3.1

Binomial Series

Learning Objectives:

- To define the Binomial Series when the index is negative and fractional values
- To find the general term of the Binomial series

AND

- To solve problems related to them

We shall consider the binomial theorem when the index has negative and fractional values. It gives rise to an infinite series, called ***binomial series***.

The expansion of a binomial may always be made to depend upon the case in which the first term is unity.

$$(x+y)^n = \left\{x\left(1+\frac{y}{x}\right)\right\}^n = x^n(1+z)^n$$

where, $z=\frac{y}{x}$. Therefore, it is sufficient to confine our attention to binomials of the form $(1+x)^n$. By actual division, we find

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

In each of these series, the number of terms is unlimited. They are only particular cases of the general formula for the expansion of $(1+x)^n$, where n is any rational quantity.

For a positive integral index n the expansion of $(1+x)^n$ is

$$(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{r}x^r+\cdots+\binom{n}{n}x^n$$

When n is fractional or negative, the symbol $\binom{n}{r}$ is no longer applicable. We write the above expansion as

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

It can be proved that the above expansion into infinite series holds only when the modulus of x is less than 1.

The binomial theorem for any index may then be stated as follows.

If x is a real number and n is a rational number, then the following formula, known as the binomial expansion,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots \quad \ldots\ldots\ldots(1)$$

holds whenever $|x| < 1$ i.e., $x \in (-1,1)$.

The **general term** of the expansion is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

where $r = 1, 2, 3 \ldots$

The coefficient of the general term vanishes only when one of the factors of its numerator is zero. The series will therefore stop at the r^{th} term when $n-r+1$ is zero; that is when $r=n+1$; but since r is a positive integer this equality can never hold except when the index n is positive and integral. Thus, **the expansion by the binomial theorem extends to an infinite number of terms when n is negative or fractional**. (If n is zero or a positive integer, then the series is finite)

If we apply the binomial expansion formula to $(1-2)^{-2}$, we will have

$$(1-2)^{-2} = 1 + (-2)(-2) + \frac{(-2)(-3)}{1.2}(-2)^2 + \cdots$$

$$1 = 1 + 4 + 12 + \ldots$$

which is impossible.

It is important to remember that the *condition* $|x| < 1$ *is necessary for the binomial expansion when* n *is fractional or negative.*

Example

Expand $\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}$ when $|x|<2$

Solution

Let $z=-\frac{x}{2}$. Then $|z|=\frac{1}{2}|x|<\frac{2}{2}=1$

Therefore, $(1+z)^{-\frac{1}{2}}=1+\frac{\left(-\frac{1}{2}\right)}{1}z+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{2}-1\right)}{1\cdot 2}z^2+\cdots$

$$\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}=1+\frac{x}{4}+\frac{3x^2}{32}+\cdots$$

The following particular cases of binomial series occur in science and engineering often, and they are valid only when $|x| < 1$.

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Theorem:

If $|x| < 1$ and $n \in N$, then

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots + (-1)^r \frac{n(n+1)\dots(n+r-1)}{r!}x^r + \cdots$$

Proof: We have, by Binomial expansion (1)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \cdots$$

Replacing n by $-n$, we get

$$(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3 + \cdots + \frac{(-n)(-n-1)\dots(-n-r+1)}{r!}x^r + \cdots$$

$$= 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots + (-1)^r \frac{n(n+1)\dots(n+r-1)}{r!}x^r + \cdots$$

Note (1):

a) If $|x| < 1, n \in N$, then

$$(1+x)^{-n} = 1 - {}^nC_1x + {}^{(n+1)}C_2x^2 - {}^{(n+2)}C_3x^3 + \cdots + (-1)^r\, {}^{(n+r-1)}C_r\, x^r + \ldots (2)$$

$$= \sum_{r=0}^{\infty} (-1)^r\, {}^{(n+r-1)}C_r\, x^r$$

b) Its general term is $T_{r+1} = (-1)^r\, {}^{(n+r-1)}C_r x^r$

Note (2):

a) If $|x| < 1, n \in N$, then (Replacing x by $-x$ in (2))

$$(1-x)^{-n} = 1 + {}^nC_1x + {}^{(n+1)}C_2x^2 + {}^{(n+2)}C_3x^3 + \cdots + {}^{(n+r-1)}C_r\, x^r + \ldots$$

$$= \sum_{r=0}^{\infty} {}^{(n+r-1)}C_r\, x^r$$

b) Its general term is $T_{r+1} = {}^{(n+r-1)}C_r\, x^r$

Some particular cases of Binomial expansion:

a)

$$(1+x)^{-1} = \sum_{r=0}^{\infty} (-1)^r\, {}^{(n+r-1)}C_r x^r \quad \text{where } n = 1$$

$$= \sum_{r=0}^{\infty} (-1)^r\, {}^rC_r x^r = \sum_{r=0}^{\infty} (-1)^r x^r$$

$$= 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots$$

Its general term is $T_{r+1} = (-1)^r x^r$

b)

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$$ (Replacing x by $-x$ in the above)

Its general term is $T_{r+1} = x^r$

c)

$$(1+x)^{-2} = \sum_{r=0}^{\infty} (-1)^r\, {}^{(n+r-1)}C_r x^r \quad \text{where } n = 2$$

$$= \sum_{r=0}^{\infty} (-1)^r\, {}^{(r+1)}C_r x^r = \sum_{r=0}^{\infty} (-1)^r\, {}^{(r+1)}C_1 x^r$$

$$= \sum_{r=0}^{\infty} (-1)^r (r+1)\, x^r$$

$$= 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^r (r+1)\, x^r + \ldots$$

Its general term is $T_{r+1} = (-1)^r (r+1)\, x^r$

d) By replacing x by $-x$ in the above, we get

$$(1-x)^{-2} = \sum_{r=0}^{\infty} (r+1)\, x^r$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+1)x^r + \ldots$$

Its general term is $T_{r+1} = (r+1)\, x^r$

e)

$$(1+x)^{-3} = \sum_{r=0}^{\infty} (-1)^r\, {}^{(n+r-1)}C_r x^r \quad \text{where } n = 3$$

$$= \sum_{r=0}^{\infty} (-1)^r\, {}^{(r+2)}C_r x^r = \sum_{r=0}^{\infty} (-1)^r\, {}^{(r+2)}C_2 x^r$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(r+1)(r+2)}{2} x^r$$

$$= \frac{1}{2}\left[1.2 - 2.3x + 3.4x^2 - 4.5x^3 + \ldots + (-1)^r (r+1)(r+2)x^r + \ldots\right]$$

Its general term is $T_{r+1} = (-1)^r \dfrac{(r+1)(r+2)}{2} x^r$

f) Replacing x by $-x$ in the above, we get

$$(1-x)^{-3} = \frac{1}{2}\left[1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \ldots + (r+1)(r+2)x^r + \ldots\right]$$

Its general term is $T_{r+1} = \dfrac{(r+1)(r+2)}{2} x^r$

IP1:

Find the general term in the expansion of $\left(3-\frac{5x}{4}\right)^{-\frac{1}{2}}$.

Solution:

Given: $\left(3-\frac{5x}{4}\right)^{-\frac{1}{2}}$

Step1:

$$\left(3-\frac{5x}{4}\right)^{-\frac{1}{2}} = 3^{-\frac{1}{2}}\left(1-\frac{5x}{12}\right)^{-\frac{1}{2}}$$

Step2:

$$T_{r+1} = 3^{-\frac{1}{2}}\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots\left(-\frac{1}{2}-r+1\right)}{r!}\left(-\frac{5x}{12}\right)^r$$

$$= \frac{3^{-\frac{1}{2}}}{r!}(-1)^{2r}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(\frac{2r-1}{2}\right)\left(\frac{5x}{12}\right)^r$$

$$= 3^{-\frac{1}{2}}\frac{1\times3\times5\times\ldots\times(2r-1)}{r!}\left(\frac{5x}{24}\right)^r$$

Step3:

The general term of $\left(3-\frac{5x}{4}\right)^{-\frac{1}{2}}$ is $3^{-\frac{1}{2}}\frac{1\times3\times5\times\ldots\times(2r-1)}{r!}\left(\frac{5x}{24}\right)^r$

IP2:

Find 8th term of $\left(1-\frac{5x}{2}\right)^{-\frac{3}{5}}$

Solution:

Given: $\left(1-\frac{5x}{2}\right)^{-\frac{3}{5}}$

$$T_8=\frac{\left(-\frac{3}{5}\right)\left(-\frac{3}{5}-1\right)\left(-\frac{3}{5}-2\right)\dots\left(-\frac{3}{5}-6\right)}{7!}\left(-\frac{5x}{2}\right)^7$$

$$=\frac{\left(\frac{3}{5}\right)\left(\frac{8}{5}\right)\left(\frac{13}{5}\right)\dots\left(\frac{33}{5}\right)}{7!}\left(\frac{5x}{2}\right)^7$$

$$=\frac{3\times 8\times 13\times \dots\times 33}{7!}\left(\frac{x}{2}\right)^7=\frac{118404}{5}\left(\frac{x}{2}\right)^7$$

IP3:

Find the coefficient of x^8 in $\frac{(1+x)^2}{(1-\frac{2}{3}x)^3}$.

Solution:

Given: $\frac{(1+x)^2}{(1-\frac{2}{3}x)^3}$

Step1:

$$\frac{(1+x)^2}{(1-\frac{2}{3}x)^3} = (1 + 2x + x^2)\left(1 - \frac{2}{3}x\right)^{-3}$$

$$= (1 + 2x + x^2)\left[1 + 3_{C_1}\left(\frac{2}{3}x\right) + 4_{C_2}\left(\frac{2}{3}x\right)^2 + 5_{C_3}\left(\frac{2}{3}x\right)^3 + \cdots\right]$$

Step2:

$$\therefore Coefficient\ of\ x^8 = 10_{C_8}\left(\frac{2}{3}\right)^8 + 2 \times 9_{C_7}\left(\frac{2}{3}\right)^7 + 8_{C_6}\left(\frac{2}{3}\right)^6$$

$$= \left(\frac{2}{3}\right)^6 (10_{C_8} \times \frac{4}{9} + 2 \times 9_{C_7} \times \frac{2}{3} + 8_{C_6})$$

$$= \left(\frac{2}{3}\right)^6 (20 + 48 + 28) = \frac{64}{729} \times 96 = \frac{2048}{243}$$

Step3:

Therefore, Coefficient of x^8 in $\frac{(1+x)^2}{(1-\frac{2}{3}x)^3}$ is $\frac{2048}{243}$

IP4:

If $x = \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \cdots$, then $9x^2 + 24x =$ ___.

Solution:

Given: $x = \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \cdots$

Adding $1 + \frac{1}{3}$ on both sides, we get

$$1 + \frac{1}{3} + x = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \cdots$$

Comparing the series with $1 + ny + \frac{n(n-1)}{2}y^2 + \cdots$, we get

$ny = \frac{1}{3}$.............. (1) and $\frac{n(n-1)}{2}y^2 = \frac{1.3}{3.6}$ .... (2)

From (2), we have: $ny(ny - y) = \frac{1}{3} \Rightarrow \frac{1}{3}\left(\frac{1}{3} - y\right) = \frac{1}{3} \Rightarrow y = -\frac{2}{3}$

From (1), we have: $n\left(-\frac{2}{3}\right) = \frac{1}{3} \Rightarrow n = -\frac{1}{2}$

Therefore, $1 + \frac{1}{3} + x = (1 + y)^n = \left(1 - \frac{2}{3}\right)^{-\frac{1}{2}} = (3)^{\frac{1}{2}}$

$$\Rightarrow \left(\frac{4}{3} + x\right)^2 = 3$$

$$\Rightarrow x^2 + \frac{8}{3}x + \frac{16}{9} = 3$$

$$\Rightarrow 9x^2 + 24x = 11$$

P1:

Find the general term in the expansion of $(2-3x)^{-\frac{1}{3}}$.

Solution:

Given: $(2-3x)^{-\frac{1}{3}}$

$$(2-3x)^{-\frac{1}{3}} = 2^{-\frac{1}{3}}\left(1-\frac{3x}{2}\right)^{-\frac{1}{3}}$$

$$T_{r+1} = 2^{-\frac{1}{3}}\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)\dots\left(-\frac{1}{3}-r+1\right)}{r!}\left(-\frac{3x}{2}\right)^{r}$$

$$= \frac{2^{-\frac{1}{3}}}{r!}(-1)^{2r}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)\dots\left(\frac{3r-2}{3}\right)\left(\frac{3x}{2}\right)^{r}$$

$$= 2^{-\frac{1}{3}}\frac{1\times4\times7\times\dots\times(3r-2)}{r!}\left(\frac{x}{2}\right)^{r}$$

P2:

Find 6th term of $\left(3+\frac{2x}{3}\right)^{\frac{3}{2}}$

Solution:

Given: $$\left(3+\frac{2x}{3}\right)^{\frac{3}{2}} = 3^{\frac{3}{2}}\left(1+\frac{2x}{9}\right)^{\frac{3}{2}}$$

$$T_6 = 3^{\frac{3}{2}}\frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)\left(\frac{3}{2}-4\right)}{5!}\left(\frac{2x}{9}\right)^5$$

$$= 3^{\frac{3}{2}}\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{5!}\left(\frac{2x}{9}\right)^5$$

$$= -3\sqrt{3}\,\frac{3\times 1\times 3\times 5}{5!}\left(\frac{x}{9}\right)^5 = -\frac{9\sqrt{3}}{8}\left(\frac{x}{9}\right)^5$$

P3:

Find the coefficient of x^{12} in $\frac{(1+3x)}{(1+4x)^4}$.

Solution:

Given: $\frac{(1+3x)}{(1-4x)^4}$

$$\frac{(1+3x)}{(1+4x)^4} = (1+3x)(1+4x)^{-4}$$

$$= (1+3x)\left[1 - 4_{C_1}(4x) + 5_{C_2}(4x)^2 - \ 6_{C_3}(4x)^3 + \cdots + (-1)^r(4+r-1)_{C_r}(4x)^r\right]$$

$$\therefore Coefficient\ of\ x^{12} = 15_{C_{12}}4^{12} + 3 \times (-1)^{11}14_{C_{11}}4^{11}$$

$$= 4^{11}(15_{C_3} \times 4 - 3 \times 14_{C_3})$$

$$= 4^{11}\left[\frac{15\times14\times13\times4}{6} - \frac{3\times14\times13\times12}{6}\right]$$

$$= 4^{11} \times 14 \times 13 \times 4$$

$$= 182 \times 4^{12}$$

P4:

If $x = \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + \cdots$, then $x^2 + 2x =$ ___.

Solution:

Given: $x = \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + \cdots$

Adding 1 on both sides, we get

$$1 + x = 1 + \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + \cdots$$

Comparing the series with $1 + ny + \frac{n(n-1)}{2}y^2 + \cdots$, we get

$ny = \frac{1}{5}$.............. (1) and $\frac{n(n-1)}{2}y^2 = \frac{1.3}{5.10}$ .... (2)

From (2), we have: $\frac{ny(ny-y)}{2} = \frac{3}{50} \Rightarrow \frac{1}{5}\left(\frac{1}{5} - y\right) = \frac{3}{25} \Rightarrow y = -\frac{2}{5}$

From (1), we have: $n\left(-\frac{2}{5}\right) = \frac{1}{5} \Rightarrow n = -\frac{1}{2}$

Therefore, $1 + x = (1 + y)^n = \left(1 - \frac{2}{5}\right)^{-\frac{1}{2}} = \left(\frac{5}{3}\right)^{\frac{1}{2}}$

$$\Rightarrow (1 + x)^2 = \frac{5}{3}$$

$$\Rightarrow x^2 + 2x = \frac{2}{3}$$

1. Find the coefficient of x in the expansion of $\frac{1+x}{1-x}$ if $|x|<1$.

2. Find the $(r+1)^{th}$ term and the coefficient of x^r in the expansion of $(1-2x)^{-\frac{3}{2}}$ where $|2x|<1$.

3. Find the range of x for which the binomial expansions of the following are valid

a. $(3-4x)^{-\frac{5}{2}}$

b. $(9+5x)^{\frac{3}{2}}$

c. $(4+7x)^{-\frac{5}{3}}$

d. $(7+3x)^{-3}$

4. Write the first 3 terms in the expansion of

a. $(4-5x)^{-\frac{1}{2}}$

b. $(2-7x)^{-\frac{3}{4}}$

c. $\left(1+\frac{x}{2}\right)^{-5}$

5. Write the general term in the expansion of

a. $(1 - 4x)^{-3}$

b. $\left(1 - \frac{5x}{3}\right)^{-3}$

c. $\left(3 + \frac{x}{2}\right)^{-\frac{2}{3}}$

6. Find the coefficients of

a. x^4 in $(1-4x)^{-\frac{3}{5}}$

b. x^6 in $(1-3x)^{-\frac{2}{5}}$

7. Find the

a. 6th term of $\left(1+\frac{x}{2}\right)^{-5}$

b. 5th term of $\left(7+\frac{8y}{3}\right)^{\frac{7}{4}}$

c. 10th term of $(3-4x)^{-\frac{2}{3}}$

8. Find the coefficient of x^n in the expansion of

$$(1 - 2x + 3x^2 - 4x^3 + \cdots \infty)^{-n}$$

9. Find the sum of the series

a. $1 + \frac{1}{4} + \frac{1.4}{4.8} + \frac{1.4.7}{4.8.12} + \cdots + \cdots$

b. $1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots + \cdots$

c. $1 + \frac{1}{3} \cdot \frac{1}{4} + \frac{1.4}{3.6} \cdot \frac{1}{4^2} + \frac{1.4.7}{3.6.9} \cdot \frac{1}{4^3} + \cdots + \cdots$

d. $1 - \frac{1}{8} + \frac{1.3}{8.16} - \frac{1.3.5}{8.16.24} + \cdots + \cdots$

10. If $y = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots + \cdots$, then find $y^2 + 2y$.

3.2
Geometric Series

Learning objectives:

- To derive a formula of sum of an infinite number of terms of a decreasing geometric series and to solve related problems

Now, consider the series $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\cdots$

Let *s* be the sum of the infinite series.

The sum to *n* terms $= \dfrac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}$

By making n sufficiently large, we can make the fraction $\frac{1}{2^{n-1}}$ as small as we please. Then the sum differs from 2 by as little as we please. Thus, by taking sufficient number of terms, we have $S = 2$.

We now consider a more general case. We have

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$= \frac{a}{1-r} - \frac{ar^n}{1-r}$$

Suppose r is a proper fraction; then the greater the value of n the smaller is the value of r^n, and consequently of $\frac{ar^n}{1-r}$; and therefore by making n sufficiently large, we can make the sum of n terms of the series differ from $\frac{a}{1-r}$ by a small quantity as we please. Thus: ***the sum of an infinite number of terms of a decreasing geometrical progression is*** $\frac{a}{1-r}$. In other words, the sum to infinity is

$$S = \frac{a}{1-r}$$

Example

The sum of an infinite number of terms in a G.P. is 15, and the sum of their squares is 45; find the series.

Solution

Let a denote the first term, r the common ratio; then the sum of the terms is $a/(1-r)$. The squares series is

$$a^2, a^2r^2, a^3r^3, \cdots$$

The common ratio is r^2. Therefore the sum of the series is $\frac{a}{1-r}$.

Hence,

$$\frac{a}{1-r} = 15 \ , \ \frac{a^2}{1-r^2} = 45$$

Dividing

$$\frac{a^2}{1-r^2} \cdot \frac{1-r}{a} = \frac{45}{15} \Rightarrow \frac{a}{1+r} = 3$$

Now, $\frac{a}{1+r} = 3, \frac{a}{1-r} = 15 \Rightarrow \frac{a}{1-r} \cdot \frac{1+r}{a} = \frac{15}{3}$

Hence

$$\frac{1+r}{1-r} = 5 \Rightarrow r = \frac{2}{3} \text{ and } a = 5$$

Thus the series is $5, \frac{10}{3}, \frac{20}{9}, \cdots$

Example

Find the value of $.4\dot{2}\dot{3}$.

Solution

$$.4\dot{2}\dot{3} = \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} + \cdots$$
$$= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \cdots$$
$$= \frac{4}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right)$$
$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{1}{1-(1/10^2)}$$
$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{100}{99}$$
$$= \frac{4}{10} + \frac{23}{990} = \frac{419}{990}$$

IP1)

Sum of $4 + 2 + 1 + .5 + .25 + \cdots + \cdots$

Solution

Step1:

$4 + 2 + 1 + .5 + .25 + \cdots$

It is a G.P.

Step2:

a= 4, $r = \frac{1}{2}$

Step3:

Sum of infinite number of terms $S = \frac{a}{1-r} = \frac{4}{1-\frac{1}{2}} = 8$

IP2)

Find a rational number which when expressed as a decimal will have $0.12\overline{76}$ as its expansion

Solution:

$$.12\overline{76}=\frac{12}{10}+\frac{76}{10000}+\frac{76}{1000000}+\cdots$$

$$=\frac{12}{10}+\frac{76}{10^4}+\frac{76}{10^6}+\cdots$$

$$=\frac{12}{10}+\frac{76}{10^4}\left(1+\frac{1}{10^2}+\frac{1}{10^4}+\cdots\right)$$

$$=\frac{12}{10}+\frac{76}{10^4}\cdot\frac{1}{1-\left(1/10^2\right)}$$

$$=\frac{12}{10}+\frac{76}{10^4}\cdot\frac{100}{99}$$

$$=\frac{12}{10}+\frac{76}{9900}=\frac{11956}{9900}=\frac{2989}{2475}$$

IP3)

The sum of an infinite G.P is 57 and the sum of their cubes is 9747, find the G.P.

Solution:

Step1:

Let a be the first term and r the common ratio of the G.P. Then,

Sum $=\ 57\ \Rightarrow\ \frac{a}{1-r}=57 \qquad \ldots..(1)$

Step2:

Sum of the cubes $=9747$

$$\Rightarrow \quad a^3+a^3r^3+a^3r^6+\cdots=9747$$

$$\Rightarrow \quad \frac{a^3}{1-r^3}=9747 \quad \ldots..(ii)$$

Step3:

Dividing the cube of (i) by(ii), we get

$$\frac{a^3}{(1-r)^3}\cdot\frac{(1-r^3)}{a^3}=\frac{(57)^3}{9747}$$

$$\Rightarrow \quad \frac{1-r^3}{(1-r)^3}=19$$

$$\Rightarrow \quad \frac{1+r+r^2}{(1-r)^2}=19$$

$$\Rightarrow \quad 18r^2-39r+18=0$$

$$\Rightarrow \quad (3r-2)(6r-9)=0$$

$$\Rightarrow \quad r=\frac{2}{3} \text{ or, } \quad r=\frac{3}{2}$$

Step4:

$$\Rightarrow \quad r=\frac{2}{3} \quad [\because \quad r\neq\frac{3}{2}, because$$
$$-1<r<1\ for\ an\ infinite\ G.P.]$$

Step5:

Putting $r=\frac{2}{3}$ in(i), we get

$$\frac{a}{1-\left(\frac{2}{3}\right)}=57\ \Rightarrow a=19$$

Hence, the G.P. is $19, \frac{38}{3}, \frac{76}{9}, \ldots..$

IP4)

Let S_n , $n = 1, 2, 3$ ……. be the sum of infinite geometric series whose first term is n and the common ratio is $\frac{1}{(n+1)}$

Then $\lim_{n\to\infty} \frac{S_1S_n+S_2S_{n-1}+S_3S_{n-2}+\cdots+S_nS_1}{S_1{}^2+S_2{}^2+\cdots+S_n{}^2} =$

Solution:

Step1:

Given that $a = n$ and $r = \frac{1}{(n+1)}$

Then the sum of the terms in infinite geometric series is

$$S_n = \frac{a}{1-r} \text{ , as } n \to \infty$$

$$\therefore\ S_n = \frac{n}{1 - \dfrac{1}{(n+1)}} = (n+1)$$

Step2:

Now , $S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \cdots + S_nS_1$

$$= 2(n+1) + 3n + 4(n-1) + \cdots + (n+1)2$$

$$= \sum_{r=1}^{n}(r+1)(n-r+2)$$

$$= \sum_{r=1}^{n}[nr - r^2 + 2r + n - r + 2]$$

$$= \sum_{r=1}^{n}[nr - r^2 + r + n + 2] = \sum_{r=1}^{n}[r(n+1) - r^2 + (n+2)]$$

$$= \sum_{r=1}^{n}[r(n+1)] - \sum_{r=1}^{n}[r^2] + (n+2)\sum_{r=1}^{n}1$$

$$= (n+1)\sum n - \sum n^2 + (n+2)n$$

$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + n(n+2)$$

By simplifying, we get

$$S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \cdots + S_nS_1 = \frac{n}{6}[n^2 + 9n + 14]$$

…………………(1)

Step3:

$$S_1{}^2 + S_2{}^2 + \cdots + S_n{}^2 = 2^2 + 3^2 + 4^2 + 5^2 + \cdots + (n+1)^2$$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + (n+1)^2 - 1$$

$$= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6} - 1$$

By simplifying, we get

$S_1{}^2 + S_2{}^2 + \cdots + S_n{}^2 = \frac{n}{6}[2n^2 + 9n + 13]$ ………….(2)

Step4:

Now $\lim_{n\to\infty} \frac{S_1S_n+S_2S_{n-1}+S_3S_{n-2}+\cdots+S_nS_1}{S_1{}^2+S_2{}^2+\cdots+S_n{}^2}$

$$= \lim_{n\to\infty} \frac{\frac{n}{6}[n^2 + 9n + 14]}{\frac{n}{6}[2n^2 + 9n + 13]}$$

$$= \lim_{n\to\infty} \frac{n^2\left[1 + \frac{9}{n} + \frac{14}{n^2}\right]}{n^2\left[2 + \frac{9}{n} + \frac{13}{n^2}\right]} = \frac{1}{2}$$

P1)

Sum $0.4 + 0.04 + 0.004 + \cdots + \cdots$

Solution:

$$0.4 + 0.04 + 0.004 + \cdots + \cdots$$

It is a G.P with first term 0.4 and common ratio $\frac{1}{10}$.

a = 0.4, $r = \frac{1}{10}$

$$\therefore s = \frac{a}{1-r} = \frac{0.4}{1-\frac{1}{10}} = \frac{4}{9}$$

P2)

Find a rational number which when expressed as a decimal will have $0.1\overline{57}$ as its expansion.

Solution:

$$
\begin{aligned}
.1\overline{57} &= \frac{1}{10} + \frac{57}{1000} + \frac{57}{100000} + \cdots \\
&= \frac{1}{10} + \frac{57}{10^3} + \frac{57}{10^5} + \cdots \\
&= \frac{1}{10} + \frac{57}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) \\
&= \frac{1}{10} + \frac{57}{10^3} \cdot \frac{1}{1-\left(1/10^2\right)} \\
&= \frac{1}{10} + \frac{57}{10^3} \cdot \frac{100}{99} \\
&= \frac{1}{10} + \frac{57}{990} = \frac{99+57}{990} = \frac{156}{990} = \frac{26}{195}
\end{aligned}
$$

P3)

The sum of an infinite geometric series is $3\sqrt{5}$ and the sum of the squares of these terms is 9. Find the series.

Solution:

Let a be the first term and r be the common ratio of the infinite geometric series.

Sum $= 3\sqrt{5} \Rightarrow (a + ar + ar^2 + \cdots \infty) = 3\sqrt{5} \Rightarrow \frac{a}{1-r} = 3\sqrt{5}$ ... (i)

Sum if the squares $=$ 9

$$\Rightarrow \quad a^2 + a^2r^2 + a^2r^4 + \cdots \infty = 9$$

$$\Rightarrow \frac{a^2}{1-r^2} = 9 \quad \ldots (ii)$$

Dividing the square of (i) by (ii), we get

$$\frac{a^2}{(1-r)^2} \times \frac{1-r^2}{a^2} = 5$$

$$\Rightarrow \quad \frac{1+r}{1-r} = 5 \Rightarrow 1 + r = 5 - 5r \Rightarrow \; r = \frac{2}{3}$$

Putting $r = \frac{2}{3}$ in (i), we get

$$\frac{a}{1-\left(\frac{2}{3}\right)} = 3\sqrt{5} \; \Rightarrow a = \sqrt{5}$$

P4)

Let S_n , $n = 1, 2, 3$ ……. be the sum of infinite geometric series whose first term is $(n + 1)$ and the common ratio is $\frac{1}{(n+2)}$

Then $\lim_{n\to\infty} \frac{S_1 S_n + S_2 S_{n-1} + S_3 S_{n-2} + \cdots + S_n S_1}{{S_1}^2 + {S_2}^2 + \cdots + {S_n}^2} =$

Solution:

Given that $a = (n+1)$ and $r = \frac{1}{(n+2)}$

Then the sum of the terms in infinite geometric series is

$$S_n = \frac{a}{1-r}\ , \text{ as } n \to \infty$$

$$\therefore\ S_n = \frac{(n+1)}{1 - \frac{1}{(n+2)}} = (n+2)$$

Now, $S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \cdots + S_nS_1$

$$= 3(n+2) + 4(n+1) + 5n + \cdots + (n+2)3$$

$$= \sum_{r=1}^{n}(r+2)(n-r+3)$$

$$= \sum_{r=1}^{n}[nr - r^2 + 3r + 2n - 2r + 6]$$

$$= \sum_{r=1}^{n}[nr - r^2 + r + 2n + 6]$$

$$= \sum_{r=1}^{n}[r(n+1) - r^2 + 2n + 6]$$

$$\sum_{r=1}^{n}[r(n+1)] - \sum_{r=1}^{n}[r^2] + (2n+6)\sum_{r=1}^{n} 1$$

$$= (n+1)\sum n - \sum n^2 + (2n+6)n$$

$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + (2n+6)n$$

By simplifying, we get

$$S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \cdots + S_nS_1 = \frac{n}{6}[n^2 + 15n + 38]$$

…………………(1)

$$S_1{}^2 + S_2{}^2 + \cdots + S_n{}^2 = 3^2 + 4^2 + 5^2 + \cdots + (n+2)^2$$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + (n+2)^2 - 1^2 - 2^2$$

$$= \frac{(n+2)[(n+2)+1][2(n+2)+1]}{6} - 5$$

By simplifying, we get

$$S_1{}^2 + S_2{}^2 + \cdots + S_n{}^2 = \frac{n}{6}[2n^2 + 15n + 37]\ \text{………….(2)}$$

Now

$$\lim_{n\to\infty} \frac{S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \cdots + S_nS_1}{S_1{}^2 + S_2{}^2 + \cdots + S_n{}^2}$$

$$= \lim_{n\to\infty} \frac{\frac{n}{6}[n^2 + 15n + 38]}{\frac{n}{6}[2n^2 + 15n + 37]}$$

$$= \lim_{n\to\infty} \frac{n^2\left[1 + \frac{15}{n} + \frac{38}{n^2}\right]}{n^2\left[2 + \frac{15}{n} + \frac{37}{n^2}\right]} = \frac{1}{2}$$

1. Find the sum of the following series to infinity:

$$\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \cdots \infty.$$

2. Prove that $(9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots \infty) = 3.$

3. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \cdots$ to ∞ and T_p the sum of the series $1 - r^p + r^{2p} - \cdots$ to ∞, prove that $S_p + T_p = 2.S_{2p}$.

4. The sum of first two terms of an infinite G.P. is 5 and each term is three times the sum of the succeeding terms. Find the G.P.

5. If second term of a G.P. is 2 and the sum of its infinite terms is 8, then its first term is

6. The sum of an infinite G.P. is 4 and the sum of the cubes of its terms is 92. The common ratio of the original G.P. is.

7. If the sum of infinite Geometric series $P, 1, \frac{1}{P}, \frac{1}{P^2}, \ldots$ is $\frac{9}{2}$.

Then the value of P?

8. The first term of a G.P is 2 more than the second term and the sum to infinity is 50. Find the G.P.

9. Find a rational number which when expressed as a decimal will have $1.2\overline{56}$ as its expansion.

3.3

Arithmetic-Geometric Series

Learning objectives:

- To define an Arithmetic-Geometric series (AGS).
- To develop a method to find the sum of n terms of a given AGS.
- To find the sum of the infinite number of the terms of a given AGS.
- To solve related problems.

The series

$$a+(a+d)r+(a+2d)r^2+(a+3d)r^3+\cdots\cdots$$

in which each term is the product of corresponding terms in an arithmetic and geometric series is called Arithmetic-Geometric series.

Let S be the sum of the n terms of the above series. Then

$$S=a+(a+d)r+(a+2d)r^2+\cdots+(a+\overline{n-1}\,d)r^{n-1}$$

$$rS=\quad ar+\quad (a+d)r^2+\cdots+(a+\overline{n-2}\,d)r^{n-1}+(a+\overline{n-1}\,d)r^n$$

By subtraction

$$S(1-r)=a+(dr+dr^2+\cdots+dr^{n-1})-(a+\overline{n-1}\,d)r^n$$

$$=a+\frac{dr(1-r^{n-1})}{1-r}-(a+\overline{n-1}\,d)r^n$$

Therefore

$$S=\frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{(a+\overline{n-1}\,d)r^n}{1-r}$$

We write S in the form

$$\boldsymbol{S}=\frac{\boldsymbol{a}}{1-\boldsymbol{r}}+\frac{\boldsymbol{dr}}{(1-\boldsymbol{r})^2}-\frac{\boldsymbol{dr}^{\boldsymbol{n}}}{(1-\boldsymbol{r})^2}-\frac{(\boldsymbol{a}+\overline{\boldsymbol{n}-1}\,\boldsymbol{d})\boldsymbol{r}^{\boldsymbol{n}}}{1-\boldsymbol{r}}$$

If $r<1$, we can make r^n as small as we please by taking n sufficiently large. In this case, we obtain

$$\boldsymbol{S}_\infty=\frac{\boldsymbol{a}}{1-\boldsymbol{r}}+\frac{\boldsymbol{dr}}{(1-\boldsymbol{r})^2}$$

for the sum to infinity.

Example

If $x < 1$, sum the series $1 + 2x + 3x^2 + 4x^3 + \cdots$ to infinity.

Solution

Let S denote the sum of the series; then

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$xS = x + 2x^2 + 3x^3 + \cdots$$

$$S(1 - x) = 1 + x + x^2 + x^3 + \cdots$$

$$= \frac{1}{1 - x}$$

Therefore

$$S = \frac{1}{(1 - x)^2}$$

IP1:

Find the sum of n terms of an Arithmetic-Geometric series

$\mathbf{1 + 4x + 7x^2 + 10x^3 + 13x^4 + \cdots} \ldots\ldots\ldots.$

Solution:

Step1:

Given Arithmetic-Geometric series is

$$S = 1 + 4x + 7x^2 + 10x^3 + 13x^4 + \ldots\ldots + (3n-2)x^{n-1}$$

$$xS = \quad x + 4x^2 + 7x^3 + 10x^4 + \ldots\ldots + (3n-5)x^{n-1} + (3n-2)x^n$$

Step2:

Now, $(1) - (2)$

$$(1-x)S = 1 + 3x + 3x^2 + 3x^3 + 3x^4 + \ldots\ldots + 3x^{n-1} + (3n-2)x^n$$

$$(1-x)S = 1 + 3(x + x^2 + x^3 + x^4 + \ldots\ldots + x^{n-1}) + (3n-2)x^n$$

$$(1-x)S = 1 + \frac{3x(1-x^{n-1})}{(1-x)} - (3n-2)x^n$$

$$\Rightarrow S = \frac{1}{(1-x)} + \frac{3x(1-x^{n-1})}{(1-x)^2} - \frac{(3n-2)x^n}{(1-x)}$$

$$\Rightarrow S = \frac{1}{(1-x)} + \frac{3(x-x^n)}{(1-x)^2} - \frac{(3n-2)x^n}{(1-x)}$$

$$\Rightarrow S = \frac{(1-x)+3(x-x^n)-(3n-2)x^n(1-x)}{(1-x)^2}$$

$$\Rightarrow S = \frac{1-x+3x-3x^n-\{(3n-2)(x^n-x^{n+1})\}}{(1-x)^2}$$

$$\Rightarrow S = \frac{1+2x-3x^n-\{3nx^n-3nx^{n+1}-2x^n+2x^{n+1}\}}{(1-x)^2}$$

$$\Rightarrow S = \frac{1+2x-x^n(3n+1)+x^{n+1}(3n-2)}{(1-x)^2}$$

which is the sum of n terms of the given series.

IP2:

Find the sum of the n terms of the Arithmetic- Geometric series
$$1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\frac{13}{5^4}+\cdots\ldots\ldots\ldots\ldots$$

Solution:

Step1:

Given Arithmetic-Geometric series is

$$1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\frac{13}{5^4}+\cdots\ldots\ldots\ldots\ldots(1)$$

Step2:

To manipulate the sum of the n terms of the given series,

we have

$$S=\frac{a}{(1-r)}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{(a+(n-1)d)r^n}{(1-r)}$$

Step3:

From (1), $a=1,\ d=4-1=3,\ r=\frac{\frac{1}{5}}{1}=\frac{1}{5}$

$$\therefore S=\frac{a}{(1-r)}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{(a+(n-1)d)r^n}{(1-r)}$$

$$\Rightarrow S=\frac{1}{\left(1-\frac{1}{5}\right)}+\frac{3\left(\frac{1}{5}\right)\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{\left(1-\frac{1}{5}\right)^2}-\frac{(1+(n-1)3)\left(\frac{1}{5}\right)^n}{\left(1-\frac{1}{5}\right)}$$

$$\Rightarrow S=\frac{1}{\left(\frac{4}{5}\right)}+\frac{\left(\frac{3}{5}\right)\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{\left(\frac{4}{5}\right)^2}-\frac{(1+(n-1)3)\left(\frac{1}{5}\right)^n}{\left(\frac{4}{5}\right)}$$

$$\Rightarrow S=\left[\frac{4}{5}+\left(\frac{3}{5}-\frac{3}{5^n}\right)-\frac{4}{5}\left(\frac{1}{5^n}\right)(3n-2)\right]\left(\frac{25}{16}\right)$$

$$\Rightarrow S=\left[\left(\frac{4}{5}+\frac{3}{5}\right)-\frac{3}{5^n}-\left(\frac{4}{5^{n+1}}\right)(3n-2)\right]\left(\frac{25}{16}\right)$$

$$\Rightarrow S=\left[\left(\frac{7}{5}\right)-\frac{3}{5^n}-\left(\frac{12n}{5^{n+1}}-\frac{8}{5^{n+1}}\right)\right]\left(\frac{25}{16}\right)$$

$$\Rightarrow S=\left[\left(\frac{7}{5}\right)\left(\frac{25}{16}\right)-\frac{3}{5^n}\left(\frac{25}{16}\right)-\left(\frac{25}{16}\right)\left(\frac{12n}{5^{n+1}}-\frac{8}{5^{n+1}}\right)\right]$$

$$\Rightarrow S=\left[\left(\frac{35}{16}\right)-\frac{15}{16\,(5^{n-1})}-\left(\frac{12n-8}{16\,(5^{n+1-2})}\right)\right]$$

$$\Rightarrow S=\left[\left(\frac{35}{16}\right)-\frac{15}{16\,(5^{n-1})}-\left(\frac{12n-8}{16\,(5^{n-1})}\right)\right]$$

$$\Rightarrow S=\left[\left(\frac{35}{16}\right)-\frac{(12n+7)}{16(5^{n-1})}\right]$$

which gives the sum of n terms of the series (1)

IP3:

Find the sum of the terms of the Arithmetic- Geometric series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\frac{13}{5^4}+\cdots$ ... ... to infinity.

Solution:

Method:1

Step1:

Given Arithmetic-Geometric series is

$$1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\frac{13}{5^4}+\dots\dots\dots\infty \longrightarrow (1)$$

Step2:

We know that the sum of the series up to infinity is

$$S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

Step3:

From (1), $a=1,\ d=3, r=\frac{1}{5}$

$$\therefore S_\infty=\frac{1}{\left(1-\frac{1}{5}\right)}+\frac{3\left(\frac{1}{5}\right)}{\left(1-\frac{1}{5}\right)^2}$$

$$\Rightarrow S_\infty=\frac{1}{\left(\frac{4}{5}\right)}+\frac{\left(\frac{3}{5}\right)}{\left(\frac{4}{5}\right)^2}$$

$$\Rightarrow\ S_\infty=\left(\frac{25}{16}\right)\left(\frac{4}{5}+\frac{3}{5}\right)=\frac{35}{16}$$

Method:2

Step1:

$$S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\frac{13}{5^4}+\dots\dots \longrightarrow (1)$$

Step2:

$$\frac{S}{5}=\quad \frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+\frac{10}{5^4}+\dots\dots \longrightarrow (2)$$

Step3:

Now, $(1)-(2)$

$$S\left(1-\frac{1}{5}\right)=1+\left(\frac{3}{5}+\frac{3}{5^2}+\frac{3}{5^3}+\frac{3}{5^4}+\cdots\right)$$

$$\Rightarrow S\left(\frac{4}{5}\right)=1+3\left(\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\cdots\right)$$

$$\Rightarrow S\left(\frac{4}{5}\right)=1+3\,\frac{\frac{1}{5}}{\left(1-\frac{1}{5}\right)} \quad \text{(By the geometric series)}$$

$$\Rightarrow S=\frac{35}{16}$$

IP4:

The sum of the series $2^2 + 4^2x + 6^2x^2 + 8^2x^3 + 10^2x^4$..............$\infty$, $|x| < 1$ is

Solution:

Step1:

Given series is

$$S = 2^2 + 4^2x + 6x^2 + 8^2x^3 + 10^2x^4 + \ldots\ldots\ldots\ldots\infty$$

Notice that the above series is not an Arithmetic-Geometric Series.

Here $2^2,\ 4^2,\ 6^2,\ 8^2,\ 10^2$......... are not in A.P

$i, e.,\ 4,\ 16,\ 36,\ 64,\ 100$,........... are not in A.P

But $4,\ 16-4,\ 36-16,\ 64-36,\ 100-64,$ .... are in A.P

$i.e.,\ 4,\ 12,\ 20,\ 28,\ 36$........... are in A.P

Step2:

Now,

$$S = 2^2 + 4^2x + 6x^2 + 8^2x^3 + 10^2x^4 + \ldots\ldots\ldots\ldots\infty$$

$$S = 4 + 16x + 36x^2 + 64x^3 + 100x^4 + \ldots\ldots\ldots\ldots\infty \longrightarrow (1)$$

$$xS = 4x + 16x^2 + 36x^3 + 64x^4 + 100x^5 + \ldots\ldots\ldots\ldots\infty \longrightarrow (2)$$

Step3:

Subtracting (2) from (1), we get

$$(1-x)S = 4 + 12x + 20x^2 + 28x^3 + 36x^4 + \ldots\ldots\ldots\ldots\infty \longrightarrow (3)$$

Now (3) is an Arithmetic-Geometric Series.

$$x(1-x)S = \quad 4x + 12x^2 + 20x^3 + 28x^4 + 36x^5 + \ldots\ldots\infty \longrightarrow (4)$$

Step4:

Subtracting (3) from (4), we get

$$(1-x)(1-x)S = 4 + 8x + 8x^2 + 8x^3 + 8x^4 + \ldots\ldots\ldots\ldots\infty$$

$$\Rightarrow (1-x)^2S = 4 + 8(x + x^2 + x^3 + x^4 + \ldots\ldots\ldots\ldots\infty)$$

$$\Rightarrow (1-x)^2S = 4 + \frac{8x}{1-x}$$

$$\Rightarrow (1-x)^2S = \frac{4+4x}{1-x}$$

$$\Rightarrow S = \frac{4(1+x)}{(1-x)^3}$$

Solution:

Given Arithmetic-Geometric series is

$$S = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \dots\dots\dots + (2n-1)x^{n-1} \longrightarrow (1)$$

$$xS = \quad x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \dots\dots\dots$$

$$+(2n-3)x^{n-1} + (2n-1)x^n \longrightarrow (2)$$

Now, $(1) - (2)$

$$(1-x)S = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \dots\dots + 2x^{n-1} - (2n-1)x^n$$

$$(1-x)S = 1 + 2(x + x^2 + x^3 + x^4 + \dots\dots + x^{n-1}) - (2n-1)x^n$$

$$(1-x)S = 1 + \frac{2x(1-x^{n-1})}{(1-x)} - (2n-1)x^n$$

$$\Rightarrow S = \frac{1}{(1-x)} + \frac{2x(1-x^{n-1})}{(1-x)^2} - \frac{(2n-1)x^n}{(1-x)}$$

$$\Rightarrow S = \frac{1}{(1-x)} + \frac{2(x-x^n)}{(1-x)^2} - \frac{(2n-1)x^n}{(1-x)}$$

$$\Rightarrow S = \frac{(1-x)+2(x-x^n)-(2n-1)x^n(1-x)}{(1-x)^2}$$

$$\Rightarrow S = \frac{1-x+2x-2x^n-\{(2n-1)(x^n-x^{n+1})\}}{(1-x)^2}$$

$$\Rightarrow S = \frac{1+x-2x^n-\{2nx^n-2nx^{n+1}-x^n+x^{n+1}\}}{(1-x)^2}$$

$$\Rightarrow S = \frac{1+x-x^n(2n+1)+x^{n+1}(2n-1)}{(1-x)^2}$$

which is the sum of n terms of the given series.

Solution:

Given Arithmetic-Geometric series is

$$1+\frac{3}{10}+\frac{5}{10^2}+\frac{7}{10^3}+\frac{9}{10^4}+\cdots \dots\dots\dots\dots (1)$$

To manipulate the sum of the n terms of the given series,

we have

$$S=\frac{a}{(1-r)}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{(a+(n-1)d)r^n}{(1-r)}$$

From (1), $a=1,\ d=3-1=2,\ r=\frac{\frac{1}{10}}{1}=\frac{1}{10}$

$$\therefore S=\frac{a}{(1-r)}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{(a+(n-1)d)r^n}{(1-r)}$$

$$\Rightarrow S=\frac{1}{\left(1-\frac{1}{10}\right)}+\frac{2\left(\frac{1}{10}\right)\left(1-\left(\frac{1}{10}\right)^{n-1}\right)}{\left(1-\frac{1}{10}\right)^2}-\frac{(1+(n-1)2)\left(\frac{1}{10}\right)^n}{\left(1-\frac{1}{10}\right)}$$

$$\Rightarrow S=\frac{1}{\left(\frac{9}{10}\right)}+\frac{2\left(\frac{1}{10}\right)\left(1-\frac{1}{10^{n-1}}\right)}{\left(\frac{9}{10}\right)^2}-\frac{(1+(n-1)2)\frac{1}{10^n}}{\left(\frac{9}{10}\right)}$$

$$\Rightarrow S=\left[\left(\frac{9}{10}\right)+\left(\frac{2}{10}-\frac{2}{10^n}\right)-\frac{9}{10}\left(\frac{1}{10^n}\right)(2n-1)\right]\left(\frac{100}{81}\right)$$

$$\Rightarrow S=\left[\left(\frac{9}{10}+\frac{2}{10}\right)-\frac{2}{10^n}-\left(\frac{9}{10^{n+1}}\right)(2n-1)\right]\left(\frac{100}{81}\right)$$

$$\Rightarrow S=\left[\left(\frac{11}{10}\right)-\frac{2}{10^n}-\left(\frac{18n}{10^{n+1}}-\frac{9}{10^{n+1}}\right)\right]\left(\frac{100}{81}\right)$$

$$\Rightarrow S=\left[\left(\frac{11}{10}\right)\left(\frac{100}{81}\right)-\frac{2}{10^n}\left(\frac{100}{81}\right)-\left(\frac{100}{81}\right)\left(\frac{18n}{10^{n+1}}-\frac{9}{10^{n+1}}\right)\right]$$

$$\Rightarrow S=\left[\left(\frac{110}{81}\right)-\frac{20}{81.10^{n-1}}-\left(\frac{18n-9}{81.(10^{n+1-2})}\right)\right]$$

$$\Rightarrow S=\left[\left(\frac{110}{81}\right)-\frac{20}{81\,(10^{n-1})}-\left(\frac{18n-9}{81\,(10^{n-1})}\right)\right]$$

$$\Rightarrow S=\left[\left(\frac{110}{81}\right)-\frac{(18n+11)}{81\,(10^{n-1})}\right]$$

which gives the sum of n terms of the series (1)

Solution:

Method:1

Given Arithmetic-Geometric series is

$$\mathbf{1} + \frac{\mathbf{3}}{\mathbf{10}} + \frac{\mathbf{5}}{\mathbf{10^2}} + \frac{\mathbf{7}}{\mathbf{10^3}} + \frac{\mathbf{9}}{\mathbf{10^4}} + \cdots \ldots \ldots \ldots \infty \longrightarrow (1)$$

We know that the sum of the series up to infinity is

$$\boldsymbol{S_\infty} = \frac{\boldsymbol{a}}{\mathbf{1}-\boldsymbol{r}} + \frac{\boldsymbol{dr}}{(\mathbf{1}-\boldsymbol{r})^2}$$

From (1), $a = 1,\ d = 2, r = \frac{1}{10}$

$$\therefore S_\infty = \frac{1}{\left(1-\frac{1}{10}\right)} + \frac{2\left(\frac{1}{10}\right)}{\left(1-\frac{1}{10}\right)^2}$$

$$\Rightarrow S_\infty = \frac{1}{\left(\frac{9}{10}\right)} + \frac{2\left(\frac{1}{10}\right)}{\left(\frac{9}{10}\right)^2}$$

$$\Rightarrow\ S_\infty = \left(\frac{100}{81}\right)\left(\frac{9}{10} + \left(\frac{2}{10}\right)\right) = \frac{110}{81}$$

Method: 2

$$S = \mathbf{1} + \frac{\mathbf{3}}{\mathbf{10}} + \frac{\mathbf{5}}{\mathbf{10^2}} + \frac{\mathbf{7}}{\mathbf{10^3}} + \frac{\mathbf{9}}{\mathbf{10^4}} + \cdots \longrightarrow (1)$$

$$\frac{S}{10} = \frac{\mathbf{1}}{\mathbf{10}} + \frac{\mathbf{3}}{\mathbf{10^2}} + \frac{\mathbf{5}}{\mathbf{10^3}} + \frac{\mathbf{7}}{\mathbf{10^4}} + \cdots \longrightarrow (2)$$

Now, $(1) - (2)$

$$S\left(1 - \frac{1}{10}\right) = 1 + \left(\frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \frac{2}{10^4} + \cdots\right)$$

$$\Rightarrow S\left(\frac{9}{10}\right) = 1 + 2\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots\right) - 2$$

$$\Rightarrow S\left(\frac{9}{10}\right) = -1 + 2\frac{1}{\left(1-\frac{1}{10}\right)} \quad \text{(By the geometric series)}$$

$$\Rightarrow S = \frac{110}{81}$$

Solution:

Given series is

$$S = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + \ldots\ldots\ldots\ldots\infty,$$

Notice that the above series is not an Arithmetic-Geometric Series.

Here $1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2$……… are not in A.P

$i,e.,1,\ 4,\ 9,\ 16,\ 25,$……….. are not in A.P

But $1,\ 4-1,\ 9-4,\ 16-9,\ 25-16,$ ………… are in A.P

$i.e.,\ 1,\ 3,\ 5,\ 7,\ 9$……….. are in A.P

Now,

$$S = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + \ldots\ldots\ldots\ldots\infty$$

$$S = 1 + 4x + 9x^2 + 16x^3 + 25x^4 + \ldots\ldots\ldots\ldots\infty \longrightarrow (1)$$

$$xS = \quad x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \ldots\ldots\ldots\ldots\infty \longrightarrow (2)$$

Subtracting (2) from (1), we get

$$(1-x)S = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \ldots\ldots\ldots\ldots\infty \longrightarrow (3)$$

Now (3) is an Arithmetic-Geometric Series.

$$x(1-x)S = \quad x + 3x^2 + 5x^3 + 7x^4 + 25x^5 + \ldots..\infty \longrightarrow (4)$$

Subtracting (3) from (4), we get

$$(1-x)(1-x)S = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \ldots\ldots\ldots\ldots\infty$$

$$\Longrightarrow (1-x)^2S = 1 + 2(x + x^2 + x^3 + x^4 + \ldots\ldots\ldots\ldots\infty)$$

$$\Longrightarrow (1-x)^2S = 1 + \frac{2x}{1-x}$$

$$\Longrightarrow (1-x)^2S = \frac{1+x}{1-x}$$

$$\Longrightarrow S = \frac{(1+x)}{(1-x)^3}$$

1. Find the sum of the following series
   I. Up to n terms
   II. Up to infinity

2. $1 + 4x + 7x^2 + 10x^3 + \ldots\ldots\ldots\ldots , |x| < 1$

3. $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots\ldots\ldots\ldots$

4. $2 + 3.3 + 4.3^2 + 5.3^3 + \ldots\ldots\ldots\ldots$

5. $4.7^2 + 7.7^2 + 10.7^3 + 13.7^4 + \ldots\ldots\ldots\ldots$

6. $2 + \frac{4}{5} + \frac{6}{5^2} + \frac{8}{5^3} + \ldots\ldots\ldots$

7. $1 + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots\ldots\ldots\ldots$

$8.\ 2. + 4.\ 2^2 + 6.\ 2^3 + 8.\ 2^4 + \ldots\ldots\ldots\ldots$

9. $3 + 9a^2 + 12a^4 + 15a^6 + \ldots\ldots\ldots \quad |a| < 1$

10. $1 + 3x + 5x^2 + 7x^3 + 9x^4$.........., $|x| < 1$

11. $1+\frac{2}{2}+\frac{3}{2^2}+\frac{5}{3^3}+$.........

12. $1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\ldots, |x| < 1$

13. $1 + 3x + 5x^2 + 7x^3 + \dots\dots\dots\dots, |x| < 1$

14. If the sum of the series $1 + 4x + 7x^2 + 10x^3 + \ldots\ldots\ldots\ldots$ is

a. $\frac{35}{16}$ Then find the value of x.

15. $1 + 2a + 3a^2 + 4a^3 + \ldots\ldots\ldots\ldots \quad |a| < 1$

3.4

Exponential Number

Learning Objectives:

- To define the exponential number (e)
- To study its properties

AND

- To solve problems related to e

We expand the quantity $\left(1+\frac{1}{n}\right)^n$ by the binomial theorem where n is a positive integer.

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{1}{n^3}+\cdots$$

$$=1+1+\frac{1-\frac{1}{n}}{1\cdot 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\cdot 2\cdot 3}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{1\cdot 2\cdot 3\cdot 4}+\cdots$$

The series is true for all values of n.

When n is indefinitely large, we usually denote the left-hand side as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

The sum of the series on the right-hand side is denoted by the quantity e. Hence we have

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

where, $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$

If we put $n=\frac{1}{m}$, we have

$$\lim_{m\to 0}(1+m)^{\frac{1}{m}} = e$$

Properties of *e*

- $2 < e < 3$

We first note that $e > 2$.

Since

$$\frac{1}{3!} < \frac{1}{2\cdot 2} = \frac{1}{2^2}$$
$$\frac{1}{4!} < \frac{1}{2\cdot 2\cdot 2} = \frac{1}{2^3}$$
$$\vdots$$

we have

$$e < 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$
$$< 1+\frac{1}{1-\frac{1}{2}} < 1+2 = 3$$

Hence e lies between 2 and 3.

- By taking sufficient number of terms in the series, it can be shown that

$$e = 2.7182818285 \cdots$$

- It can also be shown that the quantity e is an *irrational number.*
- In the next module, we show that

$$e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots$$

✓ Note:

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}=\sum_{n=2}^{\infty}\frac{1}{(n-2)!}$$
$$e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$$

Example: Prove that $\frac{1}{2}\left(e-\frac{1}{e}\right)=1+\frac{1}{3!}+\frac{1}{5!}+\cdots$

***Solution*:** Using the relations

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots \qquad \text{and}$$

$$e^{-1}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots,$$

we obtain by subtraction

$$e-e^{-1}=2\left(1+\frac{1}{3!}+\frac{1}{5!}+\cdots\right)$$

and then

$$\frac{1}{2}\left(e-\frac{1}{e}\right)=1+\frac{1}{3!}+\frac{1}{5!}+\cdots$$

IP1:

$$\mathbf{1} + \frac{\mathbf{3}}{\mathbf{1}!} + \frac{\mathbf{5}}{\mathbf{2}!} + \frac{\mathbf{7}}{\mathbf{3}!} + \cdots = \_\_\_\_\_$$

Solution:

We have, $1 + \frac{3}{1!} + \frac{5}{2!} + \frac{7}{3!} + \cdots$ .... (1)

Step1:

The n^{th} term of the series (1) is $\frac{2n-1}{(n-1)!}$

Step2:

$$\therefore 1 + \frac{3}{1!} + \frac{5}{2!} + \frac{7}{3!} + \cdots + \frac{2n-1}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(2n-1)}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(2n+1)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= 2\sum_{n=0}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= 2e + e = 3e$$

Step3:

Therefore,

$$1 + \frac{3}{1!} + \frac{5}{2!} + \frac{7}{3!} + \cdots + \frac{2n-1}{(n-1)!} = 3e$$

IP2:

$$\frac{5}{1!}+\frac{11}{2!}+\frac{17}{3!}+\frac{23}{4!}+\cdots=\_\_\_\_\_$$

Solution:

We have, $\frac{5}{1!}+\frac{11}{2!}+\frac{17}{3!}+\frac{23}{4!}+\cdots$ .... (1)

Step1:

The numerators of the series 5, 11, 17, 23 ... are in A.P. with first term 5 and common difference 6.

So n^{th} term $=5+(n-1)6=6n-1$

And denominators, 1, 2, 3, 4, ... are also in A.P, then

n^{th} term $=1+(n-1)1=n$

Step2:

Therefore, n^{th} term of the series (1) is $\frac{6n-1}{n!}$

Step3:

$$\therefore \frac{5}{1!}+\frac{11}{2!}+\frac{17}{3!}+\frac{23}{4!}+\cdots+\frac{6n-1}{n!}=\sum_{n=1}^{n}\frac{6n-1}{n!}$$

$$=6\sum_{n=1}^{n}\frac{n}{n!}-\sum_{n=1}^{n}\frac{1}{n!}$$

$$=6\sum_{n=1}^{n}\frac{1}{(n-1)!}-\sum_{n=1}^{n}\frac{1}{n!}$$

$$=6e-(e-1)$$

$$=5e+1$$

Step4:

Therefore, $\frac{5}{1!}+\frac{11}{2!}+\frac{17}{3!}+\frac{23}{4!}+\cdots+\frac{6n-1}{n!}=5e+1$

IP3:

$$\mathbf{1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots =} \_\_\_\_\_\_$$

Solution:

We have, $\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$ .... (1)

Step1:

The n^{th} term of the series (1) is $\frac{n^2}{n!}$

Step2:

$$\therefore \frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots + \frac{n^2}{n!} = \sum_{n=1}^{n} \frac{n^2}{n!}$$

$$= \sum_{n=1}^{n} \frac{n(n-1) + n}{n!}$$

$$= \sum_{n=1}^{n} \frac{n(n-1)}{n!} + \sum_{n=1}^{n} \frac{n}{n!}$$

$$= \sum_{n=2}^{n} \frac{1}{(n-2)!} + \sum_{n=1}^{n} \frac{1}{(n-1)!}$$

$$= e + e = 2e$$

Step3: Therefore, $\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots + \frac{n^2}{n!} = 2e$

IP4:

$$\mathbf{1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots=}\_\_\_\_\_$$

Solution:

We have, $\frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots$ .... (1)

Step1:

The n^{th} term of the series (1) is $\frac{1+2+3+4+\cdots+n}{n!}$

Step2:

$$\therefore \frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots+\frac{1+2+3+4+\cdots+n}{n!}$$

$$=\sum_{n=1}^{n}\frac{1+2+3+4+\cdots+n}{n!}$$

$$=\sum_{n=1}^{n}\frac{n(n+1)}{2n!}$$

$$=\frac{1}{2}\sum_{n=1}^{n}\frac{n(n+1)}{n(n-1)!}$$

$$=\frac{1}{2}\sum_{n=1}^{n}\frac{(n-1)+2}{(n-1)!}$$

$$=\frac{1}{2}\sum_{n=2}^{n}\frac{1}{(n-2)!}+\sum_{n=1}^{n}\frac{1}{(n-1)!}$$

$$=\frac{1}{2}e+e$$

$$=\frac{3}{2}e$$

Step3:

Therefore,

$$\frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots+\frac{1+2+3+4+\cdots+n}{n!}=\frac{3}{2}e$$

Solution:

We have, $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\cdots$ .... (1)

The n^{th} term of the series (1) is $\frac{2n}{(2n+1)!}$

$\therefore \frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\cdots=\sum_{n=1}^{\infty}\frac{2n}{(2n+1)!}$

$$=\sum_{n=1}^{\infty}\frac{(2n+1)}{(2n+1)!}-\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}$$

$$=\sum_{n=1}^{\infty}\frac{1}{(2n)!}-\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}$$

$$=\sum_{n=0}^{\infty}\frac{1}{(2n)!}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}=\frac{1}{e}$$

Therefore, $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\cdots+\frac{2n}{(2n+1)!}=\frac{1}{e}$

Solution:

We have, $\frac{1}{1!}+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+\cdots$..................... (1)

The numerators of the series 1, 4, 7, 10... are in A.P. with the first term 1 and common difference 3.

So n^{th} term $=1+(n-1)3=3n-2$

And denominators, 1, 2, 3, 4, ... are also in A.P, then

n^{th} term $=1+(n-1)1=n$

Therefore, n^{th} term of the series (1) is $\frac{3n-2}{n!}$

$$\therefore \frac{1}{1!}+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+\cdots\frac{3n-2}{n!}=\sum_{n=1}^{n}\frac{3n-2}{n!}$$

$$=3\sum_{n=1}^{n}\frac{n}{n!}-2\sum_{n=1}^{n}\frac{1}{n!}$$

$$=3\sum_{n=1}^{n}\frac{1}{(n-1)!}-2\sum_{n=1}^{n}\frac{1}{n!}$$

$$=3e-2(e-1)$$

$$=e+2$$

Therefore, $\frac{1}{1!}+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+\cdots\frac{3n-2}{n!}=e+2$

Solution:

We have, $\frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\cdots$ .... (1)

The n^{th} term of the series (1) is $\frac{n^3}{n!}$

$$\therefore \frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\cdots+\frac{n^3}{n!}=\sum_{n=1}^{n}\frac{n^3}{n!}$$

$$=\sum_{n=1}^{n}\frac{n(n^2-1)+n}{n!}$$

$$=\sum_{n=1}^{n}\frac{n(n-1)(n+1)}{n!}+\sum_{n=1}^{n}\frac{n}{n!}$$

$$=\sum_{n=2}^{n}\frac{n+1}{(n-2)!}+\sum_{n=1}^{n}\frac{1}{(n-1)!}$$

$$=\sum_{n=2}^{n}\frac{(n-2)+3}{(n-2)!}+\sum_{n=1}^{n}\frac{1}{(n-1)!}$$

$$=\sum_{n=3}^{n}\frac{1}{(n-3)!}+3\sum_{n=2}^{n}\frac{1}{(n-2)!}+\sum_{n=1}^{n}\frac{1}{(n-1)!}$$

$$=e+3e+e=5e$$

Therefore, $\frac{1^3}{1!}+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\cdots+\frac{n^3}{n!}=5e$

Solution:

We have, $\frac{1}{1!}+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\cdots$ .... (1)

The n^{th} term of the series (1) is $\frac{1^2+2^2+3^2+\cdots+n^2}{n!}$

$$\therefore \frac{1}{1!}+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\cdots+\frac{1^2+2^2+3^2+\cdots+n^2}{n!}$$

$$=\sum_{n=1}^{n}\frac{1^2+2^2+3^2+\cdots+n^2}{n!}$$

$$=\sum_{n=1}^{n}\frac{n(n+1)(2n+1)}{6n!}$$

$$=\frac{1}{6}\sum_{n=1}^{n}\frac{2n^2+3n+1}{(n-1)!}$$

$$=\frac{1}{6}\sum_{n=1}^{n}\frac{n(2n+3)+1}{(n-1)!}$$

$$=\frac{1}{6}\sum_{n=1}^{n}\frac{n(2n-2+5)+1}{(n-1)!}+\frac{1}{6}\sum_{n=1}^{n}\frac{1}{(n-1)!}$$

$$=\frac{2}{6}\sum_{n=1}^{n}\frac{n(n-1)}{(n-1)!}+\frac{5}{6}\sum_{n=1}^{n}\frac{n}{(n-1)!}+\frac{1}{6}\sum_{n=1}^{n}\frac{1}{(n-1)!}$$

$$=\frac{1}{3}\sum_{n=2}^{n}\frac{(n-2)+2}{(n-2)!}+\frac{5}{6}\sum_{n=1}^{n}\frac{(n-1)+1}{(n-1)!}+\frac{1}{6}\sum_{n=1}^{n}\frac{1}{(n-1)!}$$

$$=\frac{1}{3}\sum_{n=3}^{n}\frac{1}{(n-3)!}+\frac{2}{3}\sum_{n=2}^{n}\frac{1}{(n-2)!}+\frac{5}{6}\sum_{n=2}^{n}\frac{1}{(n-2)!}$$

$$+\frac{5}{6}\sum_{n=1}^{n}\frac{1}{(n-1)!}+\frac{1}{6}\sum_{n=1}^{n}\frac{1}{(n-1)!}$$

$$=\frac{1}{3}e+\frac{2}{3}e+\frac{5}{6}e+\frac{5}{6}e+\frac{1}{6}e$$

$$=\frac{17}{6}e$$

Therefore,

$$\frac{1}{1!}+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\cdots+\frac{1^2+2^2+3^2+\cdots+n^2}{n!}=\frac{17}{6}e$$

Show that

1. $\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots\right)=1$

Show that

2. $$\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right)^2=1+\left(1+\frac{1}{3!}+\frac{1}{5!}+\cdots\right)^2$$

Show that

3. $$\frac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{1+\frac{1}{3!}+\frac{1}{5!}+\cdots}=\frac{e-1}{e+1}$$

Show that

4. $\frac{1.2}{1!} + \frac{2.3}{2!} + \frac{3.4}{3!} + \cdots = 3e$

Show that

5. $\frac{4}{1!} + \frac{11}{2!} + \frac{22}{3!} + \frac{37}{4!} + \frac{56}{5!} + \cdots = 6e - 1$

Show that

6. $\frac{1^2.2^2}{1!}+\frac{2^2.3^2}{2!}+\frac{3^2.4^2}{3!}+\cdots=27e$

3.5

Exponential Series

Learning objectives:

1. To derive exponential series
2. To prove exponential theorem
    a. And
3. To solve problems related to the above.

When n is greater than unity and $x \in \mathbf{R}$, we have

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}^x=\left(1+\frac{1}{n}\right)^{nx}=1+nx\cdot\frac{1}{n}+\frac{nx(nx-1)}{1\cdot 2}\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{1\cdot 2\cdot 3}\frac{1}{n^3}+\cdots$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{1\cdot 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1\cdot 2\cdot 3}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\left(x-\frac{3}{n}\right)}{1\cdot 2\cdot 3\cdot 4}+\cdots$$

For an indefinitely large n, (*i.e, as* $n\to\infty$) the left-hand side is e^x and the right hand side becomes

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

Hence we have

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

Let $a>0$ and $a=e^c$, then $c=\log_e a$ and

$$a^x=e^{cx}=1+\frac{cx}{1!}+\frac{c^2x^2}{2!}+\frac{c^3x^3}{3!}+\frac{c^4x^4}{4!}+\cdots$$

Thus for $a>0, x\in R$

$$a^x=1+\frac{x}{1!}\log_e a+\frac{x^2}{2!}(\log_e a)^2+\frac{x^3}{3!}(\log_e a)^3+\frac{x^4}{4!}(\log_e a)^4+\cdots,$$

This is known as ***Exponential Theorem***.

We note the following:

1) $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ ...(1)

Replacing x by −x, we get

2) $e^x=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ ...(2)

Taking x=1 in (1) and (2) we get

3) $e^1=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{1}{n!}$

4) $e^{-1}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$

Adding (1) and (2)

5) $e^x+e^{-x}=2\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)$ and

$$\cosh x=\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$$

Subtracting (2) from (1) we get

6) $e^x-e^{-x}=2\left(\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\right)$ and

$$\sinh x=\frac{e^x-e^{-x}}{2}=\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$$

7) $e^x>0,\ \forall\ x\in R$

8) if $x,y\in R$ and $x>y$ then $e^x>e^y$

9) $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}=\sum_{n=2}^{\infty}\frac{x^{n-2}}{(n-2)!}=\ldots$

IP1)

Show that $1+\frac{1+a}{2!}+\frac{1+a+a^2}{3!}+\frac{1+a+a^2+a^3}{4!}+\cdots\infty=\frac{e^a-e}{a-1}$

Solution:

STEP1:

In the given series

$$T_n=\frac{1+a+a^2+a^3+\cdots+a^{n-1}}{n!}+\cdots\infty=\left(\frac{a^n-1}{a-1}\right).\frac{1}{n!}$$

STEP2:

Sum of the given series

$$=\frac{1}{(a-1)}\left[\frac{a-1}{1!}+\frac{a^2-1}{2!}+\frac{a^3-1}{3!}+\cdots\right]$$

STEP3:

$$=\frac{1}{(a-1)}\left[\left(\frac{a}{1!}+\frac{a^2}{2!}+\frac{a^3}{3!}+\cdots\right)\right]-\left[\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)\right]$$

STEP4:

$$=\frac{1}{(a-1)}\left[(e^a-1)-(e-1)\right]$$

$$=\frac{1}{(a-1)}(e^a-e).$$

STEP5:

$$1+\frac{1+a}{2!}+\frac{1+a+a^2}{3!}+\frac{1+a+a^2+a^3}{4!}+\cdots\infty=\frac{e^a-e}{a-1}$$

IP2)

Find the coefficient of x^n in the series expansion of $\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2$ when n is even

Solution:

STEP1:

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2 = \left(\frac{e^x+e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}(e^{2x}+e^{-2x}+2)$$

$$= \frac{1}{4}\left[\begin{array}{l}\left(1+\frac{(2x)}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots\right)+ \\ \left(1-\frac{(2x)}{1!}+\frac{(2x)^2}{2!}-\frac{(2x)^3}{3!}+\cdots\right)+2\end{array}\right]$$

STEP2:

Coefficient of x^n in the series expansion is

$$= \frac{1}{4}\left[\frac{2^n}{n!}+\frac{(-1)^n.2^n}{n!}\right]$$

STEP3:

When n is even, coefficient of x^n

$$= \frac{1}{4}\left[\frac{2^n}{n!}+\frac{2^n}{n!}\right]$$

$$= \frac{2^{n-1}}{n!}$$

IP3)

Find the coefficient of x^n in the expansion of e^{e^x}

Solution:

STEP1:

Let $e^x = y$

$$e^{e^x} = e^y = \left\{1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots \infty\right\}$$

$$= \left\{1 + \frac{e^n}{1!} + \frac{e^{2n}}{2!} + \frac{e^{3x}}{3!} + \cdots \infty\right\}$$

STEP2:

$$e^{e^x} = 1 + \frac{1}{1!}\left\{1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty\right\}$$

$$+ \frac{1}{2!}\left\{1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots \infty\right\}$$

$$+ \frac{1}{3!}\left\{1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \cdots \infty\right\}$$

STEP3:

Hence, coefficient of x^n in e^{e^x} is

$$= \frac{1}{n!} + \frac{1}{2!} \cdot \frac{2^n}{n!} + \frac{1}{3!} \cdot \frac{3^n}{n!} \ldots \infty$$

$$= \frac{1}{n!}\left[1 + \frac{2^n}{2!} + \frac{3^n}{3!} + \cdots \infty\right]$$

STEP4:

Coefficient of x^n in e^{e^x} is $= \frac{1}{n!}\left[1 + \frac{2^n}{2!} + \frac{3^n}{3!} + \cdots \infty\right]$

IP4)

Find the sum of the series $\frac{3.5}{1!}x + \frac{4.6}{2!}x^2 + \frac{5.7}{3!}x^3 + \cdots$

Solution:

STEP1:

$$\frac{3.5}{1!}x + \frac{4.6}{2!}x^2 + \frac{5.7}{3!}x^3 + \cdots = \sum_{n=1}^{\infty} \frac{(n+2)(n+4)}{n!} x^n$$

STEP2:

$$= \sum_{n=1}^{\infty} \frac{n^2 + 6n + 8}{n!} x^n$$

STEP3:

$$= \sum_{n=1}^{\infty} \left[\frac{n(n+6)}{n!} + \frac{8}{n!}\right] x^n = \sum_{n=1}^{\infty} \left[\frac{n+6}{(n-1)!} + \frac{8}{n!}\right] x^n = \sum_{n=1}^{\infty} \left[\frac{n-1+7}{(n-1)!} + \frac{8}{n!}\right] x^n$$

STEP4:

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(n-2)!} + \frac{7}{(n-1)!} + \frac{8}{n!}\right] x^n$$

STEP5:

$$= \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} + 7\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} + 8\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

STEP6:

$$= x^2 \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} + 7x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} + 8\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

STEP7:

$$= x^2 e^x + 7xe^x + 8(e^x - 1)$$

STEP8:

$$= (x^2 + 7x + 8)e^x - 8$$

P1)

Sum the series $1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \frac{1+3+3^2+3^3}{4!} + \cdots$

Solution:

Let T_n be the n^{th} term of the series and S be the sum of the series.

Then $T_n = \dfrac{1+3+3^2+3^3+\cdots+3^{n-1}}{n!}$

$$= \frac{3^n-1}{3-1} \cdot \frac{1}{n!} = \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

$$\therefore \; S = \sum_{n=1}^{\infty} T_n = \frac{1}{2}\left[\sum_{n=1}^{\infty} \frac{3^n}{n!} - \sum_{n=1}^{\infty} \frac{1}{n!}\right]$$

$$= \frac{1}{2}\left[(e^3 - 1) - (e - 1)\right]$$

$$= \frac{1}{2}\left[(e^3 - e)\right]$$

$$= \frac{1}{2}\, e(e^2 - 1).$$

P2)

Find the coefficient of x^n in the series expansion of

$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2$ when n is odd

Solution:

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2=\left(\frac{e^x+e^{-x}}{2}\right)^2$$

$$=\frac{1}{4}\left(e^{2x}+e^{-2x}+2\right)$$

$$=\frac{1}{4}\left[\left(1+\frac{(2x)}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots\right)+\left(1-\frac{(2x)}{1!}+\frac{(2x)^2}{2!}-\frac{(2x)^3}{3!}+\cdots\right)+2\right]$$

Coefficient of x^n in the series expansion is

$$=\left[\frac{2^n}{n!}+\frac{(-1)^n.2^n}{n!}\right]$$

When n is odd, the coefficient of x^n in the expansion = 0

P3)

Find the coefficient of x^n in the expansion of $\left(\frac{a+bx+cx^2}{e^x}\right)$

Solution:

$$\frac{a+bx+cx^2}{e^x} = (a+bx+cx^2)\,e^{-x}$$

$$(a+bx+cx^2)\left[1+(-x)+\frac{(-x)^2}{2!}+\cdots+\frac{(-x)^{n-2}}{(n-2)!}+\frac{(-x)^{n-1}}{(n-1)!}+\frac{(-x)^n}{n!}+\cdots\right]$$

Hence coefficient of x^n in the expansion $\left(\frac{a+bx+cx^2}{e^x}\right)$ is

$$= \frac{a(-1)^n}{n!}+\frac{b(-1)^{n-1}}{(n-1)!}+\frac{c(-1)^{n-2}}{(n-2)!}$$

$$= \frac{a(-1)^n}{n!}-\frac{b(-1)^n.n}{n.(n-1)!}+\frac{c(-1)^n.n(n-1)}{n(n-1)(n-2)!}$$

$$= \frac{a(-1)^n}{n!}-\frac{b(-1)^n.n}{n!}+\frac{c(-1)^n.n(n-1)}{n!}$$

$$= \frac{(-1)^n}{n!}(a-nb+cn^2-cn)$$

$$= \frac{(-1)^n}{n!}\,[cn^2-(b+c)n+a]$$

P4)

Find the sum of the series $1 + \frac{2^3}{1!}x + \frac{3^3}{2!}x^2 + \cdots$

Solution:

$$1+\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+\cdots=\sum_{n=0}^{\infty}\frac{(n+1)^3}{n!}x^n.$$

$$=\sum_{n=0}^{\infty}\frac{n^3+3n^2+3n+1}{n!}x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{n^2(n+3)}{n!}+\frac{3}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{n(n+3)}{(n-1)!}+\frac{3}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{(n-1)(n+4)+4}{(n-1)!}+\frac{3}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{(n+4)}{(n-2)!}+\frac{7}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{(n-2)+6}{(n-2)!}+\frac{7}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{1}{(n-3)!}+\frac{6}{(n-2)!}+\frac{7}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=3}^{\infty}\frac{x^n}{(n-3)!}$$

$$+6\sum_{n=2}^{\infty}\frac{x^n}{(n-2)!}+7\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$=x^3\sum_{n=3}^{\infty}\frac{x^{n-3}}{(n-3)!}$$

$$+6x^2\sum_{n=2}^{\infty}\frac{x^{n-2}}{(n-2)!}+7x\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$=(x^3+6x^2+7x+1)e^x$$

1. Find the coefficient of x^r in the expansion of $\frac{a-bx}{e^x}$.

2. Find the sum of the series $$1+\frac{2x}{1!}+\frac{3x^2}{2!}+\frac{4x^3}{3!}+\cdots$$

3. Show that $$\frac{1+\frac{2^2}{2!}+\frac{2^4}{3!}+\frac{2^6}{4!}+\cdots}{1+\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+\cdots}=e^2-1$$

4. The sum of the series $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\cdots$ is equal to

(a) $2e^{-2}$ (b) e^{-2} (c) e^{-1} (d) $2e^{-1}$

5. If $a = \sum_{n=1}^{\infty} \frac{2n}{(2n-1)!}$, $b = \sum_{n=1}^{\infty} \frac{2n}{(2n+1)!}$, then ab equals

(a) 1 (b) e^2 (c) $\frac{e-1}{e+1}$ (d) $\frac{e+1}{e-1}$

6. If $\frac{e^{5x}+e^{x}}{e^{3x}}$ is expanded in a series of ascending powers of x and n is an odd natural number, then the coefficient of x^{n} is

7. The product of the following series

$$\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)\times\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots\right) \text{ is}$$

(a) 1 (b) e^{-2} (c) $-e^{2}$ (d) -1

8. If $\frac{e^x}{1-x} = B_0 + B_1 x + B_2 x^2 + \cdots + B_n x^n + \cdots$, find the value of $B_n - B_{n-1} =$

9. By comparing two different expansions of $\frac{1}{2}(e^{x} - e^{-x})$. prove that

$$n - \frac{n(n-2)^n}{1!} + \frac{n(n-1)(n-4)^n}{2!} - \cdots = 2^n . n^n.$$

10. $1.2 + \frac{2.3}{1!}x + \frac{3.4}{2!}x^2 + \cdots = (x^2 + 4x + 2)e^x$

11. Show that $\frac{1}{1!}+\frac{1+5}{2!}+\frac{1+5+5^2}{3!}+\frac{1+5+5^2+5^3}{4!}+\cdots=\frac{e^5-e}{4}$

12. Find the coefficient of x^n in the series expansion of

$\frac{1+2x+3x^2}{e^x}$ in powers of x.

13. Find the coefficient of x^n in the series expansion of $(3 + 2x)e^{3x}$ in powers of x.

14. Show that the coefficient of x^n in the series expansion of

$\dfrac{2+3x}{e^{2x}}$ is $\dfrac{(-1)^n.2^{n-1}.(4-3n)}{n!}$.

3.6
Logarithmic Series

Learning objectives:

1. To derive the logarithmic series from exponential series.
2. To express the given series in terms of logarithmic functions.
   And
   Solve the problems related to the above concepts.

Logarithmic Series

From the exponential series, we have for $a > 0$,

$$a^y = 1 + \frac{y}{1!}\log_e a + \frac{y^2}{2!}(\log_e a)^2 + \frac{y^3}{3!}(\log_e a)^3 + \frac{y^4}{4!}(\log_e a)^4 + \cdots$$

In this series, we write $1+x$ for a; thus

$$(1+x)^y = 1 + \frac{y}{1!}\log_e(1+x) + \frac{y^2}{2!}(\log_e(1+x))^2 + \frac{y^3}{3!}(\log_e(1+x))^3 + \frac{y^4}{4!}(\log_e(1+x))^4 + \ldots\ldots\ldots(1)$$

When x is numerically less than unity, we have by the binomial theorem

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{2!}x^2 + \frac{y(y-1)(y-2)}{3!}x^3 + \ldots\ldots \longrightarrow (2)$$

Equating the power of y in (1) and (2), we obtain,

$$\boxed{\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots,\quad |x|<1} \longrightarrow (3)$$

This is known as ***Logarithmic Series***.

Replacing x by $-x$ in (3), we get

$$\boxed{\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots,\quad |x| < 1} \longrightarrow (4)$$

Note:

The logarithmic series

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots,$$

exists for $x \in \mathbb{R}$ such that $-1 < x \le 1$

Example1:

Show that $\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$ *if* $|x|<1$

We have the Logarithmic Series

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots \quad \textit{and}$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots\ldots$$

By subtraction

$$\log_e(1+x) - \log_e(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

Therefore,

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

Example2:

If $x > 1$ then $\log_e\left(\frac{x+1}{x-1}\right) = 2\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots\right)$

We have from example (1)

$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$ valid when $|x| < 1$

Suppose that $x > 1$ .Let $y = \frac{1}{x}$ then $y < 1$

$\therefore \log_e\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right)$

$\Rightarrow \log_e\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right) = 2\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots\right)$

$\Rightarrow \log_e\left(\frac{x+1}{x-1}\right) = 2\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots\right)$, where $x > 1$

Note:

$$\log_e 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \ldots\ldots = 1 - \frac{1}{2.3} - \frac{1}{4.5} - \frac{1}{6.7} - \frac{1}{8.9} - \ldots\ldots$$

We have

$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots$ valid in $-1 < x \le 1$

Put $x = 1$,

$$\boxed{\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ldots\ldots}$$

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ldots\ldots\ldots\ldots$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots\ldots\ldots\ldots$$

$$\boxed{\log_e 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \ldots\ldots\ldots\ldots}$$

Also, $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \ldots\ldots\ldots\ldots$

$$= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \ldots\ldots\ldots\ldots$$

$$\boxed{\log_e 2 = 1 - \frac{1}{2.3} - \frac{1}{4.5} - \frac{1}{6.7} - \ldots\ldots\ldots\ldots}$$

IP1.

Show that $\log_e 2 - \frac{1}{2} = \frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \cdots \infty$

Solution:

Step1:

$$L.H.S = \log_e 2 - \frac{1}{2} = \frac{2\log_e 2 - 1}{2}$$

Step 2:

We have,

$$\log_e 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots \longrightarrow (1)$$

$$\log_e 2 = 1 - \frac{1}{2.3} - \frac{1}{4.5} - \frac{1}{6.7} - \cdots \longrightarrow (2)$$

Step 3:

$(2) + (3)$

$$\log_e 2 + \log_e 2 = \left(\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots\right) + \left(1 - \frac{1}{2.3} - \frac{1}{4.5} - \frac{1}{6.7} - \cdots\right)$$

$$\Rightarrow 2\log_e 2 = 1 + \left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{3.4} - \frac{1}{4.5}\right) + \left(\frac{1}{5.6} - \frac{1}{6.7}\right) + \cdots$$

$$\Rightarrow 2\log_e 2 - 1 = \frac{2}{1.2.3} + \frac{2}{3.4.5} + \frac{2}{5.6.7} + \cdots$$

$$\Rightarrow \log_e 2 - \frac{1}{2} = \frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \cdots \infty$$

I.P2:

Show that for $|x| < 1$,

$$\frac{1}{1.2} + \frac{x}{3.4} + \frac{x^2}{5.6} + \frac{x^3}{7.8} + \cdots = \frac{1}{2\sqrt{x}} log \left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) + \frac{1}{2x}. \log(1-x),$$

Solution:

Step1:

Given series is $\frac{1}{1.2} + \frac{x}{3.4} + \frac{x^2}{5.6} + \frac{x^3}{7.8} + \cdots \longrightarrow (1)$

Step2:

The n^{th} term of (1) is $T_n = \frac{x^{n-1}}{(2n-1)(2n)}$, $n \in N$

Step3:

$$\frac{1}{1.2} + \frac{x}{3.4} + \frac{x^2}{5.6} + \frac{x^3}{7.8} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(2n-1)(2n)}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)} - \frac{1}{(2n)}\right) x^{n-1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)}\right) x^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{(2n)}\right) x^{n-1}$$

$$= \left(1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \cdots\right) - \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{3} + \cdots\right)$$

$$= \frac{1}{\sqrt{x}}\left[\sqrt{x} + \frac{x\sqrt{x}}{3} + \frac{x^2\sqrt{x}}{5} + \frac{x^3\sqrt{x}}{7} + \cdots\right] - \frac{1}{2x}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \ldots\right)$$

$$= \frac{1}{\sqrt{x}}\left[\sqrt{x} + \frac{(\sqrt{x})^3}{3} + \frac{(\sqrt{x})^5}{5} + \frac{(\sqrt{x})^7}{7} + \cdots\right] - \frac{1}{2x}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \ldots\right)$$

$$= \frac{1}{\sqrt{x}} . \frac{1}{2}. \log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) + \frac{1}{2x}. \log(1-x) \quad (\because |x| < 1)$$

$$= \frac{1}{2\sqrt{x}} log \left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) + \frac{1}{2x}. \log(1-x)$$

Step4:

Hence

$$\frac{1}{1.2} + \frac{x}{3.4} + \frac{x^2}{5.6} + \frac{x^3}{7.8} + \cdots = \frac{1}{2\sqrt{x}} log \left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) + \frac{1}{2x}. \log(1-x)$$

IP3.

If α, β are roots of the equation $ax^2 + bx + c = 0$ then prove that

$$\log_e(ax^2 + bx + c) = \log_e a + 2\log_e x - \frac{1}{x}(\alpha+\beta) - \frac{1}{2x^2}(\alpha^2+\beta^2) - \frac{1}{3x^2}(\alpha^3+\beta^3) - \cdots$$

Solution:

Step 1:

By the Hypothesis,

α, β are roots of the equation $ax^2 + bx + c = 0$

Sum of the roots $= \alpha + \beta = \frac{-b}{a}$

Product of the roots $= \alpha\beta = \frac{c}{a}$

Step 2:

$$\log_e a + 2\log_e x - \frac{1}{x}(\alpha+\beta) - \frac{1}{2x^2}(\alpha^2+\beta^2) - \frac{1}{3x^2}(\alpha^3+\beta^3) - \cdots$$

$$= \log_e a + \log_e x^2 - \left(\frac{\alpha}{x} + \frac{\alpha^2}{2x^2} + \frac{\alpha^3}{3x^3} + \cdots\right) - \left(\frac{\beta}{x} + \frac{\beta^2}{2x^2} + \frac{\beta^3}{3x^3} + \cdots\right)$$

$$= log_e^{a} + log_e^{x^2} + log_e\left(1 - \frac{\alpha}{x}\right) + log_e\left(1 - \frac{\beta}{x}\right)$$

$$= log_e\left\{a.x^2.\left(1 - \frac{\alpha}{x}\right).\left(1 - \frac{\beta}{x}\right)\right\}$$

$$= log_e\{a.\ (x-\alpha).(x-\beta)\}$$

$$= log\ _e\{ax^2 + bx + c\}$$

I.P4

If $x^2 y = 2x - y$ and $|x| < 1$ then

show that $\boldsymbol{y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots \ldots \ldots . \infty = \log_e\left(\frac{1+x}{1-x}\right)}$

Solution:

Step1:

Given that $x^2 y = 2x - y \Rightarrow y = \frac{2x}{1+x^2}$

Step2:

$$y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots \ldots \ldots . \infty$$

$$= \frac{1}{2}\left[2\left((y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots \ldots \ldots . \infty\right)\right]$$

$$= \frac{1}{2}\log_e\left(\frac{1+y}{1-y}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{x^2+1+2x}{x^2+1-2x}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)^2$$

$$= \frac{1}{2}.2\log_e\left(\frac{1+x}{1-x}\right) = \log_e\left(\frac{1+x}{1-x}\right)$$

Step3:

Hence $y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots \ldots \ldots . \infty = \log_e\left(\frac{1+x}{1-x}\right)$, $|x| < 1$

P1:

$$\frac{5}{2} - \log_e 8 =$$

A. $\frac{1}{1.2} + \frac{3}{2.3} + \frac{5}{3.4} + \frac{7}{4.5} + \cdots \infty$

B. $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + \cdots \infty$

C. $\frac{1}{1.2.3} + \frac{5}{3.4.5} + \frac{9}{5.6.7} + \cdots \infty$

D. $\frac{1}{1.2.3} + \frac{3}{3.4.5} + \frac{5}{5.6.7} + \cdots \infty$

Solution:

$$\frac{5}{2} - \log_e 8 = \frac{5}{2} - \log_e 2^3 = \frac{5}{2} - 3\log_e 2 = \frac{1}{2}(5 - 6\log_e 2)$$

We have $\log_e 2 = \frac{1}{1\,.\,2} + \frac{1}{3\,.\,4} + \frac{1}{5\,.\,6} + \cdots \longrightarrow (1)$

$$\log_e 2 = 1 - \frac{1}{2\,.\,3} - \frac{1}{4\,.\,5} - \frac{1}{6\,.\,7} - \cdots \longrightarrow (2)$$

$$(2)\times 5 \Rightarrow 5\log_e 2 = 5\left[1 - \frac{1}{2\,.\,3} - \frac{1}{4\,.\,5} - \frac{1}{6\,.\,7} - \cdots\right]$$

$$= 5 - \frac{5}{2\,.\,3} - \frac{5}{4\,.\,5} - \frac{5}{6\,.\,7} - \cdots \longrightarrow (3)$$

Now $(1) + (3)$

$$\log_e 2 + 5\log_e 2 = \left(\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots\right) + \left(5 - \frac{5}{2.3} - \frac{5}{4.5} - \frac{5}{6.7} - \cdots\right)$$

$$\Rightarrow 6\log_e 2 = 5 - \left(\frac{5}{2.3} - \frac{1}{1.2}\right) - \left(\frac{5}{4.5} - \frac{1}{3.4}\right) - \left(\frac{5}{6.7} - \frac{1}{5.6}\right) - \cdots$$

$$\Rightarrow 6\log_e 2 = 5 - \frac{2}{1\,.\,2\,.\,3} - \frac{10}{3\,.\,4\,.\,5} - \frac{18}{5\,.\,6\,.\,7} - \cdots$$

$$\Rightarrow -\frac{6\log_e 2}{2} = -\frac{1}{2}\left[5 - \frac{2}{1\,.\,2\,.\,3} - \frac{10}{3\,.\,4\,.\,5} - \frac{18}{5\,.\,6\,.\,7} - \cdots\right]$$

$$\Rightarrow -3\log_e 2 = -\frac{5}{2} + \frac{1}{1\,.\,2\,.\,3} + \frac{5}{3\,.\,4\,.\,5} + \frac{9}{5\,.\,6\,.\,7} + \cdots$$

$$\Rightarrow \frac{5}{2} - \log_e 2^3 = \frac{1}{1\,.\,2\,.\,3} + \frac{5}{3\,.\,4\,.\,5} + \frac{9}{5\,.\,6\,.\,7} + \cdots$$

$$\therefore \frac{5}{2} - \log_e 8 = \frac{1}{1\,.\,2\,.\,3} + \frac{5}{3\,.\,4\,.\,5} + \frac{9}{5\,.\,6\,.\,7} + \cdots$$

P2.

For $|x| < 1$, $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \cdots =$

A. $\frac{x}{x-1} + \log(x-1)$

B. $\frac{x-1}{x} - \log(x-1)$

C. $\frac{x}{1-x} + \log(1-x)$

D. $\frac{x}{x-1} + \log(1-x)$

Solution:

Given series is

$$\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \cdots \longrightarrow (1)$$

The n^{th} term of (1) is $T_n = \frac{n}{n+1} \cdot x^{n+1}$, $n \in N$

$$\therefore \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \cdots$$

$$= \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right) x^{n+1}$$

$$= \sum_{n=1}^{\infty} \left(1 - \frac{1}{n+1}\right) x^{n+1}$$

$$= \sum_{n=1}^{\infty} (x^{n+1}) - \sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right) x^{n+1}$$

$$= (x^2 + x^3 + x^4 + \cdots) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

$$= (x^2 + x^3 + x^4 + \cdots) + x - \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

$$= \left(\frac{x^2}{1-x}\right) + x + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) \qquad (\because |x| < 1)$$

$$= \frac{x^2 + x - x^2}{1-x} + \log(1-x) \qquad (\because |x| < 1)$$

$$= \frac{x}{1-x} + \log(1-x)$$

$$\therefore \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \cdots = \frac{x}{1-x} + \log(1-x), |x| < 1$$

P3:

If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are roots of the equation $\boldsymbol{x^2 - px + q = 0}$ then

$$(\alpha+\beta)x - \left(\frac{\alpha^2+\beta^2}{2}\right)x^2 + \left(\frac{\alpha^3+\beta^3}{3}\right)x^3 - \ldots = \_\_\_\_\_$$

A. $log\,(qx^2 + px + 1)$
B. $log\,(px^2 + qx + 1)$
C. $log\,\,(1 - px + qx^2)$
D. $log\,(qx^2 - px - 1)$

Answer: $\boldsymbol{A}$

Solution:

Given that α, β are roots of $x^2 - px + q = 0$

Sum of the roots $= \alpha + \beta = p$

Product of the roots $= \alpha\beta = q$

By Hypothesis,

$$(\alpha+\beta)x - \left(\frac{\alpha^2+\beta^2}{2}\right)x^2 + \left(\frac{\alpha^3+\beta^3}{3}\right)x^3 - \ldots\ldots$$

$$= \left(\alpha x - \frac{(\alpha x)^2}{2} + \frac{(\alpha x)^3}{3} - \cdots\right) + \left(\beta x - \frac{(\beta x)^2}{2} + \frac{(\beta x)^3}{3} - \cdots\right)$$

$$= log_e\,(1 + \alpha x) + log\,(1 + \beta x)$$

$$= log\,[(1 + \alpha x)\,.\,\,(1 + \beta x)]$$

$$= log_e\,[1 + (\alpha + \beta)x + \alpha\beta x^2\,]$$

$$= log_e\,(1 + px + qx^2)$$

$$\therefore (\alpha+\beta)x - \left(\frac{\alpha^2+\beta^2}{2}\right)x^2 + \left(\frac{\alpha^3+\beta^3}{3}\right)x^3 - \ldots\ldots = log_e\,(1 + px + qx^2)$$

P4:

If $x = \frac{\sqrt{2}-1}{\sqrt{2}}$ then $1 + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \frac{1}{20}x^4 + \ldots\ldots\ldots\ldots\infty =$

A. $2 + \frac{1}{2}(\sqrt{2}+1)\log_e 2$

B. $2 + \frac{1}{2}(\sqrt{2}-1)\log_e 2$

C. $2 - \frac{1}{2}(\sqrt{2}+1)\log_e 2$

D. $2 - \frac{1}{2}(\sqrt{2}-1)\log_e 2$

Solution:

Given that $x = \frac{\sqrt{2}-1}{\sqrt{2}}$...............(1)

$$1 + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \frac{1}{20}x^4 + \ldots\ldots\ldots\infty$$

$$= 1 + \left(1 - \frac{1}{2}\right)x + \left(\frac{1}{2} - \frac{1}{3}\right)x^2 + \left(\frac{1}{3} - \frac{1}{4}\right)x^3 + \left(\frac{1}{4} - \frac{1}{5}\right)x^4 + \ldots\ldots\infty$$

$$= \left(1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} \ldots\ldots\ldots\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \ldots\ldots\ldots\right)$$

$$= \frac{1}{x}\left(x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots\ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots\ldots\right)$$

$$= \frac{1}{x}\left(2x - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots\ldots\right) - \log_e(1 - x)$$

$$= \frac{1}{x}(2x + \log_e(1 - x)) - \log_e(1 - x)$$

$$= 2 + \left(\frac{1}{x} - 1\right)\log_e(1 - x)$$

$$= 2 + \left(\frac{1}{\frac{\sqrt{2}-1}{\sqrt{2}}} - 1\right)\log_e\left(1 - \frac{\sqrt{2}-1}{\sqrt{2}}\right) \qquad \text{(From (1))}$$

$$= 2 + \frac{1}{\sqrt{2}-1}\log_e\left(2^{-\frac{1}{2}}\right)$$

$$= 2 + (\sqrt{2} + 1)\left(-\frac{1}{2}\right)\log_e 2$$

$$= 2 - \frac{1}{2}(\sqrt{2} + 1)\log_e 2$$

Hence

$$1 + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \ldots\ldots\ldots\infty = 2 - \frac{1}{2}(\sqrt{2} + 1)\log_e 2$$