4.1

Vectors in Plane

Learning Objectives:

- To define a vector in a plane, scalar multiples of a vector and geometric addition of two vectors in a plane.

AND

- To practice related problems.

Some of the things we measure are determined by their magnitudes. To record mass, length, or time, for example, we need only to write down a number and name an appropriate unit of measure. But we need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moves as well as how far. To describe a body's velocity, we have to know where the body is headed as well as how fast it is going.

Quantities that have direction as well as magnitude are usually represented by arrows that point in the direction of the action and whose lengths give the magnitude of the action in terms of a suitably chosen unit.

When we discuss these arrows abstractly, we think of them as directed line segments and we call them *vectors*.

Definitions

A **vector** in the plane is a directed line segment. Two vectors are **equal** or **the same** if they have the same length and direction.

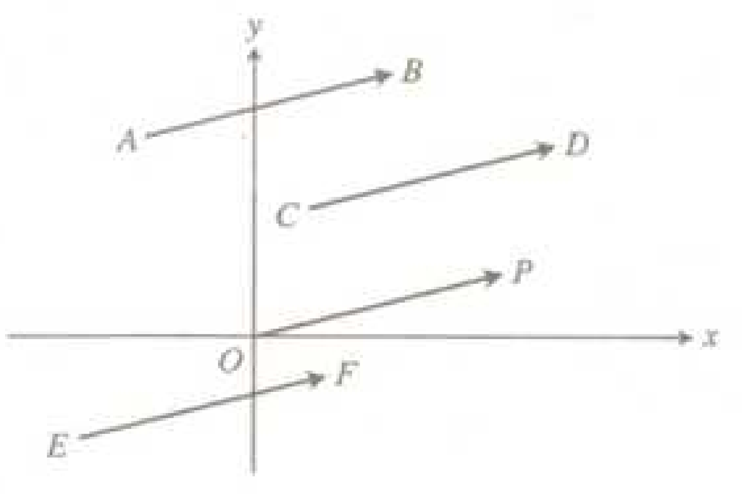
Thus, the arrows we use when we draw vectors are understood to represent the same vector if they have the same length, are parallel, and point in the same direction.

In print, vectors are usually described with single boldface roman letters, as $\mathbf{v}$. The vector defined by the directed line segment from point A to point B is written as $\overrightarrow{AB}$.

Example 1

The four arrows in figure below have the same length and direction. They therefore represent the same vector, and we write

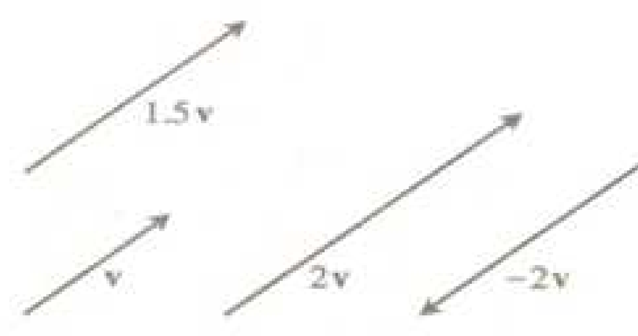
$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$$



Arrows with the same length and direction represent the same vector.

Scalars and Scalar Multiples

We multiply a vector by a positive real number by multiplying its length by the number.



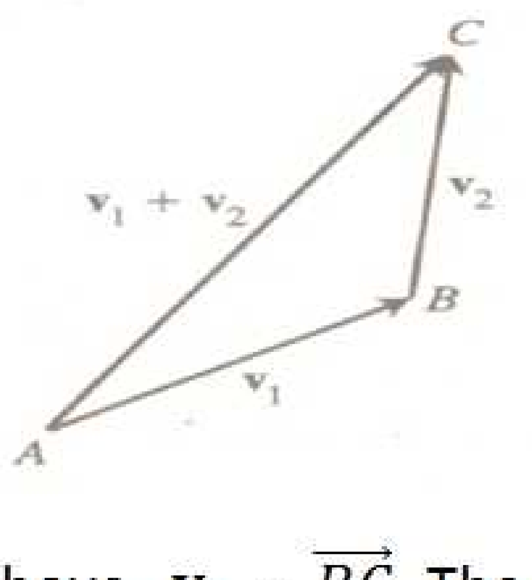
To multiply a vector by 2, we double its length. To multiply a vector by 1.5, we increase its length by 50%, and so on. We multiply a vector by a negative number by reversing the vector's direction and multiplying the length by the number's absolute value.

If c is a nonzero real number and $\mathbf{v}$ is a vector, the direction of $c\mathbf{v}$ agrees with that of $\mathbf{v}$ if c is positive and is opposite to that of $\mathbf{v}$ if c is negative. Since real numbers work like scaling factors in this context, we call them ***scalars*** and call multiples like $c\mathbf{v}$ ***scalar multiples*** of $\mathbf{v}$.

To include multiplication by zero, we adopt the convention that multiplying a vector by zero produces the ***zero vector*** $\mathbf{0}$, consisting of points that are degenerate line segments of zero length. Unlike other vectors, the vector $\mathbf{0}$ has no direction.

Geometric Addition: The Parallelogram Law

Two nonzero vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ can be added geometrically by drawing a representative of $\mathbf{v_1}$, say from A to B as in figure below, and then a representative of $\mathbf{v_2}$ starting from the terminal point B of $\mathbf{v_1}$.



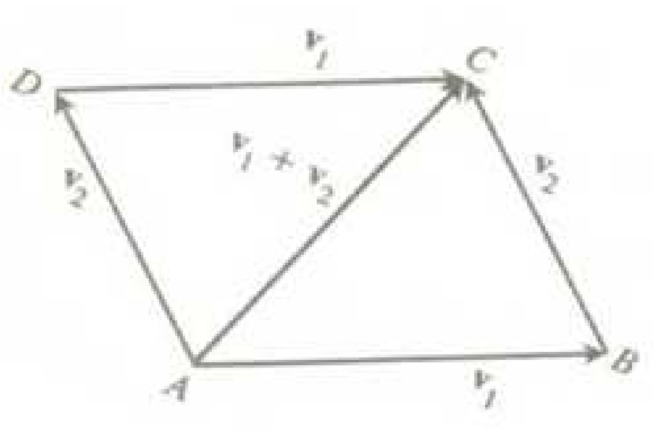
In the figure above, $\mathbf{v_2} = \overrightarrow{BC}$. The sum $\mathbf{v_1} + \mathbf{v_2}$ is then the vector represented by the arrow from the initial point A of $\mathbf{v_1}$ to the terminal point C of $\mathbf{v_2}$. That is, if

$$\mathbf{v_1} = \overrightarrow{AB} \quad \text{and} \quad \mathbf{v_2} = \overrightarrow{BC}$$

Then

$$\mathbf{v_1} + \mathbf{v_2} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

This description of addition is called the ***Parallelogram Law*** of addition because $\mathbf{v_1} + \mathbf{v_2}$ is given by the diagonal of the parallelogram determined by $\mathbf{v_1}$ and $\mathbf{v_2}$.



IP1:

If $\boldsymbol{a}$ is a vector as shown in figure given below. Then find the following scalar multiples of the vector $\boldsymbol{a}$.

a) $2\boldsymbol{a}$

b) $-\boldsymbol{a}$

c) $\frac{1}{2}\boldsymbol{a}$

Solution:

a. **Finding the vector $2\boldsymbol{a}$:**

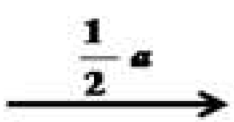
If we double the magnitude (length) of $\boldsymbol{a}$ in the same direction, then we get the vector $2\boldsymbol{a}$.

b. **Finding the vector $-\boldsymbol{a}$:**

If we take vector $\boldsymbol{a}$ in the opposite direction, then we get the vector $-\boldsymbol{a}$.

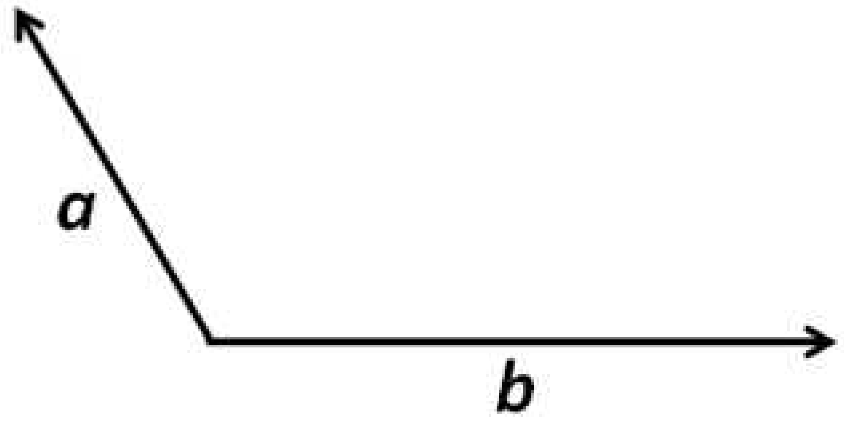
c. **Finding the vector $\frac{1}{2}\boldsymbol{a}$:**

If we reduce the magnitude of the vector $\boldsymbol{a}$ by 50%, then we get the vector $\frac{1}{2}\boldsymbol{a}$ in the same direction.



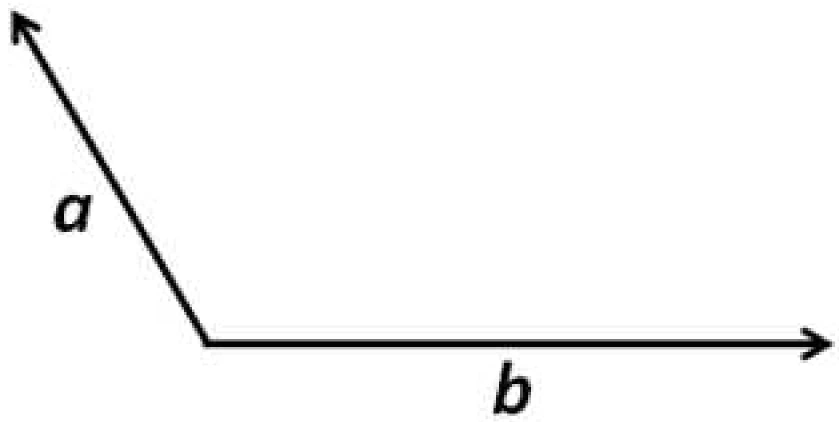
IP2:

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors in the figure given below, then find $\boldsymbol{a} + \boldsymbol{b}$.

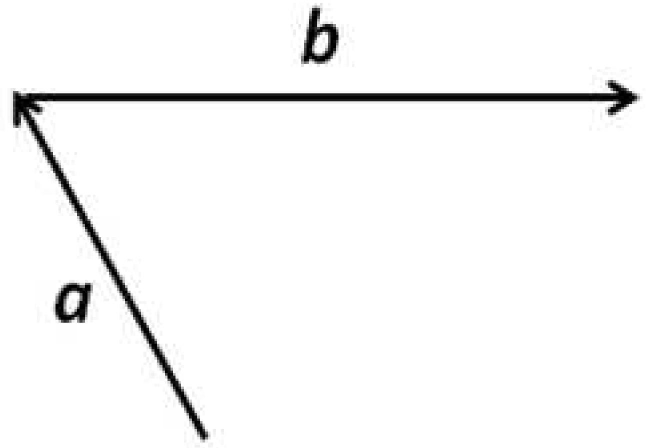


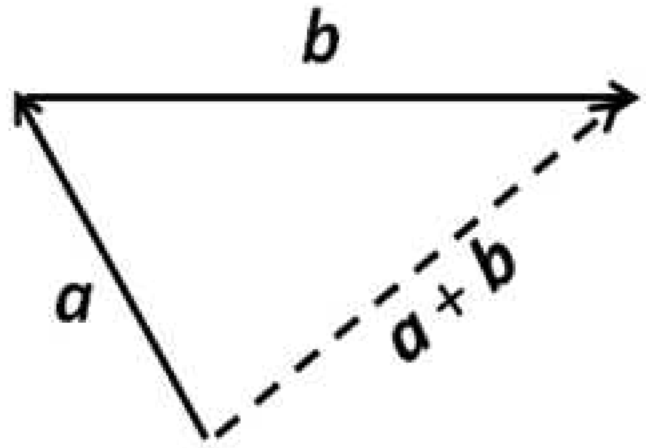
Solution:

Given:



Finding $\boldsymbol{a} + \boldsymbol{b}$:

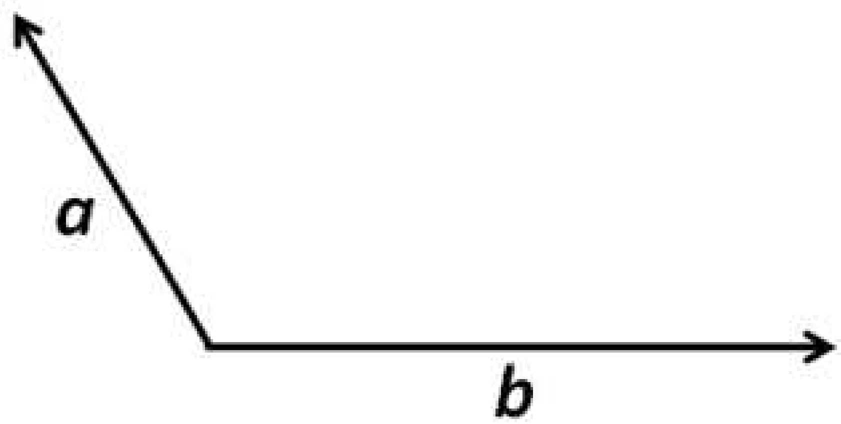




The sum $\boldsymbol{a} + \boldsymbol{b}$ is the vector represented by arrow from the initial point of $\boldsymbol{a}$ to the terminal point of the $\boldsymbol{b}$.

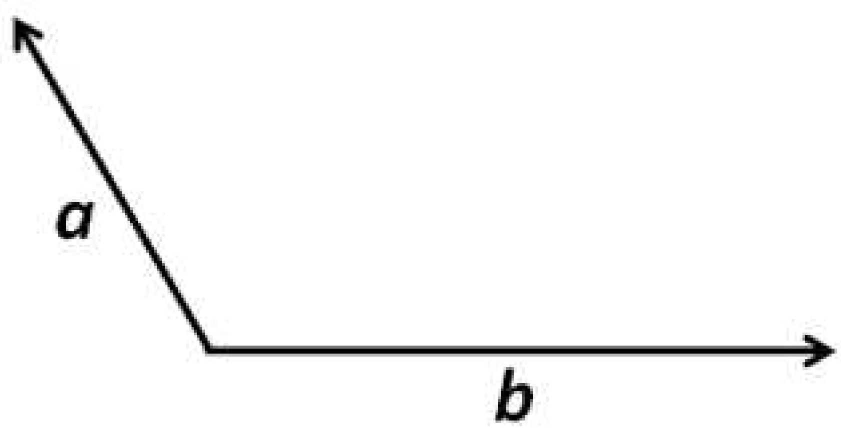
IP3:

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors in the figure given below, then find $\boldsymbol{b}-\boldsymbol{a}$.

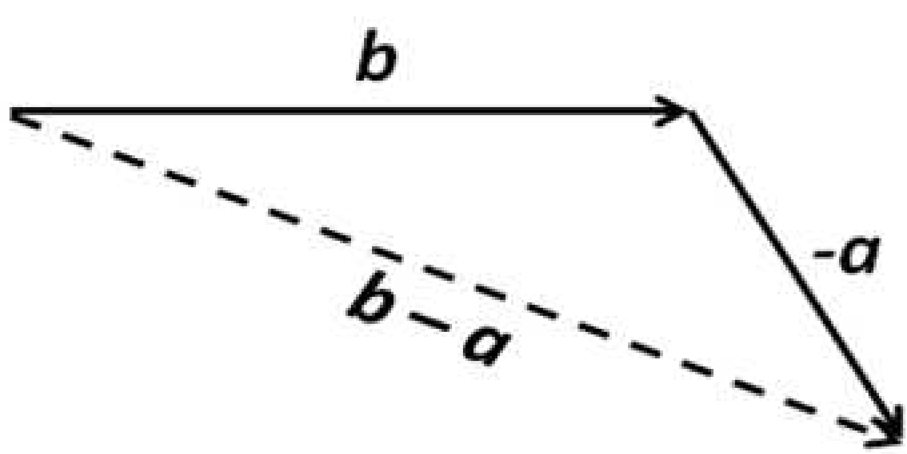


Solution:

Given:

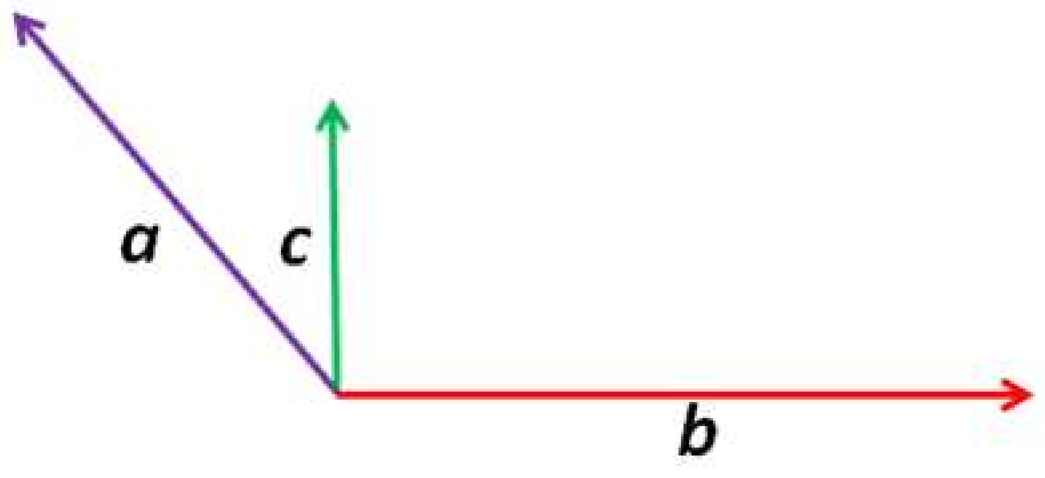


Finding $\boldsymbol{a}-\boldsymbol{b}$:



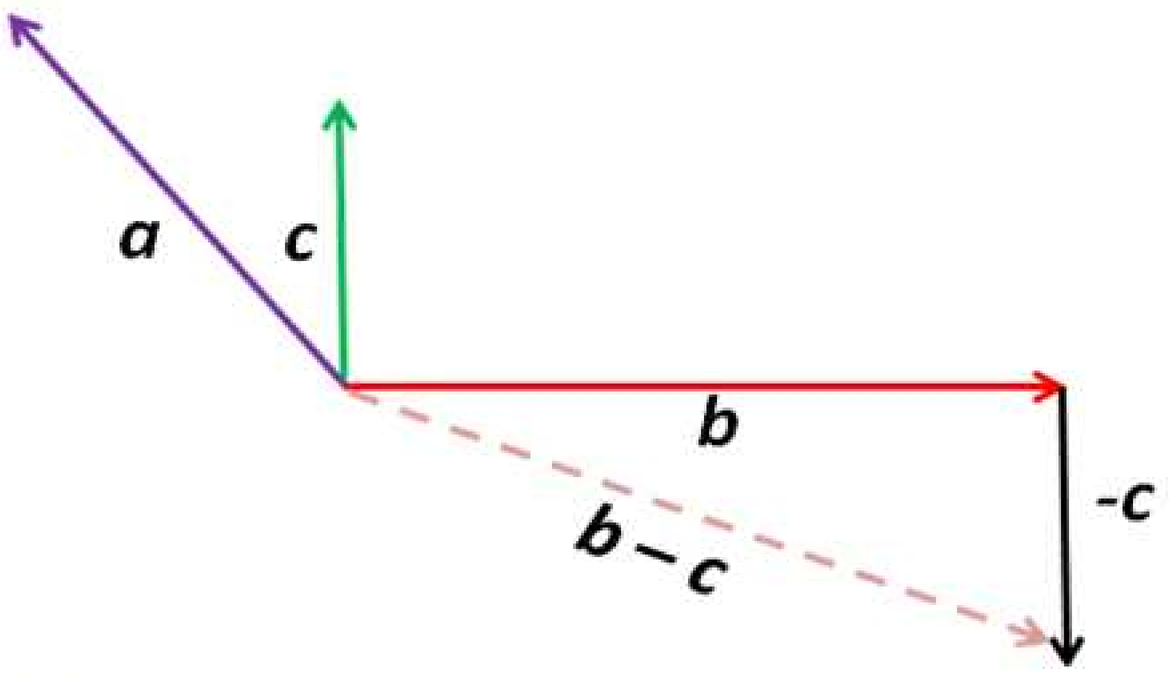
IP4:

If a, b and c are three vectors in the figure given below, then find $a + (b - c)$.

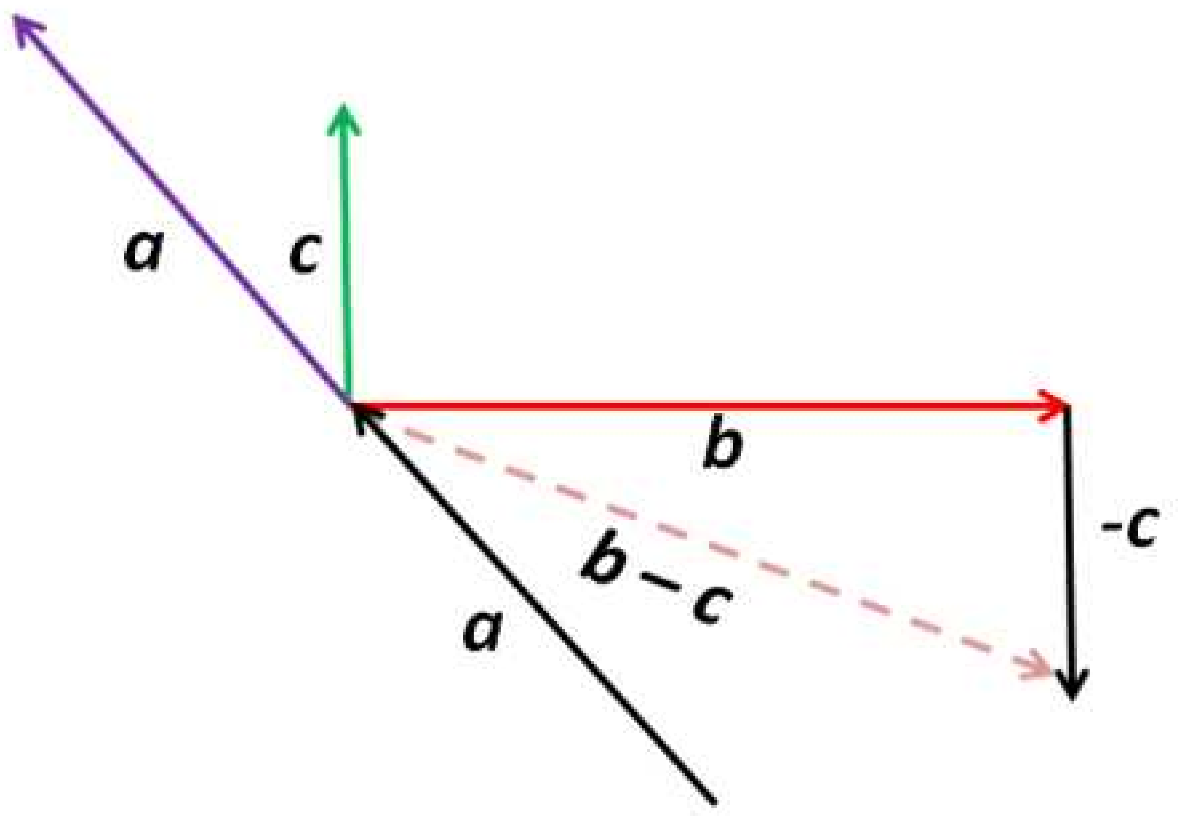


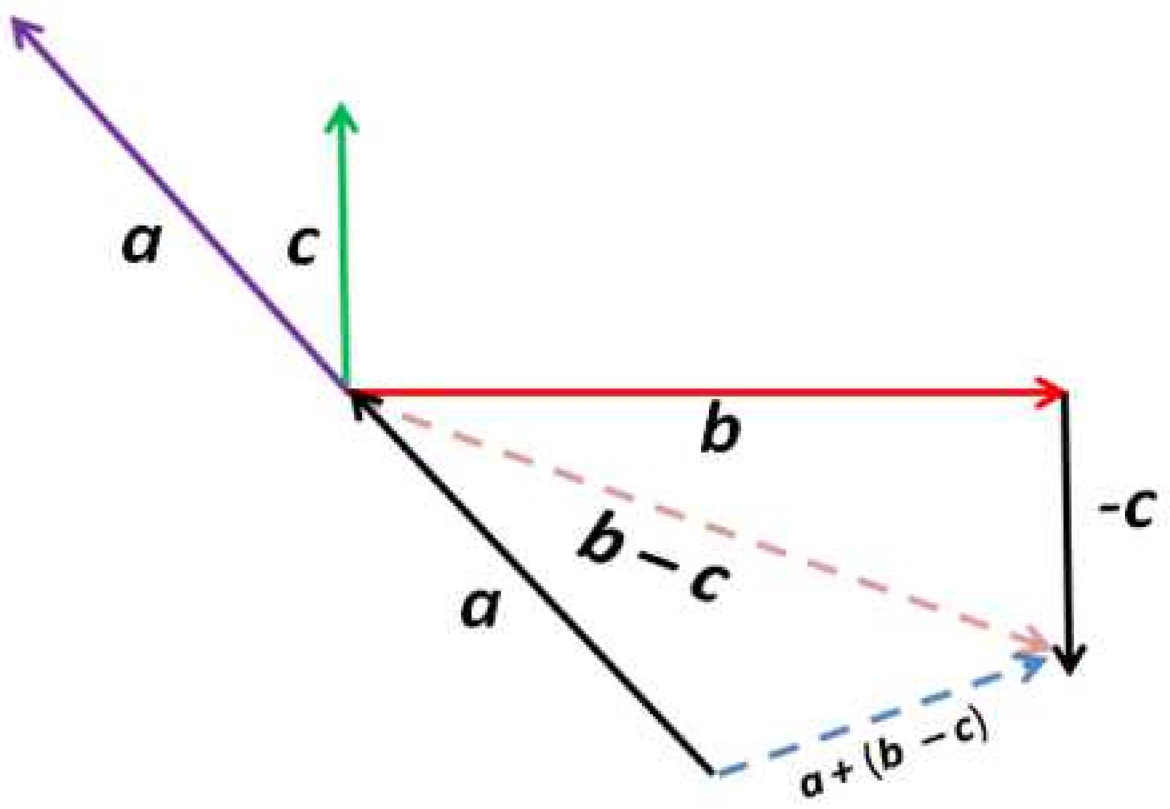
Solution:

Finding $b - c$:



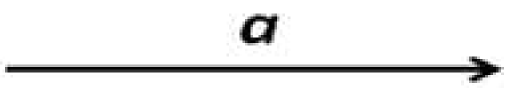
Finding $a + (b - c)$:





P1:

If $\boldsymbol{a}$ is a vector as shown in figure given below. Then find the following scalar multiples of the vector $\boldsymbol{a}$.



a) $3\boldsymbol{a}$

b) $-2\,\boldsymbol{a}$

c) $\frac{3}{2}\,\boldsymbol{a}$

Solution:

a. **Finding the vector $3a$:**

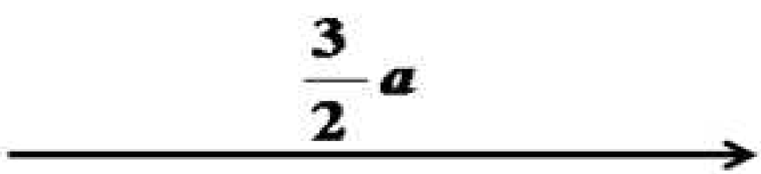
If we make three times the magnitude (length) of a in the same direction, then we get the vector $3a$.

b. **Finding the vector $-2a$:**

If we double the vector a and take it in the opposite direction, then we get the vector $-2a$.

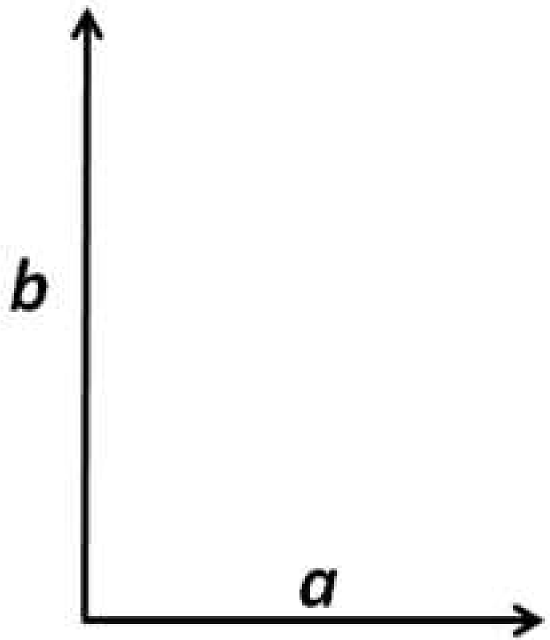
c. **Finding the vector $\frac{3}{2}a$:**

If we increase the magnitude of the vector a by 50%, then we get the vector $\frac{3}{2}a$ in the same direction.



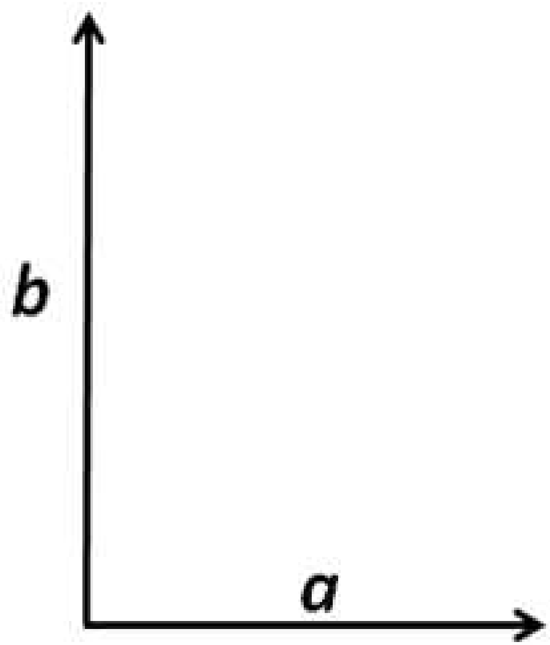
P2:

If a and b are two vectors in the figure given below, then find $b + a$.

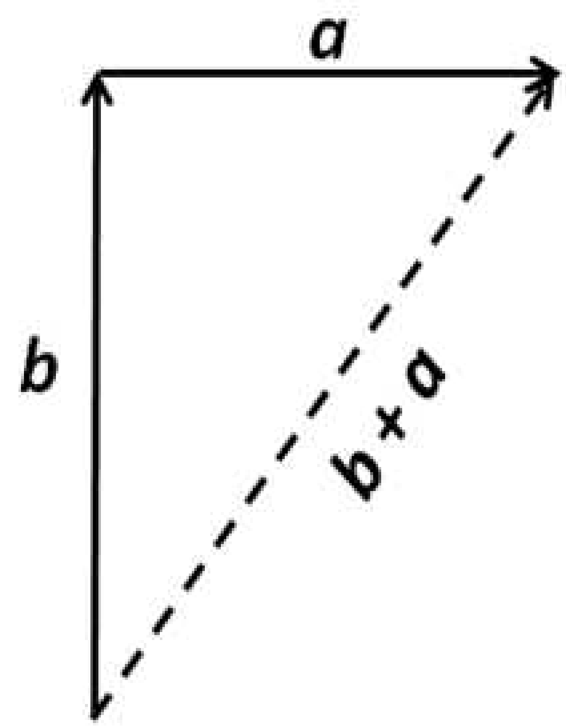


Solution:

Given:



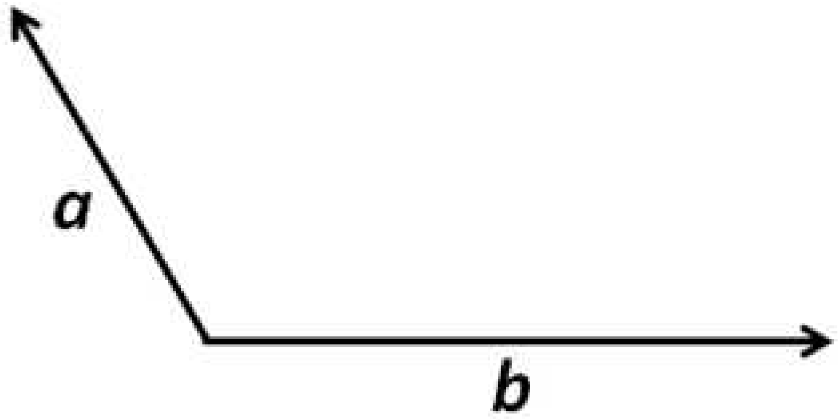
Finding $\boldsymbol{a} + \boldsymbol{b}$



The sum $\boldsymbol{b} + \boldsymbol{a}$ is the vector represented by arrow from the initial point of $\boldsymbol{b}$ to the terminal point of the $\boldsymbol{a}$.

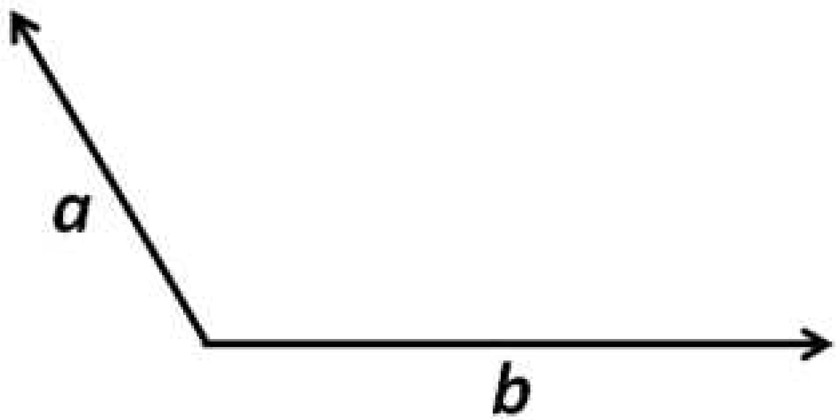
P3:

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors in the figure given below, then find $\boldsymbol{a} - \boldsymbol{b}$.

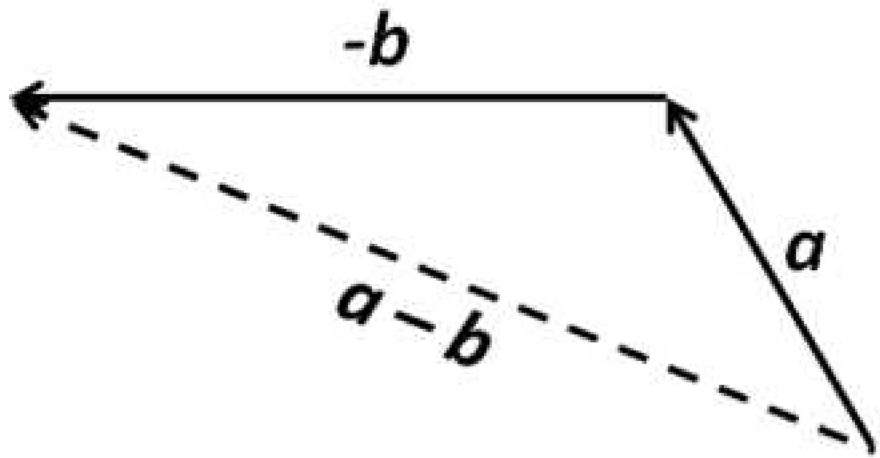


Solution:

Given:

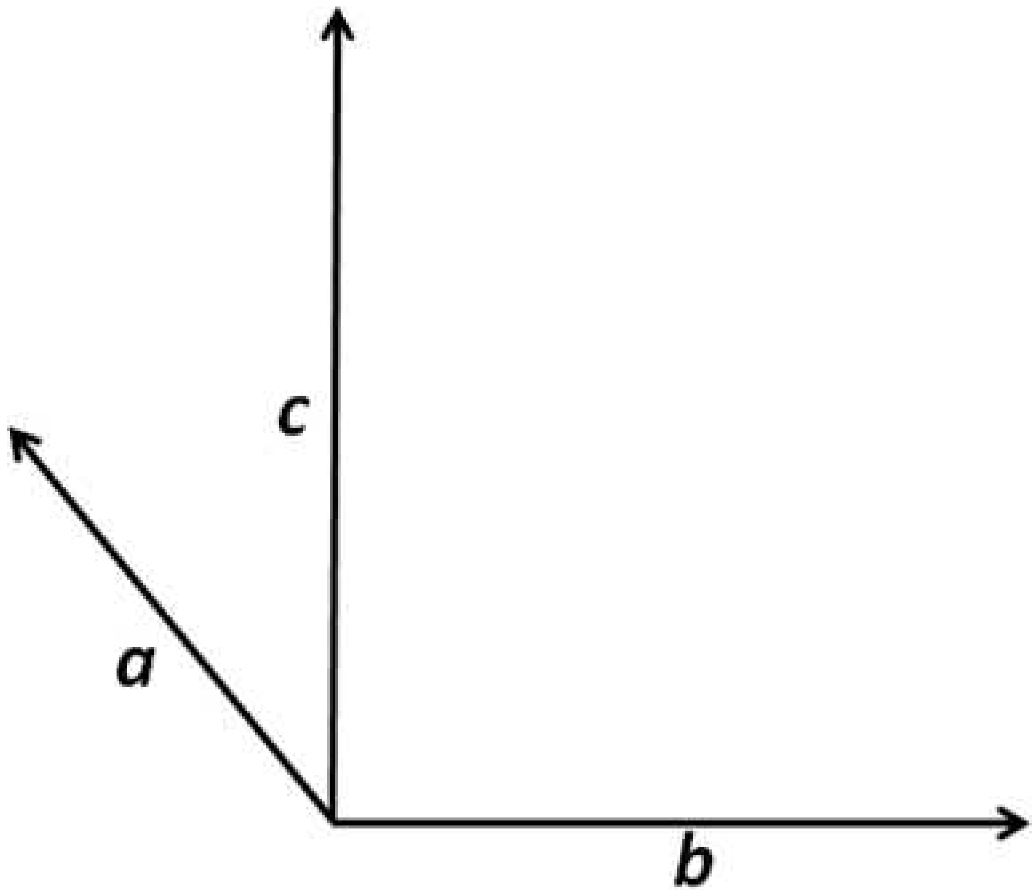


Finding $a - b$:



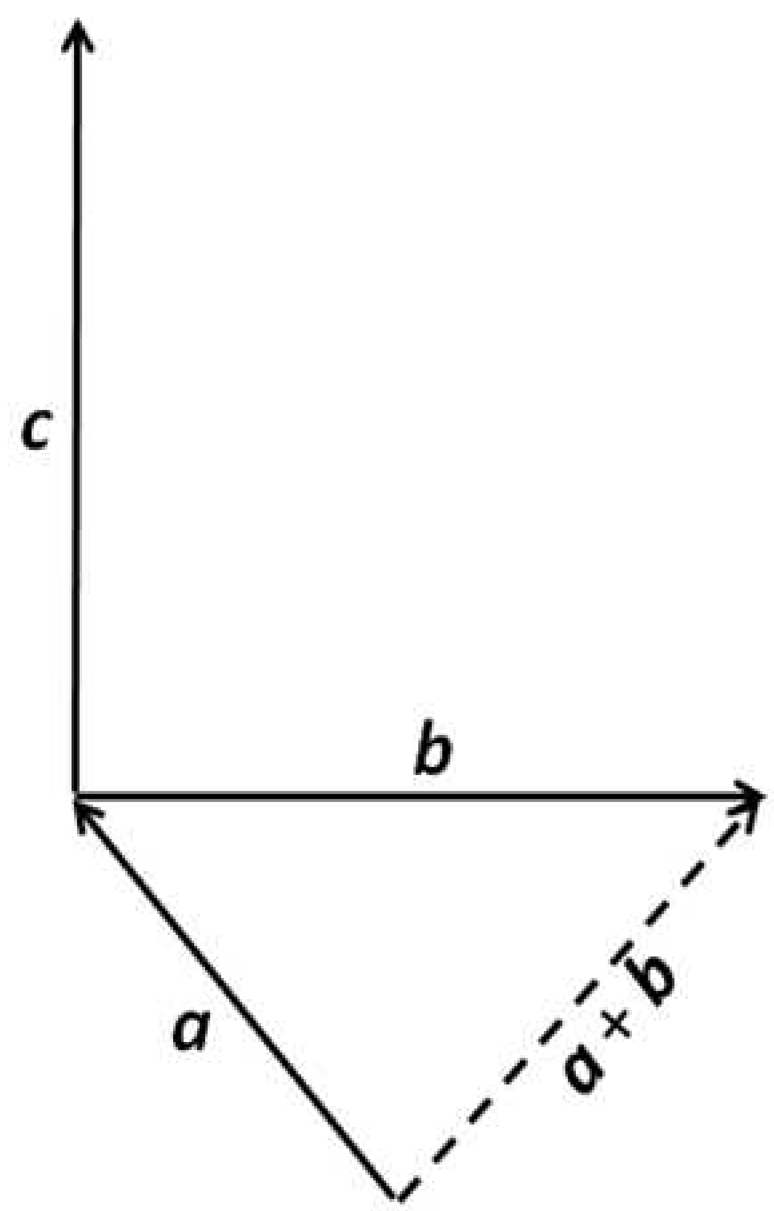
P4:

If a, b and c are three vectors in the figure given below, then find $(a + b) + c$.

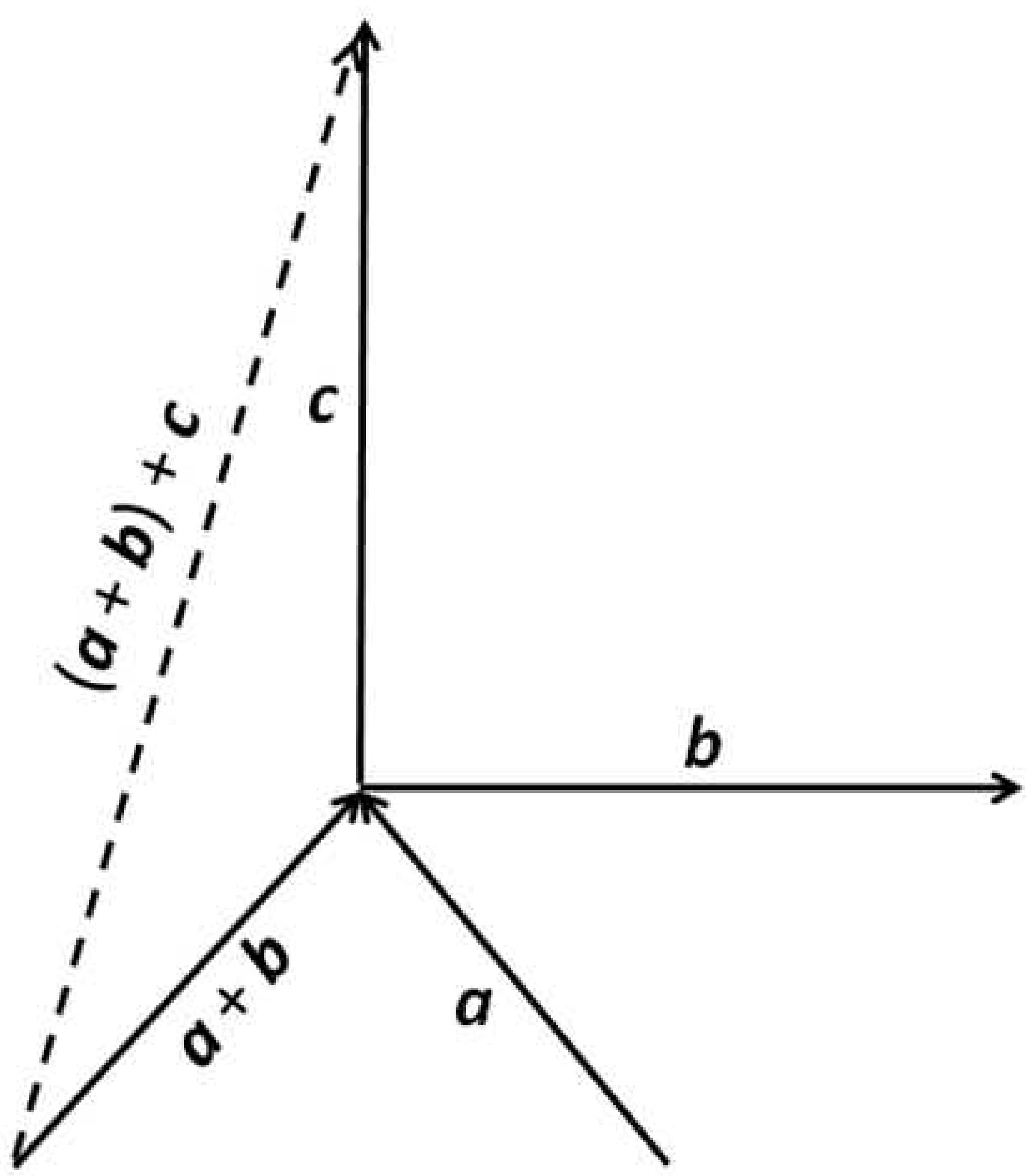


Solution:

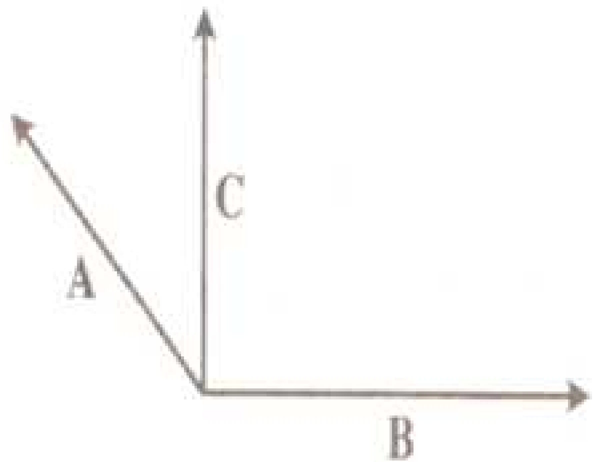
Finding $\boldsymbol{a} + \boldsymbol{b}$:



Finding $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$:



1. The vectors A , B and C in the figure here lie in a plane.



By arranging vectors head to tail, sketch

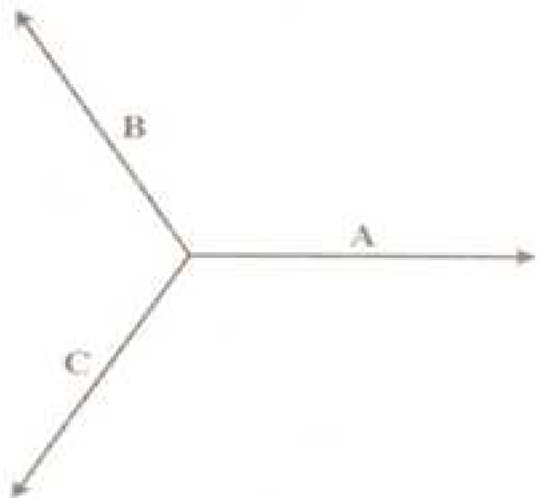
(a) $\boldsymbol{A} + \boldsymbol{B}$

(b) $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$

(c) $\boldsymbol{A} - 2\boldsymbol{B}$

(d) $\frac{1}{2}\boldsymbol{A} - \boldsymbol{C}$

2. The vectors A , B and C in the figure here lie in a plane.



By arranging vectors head to tail, sketch

(a) $\boldsymbol{A} - \boldsymbol{B}$

(b) $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$

(c) $2\boldsymbol{A} - \frac{1}{2}\boldsymbol{B}$

(d) $\boldsymbol{A} - (\boldsymbol{B} - \boldsymbol{C})$

4.2
Components of a Vector

Learning objectives:

1) To represent a vector in a plane in terms of components parallel to the Cartesian coordinate axes and to write each component as an appropriate multiple of a basic vector of length 1.
2) To define equality of vectors in a plane.
3) To derive the magnitude or length of a vector.
4) To define algebraic addition, subtraction and scalar multiplication of vectors in the plane.
And
5) To practice the related problems.

Two vectors are said to be ***parallel*** if they are nonzero scalar multiples of one another or, equivalently, if the line segments representing them are parallel.

Whenever a vector $\boldsymbol{v}$ can be written as a sum

$$\boldsymbol{v} = \boldsymbol{v_1} + \boldsymbol{v_2}$$

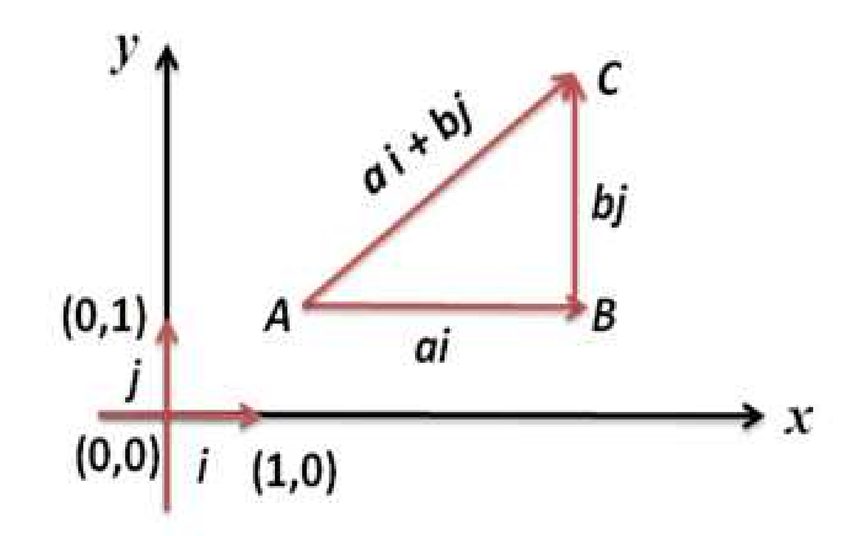
of two *nonparallel* vectors, the vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are said to be ***components*** of $\boldsymbol{v}$. We also say that $\boldsymbol{v}$ is ***represented*** or ***resolved*** $\boldsymbol{v}$ in terms of vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

The algebra of vectors is based on representing each vector in terms of components parallel to the Cartesian coordinate axes and writing each component as an appropriate multiple of a basic vector of length 1.

The basic vector in the positive x-direction is the vector $\boldsymbol{i}$ determined by the directed line segment that runs from (0,0) to (1,0).

The basic vector in the positive y-direction is the vector $\boldsymbol{j}$ determined by the directed line segment that runs from (0,0) to (0,1).

Then $a\boldsymbol{i}$, a being a scalar, represents a vector of length $|a|$ parallel to the x -axis, pointing to the right if $a > 0$ and to the left if $a < 0$. Similarly, $b\boldsymbol{j}$ is a vector of length $|b|$ parallel to the y-axis, pointing up if $b > 0$ and down if $b < 0$. Figure below shows a vector $\boldsymbol{v} = \overrightarrow{AC}$ resolved into $\boldsymbol{i}$- and $\boldsymbol{j}$-components as the sum $\qquad \boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$



Definitions

If $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, the vectors $a\boldsymbol{i}$ and $b\boldsymbol{j}$ are the vector components of $\boldsymbol{v}$ in the directions of $\boldsymbol{i}$ and $\boldsymbol{j}$. The numbers a and b are the scalar components of $\boldsymbol{v}$ in the directions of $\boldsymbol{i}$ and $\boldsymbol{j}$.

Components enable us to define the equality of vectors algebraically.

Definition

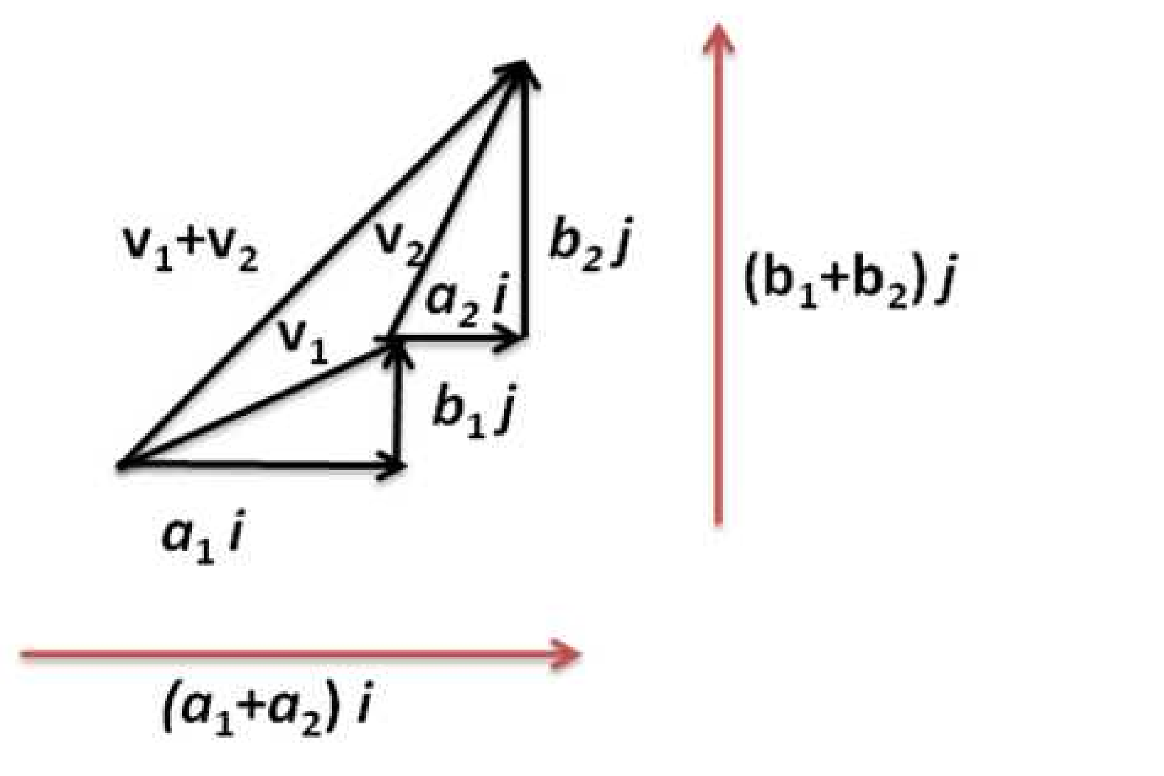
Equality of vectors

$$a\boldsymbol{i} + b\boldsymbol{j} = a'\boldsymbol{i} + b'\boldsymbol{j} \Leftrightarrow a = a' \text{ and } b = b' \qquad (1)$$

Two vectors are equal if and only if their scalar components in the directions of $\boldsymbol{i}$ *and* $\boldsymbol{j}$ *are identical.*

Algebraic Addition

Vectors may be added algebraically by adding their corresponding scalar components, as shown in figure below.



If $\boldsymbol{v}_1 = a_1\boldsymbol{i} + b_1\boldsymbol{j}$ and $\boldsymbol{v}_2 = a_2\boldsymbol{i} + b_2\boldsymbol{j}$, then

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 = (a_1 + a_2)\boldsymbol{i} + (b_1 + b_2)\boldsymbol{j} \qquad (2)$$

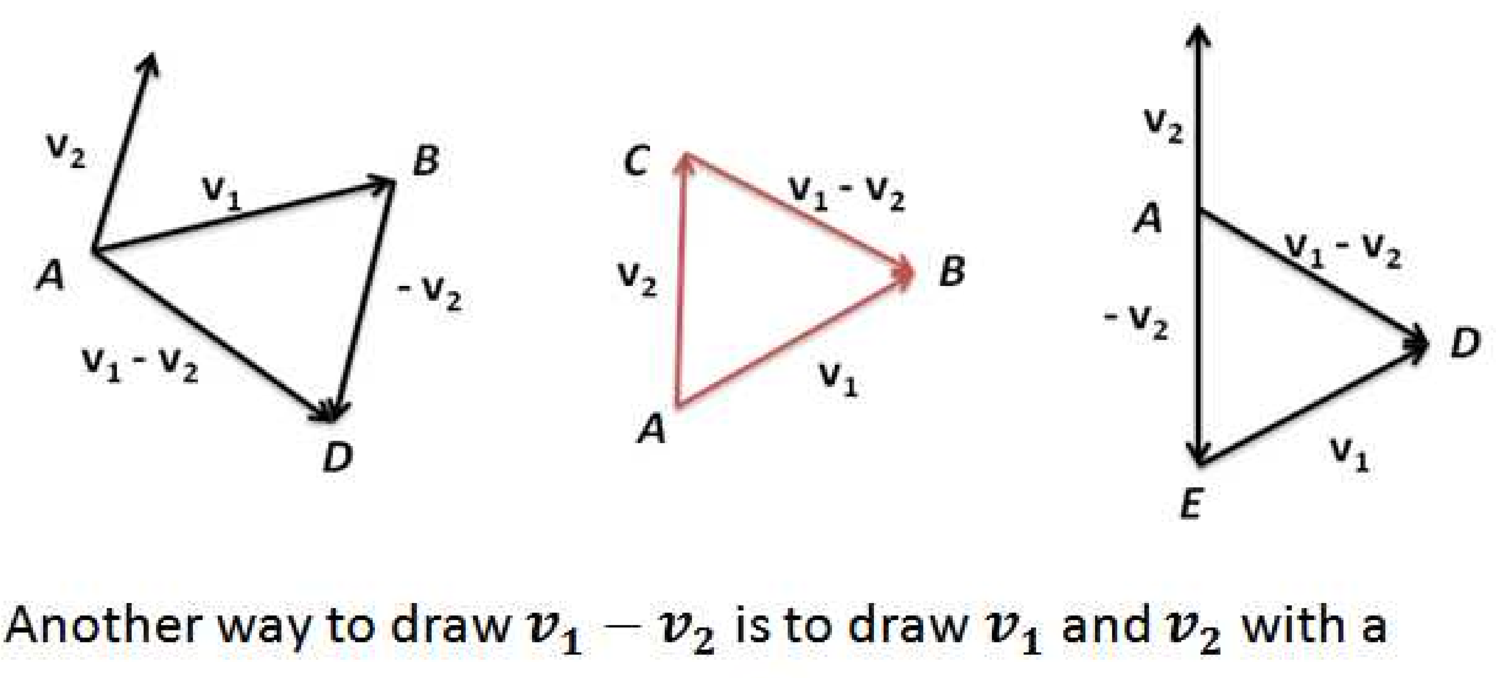
Example 1:

$$(2\boldsymbol{i} - 4\boldsymbol{j}) + (5\boldsymbol{i} + 3\boldsymbol{j}) = (2+5)\boldsymbol{i} + (-4+3)\boldsymbol{j} = 7\boldsymbol{i} - \boldsymbol{j}$$

Subtraction

The negative of a vector $\boldsymbol{v}$ is the vector $-\boldsymbol{v} = (-\boldsymbol{1})\boldsymbol{v}$. It has the same length as $\boldsymbol{v}$ but points in the opposite direction. To subtract a vector $\boldsymbol{v_2}$ from a vector $\boldsymbol{v_1}$, we add $-\boldsymbol{v_2}$ to $\boldsymbol{v_1}$. This can be done geometrically by drawing $-\boldsymbol{v_2}$ from the tip of $\boldsymbol{v_1}$ and then drawing the vector from the initial point of $\boldsymbol{v_1}$ to the tip of $-\boldsymbol{v_2}$, as shown in figure (a) below, where

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \boldsymbol{v}_1 + (-\boldsymbol{v}_2) = \boldsymbol{v}_1 - \boldsymbol{v}_2$$



Another way to draw $\boldsymbol{v_1} - \boldsymbol{v_2}$ is to draw $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ with a common initial point and then draw $\boldsymbol{v_1} - \boldsymbol{v_2}$ as the vector from the tip of $\boldsymbol{v_2}$ to the tip of $\boldsymbol{v_1}$. This is illustrated in figure (b), where

$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\boldsymbol{v_2} + \boldsymbol{v_1} = \boldsymbol{v_1} - \boldsymbol{v_2}$$

Still another way is to draw $\boldsymbol{v_1}$ from the tip of $-\boldsymbol{v_2}$ (figure c).

In terms of components, vector subtraction follows the algebraic law

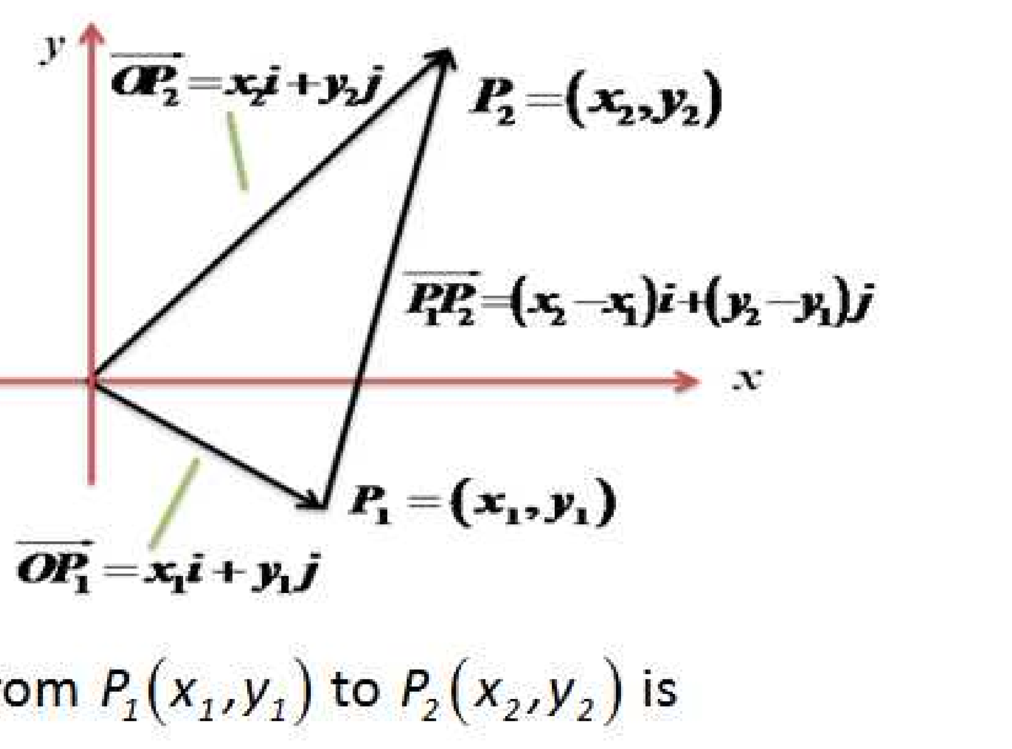
$$\boldsymbol{v_1} - \boldsymbol{v_2} = (a_1 - a_2)\boldsymbol{i} + (b_1 - b_2)\boldsymbol{j} \qquad (3)$$

which says that the corresponding scalar components are subtracted.

Example 2:

$$(6\boldsymbol{i} + 2\boldsymbol{j}) - (3\boldsymbol{i} - 5\boldsymbol{j}) = (6-3)\boldsymbol{i} + (2-(-5))\boldsymbol{j} = 3\boldsymbol{i} + 7\boldsymbol{j}$$

We find the components of the vector from a point $P_1(x_1, y_1)$ to a point $P_2(x_2, y_2)$ by subtracting the components $\overrightarrow{\boldsymbol{OP_1}} = x_1\boldsymbol{i} + y_1\boldsymbol{j}$ from the components of $\overrightarrow{\boldsymbol{OP_2}} = x_2\boldsymbol{i} + y_2\boldsymbol{j}$.



The vector from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

$$\overrightarrow{\boldsymbol{P_1P_2}} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} \qquad (4)$$

Example 3

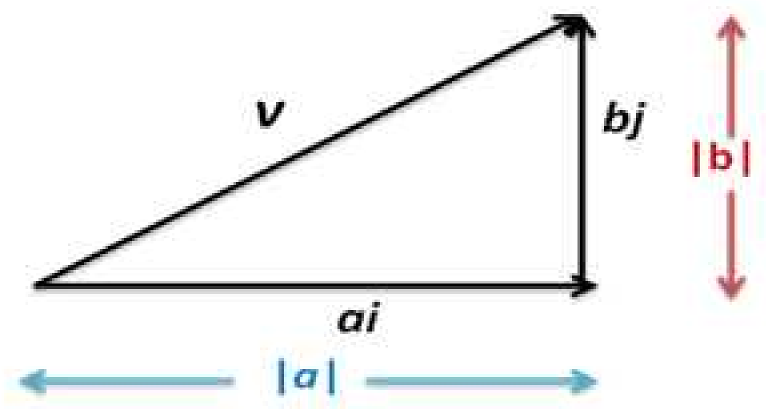
The vector from $P_1(3,4)$ to $P_2(5,1)$ is

$$\overrightarrow{\boldsymbol{P_1P_2}} = (5-3)\boldsymbol{i} + (1-4)\boldsymbol{j} = 2\boldsymbol{i} - 3\boldsymbol{j}$$

Magnitude

The ***magnitude*** or ***length*** of $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ is

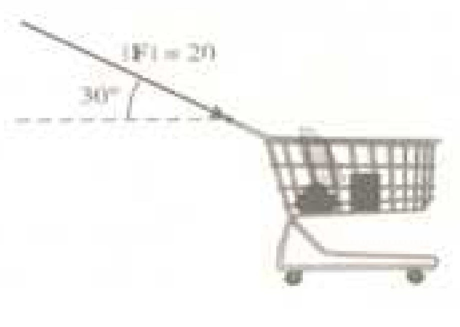
$|\boldsymbol{v}| = \sqrt{\boldsymbol{a^2} + \boldsymbol{b^2}}$. We arrive at this number by applying the Pythagorean Theorem to the right triangle determined by $\boldsymbol{v}$ and its two vector components.



The magnitude or length of

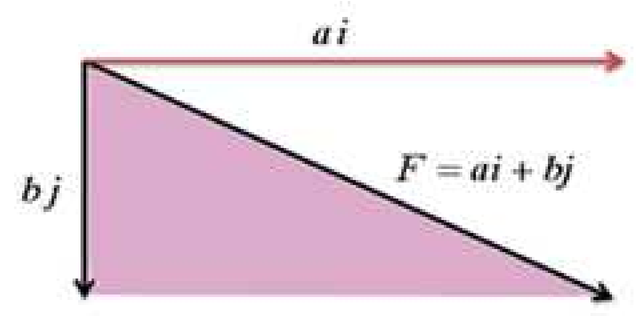
$\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ is $|\boldsymbol{v}| = \sqrt{\boldsymbol{a^2} + \boldsymbol{b^2}}$. (5)

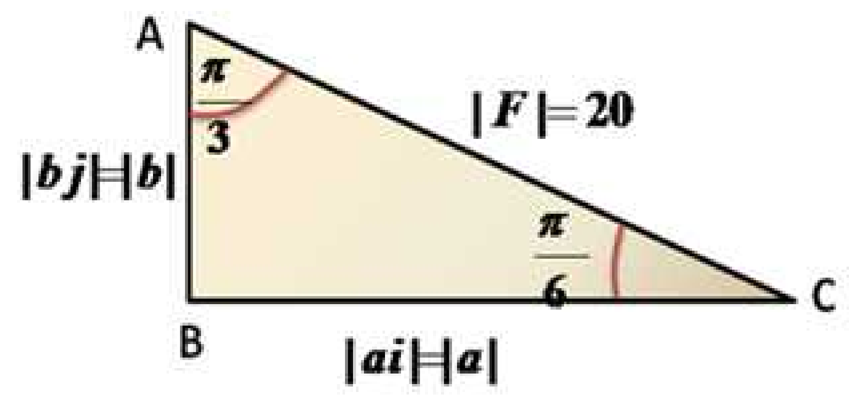
Example 4 You push a loaded supermarket cart by applying a 20-N force $\boldsymbol{F}$ that makes a $30°$ angle with the horizontal. Resolve $\boldsymbol{F}$ into its horizontal and vertical components. (The horizontal component is the effective force in the direction of motion. The vertical component just adds weight to the cart.)



Solution:

We draw a vector triangle for $\boldsymbol{F} = a\boldsymbol{i} + b\boldsymbol{j}$ and its vector components along with the right triangle determined by their magnitudes.





The triangle is a 30-60-90 triangle,

so $|\boldsymbol{a}| = |\boldsymbol{F}|\cos\frac{\pi}{6} = 10\sqrt{3}$ and $|\boldsymbol{b}| = |\boldsymbol{F}|\sin\frac{\pi}{6} = 10$. The horizontal component of $\boldsymbol{F}$ is $10\sqrt{3}\boldsymbol{i}$. The vertical component is $-10\boldsymbol{j}$ (negative because it points down). That is, $\boldsymbol{F} = 10\sqrt{3}\boldsymbol{i} - 10\boldsymbol{j}$

Scalar Multiplication:

Scalar multiplication can be accomplished component by component.

If c is a scalar and $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ is a vector, then

$$c\boldsymbol{v} = c(a\boldsymbol{i} + b\boldsymbol{j}) = (ca)\boldsymbol{i} + (cb)\boldsymbol{j} \qquad (6)$$

The length of $c\boldsymbol{v}$ is $|c|$ times the length of $\boldsymbol{v}$:

$$\begin{aligned} |c\boldsymbol{v}| &= |(ca)\boldsymbol{i} + (cb)\boldsymbol{j}| \\ &= \sqrt{(ca)^2 + (cb)^2} \\ &= \sqrt{c^2(a^2 + b^2)} \\ &= |c||\boldsymbol{v}| \end{aligned}$$

If c is a scalar and $\boldsymbol{v}$ is a vector, then $|c\boldsymbol{v}| = |c||\boldsymbol{v}|$.

Example 5

If $c = -2$ and $\boldsymbol{v} = -3\boldsymbol{i} + 4\boldsymbol{j}$ then

$$|\boldsymbol{v}| = |-3\boldsymbol{i} + 4\boldsymbol{j}| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\begin{aligned} |-2\boldsymbol{v}| &= |(-2)(-3\boldsymbol{i} + 4\boldsymbol{j})| = |6\boldsymbol{i} - 8\boldsymbol{j}| = \sqrt{6^2 + (-8)^2} \\ &= \sqrt{36 + 64} \\ &= \sqrt{100} = 10 = |-2|5 = |c||\boldsymbol{v}| \end{aligned}$$

The Zero Vector

In terms of components, the zero vector is the vector

$$\boldsymbol{0} = 0\boldsymbol{i} + 0\boldsymbol{j}$$

It is the only vector whose length is zero, as we can see from the fact that

$$|a\boldsymbol{i} + b\boldsymbol{j}| = \sqrt{a^2 + b^2} = 0 \Leftrightarrow a = b = 0$$

IP1)

Find p if $\frac{2p}{3}\boldsymbol{i} + 4\boldsymbol{j}$ is parallel to $2\boldsymbol{i} + \boldsymbol{j}$.

Solution:

$\frac{2p}{3}\boldsymbol{i} + 4\boldsymbol{j}$ is parallel to $2\boldsymbol{i} + \boldsymbol{j}$

$\Longrightarrow \frac{2p}{3}\boldsymbol{i} + 4\boldsymbol{j} = m(2\boldsymbol{i} + \boldsymbol{j})$ for some scalar m.

$\Longrightarrow 4 = m,$

$\frac{2p}{3} = 2m \Longrightarrow p = 3m = 12.$

$p = 12$

IP2)

If $\boldsymbol{a}+\boldsymbol{b}=7\boldsymbol{i}+4\boldsymbol{j}$ and $\boldsymbol{a}=7\boldsymbol{i}+4\boldsymbol{j}$. Find $\boldsymbol{b}$

Solution:

Given, $\boldsymbol{a}=7\boldsymbol{i}+4\boldsymbol{j}$ and $\boldsymbol{a}+\boldsymbol{b}=7\boldsymbol{i}+4\boldsymbol{j}$

We have to find $\boldsymbol{b}$

Subtract $\boldsymbol{a}$ from $+\boldsymbol{b}$.

$$\boldsymbol{b}=\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{a}=(7\boldsymbol{i}+4\boldsymbol{j})-(7\boldsymbol{i}+4\boldsymbol{j})$$

$$=(7-7)\boldsymbol{i}+(4-4)\boldsymbol{j}=\mathbf{0}$$

Therefore, $\boldsymbol{b}$ is a zero vector.

IP3)

Prove that the vectors $3\boldsymbol{i} + 5\boldsymbol{j}$, $5\boldsymbol{i} - 3\boldsymbol{j}$, and $-5\boldsymbol{i} + 3\boldsymbol{j}$ form the sides of an equilateral triangle.

Solution:

Let $\boldsymbol{a} = 3\boldsymbol{i} + 5\boldsymbol{j}$, $\boldsymbol{b} = 5\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{c} = -5\boldsymbol{i} + 3\boldsymbol{j}$

$\therefore$ the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are represented by three sides of a triangle.

Now $|\boldsymbol{a}| = |3\boldsymbol{i} + 5\boldsymbol{j}|$

$$= \sqrt{3^2 + 5^2} = \sqrt{34}$$

$$|\boldsymbol{b}| = 5\boldsymbol{i} - 3\boldsymbol{j}$$

$$= \sqrt{5^2 + (-3)^2} = \sqrt{34}$$

$$|\boldsymbol{c}| = -5\boldsymbol{i} + 3\boldsymbol{j}$$

$$= \sqrt{25 + 9} = \sqrt{34}$$

$\therefore$ $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$;

Thus, three vectors represent sides of equilateral triangle.

IP4)

If m is a scalar and $\boldsymbol{a}$ is vector, then

$$(-m)\boldsymbol{a} = m(-\boldsymbol{a}) = -(m\boldsymbol{a})$$

Proof:

Let $m > 0$,

$$|(-m)\boldsymbol{a}| = |-m|\,|\boldsymbol{a}\,| = |m|\,|\boldsymbol{a}|.$$

$$|m(-\boldsymbol{a})| = |m|\,|-\boldsymbol{a}| = |m|\,|\boldsymbol{a}|.$$

$$|-(m\boldsymbol{a})| = |m\boldsymbol{a}| = |m|\,|\boldsymbol{a}|.$$

$\therefore$ The length of $(-m)\boldsymbol{a}, m(-\boldsymbol{a}), -m\boldsymbol{a}$ are equal.

Direction of $(-m)\boldsymbol{a}$ = opposite to the direction of $\boldsymbol{a}$

Direction of $m(-\boldsymbol{a})$ = Direction of $(-\boldsymbol{a}) =$ Opposite to the direction of $\boldsymbol{a}$

Direction of – $(m\boldsymbol{a})$ = Opposite to the direction of $m\boldsymbol{a}$ = opposite to the direction of $\boldsymbol{a}$

$\therefore$ Direction of $(-m)\boldsymbol{a}, m(-\boldsymbol{a}), -(m\boldsymbol{a})$ are the same.

Hence $(-m)\boldsymbol{a} = m(-\boldsymbol{a}) = -(m\boldsymbol{a})$.

The result follows on similar lines when $m < 0$, hence the result.

P1)

Find the values of x,and y so that the vectors

$$\boldsymbol{a} = x\boldsymbol{i} + 2\boldsymbol{j} \text{ and } \boldsymbol{b} = 2\boldsymbol{i} + y\boldsymbol{j} \text{ are equal.}$$

Solution:

Given, $\boldsymbol{a} = x\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + y\boldsymbol{j}$

Note that two vectors are equal if and only if their corresponding components are equal.

Thus, the given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal if and only if

$$x = 2, y = 2$$

P2)

Find the sum of the vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$

Solution:

$\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$

The sum of the given vectors is

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{i} + \boldsymbol{j} \\ &= (2+2)\boldsymbol{i} + (2+1)\boldsymbol{j} \\ &= 4\boldsymbol{i} + 3\boldsymbol{j} \end{aligned}$$

P3)

Use vectors to decide whether the triangle with vertices

$A(2,-1), B(1,-3)$ *and* $C(3,-4)$ is a right angled triangle.

Solution:

We have $A(2,-1), B(1,-3)$ and $C(3,-4)$, then

$\overrightarrow{\boldsymbol{AB}} = (1-2)\boldsymbol{i} + (-3+1)\boldsymbol{j} = -\boldsymbol{i} - 2\boldsymbol{j}$

$\overrightarrow{\boldsymbol{BC}} = (3-1)\boldsymbol{i} + (-4+3)\boldsymbol{j} = 2\boldsymbol{i} - \boldsymbol{j}$ and

$\overrightarrow{\boldsymbol{CA}} = (2-3)\boldsymbol{i} + (-1+4)\boldsymbol{j} = -\boldsymbol{i} + 3\boldsymbol{j}$

Further, note that

$|AB|^2 = 1 + 4 = 5,$

$|BC|^2 = 4 + 1 = 5$ and $|CA|^2 = 1 + 9 = 10,$

$|CA|^2 = |AB|^2 + |BC|^2$

Hence, the triangle is a right angled triangle.

P4)

If $\boldsymbol{a}, \boldsymbol{b}$ are two vectors then

$i)\quad |\boldsymbol{a}+\boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$

$ii)\quad |\boldsymbol{a}-\boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$

$iii)\quad |\boldsymbol{a}-\boldsymbol{b}| \geq \big||\boldsymbol{a}| - |\boldsymbol{b}|\big|$

Proof:

$i)$ Let $\boldsymbol{a} = \overrightarrow{AB}$, $\boldsymbol{b} = \overrightarrow{BC}$.

Now $\boldsymbol{a} + \boldsymbol{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CA}$. and ABC is a triangle.

$\therefore |\boldsymbol{a} + \boldsymbol{b}| = AC,\ |\boldsymbol{a}| = AB,\ |\boldsymbol{b}| = BC$

In a triangle we have $AC \leq AB + BC \Longrightarrow |\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$.

$ii)$ $|\boldsymbol{a} - \boldsymbol{b}| = |\boldsymbol{a} + (-\boldsymbol{b})| \leq |\boldsymbol{a}| + |-\boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$

$iii)$ $|\boldsymbol{a}| = |(\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{b}| \leq |\boldsymbol{a} - \boldsymbol{b}| + |\boldsymbol{b}|$

$$\Longrightarrow |\boldsymbol{a}| - |\boldsymbol{b}| \leq |\boldsymbol{a} - \boldsymbol{b}|$$

$$|\boldsymbol{b}| = |(\boldsymbol{b} - \boldsymbol{a}) + \boldsymbol{a}| \leq |\boldsymbol{b} - \boldsymbol{a}| + |\boldsymbol{a}|$$

$$\Longrightarrow |\boldsymbol{b}| - |\boldsymbol{a}| \leq |\boldsymbol{b} - \boldsymbol{a}| = |\boldsymbol{a} - \boldsymbol{b}|$$

$$\therefore \big|\,|\boldsymbol{a}| - |\boldsymbol{b}|\,\big| \leq |\boldsymbol{a} - \boldsymbol{b}|.$$

1. Find the values of x and y so that the vectors $2\boldsymbol{i} + 3\boldsymbol{j}$ and $x\boldsymbol{i} - y\boldsymbol{j}$ are equal.

2. Find the sum of the vectors $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j}$;

$\boldsymbol{b} = -2\boldsymbol{i} + 4\boldsymbol{j}$; $\boldsymbol{c} = \boldsymbol{i} - 6\boldsymbol{j}$;

3. Compute the magnitude of the following vectors:

$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j};\ \boldsymbol{b} = 2\boldsymbol{i} - 7\boldsymbol{j}\ and\ \ \boldsymbol{c} = \frac{1}{\sqrt{3}}\boldsymbol{i} + \frac{1}{\sqrt{3}}\boldsymbol{j}$$

4. Write two different vectors having same magnitude.

5. Sum of two vectors is $4\boldsymbol{i} + 2\boldsymbol{j}$. if one of the vectors is $3\boldsymbol{j} - 2\boldsymbol{i}$ then another vector is

6. If $\boldsymbol{a} = -2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 6\boldsymbol{i} - 8\boldsymbol{j}$ then find

$3\boldsymbol{a} + 4\boldsymbol{b}$ and $\boldsymbol{a} - 6\boldsymbol{b}$

7. If $= 2\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{c} = 3\boldsymbol{i} + \boldsymbol{j}$ are sides of a triangle then prove that it is a right angled triangle.

8. If $= 3\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} + \sqrt{6}\boldsymbol{j}$ are sides of a triangle then prove that it is a equilateral triangle.

4.3

Slopes, Tangents and Normals

Learning Objectives:

- To introduce the concept of unit vectors in the plane
- To learn how to express a given non-zero vector in terms of its length and direction
- To define a vector parallel to a non-vertical line
- To find tangent and normal vectors to a curve at a point

AND

- To practice the related problems

Unit Vectors

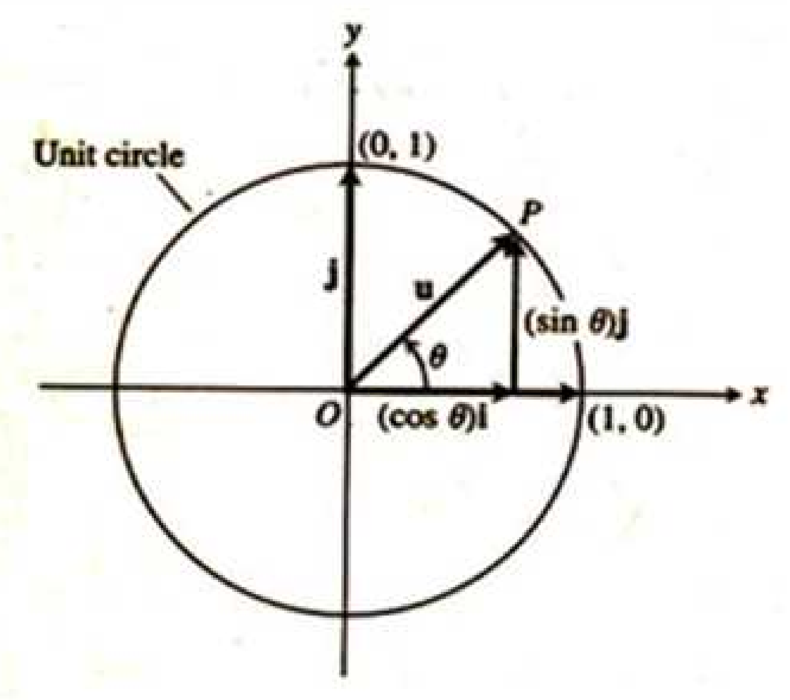
Any vector whose length is 1 is a ***unit vector***. The vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors.

$$|\boldsymbol{i}| = |1\boldsymbol{i} + 0\boldsymbol{j}| = \sqrt{1^2 + 0^2} = 1$$

$$|\boldsymbol{j}| = |0\boldsymbol{i} + 1\boldsymbol{j}| = \sqrt{0^2 + 1^2} = 1$$

If $\boldsymbol{u}$ is the unit vector obtained by rotating $\boldsymbol{i}$ through an angle θ in the positive direction, then $\boldsymbol{u}$ has a horizontal component of $\cos\theta$ and vertical component of $\sin\theta$, so that

$$\boldsymbol{u} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j} \text{ ------- ------- --- } (1)$$



As θ varies from 0 to 2π, the point P traces the circle $x^2 + y^2 = 1$ counterclockwise. This takes on all possible directions, so equation (1) gives every unit vector in the plane.

Length and Direction

If $\boldsymbol{v} \neq 0$, then

$$\left|\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right| = \left|\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}\right| = \frac{1}{|\boldsymbol{v}|}|\boldsymbol{v}| = 1$$

so $\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is a unit vector in the direction of $\boldsymbol{v}$. We can therefore express $\boldsymbol{v}$ in terms of its two important features, length and direction, by writing $\boldsymbol{v} = |\boldsymbol{v}|\left(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right)$.

If $\boldsymbol{v} \neq 0$, then

1. $\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is a unit vector in the direction of $\boldsymbol{v}$;
2. the equation $\boldsymbol{v} = |\boldsymbol{v}|\left(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right)$ expresses $\boldsymbol{v}$ in terms of its length and direction.

Example 1

Express $\boldsymbol{v} = 3\boldsymbol{i} - 4\boldsymbol{j}$ as a product of its length and direction.

Solution

Length of $\boldsymbol{v}$: $\quad |\boldsymbol{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5$

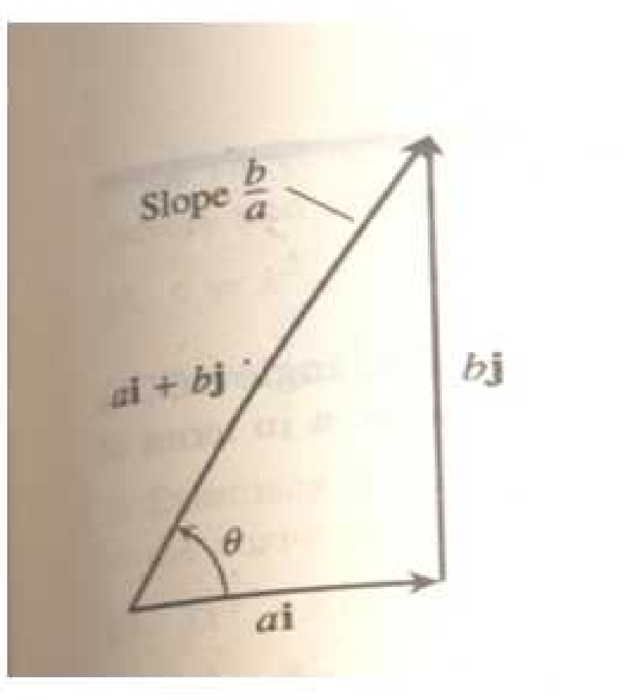
Direction of $\boldsymbol{v}$: $\quad \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{3\boldsymbol{i} - 4\boldsymbol{j}}{5} = \frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}$

$$\boldsymbol{v} = 3\boldsymbol{i} - 4\boldsymbol{j} = 5\left(\frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}\right)$$

Slopes

A vector is ***parallel*** to a line if the line segments that represent the vector and the line are parallel.

The ***slope*** of a vector that is not vertical is the slope of the lines parallel to the vector. Thus, if $a \neq 0$, the vector $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ has a well-defined slope, which can be calculated from the components of $\boldsymbol{v}$ as the number $\frac{b}{a}$.



Tangents and Normals

A vector is ***tangent*** or ***normal*** to a curve at a point if it is parallel or normal to the line that is tangent to the curve at that point.

In the next example, we learn how to find such vectors.

Example 2

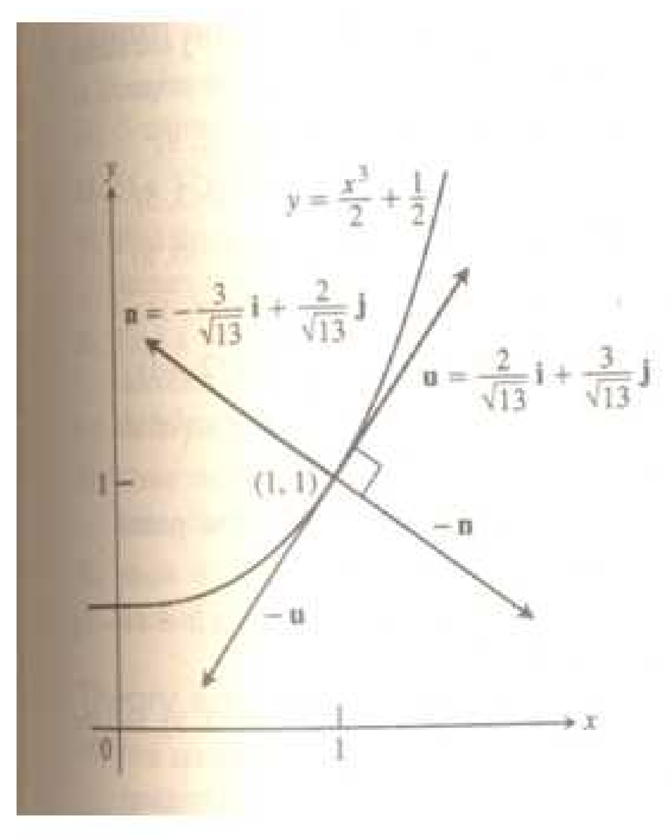
Find unit vectors tangent and normal to the curve

$$y = \frac{x^3}{2} + \frac{1}{2}$$

at the point (1,1).

Solution

We find the unit vectors that are parallel and normal to the curve's tangent line at (1,1).



The slope of the line tangent to the curve at (1,1) is

$$y' = \left.\frac{3x^2}{2}\right|_{x=1} = \frac{3}{2}$$

We look for a unit vector with this slope. The vector $\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j}$ has slope $\frac{3}{2}$, as does every nonzero multiple of $\boldsymbol{v}$. To find a multiple of $\boldsymbol{v}$ that is a unit vector, we divide $\boldsymbol{v}$ by

$$|\boldsymbol{v}| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

obtaining

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{2}{\sqrt{13}}\boldsymbol{i} + \frac{3}{\sqrt{13}}\boldsymbol{j}$$

The vector $\boldsymbol{u}$ is tangent to the curve at (1, 1) because it has the same direction as $\boldsymbol{v}$. Of course,

$$-\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = -\frac{2}{\sqrt{13}}\boldsymbol{i} - \frac{3}{\sqrt{13}}\boldsymbol{j}$$

which points in the opposite direction, is also tangent to the curve at (1,1). Without some additional requirement, there is no reason to prefer one of these vectors to the other.

To find unit vectors normal to the curve at (1,1), we look for unit vectors whose slopes are the negative reciprocal of the slope of $\boldsymbol{u}$. This is quickly done by interchanging the scalar components of $\boldsymbol{u}$ and changing the sign of one of them. We obtain

$$\boldsymbol{n} = -\frac{3}{\sqrt{13}}\boldsymbol{i} + \frac{2}{\sqrt{13}}\boldsymbol{j} \text{ and } -\boldsymbol{n} = \frac{3}{\sqrt{13}}\boldsymbol{i} - \frac{2}{\sqrt{13}}\boldsymbol{j}$$

Again, either one will do. The vectors have opposite directions but both are normal to the curve at (1,1).

If $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, then $\boldsymbol{p} = -b\boldsymbol{i} + a\boldsymbol{j}$ and $\boldsymbol{q} = b\boldsymbol{i} - a\boldsymbol{j}$ are perpendicular to $\boldsymbol{v}$ because their slopes are both $-\frac{a}{b}$, the negative reciprocal of the slope of $\boldsymbol{v}$.

IP1.

Find the unit vector in the direction of the vector $\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j}$.

Solution:

The given vector is $\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $|\boldsymbol{v}| = \sqrt{2^2 + 3^2} = \sqrt{13}$

The unit vector $\boldsymbol{u}$ in the direction of the vector $\boldsymbol{v}$ is,

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

Therefore, the required unit vector is

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{2\boldsymbol{i}+3\boldsymbol{j}}{\sqrt{13}} = \frac{2}{\sqrt{13}}\boldsymbol{i} + \frac{3}{\sqrt{13}}\boldsymbol{j}$$

IP2:

Find a vector in the direction of the vector $\boldsymbol{a} = 5\boldsymbol{i} - 3\boldsymbol{j}$ that has magnitude 8 units.

Solution:

The given vector is, $\boldsymbol{a} = 5\boldsymbol{i} - 3\boldsymbol{j}$

and $|\boldsymbol{a}| = \sqrt{5^2 + (-3)^2} = \sqrt{34}$

The unit vector $\boldsymbol{u}$ in the direction of the given vector $\boldsymbol{a}$ is

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

$$= \frac{5\boldsymbol{i}-3\boldsymbol{j}}{\sqrt{34}} = \frac{5}{\sqrt{34}}\boldsymbol{i} - \frac{3}{\sqrt{34}}\boldsymbol{j}$$

Now, the vector having magnitude 8 units in the direction of $\boldsymbol{a}$ is $8\boldsymbol{u}$ and $-8\boldsymbol{u}$, where

$$8\boldsymbol{u} = 8\left(\frac{5}{\sqrt{34}}\boldsymbol{i} - \frac{3}{\sqrt{34}}\boldsymbol{j}\right) = \frac{40}{\sqrt{34}}\boldsymbol{i} - \frac{24}{\sqrt{34}}\boldsymbol{j}$$

$$-8\boldsymbol{u} = -8\left(\frac{5}{\sqrt{34}}\boldsymbol{i} - \frac{3}{\sqrt{34}}\boldsymbol{j}\right) = -\frac{40}{\sqrt{34}}\boldsymbol{i} + \frac{24}{\sqrt{34}}\boldsymbol{j}$$

IP3:

Find the unit vector tangent to the curve $y = \frac{x-1}{x+1}$ at the point $(0, -1)$.

Solution:

The given curve is, $y = \frac{x-1}{x+1}$

Differentiating w.r.t , $y' = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$

$$= \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

The slope of the line tangent to the curve at $(0, -1)$ is

$$y' = \left[\frac{2}{(x+1)^2}\right]_{x=0} = \frac{2}{(0+1)^2} = 2$$

Therefore, the vectors $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j}$ and $-\boldsymbol{v}$ have a slope 2; and they are parallel to the tangent line of the curve at (0, -1).

Now, the unit vectors tangent to the curve at (0,-1) are $\boldsymbol{u}$ and $-\boldsymbol{u}$, where

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{\boldsymbol{i}+2\boldsymbol{j}}{\sqrt{1^2+2^2}} = \frac{\boldsymbol{i}+2\boldsymbol{j}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\boldsymbol{i} + \frac{2}{\sqrt{5}}\boldsymbol{j}$$

IP4:

Find the unit vector normal to the curve $y = x^3 + 4x^2$ at the point $(-1, 3)$.

Solution:

The given curve is, $y = x^3 + 4x^2$

Differentiating w.r.t x, we get $y' = 3x^2 + 8x$

The slope of the line tangent to the curve at $(-1, 3)$ is

$$y' = [3x^2 + 8x]_{x=-1} = 3 - 8 = -5$$

Slope of the normal $= \frac{1}{5}$

Therefore, the vector $\boldsymbol{v} = 5\boldsymbol{i} + \boldsymbol{j}$ and $-\boldsymbol{v}$ have slope $\frac{1}{5}$, and they are parallel to the normal of the curve at (-1, 3).

Now, the unit vectors normal to the curve at (-1, 3) are $\boldsymbol{n}$ and $-\boldsymbol{n}$, where

$$\boldsymbol{n} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{5\boldsymbol{i}+\boldsymbol{j}}{\sqrt{5^2+1^2}} = \frac{5\boldsymbol{i}+\boldsymbol{j}}{\sqrt{26}} = \frac{5}{\sqrt{26}}\boldsymbol{i} + \frac{1}{\sqrt{26}}\boldsymbol{j}$$

P1:

Find the unit vector in the direction of the sum of the vectors $\boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j}$.

Solution:

Given vectors are, $\boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j}$

Sum of the vectors is, $\boldsymbol{u} + \boldsymbol{v} = (2\boldsymbol{i} + 2) + (2\boldsymbol{i} + \boldsymbol{j})$

$$= (2 + 2)\boldsymbol{i} + (2 + 1)\boldsymbol{j}$$

$$= 4\boldsymbol{i} + 3\boldsymbol{j}$$

and $|\boldsymbol{u} + \boldsymbol{v}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

Therefore, the required unit vector is

$$\boldsymbol{u} + \boldsymbol{v} = \frac{\boldsymbol{u}+\boldsymbol{v}}{|\boldsymbol{u}+\boldsymbol{v}|} = \frac{(4\boldsymbol{i}+3\boldsymbol{j})}{5} = \frac{4}{5}\boldsymbol{i} + \frac{3}{5}\boldsymbol{j}$$

P2.

Find a vector in the direction of the vector $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j}$ that has length 7 units.

Solution:

The given vector is, $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j}$ and $|\boldsymbol{a}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

The unit vector $\boldsymbol{u}$ in the direction of the given vector $\boldsymbol{a}$ is

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

$$= \frac{\boldsymbol{i} - 2\boldsymbol{j}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\boldsymbol{i} - \frac{2}{\sqrt{5}}\boldsymbol{j}$$

Now, the vector having magnitude 7 units in the direction of $\boldsymbol{a}$ is $7\boldsymbol{u}$ and $-7\boldsymbol{u}$, where

$$7\boldsymbol{u} = 7\left(\frac{1}{\sqrt{5}}\boldsymbol{i} - \frac{2}{\sqrt{5}}\boldsymbol{j}\right) = \frac{7}{\sqrt{5}}\boldsymbol{i} - \frac{14}{\sqrt{5}}\boldsymbol{j}$$

$$-7\boldsymbol{u} = -7\left(\frac{1}{\sqrt{5}}\boldsymbol{i} - \frac{2}{\sqrt{5}}\boldsymbol{j}\right) = -\frac{7}{\sqrt{5}}\boldsymbol{i} + \frac{14}{\sqrt{5}}\boldsymbol{j}$$

P3.

Find the unit vector tangent to the curve $y = x^2 - \frac{2}{3}x + 5$ at the point $(1, 2)$.

Solution:

The given curve is, $y = x^2 - \frac{2}{3}x + 5$

Differentiating w.r.t x, $y' = 2x - \frac{2}{3}$

The slope of the line tangent to the curve at $(1, 2)$ is

$$y' = \left[2x - \frac{2}{3}\right]_{x=1} = 2 - \frac{2}{3} = \frac{4}{3}$$

Therefore, the vectors $\boldsymbol{v} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $-\boldsymbol{v}$ have a slope $\frac{4}{3}$; and they are parallel to the tangent line of the curve at (1, 2).

Now, the unit vectors tangent to the curve at (1, 2) are $\boldsymbol{u}$ and $-\boldsymbol{u}$, where

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{3\boldsymbol{i}+4\boldsymbol{j}}{\sqrt{3^2+4^2}} = \frac{3\boldsymbol{i}+4\boldsymbol{j}}{\sqrt{25}} = \frac{3\boldsymbol{i}+4\boldsymbol{j}}{5} = \frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$$

P4.

Find the unit vector normal to the curve

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5$$

at the point $(0, 5)$.

Solution:

The given curve is, $y = x^4 - 6x^3 + 13x^2 - 10x + 5$

Differentiating w.r.t x, we get

$$y' = 4x^3 - 18x^2 + 26x - 10$$

The slope of the line tangent to the curve at $(0, 5)$ is

$$y' = [4x^3 - 18x^2 + 26x - 10]_{x=0} = -10$$

Slope of the normal $= -\frac{1}{10}$

Therefore, the vectors $\boldsymbol{v} = 10\boldsymbol{i} - \boldsymbol{j}$ and $-\boldsymbol{v}$ have slope $-\frac{1}{10}$,

and they are parallel to the normal of the curve at (0, 5).

Now, the unit vectors normal to the curve at (0, 5) are $\boldsymbol{n}$ and

$-\boldsymbol{n}$, where

$$\boldsymbol{n} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{10\boldsymbol{i} - \boldsymbol{j}}{\sqrt{10^2 + 1^2}} = \frac{10\boldsymbol{i} - \boldsymbol{j}}{\sqrt{101}} = \frac{10}{\sqrt{101}}\boldsymbol{i} - \frac{1}{\sqrt{101}}\boldsymbol{j}$$

Compute the magnitude of the following vectors.

I. $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$

II. $\boldsymbol{b} = 2\boldsymbol{i} - 7\boldsymbol{j}$

III. $\boldsymbol{c} = \frac{1}{\sqrt{3}}\boldsymbol{i} - \frac{1}{\sqrt{3}}\boldsymbol{j}$

Find the unit vectors in the direction of the following vectors and express each vector as product of its length and direction.

A. $\boldsymbol{i} - 3\boldsymbol{j}$

B. $-3\boldsymbol{i} - \sqrt{5}\boldsymbol{j}$

C. $-\boldsymbol{i}$

D. $-5\boldsymbol{j}$

Find the slopes of the unit vector Tangents and Normals to the following curves at given points.

a. $y = x^3$: $(1,1)$

b. $y = x^2$: $(0,0)$

c. $y = 3x^4 - 4x$: at $x = 4$

d. $y = x^3 - x + 1$ at the point whose $x -$ coordinate is 2

Find the unit vector tangents and Normals to the following curves at the given points.

i. $y = x^3 + 4x^2$: $(-1,3)$

ii. $xy = 10$: $(2,5)$

iii. $y = 5x^4$: $(1,5)$

4.4

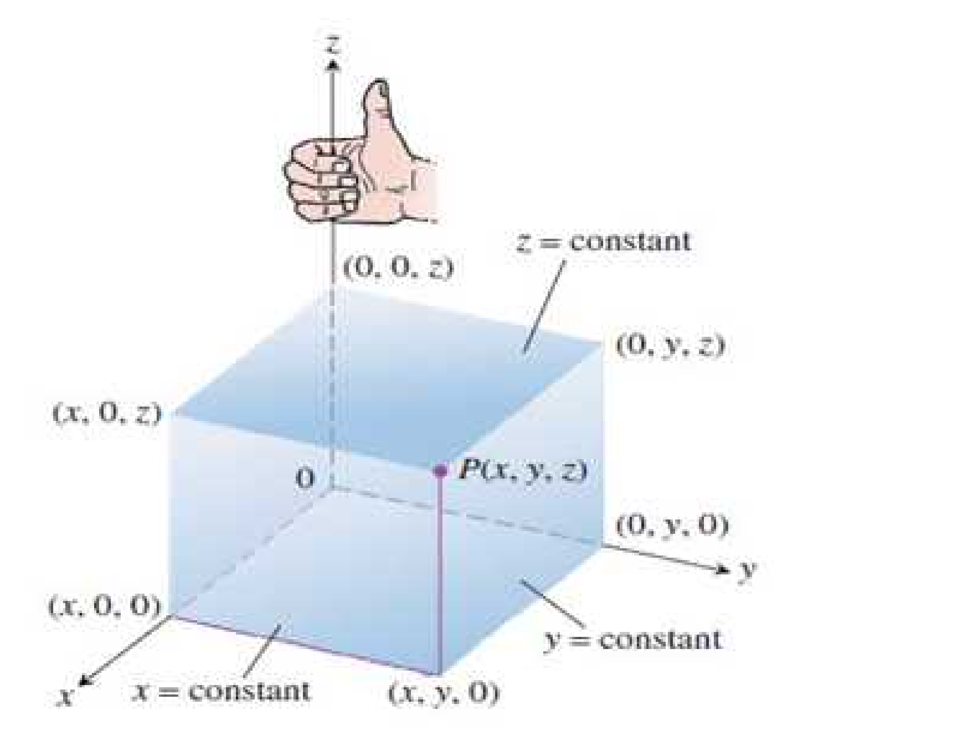
Cartesian Coordinates

Learning Objectives:

- To introduce the three-dimensional rectangular Cartesian coordinate system and to describe the points in space.
- To write verbal descriptions of defining inequalities or equations in x, y and z.

In this module we describe the three-dimensional Cartesian coordinate system and learn about the space around us. This means defining distance, practicing with the arithmetic of vectors in space, and making connections between sets of points and equations and inequalities.

To locate points in space, we use three mutually perpendicular coordinate axes, arranged as in figure below.



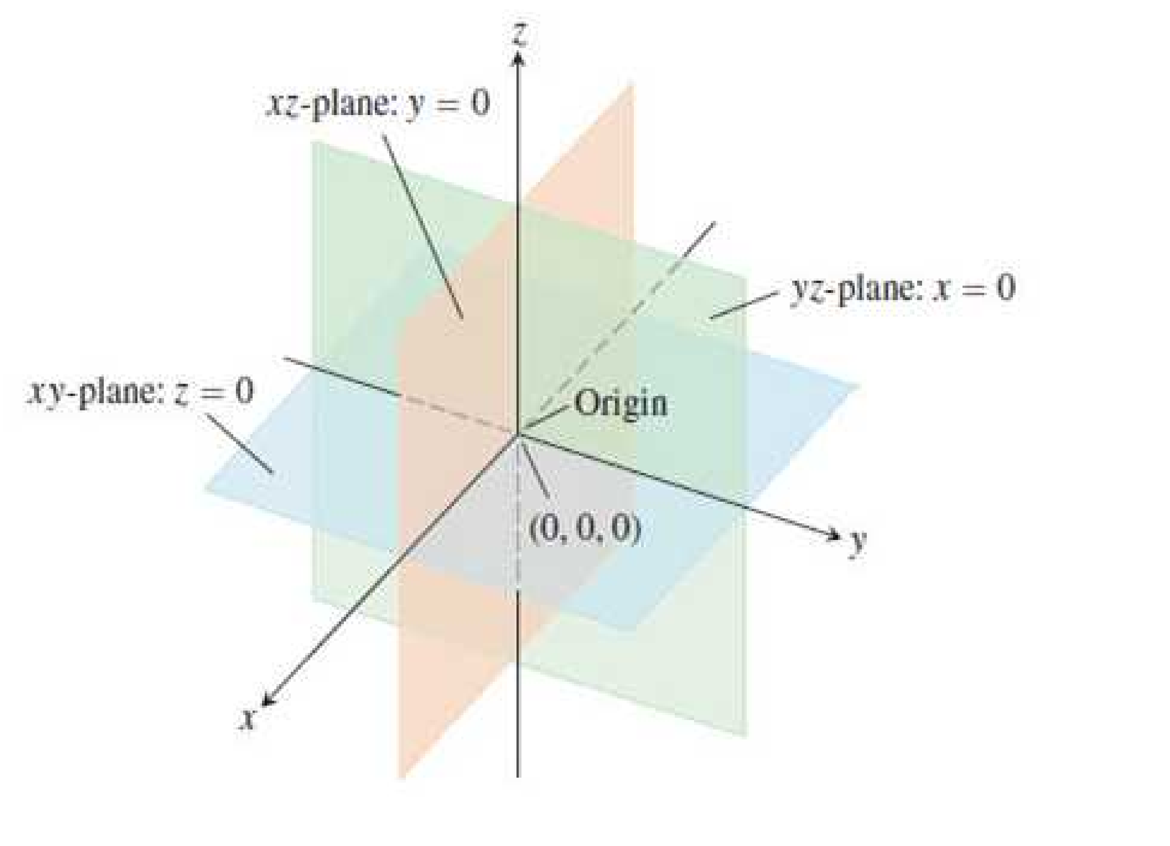
The axes Ox, Oy, and Oz make a ***right-handed*** coordinate frame. When you hold your right-hand so that the fingers curl from the positive x-axis toward the positive y-axis, your thumb points along the positive z-axis.

The Cartesian coordinates (x, y, z) of a point P in space are the numbers at which the planes through P perpendicular to the axes cut the axes.

Cartesian coordinates for space are also called **rectangular coordinates** because the axes that define them meet at right angles.

Points on the x-axis have y- and z-coordinates equal to zero. That is, they have coordinates of the form $(x, 0,0)$. Similarly, points on the y-axis have coordinates of the form $(0, y, 0)$. Points on the z-axis have coordinates of the form $(0,0, z)$.

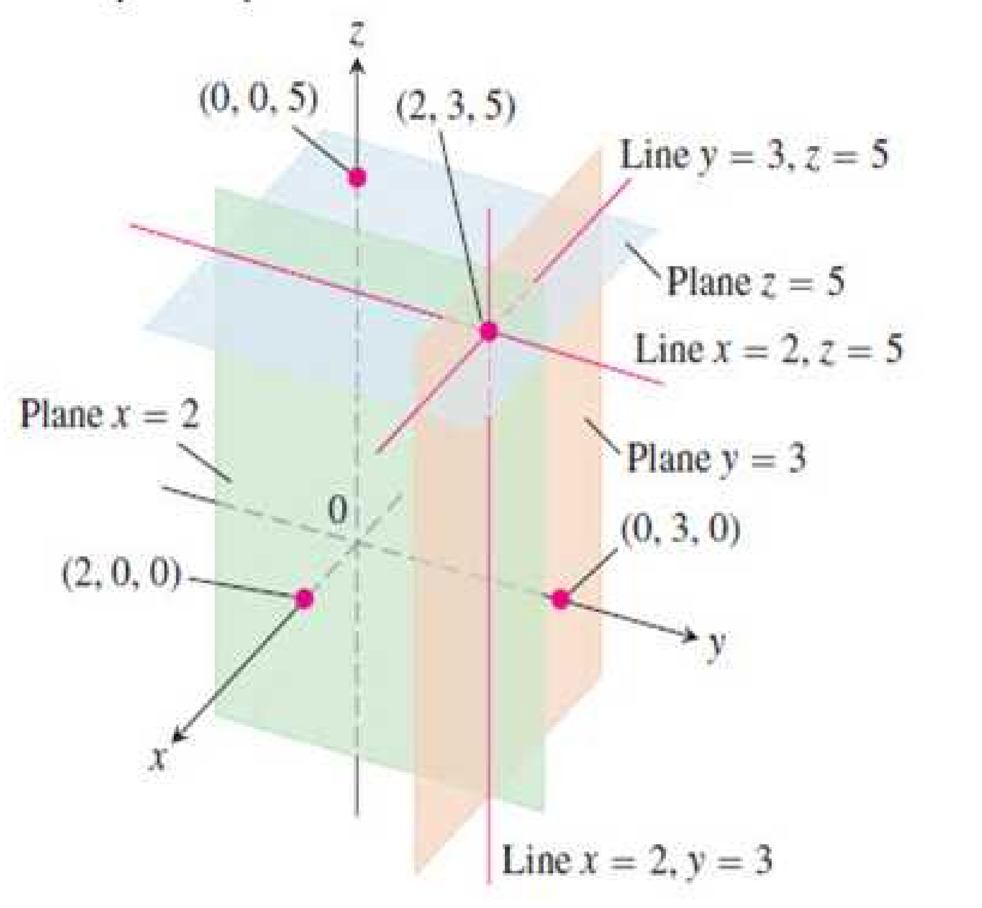
The planes determined by the coordinate axes are the $\boldsymbol{xy}$**-plane**, whose standard equation is $z = 0$; the $\boldsymbol{yz}$**-plane**, whose standard equation is $x = 0$; and the $\boldsymbol{xz}$**-plane**, whose standard equation is $y = 0$. They meet at the **origin**$(0,0,0)$.



The three **coordinate planes** $x = 0, y = 0$, and $z = 0$ divide space into eight cells called **octants**. The octant in which the point coordinates are all positive is called the **first octant**; there is no conventional numbering for the other seven octants.

The points in a plane perpendicular to the x-axis all have the same x-coordinate, this being the number at which that plane cuts the x-axis. The y- and z-coordinates can be any numbers. Similarly, the points in a plane perpendicular to the y-axis all have a common y-coordinate and the points in a plane perpendicular to the z-axis all have a common z-coordinate.

To write equations for these planes, we name the common coordinate's value. The plane $x = 2$ is the plane perpendicular to the x-axis at $x = 2$. The plane $y = 3$ is the plane perpendicular to the y-axis at $y = 3$. The plane $z = 5$ is the plane perpendicular to the z-axis at $z = 5$. Figure below shows the planes $x = 2, y = 3$, and $z = 5$ together with their intersection point $(2,3,5)$.



The planes $x = 2$ and $y = 3$ in figure above intersect in a line parallel to the z-axis. This line is described by the *pair* of equations $x = 2$, $y = 3$. A point (x, y, z) lies on the line if and only if $x = 2$ and $y = 3$. Similarly, the line of intersection of the planes $y = 3$ and $z = 5$ is described by the equation pair $y = 3, z = 5$. This line runs parallel to the x-axis. The line of intersection of the planes $x = 2$ and $z = 5$, parallel to the y-axis, is described by the equation pair $x = 2$, $z = 5$.

In the following examples, we match coordinate equations and inequalities with the set of points they define in space.

Example 1

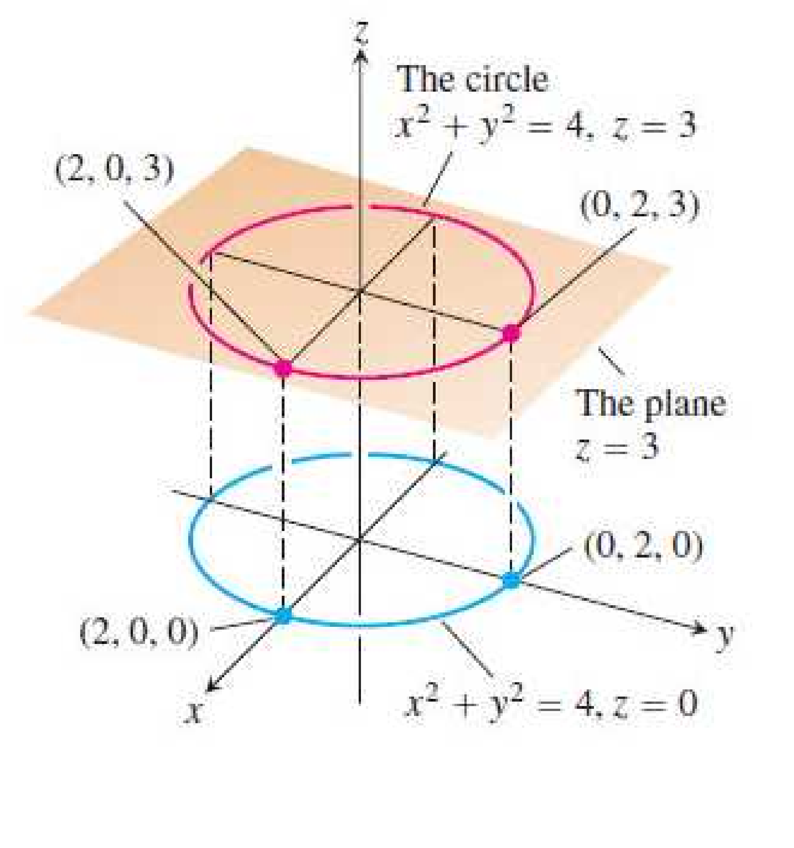
Defining equations and inequalities	Verbal description
$z \geq 0$	The half-space consisting of the points on and above the xy-plane.
$x = -3$	The plane perpendicular to the x-axis at $x = -3$. This plane lies parallel to the yz-plane and 3 units behind it.
$z = 0, x \leq 0, y \geq 0$	The second quadrant of the xy-plane.
$x \geq 0, y \geq 0, z \geq 0$	The first octant.
$-1 \leq y \leq 1$	The slab between the planes $y = -1$ and $y = 1$ (planes included)
$y = -2 , z = 2$	The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2,2)$ parallel to the x-axis.

Example 2

What points $P(x, y, z)$ satisfy the equations

$$x^2 + y^2 = 4 \quad \text{and} \quad z = 3\,?$$

Solution The points lie in the horizontal plane $z = 3$ and, in this plane, make up the circle $x^2 + y^2 = 4$. We call this set of points "the circle $x^2 + y^2 = 4$ in the plane $z = 3$" or, more simply, "the circle $x^2 + y^2 = 4 , z = 3$".



IP1:

Give a geometric description of the set of points in space whose coordinates satisfy the given pair of equations.

$$x^2 + y^2 = 4 \quad , \qquad z = -2$$

Solution:

We have pair of equations: $x^2 + y^2 = 4 \quad , z = -2$

It is the set of points on the circle $x^2 + y^2 = 4$ in the plane $z = -2$.

IP2:

Give a geometric description of the set of points in space whose coordinates satisfy the given pair of equations.

$$x^2 + y^2 + z^2 = 25 \quad , \qquad y = -4$$

Solution:

We have pair of equations: $x^2 + y^2 + z^2 = 25 \quad , \ y = -4$

When $y = -4$ we have $x^2 + z^2 = 9$

We make the circle $x^2 + z^2 = 9$ in the vertical plane $y = -4$.

We call this as "the circle $x^2 + z^2 = 9$ in the plane $y = -4$".

IP3:

Describe the set of points in space whose coordinates satisfy the given inequalities.

a. $0 \le x \le 1$
b. $0 \le x \le 1$, $0 \le y \le 1$
c. $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

Solution:

a. We have the inequality: $0 \le x \le 1$

Verbal description: The slab bounded between the planes $x = 0$ and $x = 1$ and included these planes.

b. We have the combination of inequalities:

$$0 \le x \le 1 , \qquad 0 \le y \le 1$$

Verbal description: The square column bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 1.$

c. We have the combination of inequalities:

$$0 \le x \le 1 , \qquad 0 \le y \le 1 , \qquad 0 \le y \le 1$$

Verbal description: The unit cube in the first octant having one vertex at the origin.

IP4:

Describe the given set with a single equation or with a pair of equations.

The circle of radius 2 centered at (0, 0, 0) and lying in the

a. xy-plane b. yz-plane c. xz-plane

Solution:

a. The circle of radius 2 centered at (0, 0, 0) and lying in the xy-plane is $x^2 + y^2 = 4$, $z = 0$.
b. The circle of radius 2 centered at (0, 0, 0) and lying in the yz-plane is $y^2 + z^2 = 4$, $x = 0$.
c. The circle of radius 2 centered at (0, 0, 0) and lying in the xz-plane is $x^2 + z^2 = 4$, $y = 0$.

P1:

Give a geometric description of the set of points in space whose coordinates satisfy the given pair of equations.

$$x^2 + z^2 = 4 \quad , \qquad y = 0$$

Solution:

We have pair of equations: $x^2 + z^2 = 4 \quad , \; y = 0$

The points lie in the xz-plane and on the circle $x^2 + z^2 = 4$ in it.

That is, the set of points of the circle $x^2 + z^2 = 4$ in the xz-plane.

P2:

Give a geometric description of the set of points in space whose coordinates satisfy the given pair of equations.

$$x^2 + y^2 + (z+3)^2 = 25 \quad , \qquad z = 0$$

Solution:

We have pair of equations: $x^2 + y^2 + (z+3)^2 = 25$, $z = 0$

When $z = 0$ we have $x^2 + y^2 = 16$

It is the set of points on the circle $x^2 + y^2 = 16$ in the xy-plane.

P3:

Describe the set of points in space whose coordinates satisfy the combination of inequalities and equations.

a. $x^2 + y^2 \leq 1$, $z = 0$
b. $x^2 + y^2 \leq 1$, $z = 3$
c. $x^2 + y^2 \leq 1$, no restriction on z

Solution:

a. We have $x^2 + y^2 \leq 1$, $z = 0$

Verbal description: The circumference and interior of the circle $x^2 + y^2 = 1$ in the xy-plane.

b. We have $x^2 + y^2 \leq 1$, $z = 3$

Verbal description: The circumference and interior of the circle $x^2 + y^2 = 1$ in the plane $z = 3$.

c. We have $x^2 + y^2 \leq 1$, no restriction on z

Verbal description: A solid cylindrical column of radius 1 whose axis is the z-axis.

P4:

Describe the given set with a single equation or with a pair of equations.

The circle of radius 2 centered at (0, 2, 0) and lying in the

a. xy-plane b. yz-plane c. plane $y = 2$

Solution:

a. The circle of radius 2 centered at (0, 2, 0) and lying in the xy-plane is $x^2 + (y-2)^2 = 4$, $z = 0$.

b. The circle of radius 2 centered at (0, 2, 0) and lying in the yz-plane is $(y-2)^2 + z^2 = 4$, $x = 0$.

c. The circle of radius 2 centered at (0, 2, 0) and lying in the xz-plane is $x^2 + z^2 = 4$, $y = 2$.

1. Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations.

a. $x = 2, y = 3$

b. $x = -1, z = 0$

c. $y = 0, z = 0$

d. $x = 1, y = 0$

e. $x^2 + y^2 = 4$, $z = 0$

f. $y^2 + z^2 = 1$, $x = 0$

g. $x^2 + y^2 + z^2 = 1$, $x = 0$

h. $x^2 + (y - 1)^2 + z^2 = 4$, $y = 0$

2. Describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequations.

a.

i. $x \geq 0 , y \geq 0 , z = 0$

ii. $x \geq 0 , y \leq 0 , z = 0$

b.

i. $x^2 + y^2 + z^2 \leq 1$

ii. $x^2 + y^2 + z^2 > 1$

c.

i. $x^2 + y^2 + z^2 = 1 , z \geq 0$

ii. $x^2 + y^2 + z^2 \leq 1 , z \geq 0$

d.

i. $x = y , z = 0$

ii. $x = y$, no restriction on z

3. Describe the given set with a single equation or with a pair of equations.

   a. The plane perpendicular to the
      i. x-axis at (3,0,0)
      ii. y-axis at (0,-1,0)
      iii. z-axis at (0,0,-2)

   b. The plane through the point (3,-1,2) perpendicular to the
      i. x-axis
      ii. y-axis
      iii. z-axis

   c. The plane through the point (3,-1,1) parallel to the
      i. xy-plane
      ii. yz-plane
      iii. xz-plane

   d. The circle of radius 1 centered at (-3,4,1) and lying in a plane parallel to the
      i. xy-plane
      ii. yz-plane
      iii. xz-plane

   e. The line through the point (1,3,-1) parallel to the
      i. x-axis
      ii. y-axis
      iii. z-axis

   f. The set of points in space equidistant from the origin and the point (0,2,0).

   g. The circle in which the plane through the point (1,1,3) perpendicular to the z-axis meets the sphere of radius 5 centered at the origin.

   h. The set of points in space that lie 2 units from the point (0,0,1) and , at the same time, 2 units from the point (0,0,-1).

4. Write inequalities to describe the sets given below.

a. The slab bounded by the planes $z = 0$ and $z = 1$ (planes included).

b. The solid cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$.

c. The half-space consisting of the points on and below the xy-plane.

d. The upper hemisphere of the sphere of radius 1 centered at the origin.

e. The (i) interior and (ii) exterior of the sphere of radius 1 centered at the point (1,1,1).

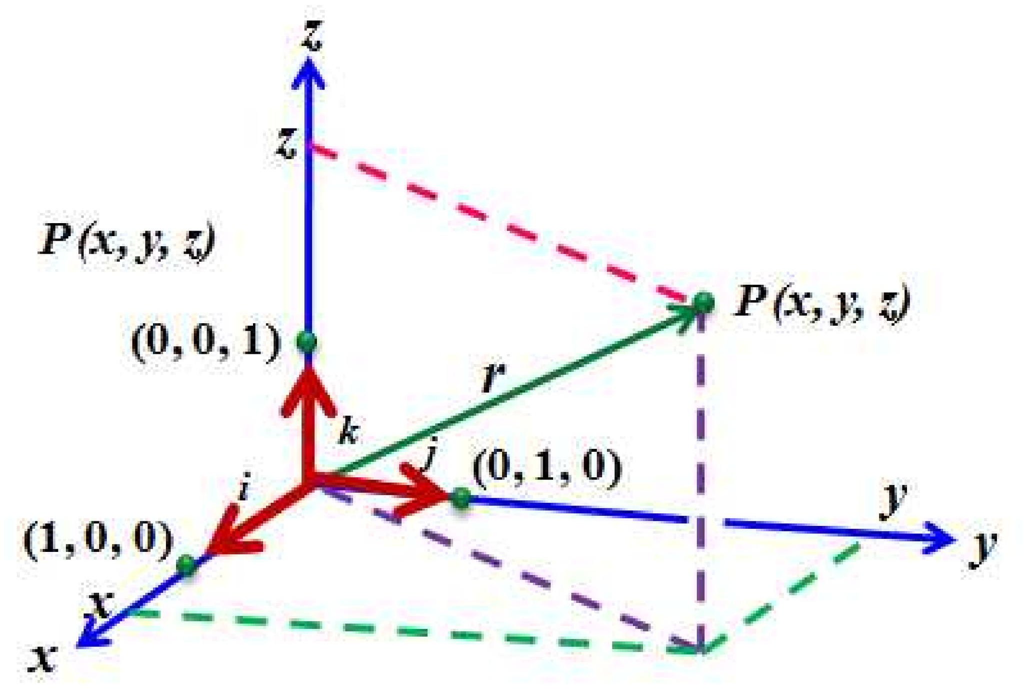
4.5
Vectors in Space

Learning objectives:

1) To introduce the concept of the position vector of a point in space.
2) To define the magnitude or length of a vector.
3) To define addition, subtraction and scalar multiplication of vectors in the space.
   And
4) To practice the related problems.

Just as in the plane, a directed line segment in space is called a ***vector***. The same rules of addition, subtraction, and scalar multiplication apply.

The vectors represented by the directed line segments from the origin to the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ are the **basic vectors.**



We denote them by $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. The **position vector** r from the origin O to the typical point $P(x, y, z)$ is

$$r = \overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \qquad (1)$$

Definition
Addition and Subtraction for Vectors in Space

For any vectors $\boldsymbol{A} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and

$$\boldsymbol{B} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

$$\boldsymbol{A} + \boldsymbol{B} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$$

$$\boldsymbol{A} - \boldsymbol{B} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k} \qquad (2)$$

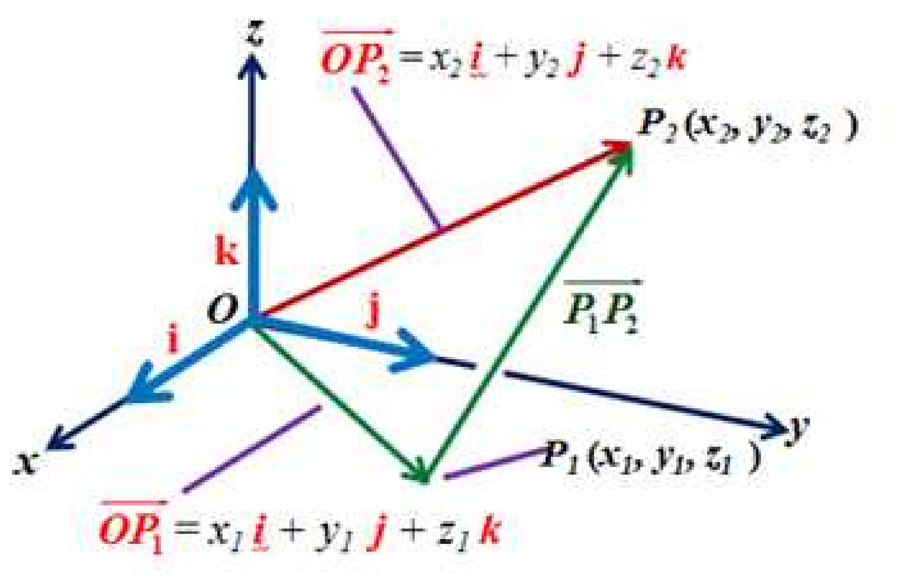
The Vector between Two Points

We can express the vector $\overrightarrow{P_1P_2}$ from the point $P_1(x_1,\ y_1, z_1)$ to the point $P_2(x_2,\ y_2, z_2)$ in terms of the coordinates of P_1 and P_2 because

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$= (x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}) - (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k})$$

$$= (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}$$

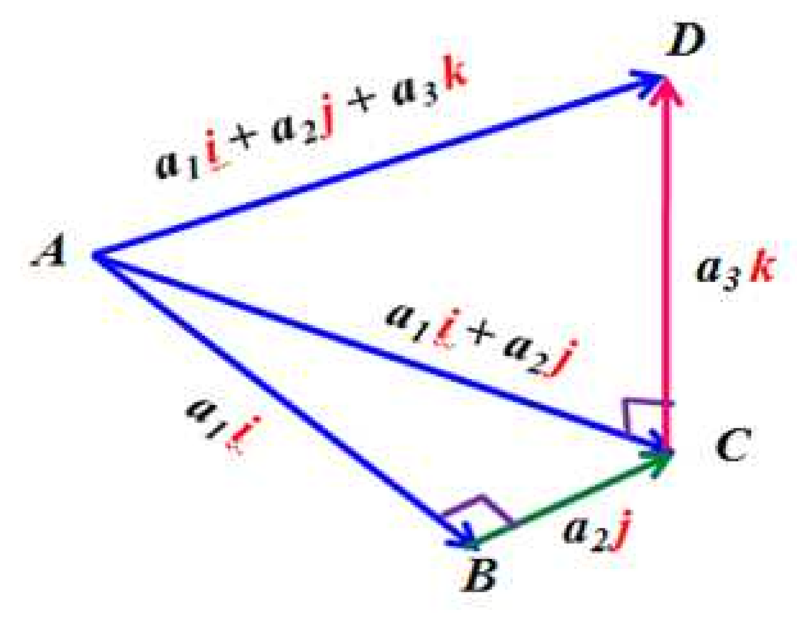


The vector from point $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k} \qquad (3)$$

Magnitude

The important features of a vector are its magnitude and direction. We find a formula for the magnitude (length) of $a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ by applying the Pythagorean theorem to the right triangles in figure below.



From triangle ABC, $|\overrightarrow{AC}| = \sqrt{a_1{}^2 + a_2{}^2}$

and from triangle ACD,

$$|\overrightarrow{AD}| = |a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}|$$

$$|\overrightarrow{AD}| = \sqrt{|\overrightarrow{AC}|^2 + |\overrightarrow{CD}|^2} = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$$

The magnitude (length) of $\boldsymbol{A} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ is

$$|\boldsymbol{A}| = |a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2} \qquad (4)$$

Scalar Multiplication

Definition

If c is a scalar and $\boldsymbol{A} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ is a vector, then

$$c\boldsymbol{A} = (ca_1)\boldsymbol{i} + (ca_2)\boldsymbol{j} + (ca_3)\boldsymbol{k}$$

Example 1 The length of $\vec{A} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ is

$$|\boldsymbol{A}| = \sqrt{1^2 + (-2)^2 + 3^2}$$

$$= \sqrt{1 + 4 + 9} = \sqrt{14}$$

If we multiply $\boldsymbol{A} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ by a scalar c, the length of cA is $|c|$ times the length of A, as in the plane. The reason is the same, as well: $c\boldsymbol{A} = (ca_1)\boldsymbol{i} + (ca_2)\boldsymbol{j} + (ca_3)\boldsymbol{k}$

$$|c\boldsymbol{A}| = \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2}$$

$$= \sqrt{c^2(a_1{}^2 + a_2{}^2 + a_3{}^2)}$$

$$= |c|\sqrt{(a_1{}^2 + a_2{}^2 + a_3{}^2)} = |c|\ |\boldsymbol{A}| \quad (5)$$

Example 2

If $\boldsymbol{A}$ is the vector of example 1, then the length of

$2\boldsymbol{A} = 2(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) = 2\boldsymbol{i} - 4\boldsymbol{j} + 6\boldsymbol{k}$ is

$$\sqrt{(2)^2 + (-4)^2 + (6)^2} = \sqrt{4 + 16 + 36} = \sqrt{56}$$

$$= \sqrt{4.14} = 2\sqrt{14} = 2\,|\vec{A}|$$

P1:

Find the vector $\overrightarrow{AB}$

if A is the point $(4,0,-2)$ and B is the point $(4,2,1)$

Solution:

We have $\overrightarrow{OA} = 4\boldsymbol{i} - 2\boldsymbol{j}$

$$\overrightarrow{OB} = 4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) - (4\boldsymbol{i} + 2\boldsymbol{k})$$

$$= (4-4)\boldsymbol{i} + (2-0)\boldsymbol{j} + (1-2)\boldsymbol{k}$$

$$= 2\boldsymbol{j} - \boldsymbol{k}$$

P2:

Find $5\boldsymbol{A} + 2\boldsymbol{B}$ if $\boldsymbol{A} = 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{B} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$

Solution:

Given,

$$\boldsymbol{A} = 3\boldsymbol{j} + \boldsymbol{k} \text{ and } \boldsymbol{B} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$$

$$5\boldsymbol{A} = 5(3\boldsymbol{j} + \boldsymbol{k}) = 15\boldsymbol{j} + 5\boldsymbol{k}$$

$$2\boldsymbol{B} = 2(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) = 4\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$$

$$5\boldsymbol{A} + 2\boldsymbol{B} = 15\boldsymbol{j} + 5\boldsymbol{k} + 4\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k} = 4\boldsymbol{i} + 21\boldsymbol{j} + 3\boldsymbol{k}$$

P3:

Find $|2\boldsymbol{a} + 3\boldsymbol{b}|$ where $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k},\ \boldsymbol{b} = 2\boldsymbol{j} - \boldsymbol{k}$

Solution:

Given, $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$

$$\boldsymbol{b} = 2\boldsymbol{j} - \boldsymbol{k}$$

$$2\boldsymbol{a} = 2(2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}\,) = 4\boldsymbol{i} + 8\boldsymbol{j} + 8\boldsymbol{k}$$

$$3\boldsymbol{b} = 3(2\boldsymbol{j} - \boldsymbol{k}) = 6\boldsymbol{j} - 3\boldsymbol{k}$$

$$2\boldsymbol{a} + 3\boldsymbol{b} = 4\boldsymbol{i} + 8\boldsymbol{j} + 8\boldsymbol{k} + 6\boldsymbol{j} - 3\boldsymbol{k}$$

$$= 4\boldsymbol{i} + 14\boldsymbol{j} + 5\boldsymbol{k}$$

$$|2\boldsymbol{a} + 3\boldsymbol{b}| = \sqrt{4^2 + 14^2 + 5^2} = \sqrt{16 + 196 + 25} = 237$$

P4:

Is the sides of the triangle formed by the vectors

$$3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}, \;\; 2\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k} \text{ and } -5\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$$

equilateral triangle?

Solution:

Let

$$\boldsymbol{a} = 3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}\,,\ \boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}\ \text{ and } \boldsymbol{c} = -5\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$$

$$|\boldsymbol{a}| = \sqrt{9 + 25 + 4} = \sqrt{38}$$

$$|\boldsymbol{b}| = \sqrt{4 + 9 + 25} = \sqrt{38}$$

$$|\boldsymbol{c}| = \sqrt{25 + 4 + 9} = \sqrt{38}$$

$$\therefore |\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$$

Thus, the given triangle is an equilateral triangle.

IP1:

Find the vector $\overrightarrow{BA}$,

if A is the point $(3, -1, -2)$ and B is the point $(3,1,1)$

Solution:

We have $\overrightarrow{OA} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$

$$\overrightarrow{OB} = 3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

$$\begin{aligned}\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} &= (3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) - (3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \\ &= (3-3)\boldsymbol{i} + (-1-1)\boldsymbol{j} + (-2-1)\boldsymbol{k} \\ &= -2\boldsymbol{j} - 3\boldsymbol{k}\end{aligned}$$

IP2:

Find $3\boldsymbol{A} - 2\boldsymbol{B}$ if $\boldsymbol{A} = 4\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{B} = \boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$

Solution:

Given,

$$\boldsymbol{A} = 4\boldsymbol{i} - 3\boldsymbol{j} \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$$

$$3\boldsymbol{A} = 3(4\boldsymbol{i} - 3\boldsymbol{j}) = 12\boldsymbol{i} - 9\boldsymbol{j}$$

$$2\boldsymbol{B} = 2(\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}) = 2\boldsymbol{i} - 4\boldsymbol{j} - 6\boldsymbol{k}$$

$$3\boldsymbol{A} - 2\boldsymbol{B} = (12\boldsymbol{i} - 9\boldsymbol{j}) - (2\boldsymbol{i} - 4\boldsymbol{j} - 6\boldsymbol{k}) = 10\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$$

IP3:

Find $|3\boldsymbol{a} - 2\boldsymbol{b}|$ where $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k},\ \boldsymbol{b} = 2\boldsymbol{j} - \boldsymbol{k}$

Solution:

Given, $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$

$$\boldsymbol{b} = 2\boldsymbol{j} - \boldsymbol{k}$$

$$3\boldsymbol{a} = 3(2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}\,) = 6\boldsymbol{i} + 12\boldsymbol{j} + 12\boldsymbol{k}$$

$$2\boldsymbol{a} = 2(2\boldsymbol{j} - \boldsymbol{k}) = 4\boldsymbol{j} - 2\boldsymbol{k}$$

$$3\boldsymbol{a} - 2\boldsymbol{b} = 6\boldsymbol{i} + 12\boldsymbol{j} + 12\boldsymbol{k} - (4\boldsymbol{j} - 2\boldsymbol{k})$$

$$= 6\boldsymbol{i} + 12\boldsymbol{j} + 12\boldsymbol{k} - 4\boldsymbol{j} + 2\boldsymbol{k} = 6\boldsymbol{i} + 8\boldsymbol{j} + 14\boldsymbol{k}$$

$$|3\boldsymbol{a} - 2\boldsymbol{b}| = \sqrt{6^2 + 8^2 + 14^2} = \sqrt{36 + 64 + 196} = 296$$

IP4:

Using vectors, Show that

$P(1,-3,-2), Q(2,0,-4)$, and $R(6,-2,-5)$

are vertices of a right angled triangle.

Solution:

$$P = (1,-3,-2)$$

$$Q = (2,0,-4)$$

$$R = (6,-2,-5)$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (2\boldsymbol{i} - 4\boldsymbol{k}) - (\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}) = (\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k})$$

$$|\overrightarrow{PQ}| = \sqrt{1+9+4} = \sqrt{14}$$

$$\overrightarrow{QR} = (6-2)\boldsymbol{i} + (-2-0)\boldsymbol{j} + (-5+4)\boldsymbol{k} = 4\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$$

$$|\overrightarrow{QR}| = \sqrt{16+4+1} = \sqrt{21}$$

$$\overrightarrow{PR} = (6-1)\boldsymbol{i} + (-2+3)\boldsymbol{j} + (-5+2)\boldsymbol{k} = 5\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$$

$$|\overrightarrow{PR}| = \sqrt{25+1+9} = \sqrt{35}$$

Notice that $|\overrightarrow{PR}|^2 = |\overrightarrow{PQ}|^2 + |\overrightarrow{QR}|^2$

$\therefore PQR$ is a right angled triangle.

1. Find vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ if the points

i) $A(1,2,2),\ B(4,0,-3)$

ii) $A(-3,4,0),\ B(2,-1,3)$

2. Find $\boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{A} - \boldsymbol{B}$ when

I) $\boldsymbol{A} = 3\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}, \;\; \boldsymbol{B} = \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$

II) $\boldsymbol{A} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}, \quad \boldsymbol{B} = -3\boldsymbol{i} - 9\boldsymbol{j} + 6\boldsymbol{k}$

3. Find $|2\boldsymbol{a} + 3\boldsymbol{b}|, |\boldsymbol{a} - 2\boldsymbol{b}|$ when

$(i)\ \boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k},\ \boldsymbol{b} = -2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$

$(ii)\boldsymbol{a} = 2\boldsymbol{i} - 4\boldsymbol{j} + 4\boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{k}$

4. If $\overrightarrow{OA} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{AB} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}, \overrightarrow{BC} = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$ and $\overrightarrow{CD} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ then find the vector $\overrightarrow{OD}$

4.6

Unit Vectors, Distance and Midpoints

Learning objectives:

- To express a vector as a product of its magnitude and direction.
- To define the distance between two given points in space and to find the midpoint of the line segment joining two given points.
- To derive the standard equation of a sphere with a given center and radius.

  AND
- To practice the related problems.

The zero Vector

The **zero vector** in space is the vector $\boldsymbol{0} = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$. As in the plane, $\boldsymbol{0}$ has zero length, and no direction.

Unit Vectors

A **unit vector** in space is a vector of length 1. The basic vectors are unit vectors because

$$|\boldsymbol{i}| = |1\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

$$|\boldsymbol{j}| = |0\boldsymbol{i} + 1\boldsymbol{j} + 0\boldsymbol{k}| = \sqrt{0^2 + 1^2 + 0^2} = 1$$

$$|\boldsymbol{k}| = |0\boldsymbol{i} + 0\boldsymbol{j} + 1\boldsymbol{k}| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

Magnitude and Direction

If $\boldsymbol{A} \neq \boldsymbol{0}$, then

(i) $\boldsymbol{A}/|\boldsymbol{A}|$ is a unit vector in the direction of $\boldsymbol{A}$ and

(ii) $\boldsymbol{A}$ is expressed as a product of its magnitude and direction

$$\boldsymbol{A} = |\boldsymbol{A}|\frac{\boldsymbol{A}}{|\boldsymbol{A}|}$$

Example 1

Express $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ as a product of its magnitude and direction.

Solution

$$\begin{aligned} \boldsymbol{A} &= |\boldsymbol{A}| \cdot \frac{\boldsymbol{A}}{|\boldsymbol{A}|} \\ &= \sqrt{14} \cdot \frac{\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}}{\sqrt{14}} \\ &= \sqrt{14}\left(\frac{1}{\sqrt{14}}\boldsymbol{i} - \frac{2}{\sqrt{14}}\boldsymbol{j} + \frac{3}{\sqrt{14}}\boldsymbol{k}\right) \\ &= (\text{length of } \boldsymbol{A}) \cdot (\text{direction of } \boldsymbol{A}) \end{aligned}$$

Example 2

Find a unit vector $\boldsymbol{u}$ in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution: We divide $\overrightarrow{P_1P_2}$ by its length, where

$$\overrightarrow{P_1P_2} = (3-1)\boldsymbol{i} + (2-0)\boldsymbol{j} + (0-1)\boldsymbol{k} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\boldsymbol{u} = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}}{3} = \frac{2}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} - \frac{1}{3}\boldsymbol{k}$$

Example 3

Find a vector of length 6 units in the direction of $\boldsymbol{A} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

Solution

The vector we want is

$$6\frac{\boldsymbol{A}}{|\boldsymbol{A}|} = 6\frac{2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}}{3} = 4\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$$

Distance in Space

The distance between two points P_1 and P_2 in space is the length of $\overrightarrow{P_1P_2}$.

The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad \text{...... (1)}$$

Example 4

The distance between $P_1(2,1,5)$ and $P_2(-2,3,0)$ is

$$\begin{aligned} \left|\overrightarrow{P_1P_2}\right| &= \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2} \\ &= \sqrt{16+4+25} = \sqrt{45} = 3\sqrt{5} \end{aligned}$$

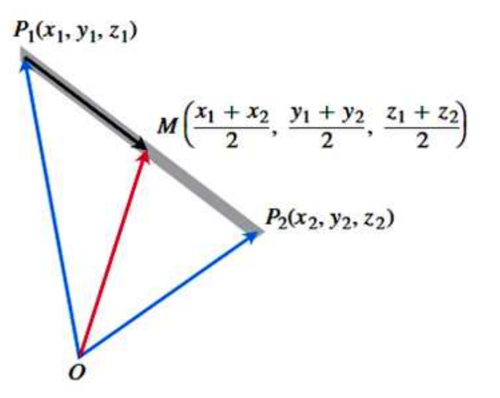
Midpoints

The coordinates of the midpoint of a line segment are found by averaging.

The midpoint M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

We can see this from the following. From the figure below



$$\begin{aligned} \overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2} \\ &= \overrightarrow{OP_1} + \frac{1}{2}\left(\overrightarrow{OP_2} - \overrightarrow{OP_1}\right) = \frac{1}{2}\left(\overrightarrow{OP_1} + \overrightarrow{OP_2}\right) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k} \end{aligned}$$

Example 5

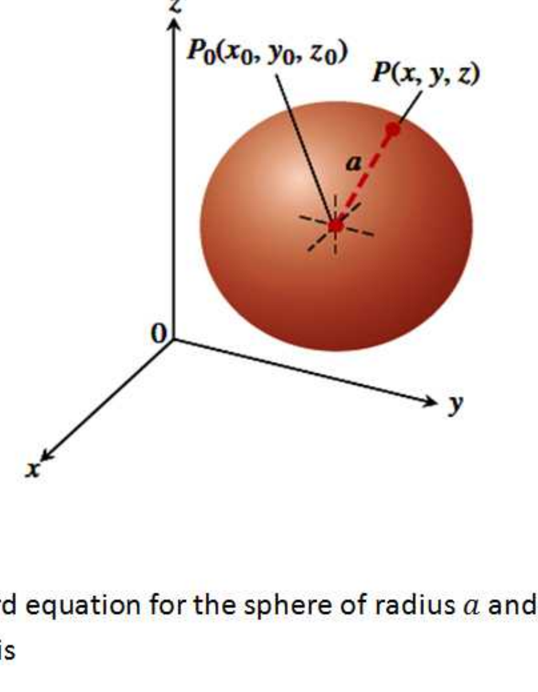
The midpoint of the segment joining $P_1(3,-2,0)$ and $P_2(7,4,4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right) = (5,1,2).$$

Spheres

We use the distance formula (1) to write equations for spheres in space. A point $P(x,y,z)$ lies on the sphere of radius a centered at $P_0(x_0, y_0, z_0)$ precisely when $|\overrightarrow{P_0P}| = a$ or

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$



The standard equation for the sphere of radius a and center (x_0, y_0, z_0) is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2 \quad \text{........ (2)}$$

Example 6

Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

Solution

We find the center and radius of a sphere the way we find the center and radius of a circle: Complete the squares on the x- y- and z-terms as necessary and write each quadratic as a squared linear expression. Then, from the equation in standard form, read off the center and radius. For the sphere here, we have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 = \frac{21}{4}$$

This is equation (2) with $x_0 = -\frac{3}{2}, y_0 = 0,\ z_0 = 0$, and

$a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$.

Example 7

Set bounded by spheres or portions of spheres

Defining equations and inequalities	Description
$x^2 + y^2 + z^2 < 4$	The interior of the sphere $x^2 + y^2 + z^2 = 4$.
$x^2 + y^2 + z^2 \le 4$	The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.
$x^2 + y^2 + z^2 > 4$	The exterior of the sphere $x^2 + y^2 + z^2 = 4$.
$x^2 + y^2 + z^2 = 4,\ z \le 0$	The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy-plane (the plane $z = 0$).

IP1.

Find a unit vector $\boldsymbol{u}$ in the direction of the vector from $\boldsymbol{P_1(1, 2, -3)}$ and $\boldsymbol{P_2(-1, -2, 1)}$.

Solution:

Given $P_1(1, 2, -3)$ and $P_2(-1, -2, 1)$.

Now, $\overrightarrow{P_1P_2} = (-1-1)\boldsymbol{i} + (-2-2)\boldsymbol{j} + (1-3)\boldsymbol{k}$

$$= -2\boldsymbol{i} - 4\boldsymbol{j} + 4\boldsymbol{k}$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(-2)^2 + (-4)^2 + (4)^2} = \sqrt{36} = 6$$

Therefore, the unit vector $\boldsymbol{u}$ in the direction of $\overrightarrow{P_1P_2}$ is

$$\boldsymbol{u} = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{-2\boldsymbol{i}-4\boldsymbol{j}+4\boldsymbol{k}}{6} = -\frac{1}{3}\boldsymbol{i} - \frac{2}{3}\boldsymbol{j} + \frac{2}{3}\boldsymbol{k}$$

IP2.

The position vectors of the points A, B and C are $2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$, $-\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$ and $4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ respectively, then show that the points A, B and C are the vertices of a right angled isosceles triangle.

Solution:

We have

$$\overrightarrow{AB} = (-1-2)\boldsymbol{i} + (5-3)\boldsymbol{j} + (-1-5)\boldsymbol{k} = -3\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}$$

$$\overrightarrow{BC} = (4+1)\boldsymbol{i} + (-3-5)\boldsymbol{j} + (2+1)\boldsymbol{k} = 5\boldsymbol{i} - 8\boldsymbol{j} + 3\boldsymbol{k}$$

$$\overrightarrow{CA} = (2-4)\boldsymbol{i} + (3+3)\boldsymbol{j} + (5-2)\boldsymbol{k} = -2\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k}$$

Now,

$$\left|\overrightarrow{AB}\right| = \sqrt{(-3)^2+(2)^2+(-6)^2} = \sqrt{9+4+36} = 7$$

$$\left|\overrightarrow{BC}\right| = \sqrt{(5)^2+(-8)^2+(3)^2} = \sqrt{25+64+9} = \sqrt{98}$$

$$\left|\overrightarrow{CA}\right| = \sqrt{(-2)^2+(6)^2+(3)^2} = \sqrt{4+36+9} = 7$$

Here $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CA}\right|$, two sides are equal and

$$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 = \left|\overrightarrow{BC}\right|^2$$

Thus, ΔABC is a right angled isosceles triangle.

IP3.

i. **Find the equation of the sphere with center $(\mathbf{1}, \mathbf{2}, \mathbf{3})$ and radius $\sqrt{\mathbf{14}}$.**

Solution:

We know that the equation of the sphere with center (x_0, y_0, z_0) and radius a is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = (a)^2$$

Therefore, the equation of the sphere with center at $(1, 2, 3)$ and radius $\sqrt{14}$ is

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = \left(\sqrt{14}\right)^2$$

i.e., $x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9 = 14$

i.e., $x^2 + y^2 + z^2 - 2x - 4y - 6z = 0$

ii. **Describe the sets of points in space whose coordinates satisfy the combination of the equation and inequality**

$$\boldsymbol{x^2 + y^2 + z^2 \leq 1, \quad z \geq 0}$$

Solution:

The given combination of the equation and inequality $x^2 + y^2 + z^2 \leq 1, \quad z \geq 0$

denotes the solid upper hemisphere of radius 1 centered at origin, cut by the xy −plane (i.e., $z = 0$).

IP4.

Find the midpoint of the line segment joining points $\boldsymbol{P_1(3,4,5)}$ and $\boldsymbol{P_2(2,3,4)}$.

Solution:

The midpoint of the line segment joining points $P_1(3,4,5)$ and $P_2(2,3,4)$ is

$$\left(\frac{3+2}{2},\frac{4+3}{2},\frac{5+4}{2}\right)=\left(\frac{5}{2},\frac{7}{2},\frac{9}{2}\right)$$

P1.

Find a vector in the direction of a vector $\boldsymbol{u} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ that has a magnitude 7.

Solution:

The given vector is $\boldsymbol{u} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$.

Now, $|\boldsymbol{u}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

The unit vector in the direction of the given vector $\boldsymbol{u}$ is

$$\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}}{\sqrt{14}} = \frac{2}{\sqrt{14}}\boldsymbol{i} + \frac{3}{\sqrt{14}}\boldsymbol{j} + \frac{1}{\sqrt{14}}\boldsymbol{k}$$

The vector having magnitude equal to 7 and in the direction of $\boldsymbol{u}$ is

$$7\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = 7\left(\frac{2}{\sqrt{14}}\boldsymbol{i} + \frac{3}{\sqrt{14}}\boldsymbol{j} + \frac{1}{\sqrt{14}}\boldsymbol{k}\right)$$

$$= \left(\frac{14}{\sqrt{14}}\boldsymbol{i} + \frac{21}{\sqrt{14}}\boldsymbol{j} + \frac{7}{\sqrt{14}}\boldsymbol{k}\right)$$

$$= \sqrt{14}\,\boldsymbol{i} + \frac{21}{\sqrt{14}}\boldsymbol{j} + \sqrt{\frac{7}{2}}\,\boldsymbol{k}$$

P2.

The position vectors of the points A, B and C are $2i - j + k$, $-3j - 5k$, $3i - 4j - 4k$ respectively then show that the points A, B and C are the vertices of a right angled triangle.

Solution:

We have

$$\overrightarrow{AB} = (1-2)\boldsymbol{i} + (-3+1)\boldsymbol{j} + (-5-1)\boldsymbol{k} = -\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}$$

$$\overrightarrow{BC} = (3-1)\boldsymbol{i} + (-4+3)\boldsymbol{j} + (-4+5)\boldsymbol{k} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

$$\overrightarrow{CA} = (2-3)\boldsymbol{i} + (-1+4)\boldsymbol{j} + (1+4)\boldsymbol{k} = -\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$$

Now,

$$|\overrightarrow{AB}| = \sqrt{(-1)^2+(-2)^2+(-6)^2} = \sqrt{1+4+36} = \sqrt{41}$$

$$|\overrightarrow{BC}| = \sqrt{(2)^2+(-1)^2+(1)^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$|\overrightarrow{CA}| = \sqrt{(-1)^2+(3)^2+(5)^2} = \sqrt{1+9+25} = \sqrt{35}$$

$$\therefore\ |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 = (\sqrt{6})^2 + (\sqrt{35})^2 = 6+35 = 41 = |\overrightarrow{AB}|^2$$

Hence, ΔABC is a right angled triangle.

P3.

i. Find the center and radius of the sphere

$$2x^2 + 2y^2 + 2z^2 + x + y + z = 9$$

ii. Describe the sets of points in space whose coordinates satisfy the combination of the equation and inequality

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0$$

i.Solution:

The equation of the sphere is

$$2x^2 + 2y^2 + 2z^2 + x + y + z = 9$$

$$\Rightarrow x^2 + y^2 + z^2 + \frac{x}{2} + \frac{y}{2} + \frac{z}{2} = \frac{9}{2}$$

$$\Rightarrow \left[x^2 + \frac{x}{2}\right] + \left[y^2 + \frac{y}{2}\right] + \left[z^2 + \frac{z}{2}\right] = \frac{9}{2}$$

$$\Rightarrow \left[x^2 + 2.x.\frac{1}{4} + \left(\frac{1}{4}\right)^2\right] + \left[y^2 + 2.y.\frac{1}{4} + \left(\frac{1}{4}\right)^2\right] +$$

$$\left[z^2 + 2.z.\frac{1}{4} + \left(\frac{1}{4}\right)^2\right] = \frac{9}{2} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$\Rightarrow \left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \frac{75}{16}$$

$$\Rightarrow \left(x - \left(-\frac{1}{4}\right)\right)^2 + \left(y - \left(-\frac{1}{4}\right)\right)^2 + \left(z - \left(-\frac{1}{4}\right)\right)^2 = \left(\frac{5\sqrt{3}}{4}\right)^2$$

Therefore, the center is $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ and radius is $\frac{5\sqrt{3}}{4}$

ii. Solution:

The given combination of the equation and inequality

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0$$

denotes a closed upper hemisphere of radius 1 centered at origin, cut by the xy $-$plane (i.e., $z = 0$)

P4.

Find the midpoint of the line segment joining points $\boldsymbol{P_1(-1,1,5)}$ and $\boldsymbol{P_2(2,5,0)}$.

Solution:

The midpoint of the line segment joining points $P_1(-1,1,5)$ and $P_2(2,5,0)$ is

$$\left(\frac{-1+2}{2},\frac{1+5}{2},\frac{5+0}{2}\right)=\left(\frac{1}{2},\frac{6}{2},\frac{5}{2}\right)=\left(\frac{1}{2},\ 3,\ \frac{5}{2}\right)$$

EXERCISES

1. Express each vector as a product of its length and direction

   a. $2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

   b. $9\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}$

   c. $5\boldsymbol{k}$

   d. $\frac{1}{\sqrt{6}}\boldsymbol{i} - \frac{1}{\sqrt{6}}\boldsymbol{j} - \frac{1}{\sqrt{6}}\boldsymbol{k}$

   e. $\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$

2. Find the vector whose length and directions are given below

Length	Direction
a. $\frac{1}{2}$	$\frac{3}{5}\boldsymbol{j} + \frac{4}{5}\boldsymbol{k}$
b. 7	$\frac{6}{7}\boldsymbol{i} - \frac{2}{7}\boldsymbol{j} + \frac{3}{7}\boldsymbol{k}$
c. $\sqrt{3}$	$-\boldsymbol{k}$
d. 2	$\boldsymbol{i}$
e. $a > 0$	$\frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{3}}\boldsymbol{j} - \frac{1}{\sqrt{6}}\boldsymbol{k}$
f. 7	$12\boldsymbol{i} - 5\boldsymbol{k}$
g. 3	$\frac{\boldsymbol{i}}{2} - \frac{\boldsymbol{j}}{2} - \frac{\boldsymbol{k}}{2}$

3. Find the distance between the points P_1, P_2 given below.

a. $P_1(1,1,1)$; $P_2(3,3,0)$

b. $P_1(0,0,0)$; $P_2(2,-2,-2)$

c. $P_1(5,3,-2)$; $P_2(0,0,0)$

d. $P_1(3,1,0)$; $P_2(3,3,3)$

4. Find the midpoint of line segment $P_1,\ P_2$

a. $P_1(1,4,5)$; $P_2(4,-2,7)$

b. $P_1(0,0,0)$; $P_2(3,3,0)$

5. Find the equation of the sphere whose centers and radii are given below.

a. $(0,-1,5)$; 2

b. $(-2,0,0)$; $\sqrt{3}$

c. $(0,-7,0)$; 7

6. Find the center and radius of the following spheres.

a. $x^2 + y^2 + z^2 + 4x - 4y = 0$

b. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 0$

c. $x^2 + y^2 + z^2 - 6y + 8z = 0$

7. Describe the set of points in space whose coordinates satisfy the given inequalities.

a. $x^2 + y^2 + z^2 \leq 1$

b. $x^2 + y^2 + z^2 > 1$

4.7

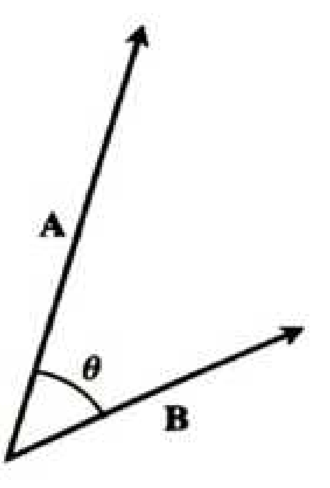
Dot Product

Learning Objectives:

- To define the dot(scalar) product of two vectors
- To derive a formula for the angle between two nonzero vectors
- To study the properties of dot product
- To define the orthogonality of two nonzero vectors AND
- To solve related problems

In this module, we learn the *dot product*, a method of multiplying two vectors.

When two nonzero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are placed so that their initial points coincide, they form an angle θ of measure $0 \leq \theta \leq \pi$. This angle is called the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.



Definition

The dot product (scalar product) $\boldsymbol{A} \cdot \boldsymbol{B}$ of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is the number

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta \text{ -------- (1)}$$

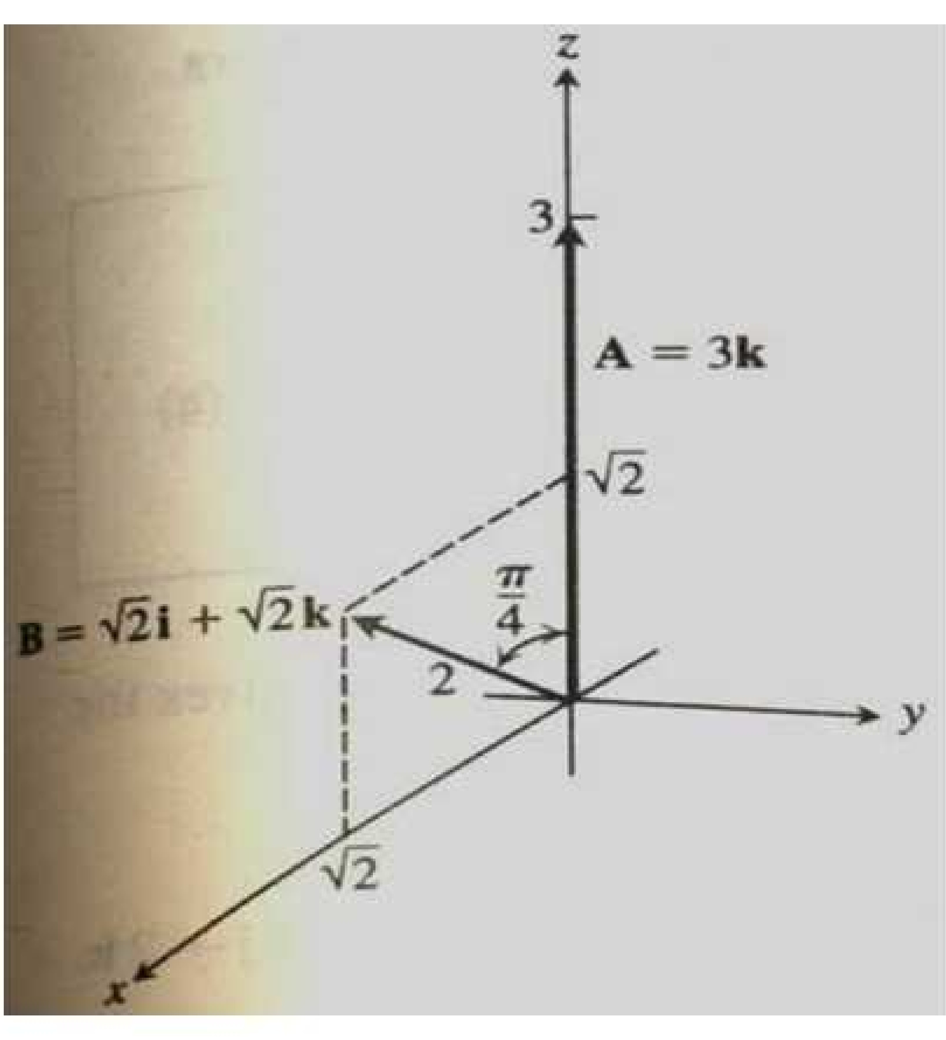
where θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

In words, $\boldsymbol{A} \cdot \boldsymbol{B}$ is the length of $\boldsymbol{A}$ times the length of $\boldsymbol{B}$ times the cosine of the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

Dot product is also called a ***scalar product*** because the multiplication results in a scalar.

Example 1 If $\boldsymbol{A} = 3\boldsymbol{k}$ and $\boldsymbol{B} = \sqrt{2}\boldsymbol{i} + \sqrt{2}\boldsymbol{k}$, then

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta = (3)(2)\cos\frac{\pi}{4} = 6 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}$$



Since the sign of $\boldsymbol{A} \cdot \boldsymbol{B}$ is determined by $\cos\theta$, *the scalar product is positive if the angle between the vectors is acute, negative if the angle is obtuse.*

Since the angle a vector $\boldsymbol{A}$ makes with itself is zero, and $\cos 0 = 1$,

$$\boldsymbol{A} \cdot \boldsymbol{A} = |\boldsymbol{A}||\boldsymbol{A}| \cos\theta = |\boldsymbol{A}||\boldsymbol{A}|(1) = |\boldsymbol{A}|^2$$

$$\Rightarrow |\boldsymbol{A}| = \sqrt{\boldsymbol{A}.\boldsymbol{A}} \text{ ------------------ (2)}$$

Calculation

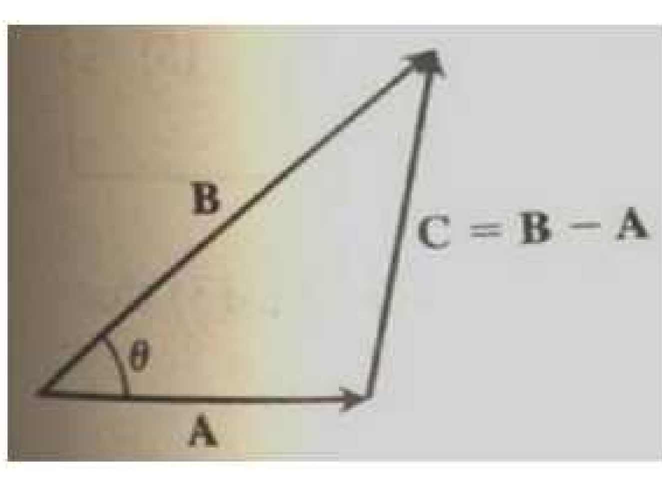
In a Cartesian coordinate system with unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, we let

$$\boldsymbol{A} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$$

$$\boldsymbol{B} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

and

$$\boldsymbol{C} = \boldsymbol{B} - \boldsymbol{A} = (b_1 - a_1)\boldsymbol{i} + (b_2 - a_2)\boldsymbol{j} + (b_3 - a_3)\boldsymbol{k}$$



The law of cosines for the triangle whose sides represent $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ is

$$|\boldsymbol{C}|^2 = |\boldsymbol{A}|^2 + |\boldsymbol{B}|^2 - 2|\boldsymbol{A}||\boldsymbol{B}| \cos\theta$$

$$\Rightarrow |\boldsymbol{A}||\boldsymbol{B}| \cos\theta = \frac{|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2 - |\boldsymbol{C}|^2}{2}$$

The left side of this equation is $\boldsymbol{A} \cdot \boldsymbol{B}$ and the above equation becomes,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \frac{(a_1{}^2 + a_2{}^2 + a_3{}^2) + (b_1{}^2 + b_2{}^2 + b_3{}^2) - [(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2]}{2}$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Therefore, $\boldsymbol{A} \cdot \boldsymbol{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$ ------------ (3)

Thus, to find the scalar product of two given vectors we multiply their corresponding $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$-components and add the results.

Solving equation (1) for θ, we get a formula for finding angles between vectors.

The angle between two nonzero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right) \text{ --------- (4)}$$

Since the values of the *arc cosine* lie in $[0, \pi]$, equation (4) automatically gives the angle made by $\boldsymbol{A}$ and $\boldsymbol{B}$.

Example 2 Find the angles between $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{B} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

Solution We use equation (4):

$$\boldsymbol{A} \cdot \boldsymbol{B} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\boldsymbol{A}| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\boldsymbol{B}| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right)$$

$$= \cos^{-1}\left(-\frac{4}{21}\right) \quad (\approx 1.76 \textit{ rad})$$

Laws of the Dot Product

From the equation $\boldsymbol{A} \cdot \boldsymbol{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$, we can see that

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A} \text{ ------------ (5)}$$

In other words, the dot product is commutative. We can also see that if c is any number, then

$$(c\boldsymbol{A}) \cdot \boldsymbol{B} = \boldsymbol{A} \cdot (c\boldsymbol{B}) = c(\boldsymbol{A} \cdot \boldsymbol{B}) \text{ ------------ (6)}$$

If $\boldsymbol{C} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$ is any third vector, then

$$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3)$$

$$= \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$$

Hence, *dot products obey the distributive law*:

$$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C} \text{ ------------ (7)}$$

If we combine this with commutative law, it is also evident that

$$(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C} \text{ ------------ (8)}$$

Equations (7) and (8) together permit us to multiply sums of vectors by the familiar laws of algebra. For example,

$$(\boldsymbol{A} + \boldsymbol{B}) \cdot (\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{A} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{D} \text{ ------ (9)}$$

Perpendicular (Orthogonal) Vectors

Two nonzero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular or **orthogonal** if the angle between them is $\frac{\pi}{2}$. For such vectors, we automatically have $\boldsymbol{A} \cdot \boldsymbol{B} = 0$ because $\cos\left(\frac{\pi}{2}\right) = 0$. The converse is also true. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonzero vectors with $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta = 0$, then $\cos\theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$.

Nonzero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal (perpendicular) if and only if $\boldsymbol{A} \cdot \boldsymbol{B} = 0$.

Example 3 $\boldsymbol{A} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{B} = 2\boldsymbol{j} + 4\boldsymbol{k}$ are orthogonal because

$$\boldsymbol{A} \cdot \boldsymbol{B} = (3)(0) + (-2)(2) + (1)(4) = 0$$

P1:

If $\boldsymbol{A} = a\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{B} = \sqrt{2}\boldsymbol{i} + \sqrt{3}\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{B} \cdot \boldsymbol{A} = \sqrt{3} - \sqrt{2}$, then find the value of a.

Solution:

We have, $\boldsymbol{A} = a\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{B} = \sqrt{2}\boldsymbol{i} + \sqrt{3}\boldsymbol{j} + 2\boldsymbol{k}$.

and $\boldsymbol{B} \cdot \boldsymbol{A} = \sqrt{3} - \sqrt{2}$

$$\Rightarrow \left(\sqrt{2}\boldsymbol{i} + \sqrt{3}\boldsymbol{j} + 2\boldsymbol{k}\right) \cdot (a\boldsymbol{i} + \boldsymbol{j}) = \sqrt{3} - \sqrt{2}$$

$$\Rightarrow \left(\sqrt{2}\right).(a) + \left(\sqrt{3}\right).(1) + (2)(0) = \sqrt{3} - \sqrt{2}$$

$$\Rightarrow \quad \sqrt{2}a + \sqrt{3} = \sqrt{3} - \sqrt{2}$$

$$\Rightarrow \quad \sqrt{2}a = -\sqrt{2}$$

$$\Rightarrow \quad a = -1$$

P2:

Find the angle between the vectors

$$\boldsymbol{A} = 2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k} \text{ and } \boldsymbol{B} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}.$$

Solution:

We have, $\boldsymbol{A} = 2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}$ and $\boldsymbol{B} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

$$\boldsymbol{A} \cdot \boldsymbol{B} = (2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}\ ) \cdot (2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$$

$$= (2).(2) + (10).(2) + (-11)(1)$$

$$= 4 + 20 - 11 = 13$$

$$|\boldsymbol{A}| = \sqrt{(2)^2 + (10)^2 + (-11)^2} = \sqrt{4 + 100 + 121} = 15$$

$$|\boldsymbol{B}| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{4 + 4 + 1} = 3$$

Therefore, the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\cos^{-1}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right) = \cos^{-1}\left(\frac{13}{(15).(3)}\right) = \cos^{-1}\left(\frac{13}{45}\right)$$

P3:

If the vectors $\lambda\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ and $2\lambda\boldsymbol{i} - \lambda\boldsymbol{j} - \boldsymbol{k}$ are perpendicular to each other, then find λ.

Solution:

Let $\boldsymbol{a} = \lambda\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{b} = 2\lambda\boldsymbol{i} - \lambda\boldsymbol{j} - \boldsymbol{k}$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

$$\Rightarrow (\lambda\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}) \cdot (2\lambda\boldsymbol{i} - \lambda\boldsymbol{j} - \boldsymbol{k}) = 0$$

$$\Rightarrow (\lambda)(2\lambda) + (-3)(-\lambda) + (5)(-1) = 0$$

$$\Rightarrow 2\lambda^2 + 3\lambda - 5 = 0$$

$$\Rightarrow (2\lambda + 5)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = -\frac{5}{2} \text{ or } 1$$

IP4:

If $|\boldsymbol{a}| = 11$, $|\boldsymbol{b}| = 23$ and $|\boldsymbol{a} - \boldsymbol{b}| = 30$, then

(i) find the angle between the vectors $\boldsymbol{a}, \boldsymbol{b}$.

(ii) find $|\boldsymbol{a} + \boldsymbol{b}|$.

Solution:

Let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

We have, $|\boldsymbol{a}| = 11$, $|\boldsymbol{b}| = 23$ and $|\boldsymbol{a} - \boldsymbol{b}| = 30$

(i) Now, $|\boldsymbol{a} - \boldsymbol{b}| = 30 \Rightarrow |\boldsymbol{a} - \boldsymbol{b}|^2 = 900$

$$\Rightarrow (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) = 900 \Rightarrow \boldsymbol{a}.\boldsymbol{a} - 2(\boldsymbol{a}.\boldsymbol{b}) + \boldsymbol{b}.\boldsymbol{b} = 900$$

$$\Rightarrow |\boldsymbol{a}|^2 - 2|\boldsymbol{a}|.|\boldsymbol{b}| \cos\theta + |\boldsymbol{b}|^2 = 900$$

$$\Rightarrow 121 - 2 \times 11 \times 23 \times \cos\theta + 529 = 900$$

$$\Rightarrow 650 - 506 \cos\theta = 900 \Rightarrow \cos\theta = -\frac{125}{253}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{125}{253}\right)$$

$$\therefore \theta = \pi - \cos^{-1}\left(\frac{125}{253}\right)$$

(ii) $|\boldsymbol{a} + \boldsymbol{b}|^2 = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} + 2(\boldsymbol{a}.\boldsymbol{b}) + \boldsymbol{b}.\boldsymbol{b}$

$$= |\boldsymbol{a}|^2 + 2|\boldsymbol{a}|.|\boldsymbol{b}| \cos\theta + |\boldsymbol{b}|^2$$

$$= 121 + 2 \times 11 \times 23 \times \left(-\frac{125}{253}\right) + 529 = 400$$

$$\therefore |\boldsymbol{a} + \boldsymbol{b}| = 20$$

IP1:

If $\boldsymbol{A} = 2\boldsymbol{i} + a\boldsymbol{j} + \sqrt{5}\boldsymbol{k}$, $\boldsymbol{B} = -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}$ and $\boldsymbol{A} \cdot \boldsymbol{B} = -25$, then find a.

Solution:

We have, $\boldsymbol{A} = 2\boldsymbol{i} + a\boldsymbol{j} + \sqrt{5}\boldsymbol{k}$, $\boldsymbol{B} = -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}$.

and $\boldsymbol{A} \cdot \boldsymbol{B} = -25$

$$\Rightarrow \left(2\boldsymbol{i} + a\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right) \cdot \left(-2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}\right) = -25$$

$$\Rightarrow (2).(-2) + (a).(4) + \left(\sqrt{5}\right)\left(-\sqrt{5}\right) = -25$$

$$\Rightarrow -4 + 4a - 5 = -25$$

$$\Rightarrow 4a = -16$$

$$\Rightarrow a = -4$$

IP2:

Find the angle between the vectors

$$\boldsymbol{A} = \frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k} \text{ and } \boldsymbol{B} = 5\boldsymbol{i} + 12\boldsymbol{j}.$$

Solution:

We have, $\boldsymbol{A} = \frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k}$ and $\boldsymbol{B} = 5\boldsymbol{i} + 12\boldsymbol{j}$.

$$\boldsymbol{A} \cdot \boldsymbol{B} = \left(\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k}\right) \cdot (5\boldsymbol{i} + 12\boldsymbol{j})$$

$$= \left(\frac{3}{5}\right).(5) + (0).(12) + \left(\frac{4}{5}\right)(0) = 3$$

$$|\boldsymbol{A}| = \sqrt{\left(\frac{3}{5}\right)^2 + 0 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$$

$$|\boldsymbol{B}| = \sqrt{(5)^2 + (12)^2 + 0} = \sqrt{25 + 144} = 13$$

Therefore, the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\cos^{-1}\left(\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right) = \cos^{-1}\left(\frac{3}{(1).(13)}\right) = \cos^{-1}\left(\frac{3}{13}\right)$$

IP3:

If the vectors $2\boldsymbol{i} + \lambda\boldsymbol{j} - \boldsymbol{k}$ and $4\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ are orthogonal, then find λ.

Solution:

Let $\boldsymbol{a} = 2\boldsymbol{i} + \lambda\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

$$\Rightarrow (2\boldsymbol{i} + \lambda\boldsymbol{j} - \boldsymbol{k}) \cdot (4\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) = 0$$

$$\Rightarrow (2)(4) + (\lambda)(-2) + (-1)(2) = 0$$

$$\Rightarrow 8 - 2\lambda - 2 = 0$$

$$\Rightarrow 2\lambda = 6$$

$$\Rightarrow \lambda = 3$$

IP4:

If $|\boldsymbol{a}| = 11$, $|\boldsymbol{b}| = 23$ and $|\boldsymbol{a} - \boldsymbol{b}| = 30$, then

(i) find the angle between the vectors $\boldsymbol{a}, \boldsymbol{b}$.

(ii) find $|\boldsymbol{a} + \boldsymbol{b}|$.

Solution:

Let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

We have, $|\boldsymbol{a}| = 11$, $|\boldsymbol{b}| = 23$ and $|\boldsymbol{a} - \boldsymbol{b}| = 30$

(i) Now, $|\boldsymbol{a} - \boldsymbol{b}| = 30 \Rightarrow |\boldsymbol{a} - \boldsymbol{b}|^2 = 900$

$$\Rightarrow (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) = 900 \Rightarrow \boldsymbol{a}.\boldsymbol{a} - 2(\boldsymbol{a}.\boldsymbol{b}) + \boldsymbol{b}.\boldsymbol{b} = 900$$

$$\Rightarrow |\boldsymbol{a}|^2 - 2|\boldsymbol{a}|.|\boldsymbol{b}| \cos\theta + |\boldsymbol{b}|^2 = 900$$

$$\Rightarrow 121 - 2 \times 11 \times 23 \times \cos\theta + 529 = 900$$

$$\Rightarrow 650 - 506\cos\theta = 900 \Rightarrow \cos\theta = -\frac{125}{253}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{125}{253}\right)$$

$$\therefore \theta = \pi - \cos^{-1}\left(\frac{125}{253}\right)$$

(ii) $|\boldsymbol{a} + \boldsymbol{b}|^2 = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} + 2(\boldsymbol{a}.\boldsymbol{b}) + \boldsymbol{b}.\boldsymbol{b}$

$$= |\boldsymbol{a}|^2 + 2|\boldsymbol{a}|.|\boldsymbol{b}| \cos\theta + |\boldsymbol{b}|^2$$

$$= 121 + 2 \times 11 \times 23 \times \left(-\frac{125}{253}\right) + 529 = 400$$

$$\therefore |\boldsymbol{a} + \boldsymbol{b}| = 20$$

1. Find the dot product and angle between the vectors given below in each problem.

a. $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

b. $\boldsymbol{a} = 2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}$, $\boldsymbol{b} = -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}$

c. $\boldsymbol{a} = 10\boldsymbol{i} + 11\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{j} + 4\boldsymbol{k}$

d. $\boldsymbol{a} = 2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

e. $\boldsymbol{a} = 5\boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

f. $\boldsymbol{a} = 5\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + \sqrt{17}\boldsymbol{j}$

g. $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

h. $\boldsymbol{a} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{k}$

i. $\boldsymbol{a} = \sqrt{3}\boldsymbol{i} - 7\boldsymbol{j}$, $\boldsymbol{b} = \sqrt{3}\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

j. $\boldsymbol{a} = \boldsymbol{i} + \sqrt{2}\boldsymbol{j} - \sqrt{2}\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

2. If $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, then show that $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$ are perpendicular to each other.

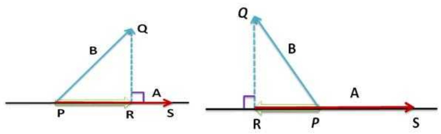
3. For what values of λ , the vectors $\boldsymbol{i} - \lambda\boldsymbol{j} + 2\boldsymbol{k}$ and $8\boldsymbol{i} + 6\boldsymbol{j} - \boldsymbol{k}$ are at right angles?

4.8
Vector Projections

Learning objectives:

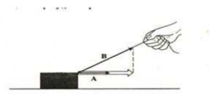
1) To define the vector projection of a vector onto another vector and to derive a formulae for it.
2) To express a given vector as a sum of orthogonal vector
3) To define the work done by a constant force during a displacement
   And
4) To practice related the related problems

The **vector projection** of $\boldsymbol{B} = \overrightarrow{PQ}$ onto a nonzero vector $\boldsymbol{A} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.

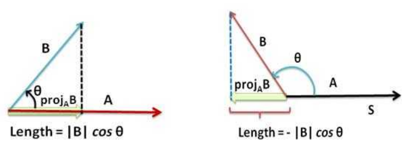


The notation for this vector is $proj_{\boldsymbol{A}}\boldsymbol{B}$ (the vector projection of B onto A)

If $\boldsymbol{B}$ represents a force then $proj_{\boldsymbol{A}}\boldsymbol{B}$ represents *the effective force in the direction of* **A**.



If the angle θ between **A** *and* $\boldsymbol{B}$ is acute, $proj_{\boldsymbol{A}}\boldsymbol{B}$ has length $|\boldsymbol{B}|cos\theta$ and direction $\frac{\mathbf{A}}{|\mathbf{A}|}$.



If θ is obtuse, $\cos\theta < 0$ and $proj_{\boldsymbol{A}}\boldsymbol{B}$ has length $-|\boldsymbol{B}|cos\theta$ and direction $-\frac{\mathbf{A}}{|\mathbf{A}|}$. In any case,

$$proj_{\boldsymbol{A}}\boldsymbol{B} = (|\boldsymbol{B}|cos\theta)\frac{\mathbf{A}}{|\mathbf{A}|}$$

$$= \left(\frac{\boldsymbol{A}.\boldsymbol{B}}{|\boldsymbol{A}|}\right)\left(\frac{\boldsymbol{A}}{|\boldsymbol{A}|}\right) = \left(\boldsymbol{B}.\frac{\boldsymbol{A}}{|\boldsymbol{A}|}\right)\frac{\boldsymbol{A}}{|\boldsymbol{A}|}$$

$$proj_{\boldsymbol{A}}\boldsymbol{B} = \left(\boldsymbol{B}.\frac{\boldsymbol{A}}{|\boldsymbol{A}|}\right)\frac{\boldsymbol{A}}{|\boldsymbol{A}|} = \left(\frac{\boldsymbol{B}.\boldsymbol{A}}{\boldsymbol{A}.\boldsymbol{A}}\right)\boldsymbol{A} \qquad (1)$$

The number $|\boldsymbol{B}|cos\theta$ is called the *scalar component of* $\boldsymbol{B}$ *in the direction of* $\boldsymbol{A}$. Since

$$|\boldsymbol{B}|cos\theta = \boldsymbol{B}.\frac{\boldsymbol{A}}{|\boldsymbol{A}|} \qquad (2)$$

we can find the scalar component by "dotting" $\boldsymbol{B}$ with the direction of $\boldsymbol{A}$. Equation (1) says that the vector projection of $\boldsymbol{B}$ onto $\boldsymbol{A}$ is the scalar component of $\boldsymbol{B}$ in the direction of $\boldsymbol{A}$ times the direction of $\boldsymbol{A}$.

While the first part of equation (1) describes the effect of $\boldsymbol{B}$ in the direction of $\boldsymbol{A}$, the second part is better for calculation because it avoids square roots.

Example 1

Find the vector projection of $\boldsymbol{B} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ onto $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ and the scalar component of $\boldsymbol{B}$ in the direction of $\boldsymbol{A}$.

Solution:

We find $proj_{\boldsymbol{A}}\boldsymbol{B}$ from equation (1):

$$proj_{\boldsymbol{A}}\boldsymbol{B} = \left(\frac{\boldsymbol{B}.\boldsymbol{A}}{\boldsymbol{A}.\boldsymbol{A}}\right)\boldsymbol{A}$$

$$= \frac{6-6-4}{1+4+4}(\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k})$$

$$= -\frac{4}{9}(\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}) = -\frac{4}{9}\boldsymbol{i} + \frac{8}{9}\boldsymbol{j} + \frac{8}{9}\boldsymbol{k}$$

We find the scalar component of $\boldsymbol{B}$ in the direction of $\boldsymbol{A}$ from equation (2):

$$|\boldsymbol{B}|cos\theta = \boldsymbol{B}.\frac{\boldsymbol{A}}{|\boldsymbol{A}|}$$

$$= (6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}) \cdot \left(\frac{1}{3}\boldsymbol{i} - \frac{2}{3}\boldsymbol{j} - \frac{2}{3}\boldsymbol{k}\right)$$

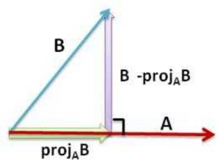
$$= 2 - 2 - \frac{4}{3} = -\frac{4}{3}$$

Writing a Vector as a Sum of Orthogonal Vectors

In mechanics, we often need to express a vector $\boldsymbol{B}$ as a sum of a vector parallel to a vector $\boldsymbol{A}$ and a vector orthogonal to $\boldsymbol{A}$. We can accomplish this with the equation

$$\boldsymbol{B} = proj_{\boldsymbol{A}}\boldsymbol{B} + (\boldsymbol{B} - proj_{\boldsymbol{A}}\boldsymbol{B}) \qquad (3)$$

shown in figure below.



The vector $\boldsymbol{B}$ is written as a vector parallel to $\boldsymbol{A}$ plus a vector orthogonal to $\boldsymbol{A}$:

$$\boldsymbol{B} = proj_{\boldsymbol{A}}\boldsymbol{B} + (\boldsymbol{B} - proj_{\boldsymbol{A}}\boldsymbol{B})$$
$$= \underbrace{\left(\frac{\boldsymbol{B.A}}{\boldsymbol{A.A}}\right)\boldsymbol{A}}_{\text{Parallel to A}} + \underbrace{\left(\boldsymbol{B} - \left(\frac{\boldsymbol{B.A}}{\boldsymbol{A.A}}\right)\boldsymbol{A}\right)}_{\text{orthogonal to A}} \qquad (4)$$

Example 2

Express $\boldsymbol{B} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ as the sum of a vector parallel to $\boldsymbol{A} = 3\boldsymbol{i} - \boldsymbol{j}$ and a vector orthogonal to $\boldsymbol{A}$.

Solution:

We use equation (4). With

$$\boldsymbol{B.A} = 6 - 1 = 5 \quad and \quad \boldsymbol{A.A} = 9 + 1 = 10$$

equation (4) gives

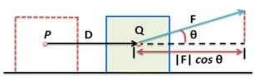
$$\boldsymbol{B} = \left(\frac{\boldsymbol{B.A}}{\boldsymbol{A.A}}\right)\boldsymbol{A} + \left(\boldsymbol{B} - \left(\frac{\boldsymbol{B.A}}{\boldsymbol{A.A}}\right)\boldsymbol{A}\right)$$
$$= \frac{5}{10}(3\boldsymbol{i} - \boldsymbol{j}) + \left(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k} - \frac{5}{10}(3i - j)\right)$$
$$= \left(\frac{3}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}\right) + \left(\frac{1}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} - 3\boldsymbol{k}\right)$$

Check: The first vector in the sum is parallel to $\boldsymbol{A}$ because it is $\left(\frac{1}{2}\right)\boldsymbol{A}$. The second vector in the sum is orthogonal to $\boldsymbol{A}$ because $\left(\frac{1}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} - 3\boldsymbol{k}\right) \cdot (3\boldsymbol{i} - \boldsymbol{j}) = \left(\frac{3}{2} - \frac{3}{2}\right) = 0$

Work

In an earlier module, we calculated the work done by a constant force of magnitude F in moving an object through a distance d as $W = Fd$. That formula holds only if the force is directed along the line of motion. If a force $\boldsymbol{F}$ moving an object

through a displacement $\boldsymbol{d} = \overrightarrow{PQ}$ has some other direction, the work is performed by the component of $\boldsymbol{F}$ in the direction of $\boldsymbol{d}$.



If θ is the angle between $\boldsymbol{F}$ and $\boldsymbol{d}$, then

Work=(scalar component of ***F*** in the direction of ***d***)(length of ***d***)

$$= (|\boldsymbol{F}|cos\theta)|\boldsymbol{d}|$$
$$= \boldsymbol{F.d}$$

Definition

The work done by a constant force $\boldsymbol{F}$ acting through a displacement $\boldsymbol{d} = \overrightarrow{PQ}$ is

$$\boldsymbol{w} = \boldsymbol{F.d} = |\boldsymbol{F}||\boldsymbol{d}|cos\theta \qquad (5)$$

where θ is the angle between $\boldsymbol{F}$and $\boldsymbol{d}$

Example 3 If $|\boldsymbol{F}| = 40\,N, |\boldsymbol{d}| = 3m, and\ \theta = 60^\circ$, the work done by $\boldsymbol{F}$ in acting from P to Q is

$$W = |\boldsymbol{F}||\boldsymbol{d}|cos\theta$$
$$= (40)(3)cos60^\circ$$
$$= 120 \times (1/2)$$
$$= 60\,J$$

IP1)

Find the vector projection and scalar projection of

$\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ onto $\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$

Solution:

Vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$: $proj_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{a}}\right)\boldsymbol{a}$

$$proj_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{(2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) \cdot (3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})}{(3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) \cdot (3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})}\right)(3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$$

$$= \frac{6 + 2 + 1}{9 + 4 + 1}(3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$$

$$= \frac{27}{14}\boldsymbol{i} - \frac{9}{7}\boldsymbol{j} - \frac{9}{14}\boldsymbol{k}$$

Scalar component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$

$$|\boldsymbol{b}|cos\theta = \boldsymbol{b} \cdot \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = (2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) \cdot \frac{(3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})}{\sqrt{14}} = \frac{9}{\sqrt{14}}$$

IP2)

Find the vector projection and scalar projection of

$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ onto $\boldsymbol{b} = 5\boldsymbol{j} - 3\boldsymbol{k}$

Solution:

Vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$: $proj_{\boldsymbol{b}}\boldsymbol{a} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\right)\boldsymbol{b}$

$$proj_{\boldsymbol{b}}\boldsymbol{a} = \left(\frac{(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\cdot(5\boldsymbol{j}-3\boldsymbol{k})}{(5\boldsymbol{j}-3\boldsymbol{k})\cdot(5\boldsymbol{j}-3\boldsymbol{k})}\right)(5\boldsymbol{j}-3\boldsymbol{k})$$

$$= \frac{5-3}{25+9}(5\boldsymbol{j}-3\boldsymbol{k})$$

$$= \frac{5}{17}\boldsymbol{j} - \frac{3}{17}\boldsymbol{k}$$

Scalar component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

$$|\boldsymbol{a}|cos\theta = \boldsymbol{a}\cdot\frac{\boldsymbol{b}}{|\boldsymbol{b}|} = (\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\,)\cdot\frac{(5\boldsymbol{j}-3\boldsymbol{k})}{\sqrt{34}} = \frac{5-3}{\sqrt{34}} = \frac{2}{\sqrt{34}}$$

IP3)

Express $\boldsymbol{A} = \boldsymbol{j} + \boldsymbol{k}$ as the sum of a vector parallel to $\boldsymbol{B} = \boldsymbol{i} + \boldsymbol{j}$ and a vector orthogonal to $\boldsymbol{B}$.

Solution:

We know that, we can express A as a vector parallel to $\boldsymbol{B}$ plus a vector orthogonal to $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{proj_B A} + (\boldsymbol{A} - \boldsymbol{proj_B A})$$

$$\boldsymbol{A} = \left(\frac{\boldsymbol{A\cdot B}}{\boldsymbol{B\cdot B}}\right)\boldsymbol{B} + \left(\boldsymbol{A} - \left(\frac{\boldsymbol{A\cdot B}}{\boldsymbol{B\cdot B}}\right)\boldsymbol{B}\right)$$

We have, $\boldsymbol{A}\cdot\boldsymbol{B} = 1 \quad and \quad \boldsymbol{B}\cdot\boldsymbol{B} = 2$

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{i} + \boldsymbol{j}) + \left(\boldsymbol{j} + \boldsymbol{k} - \frac{1}{2}(\boldsymbol{i} + \boldsymbol{j})\right)$$

$$= \left(\frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}\right) + \left(-\frac{1}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + \boldsymbol{k}\right)$$

IP4)

How much work does it take to slide a crate *20m* along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?

Solution:

We have, $|\boldsymbol{F}| = 200\ N, |\boldsymbol{d}| = 20m, and\ \theta = 30^{\circ}$

$$W = |\boldsymbol{F}||\boldsymbol{d}|cos\theta$$

$$= (200)(20)cos30^{\circ}$$

$$= 4000 \times (\frac{\sqrt{3}}{2})$$

$$= 2000\sqrt{3}\ \mathrm{J}$$

P1)

Find the vector projection and scalar projection of

$\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ onto $\boldsymbol{a} = -2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$

Solution:

Vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ $\quad proj_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{\boldsymbol{b}\cdot\boldsymbol{a}}{\boldsymbol{b}\cdot\boldsymbol{a}}\right)\boldsymbol{a}$

$$proj_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\cdot(-2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})}{(-2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})\cdot(-2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})}\right)(-2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})$$

$$= \frac{-2+3+1}{4+9+1}(-2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})$$

$$= -\frac{2}{7}\boldsymbol{i} + \frac{3}{7}\boldsymbol{j} + \frac{1}{7}\boldsymbol{k}$$

Scalar component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$

$$|\boldsymbol{b}|cos\theta = \boldsymbol{b}\cdot\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = (\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\cdot\frac{(-2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})}{\sqrt{14}} = \frac{-2+3+1}{\sqrt{14}} = \frac{2}{\sqrt{14}}$$

P2)

Find the vector projection and scalar projection of

$\boldsymbol{a} = 3\boldsymbol{j} + 4\boldsymbol{k}$ onto $\boldsymbol{b} = 10\boldsymbol{i} + 11\boldsymbol{j} - 2\boldsymbol{k}$

Solution:

Vector projection of ***a*** onto ***b***: $proj_{\boldsymbol{b}}\boldsymbol{a} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\right)\boldsymbol{b}$

$$proj_{\boldsymbol{b}}\boldsymbol{a} = \left(\frac{(3\boldsymbol{j}+4\boldsymbol{k})\cdot(10\boldsymbol{i}+11\boldsymbol{j}-2\boldsymbol{k})}{(10\boldsymbol{i}+11\boldsymbol{j}-2\boldsymbol{k})\cdot(10\boldsymbol{i}+11\boldsymbol{j}-2\boldsymbol{k})}\right)(10\boldsymbol{i}+11\boldsymbol{j}-2\boldsymbol{k})$$

$$= \frac{33-8}{100+121+4}(10\boldsymbol{i}+11\boldsymbol{j}-2\boldsymbol{k})$$

$$= (\tfrac{10}{9}\boldsymbol{i}+\tfrac{11}{9}\boldsymbol{j}-\tfrac{2}{9}\boldsymbol{k})$$

Scalar component of ***a*** in the direction of ***b***

$$|\boldsymbol{a}|cos\theta = \boldsymbol{a}\cdot\frac{\boldsymbol{b}}{|\boldsymbol{b}|} = (3\boldsymbol{j}+4\boldsymbol{k})\cdot\frac{(10\boldsymbol{i}+11\boldsymbol{j}-2\boldsymbol{k})}{\sqrt{225}} = \frac{33-8}{15} = \frac{5}{3}$$

P3)

Express $\boldsymbol{B} = 3\boldsymbol{i} + 4\boldsymbol{k}$ as the sum of a vector parallel to $\boldsymbol{A} = \boldsymbol{i} + \boldsymbol{j}$ and a vector orthogonal to $\boldsymbol{A}$.

Solution:

We know that, we can express $\boldsymbol{B}$ as a vector parallel to $\boldsymbol{A}$ plus a vector orthogonal to $\boldsymbol{A}$:

$$\boldsymbol{B} = \boldsymbol{proj_A B} + (\boldsymbol{B} - \boldsymbol{proj_A B})$$

$$\boldsymbol{B} = \left(\frac{\boldsymbol{B}\cdot\boldsymbol{A}}{\boldsymbol{A}\cdot\boldsymbol{A}}\right)\boldsymbol{A} + \left(\boldsymbol{B} - \left(\frac{\boldsymbol{B}\cdot\boldsymbol{A}}{\boldsymbol{A}\cdot\boldsymbol{A}}\right)\boldsymbol{A}\right)$$

We have, $\boldsymbol{A}\cdot\boldsymbol{B} = 3 \quad and \quad \boldsymbol{A}\cdot\boldsymbol{A} = 2$

$$\boldsymbol{B} = \frac{3}{2}(\boldsymbol{i}+\boldsymbol{j}) + \left(3\boldsymbol{i} + 4\boldsymbol{k} - \frac{3}{2}(\boldsymbol{i}+\boldsymbol{j})\right)$$

$$= \left(\frac{3}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j}\right) + \left(\frac{3}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + 4\boldsymbol{k}\right)$$

P4)

If $|\boldsymbol{F}| = 30\ N, |\boldsymbol{d}| = 2m, and\ \theta = 30^{\circ}$,Find the work done by ***F*** in acting from R to S?

Solution:

We have, $|\boldsymbol{F}| = 30\ N, |\boldsymbol{d}| = 2m, and\ \theta = 30^\circ$

$$\begin{aligned} W &= |\boldsymbol{F}||\boldsymbol{d}|cos\theta \\ &= (30)(2)cos30^\circ \\ &= 60 \times \left(\frac{\sqrt{3}}{2}\right) \\ &= 30\sqrt{3}\,\mathrm{J} \end{aligned}$$

Exercises:

I. Find the scalar and vector projections of

i) $\boldsymbol{b}$ onto $\boldsymbol{a}$

ii) a onto b

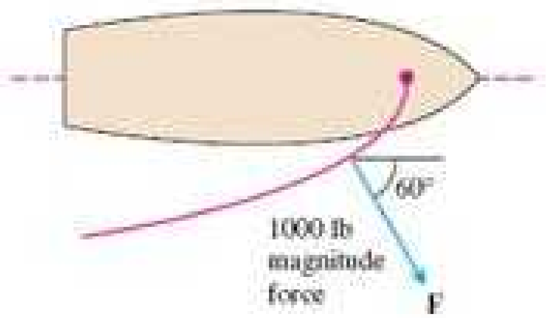
1) $\boldsymbol{a} = 3\boldsymbol{i} - 4\boldsymbol{j}$, $\boldsymbol{b} = 5\boldsymbol{i}$
2) $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = -4\boldsymbol{i} + \boldsymbol{j}$
3) $\boldsymbol{a} = 3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$
4) $\boldsymbol{a} = -2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$, $\boldsymbol{b} = 5\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$
5) $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$
6) $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$
7) $\boldsymbol{a} = 2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}$, $\boldsymbol{b} = -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}$
8) $\boldsymbol{a} = \left(\frac{3}{5}\right)\boldsymbol{i} + \left(\frac{4}{5}\right)\boldsymbol{k}$, $\boldsymbol{b} = 5\boldsymbol{i} + 12\boldsymbol{j}$
9) $\boldsymbol{a} = 2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$
10) $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = \sqrt{2}\boldsymbol{i} + \sqrt{3}\boldsymbol{j} + 2\boldsymbol{k}$

II. If $\boldsymbol{a} = 8\boldsymbol{i} + 4\boldsymbol{j} - 12\boldsymbol{k}$ and $b = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

1) Write ***a*** as the sum of a vector parallel to ***b*** and a vector orthogonal to ***b***.
2) Write ***b*** as the sum of a vector parallel to ***a*** and a vector orthogonal to ***a***.

III.

1. Find the work done by a force $F = 5\boldsymbol{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters).
2. The wind passing over a boat's sail exerted a 1000-lb magnitude force F as shown here. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.



4.9

Cross Product

Learning Objectives:

- To define the cross product (vector product) of two vectors in space
- To derive the properties of cross product of vectors
- To define Torque

AND

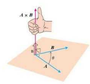
- To solve related problems

In this module, we learn the cross product, a second method of multiplying two vectors. *Cross product* is also called a *vector product* because the multiplication results in a vector.

Cross products are widely used to describe the effects of forces in studies of electricity, magnetism, fluid flows, and orbital mechanics.

The Cross Product of Two Vectors in Space

If two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in space are not parallel, they determine a plane. We select a unit vector $\boldsymbol{n}$ perpendicular to the plane by **right-hand rule**. This means we choose $\boldsymbol{n}$ to be the *unit* (*normal*) *vector* that points the way your right thumb points when your fingers curl through an angle θ from $\boldsymbol{A}$ to $\boldsymbol{B}$.



We then define the **cross product** $\boldsymbol{A} \times \boldsymbol{B}$ (read as $\boldsymbol{A}$ cross $\boldsymbol{B}$) to be a vector as follows:

Definition

$$\boldsymbol{A} \times \boldsymbol{B} = (|\boldsymbol{A}||\boldsymbol{B}| sin\,\theta)\boldsymbol{n} \text{ ---------- (1)}$$

The vector $\boldsymbol{A} \times \boldsymbol{B}$ is *orthogonal* to both $\boldsymbol{A}$ and $\boldsymbol{B}$ because it is a scalar multiple of $\boldsymbol{n}$.

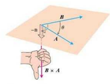
Since the sines of 0 and π are both zero in equation (1), it makes sense to define the cross product of two parallel nonzero vectors to be $\boldsymbol{0}$.

If one or both of $\boldsymbol{A}$ and $\boldsymbol{B}$ are zero, we also define $\boldsymbol{A} \times \boldsymbol{B}$ to be zero. This way, *the cross product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is zero if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel or one or both of them are zero.*

Parallel Vectors:

Nonzero vectors A and B are parallel if and only if $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{0}$.

Reversing the order of the factors in a nonzero cross product reverses the direction of the product. When the fingers of our right hand curl through the angle θ from $\boldsymbol{B}$ to $\boldsymbol{A}$, our thumb points the opposite way and the unit vector we choose in forming $\boldsymbol{B} \times \boldsymbol{A}$ is the negative of the one we choose in forming $\boldsymbol{A} \times \boldsymbol{B}$.

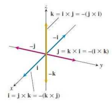


Thus, for all vectors $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\boldsymbol{B} \times \boldsymbol{A} = -(\boldsymbol{A} \times \boldsymbol{B}) \text{ ------------ (2)}$$

Unlike the dot product, the cross product is not commutative.

When we apply the definition to calculate the pairwise cross products of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, we find



$$\left.\begin{aligned} \boldsymbol{i} \times \boldsymbol{j} &= -(\boldsymbol{j} \times \boldsymbol{i}) = \boldsymbol{k} \\ \boldsymbol{j} \times \boldsymbol{k} &= -(\boldsymbol{k} \times \boldsymbol{j}) = \boldsymbol{i} \\ \boldsymbol{k} \times \boldsymbol{i} &= -(\boldsymbol{i} \times \boldsymbol{k}) = \boldsymbol{j} \end{aligned}\right| \text{ ------------- (3)}$$

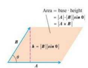
and

$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$$

Because $\boldsymbol{n}$ is a unit vector, the magnitude of $\boldsymbol{A} \times \boldsymbol{B}$ is

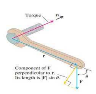
$$|\boldsymbol{A} \times \boldsymbol{B}| = |\boldsymbol{A}||\boldsymbol{B}||\sin\theta||\boldsymbol{n}| = |\boldsymbol{A}||\boldsymbol{B}|\sin\theta \text{ ----------- (4)}$$

This is the area of the parallelogram determined by $\boldsymbol{A}$ and $\boldsymbol{B}$, $|\boldsymbol{A}|$ being the base of the parallelogram and $|\boldsymbol{B}||\sin\theta|$ the height.



Torque

When we turn a bolt by applying a force $\boldsymbol{F}$ to a wrench, the torque we produce acts along the axis of the bolt to drive the bolt forward.



The magnitude of the torque depends on how far out on the wrench the force is applied and on how much of the force is perpendicular to the wrench at the point of application. The number we use to measure the torque's magnitude is the product of the length of the lever arm $\boldsymbol{r}$ and the scalar component of $\boldsymbol{F}$ perpendicular to $\boldsymbol{r}$.

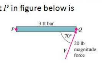
$$\text{Magnitude of torque vector } = |\boldsymbol{r}||\boldsymbol{F}|\sin\theta \text{ or } |\boldsymbol{r} \times \boldsymbol{F}|.$$

If we let $\boldsymbol{n}$ be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is $\boldsymbol{r} \times \boldsymbol{F}$, or

$$\textbf{Torque vector} = (|\boldsymbol{r}||\boldsymbol{F}|\sin\theta)\boldsymbol{n}$$

We defined $\boldsymbol{A} \times \boldsymbol{B}$ to be $\boldsymbol{0}$ when $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel. This is consistent with the torque interpretation as well. If the force $\boldsymbol{F}$ is parallel to the wrench, meaning that we are trying to turn the bolt by pushing or pulling along the line of wrench's handle, the torque produced is zero.

Example The magnitude of the torque exerted by the force $\boldsymbol{F}$ at the pivot point P in figure below is



$$|\overrightarrow{PQ} \times \boldsymbol{F}| = |\overrightarrow{PQ}||\boldsymbol{F}|\sin 70^0 \approx (3)(20)(0.94) \approx 56.4$$

The Associative and Distributive Laws

As a rule, cross-product multiplication is *not associative* because $(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$ lies in the plane of $\boldsymbol{A}$ and $\boldsymbol{B}$ whereas $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$ lies in the plane of $\boldsymbol{B}$ and $\boldsymbol{C}$. However, the following laws do hold.

Scalar Distributive Law

$$(r\boldsymbol{A}) \times (s\boldsymbol{B}) = (rs)(\boldsymbol{A} \times \boldsymbol{B}) \text{ ----------- (5)}$$

Vector Distributive Laws

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C} \text{ -------------- (6)}$$

$$(\boldsymbol{B} + \boldsymbol{C}) \times \boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{A} + \boldsymbol{C} \times \boldsymbol{A} \text{ -------------- (7)}$$

As a special case of equation (5), we also have

$$(-\boldsymbol{A}) \times \boldsymbol{B} = \boldsymbol{A} \times (-\boldsymbol{B}) = -(\boldsymbol{A} \times \boldsymbol{B}) \text{ ----------- (8)}$$

The *Scalar Distributive Law* can be verified by applying equation (1) to the products on both sides of equation (5) and comparing the results. The proof for the *Vector Distributive Law* in equation (6) is involved. Equation (7) follows from equation (6): Multiply both sides of equation (6) by -1 and reverse the orders of the products.

IP1:

Let $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ be such that $\boldsymbol{c} \neq \mathbf{0}$, $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}, \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a}$. Show that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are pair wise orthogonal vectors and $|\boldsymbol{b}| = 1$, $|\boldsymbol{c}| = |\boldsymbol{a}|$.

Solution:

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \Rightarrow \boldsymbol{c}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

$\boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a} \Rightarrow \boldsymbol{a}$ is perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$

$\therefore$ $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are mutually orthogonal vectors

$\therefore |\boldsymbol{c}| = |\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin 90^0 = |\boldsymbol{a}||\boldsymbol{b}|$ ------ (1)

$|\boldsymbol{a}| = |\boldsymbol{b} \times \boldsymbol{c}| = |\boldsymbol{b}||\boldsymbol{c}| \sin 90^0 = |\boldsymbol{b}||\boldsymbol{c}|$ ------ (2)

From (1) and (2) $|\boldsymbol{c}||\boldsymbol{a}| = |\boldsymbol{c}||\boldsymbol{a}||\boldsymbol{b}|^2 \Rightarrow |\boldsymbol{b}|^2 = \mathbf{1}$

$\therefore |\boldsymbol{b}| = 1$ and from (1), $|\boldsymbol{c}| = |\boldsymbol{a}|$.

IP2:

In ΔABC, if $\overrightarrow{BC} = \boldsymbol{a}, \overrightarrow{CA} = \boldsymbol{b}$ and $\overrightarrow{AB} = \boldsymbol{c}$ then show that

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$$

Solution:

We have $\overrightarrow{BC} = \boldsymbol{a}, \overrightarrow{CA} = \boldsymbol{b}$ and $\overrightarrow{AB} = \boldsymbol{c}$

$$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{BB} = \boldsymbol{0}$$

$$\therefore \boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{c}$$

$$\therefore \boldsymbol{a} \times (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a} \times (-\boldsymbol{c})$$

$$\therefore \boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{a} \times \boldsymbol{c}) = \boldsymbol{c} \times \boldsymbol{a} \quad [\text{ since } \boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}\,]$$

Also $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{b} = (-\boldsymbol{c}) \times \boldsymbol{b}$

$$\therefore \boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{c} \times \boldsymbol{b}) = \boldsymbol{b} \times \boldsymbol{c} \quad [\text{ since } \boldsymbol{b} \times \boldsymbol{b} = \boldsymbol{0}\,]$$

$$\therefore \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$$

IP3:

If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are vectors such that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{d}$ and $\boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{d}$, then show that the vectors $\boldsymbol{a} - \boldsymbol{d}$ and $\boldsymbol{b} - \boldsymbol{c}$ are parallel.

Solution:

We have $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{d}$ ------- (1)

and $\boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{d}$ ------- (2)

(1) – (2) $\Rightarrow (\boldsymbol{a} \times \boldsymbol{b}) - (\boldsymbol{a} \times \boldsymbol{c}) = (\boldsymbol{c} \times \boldsymbol{d}) - (\boldsymbol{b} \times \boldsymbol{d})$

$\Rightarrow \boldsymbol{a} \times (\boldsymbol{b} - \boldsymbol{c}) = (\boldsymbol{c} - \boldsymbol{b}) \times \boldsymbol{d}$ [Vector Distributive Law]

$\Rightarrow \boldsymbol{a} \times (\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{d} \times (\boldsymbol{b} - \boldsymbol{c})$

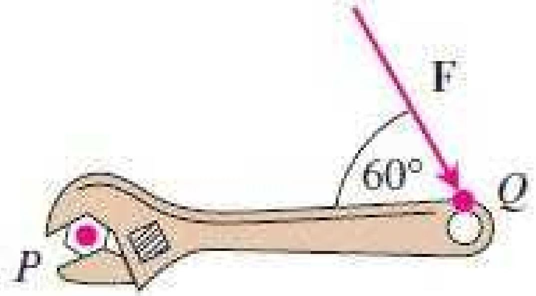
$\Rightarrow \boldsymbol{a} \times (\boldsymbol{b} - \boldsymbol{c}) - \boldsymbol{d} \times (\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{0}$

$\Rightarrow (\boldsymbol{a} - \boldsymbol{d}) \times (\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{0}$

Therefore, $\boldsymbol{a} - \boldsymbol{d}$ and $\boldsymbol{b} - \boldsymbol{c}$ are parallel vectors.

IP4:

The magnitude of the torque exerted by the force $\boldsymbol{F}$ on the bolt at P if $|\overrightarrow{PQ}| = 8$ *in*. and $|\boldsymbol{F}| = 30$ *lb*. Answer in foot-pounds.



Solution:

We have $|\overrightarrow{PQ}| = 8\ in. = \frac{8}{12} = \frac{2}{3}\ foot$, $|\boldsymbol{F}| = 30\ lb.$ and $\theta = 60^0$.

$$\text{Torque} = |\overrightarrow{PQ} \times \boldsymbol{F}| = |\overrightarrow{PQ}||\boldsymbol{F}| \sin 60^0$$

$$= \left(\frac{2}{3}\right)(30)\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}\ ft\text{-}lb.$$

P1:

Let $\boldsymbol{a}$ is a non-zero vector and $\boldsymbol{b}, \boldsymbol{c}$ are two vectors such that, $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c}$ and $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c}$, then prove that $\boldsymbol{b} = \boldsymbol{c}$.

Solution:

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c} \Rightarrow \boldsymbol{a} \times (\boldsymbol{b} - \boldsymbol{c}) = \mathbf{0}$$

$\Rightarrow$ either $\boldsymbol{b} = \boldsymbol{c}$ or $\boldsymbol{b} - \boldsymbol{c}$ is parallel to $\boldsymbol{a}$

Again $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c} \Rightarrow \boldsymbol{a} \cdot (\boldsymbol{b} - \boldsymbol{c}) = \mathbf{0}$

$\Rightarrow \boldsymbol{b} = \boldsymbol{c}$ or $\boldsymbol{b} - \boldsymbol{c}$ is perpendicular to $\boldsymbol{a}$

$\therefore$ If $\boldsymbol{b} \neq \boldsymbol{c}$, then $\boldsymbol{b} - \boldsymbol{c}$ is parallel to $\boldsymbol{a}$ and is perpendicular to $\boldsymbol{a}$ which is impossible.

$$\therefore \boldsymbol{b} = \boldsymbol{c}.$$

P2:

Compute $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} + \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b})$.

Solution:

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} + \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b})$$

$$= \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b}$$

[Distributive Law]

$$= \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c} - \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{a} \times \boldsymbol{c} - \boldsymbol{b} \times \boldsymbol{c}$$

$$= \mathbf{0}$$

P3:

If $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 3$ and the angle between $\boldsymbol{a}$ to $\boldsymbol{b}$ is $\frac{\pi}{6}$, then find $|\boldsymbol{a} \times \boldsymbol{b}|^2$.

Solution:

We have $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 3$ and $\theta = \frac{\pi}{6}$

Since $\boldsymbol{a} \times \boldsymbol{b} = (|\boldsymbol{a}||\boldsymbol{b}| sin\,\theta)\boldsymbol{n}$

$$\Rightarrow |\boldsymbol{a} \times \boldsymbol{b}| = \left||\boldsymbol{a}||\boldsymbol{b}| sin\,\theta\right||\boldsymbol{n}|$$

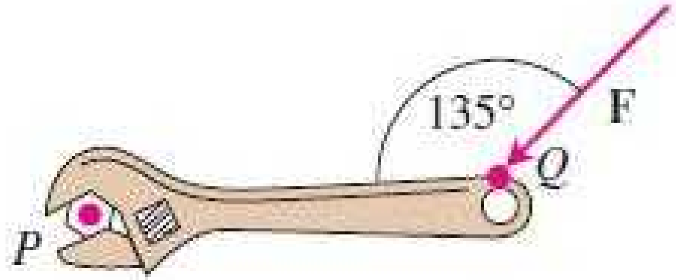
$$\Rightarrow |\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| sin\,\theta \quad (\boldsymbol{n} \text{ is a unit vector})$$

$$\Rightarrow |\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 sin^2\,\theta = (2^2)(3^2) sin^2 \frac{\pi}{6}$$

$$= (4)(9)\left(\frac{1}{4}\right) = 9$$

P4:

The magnitude of the torque exerted by the force $\boldsymbol{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8$ *in*. and $|\boldsymbol{F}| = 30$ *lb*. Answer in foot-pounds.



Solution:

We have, $\left|\overrightarrow{PQ}\right| = 8 \; in. = \frac{8}{12} = \frac{2}{3} \; foot$, $|\boldsymbol{F}| = 30 \; lb.$

and $\theta = 135^0$.

Torque $= \left|\overrightarrow{PQ} \times \boldsymbol{F}\right| = \left|\overrightarrow{PQ}\right| |\boldsymbol{F}| \sin 135^0$

$$= \left(\frac{2}{3}\right)(30)\left(\frac{1}{\sqrt{2}}\right) = 10\sqrt{2} \; ft\text{-}lb.$$

Exercises:

1. If $\boldsymbol{p} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then find $|\boldsymbol{p} \times \boldsymbol{k}|^2$.

2. Let $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$. If $\boldsymbol{c}$ is a vector such that $\boldsymbol{a}.\boldsymbol{c} = |\boldsymbol{c}|, |\boldsymbol{c} - \boldsymbol{a}| = 2\sqrt{2}$, then find $|\boldsymbol{c}|$.

4.10.

Calculation of Cross Products

Learning objectives:

- To derive a determinant formula for the cross product of two given vectors.
- To define the scalar triple product (box product) of three vectors.
  AND
- To practice the related problems.

We develop a formula to calculate $\boldsymbol{A}\times\boldsymbol{B}$ from the components of $\boldsymbol{A}$ and $\boldsymbol{B}$ relative to a Cartesian coordinate system.

Determinant formula for $\boldsymbol{A}\times\boldsymbol{B}$

If $\boldsymbol{A}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$ and $\boldsymbol{B}=b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k}$, then

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \quad \text{........ (1)}$$

By the distributive laws and the rules for multiplying $\boldsymbol{i},\boldsymbol{j},$ and $\boldsymbol{k}$ we get,

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B}&=\left(a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}\right)\times\left(b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k}\right)\\ &=a_1b_1\boldsymbol{i}\times\boldsymbol{i}+a_1b_2\boldsymbol{i}\times\boldsymbol{j}+a_1b_3\boldsymbol{i}\times\boldsymbol{k}\\ &\quad+a_2b_1\boldsymbol{j}\times\boldsymbol{i}+a_2b_2\boldsymbol{j}\times\boldsymbol{j}+a_2b_3\boldsymbol{j}\times\boldsymbol{k}\\ &\quad+a_3b_1\boldsymbol{k}\times\boldsymbol{i}+a_3b_2\boldsymbol{k}\times\boldsymbol{j}+a_3b_3\boldsymbol{k}\times\boldsymbol{k}\\ &=\left(a_2b_3-a_3b_2\right)\boldsymbol{i}-\left(a_1b_3-a_3b_1\right)\boldsymbol{j}+\left(a_1b_2-a_2b_1\right)\boldsymbol{k}\end{aligned}$$

The terms in the last line are the same as the terms in the expansion of the determinant

$$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Example 1

Find $\boldsymbol{A}\times\boldsymbol{B}$ and $\boldsymbol{B}\times\boldsymbol{A}$ if $\boldsymbol{A}=2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{B}=-4\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$

Solution

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B}&=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 1 & 1\\ -4 & 3 & 1\end{vmatrix}\\ &=\begin{vmatrix}1 & 1\\ 3 & 1\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}2 & 1\\ -4 & 1\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}2 & 1\\ -4 & 3\end{vmatrix}\boldsymbol{k}\\ &=-2\boldsymbol{i}-6\boldsymbol{j}+10\boldsymbol{k}\end{aligned}$$

$$\boldsymbol{B}\times\boldsymbol{A}=-\left(\boldsymbol{A}\times\boldsymbol{B}\right)=2\boldsymbol{i}+6\boldsymbol{j}-10\boldsymbol{k}$$

Example 2

Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

Solution

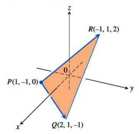
The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ}=(2-1)\boldsymbol{i}+(1+1)\boldsymbol{j}+(-1-0)\boldsymbol{k}=\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$$
$$\overrightarrow{PR}=(-1-1)\boldsymbol{i}+(1+1)\boldsymbol{j}+(2-0)\boldsymbol{k}=-2\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$$

$$\begin{aligned}\overrightarrow{PQ}\times\overrightarrow{PR}&=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2 & -1\\ -2 & 2 & 2\end{vmatrix}\\ &=\begin{vmatrix}2 & -1\\ 2 & 2\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}1 & -1\\ -2 & 2\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}1 & 2\\ -2 & 2\end{vmatrix}\boldsymbol{k}\\ &=6\boldsymbol{i}+6\boldsymbol{k}\end{aligned}$$

Example 3

Find the area of the triangle with vertices $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.



Solution

The area of the parallelogram determined by P, Q, and R is

$$\begin{aligned}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|&=\left|6\boldsymbol{i}+6\boldsymbol{k}\right|\\ &=\sqrt{6^2+6^2}=\sqrt{2\cdot 36}=6\sqrt{2}\end{aligned}$$

The triangle area is half of this i.e., $3\sqrt{2}$.

Example 4

Find a unit vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

Solution

Since $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane, its direction $\boldsymbol{n}$ is a unit vector perpendicular to the plane. Taking values from examples 2 and 3, we have

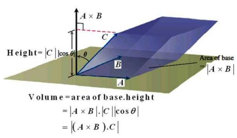
$$\boldsymbol{n}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\frac{6\boldsymbol{i}+6\boldsymbol{k}}{6\sqrt{2}}=\frac{1}{\sqrt{2}}\boldsymbol{i}+\frac{1}{\sqrt{2}}\boldsymbol{k}$$

The Scalar Triple Product

The product $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$ is called the *scalar triple product of* $\boldsymbol{A},\boldsymbol{B},$ and $\boldsymbol{C}$ (in that order). We can see from the formula

$$\left|(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}\right|=\left|\boldsymbol{A}\times\boldsymbol{B}\right|\left|\boldsymbol{C}\right|\left|\cos\theta\right| \quad \text{........ (2)}$$

that the absolute value of the product is the volume of the parallelepiped determined by $\boldsymbol{A},\boldsymbol{B},$ and $\boldsymbol{C}$ (from figure).



The number $\left|\boldsymbol{A}\times\boldsymbol{B}\right|$ is the area of the base parallelogram. The number $\left|\boldsymbol{C}\right|\left|\cos\theta\right|$ is the parallelepiped's height. Because of this geometry, $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$ is also called the **box product** of $\boldsymbol{A},\boldsymbol{B},$ and $\boldsymbol{C}$.

By treating the planes of $\boldsymbol{B}$ and $\boldsymbol{C}$ and of $\boldsymbol{C}$ and $\boldsymbol{A}$ as the base planes of the parallelepiped determined by $\boldsymbol{A},\boldsymbol{B},$ and $\boldsymbol{C}$, we see that

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=(\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{A}=(\boldsymbol{C}\times\boldsymbol{A})\cdot\boldsymbol{B} \quad \text{....... (3)}$$

Since the dot product is commutative, equation (3) also gives

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) \quad \text{....... (4)}$$

The dot and cross may be interchanged in a scalar triple product without altering its value.

The scalar triple product can be evaluated as a determinant.

$$\begin{aligned}\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})&=\boldsymbol{A}\cdot\left[\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}\boldsymbol{k}\right]\\ &=a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix}-a_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix}+a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}\\ &=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\end{aligned}$$

Thus: $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$ .... (5)

Example 5

Find the volume of the box (parallelepiped) determined by $\boldsymbol{A}=\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$, $\boldsymbol{B}=-2\boldsymbol{i}+3\boldsymbol{k}$, and $\boldsymbol{C}=7\boldsymbol{j}-4\boldsymbol{k}$.

Solution

$$\begin{aligned}\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})&=\begin{vmatrix}1 & 2 & -1\\ -2 & 0 & 3\\ 0 & 7 & -4\end{vmatrix}=\begin{vmatrix}0 & 3\\ 7 & -4\end{vmatrix}-2\begin{vmatrix}-2 & 3\\ 0 & -4\end{vmatrix}-\begin{vmatrix}-2 & 0\\ 0 & 7\end{vmatrix}\\ &=-21-16+14=-23\end{aligned}$$

The volume is $\left|\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})\right|=\left|-23\right|=23$ units cubied.

IP1.

If $\boldsymbol{u} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, then find $(\boldsymbol{u} \times \boldsymbol{v})$ and $(\boldsymbol{v} \times \boldsymbol{u})$.

Solution:

$$(\boldsymbol{u} \times \boldsymbol{v}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= \boldsymbol{i}(2-1) - \boldsymbol{j}(-2-2) + \boldsymbol{k}(1+2)$$

$$= \boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$$

$$(\boldsymbol{v} \times \boldsymbol{u}) = -(\boldsymbol{u} \times \boldsymbol{v}) = -\boldsymbol{i} - 4\boldsymbol{j} - 3\boldsymbol{k}$$

IP2.

Find a vector perpendicular to the plane $\boldsymbol{PQR}$ where $\boldsymbol{P}(\mathbf{1},\mathbf{4},\mathbf{6})$, $\boldsymbol{Q}(-\mathbf{2},\mathbf{5},-\mathbf{1})$ and $\boldsymbol{R}(\mathbf{1},-\mathbf{1},\mathbf{1})$?

Solution:

Since the vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, it is perpendicular to the plane through P, Q and R

Now,

$$\overrightarrow{PQ} = (-2-1)\boldsymbol{i} + (5-4)\boldsymbol{j} + (-1-6)\boldsymbol{k} = -3\boldsymbol{i} + \boldsymbol{j} - 7\boldsymbol{k}$$

$$\overrightarrow{PR} = (1-1)\boldsymbol{i} + (-1-4)\boldsymbol{j} + (1-6)\boldsymbol{k} = -5\boldsymbol{j} - 5\boldsymbol{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= \boldsymbol{i}(-5-35) - \boldsymbol{j}(15-0) + \boldsymbol{k}(15-0)$$

$$= -40\boldsymbol{i} - 15\boldsymbol{j} + 15\boldsymbol{k}$$

Thus, the vector $-40\boldsymbol{i} - 15\boldsymbol{j} + 15\boldsymbol{k}$ is perpendicular to the given plane.

Any non-zero scalar multiple of this vector such as $-8\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$ is also perpendicular to the plane.

IP3.

Find the area of the triangle with vertices $\boldsymbol{P(2,-2,1)}$, $\boldsymbol{Q(3,-1,2)}$ and $\boldsymbol{R(3,-1,1)}$?

Solution:

The given vertices of a triangle are $P(2,-2,1), Q(3,-1,2)$ and $R(3,-1,1)$.

We know that the area A of the triangle formed by the vertices P, Q and R is

$$A = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

Now,

$$\overrightarrow{PQ} = (3-2)\boldsymbol{i} + (-1+2)\boldsymbol{j} + (2-1)\boldsymbol{k} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

$$\overrightarrow{PR} = (3-2)\boldsymbol{i} + (-1+2)\boldsymbol{j} + (1-1)\boldsymbol{k} = \boldsymbol{i} + \boldsymbol{j}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \boldsymbol{i}(0-1) - \boldsymbol{j}(0-1) + \boldsymbol{k}(1-1)$$

$$= -\boldsymbol{i} + \boldsymbol{j}$$

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

Therefore, Area is $A = \frac{1}{2}\left(\sqrt{2}\right) = \frac{\sqrt{2}}{2}$

IP4.

Find the volume of the parallelepiped (box) whose coterminous edges are represented by the vectors $2\boldsymbol{i} - 3\boldsymbol{j}, \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $3\boldsymbol{i} - \boldsymbol{k}$?

Solution:

Let $A = 2\boldsymbol{i} - 3\boldsymbol{j}$, $B = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $C = 3\boldsymbol{i} - \boldsymbol{k}$

Now,

$$(A \times B).C = \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= 2(-1 - 0) + 3(-1 + 3) + 0$$

$$= -2 + 6 = 4$$

Therefore, the volume of the parallelepiped (box) is

$$|(A \times B).C| = |4| = 4$$

P1.

If $\boldsymbol{A} = 2\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{B} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ then find $(\boldsymbol{A} \times \boldsymbol{B})$ and $(\boldsymbol{B} \times \boldsymbol{A})$

Solution:

$$(\boldsymbol{A} \times \boldsymbol{B}) = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= (1 - 0)\boldsymbol{i} - (2 - 0)\boldsymbol{j} + (-2 - 2)\boldsymbol{k}$$

$$= \boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$$

$$\boldsymbol{B} \times \boldsymbol{A} = -(\boldsymbol{A} \times \boldsymbol{B}) = -\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$$

P2.

Find a unit vector perpendicular to the plane $\boldsymbol{PQR}$ where $\boldsymbol{P(1,-1,2)}$, $\boldsymbol{Q(2,0,-1)}$ and $\boldsymbol{R(0,2,1)}$.

Solution:

Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, the unit vector $\boldsymbol{n}$ perpendicular to the plane is given by

$$\boldsymbol{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$

Now,

$$\overrightarrow{PQ} = (2-1)\boldsymbol{i} + (0+1)\boldsymbol{j} + (-1-2)\boldsymbol{k} = \boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$$

$$\overrightarrow{PR} = (0-1)\boldsymbol{i} + (2+1)\boldsymbol{j} + (1-2)\boldsymbol{k} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= \boldsymbol{i}(-1+9) - \boldsymbol{j}(-1-3) + \boldsymbol{k}(3+1)$$

$$= 8\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$$

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{8^2 + 4^2 + 4^2} = \sqrt{64 + 16 + 16} = 4\sqrt{6}$$

$$\boldsymbol{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|} = \frac{8\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}}{4\sqrt{6}} = \frac{2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}}{\sqrt{6}}$$

P3.

Find the area of the triangle with vertices $\boldsymbol{P}(\mathbf{1}, \mathbf{1}, \mathbf{1}), \boldsymbol{Q}(\mathbf{2}, \mathbf{1}, \mathbf{3})$ and $\boldsymbol{R}(\mathbf{0}, \mathbf{2}, \mathbf{1})$?

Solution:

The given vertices of a triangle are $P(1, 1, 1), Q(2, 1, 3)$ and $R(0, 2, 1)$.

We know that the area A of the triangle formed by the vertices P, Q and R is

$$A = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

Now,

$$\overrightarrow{PQ} = (2-1)\boldsymbol{i} + (1-1)\boldsymbol{j} + (3-1)\boldsymbol{k} = \boldsymbol{i} + 2\boldsymbol{k}$$

$$\overrightarrow{PR} = (0-1)\boldsymbol{i} + (2-1)\boldsymbol{j} + (1-1)\boldsymbol{k} = -\boldsymbol{i} + \boldsymbol{j}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \boldsymbol{i}(0-2) - \boldsymbol{j}(0+2) + \boldsymbol{k}(1-0)$$

$$= -2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{4+4+1} = 3$$

Therefore, Area is $A = \frac{1}{2}(3) = \frac{3}{2}$

P4.

Find the volume of the parallelepiped (box) whose coterminous edges are represented by the vectors $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$?

Solution:

Let $A = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $B = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $C = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

Now,

$$(A \times B).C = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 2(1-2) + 3(-1-4) + 1(1+2)$$

$$= -2 - 15 + 3 = -14$$

Therefore, the volume of the parallelepiped (box) is

$$|(A \times B).C| = |-14| = 14$$

1. Find $(\boldsymbol{u} \times \boldsymbol{v})$ and $(\boldsymbol{v} \times \boldsymbol{u})$

i. $\boldsymbol{u} = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{k}$

ii. $\boldsymbol{u} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j}$

iii. $\boldsymbol{u} = 2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

iv. $\boldsymbol{u} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{0}$

v. $\boldsymbol{u} = \frac{3}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

2. If $\boldsymbol{u} = 3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{w} = -4\boldsymbol{k}$ the find $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$ and $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$.

3. In I --- V problems

a. Find the area of the triangle determined by the points P, Q and R

b. Find a unit vector perpendicular to the plane PQR

I. $P(3,-1,2)$, $Q(2,0,-1)$, $R(0,2,1)$

II. $P(6,2,5)$, $Q(5,4,9)$, $R(-3,5,4)$

III. $P(-2,4,1)$, $Q(3,1,-1)$, $R(-1,2,-2)$

IV. $P(1,-1,2)$, $Q(2,0,-1)$, $R(0,2,1)$

V. $P(-2,2,0)$, $Q(0,1,-1)$, $R(-1,2,-2)$

4. In a ---- e problems verify that

$$(\boldsymbol{u} \times \boldsymbol{v}).\boldsymbol{w} = (\boldsymbol{v} \times \boldsymbol{w}).\boldsymbol{u} = (\boldsymbol{w} \times \boldsymbol{u}).\boldsymbol{v}$$

	$\boldsymbol{u}$	$\boldsymbol{v}$	$\boldsymbol{w}$
a.	$2\boldsymbol{i}$	$2\boldsymbol{j}$	$2\boldsymbol{k}$
b.	$\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$	$-\boldsymbol{i} - \boldsymbol{k}$	$2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$
c.	$-2\boldsymbol{j} + \boldsymbol{k}$	$-9\boldsymbol{k}$	$4\boldsymbol{i} + \boldsymbol{j}$
d.	$-10\boldsymbol{i} + 5\boldsymbol{k}$	$10\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$	$11\boldsymbol{i} + 12\boldsymbol{j} + 13\boldsymbol{k}$
e.	$-14\boldsymbol{i} + 15\boldsymbol{j} - \boldsymbol{k}$	$12\boldsymbol{i} - \boldsymbol{j}$	$8\boldsymbol{i} + 9\boldsymbol{j} + 10\boldsymbol{k}$

5. Find the volume of the parallelepiped whose coterminous edges are $3\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

6. Find the volume of the parallelepiped whose coterminous edges are $\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $2\boldsymbol{i} - 7\boldsymbol{j} + \boldsymbol{k}$.