pi}{4}$

We now compute the corresponding Riemann sum.

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4$$
$$= \left(\frac{\pi}{2}\right)\left[\sin\left(-\frac{3\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\right]$$
$$= \left(\frac{\pi}{2}\right)\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 0$$

IP2:

Find the norm of the partition $\boldsymbol{P = \{-2, -1.6, -0.5, 0, 0.8, 1\}}$ of the interval $\boldsymbol{[-2, 1]}$?

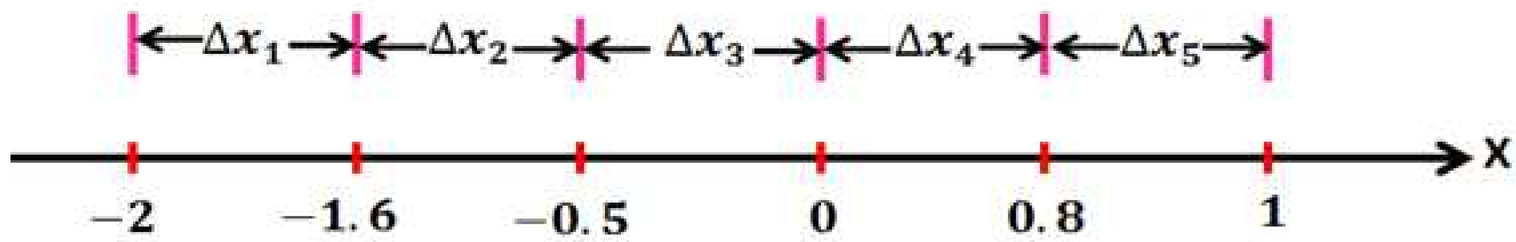
Solution:

The set $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$ is a partition of the interval $[-2, 1]$, since

$$-2 < -1.6 < -0.5 < 0 < 0.8 < 1$$

There are 5 subintervals of P:

$[-2, -1.6], [-1.6, -0.5], [-0.5, 0], [0, 0.8], [0.8, 1]$



The lengths of the subintervals are

$$\Delta x_1 = -1.6 - (-2) = 0.4$$

$$\Delta x_2 = -0.5 - (-1.6) = 1.1$$

$$\Delta x_3 = 0 - (-0.5) = 0.5$$

$$\Delta x_4 = 0.8 - 0 = 0.8$$

$$\Delta x_5 = 1 - 0.8 = 0.2$$

The longest subinterval length is 1.1. So, the norm of the partition is $\|P\| = 1.1$

IP3:

Express the limits in the problems a and b as definite integrals.

a. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4-c_k^2}\,.\Delta x_k$,where P is a partition of $[0, 1]$

b. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \tan(c_k).\Delta x_k$, where P is a partition of $\left[0, \frac{\pi}{4}\right]$

Solution:

a. The function evaluated at c_k in each term of the sum is $f(x) = \sqrt{4 - x^2}$. The interval partitioned is $[0, 1]$.

Notice that $f(x)$ is a continuous function on $[0, 1]$.

The limit is therefore the integral of f from 0 to 1.

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4-c_k^2}\,.\Delta x_k = \int_0^1 \sqrt{4-x^2}\,dx$$

b. The function being evaluated at c_k in each term of the sum is $f(x) = \tan x$. The interval partitioned is $\left[0, \frac{\pi}{4}\right]$.

Notice that $f(x)$ is a continuous function on $\left[0, \frac{\pi}{4}\right]$.

The limit is therefore the integral of f from 0 to $\frac{\pi}{4}$.

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \tan(c_k).\Delta x_k = \int_0^{\pi/4} \tan x\,dx$$

IP4.

If the function $f(x)$ is defined on the interval $[-1.05, 1.5]$ as $f(x) = 3.5$ then find $\int_{-1.05}^{1.5} f(x)\, dx$

Solution:

The function f defined on the interval $[-1.05, 1.5]$ is $(x) = 3.5$, which is a constant function and therefore,

$$\int_{-1.05}^{1.5} f(x)\, dx = \int_{-1.05}^{1.5} 3.5\, dx = 3.5[1.5-(-1.05)] = 8.925$$

P1:

Let $f(x) = x^2 - 1$ be a function defined over the interval $[0, 2]$. Partition the interval into 4 subintervals of equal length. Then compute the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x_k$.

Given that c_k is the

a. Left hand endpoint
b. Right hand endpoint
c. Midpoint of the k^{th} subinterval

Solution:

The given function is $f(x) = x^2 - 1, x \in [0, 2]$

By partitioning the interval $[0, 2]$ into 4 subintervals with $\Delta x = \frac{2-0}{4} = 0.5$

We have, $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$

Left hand endpoints:

The left hand endpoints of the subintervals are $c_1 = 0$, $c_2 = 0.5$ $c_3 = 1$, $c_4 = 1.5$

We now compute the corresponding Riemann sum.

$$\begin{aligned}\sum_{k=1}^{4} f(c_k)\Delta x_k &= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4 \\ &= (0.5)\left[\left(0^2-1\right)+\left((0.5)^2-1\right)+\left(1^2-1\right)+\left((1.5)^2-1\right)\right] \\ &= (0.5)\left[-1-0.75+0+1.25\right] = -0.25\end{aligned}$$

Right hand endpoints:

The right hand endpoints of the subintervals are $c_1 = 0.5$, $c_2 = 1$ $c_3 = 1.5$, $c_4 = 2$

We now compute the corresponding Riemann sum.

$$\begin{aligned}\sum_{k=1}^{4} f(c_k)\Delta x_k &= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4 \\ &= (0.5)\left[\left((0.5)^2-1\right)+\left(1^2-1\right)+\left((1.5)^2-1\right)+\left(2^2-1\right)\right] \\ &= (0.5)\left[-0.75+0+1.25+3\right] = 1.75\end{aligned}$$

C. Midpoints of the subintervals:

The midpoints of the subintervals are $c_1 = 0.25$, $c_2 = 0.75$, $c_3 = 1.25$, $c_4 = 1.75$

We now compute the corresponding Riemann sum.

$$\begin{aligned}\sum_{k=1}^{4} f(c_k)\Delta x_k &= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4 \\ &= (0.5)\left[\left((0.25)^2-1\right)+\left((0.75)^2-1\right)+\left((1.25)^2-1\right)+\left((1.75)^2-1\right)\right] \\ &= (0.5)\left[-0.9375-0.4375+0.5625+2.0625\right] = 0.625\end{aligned}$$

P2:

Find the norm of the partition $\boldsymbol{P = \{-1, -0.5, -0.1, 1, 1.05, 1.5, 2\}}$ of the interval $\boldsymbol{[-1, 2]}$?

Solution:

The set $P = \{-1, \ -0.5, \ -0.1, \ 1, \ 1.05, \ 1.5, \ 2\}$ is a partition of the interval $[-1, 2]$, since

$$-1 < -0.5 < -0.1 < 1 < 1.05 < 1.5 < 2$$

There are 6 subintervals of P:

$[-1, -0.5], [-0.5, -0.1], [-0.1, 1], [1, 1.05], [1.05, 1.5], [1.5, 2]$

The lengths of the subintervals are

$$\Delta x_1 = -0.5 - (-1) = 0.5$$

$$\Delta x_2 = -0.1 - (-0.5) = 0.4$$

$$\Delta x_3 = 1 - (-0.1) = 1.1$$

$$\Delta x_4 = 1.05 - 1 = 0.05$$

$$\Delta x_5 = 1.5 - 1.05 = 0.45$$

$$\Delta x_6 = 2 - 1.5 = 0.5$$

The longest subinterval length is 1.1. So, the norm of the partition is $\|P\| = 1.1$

P3:

Express the limits in the problems a, b as definite integrals.

a. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^3 . \Delta x_k$,where P is a partition of $[-1, 0]$

b. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^3 . \Delta x_k$, where P is a partition of $[1, 4]$

Solution:

a. The function evaluated at c_k in each term of the sum is $f(x) = 2x^3$ and it is a continuous function on $[-1, 0]$. The interval partitioned is $[-1, 0]$.

The limit is therefore the integral of f from -1 to 0.

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^3 . \Delta x_k = \int_{-1}^{0} 2x^3 \, dx$$

b. The function being evaluated at c_k in each term of the sum is $f(x) = 2x^3$ and it is a continuous function on $[1, 4]$. The interval partitioned is $[1, 4]$.

The limit is therefore the integral of f from 1 to 4.

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^3 . \Delta x_k = \int_{1}^{4} 2x^3 \, dx$$

P4:

If the function $f(x)$ is defined on the interval $[3,5]$ as $f(x) = 3$ then find $\int_3^5 f(x)dx$

Solution:

The function f defined on the interval $[3,5]$ is $f(x) = 3$, which is a constant function and therefore,

$$\int_3^5 f(x)\,dx = \int_3^5 3\,dx = 3[5-3] = 6$$

2.6. Limit of Riemann Sums

Exercise:

1. Partition the interval into four subintervals of equal length. Then compute the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x_k$, given that c_k is the (a) left-hand endpoint, (b) right-hand endpoint, (c) midpoint of the k^{th} subinterval.

   $a.\ f(x)=x^2-1 \quad , \quad [0,2]$

   $b.\ f(x)=\sin x \quad , \quad [-\pi,\pi]$

2. Find the norm of the partition

   a. $P=\{0,1.2,1.5,2.3,2.6,3\}$

   b. $P=\{0,\ 0.2,\ 0.5,0.7,\ 0.9,\ 1\}$

3. Express the limits as definite integrals.

   a. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} c_k^2\Delta x_k$, where P is a partition of $[0,2]$

   b. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} \frac{1}{1-c_k}\Delta x_k$, where P is a partition of $[2,3]$

   c. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} \left(c_k^2-3c_k\right)\Delta x_k$, where P is a partition of $[-7,5]$

   d. $\lim_{\|P\|\to 0}\sum_{k=1}^{n} (\sec c_k)\Delta x_k$, where P is a partition of $[-\pi/4,0]$

4. Evaluate the integrals

   $a.\ \int_{-2}^{1} 5\,dx$

   $b.\ \int_{0}^{3} (-160)\,dt$

   $c.\ \int_{-2.1}^{3.4} 0.5\,ds$