2.7

Definite integrals

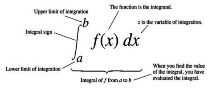
Learning objectives:

- To define the area under a curve as a definite integral.
- To evaluate definite integrals

  $$\int_a^b c\,dx\,,\ \int_a^b x\,dx\,,\ \int_a^b x^2\,dx \quad (a<b)$$

  AND
- To practice the related problems.

The terminology associated with symbol $\int_a^b f(x)dx$ is illustrated below.



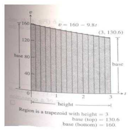
The value of the definite integral of a function over any particular interval depends on the function and not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)dt \ \text{ or } \int_a^b f(u)du \ \text{ instead of } \int_a^b f(x)dx$$

No matter how we write the integral, it is still the same number defined as a limit of Riemann sums. Since it does not matter what letter we use, the variable of integration is called a ***dummy variable***.

The Area under the Graph of a Non-negative Function

The sums, we used to estimate the height of the projectile in the projectile example in Module 3.3 (example 1), were Riemann sums for the projectile's velocity function $v=f(t)=160-9.8t$ on the interval $[0,3]$.



From the above figure, we see how the associated rectangles approximate the trapezoid between the t-axis and the curve $v=160-9.8t$. As the norm of the partition goes to zero, the rectangles fit the trapezoid with increasing accuracy and the sum of the areas they enclose approaches the trapezoid's area, which is

$$\text{Trapezoid area } = 3\cdot\frac{160+130.6}{2} = 435.9$$

Thus the sums we constructed in the projectile example approached a limit of 435.9. Since the limit of these sums is also the integral of f from 0 to 3, we know the value of the integral as well:

$$\int_0^3 (160-9.8t)dt = \text{trapezoid area} = 435.9$$

Definition

Let $f(x)\geq 0$ be continuous on $[a,b]$. The area of the region between the graph of f, the x-axis and the vertical lines $x=a$, $x=b$ is

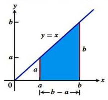
$$A=\int_a^b f(x)dx$$

Example 1:

Evaluate $\int_a^b x\,dx \qquad 0<a<b.$

Solution:

We sketch the region under the curve $y=x,\ a\leq x\leq b$.



The region is a trapezoid with height $(b-a)$ and bases a and b.

The area of the trapezoid $=(b-a).\frac{a+b}{2}=\frac{b^2}{2}-\frac{a^2}{2}$

The value of the integral is the area of the trapezoid:

$$\int_a^b x\,dx=\frac{b^2}{2}-\frac{a^2}{2}$$

For example,

$$\int_1^{\sqrt{5}} x\,dx=\frac{\left(\sqrt{5}\right)^2}{2}-\frac{(1)^2}{2}=2$$

We notice that $\frac{x^2}{2}$ is an anti-derivative of x, indicating a possible connection between anti-derivatives and summation.

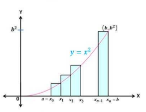
Example 2:

Find the area of the region between the parabola $y=x^2$ and the x-axis on the interval $[a,b]$, $0<a<b$.

Solution:

We evaluate the integral for the area as a limit of Riemann sums. We sketch the region and partition $[a,b]$ into n subintervals of length

$$\Delta x=\frac{(b-a)}{n}$$



The points of the partition are

$$x_0=a,\ x_1=a+\Delta x,\ x_2=a+2\Delta x,\ \cdots,$$
$$x_{n-1}=a+(n-1)\Delta x,\ x_n=a+n\Delta x=b$$

We are free to choose c_k's any way we please. We choose each c_k to be the right-hand endpoint of its subinterval. Thus $c_1=x_1, c_2=x_2, \dots, c_n=x_n$. The rectangles defined by these choices have areas

$$f(c_1)\Delta x=f(a+\Delta x)\Delta x=(a+\Delta x)^2\,\Delta x$$
$$f(c_2)\Delta x=f(a+2\Delta x)\Delta x=(a+2\Delta x)^2\,\Delta x$$
$$\vdots$$
$$f(c_n)\Delta x=f(a+n\Delta x)\Delta x=(a+n\Delta x)^2\,\Delta x$$

The sum of these areas

$$S_n=\sum_{k=1}^{n} f(c_k)\Delta x \quad =\sum_{k=1}^{n}(a+k\Delta x)^2\Delta x$$
$$=\sum_{k=1}^{n}\left(a^2+k^2(\Delta x)^2+2ak\Delta x\right)\Delta x$$
$$=\sum_{k=1}^{n}\left(a^2\Delta x+k^2(\Delta x)^3+2ak(\Delta x)^2\right)$$
$$=a^2\Delta x.n+(\Delta x)^3\sum_{k=1}^{n}k^2+2a(\Delta x)^2\sum_{k=1}^{n}k$$
$$=a^2\left(\frac{b-a}{n}\right).n+\left(\frac{b-a}{n}\right)^3\frac{n(n+1)(2n+1)}{6}+2a\left(\frac{b-a}{n}\right)^2.\frac{n(n+1)}{2}$$
$$=a^2(b-a)+(b-a)^3\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+a(b-a)^2\left(1+\frac{1}{n}\right)$$

We now use the definition of definite integral

$$\int_a^b f(x)dx=\lim_{\|P\|\to 0}\sum_{k=1}^{n}f(c_k)\Delta x$$

to find the area under the parabola from $x=a$ to $x=b$ a

$$\int_0^b x^2\,dx=\lim_{n\to\infty}S_n$$

$$=a^2(b-a)+\frac{(b-a)^3}{6}\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]+a(b-a)^2\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$=a^2(b-a)+\frac{(b-a)^3}{3}+a(b-a)^2=\frac{b^3}{3}-\frac{a^3}{3}$$

Again, we notice that $\frac{x^3}{3}$ is an anti-derivative of x^2, indicating a possible connection between anti-derivatives and definite integrals.

With different values of b, we get

$$\int_0^1 x^2\,dx=\frac{1^3}{3}=\frac{1}{3},\quad \int_0^{1.5}x^2\,dx=\frac{(1.5)^3}{3}=\frac{3.375}{3}=1.125,$$

and so on.

IP1:

i. $\int_3^5 x\,dx =$

A. 16 B. $\frac{1}{16}$ C. $\frac{3}{16}$ D. 8

Answer: **D**

Solution:

We have $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2},\ a < b$

$$\therefore \int_3^5 x\,dx = \frac{5^2}{2} - \frac{3^2}{2} = 8$$

ii. $\int_0^{\pi/2} \theta^2 d\theta =$

A. $\frac{\pi^2}{24}$ B. $\frac{\pi^3}{24}$ C. $\frac{\pi^2}{3}$ D. $\frac{\pi^2}{8}$

Answer: **B**

Solution:

We have $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3},\ a < b$

$$\therefore \int_0^{\pi/2} \theta^2 d\theta = \frac{(\pi/2)^3}{3} - 0 = \frac{\pi^3}{24}$$

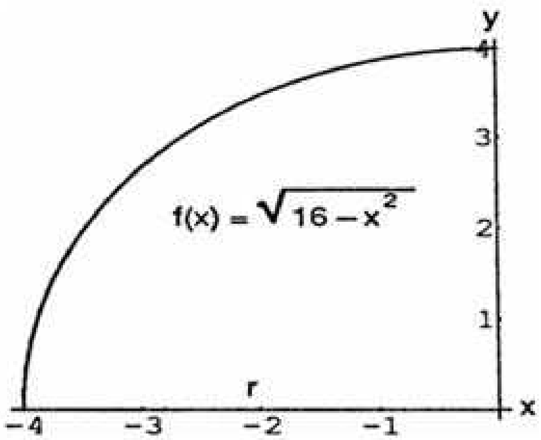
IP2:

Graph the integrand and use areas to evaluate the integral

$$\int_{-4}^{0} \sqrt{16-x^2}\ dx$$

Solution:

We have to compute the integral $\int_{-4}^{0} \sqrt{16-x^2}\ dx$ by using areas. Here the integrand is $f(x) = \sqrt{16-x^2}$, its graph is a circle with center at the origin and radius 4.



We know that area of the quarter circle is $A = \frac{\pi r^2}{4}$, where r is the radius of the circle.

$$\therefore A = \frac{\pi(4)^2}{4} = 4\pi \text{ Square units}$$

$$\therefore \int_{-4}^{0} \sqrt{16-x^2}\ dx = 4\pi$$

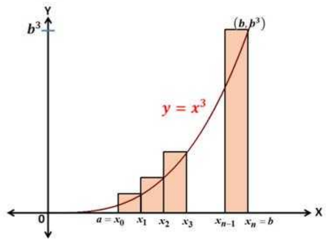
IP3:

Use a definite integral to find the area of the region between the curves $y = x^3$ and the x –axis on the interval $[a, b]$.

Solution:

We have to compute the area of under the graph of the function $y = x^3$ and the x –axis on the interval $[a, b]$.

Now, we sketch the region and partition $[a, b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$



The points of the partition are

$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ldots\ldots.. x_n = b$

Choose each c_k to be the right hand endpoint of its subintervals.

Thus $c_1 = x_1, \quad c_2 = x_2, \quad c_3 = x_3, \ldots, c_n = x_n$. The rectangles defined by these choices have areas

$$f(c_1)\Delta x = f(a+\Delta x)\Delta x = \left[(a+\Delta x)^3\right]\Delta x$$

$$f(c_2)\Delta x = f(a+2\Delta x)\Delta x = \left[(a+2\Delta x)^3\right]\Delta x$$

$$f(c_3)\Delta x = f(a+3\Delta x)\Delta x = \left[(a+3\Delta x)^3\right]\Delta x$$

$$\vdots$$

$$\vdots$$

$$f(c_n)\Delta x = f(a+n\Delta x)\Delta x = \left[(a+n\Delta x)^3\right]\Delta x$$

Then the sum of these areas is

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x \quad = \sum_{k=1}^{n}\left[(a+k\Delta x)^3\right]\Delta x$$

$$= \sum_{k=1}^{n}\left[a^3 + k^3(\Delta x)^3 + 3a^2k(\Delta x) + 3ak^2(\Delta x)^2\right]\Delta x$$

$$= a^3\Delta x + (\Delta x)^4\sum_{k=1}^{n}k^3 + 3a^2(\Delta x)^2\sum_{k=1}^{n}k + 3a(\Delta x)^3\sum_{k=1}^{n}k^2$$

$$= a^3\left(\frac{b-a}{n}\right).n + \left(\frac{b-a}{n}\right)^4.\frac{n^2(n+1)^2}{4} + 3a^2\left(\frac{b-a}{n}\right)^2.\frac{n(n+1)}{2}$$
$$+ 3a\left(\frac{b-a}{n}\right)^3\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= a^3(b-a) + \frac{(b-a)^4}{4}.\left(1+\frac{1}{n}\right)^2 + \frac{3a^2(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$
$$+ \frac{3a(b-a)^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

We now use the definition of definite integral

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x$$

to find the area under the graph from $x = a$ to $x = b$ as

$$\int_a^b x^3\,dx = \lim_{n\to\infty} S_n$$

$$= a^3(b-a) + \frac{(b-a)^4}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 + \frac{3a^2(b-a)^2}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$+ \frac{3a(b-a)^3}{6}\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= a^3(b-a) + \frac{(b-a)^4}{4} + \frac{3a^2(b-a)^2}{2} + a(b-a)^3$$

$$= \frac{b^4}{4} - \frac{a^4}{4}$$

$$\boxed{\therefore \int_a^b x^3\,dx = \frac{b^4}{4} - \frac{a^4}{4}}$$

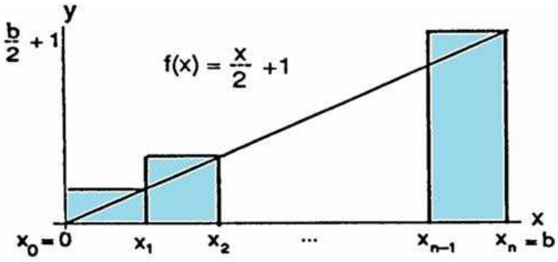
IP4:

Use a definite integral to find the area of the region between the curves $y = \frac{x}{2} + 1$ and the x −axis on the interval $[0, b]$.

Solution:

We have to compute the area of under the graph of the function $y = \frac{x}{2} + 1$ and the x −axis on the interval $[0, b]$.

Now, we sketch the region and partition $[0, b]$ into n subintervals of length $\Delta x = \frac{b-0}{n} = \frac{b}{n}$.



The points of the partition are

$x_0 = 0, \quad x_1 = \Delta x, \quad x_2 = 2\Delta x, \ldots\ldots\ x_n = n\Delta x = b$

Choose each c_k to be the right hand endpoint of its subintervals.

Thus $c_1 = x_1, \quad c_2 = x_2, \quad c_3 = x_3$ and so on. The rectangles defined by these choices have areas

$$f(c_1)\Delta x = f(\Delta x)\Delta x = \left[\frac{\Delta x}{2} + 1\right]\Delta x = \frac{1}{2}(\Delta x)^2 + \Delta x$$

$$f(c_2)\Delta x = f(2\Delta x)\Delta x = \left[\frac{2\Delta x}{2} + 1\right]\Delta x = \frac{1}{2}.2(\Delta x)^2 + \Delta x$$

$$f(c_3)\Delta x = f(3\Delta x)\Delta x = \left[\frac{3\Delta x}{2} + 1\right]\Delta x = \frac{1}{2}.3(\Delta x)^2 + \Delta x$$

$$\vdots$$

$$\vdots$$

$$f(c_n)\Delta x = f(n\Delta x)\Delta x = \left[\frac{n\Delta x}{2} + 1\right]\Delta x = \frac{1}{2}.n(\Delta x)^2 + \Delta x$$

Then the sum of these areas is

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x \quad = \sum_{k=1}^{n}\left[\frac{1}{2}k(\Delta x)^2 + \Delta x\right]$$

$$= \frac{(\Delta x)^2}{2}\sum_{k=1}^{n} k + \Delta x \sum_{k=1}^{n} 1$$

$$= \frac{b^2}{2n^2}\frac{n(n+1)}{2} + \frac{b}{n}.n = \frac{b^2}{4}\left(1 + \frac{1}{n}\right) + b$$

We now use the definition of definite integral

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x$$

to find the area under the graph from $x = 0$ to $x = 3$ as

$$\int_0^b \left(\frac{x}{2} + 1\right) dx = \lim_{n \to \infty} S_n$$

$$= \lim_{n \to \infty} \frac{b^2}{4}\left(1 + \frac{1}{n}\right) + b$$

$$= \frac{b^2}{4} + b$$

P1:

Find the following :

i. $\int_{a}^{\sqrt{3}a} x\ dx$

ii. $\int_{0}^{3b} x^2 dx$

Solution:

i. We have $\int_{a}^{b} x\ dx = \frac{b^2}{2} - \frac{a^2}{2},\ a < b$

$$\therefore \int_{a}^{\sqrt{3}a} x\,dx = \frac{\left(\sqrt{3}a\right)^2}{2} - \frac{a^2}{2} = a^2$$

ii. We have $\int_{a}^{b} x^2\ dx = \frac{b^3}{3} - \frac{a^3}{3},\ a < b$

$$\therefore \int_{0}^{3b} x^2 dx = \frac{(3b)^3}{3} - 0 = 9b^3$$

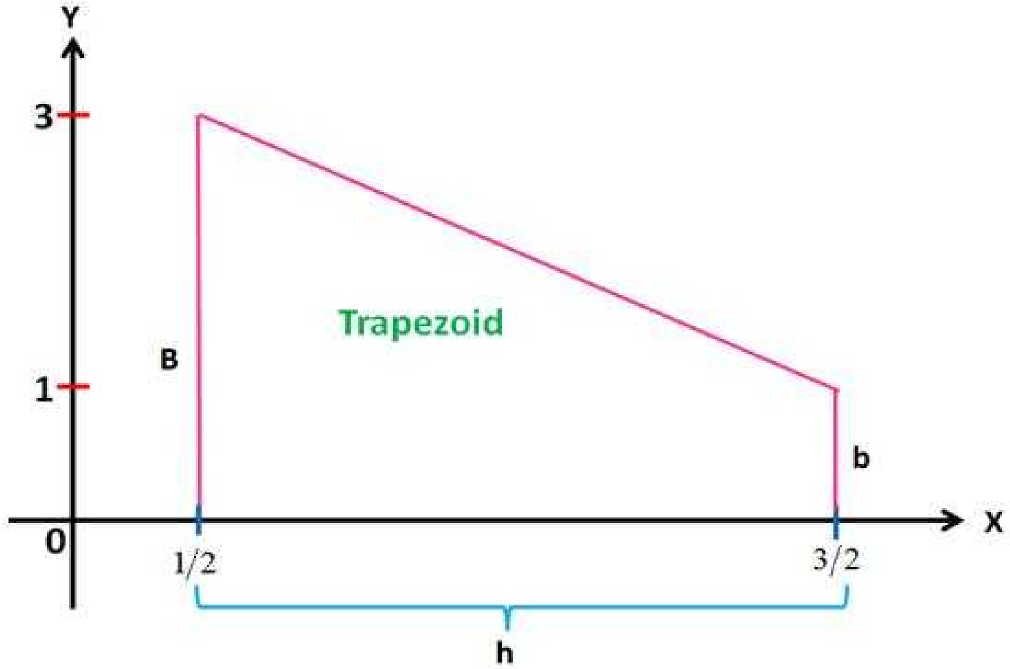
P2:

Graph the integrand and use areas to evaluate the integral

$$\int_{1/2}^{3/2} (-2x+4)\,dx$$

Solution:

We have to compute the integral $\int_{1/2}^{3/2} (-2x+4)\,dx$ by using the areas. Here integrand is $f(x) = (-2x+4)$ and its graph is a straight line with $f\left(\frac{1}{2}\right) = 3,\ f\left(\frac{3}{2}\right) = 1$.



We know the area of the trapezoid is $A = \frac{1}{2}h[B+b]$

From the graph $h = \frac{3}{2} - \frac{1}{2} = 1,\ B = f\left(\frac{1}{2}\right) = 3,\ b = f\left(\frac{3}{2}\right) = 1$

$\therefore A = \frac{1}{2}(1)[3+1] = 2$ Square units

$$\int_{1/2}^{3/2} (-2x+4)\,dx = 2$$

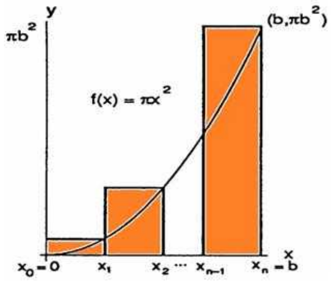
P3:

Use a definite integral to find the area of the region between the curves $y = \pi x^2$ and the x −axis on the interval $[0, b]$.

Solution:

We have to compute the area of under the graph of the function $y = \pi x^2$ and the x −axis on the interval $[0, b]$.

Now, we sketch the region and partition $[0, b]$ into n subintervals of length $\Delta x = \frac{b-0}{n} = \frac{b}{n}$.



The points of the partition are

$x_0 = 0, \quad x_1 = \Delta x, \quad x_2 = 2\Delta x, \ldots\ldots\, x_n = n\Delta x = b$

Choose each c_k to be the right hand endpoint of its subintervals.

Thus, $c_1 = x_1, \quad c_2 = x_2, \quad c_3 = x_3$ and so on. The rectangles defined by these choices have areas

$$f(c_1)\Delta x = f(\Delta x)\Delta x = \pi(\Delta x)^2\,\Delta x = \pi.1^2(\Delta x)^3$$
$$f(c_2)\Delta x = f(2\Delta x)\Delta x = \pi(2\Delta x)^2\,\Delta x = \pi.2^2(\Delta x)^3$$
$$f(c_3)\Delta x = f(3\Delta x)\Delta x = \pi(3\Delta x)^2\,\Delta x = \pi.3^2(\Delta x)^3$$
$$\vdots$$
$$\vdots$$
$$f(c_n)\Delta x = f(n\Delta x)\Delta x = \pi(n\Delta x)^2\,\Delta x = \pi.n^2(\Delta x)^3$$

Then the sum of these areas is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} \pi k^2(\Delta x)^3 = \pi(\Delta x)^3 \sum_{k=1}^{n} k^2 \\ &= \pi\frac{b^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{\pi b^3}{6}\cdot\frac{(n+1)(2n+1)}{n^2} \\ &= \frac{\pi b^3}{6}\cdot\left(2+\frac{3}{n}+\frac{1}{n^2}\right) \end{aligned}$$

We now use the definition of definite integral

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x$$

to find the area under the graph from $x = a$ to $x = b$ as

$$\begin{aligned} \int_0^b \pi x^2\,dx &= \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{\pi b^3}{6}\cdot\left(2+\frac{3}{n}+\frac{1}{n^2}\right) \\ &= \frac{\pi b^3}{6}\cdot(2+0+0) = \frac{\pi b^3}{3} \end{aligned}$$

$$\boxed{\therefore \int_0^b \pi x^2\,dx = \frac{\pi b^3}{3}}$$

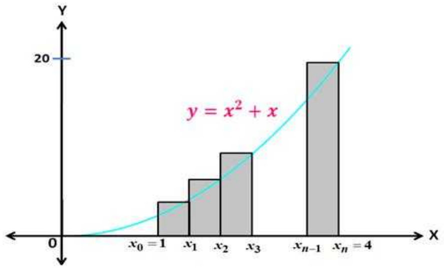
P4:

Use a definite integral to find the area of the region between the curves $y = x^2 + x$ and the x −axis on the interval $[1, 4]$.

Solution:

We have to compute the area of under the graph of the function $y = x^2 + x$ and the x −axis on the interval $[1, 4]$.

Now, we sketch the region and partition $[1, 4]$ into n subintervals of length $\Delta x = \frac{4-1}{n} = \frac{3}{n}$.



The points of the partition are

$$x_0 = 1, \quad x_1 = 1 + \frac{3}{n}, \quad x_2 = 1 + 2\left(\frac{3}{n}\right), \dots x_n = 1 + n\left(\frac{3}{n}\right) = 4$$

Choose each c_k to be the right hand endpoint of its subintervals.

Thus $c_1 = x_1, \quad c_2 = x_2, \quad c_3 = x_3$ and so on. The rectangles defined by these choices have areas

$$f(c_1)\Delta x = f\left(1+\frac{3}{n}\right)\Delta x = \left[\left(1+\frac{3}{n}\right)^2 - \left(1+\frac{3}{n}\right)\right]\Delta x$$

$$f(c_2)\Delta x = f\left(1+2\left(\frac{3}{n}\right)\right)\Delta x = \left[\left(1+2\left(\frac{3}{n}\right)\right)^2 - \left(1+2\left(\frac{3}{n}\right)\right)\right]\Delta x$$

$$f(c_3)\Delta x = f\left(1+3\left(\frac{3}{n}\right)\right)\Delta x = \left[\left(1+3\left(\frac{3}{n}\right)\right)^2 - \left(1+3\left(\frac{3}{n}\right)\right)\right]\Delta x$$

$$\vdots$$

$$\vdots$$

$$f(c_n)\Delta x = f\left(1+n\left(\frac{3}{n}\right)\right)\Delta x = \left[\left(1+n\left(\frac{3}{n}\right)\right)^2 - \left(1+n\left(\frac{3}{n}\right)\right)\right]\Delta x$$

Then the sum of these areas is

$$\begin{aligned}
S_n = \sum_{k=1}^{n} f(c_k)\Delta x \quad &= \Delta x \sum_{k=1}^{n}\left[\left(1+k\left(\frac{3}{n}\right)\right)^2 + \left(1+k\left(\frac{3}{n}\right)\right)\right] \\
&= (\Delta x)\sum_{k=1}^{n}\left[1+k^2\left(\frac{3}{n}\right)^2 + 2k\left(\frac{3}{n}\right) + 1 + k\left(\frac{3}{n}\right)\right] \\
&= \left(\frac{3}{n}\right)\sum_{k=1}^{n}\left[2+k^2\left(\frac{3}{n}\right)^2 + 3k\left(\frac{3}{n}\right)\right] \\
&= \left(\frac{3}{n}\right)\left\{2n+\left(\frac{3}{n}\right)^2\sum_{k=1}^{n}k^2 + \left(\frac{9}{n}\right)\sum_{k=1}^{n}k\right\} \\
&= \left(\frac{3}{n}\right)\left[2n+\left(\frac{9}{n^2}\right)\frac{n(n+1)(2n+1)}{6} + \left(\frac{9}{n}\right)\frac{n(n+1)}{2}\right] \\
&= 3\left[2+\frac{3}{2}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\frac{9}{2}\left(1+\frac{1}{n}\right)\right]
\end{aligned}$$

We now use the definition of definite integral

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x$$

to find the area under the graph from $x = 1$ to $x = 4$ as

$$\begin{aligned}
\int_1^4 \left(x^2+x\right)dx &= \lim_{n\to\infty} S_n \\
&= \lim_{n\to\infty} 3\left[2+\frac{3}{2}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\frac{9}{2}\left(1+\frac{1}{n}\right)\right] \\
&= \frac{57}{2}
\end{aligned}$$

2.7. Definite integrals

Exercise:

1. Graph the integrands and use areas to evaluate the integrals.

   a. $\int_{-2}^{4}\left(\frac{x}{2}+3\right)dx$

   b. $\int_{-3}^{3}\sqrt{9-x^2}\ dx$

   c. $\int_{-2}^{1}|x|\ dx$

   d. $\int_{-1}^{1}(2-|x|)\ dx$

   e. $\int_{0}^{b} x\ dx\ ,\qquad b>0$

   f. $\int_{a}^{b} 2s\ ds\ ,\qquad 0<a<b$

2. Evaluate the following integrals :

   a. $\int_{1}^{\sqrt{2}} x\ dx$

   b. $\int_{\pi}^{2\pi} \theta\ d\theta$

   c. $\int_{0}^{\sqrt[3]{7}} x^2\ dx$

   d. $\int_{0}^{1/2} t^2\ dt$

   e. $\int_{a}^{2a} x\ dx$

   f. $\int_{0}^{\sqrt[3]{b}} x^2\ dx$

3. use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$.

   a. $y = 3x^2$

   b. $y = 2x$

2.8

Properties of Definite Integrals

Learning objectives:

- To study the properties of the definite integrals.

AND

- To practice the related problems.

In this module, we describe working rules for integrals.

The following rules hold for definite integrals.

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — Also a Definition
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any Number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

These rules will enable us to add and subtract definite integrals, multiply their integrands by constants, and compare them with other definite integrals.

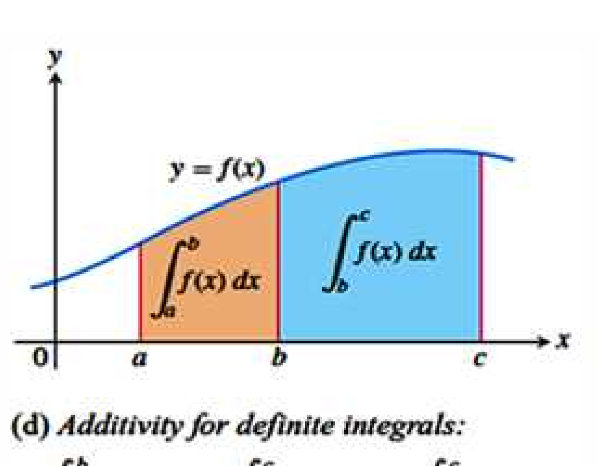
All the rules except the first two *follow from the way the integrals are defined with Riemann sums.* We omit the proofs. As the sums have these properties so their rules should have them too. Although this is not that simple, for the present we contend with this argument. The proofs will be covered in advanced calculus.

Rules 1 and 2 are definitions. We want every integral over an interval of zero length to be zero. Rule 1 extends the definition of definite integral to allow for the case $a = b$.

Rules 3 and 4 are like the analogous rules for limits and indefinite integrals. Once we know the integrals of two functions, we automatically know the integrals of all constant multiples of these functions and their sums and differences. We can also use Rules 3 and 4 repeatedly to evaluate integrals of arbitrary finite linear combinations of integrable functions term by term. For any constants $c_1, \cdots, c_n$, regardless of sign, and functions $f_1(x), \cdots, f_n(x)$, integrable on $[a, b]$,

$$\int_a^b \left(c_1 f_1(x) + \cdots + c_n f_n(x)\right) dx$$
$$= c_1 \int_a^b f_1(x)\,dx + \cdots + c_n \int_a^b f_n(x)\,dx$$

The figure below illustrates Rule 5 with a positive function, but the rule applies to any integrable function.



(d) *Additivity for definite integrals:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

The Additivity for definite integrals:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$
$$\int_b^c f(x)\,dx = \int_a^c f(x)\,dx - \int_a^b f(x)\,dx$$

Example 1:

Suppose that

$\int_{-1}^1 f(x)\,dx = 5$, $\int_{-1}^1 h(x)\,dx = 7$, $\int_1^4 f(x)\,dx = -2$. Then

(i) $\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -(-2) = 2$

(ii) $\int_{-1}^1 [2f(x) + 3h(x)]\,dx = 2\int_{-1}^1 f(x)\,dx + 3\int_{-1}^1 h(x)\,dx$
$= 2(5) + 3(7) = 31$

(iii) $\int_{-1}^4 f(x)\,dx = \int_{-1}^1 f(x)\,dx + \int_1^4 f(x)\,dx$
$= 5 + (-2) = 3$

In the previous module and in this module, we learned to evaluate three general integrals:

$$\int_a^b c\,dx = c(b-a) \qquad (\text{Any constant } c)$$

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2} \qquad (0 < a < b)$$

$$\int_0^b x^2\,dx = \frac{b^3}{3} \qquad (b > 0)$$

The rules stated earlier for the definite integrals will enable us to build on these results.

Example 2:

$$\int_0^2 \left(\frac{t^2}{4} - 7t + 5\right) dt = \frac{1}{4}\int_0^2 t^2\,dt - 7\int_0^2 t\,dt + \int_0^2 5\,dt$$
$$= \frac{1}{4}\frac{2^3}{3} - 7\frac{2^2}{2} + 5(2-0) = -\frac{10}{3}$$

Example 3:

Evaluate $\int_2^3 x^2\,dx$

Solution:

We cannot apply the equation $\int_0^b x^2\,dx = \frac{b^3}{3}$ directly because the lower limit of integration is different from 0. We can, however, use the Additivity Rule to express $\int_2^3 x^2\,dx$ as a difference of two integrals that can be evaluated with this equation.

$$\int_0^2 x^2\,dx + \int_2^3 x^2\,dx = \int_0^3 x^2\,dx$$
$$\int_2^3 x^2\,dx = \int_0^3 x^2\,dx - \int_0^2 x^2\,dx = \frac{3^3}{3} - \frac{2^3}{3} = \frac{19}{3}$$

The Max-Min inequality for definite integrals says that $m(b-a)$ is a ***lower bound*** for the value of $\int_a^b f(x)\,dx$ and $M(b-a)$ is an ***upper bound***.

Example 4:

Show that the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ cannot possibly be 2.

Solution:

The maximum value of $\sqrt{1+\cos x}$ on [0, 1] is $\sqrt{1+1} = \sqrt{2}$, so

$$\int_0^1 \sqrt{1+\cos x}\,dx \le max.\sqrt{1+\cos x}\,.\,(1-0) \le \sqrt{2}.1 = \sqrt{2}$$

The integral cannot exceed $\sqrt{2}$, so it cannot possibly equal 2.

Example 5:

Use the inequality $\cos x \ge \left(1 - x^2/2\right)$, which holds for all x, to find a lower bound for the value of $\int_0^1 \cos x\,dx$.

Solution:

$$\int_0^1 \cos x\,dx \ge \int_0^1 \left(1 - \frac{x^2}{2}\right) dx$$
$$\ge \int_0^1 1\,dx - \frac{1}{2}\int_0^1 x^2\,dx$$
$$\ge 1 \cdot (1-0) - \frac{1}{2} \cdot \left(1^3/3\right) = 5/6 \approx 0.83$$

The value of the integral is at least $\frac{5}{6}$.

IP1:

Suppose that $\boldsymbol{h}$ is integrable and that $\int_{-1}^{1} h(r)\,dr = 0$ and $\int_{-1}^{3} h(r)\,dr = 6$ then find

a. $\int_{1}^{3} h(r)dr$ b. $-\int_{3}^{1} h(r)dr$

Solution:

$$(a)\ \int_{-1}^{3} h(r)\,dr = \int_{-1}^{1} h(r)\,dr + \int_{1}^{3} h(r)\,dr$$

$$(Additivity)$$

$$\therefore\ \int_{1}^{3} h(r)\,dr = \int_{-1}^{3} h(r)\,dr - \int_{-1}^{1} h(r)\,dr = 6 - 0 = 6$$

$$(b)\ -\int_{3}^{1} h(r)\,dr = -(-1)\int_{1}^{3} h(u)\,du$$

$$(order\ \ of\ \ int\,egration)$$

$$= \int_{1}^{3} h(r)\,dr \qquad (\because u\ is\ \ dummy)$$

$$= 6 \qquad (by(a)\ \ above)$$

IP2:

Find the following:

I. $\int_2^1 \left(1+\frac{u}{2}\right) du$

II. $\int_0^{\sqrt{2}} \left(t-\sqrt{2}\right) dt$

Solution:

$$i.\ \int_2^1 \left(1+\frac{u}{2}\right) du = -\int_1^2 \left(1+\frac{u}{2}\right) du$$

$$= -\int_1^2 du - \frac{1}{2}\int_1^2 u\, du$$

$$= -[u]_1^2 - \frac{1}{2}\left[\frac{u^2}{2}\right]_1^2$$

$$= -[2-1] - \frac{1}{2}\left[\frac{2^2}{2} - \frac{1^2}{2}\right] = -\frac{7}{4}$$

$$ii.\ \int_0^{\sqrt{2}} \left(t-\sqrt{2}\right) dt = \int_0^{\sqrt{2}} t\ dt - \int_0^{\sqrt{2}} \sqrt{2}\ dt$$

$$= \int_0^{\sqrt{2}} t\ dt - \sqrt{2}\int_0^{\sqrt{2}} dt$$

$$= \left[\frac{t^2}{2}\right]_0^{\sqrt{2}} - \sqrt{2}\,[t]_0^{\sqrt{2}}$$

$$= \left[\frac{\left(\sqrt{2}\right)^2}{2} - 0\right] - \sqrt{2}\left[\sqrt{2}-0\right] = -1$$

IP3:

Show that the value $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2}$ and 3.

Solution:

Given $\int_0^1 \sqrt{x+8}\,dx$

The integrand $f(x) = \sqrt{x+8}$ is increasing on $[0, 1]$

$\Rightarrow max\, f = f(1) = \sqrt{1+8} = 3$ and

$min\, f = f(0) = \sqrt{0+8} = 2\sqrt{2}$

From the Max-Min inequality, we have

If f has maximum value $max\, f$ and minimum value $min\, f$ on $[a, b]$, then

$$min\, f\,(b-a) \le \int_a^b f(x)\,dx \le max\, f\,(b-a)$$

$$\Rightarrow 2\sqrt{2}(1-0) \le \int_0^1 \sqrt{x+8}\,dx \le 3(1-0)$$

$$\Rightarrow 2\sqrt{2} \le \int_0^1 \sqrt{x+8}\,dx \le 3$$

IP4:

Use Max-Min inequality to find upper and lower bounds for the integrals $\int_0^{0.5} \frac{1}{1+x^2}\,dx$ and $\int_{0.5}^{1} \frac{1}{1+x^2}\,dx$

Add these two integrals to arrive at an improved estimate of $\int_0^{1} \frac{1}{1+x^2}\,dx$

Solution:

We have to compute the upper and lower bounds of the integral $\int_0^{0.5} \frac{1}{1+x^2}\,dx$. Here integrand is $f(x) = \frac{1}{1+x^2}$,

$\Rightarrow f$ is decreasing function on $[0, 0.5]$

$\therefore$ Maximum value of f occurs at $x = 0$

$$\Rightarrow max\, f = f(0) = \frac{1}{1+0^2} = 1$$

$\therefore$ Minimum value of f occurs at $x = 0.5$

$$\Rightarrow min\, f = f(0) = \frac{1}{1+(0.5)^2} = 0.8$$

By Max-Min inequality, we have

$$(0.5-0)\,min\, f \le \int_0^{0.5} \frac{1}{1+x^2}\,dx \le (0.5-0)\,max\, f$$

$$\Rightarrow 0.5(0.8) \le \int_0^{0.5} \frac{1}{1+x^2}\,dx \le 0.5(1)$$

$$\Rightarrow 0.4 \le \int_0^{0.5} \frac{1}{1+x^2}\,dx \le 0.5 \qquad \ldots(1)$$

$\Rightarrow$ Upper bound is 0.5 and Lower bound is 0.4

Again, we have to compute the upper and lower bounds of the integral $\int_{0.5}^{1} \frac{1}{1+x^2}\,dx$. Here integrand is $f(x) = \frac{1}{1+x^2}$,

$\Rightarrow f$ is decreasing function on $[0.5, 1]$

$\therefore$ Maximum value of f occurs at $x = 0.5$

$$\Rightarrow \max f = f(0.5) = \frac{1}{1+(0.5)^2} = 0.8$$

$\therefore$ Minimum value of f occurs at $x = 1$

$$\Rightarrow \min f = f(0) = \frac{1}{1+(1)^2} = 0.5$$

By Max-Min inequality, we have

$$(1-0.5)\,min\, f \le \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le (1-0.5)\,max\, f$$

$$\Rightarrow 0.5(0.5) \le \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le 0.5(0.8)$$

$$\Rightarrow 0.25 \le \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le 0.4 \qquad \ldots\ldots(2)$$

$\Rightarrow$ Upper bound is 0.4 and Lower bound is 0.25

By adding (1) and (2), we get

$$0.25 + 0.4 \le \int_0^{0.5} \frac{1}{1+x^2}\,dx + \int_{0.5}^{1} \frac{1}{1+x^2}\,dx \le 0.5 + 0.4$$

$$\Rightarrow 0.65 \le \int_0^{1} \frac{1}{1+x^2}\,dx \le 0.9$$

$\Rightarrow$ Upper bound is 0.9 and Lower bound is 0.65

P1:

Suppose that *f* and *h* are integrable and that $\int_1^9 f(x) = -1$

$\int_7^9 f(x)\,dx = 5$ **and** $\int_7^9 h(x)\,dx = 4$, **then find**

i. $\int_7^9 [2f(x) - 3h(x)]\,dx$

ii. $\int_9^7 [h(x) - f(x)]\,dx$

iii. $\int_1^7 -2f(x)\,dx$

P1:

Suppose that f and h are integrable and that $\int_1^9 f(x) = -1$

$\int_7^9 f(x)\,dx = 5$ and $\int_7^9 h(x)\,dx = 4$, then find

i. $\int_7^9 \left[2f(x) - 3h(x)\right] dx$

ii. $\int_9^7 \left[h(x) - f(x)\right] dx$

iii. $\int_1^7 -2f(x)\,dx$

Solution:

$$i.\quad \int_7^9 \left[2f(x) - 3h(x)\right] dx = \int_7^9 2f(x)\,dx - \int_7^9 3h(x)\,dx$$

$$= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx$$

$$(\textit{By Difference rule and constant rule})$$

$$= 2(5) - 3(4) = -2$$

$$ii.\quad \int_9^7 \left[h(x) - f(x)\right] dx = \int_9^7 h(x)\,dx - \int_9^7 f(x)\,dx$$

$$(\textit{Difference rule})$$

$$= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx$$

$$(\textit{order of integration})$$

$$= -4 + 5 = 1$$

$$iii.\quad \int_1^9 -2f(x)\,dx = \int_1^7 -2f(x)\,dx + \int_7^9 -2f(x)\,dx$$

$$(\textit{Additivity})$$

$$\Rightarrow \int_1^7 -2f(x)\,dx = \int_1^9 -2f(x)\,dx - \int_7^9 -2f(x)\,dx$$

$$= -2\int_1^9 f(x)\,dx + 2\int_7^9 f(x)\,dx$$

$$(\textit{Constant multiple rule})$$

$$= -2(-1) + 2(5) = 12$$

P2:

Find the following

$$i. \int_1^4 \left(2x^2 - x + 5\right) dx$$

$$ii. \int_1^0 \left(3x^2 + x - 5\right) dx$$

Solution:

$$i. \int_1^4 \left(2x^2 - x + 5\right) dx = \int_1^4 2x^2 \; dx - \int_1^4 x \; dx + \int_1^4 5 \; dx$$

$$= 2\int_1^4 x^2 \; dx - \int_1^4 x \; dx + 5\int_1^4 dx$$

$$= 2\left[\frac{x^3}{3}\right]_1^4 - \left[\frac{x^2}{2}\right]_1^4 + 5[x]_1^4$$

$$= \frac{2}{3}[64-1] - \frac{1}{2}[16-1] + 5[4-1]$$

$$= 42 - \frac{15}{2} + 15 = \frac{99}{2}$$

$$ii. \int_1^0 \left(3x^2 + x - 5\right) dx = -\int_0^1 \left(3x^2 + x - 5\right) dx$$

$$= -3\int_0^1 x^2 \, dx - \int_0^1 x \, dx + 5\int_0^1 dx$$

$$= -3\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1 + 5[x]_0^1$$

$$= -[1-0] - \frac{1}{2}[1-0] + 5[1-0]$$

$$= -1 - \frac{1}{2} + 5 = \frac{7}{2}$$

P3:

Show that the value of $\int_0^1 sin\left(x^2\right)dx$ cannot possibly be 2.

Solution:

We have $-1 \leq sinx \leq 1 \Rightarrow -1 \leq sin(x^2) \leq 1, \;\; \forall x$

From Max-Min inequality, we have

If f has maximum value $max\, f$ and minimum value $min\, f$ on $[a, b]$, then

$$min\, f(b-a) \leq \int_a^b f(x)dx \leq max\, f(b-a)$$

$$\Rightarrow (-1)(1-0) \leq \int_0^1 sin\left(x^2\right)dx \leq (1)(1-0)$$

$$\Rightarrow -1 \leq \int_0^1 sin\left(x^2\right)dx \leq 1 \;\; or \quad \int_0^1 sin\left(x^2\right)dx \leq 1$$

$\therefore \int_0^1 sin\left(x^2\right)dx$ cannot possibly be 2.

P4:

Use Max-Min inequality to find upper and lower bounds for the interval of $\int_0^1 \frac{1}{1+x^2}\,dx$

Solution: We have to compute the upper and lower bounds of the integral $\int_0^1 \frac{1}{1+x^2}\,dx$. Here integrand is $f(x) = \frac{1}{1+x^2}$

$\Longrightarrow f$ is decreasing function on $[0, 1]$

$\therefore$ Maximum value of f occurs at $x = 0$

$$\Longrightarrow max\, f = f(0) = \frac{1}{1+0^2} = 1$$

$\therefore$ Minimum value of f occurs at $x = 1$

$$\Longrightarrow min\, f = f(0) = \frac{1}{1+1^2} = 0.5$$

By max-min inequality, we have

$$(1-0)\,min\, f \le \int_0^1 \frac{1}{1+x^2}\,dx \le (1-0)\,max\, f$$

$$\Rightarrow 0.5 \le \int_0^1 \frac{1}{1+x^2}\,dx \le 1$$

$\Longrightarrow$ Upper bound is 1 and Lower bound is 0.5

2.8. Properties of Definite Integrals

Exercise:

1. Suppose f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4,\ \int_1^5 f(x)\,dx = 6,\ \int_1^5 g(x)\,dx = 8.$$

Use properties of definite integrals and find the following integrals

a. $\int_2^2 g(x)\,dx$

b. $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$

d. $\int_2^5 f(x)\,dx$

e. $\int_1^5 \left[f(x)-g(x)\right]dx$

f. $\int_1^5 \left[4f(x)-g(x)\right]dx$

2. Suppose that $\int_1^2 f(x)\,dx = 5$. **Find**

a. $\int_1^2 f(u)\,du$

b. $\int_1^2 \sqrt{3}f(z)\,dz$

c. $\int_2^1 f(t)\,dt$

d. $\int_1^2 [-f(x)]\,dx$

3. Suppose that f is integrable and that $\int_0^3 f(z)\,dz = 3$ **and**

$\int_0^4 f(z)\,dz = 7$. **Find**

a. $\int_3^4 f(z)\,dz$ *b.* $\int_4^3 f(t)\,dt$

4. Evaluate the integrals.

a. $\int_3^1 7\,dx$

b. $\int_0^2 5x\,dx$

c. $\int_0^2 (2t-3)\,dt$

d. $\int_2^1 \left(1+\frac{z}{2}\right)dz$

e. $\int_1^2 3u^2\,du$

f. $\int_0^2 \left(3x^2+x-5\right)dx$

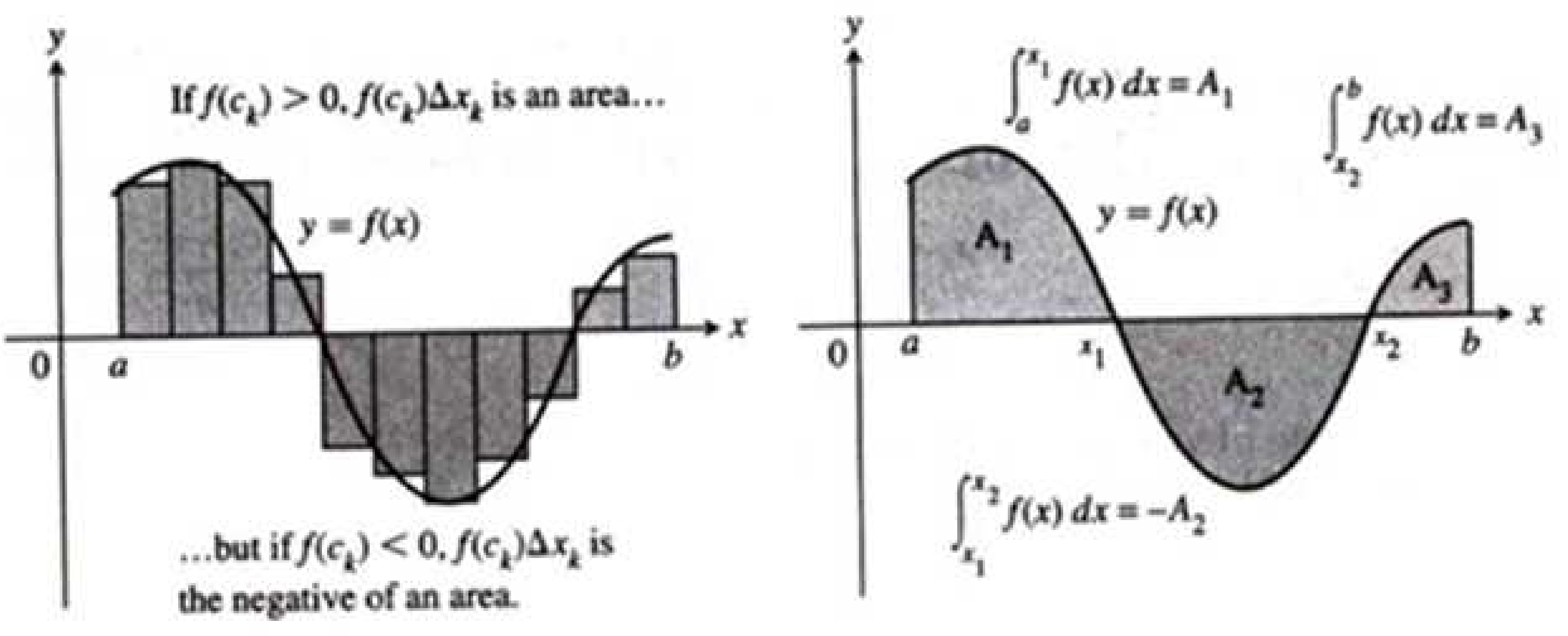
2.9

Area and Integrals

Learning objectives:

- To find the area of the region between a curve $y = f(x)$ over the interval $[a, b]$ and x –axis.
  AND
- To practice the related problems.

If an integrable function $y = f(x)$ has both positive and negative values on an interval $[a,b]$, then the Riemann sums for f on $[a,b]$ is obtained by adding the areas of the rectangles that lie above the x-axis to the negatives of the areas of the rectangles that lie below it.



The Riemann sums are algebraic sums of areas and so is the integral to which they converge.

The value of the integral of f from a to b is

$$\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{b} f(x)dx$$
$$= A_1 - A_2 + A_3$$

The resulting cancellation reduces the sums, so their limiting value is a number whose magnitude is less than the total area between the curve and the x-axis. The value of the integral is the area above the axis minus the area below the axis.

This means that *we must take special care in finding areas by integration.*

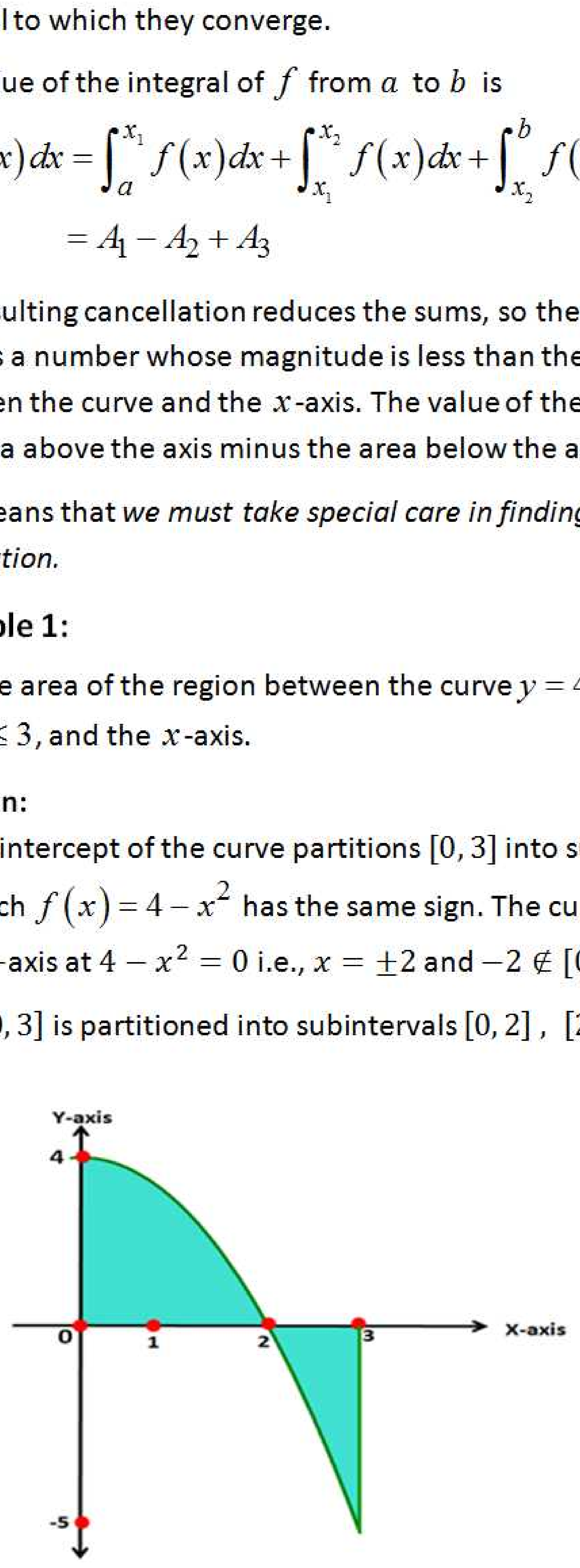
Example 1:

Find the area of the region between the curve $y = 4 - x^2$, $0 \le x \le 3$, and the x-axis.

Solution:

The x-intercept of the curve partitions $[0, 3]$ into subintervals on which $f(x) = 4 - x^2$ has the same sign. The curve crosses the x –axis at $4 - x^2 = 0$ i.e., $x = \pm 2$ and $-2 \notin [0,3]$.

Now $[0, 3]$ is partitioned into subintervals $[0, 2]$, $[2, 3]$



To find the area of the region between the graph of f and the x-axis, we integrate f over each subinterval and add the absolute values of the results.

Integral Over [0, 2]:

$$\int_0^2 \left(4 - x^2\right)dx = \int_0^2 4\,dx - \int_0^2 x^2\,dx = 4(2-0) - \frac{2^3}{3} = \frac{16}{3}$$

Integral Over [2, 3]:

$$\int_2^3 \left(4 - x^2\right)dx = \int_2^3 4\,dx - \int_2^3 x^2\,dx = 4(3-2) - \left(\frac{3^3}{3} - \frac{2^3}{3}\right) = -\frac{7}{3}$$

The region's area:

$$\text{Area} = \frac{16}{3} + \left|-\frac{7}{3}\right| = \frac{23}{3}$$

Procedure

The following is a step-by-step procedure of how to find the area of the region between a curve $y = f(x)$, $a \le x \le b$, and the x-axis.

1. Partition $[a, b]$ with the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

IP1:

Find the area under the graph $y = cosx$ over the interval $[0, 2\pi]$?

Solution:

Given $y = f(x) = cosx$. It is a continuous function on $[0, 2\pi]$.

(1) Partition $[0, 2\pi]$ with zeros of f:

The zeros of $f(x)$ on $[0, 2\pi]$ are

$$cosx = 0 \Longrightarrow x = 0, \ \frac{\pi}{2}, \ \frac{3\pi}{2}, 2\pi$$

Partition$[0, 2\pi]$ into subintervals $\left[0, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \left[\frac{3\pi}{2}, 2\pi\right]$.

(2) Integrate f over each subinterval:

Integral over $\left[0, \frac{\pi}{2}\right]$:

$$A_1 = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = [1 - 0] = 1$$

Integral over $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$:

$$A_2 = \int_{\pi/2}^{3\pi/2} \cos x \, dx = [\sin x]_{\pi/2}^{3\pi/2} = [-1 - 1] = -2$$

Integral over $\left[\frac{3\pi}{2}, 2\pi\right]$:

$$A_3 = \int_{3\pi/2}^{2\pi} \cos x \, dx = [\sin x]_{3\pi/2}^{2\pi} = [0 + 1] = 1$$

(3) Add absolute values of the integrals:

Area A of the curve $f(x) = cosx$ on $[0, 2\pi]$ is

$$A = |A_1| + |A_2| + |A_3| = |1| + |-2| + |1| = 4$$

IP2:

Find the area bounded by the curve $y^2 - 1 = 2x$ and the y −axis?

Solution:

The given curve is $x = f(y) = \frac{1}{2}(y^2 - 1)$. Notice that $f(y)$ is continuous since it is a polynomial. It crosses y −axis at $y = \pm 1$

(1) Partition $[-1, 1]$ with zeros of f:

The zeros of $f(y)$ are $\frac{1}{2}(y^2 - 1) = 0 \Rightarrow y = \pm 1$

The Partition of $[-1, 1]$ with zeros of f is itself.

(2) Integrate f over each subinterval:

Integral over $[-1, 1]$:

$$A_1 = \int_{-1}^{1} \left(\frac{y^2 - 1}{2} \right) dy = \frac{1}{2} \int_{-1}^{1} y^2 dy - \frac{1}{2} \int_{-1}^{1} dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3} \right]_{-1}^{1} - \frac{1}{2} [y]_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{1}{3} + \frac{1}{3} \right] - \frac{1}{2} [1 - (-1)] = -\frac{2}{3}$$

(3) Add absolute value of the integral:

The area bounded by the curve $y^2 - 1 = 2x$ and the y −axis is

$$A = |A_1| = \left| -\frac{2}{3} \right| = \frac{2}{3}$$

IP3:

Find the area bounded by the parabola $x^2 = 8y$, the x –axis and the lines $x = -2$, $x = 4$.

Solution:

Given $y = f(x) = \frac{x^2}{8}$. It is a continuous function on $[-2, 4]$.

(1) Partition $[-2, 4]$ with zeros of f:

The zeros of $f(x)$ are $\frac{x^2}{8} = 0 \Rightarrow x = 0$

Partition $[-2, 4]$ into subintervals $[-2, 0]$, $[0, 4]$.

(2) Integrate f over each subinterval:

Integral over $[-2, 0]$:

$$A_1 = \int_{-2}^{0} \frac{x^2}{8} dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_{-2}^{0} = \frac{1}{8}\left[0 + \frac{8}{3}\right] = \frac{1}{3}$$

Integral over $[0, 4]$:

$$A_2 = \int_{0}^{4} \frac{x^2}{8} dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_{0}^{4} = \frac{1}{8}\left[\frac{64}{3} - 0\right] = \frac{8}{3}$$

(3) Add absolute values of the integrals:

The area bounded by the parabola $x^2 = 8y$, the x –axis and the lines $x = -2$, $x = 4$ is

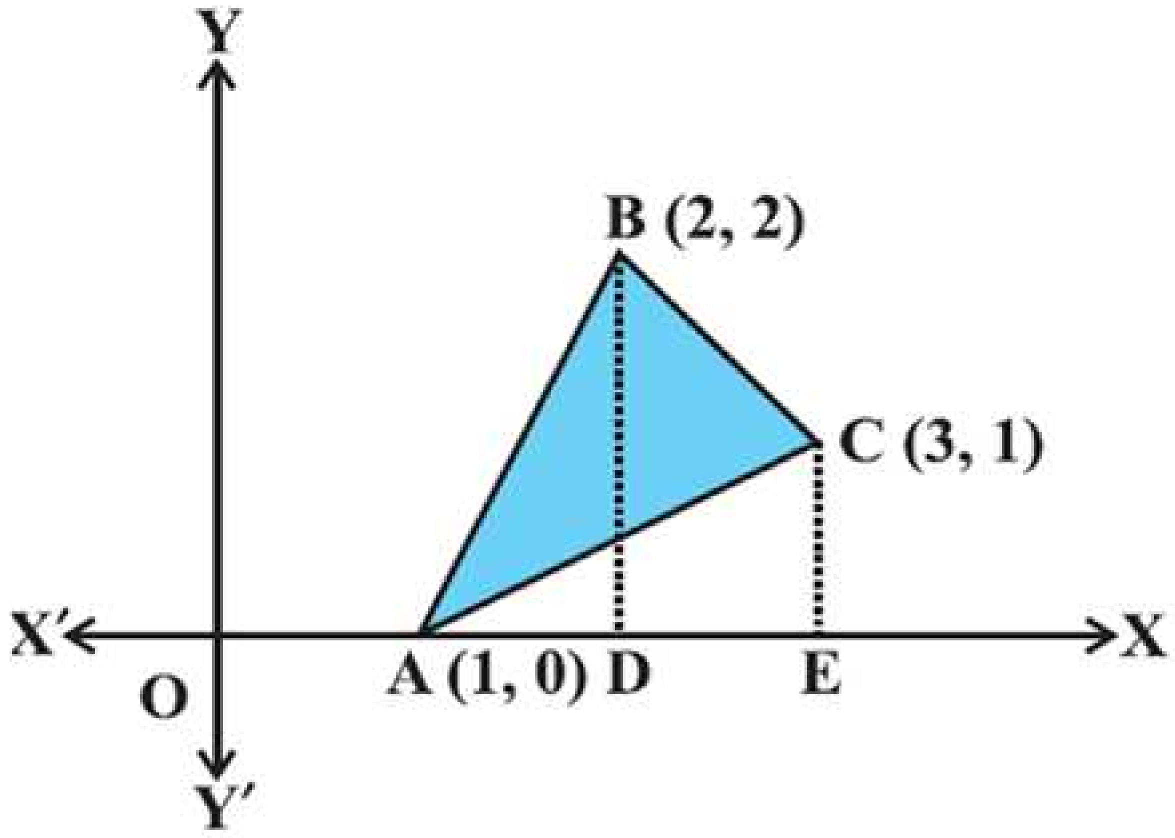
$$A = |A_1| + |A_2| = \left|\frac{1}{3}\right| + \left|\frac{8}{3}\right| = 3$$

IP4:

Using the integration find the area of region bounded by the triangle whose vertices are $(\mathbf{1},\mathbf{0}), (\mathbf{2},\mathbf{2}), (\mathbf{3},\mathbf{1})$.

Solution:

Let $A(1,0)$, $B(2,2)$, and $C(3,1)$ are the vertices of a triangle ABC which is as shown in figure.



From the graph,

Area of $\Delta ABC = \begin{pmatrix} Area\ of \\ \Delta ABD \end{pmatrix} + \begin{pmatrix} Area\ of\ the \\ trapezium\ BDEC \end{pmatrix} - \begin{pmatrix} Area\ of \\ \Delta ACE \end{pmatrix}$

Now, the equations of the sides AB, BC, and CA are given by

$y = 2(x-1)$, $y = 4-x$, $y = \frac{1}{2}(x-1)$ respectively.

Hence,

Area of ΔABC

$$= \int_1^2 2(x-1)\,dx + \int_2^3 (4-x)\,dx - \int_1^3 \left(\frac{x-1}{2}\right) dx$$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2\left[\left(\frac{2^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right]_1^2 + \left[\left(4(3) - \frac{3^2}{2}\right) - \left(4(2) - \frac{2^2}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{3^2}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{3}{2}$$

$\therefore$ Required area is $= \frac{3}{2}$

P1:

Find the area under the graph $\boldsymbol{y = sinx}$ over the interval $[\mathbf{0}, \mathbf{2\pi}]$?

Solution:

Given $y = f(x) = sinx$. It is a continuous function on $[0, 2\pi]$.

(1) Partition $[\mathbf{0}, \mathbf{2\pi}]$ with zeros of $\boldsymbol{f}$:

The zeros of $f(x)$ on $[0, 2\pi]$ are $sinx = 0 \Rightarrow x = 0, \pi, 2\pi$

Partition$[0, 2\pi]$ into subintervals $[0, \pi]$, $[\pi, 2\pi]$.

(2) Integrate $\boldsymbol{f}$ over each subinterval:

Integral over $[0, \pi]$:

$$A_1 = \int_0^{\pi} \sin x \, dx = \left[-\cos x\right]_0^{\pi} = \left[1-(-1)\right] = 2$$

Integral over $[\pi, 2\pi]$:

$$A_2 = \int_{\pi}^{2\pi} \sin x \, dx = \left[-\cos x\right]_{\pi}^{2\pi} = \left[-1-1\right] = -2$$

(3) Add absolute values of the integrals:

Area A of the curve $f(x) = sinx$ on $[0, 2\pi]$ is

$$A = |A_1| + |A_2| = |2| + |-2| = 2 + 2 = 4$$

P2:

Find the area between the x −axis and the curve $y = (x-1)^2 - 25$ over the interval $[-4, 6]$?

Solution:

Given $y = f(x) = (x-1)^2 - 25$. It is a continuous function on $[-4, 6]$.

(1) Partition $[-4, 6]$ with zeros of f:

The zeros of $f(x)$ are

$$(x-1)^2 - 25 = 0 \Rightarrow x = -4, 6$$

The Partition of $[-4, 6]$ with zeros of f is itself.

(2) Integrate f over each subinterval:

Integral over $[-4, 6]$:

$$A_1 = \int_{-4}^{6} \left[(x-1)^2 - 25\right] dx = \left[\frac{(x-1)^3}{3} - 25x\right]_{-4}^{6}$$

$$= \left[\left(\frac{(6-1)^3}{3} - 25(6)\right) - \left(\frac{(-4-1)^3}{3} - 25(-4)\right)\right]$$

$$= -\frac{500}{3}$$

(3) Add absolute values of the integrals:

The area A between the x −axis and the curve

$y = (x-1)^2 - 25$ over the interval $[-4, 6]$ is

$$\therefore A = |A_1| = \left|-\frac{500}{3}\right| = \frac{500}{3}$$

P3:

Find the area bounded by the parabola $y = x^2$, the x −axis and the lines $x = -1$, $x = 2$.

Solution:

Given $y = f(x) = x^2$. It is a continuous function on $[-1, 2]$.

(1) Partition $[-1, 2]$ with zeros of f:

The zeros of $f(x)$ are $x^2 = 0 \Rightarrow x = 0$

Partition$[-1, 2]$ into subintervals $[-1, 0]$, $[0, 2]$.

(2) Integrate f over each subinterval:

Integral over $[-1, 0]$:

$$A_1 = \int_{-1}^{0} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{0} = \left[0 - \frac{(-1)^3}{3}\right] = \frac{1}{3}$$

Integral over $[0, 2]$:

$$A_2 = \int_{0}^{2} x^2\,dx = \left[\frac{x^3}{3}\right]_{0}^{2} = \left[\frac{(2)^3}{3} - 0\right] = \frac{8}{3}$$

(3) Add absolute values of the integrals:

The area bounded by the parabola $y = x^2$, the x −axis and the lines $x = -1$, $x = 2$ is

$$A = |A_1| + |A_2| = \left|\frac{1}{3}\right| + \left|\frac{8}{3}\right| = 3$$

P4:

Find the area bounded by $y = 3x + 2$, the x −axis and the ordinates $x = -1$, $x = 1$.

Solution:

Given $y = f(x) = 3x + 2$. It is a continuous function on $[-1, 1]$.

(1) Partition $[-1, 1]$ with zeros of f:

The zeros of $f(x)$ are $3x + 2 = 0 \Rightarrow x = -\frac{2}{3}$

Partition $[-1, 1]$ into subintervals $\left[-1, -\frac{2}{3}\right], \left[-\frac{2}{3}, 1\right]$.

(2) Integrate f over each subinterval:

Integral over $\left[-1, -\frac{2}{3}\right]$:

$$A_1 = \int_{-1}^{-2/3} (3x+2)\,dx = 3\int_{-1}^{-2/3} x\,dx + 2\int_{-1}^{-2/3} dx$$

$$= 3\left[\frac{x^2}{2}\right]_{-1}^{-2/3} + 2[x]_{-1}^{-2/3}$$

$$= 3\left[\frac{(-2/3)^2}{2} - \frac{(-1)^2}{2}\right] + 2\left[-\frac{2}{3} - (-1)\right] = -\frac{1}{6}$$

Integral over $\left[-\frac{2}{3}, 1\right]$:

$$A_2 = \int_{-2/3}^{1} (3x+2)\,dx = 3\int_{-2/3}^{1} x\,dx + 2\int_{-2/3}^{1} dx$$

$$= 3\left[\frac{x^2}{2}\right]_{-2/3}^{1} + 2[x]_{-2/3}^{1}$$

$$= 3\left[\frac{(1)^2}{2} - \frac{(-2/3)^2}{2}\right] + 2[1 - (-2/3)] = \frac{25}{6}$$

(3) Add absolute values of the integrals:

The area bounded by the parabola $y = 3x + 2$, the x −axis and the ordinates $x = -1$, $x = 1$ is

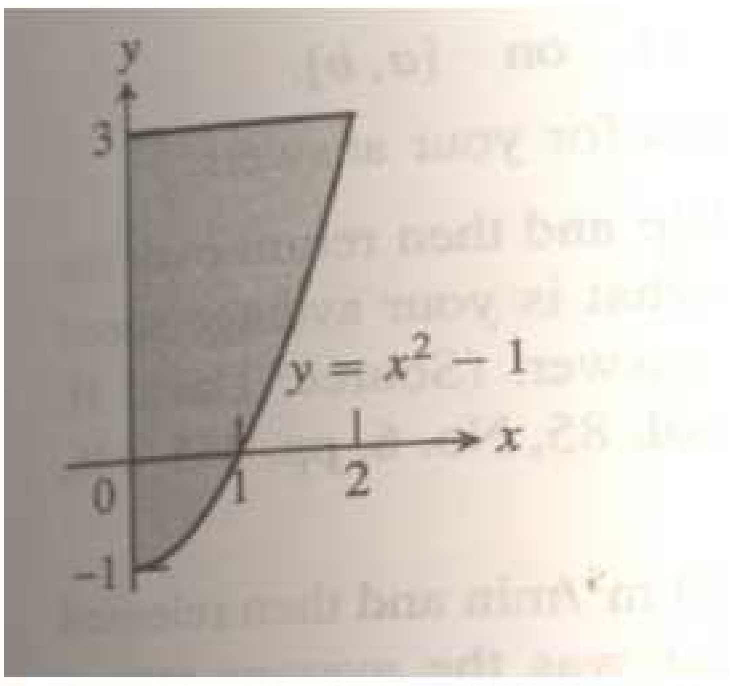
$$A = |A_1| + |A_2| = \left|-\frac{1}{6}\right| + \left|\frac{25}{6}\right| = \frac{13}{3}$$

2.9. Area and Integrals

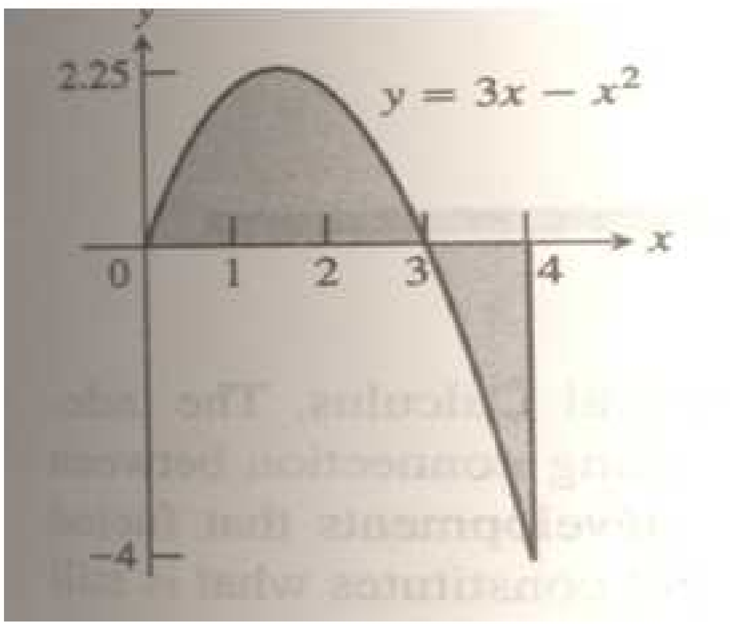
Exercise:

In problems 1, 2 and 3, find the total shaded area.

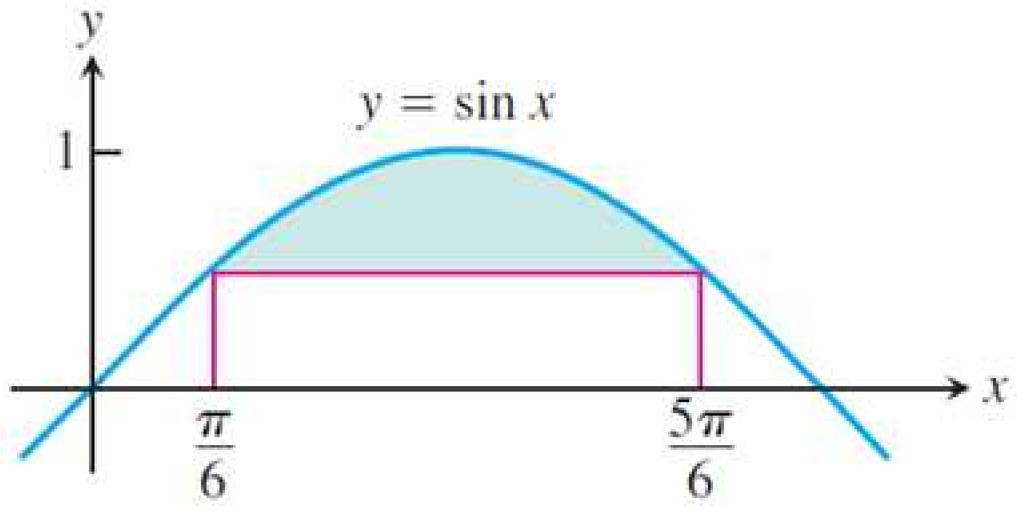
1.



2.



3.



Find the total area between the region and x –axis.

4. $y = x^2 - 6x + 8$ $\quad [0,3]$

5. $y = 2x - x^2$ $\quad [0,3]$

6. $y = -x^2 - 2x,$ $\quad -3 \le x \le 2$

7. $y = 3x^2 - 3,$ $\quad -2 \le x \le 2$

8. $y = x^3 - 3x^2 + 2x,$ $\quad 0 \le x \le 2$

9. $y = x^3 - 4x,$ $\quad -2 \le x \le 2$

10. $y = x^{1/3},$ $\quad -1 \le x \le 8$

2.10

The Mean Value Theorem for Definite Integrals

Learning objectives:

- To define the average value of an integrable function on an interval.
- To state and prove the Mean value theorem for definite integrals.

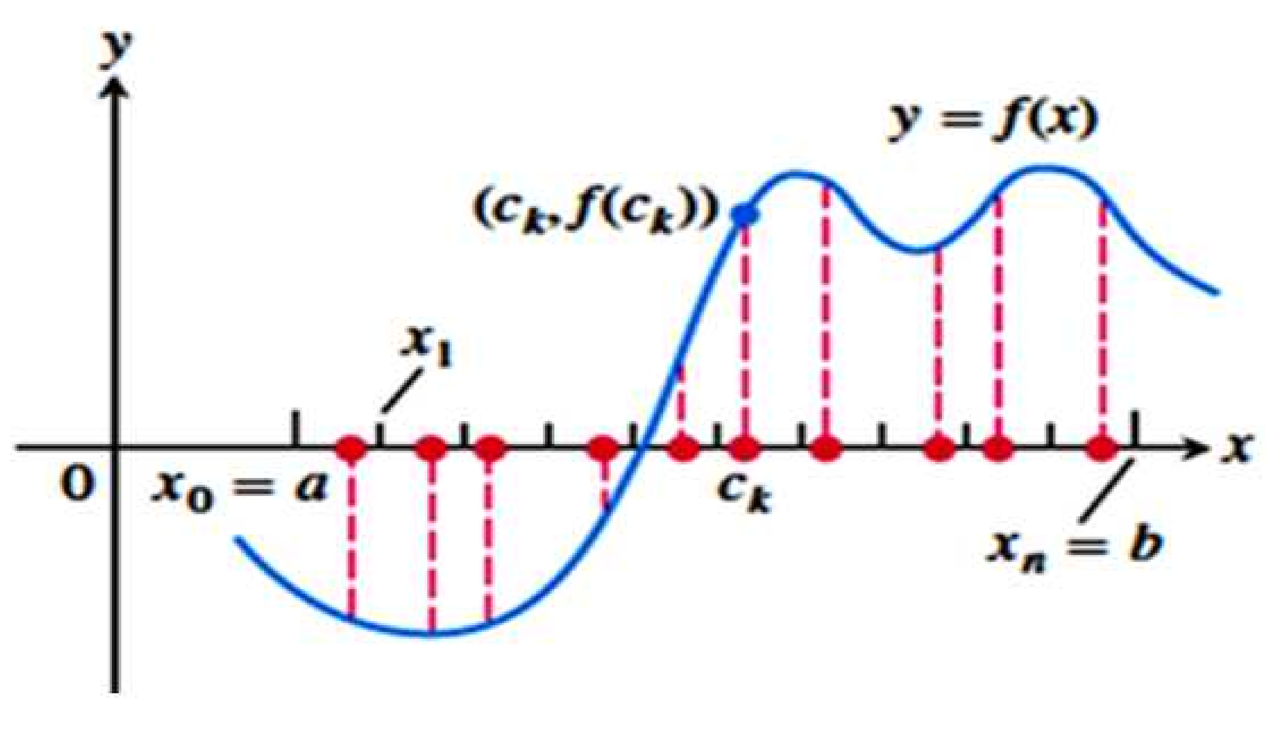
  AND

- To practice the related problems.

In an earlier module, we discussed the average value of a nonnegative continuous function. Now, we define the average value without requiring f being nonnegative, and also showing that every continuous function assumes its average value at least once.

The Average value

In arithmetic, we know that the average of n numbers is the sum of the numbers divided by n. For a continuous function f on a closed interval $[a,b]$ there may be infinitely many values to consider, but we can sample them in an orderly way. We partition $[a,b]$ into n subintervals of equal length (the length is $\Delta x = \frac{b-a}{n}$ and evaluate f at a point c_k in each subinterval.



The average of the n sampled values is

$$\frac{f(c_1)+f(c_2)+\cdots+f(c_n)}{n} = \frac{1}{n}\cdot\sum_{k=1}^{n} f(c_k)$$
$$= \frac{\Delta x}{b-a}\cdot\sum_{k=1}^{n} f(c_k)$$
$$= \frac{1}{b-a}\cdot\sum_{k=1}^{n} f(c_k)\,\Delta x$$

Thus, the average of the sampled values is always $\frac{1}{b-a}$ times a Riemann sum for f on $[a,b]$. As we increase the size of the sample and let the norm of the partition approach zero, the average must approach $\left(\frac{1}{b-a}\right)\int_a^b f(x)\,dx$.

Definition

If f is integrable on $[a,b]$, then its ***average (mean) value*** on $[a,b]$ is denoted by $\bar{f}$ or $av(f)$ and

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

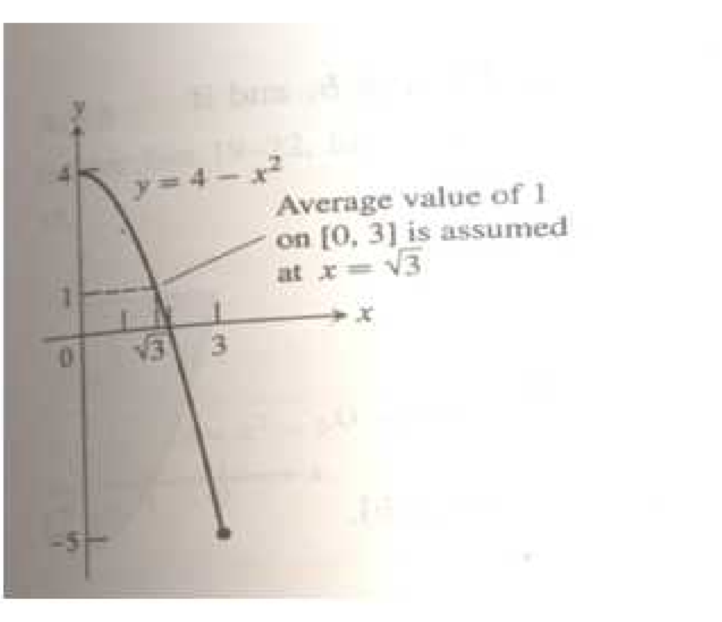
Example 1:

Find the average value of $f(x) = 4 - x^2$ on $[0, 3]$. Does f actually take on this value at some point in the given domain?

Solution:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{1}{3-0}\int_0^3 (4-x^2)\,dx = \frac{1}{3}\left(\int_0^3 4\,dx - \int_0^3 x^2\,dx\right)$$
$$= \frac{1}{3}\left(4(3-0) - \frac{3^3}{3}\right) = \frac{1}{3}(12-9) = 1$$

The average value of $f(x) = 4 - x^2$ over the interval $[0, 3]$ is 1. The function assumes this value when $4 - x^2 = 1$ or $x = \pm\sqrt{3}$. Since one of these points, $x = \sqrt{3}$, lies in $[0, 3]$, the function does assume its average value in the given domain.



The Mean Value Theorem for Definite Integrals

The statement that a continuous function on a closed interval assumes its average value at least once in the interval is known as the Mean Value theorem for Definite Integrals.

Theorem 1:

If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Proof

If we divide both sides of the Max-Min inequality by $(b-a)$, we obtain

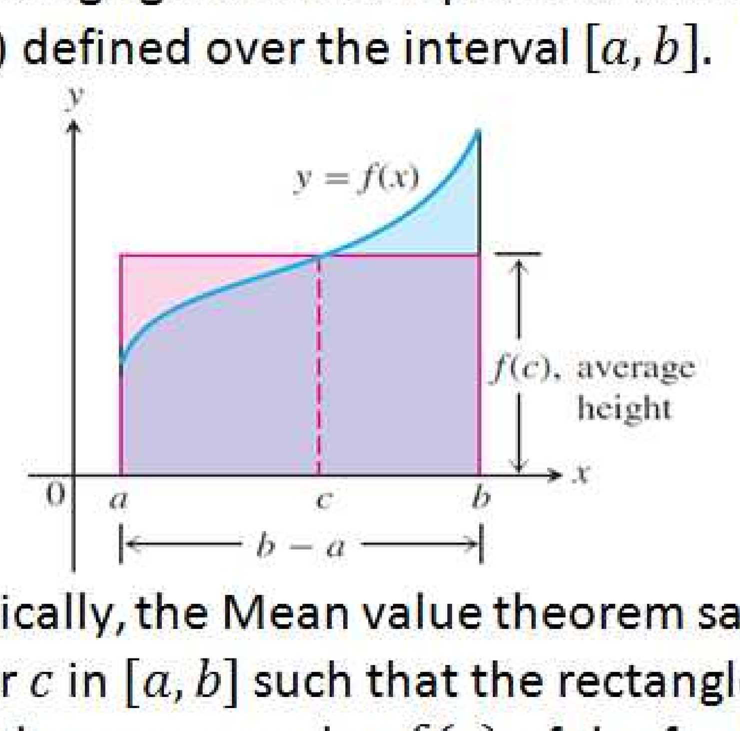
$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions says that f must assume every value between m and M. It must therefore assume the value $\frac{1}{b-a}\int_a^b f(x)\,dx$ at some point c in $[a,b]$.

Hence the theorem.

Note 1:

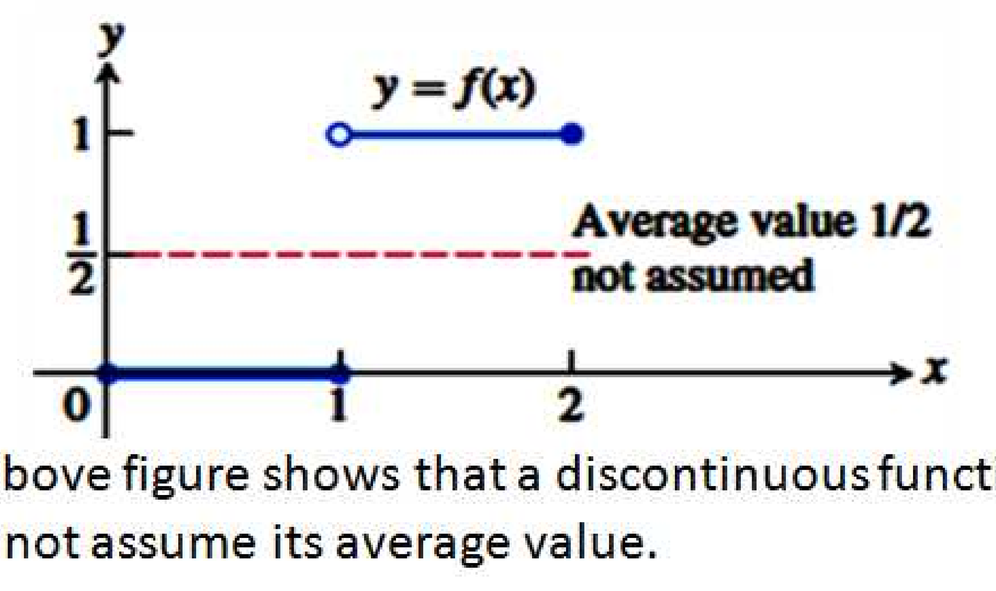
The following figure shows a positive continuous function $y = f(x)$ defined over the interval $[a, b]$.



Geometrically, the Mean value theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.

Note 2:

The continuity of f is important in the Mean value theorem of definite integrals. A discontinuous function can step over its average value.



The above figure shows that a discontinuous function need not assume its average value.

Example 2:

Show that if f is continuous on $[a,b]$, $a \ne b$, and if $\int_a^b f(x)\,dx = 0$ then $f(x) = 0$ at least once in $[a,b]$.

Solution:

The average value of f on $[a,b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot 0 = 0$$

By the Mean value theorem, f assumes this value at some point c in $[a,b]$.

IP1:

Find the average value of the function $\boldsymbol{h(x) = -|x|}$ on

a. $[-1, 0]$ b. $[0, 1]$ c. $[-1, 1]$

Solution:

Notice that $h(x) = -|x|$ is a continuous function and so its definite integral $\int_a^b h(x)\,dx$ exists.

The average value of a function $h(x)$ on $[a, b]$ is

$$av(h) = \bar{h} = \frac{1}{b-a}\int_a^b h(x)\,dx$$

a. The average value of the function $h(x) = -|x|$ on $[-1, 0]$ is

$$av(h) = \bar{h} = \frac{1}{-1-0}\int_{-1}^{0} -|x|\,dx = -\int_{-1}^{0}(-x)\,dx$$

$$= \int_{-1}^{0} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{0} = \left[0 - \frac{(-1)^2}{2}\right] = -\frac{1}{2}$$

b. The average value of the function $h(x) = -|x|$ on $[0, 1]$ is

$$av(h) = \bar{h} = \frac{1}{1-0}\int_{0}^{1} -|x|\,dx = -\int_{0}^{1} x\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{0}^{1} = -\left[\frac{(1)^2}{2} - 0\right] = -\frac{1}{2}$$

c. The average value of the function $h(x) = -|x|$ on $[-1, 1]$ is

$$av(h) = \bar{h} = \frac{1}{1-(-1)}\int_{-1}^{1} -|x|\,dx$$

$$= \frac{1}{2}\left[\int_{-1}^{0} -|x|\,dx + \int_{0}^{1} -|x|\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2} + \left(-\frac{1}{2}\right)\right] = -\frac{1}{2} \quad (\because \text{ from parts } a \text{ and } b)$$

IP2:

Find the average value of $f(x) = -\frac{x^2}{2}$ over the interval $[0,3]$. At what point or points in the given interval does the function assume its average value.

Solution:

Notice that $f(x) = -\frac{x^2}{2}$ is a continuous function and so its definite integral $\int_0^3 f(x)\,dx$ exists.

The average value of a function $f(x)$ on $[a,b]$ is

$$av(f) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$\therefore$ The average value of the function $f(x) = -\frac{x^2}{2}$ on $[0,3]$ is

$$av(f) = \overline{f} = \frac{1}{3-0}\int_0^3 \left(-\frac{x^2}{2}\right)dx = -\frac{1}{6}\int_0^3 x^2\,dx$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}\right]_0^3 = -\frac{1}{6}\left[\frac{27}{3} - 0\right] = -\frac{3}{2}$$

Notice that $-\frac{x^2}{2} = -\frac{3}{2} \Rightarrow x = \pm\sqrt{3}.$

By the mean value theorem, the function f assumes the value when $x = \sqrt{3} \in [0,3].$

IP3:

Find the average value of $f(x) = x^3 - 3x^2 + 2x$ over the interval $[0, 2]$. At what point or points in the given interval does the function assume its average value.

Solution:

Notice that $f(x) = x^3 - 3x^2 + 2x$ is a continuous function and so its definite integral $\int_0^2 f(x)\,dx$ exists.

The average value of a function $f(x)$ on $[a, b]$ is

$$av(f) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$\therefore$ The average value of the function $f(x) = x^3 - 3x^2 + 2x$ on $[0, 2]$ is

$$av(f) = \overline{f} = \frac{1}{2-0}\int_0^2 \left(x^3 - 3x^2 + 2x\right)dx$$

$$= \frac{1}{2}\left[\int_0^2 x^3\,dx - 3\int_0^2 x^2 dx + 2\int_0^2 x dx\right]$$

$$= \frac{1}{2}\left\{\left[\frac{x^4}{4}\right]_0^2 + \left[x^3\right]_0^2 + \left[x^2\right]_0^2\right\}$$

$$= \frac{1}{2}[4 - 8 + 4] = 0$$

Notice that $x^3 - 3x^2 + 2x = 0 \Rightarrow x = 0, 1, 2$

By the mean theorem, the function f assumes this value when $x = 0, 1, 2 \in [0, 2]$

IP4:

Find the average value of $f(x) = 1 + x^2$ over the interval $[-1, 2]$. At what point or points in the given interval does the function assume its average value.

Solution:

Notice that $f(x) = 1 + x^2$ is a continuous function and so its definite integral $\int_{-1}^{2} f(x)\, dx$ exists.

The average value of a function $f(x)$ on $[a, b]$ is

$$av(f) = \overline{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\, dx$$

The average value of the function $f(x) = 1 + x^2$ on $[-1, 2]$ is

$$av(f) = \overline{f} = \frac{1}{2-(-1)}\int_{-1}^{2} \left(1+x^2\right) dx = \frac{1}{3}\left[\int_{-1}^{2} dx + \int_{-1}^{2} x^2\, dx\right]$$

$$= \frac{1}{3}\left\{[x]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2}\right\} = \frac{1}{3}\left\{[2-(-1)] - \left[\frac{8}{3} + \frac{1}{3}\right]\right\}$$

$$= \frac{6}{3} = 2$$

Notice that $1 + x^2 = 2 \Rightarrow x = \pm 1$ and $x = \pm 1 \in [-1, 2]$

By the mean value theorem, the function f assumes this value when $x = \pm 1$.

P1:

Find the average value of the function $g(x) = |x| - 1$ on

a. $[-1, 1]$ b. $[1, 3]$ c. $[-1, 3]$

Solution:

Notice that $g(x) = |x| - 1$ is a continuous function and so its definite integral $\int_a^b g(x)\,dx$ exists.

The average value of a function $g(x)$ on $[a, b]$ is

$$av(g) = \bar{g} = \frac{1}{b-a}\int_a^b g(x)\,dx$$

a. The average value of the function $g(x) = |x| - 1$ on $[-1, 1]$ is

$$av(g) = \bar{g} = \frac{1}{1-(-1)}\int_{-1}^{1}(|x|-1)\,dx$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(|x|-1)\,dx + \int_{0}^{1}(|x|-1)\,dx\right]$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(-x-1)\,dx + \int_{0}^{1}(x-1)\,dx\right]$$

$$= \frac{1}{2}\left\{-\left[\frac{x^2}{2}+x\right]_{-1}^{0} + \left[\frac{x^2}{2}-x\right]_{0}^{1}\right\}$$

$$= \frac{1}{2}\left\{-\left[0-\left(\frac{1}{2}-1\right)\right] + \left[\left(\frac{1}{2}-1\right)-0\right]\right\} = -\frac{1}{2}$$

b. The average value of the function $g(x) = |x| - 1$ on $[1, 3]$ is

$$av(g) = \bar{g} = \frac{1}{3-1}\int_{1}^{3}(|x|-1)\,dx = \frac{1}{2}\int_{1}^{3}(x-1)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}-x\right]_{1}^{3} = \frac{1}{2}\left[\left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)\right] = 1$$

c. The average value of the function $g(x) = |x| - 1$ on $[-1, 3]$ is

$$av(g) = \bar{g} = \frac{1}{3-(-1)}\int_{-1}^{3}(|x|-1)\,dx = \frac{1}{4}\int_{-1}^{3}(|x|-1)\,dx$$

$$= \frac{1}{4}\int_{-1}^{1}(|x|-1)\,dx + \frac{1}{4}\int_{1}^{3}(|x|-1)\,dx$$

$$= \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{4}(1) = \frac{1}{4} \quad (\because \text{ from parts } a \text{ and } b)$$

P2:

Find the average value of $f(x) = 3x^2 - 3$ over the interval $[0, 1]$. At what point or points in the given interval does the function assume its average value.

Solution:

Notice that $f(x) = 3x^2 - 3$ is a continuous function and so its definite integral $\int_0^1 f(x)\,dx$ exists.

The average value of a function $f(x)$ on $[a, b]$ is

$$av(f) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$\therefore$ The average value of the function $f(x) = 3x^2 - 3$ on $[0, 1]$ is

$$av(f) = \overline{f} = \frac{1}{1-0}\int_0^1 \left(3x^2 - 3\right)dx$$

$$= 3\int_0^1 x^2\,dx - 3\int_0^1 dx = 3\left[\frac{x^3}{3}\right]_0^1 - 3[x]_0^1$$

$$= 3\left[\frac{1}{3} - 0\right] - 3[1-0] = -2$$

Notice that $3x^2 - 3 = -2 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

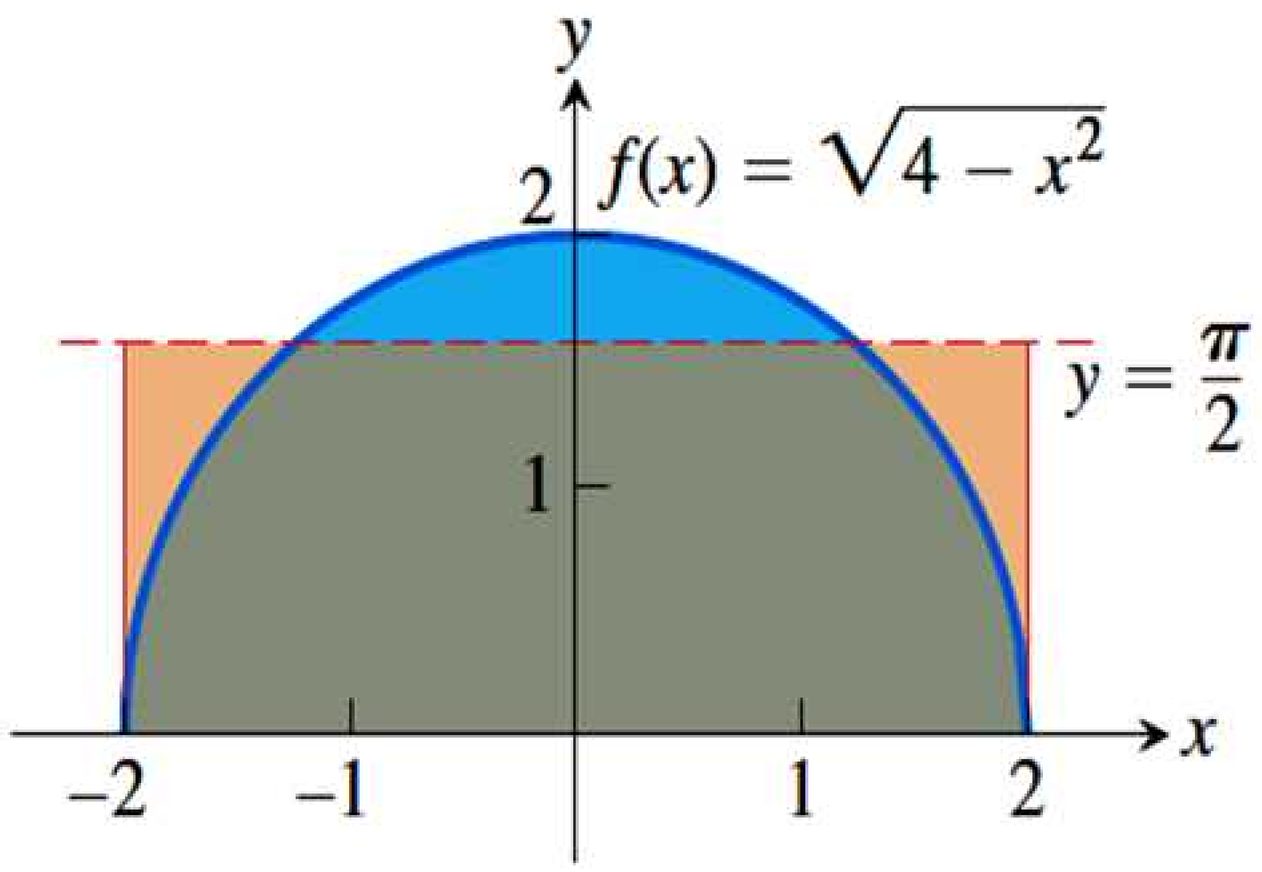
By the mean theorem, the function f assumes this value when $x = \frac{1}{\sqrt{3}} \in [0, 1]$

P3:

Find the average value of $f(x) = \sqrt{4 - x^2}$ over the interval $[-2, 2]$. At what point or points in the given interval does the function assume its average value.

Solution:

The graph of the function $f(x) = \sqrt{4 - x^2}$ over the interval is plotted over the interval $[-2, 2]$, which is shown below.



We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph is the upper semi-circle of radius centered at the origin.

The area between the semi-circle and the x –axis from -2 to 2 can be computed using the geometric formula.

$$\text{Area of a semi-circle} = \frac{\pi r^2}{2} = \frac{\pi(2)^2}{2} = 2\pi$$

Because f is non-negative, the area is also the value of the integral of f from -2 to 2

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 2\pi$$

Therefore, the average value of the function $f(x) = \sqrt{4 - x^2}$ is

$$av(f) = \frac{1}{2-(-2)} \int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

Notice that $\sqrt{4 - x^2} = \frac{\pi}{2} \Rightarrow 4 - x^2 = \frac{\pi^2}{4} \Rightarrow x^2 = 4 - \frac{\pi^2}{4}$

$$\Rightarrow x = \pm\sqrt{4 - \frac{\pi^2}{4}} \in [-2, 2]$$

By the mean value theorem, the function f assumes the value when $x = \pm\sqrt{4 - \frac{\pi^2}{4}}$.

P4:

Find the average value of $f(x) = 4 - x$ over the interval $[0, 3]$. At what point or points in the given interval does the function assume its average value.

Solution:

Notice that $f(x) = 4 - x$ is a continuous function and so its definite integral $\int_0^3 f(x)\,dx$ exists.

The average value of a function $f(x)$ on $[a, b]$ is

$$av(f) = \bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

∴ The average value of the function $f(x) = 4 - x$ on $[0, 3]$ is

$$av(f) = \bar{f} = \frac{1}{3-0}\int_0^3 (4-x)\,dx = \frac{1}{3}\left[4\int_0^3 dx - \int_0^3 x\,dx\right]$$

$$= \frac{1}{3}\left\{[4x]_0^3 - \left[\frac{x^2}{2}\right]_0^3\right\} = \frac{1}{3}\left\{[4(3)-0] - \left[\frac{3^2}{2} - 0\right]\right\}$$

$$= 4 - \frac{3}{2} = \frac{5}{2}$$

Notice that $4 - x = \frac{5}{2} \Longrightarrow x = \frac{3}{2} \in [0, 3]$

By the mean value theorem, the function f assumes this value when $x = \frac{3}{2}$

2.10. The Mean Value Theorem for Definite Integrals

Exercise:

I. Find the average value over the given interval. At what point or points in the given interval does the function assume its average value?

a. $f(x) = x^2 - 1 \quad on \quad \left[0, \sqrt{3}\right]$

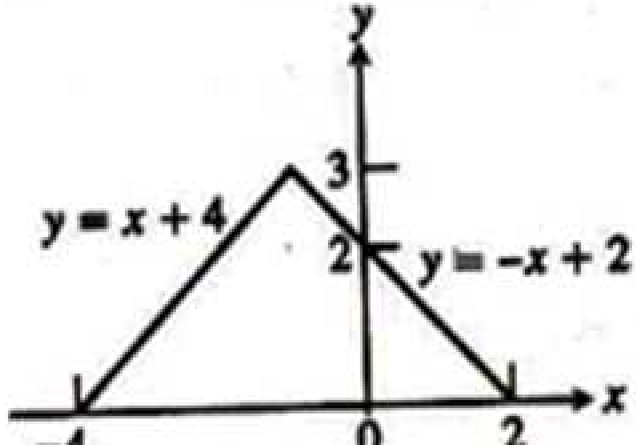
b. $f(x) = -3x^2 - 1 \quad on \quad [0,1]$

c. $f(t) = (t-1)^2 \quad on \quad [0,3]$

d. $f(t) = t^2 - t\,, \quad on \quad [-2, 1]$

II. Find the average value of the function over the given interval from the graph of f (without integrating).

a) $f(x) = \begin{cases} x+4 & -4 \le x \le -1 \\ -x+2 & -1 < x \le 2 \end{cases} \quad on \quad [-4,2]$



b) $f(t) = sint \quad on \quad [0, 2\pi]$

2.11

The Fundamental Theorem

Learning objectives:

- To state and prove the first part of the Fundamental theorem of calculus.
  AND
- To practice the related problems.

The Fundamental Theorem of Integral Calculus gives the connection between integration and differentiation. It is independently discovered by Leibniz and Newton.

The Fundamental Theorem of Calculus, Part 1

If $f(t)$ is an integral function over a finite interval I then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a function F whose value at x is

$$F(x) = \int_a^x f(t)\,dt \qquad \dots (1)$$

Equation (1) gives an important way to define new functions, but its importance now is the connection it makes between integrals and derivatives.

If f is any continuous function, then F is a differentiable function of x whose derivative is f itself. At every value of $x \in I$,

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This idea is the first part of the Fundamental Theorem of Calculus.

Theorem 1:
The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point of $[a,b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x), \qquad a \le x \le b \,\dots\dots\dots (2)$$

Proof

We prove the theorem by showing that the limit of the difference quotient

$$\frac{F(x+h) - F(x)}{h}$$

as $h \to 0$ is the number $f(x)$, where x and $(x+h)$ are in (a,b).

We replace $F(x+h)$ and $F(x)$ by their defining integrals. Then

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

The Additivity Rule for integrals simplifies the right-hand side to

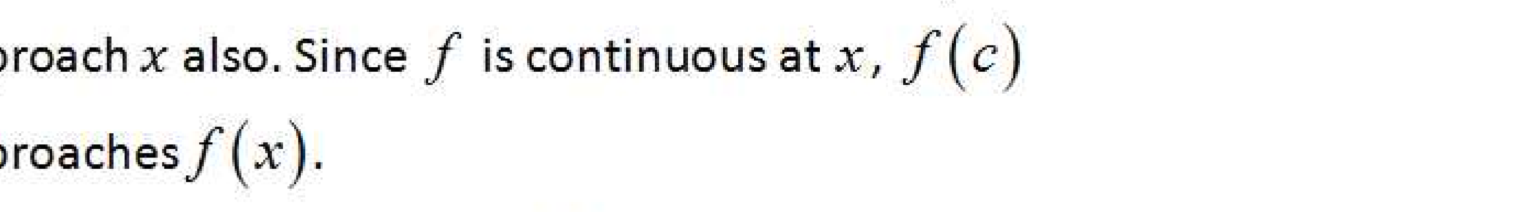
$$\int_x^{x+h} f(t)\,dt$$

Now,

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\left[F(x+h) - F(x)\right] = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

According to the Mean Value Theorem for Definite Integrals, the value of the last expression in the previous equation is one of the values taken on by f in the interval joining x and $x+h$. That is, for some number c in this interval,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c)$$



As $h \to 0$, the endpoint $x+h$ approaches to x, forcing c approach x also. Since f is continuous at x, $f(c)$ approaches $f(x)$.

$$\lim_{h \to 0} f(c) = f(x)$$

Going back to the beginning, then, we have

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h \to 0} f(c) = f(x)$$

If $x = a$ or b then the limit is interpreted as one sided limit with $h \to 0^+$ or $h \to 0^-$ respectively. Then

$$\frac{dF}{dx} = f(x), \quad a \le x \le b$$

Hence the theorem

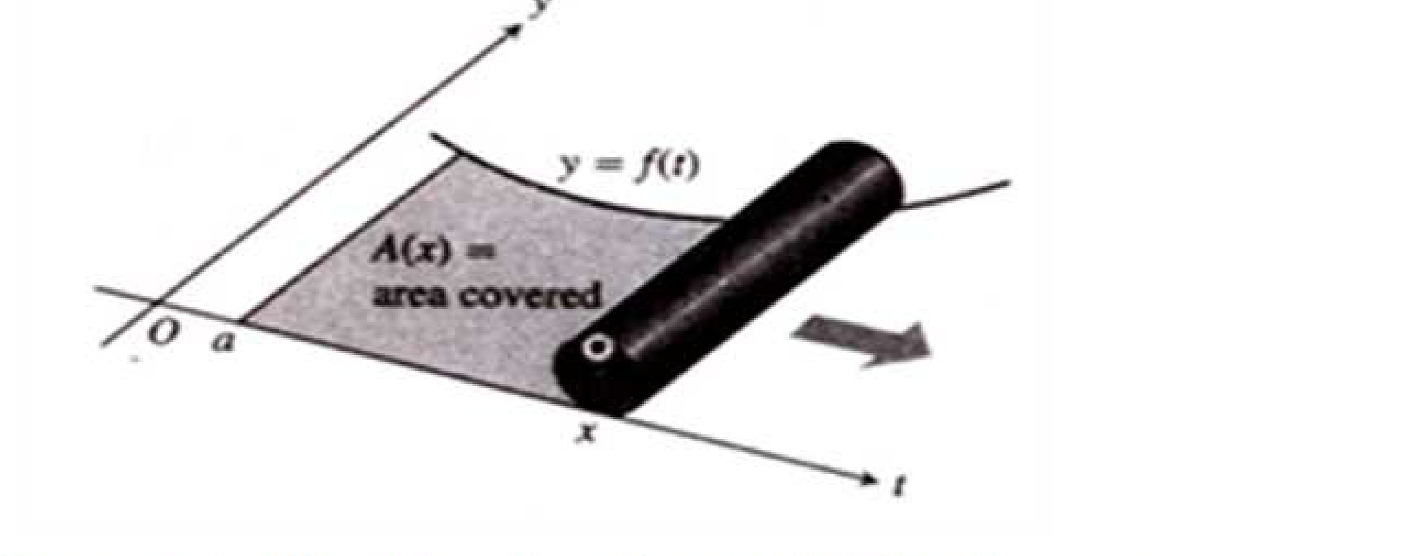
Note1: The equation (2) says that every continuous function f is the derivative of some other function, namely $\int_a^x f(t)\,dt$. It says that every continuous function has an anti-derivative. And it says that the process of integration and differentiation are inverses of one another.

Note2: If the values of f are positive, the equation

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

has a geometric interpretation.

The integral of f from a to x is the area $A(x)$ of the region between the graph of f and the x-axis from a to x. Imagine covering this region from left to right by unrolling a carpet of variable width $f(t)$.



As the carpet rolls past x, the rate at which the floor is being covered is $f(x)$.

Example 1:

$$\frac{d}{dx}\int_{-\pi}^x \cos t\,dt = \cos x$$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$$

Example 2:

Find dy/dx if $y = \int_1^{x^2} \cos t\,dt$

Solution

We treat y as the composite of the two functions,

$$y = \int_1^u \cos t\,dt, \qquad u = x^2$$

and apply the Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}\int_1^u \cos t\,dt \cdot \frac{du}{dx} = \cos u \cdot \frac{du}{dx}$$
$$= \cos x^2 \cdot 2x = 2x\cos x^2$$

Example 3:

Find a function $y = F(x)$ whose derivative is $\tan x$ and which passes through the point $(1,5)$.

Solution:

The Fundamental theorem makes it easy to construct a function with derivative $\tan x$ that equals 0 at $x = 1$

$$y = \int_1^x \tan t\,dt$$

Since $y(1) = \int_1^1 \tan t\,dt = 0$, we have to add 5 to this function to construct the function with derivative is $\tan x$ and whose value is 5 when $x = 1$

$$F(x) = \int_1^x \tan t\,dt + 5$$

Therefore, the required function is

$$y = F(x) = \int_1^x \tan t\,dt + 5$$

P1:

If $y = \int_0^x \sqrt{1+t^2}\ dt$ **then find** $\frac{dy}{dx}$

P1:

If $y=\int_0^x \sqrt{1+t^2}\ dt$ **then find** $\boldsymbol{\frac{dy}{dx}}$

Solution: Given $y=\int_0^x \sqrt{1+t^2}\ dt$

Notice that $f(t)=\sqrt{1+t^2}$ is a continuous function on $[0,b]$, $b>0$. Then by the Fundamental Theorem of calculus (part1),

$$y=F(x)=y=\int_0^x f(t)\ dt$$

is differentiable on $(0,b)$ and its derivative $F'(x)=f(x)$

$$\therefore\ \frac{dy}{dx}=\sqrt{1+x^2}$$

P2:

If $y = \int_{2x^3}^{5} 3t \sin t \, dt$ **then find** $\dfrac{dy}{dx}$

P2:

If $y=\int_{2x^3}^{5} 3t\sin t\ dt$ then find $\dfrac{dy}{dx}$

Solution:

Given $y=\int_{2x^3}^{5} 3t\sin t\ dt$

Put $u=2x^3 \Longrightarrow \frac{du}{dx}=6x^2$ then

$$y=\int_{u}^{5} 3t\sin t\,dt=\int_{5}^{u}(-3t\sin t)\ dt=F(u)$$

By the Fundamental Theorem of calculus (part1), we have

$$\frac{dy}{du}=-3u\sin u$$

By the chain rule, we have

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=(-3u\sin u).\left(6x^2\right)=-36x^5\sin\left(2x^3\right)$$

P3:

If $y = \int_0^{\tan\theta} \sec^2 t \, dt$ **then find** $\dfrac{dy}{d\theta}$

P3:

If $y = \int_0^{\tan\theta} \sec^2 t\, dt$ then find $\boldsymbol{\frac{dy}{d\theta}}$

Solution:

Given $y = \int_0^{\tan\theta} \sec^2 t\, dt$

Put $= tan\theta \Rightarrow \frac{du}{d\theta} = sec^2\theta$. Then $y = \int_0^{u} \sec^2 t\, dt = F(u)$

By the Fundamental Theorem of calculus (part1), we have

$$\frac{dy}{du} = \sec^2 u$$

By the Chain rule, we have

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} = \sec^2 u . \sec^2\theta = \sec^2(\tan\theta) . \sec^2\theta$$

P4:

Find the function $\boldsymbol{y = F(x)}$ whose derivative is $\sqrt{\mathbf{1} + \boldsymbol{x}^2}$ and which passes through $(-\mathbf{1}, \mathbf{2})$?

P4:

Find the function $y = F(x)$ whose derivative is $\sqrt{1 + x^2}$ and which passes through $(-1, 2)$?

Solution:

The Fundamental Theorem makes it easy to construct a function with derivative $\sqrt{1 + x^2}$ that equals 0 at $x = -1$.

$$y = \int_{-1}^{x} \sqrt{1+t^2}\, dt$$

Since $y = \int_{-1}^{-1} \sqrt{1+t^2}\, dt = 0$, we have to add 2 to this function to construct the function with derivative is $\sqrt{1 + x^2}$ and whose value is 2 when $x = -1$

$$F(x) = \int_{-1}^{x} \sqrt{1+t^2}\, dt + 2$$

Therefore, the required function is $F(x) = \int_{-1}^{x} \sqrt{1+t^2}\, dt + 2$

IP1:

If $y=\int_0^{x^2}\cos\sqrt{t}\ dt\ ,\ |x|<\frac{\pi}{2}$ then find $\frac{dy}{dx}$

Solution:

Given $y=\int_0^{x^2}\cos\sqrt{t}\ dt\ ,\ |x|<\frac{\pi}{2}$

Put $u=x^2 \Longrightarrow \frac{du}{dx}=2x.$

Then $y=\int_0^{u}\cos\sqrt{t}\ dt=F(u)$

By the Fundamental Theorem of calculus (part1), we have

$$\frac{dy}{du}=\cos\sqrt{u}$$

By the chain rule, we have

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=\cos\sqrt{u}.\ 2x=2x\cos|x|$$

IP2:

If $y=\int_{3}^{6x\sqrt{x}}\sqrt{t^2+t}\;dt$ then find $\dfrac{dy}{dx}$

Solution:

Given $y=\int_{3}^{6x\sqrt{x}}\sqrt{t^2+t}\;dt$

Put $u=6x\sqrt{x}\Rightarrow \frac{du}{dx}=9\sqrt{x}$ then $y=\int_{3}^{u}\sqrt{t^2+t}\;dt=F(u)$

By the fundamental theorem of calculus (part1),

$$\frac{dy}{du}=\sqrt{u^2+u}$$

By the chain rule, we have

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=\left(\sqrt{u^2+u}\right).9\sqrt{x}$$

$$=\left(\sqrt{\left(6x^{3/2}\right)^2+6x^{3/2}}\right).9\sqrt{x}$$

$$=\left(\sqrt{36x^3+6x^{3/2}}\right).9\sqrt{x}=9x\sqrt{36x^2+6x^{1/2}}$$

IP3:

If $y = \int_1^{\sin x} 3t^2\, dt$ then find $\frac{dy}{dx}$

Solution:

Given $y = \int_1^{\sin x} 3t^2\, dt$

Put $u = sinx \Rightarrow \frac{du}{dx} = cosx$. Then $y = \int_1^{u} 3t^2 dt$

By the Fundamental Theorem of calculus (part 1), we have

$$\frac{dy}{du} = 3u^2$$

By the Chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 . \cos x = 3\sin^2 x . \cos x$$

IP4:

Find the function $y = v(t)$ whose derivative is $g(t)$ and which passes through $(t_0,\ v_0)$?

Solution:

The Fundamental theorem makes it easy to construct a function with derivative $g(t)$ that equals 0 at $t = t_0$

$$v(t) = \int_{t_0}^{t} g(x)\,dx + C$$

Since $v(t_0) = \int_{t_0}^{t_0} g(x)\,dx + C$, we have to add v_0 to this function

to construct the function with derivative is $g(t)$ and whose value is v_0 when $t = t_0$

$$v(t) = \int_{t_0}^{t} g(x)\,dx + v_0$$

Therefore, the required function is $v(t) = \int_{t_0}^{t} g(x)\,dx + v_0$

2.11. The Fundamental Theorem

Exercise:

Find $\frac{dy}{dx}$ if

1. $y = \frac{d}{dx}\int_0^{\sqrt{x}} \cos t \, dt$

2. $y = \frac{d}{dt}\int_0^{t^4} \sqrt{u} \, du$

3. $y = \int_0^{x} \frac{1}{t} \, dt$

4. $y = \int_{\sqrt{x}}^{0} \sin\left(t^2\right) dt$

5. $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \qquad |x| < \frac{\pi}{2}$

6. $y = \int_{\tan x}^{0} \frac{dt}{1+t^2}$

2.12

Evaluation of Definite Integrals

(The Fundamental Theorem of Calculus part2)

Learning objectives:

- To state and prove the Fundamental Theorem of Calculus part2.
  AND
- To practice the related problems on the evaluation of definite integrals.

The second part of the Fundamental Theorem of Calculus describes how to evaluate definite integrals.

Theorem

The Fundamental Theorem of Calculus, Part 2:

If f is continuous at every point of $[a,b]$ and F is any anti-derivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof:

To prove the theorem, we use the fact that the functions with identical derivatives differ only by a constant. We know one function whose derivative equals f, namely

$$G(x) = \int_a^x f(t)\,dt$$

Therefore, if F is any other such function, then $F(x) = G(x) + C$ throughout $[a,b]$ for some constant C. We use this equation to calculate $F(b) - F(a)$. Thus,

$$\begin{aligned} F(b) - F(a) &= [G(b)+C] - [G(a)+C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 = \int_a^b f(t)\,dt \end{aligned}$$

This concludes the proof.

The usual notation for the number $F(b) - F(a)$ is $F(x)]_a^b$ when $F(x)$ has a single term or $\left[F(x)\right]_a^b$ when $F(x)$ has more than one term.

The Fundamental Theorem of Calculus (part2) says that to evaluate the definite integral of a continuous function f from a to b, we find an anti-derivative F of f and calculate the number $F(b) - F(a)$. The existence of an anti-derivative is assured by the first part of the Fundamental Theorem.

Example 1:

a) $\int_0^{\pi} \cos x\,dx = \sin x]_0^{\pi} = \sin\pi - \sin 0 = 0 - 0 = 0$

b) $\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x]_{-\pi/4}^{0} = \sec 0 - \sec(-\pi/4) = 1 - \sqrt{2}$

c)

$$\begin{aligned} \int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\ &= \left[4^{3/2} + \frac{4}{4}\right] - \left[1^{3/2} + \frac{4}{1}\right] \\ &= [8+1] - 5 = 4 \end{aligned}$$

The above theorem explains the formulas we derived for the integrals of x and x^2 derived earlier. We can now see

$$\int_a^b x\,dx = \frac{x^2}{2}\Bigg]_a^b = \frac{b^2}{2} - \frac{a^2}{2}$$

because $x^2/2$ is an anti-derivative of x.

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Bigg]_a^b = \frac{b^3}{3} - \frac{a^3}{3}$$

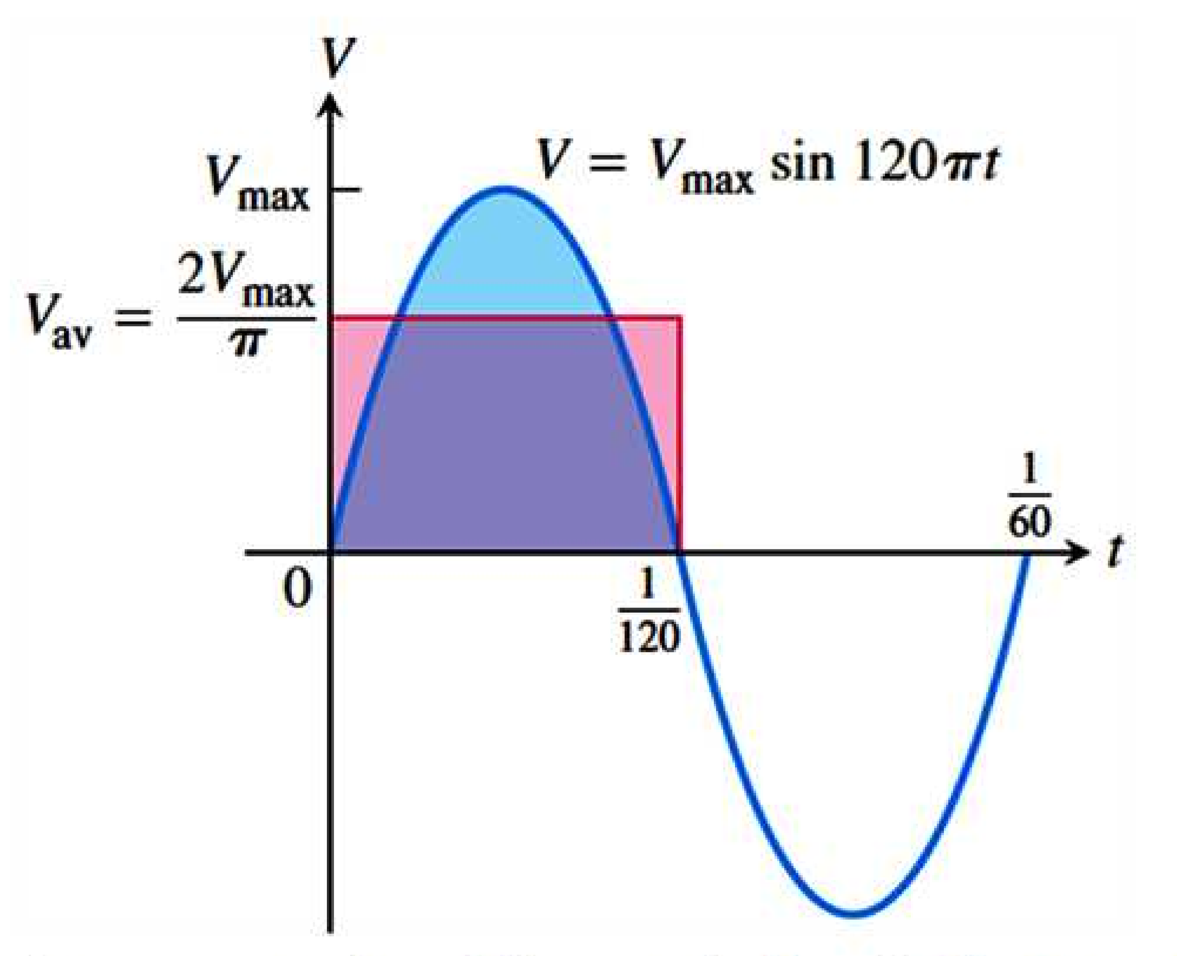
because $x^3/3$ is an anti-derivative of x^2.

Example 2:

We model the voltage in our home wiring with the sine function

$$V = V_{\max} \sin 120\pi t$$

which expresses the voltage V in volts as a function of time t in seconds. The function runs through 60 cycles each second (its frequency is 60 hertz or 60 Hz). The positive constant $V_{\max}$ is the ***peak voltage***.



The average value of V over a half-cycle (duration 1/120 second) is

$$\begin{aligned} V_{\text{av}} &= \frac{1}{(1/120) - 0}\int_0^{1/120} V_{\max} \sin 120\pi t\,dt \\ &= 120 V_{\max}\left[-\frac{1}{120\pi}\cos 120\pi t\right]_0^{1/120} \\ &= \frac{V_{\max}}{\pi}\left[-\cos\pi + \cos 0\right] = \frac{2V_{\max}}{\pi} \end{aligned}$$

The average value of the voltage over a full cycle is zero. If we measured the voltage with a standard moving-coil galvanometer, the meter would read zero.

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage, namely

$$V_{\text{rms}} = \sqrt{\left(V^2\right)_{av}}$$

The subscript "rms" stands for "root mean square." Since the average value of $V^2 = (V_{\max})^2 \sin^2 120\pi t$ over a cycle is

$$\left(V^2\right)_{\text{av}} = \frac{1}{(1/60) - 0}\int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t\,dt = \frac{(V_{\max})^2}{2}$$

, the rms voltage is

$$V_{\text{rms}} = \sqrt{\frac{(V_{\max})^2}{2}} = \frac{V_{\max}}{\sqrt{2}}$$

The values given for household currents and voltages are always rms values. Thus, "115 volts ac" means that the rms voltage is 115. The peak voltage,

$$V_{\max} = \sqrt{2}V_{\text{rms}} = \sqrt{2}\cdot 115 \approx 163 \text{ volts}$$

is considerably higher.

P1:

$$\int_{-\pi/3}^{\pi/3} \frac{1-\cos 2t}{2}\, dt =$$

A. $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ B. $\frac{\pi}{3} - \frac{3}{4}$ C. $\frac{\pi}{4} - \frac{\sqrt{3}}{4}$ D. $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

P1:

$$\int_{-\pi/3}^{\pi/3} \frac{1-\cos 2t}{2}\, dt =$$

A. $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ B. $\frac{\pi}{3} - \frac{3}{4}$ C. $\frac{\pi}{4} - \frac{\sqrt{3}}{4}$ D. $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

Answer: **A**

Solution:

$$\int_{-\pi/3}^{\pi/3} \frac{1-\cos 2t}{2}\, dt = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} dt - \frac{1}{2}\int_{-\pi/3}^{\pi/3} \cos 2t\ dt$$

$$= \frac{1}{2}[t]_{-\pi/3}^{\pi/3} - \frac{1}{2}\left[\frac{\sin 2t}{2}\right]_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] - \frac{1}{4}\left[\sin 2\left(\frac{\pi}{3}\right) - \sin 2\left(-\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{3} - \frac{1}{4}\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

P2:

$$\int_0^{\pi} \frac{1}{2}\left(\cos x + \left|\cos x\right|\right) dx =$$

A. -1 B. 0 C. 1 D. 2

P2:

$$\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right) dx =$$

A. -1 B. 0 C. 1 D. 2

Answer: **C**

Solution:

$$\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right) dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\cos x + |\cos x|\right) dx + \frac{1}{2}\int_{\pi/2}^{\pi} \left(\cos x + |\cos x|\right) dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\cos x + \cos x\right) dx + \frac{1}{2}\int_{\pi/2}^{\pi} \left(\cos x - \cos x\right) dx$$

$$= \int_0^{\pi/2} \cos x \, dx + 0 = \int_0^{\pi/2} \cos x \, dx$$

$$= [\sin x]_0^{\pi/2} = \left[\sin\frac{\pi}{2} - \sin 0\right] = (1-0) = 1$$

P3:

Evaluate $\int_9^4 \left(\frac{1-\sqrt{u}}{\sqrt{u}}\right) du$

P3:

Evaluate $\int_9^4 \left(\frac{1-\sqrt{u}}{\sqrt{u}}\right) du$

Solution:

$$\int_9^4 \left(\frac{1-\sqrt{u}}{\sqrt{u}}\right) du = -\int_4^9 \left(\frac{1-\sqrt{u}}{\sqrt{u}}\right) du$$

$$= \int_4^9 \left[1-\frac{1}{\sqrt{u}}\right] du$$

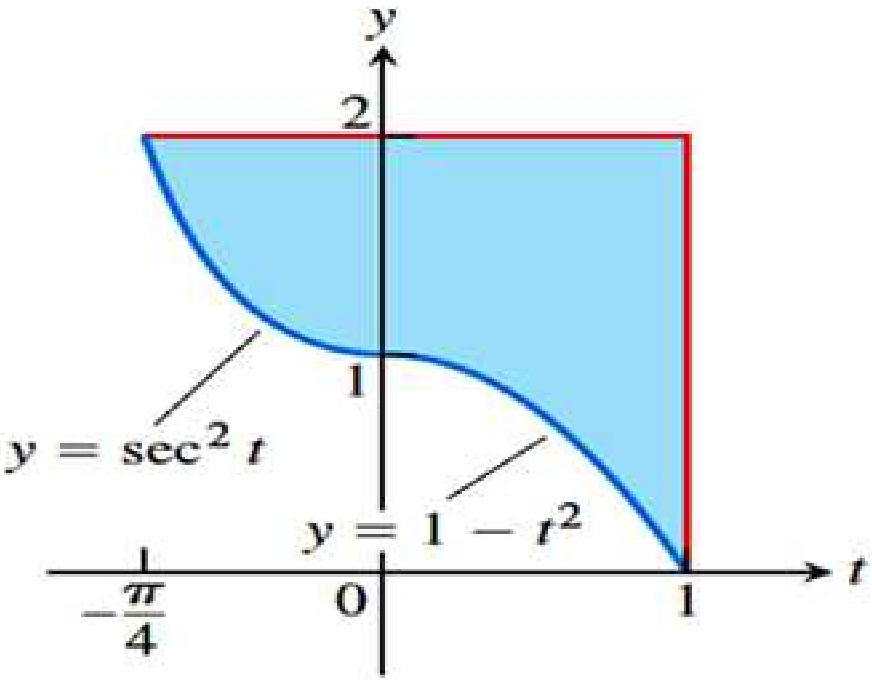
$$= \left[u - \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_4^9 = \left[u - 2u^{\frac{1}{2}}\right]_4^9$$

$$= \left[9 - 2(9)^{\frac{1}{2}}\right] - \left[4 - 2(4)^{\frac{1}{2}}\right]$$

$$= [9 - 6] - [4 - 4] = 3$$

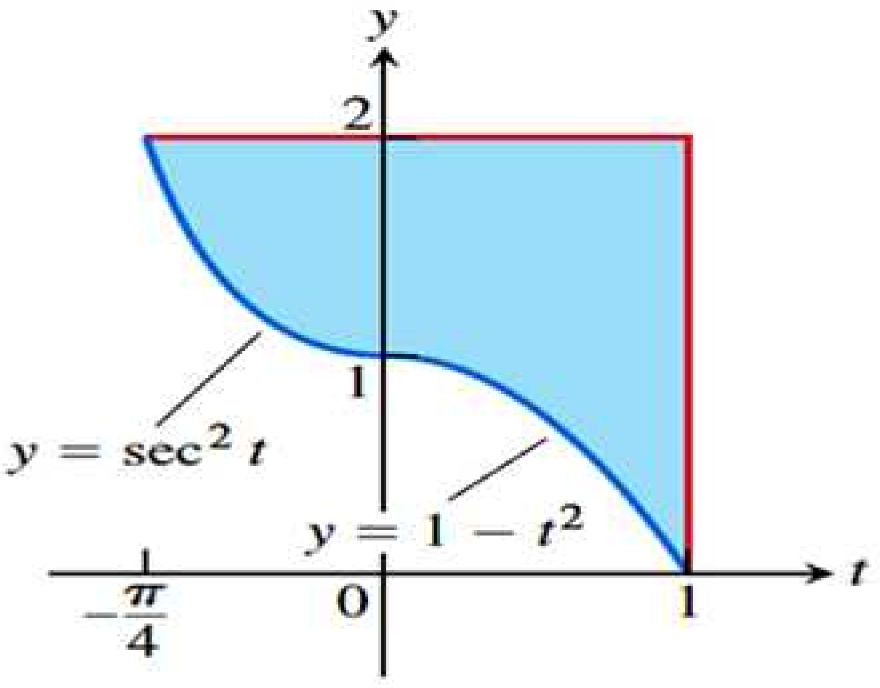
P4:

Find the area of the shaded region in the given graph



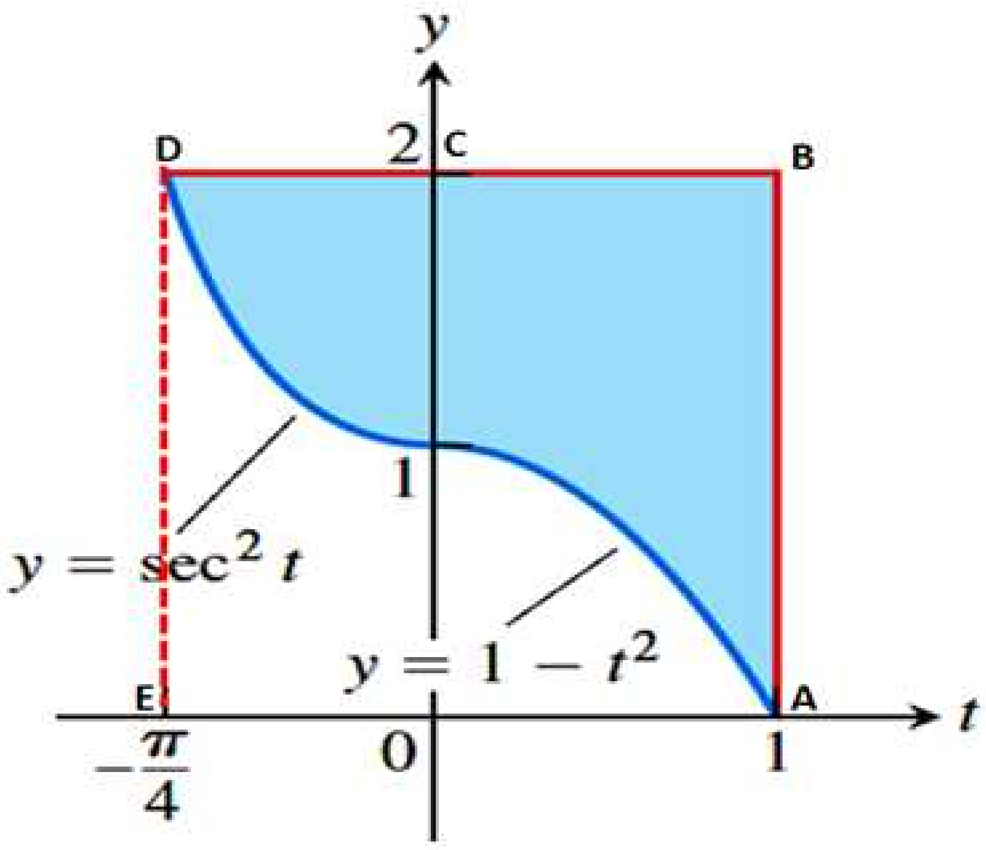
P4:

Find the area of the shaded region in the given graph



Solution:

The area of the shaded region in the given graph can be computed as follows:



The area of the rectangle bounded by the lines $y = 2$, $y = 0$, $t = -\left(\frac{\pi}{4}\right)$ and $t = 1$ is $= \left(1 + \frac{\pi}{4}\right) \times 2 = 2 + \frac{\pi}{2}$

Now, the area under the curve $y = sec^2 t$ on $\left[-\frac{\pi}{4}, 0\right]$ is

$$\int_{-\pi/4}^{0} \sec^2 t\, dt = [\tan t]_{-\pi/4}^{0}$$

$$= \left[\tan(0) - \tan\left(-\frac{\pi}{4}\right)\right] = 0 + 1 = 1$$

The area under the curve $y = 1 - t^2$ on $[0, 1]$ is

$$\int_0^1 \left(1 - t^2\right) dt = \left[t - \frac{t^3}{3}\right]_0^1 = \left[\left(1 - \frac{1^3}{3}\right) - 0\right] = \frac{2}{3}$$

Thus, the total area under curves on $\left[-\frac{\pi}{4}, 0\right]$ is $1 + \frac{2}{3} = \frac{5}{3}$

Therefore, the area of the shaded region is

$$A = \left(2 + \frac{\pi}{2}\right) - \frac{5}{3} = \left(\frac{1}{3} + \frac{\pi}{2}\right)$$

IP1:

$$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2} \right) dt =$$

A. $4\sqrt{3} - 3$ B. $4\sqrt{3} + 3$ C. $4\sqrt{3}$ D. $4\sqrt{3} - 6$

Answer: **B**

Solution:

$$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2} \right) dt$$

$$= 4\int_{-\pi/3}^{-\pi/4} \sec^2 t \; dt + \pi \int_{-\pi/3}^{-\pi/4} t^{-2} \; dt$$

$$= 4[\tan t]_{-\pi/3}^{-\pi/4} + \pi \left[\frac{t^{-2+1}}{-2+1} \right]_{-\pi/3}^{-\pi/4}$$

$$= 4\left[\tan\left(-\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{3}\right) \right] - \pi \left[-\frac{1}{(\pi/4)} - \left(-\frac{1}{(\pi/3)} \right) \right]$$

$$= 4\left[-1 + \sqrt{3} \right] + 1 = 4\sqrt{3} - 3$$

IP2:

$$\int_{-4}^{4} |x|\, dx =$$

A. 2 B. 4 C. 8 D. 16

Answer: **D**

Solution:

$$\int_{-4}^{4} |x|\, dx = \int_{-4}^{0} |x|\, dx + \int_{0}^{4} |x|\, dx$$

$$= \int_{-4}^{0} (-x)\, dx + \int_{0}^{4} x\, dx$$

$$= -\left[\frac{x^2}{2}\right]_{-4}^{0} + \left[\frac{x^2}{2}\right]_{0}^{4}$$

$$= -\left[0 - \frac{(-4)^2}{2}\right] + \left[\frac{(4)^2}{2} - 0\right] = 16$$

IP3:

Evaluate $\int_{\sqrt{2}}^{1}\left[\frac{u^{7}}{2}-\frac{1}{u^{5}}\right]du$

Solution:

$$\int_{\sqrt{2}}^{1}\left[\frac{u^{7}}{2}-\frac{1}{u^{5}}\right]du=-\int_{1}^{\sqrt{2}}\left[\frac{u^{7}}{2}-\frac{1}{u^{5}}\right]du$$

$$=\int_{1}^{\sqrt{2}}u^{-5}\ du-\frac{1}{2}\int_{1}^{\sqrt{2}}u^{7}\ du$$

$$=\left[\frac{u^{-5+1}}{-5+1}\right]_{1}^{\sqrt{2}}-\frac{1}{2}\left[\frac{u^{7+1}}{7+1}\right]_{1}^{\sqrt{2}}$$

$$=\left[\frac{u^{-4}}{-4}\right]_{1}^{\sqrt{2}}-\frac{1}{2}\left[\frac{u^{8}}{8}\right]_{1}^{\sqrt{2}}$$

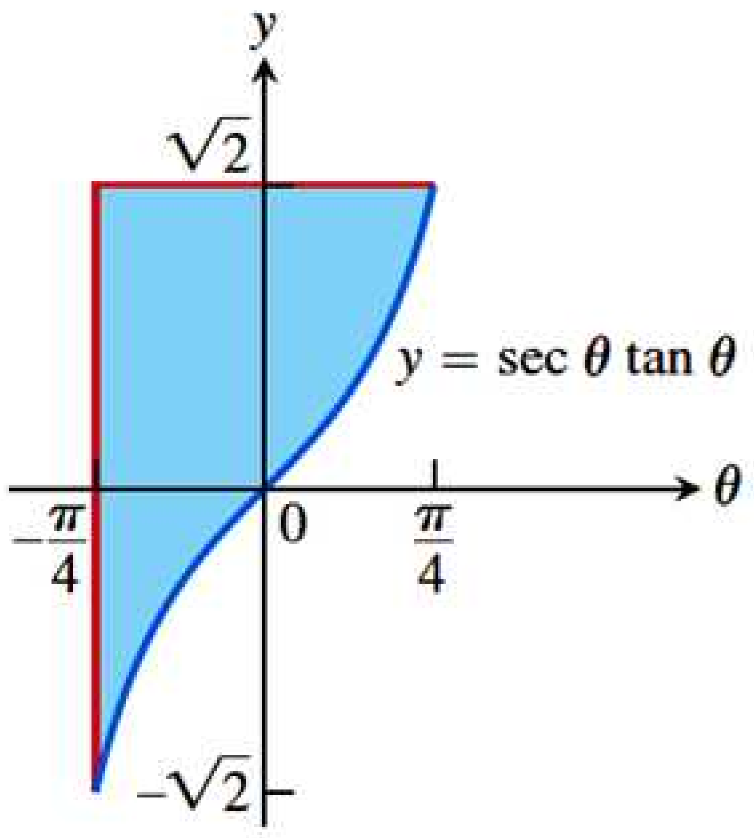
$$=-\frac{1}{4}\left[\left(\sqrt{2}\right)^{-4}-(1)^{-4}\right]-\frac{1}{16}\left[\left(\sqrt{2}\right)^{8}-(1)^{8}\right]$$

$$=-\frac{1}{4}\left[\frac{1}{4}-1\right]-\frac{1}{16}[16-1]$$

$$=\frac{3}{16}-\frac{15}{16}=-\frac{3}{4}$$

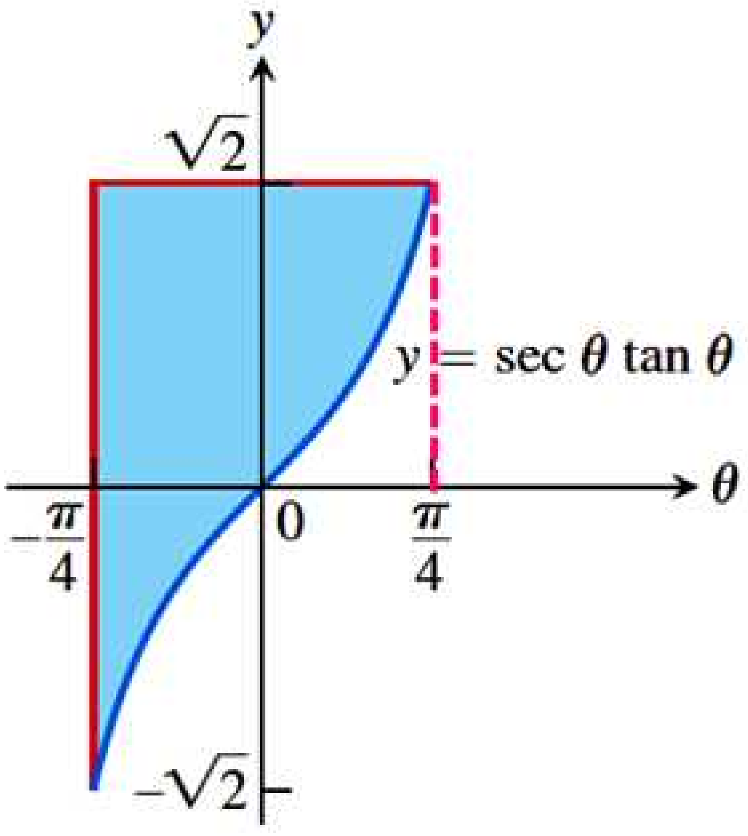
IP4:

Find the area of the shaded region in the given graph.



Solution:

The area of the shaded region in the given graph can be calculated as follows.



On $\left[-\frac{\pi}{4}, 0\right]$:

The area of the rectangle bounded by the lines $y = \sqrt{2}, y = 0,$

$\theta = 0$ and $\theta = -\frac{\pi}{4}$ is $= \sqrt{2} \times \frac{\pi}{4} = \frac{\pi\sqrt{2}}{4}$

The area between the curve $y = sec\theta.tan\theta$ and $y = 0$ is

$$-\int_{-\pi/4}^{0} \sec\theta.\tan\theta \, d\theta = \left[-\sec\theta\right]_{-\pi/4}^{0}$$

$$= \left[-\sec 0\right] - \left[-\sec(-\pi/4)\right] = -1 + \sqrt{2} = \sqrt{2} - 1$$

Therefore, the area of the shaded region on $\left[-\frac{\pi}{4}, 0\right]$ is

$= \frac{\pi\sqrt{2}}{4} + (\sqrt{2} - 1)$

On $\left[0, \frac{\pi}{4}\right]$:

The area of the rectangle bounded by the lines $\theta = 0, \theta = \frac{\pi}{4}$

$y = \sqrt{2}$ *and* $y = 0$ is $= \sqrt{2} \times \frac{\pi}{4} = \frac{\pi\sqrt{2}}{4}$

The area between the curve $y = sec\theta.tan\theta$ and $y = 0$ is

$$\int_{0}^{\pi/4} \sec\theta.\tan\theta \, d\theta = \left[\sec\theta\right]_{0}^{\pi/4}$$

$$= \left[\sec(\pi/4) - \sec 0\right] = \sqrt{2} - 1$$

Therefore, the area of the shaded region on $\left[0, \frac{\pi}{4}\right]$ is

$= \frac{\pi\sqrt{2}}{4} - (\sqrt{2} - 1)$

Thus, the area of the total shaded region on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is

$= \frac{\pi\sqrt{2}}{4} - (\sqrt{2} - 1) + \frac{\pi\sqrt{2}}{4} - (\sqrt{2} - 1) = \frac{\pi\sqrt{2}}{2}$

2.12. Evaluation of Definite Integrals

Exercise:

Evaluate the following definite integrals:

a. $\displaystyle\int_{-2}^{0}(2x+5)\,dx$

b. $\displaystyle\int_{0}^{4}\left(3x-\frac{x^3}{4}\right)dx$

c. $\displaystyle\int_{-2}^{2}\left(x^3-2x+3\right)dx$

d. $\displaystyle\int_{0}^{1}\left(x^3-\sqrt{x}\right)dx$

e. $\displaystyle\int_{-2}^{-1}\frac{2}{x^2}\,dx$

f. $\displaystyle\int_{0}^{\pi}\sin x\,dx$

g. $\displaystyle\int_{-\pi/2}^{\pi/2}\sin|x|\,dx$

h. $\displaystyle\int_{\pi/2}^{0}\left(\frac{1+cos2t}{2}\right)dt$

i. $\displaystyle\int_{0}^{\pi}(1+\cos x)\,dx$

j. $\displaystyle\int_{-1}^{1}(r+1)^2\,dr$

k. $\displaystyle\int_{-\pi/2}^{\pi/2}\left(8y^2+\sin y\right)dy$

l. $\displaystyle\int_{1/2}^{1}\left(\frac{1}{v^3}-\frac{1}{v^2}\right)dv$

m. $\displaystyle\int_{\pi/4}^{3\pi/2}\csc\theta.\cot\theta\,d\theta$

2.13

Substitution in Definite Integrals

Learning objectives:

- To evaluate the Definite integrals by the method of substitution.

  AND
- To practice the related problems.

There are two methods for evaluating a definite integral by substitution. One is to find the corresponding indefinite integral by substitution and use one of the resulting anti-derivatives to evaluate the definite integral by the Fundamental Theorem. The other is to use the following formula.

$$\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$$

We make the same u-substitution $u = g(x)$ and $du = g'(x)dx$ that we use to evaluate the corresponding indefinite integral. We then integrate with respect to u from the value u at $x = a$ to the value u at $x = b$.

Example 1:

Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

Solution

Method 1: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}du \qquad u = x^3+1, \quad du = 3x^2dx$$

$$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(x^3+1\right)^{3/2} + C$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}\left(x^3+1\right)^{3/2}\Big]_{-1}^{1}$$

$$= \frac{2}{3}\left[\left(1^3+1\right)^{3/2} - \left(\left(-1\right)^3+1\right)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral and evaluate the transformed integral with the transformed limits.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du \qquad u = x^3+1, \quad du = 3x^2dx$$

$$\text{when } x = -1,\ u = \left(-1\right)^3 + 1 = 0$$

$$\text{when } x = 1,\ u = \left(1\right)^3 + 1 = 2$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2 = \frac{2}{3}\left[2^{3/2} - 0\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Example 2:

$$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta$$

$$put\ u = \cot\theta \Rightarrow du = -\csc^2\theta\,d\theta$$

$$\text{when } \theta = \pi/4,\ u = \cot\left(\pi/4\right) = 1$$

$$\text{when } \theta = \pi/2,\ u = \cot\left(\pi/2\right) = 0$$

$$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta = \int_1^0 u\cdot\left(-du\right) = -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0 = -\left[\frac{0^2}{2} - \frac{1^2}{2}\right] = \frac{1}{2}$$

P1:

Evaluate the integrals

a. $\int_{0}^{\sqrt{7}} t\left(t^2+1\right)^{1/3}\,dt$

b. $\int_{-\sqrt{7}}^{0} t\left(t^2+1\right)^{1/3}\,dt$

P1:

Evaluate the integrals

a. $\int_0^{\sqrt{7}} t\left(t^2+1\right)^{1/3}\, dt$ b. $\int_{-\sqrt{7}}^{0} t\left(t^2+1\right)^{1/3}\, dt$

Solution:

a. To evaluate $\int_0^{\sqrt{7}} t\left(t^2+1\right)^{1/3}\, dt$

Put $u = t^2 + 1 \Rightarrow du = 2t\, dt \Rightarrow \frac{du}{2} = t\, dt$

Limits:

$t = 0 \Rightarrow u = 0 + 1 = 1$

$t = \sqrt{7} \Rightarrow u = 7 + 1 = 8$

$$\int_0^{\sqrt{7}} t\left(t^2+1\right)^{1/3}\, dt = \int_1^8 u^{1/3}.\frac{du}{2} = \frac{1}{2}\int_1^8 u^{1/3} du$$

$$= \frac{1}{2}\left[\frac{u^{1/3+1}}{1/3+1}\right]_1^8$$

$$= \frac{3}{8}\left[(8)^{4/3} - (1)^{4/3}\right] = \frac{45}{8}$$

b. To evaluate $\int_{-\sqrt{7}}^{0} t\left(t^2+1\right)^{1/3}\, dt$

Put $u = t^2 + 1 \Rightarrow du = 2t\, dt \Rightarrow \frac{du}{2} = t\, dt$

Limits:

$t = -\sqrt{7} \Rightarrow u = 7 + 1 = 8$

$t = 0 \Rightarrow u = 0 + 1 = 1$

$$\int_{-\sqrt{7}}^{0} t\left(t^2+1\right)^{1/3}\, dt = \int_8^1 u^{1/3}.\frac{du}{2} = -\frac{1}{2}\int_1^8 u^{1/3} du$$

$$= -\frac{1}{2}\left[\frac{u^{1/3+1}}{1/3+1}\right]_1^8$$

$$= -\frac{3}{8}\left[(8)^{4/3} - (1)^{4/3}\right] = -\frac{45}{8}$$

P2:

Evaluate the integrals

a. $\int_{0}^{\pi} 3\cos^2 x.\sin x \, dx$

b. $\int_{2\pi}^{3\pi} 3\cos^2 x.\sin x \, dx$

P2:

Evaluate the integrals

a. $\int_0^{\pi} 3\cos^2 x.\sin x\, dx$ b. $\int_{2\pi}^{3\pi} 3\cos^2 x.\sin x\, dx$

Solution:

a. To evaluate $\int_0^{\pi} 3\cos^2 x.\sin x\, dx$

Put $u = cosx \Rightarrow du = -sinx\ dx \Rightarrow sinx\ dx = -du$

Limits:

$x = 0 \Rightarrow u = 1$ and $x = \pi \Rightarrow u = -1$

$$\therefore \int_0^{\pi} 3\cos^2 x.\sin x\, dx = \int_1^{-1} -3u^2\ du = 3\int_{-1}^{1} u^2\, du$$

$$= 3\left[\frac{u^3}{3}\right]_{-1}^{1} = (1)^3 - (-1)^3 = 2$$

b. To evaluate $\int_{2\pi}^{3\pi} 3\cos^2 x.\sin x\, dx$

Put $u = cosx \Rightarrow du = -sinx\ dx \Rightarrow sinx\ dx = -du$

Limits:

$x = 2\pi \Rightarrow u = cos2\pi = 1$

$x = 3\pi \Rightarrow u = cos3\pi = -1$

$$\therefore \int_{2\pi}^{3\pi} 3\cos^2 x.\sin x\, dx = \int_1^{-1} -3u^2\ du = 3\int_{-1}^{1} u^2\ du$$

$$= 3\left[\frac{u^3}{3}\right]_{-1}^{1} = (1)^3 - (-1)^3 = 2$$

P3:

a. Evaluate $\displaystyle\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\, dw$

b. Evaluate $\displaystyle\int_{0}^{\pi/2} \frac{\sin w}{(3+2\cos w)^2}\, dw$

P3:

a. Evaluate $\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\, dw$

b. Evaluate $\int_{0}^{\pi/2} \frac{\sin w}{(3+2\cos w)^2}\, dw$

Solution:

a.To evaluate $\int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\, dw$

Put $u = 3 + 2cosw \Rightarrow du = -2sinw\, dw \Rightarrow sinw\, dw = -\frac{1}{2}du$

Limits:

$w = -\frac{\pi}{2} \Rightarrow u = 3 + 2cos\left(-\frac{\pi}{2}\right) = 3$

$w = 0 \Rightarrow u = 3 + 2\cos(0) = 5$

$$\therefore \int_{-\pi/2}^{0} \frac{\sin w}{(3+2\cos w)^2}\, dw$$

$$= \int_3^5 \frac{1}{u^2}\left(-\frac{1}{2}\, du\right) = -\frac{1}{2}\int_3^5 u^{-2}\, du$$

$$= -\frac{1}{2}\left[-\frac{1}{u}\right]_3^5 = \frac{1}{2}\left[\frac{1}{5}-\frac{1}{3}\right] = -\frac{1}{15}$$

b. To evaluate $\int_{0}^{\pi/2} \frac{\sin w}{(3+2\cos w)^2}\, dw$

Put $u = 3 + 2cosw \Rightarrow du = -2sinw\, dw \Rightarrow sinw\, dw = -\frac{1}{2}du$

Limits:

$w = 0 \Rightarrow u = 3 + 2\cos(0) = 5$

$w = \frac{\pi}{2} \Rightarrow u = 3 + 2cos\left(\frac{\pi}{2}\right) = 3$

$$\therefore \int_{0}^{\pi/2} \frac{\sin w}{(3+2\cos w)^2}\, dw$$

$$= \int_5^3 \frac{1}{u^2}\left(-\frac{1}{2}\, du\right) = \frac{1}{2}\int_3^5 u^{-2}\, du$$

$$= \frac{1}{2}\left[-\frac{1}{u}\right]_3^5 = -\frac{1}{2}\left[\frac{1}{5}-\frac{1}{3}\right] = \frac{1}{15}$$

P4:

$$\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right).\sec^2\left(\frac{\theta}{6}\right) d\theta =$$

A. $\sqrt{3}$ B. 2 C. $3\sqrt{2}$ D. 12

P4:

$$\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right).\sec^2\left(\frac{\theta}{6}\right)\,d\theta =$$

A. $\sqrt{3}$ B. 2 C. $3\sqrt{2}$ D. 12

Answer: **D**

Solution:

To evaluate

$$\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right).\sec^2\left(\frac{\theta}{6}\right)\,d\theta = \int_{\pi}^{3\pi/2} \frac{1}{\tan^5\left(\frac{\theta}{6}\right)}.\sec^2\left(\frac{\theta}{6}\right)\,d\theta$$

Put $u = \tan\left(\frac{\theta}{6}\right) \Rightarrow du = \frac{1}{6}\sec^2\left(\frac{\theta}{6}\right)d\theta$

$$\Rightarrow \sec^2\left(\frac{\theta}{6}\right)d\theta = 6du$$

Limits:

$\theta = \pi \Rightarrow u = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\theta = \frac{3\pi}{2} \Rightarrow u = \tan\frac{3\pi}{12} = 1$

$$\therefore \int_{\pi}^{3\pi/2} \frac{1}{\tan^5\left(\frac{\theta}{6}\right)}.\sec^2\left(\frac{\theta}{6}\right)\,d\theta$$

$$= \int_{1/\sqrt{3}}^{1} u^{-5}\,(6\,du) = 6\int_{1/\sqrt{3}}^{1} u^{-5}\,du$$

$$= 6\left[\frac{u^{-5+1}}{-5+1}\right]_{1/\sqrt{3}}^{1} = -\frac{3}{2}\left[u^{-4}\right]_{1/\sqrt{3}}^{1}$$

$$= -\frac{3}{2}[1-9] = 12$$

$$\therefore \int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right).\sec^2\left(\frac{\theta}{6}\right)\,d\theta = 12$$

IP1:

Evaluate the integrals

a. $\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$ b. $\int_1^4 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

Solution:

a. To evaluate $\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

Put $u = 1 + v^{3/2} \Rightarrow du = \frac{3}{2}\sqrt{v}\ dv$

$\Rightarrow \sqrt{v}\,dv = \frac{2}{3}du$

Limits:

$v = 0 \Rightarrow u = 0 + 1 = 1$ and $v = 1 \Rightarrow u = 1 + 1 = 2$

$$\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv = \int_1^2 \left(\frac{1}{u^2}\right)\left(\frac{20}{3}\,du\right) = \frac{20}{3}\int_1^2 u^{-2}\,du$$

$$= \frac{20}{3}\left[-\frac{1}{u}\right]_1^2 = -\frac{20}{3}\left[\frac{1}{2}-1\right] = \frac{10}{3}$$

b. To evaluate $\int_1^4 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv$

Put $u = 1 + v^{3/2} \Rightarrow du = \frac{3}{2}\sqrt{v}\ dv$

$\Rightarrow \sqrt{v}\,dv = \frac{2}{3}du$

Limits:

$v = 1 \Rightarrow u = 1 + 1 = 2$
$v = 4 \Rightarrow u = 1 + 8 = 9$

$$\int_1^4 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv = \int_2^9 \left(\frac{1}{u^2}\right)\left(\frac{20}{3}\,du\right)$$

$$= \frac{20}{3}\int_2^9 u^{-2}\,du$$

$$= \frac{20}{3}\left[-\frac{1}{u}\right]_2^9 = -\frac{20}{3}\left[\frac{1}{9}-\frac{1}{2}\right] = \frac{70}{27}$$

IP2:

Evaluate the integrals

a. $\int_0^1 r\sqrt{1-r^2}\ dr$ b. $\int_{-1}^1 r\sqrt{1-r^2}\ dr$

Solution:

a. To evaluate $\int_0^1 r\sqrt{1-r^2}\ dr$

Put $u = 1 - r^2 \Rightarrow du = -2r\ dr \Rightarrow r\ dr = -\frac{1}{2} du$

Limits:

$r = 0 \Rightarrow u = 1 - 0 = 1$

$r = 1 \Rightarrow u = 1 - 1 = 0$

$$\therefore \int_0^1 r\sqrt{1-r^2}\ dr = \int_1^0 \sqrt{u}\left(-\frac{1}{2}\ du\right)$$

$$= \frac{1}{2}\int_0^1 \sqrt{u}\ du$$

$$= \frac{1}{2}\left[\frac{u^{1/2+1}}{1/2+1}\right]_0^1 = \frac{1}{3}[1-0] = \frac{1}{3}$$

b. To evaluate $\int_{-1}^1 r\sqrt{1-r^2}\ dr$

Put $u = 1 - r^2 \Rightarrow du = -2r\ dr \Rightarrow r\ dr = -\frac{1}{2} du$

Limits:

$r = -1 \Rightarrow u = 1 - 1 = 0$

$r = 1 \Rightarrow u = 1 - 1 = 0$

$$\therefore \int_{-1}^1 r\sqrt{1-r^2}\ dr = \int_0^0 \sqrt{u}\left(-\frac{1}{2}\ du\right) = 0$$

IP3:

Evaluate the integrals

a. $\int_{-\pi/2}^{0}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

b. $\int_{-\pi/2}^{\pi/2}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

Solution:

a. To evaluate $\int_{-\pi/2}^{0}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

Put $u=2+\tan\frac{t}{2} \Rightarrow du=\frac{1}{2}\sec^2\frac{t}{2}dt \Rightarrow \sec^2\frac{t}{2}\;\; dt=2du$

Limits:

$t=-\frac{\pi}{2} \Rightarrow u=2+tan\left(-\frac{\pi}{4}\right)=1$

$t=0 \Rightarrow u=2+tan0 \Rightarrow u=2$

$$\therefore \int_{-\pi/2}^{0}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt=\int_1^2 u.(2du)=2\int_1^2 u\;du$$

$$=2\left[\frac{u^2}{2}\right]_1^2=[4-1]=3$$

b. To evaluate $\int_{-\pi/2}^{\pi/2}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

Put $u=2+\tan\frac{t}{2} \Rightarrow du=\frac{1}{2}\sec^2\frac{t}{2}dt \Rightarrow \sec^2\frac{t}{2}\;\; dt=2du$

Limits:

$t=-\frac{\pi}{2} \Rightarrow u=2+tan\left(-\frac{\pi}{4}\right)=1$

$t=\frac{\pi}{2} \Rightarrow u=2+tan\left(\frac{\pi}{4}\right)=3$

$$\therefore \int_{-\pi/2}^{\pi/2}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt=\int_1^3 u.(2du)=2\int_1^3 u\;du$$

$$=2\left[\frac{u^2}{2}\right]_1^3=[9-1]=8$$

IP4:

$$\int_0^{\pi/4} (1-\sin 2t)^{3/2} \cos 2t \ dt =$$

A. $-\frac{5}{2}$ B. 0 C. $\frac{1}{5}$ D. $\frac{2}{5}$

Answer: **C**

Solution:

To evaluate $\int_0^{\pi/4} (1-\sin 2t)^{3/2} \cos 2t \ dt$

Put $u = 1 - sin2t \Rightarrow du = -2cos2t \ dt$

$$\Rightarrow cos2t \ dt = -\frac{1}{2} \ du$$

Limits:

$t = 0 \Rightarrow u = 1 - sin2(0) = 1$

$t = \frac{\pi}{4} \Rightarrow u = 1 - sin2\left(\frac{\pi}{4}\right) = 0$

$$\therefore \int_0^{\pi/4} (1-\sin 2t)^{3/2} \cos 2t \ dt =$$

$$= \int_1^0 u^{3/2} \left(-\frac{1}{2} du\right) = -\frac{1}{2} \int_1^0 u^{3/2} \ du$$

$$= \frac{1}{2} \int_0^1 u^{3/2} \ du = \frac{1}{2} \left[\frac{u^{3/2+1}}{3/2+1}\right]_0^1 = \frac{1}{5} \left[u^{5/2}\right]_0^1$$

$$= \frac{1}{5}[1-0] = \frac{1}{5}$$

2.13. Substitution in Definite Integrals

Exercise:

Evaluate the integrals in problems 1-12.

1. a) $\int_0^3 \sqrt{y+1}\,dy$ b) $\int_{-1}^0 \sqrt{y+1}\,dy$

2. a) $\int_0^{\pi/4} \tan x \sec^2 x\,dx$ b) $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

3. a) $\int_0^1 t^3\left(1+t^4\right)^3 dt$ b) $\int_{-1}^1 t^3\left(1+t^4\right)^3 dt$

4. a) $\int_{-1}^1 \frac{5r}{\left(4+r^2\right)^2} dr$ b) $\int_0^1 \frac{5r}{\left(4+r^2\right)^2} dr$

5. a) $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx$ b) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx$

6. a) $\int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx$ b) $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}} dx$

7. a) $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$

   b) $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

8. a) $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}} dz$ b) $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}} dz$

9. $\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

10. $\int_0^1 \sqrt{t^5+2t}\left(5t^4+2\right)dt$

11. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

12. $\int_0^{\pi} 5(5-4\cos t)^{1/4} \sin t\,dt$

13. $\int_0^1 \left(4y-y^2+4y^3+1\right)^{-2/3}\left(12y^2-2y+4\right)dy$

14. $\int_0^1 \left(y^3+6y^2-12y+9\right)^{-1/2}\left(y^2+4y-4\right)dy$

15. $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right) dt$

16. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2\left(\theta^{3/2}\right)d\theta$

HWA-12 (2.13)

Answer all the questions and submit

I. Evaluate the following:

1. $\int_0^{\frac{\pi}{4}} \frac{\sin^9 x}{\cos^{11} x} dx$

2. $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}}$

3. $\int_0^1 (2x+3)\left(\sqrt{3-2x}\right) dx$

4. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta} \cos^2 \theta^{\frac{3}{2}} \, d\theta$

5. $\int_{\frac{\pi^2}{36}}^{\frac{\pi^2}{4}} \frac{\cos\sqrt{t}}{\sqrt{t \sin\sqrt{t}}} dt$

6. $\int_{1/3}^{1} \frac{\left(x-x^3\right)^{1/3}}{x^4} dx$

7. $\int_0^{\frac{\pi}{4}} \frac{\sin^2 x.\cos^2 x}{\left(\sin^3 x+\cos^3 x\right)^2} dx$

3.1

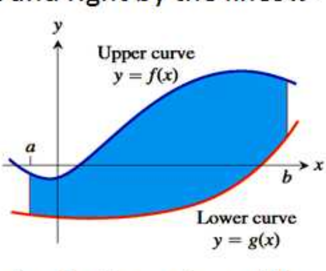
Areas between Curves

Learning objectives:

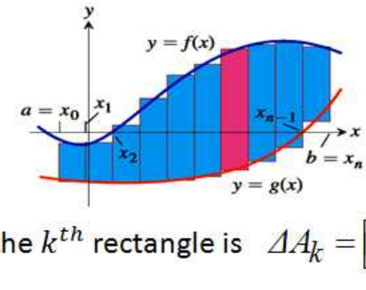
In this module, we study

- To determine the areas of the shaded regions between the given curves and lines.
  AND
- To practice the related problems.

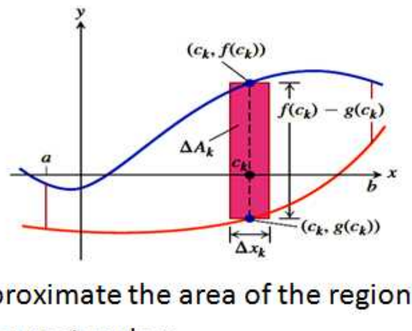
Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$.



We first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$



The area of the k^{th} rectangle is $\Delta A_k = [f(c_k) - g(c_k)]\Delta x_k$



We then approximate the area of the region by adding the areas of the n rectangles:

$$A = \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\Delta x_k \quad \text{Riemann sum}$$

As $\|P\| \to 0$ the sums on the right approach the limit

$\int_a^b [f(x) - g(x)]dx$ because f and g are continuous.

The area of the region is the value of this integral.

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\Delta x_k = \int_a^b [f(x) - g(x)]dx$$

Thus, this *basic formula* for finding the area between curves is a limit of Riemann sums.

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a,b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $[f - g]$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx \qquad \text{...(1)}$$

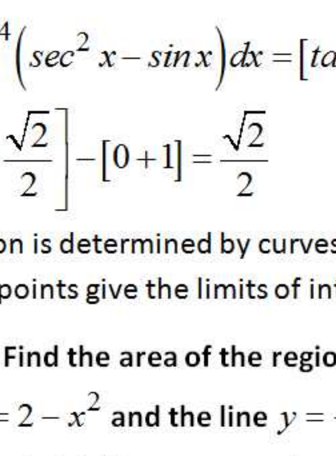
We will graph the curves and draw a representative rectangle. This reveals which curve is f (upper curve) and which is g (lower curve). It also helps identify the limits of integration. Write a formula for $f(x) - g(x)$ and simplify it. Integrate $[f(x) - g(x)]$ from a to b; this gives the area.

Example 1:

Find the area between $y = sec^2 x$ and $y = sin\, x$ from 0 to π/4.

Solution:

The curves and a vertical rectangle are sketched.



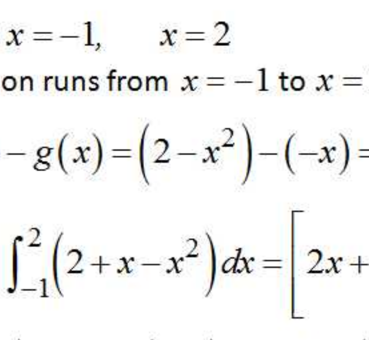
The upper curve is the graph of $f(x) = sec^2 x$; the lower is the graph of $f(x) = sinx$.

$$A = \int_0^{\pi/4} \left(sec^2 x - sin\,x\right)dx = [tan\,x + cos\,x]_0^{\pi/4}$$
$$= \left[1 + \frac{\sqrt{2}}{2}\right] - [0+1] = \frac{\sqrt{2}}{2}$$

When a region is determined by curves that intersect, the intersection points give the limits of integration.

Example 2: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution: We sketch the curves and a vertical rectangle.



The upper and lower curves are $f(x) = 2 - x^2$ and $g(x) = -x$. The x-coordinates of the intersection points are the limits of integration.

$$2 - x^2 = -x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0$$
$$\Rightarrow x = -1, \qquad x = 2$$

The region runs from $x = -1$ to $x = 2$.

$$f(x) - g(x) = (2 - x^2) - (-x) = 2 - x^2 + x$$

$$A = \int_{-1}^{2} (2 + x - x^2)dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

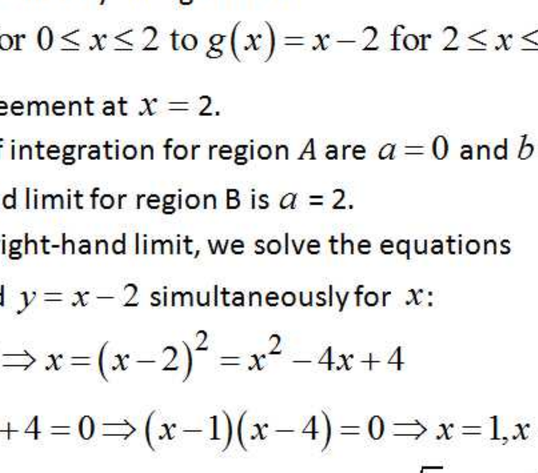
If the formula for a bounding curve changes at one or more points, we partition the region into subregions that correspond to the formula changes, calculate the area of each subregion and then add.

Example 3:

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$

Solution:

The graphs of the two functions are shown below.



The region's upper boundary is the graph of $f(x) = \sqrt{x}$.

The lower boundary changes from

$g(x) = 0$ for $0 \leq x \leq 2$ to $g(x) = x - 2$ for $2 \leq x \leq 4$.

There is agreement at $x = 2$.

The limits of integration for region A are $a = 0$ and $b = 2$.

The left-hand limit for region B is a = 2.

To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\sqrt{x} = x - 2 \Rightarrow x = (x-2)^2 = x^2 - 4x + 4$$
$$\Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0 \Rightarrow x = 1, x = 4$$

Only the value x = 4 satisfies the equation $\sqrt{x} = x - 2$.

The value x = 1 is an extraneous root introduced by squaring. The right-hand limit is $b = 4$

For $0 \leq x \leq 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \leq x \leq 4$: $\quad f(x) - g(x) = \sqrt{x} - (x-2) = \sqrt{x} - x + 2$

We add the areas of subregions A and B to find the total area:

$$\text{Total area} = \int_0^2 \sqrt{x}\, dx + \int_2^4 (\sqrt{x} - x + 2)dx$$
$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$
$$= \frac{2}{3}(2^{3/2}) - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$
$$= \frac{2}{3}(8) - 2 = \frac{10}{3}$$

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

$$A = \int_c^d [f(y) - g(y)]dy$$

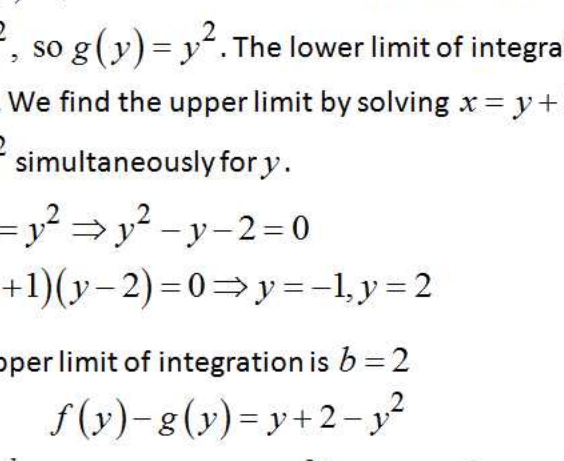
In the equation (2) above, f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Example 4:

Find the area of the region in the previous example by integrating with respect to y.

Solution:

We sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values.



The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y.

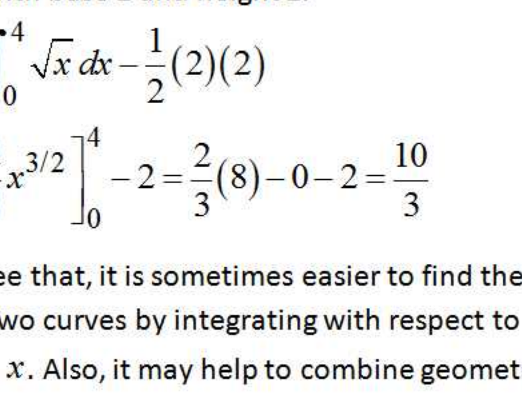
$$y + 2 = y^2 \Rightarrow y^2 - y - 2 = 0$$
$$\Rightarrow (y+1)(y-2) = 0 \Rightarrow y = -1, y = 2$$

The upper limit of integration is $b = 2$

$$f(y) - g(y) = y + 2 - y^2$$
$$A = \int_a^b [f(y) - g(y)]dy = \int_0^2 [2 + y - y^2]dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 = 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}$$

Sometimes, we can combine integrals with formulas from geometry.

The previous example can also be solved the following way.



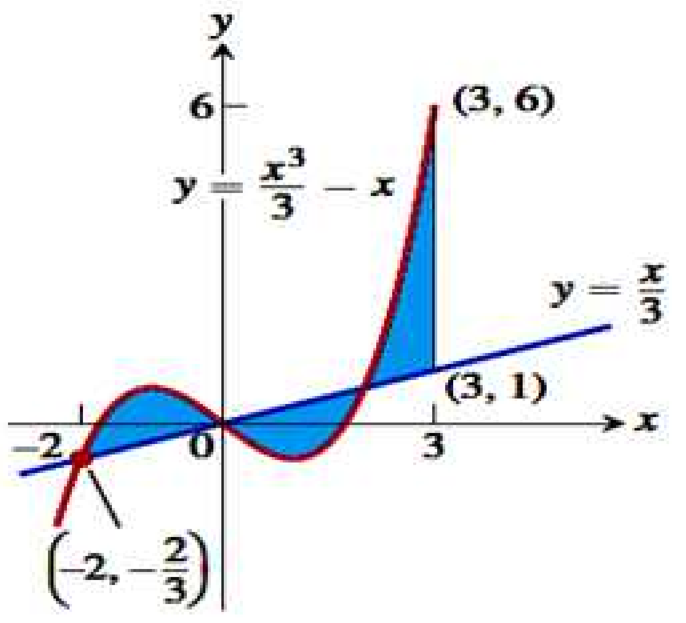
The area we want is the area between the curve $y = \sqrt{x}, 0 \leq x \leq 4$, and the x-axis, minus the area of a triangle with base 2 and height 2.

$$\text{Area} = \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2)$$
$$= \frac{2}{3}x^{3/2}\Big]_0^4 - 2 = \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}$$

We thus see that, it is sometimes easier to find the area between two curves by integrating with respect to y instead of x. Also, it may help to combine geometry and calculus.

IP1:

Find the total area of the shaded region of the graph



Solution:

To obtain the total area A of the shaded region in the given graph, we have to calculate the areas of the three shaded regions A_1, A_2, A_3 and add the absolute the values of the areas, where $A_1: -2 \le x \le 0$, upper curve $f(x) = \frac{x^3}{3} - x$ and lower curve $g(x) = \frac{x}{3}$,

$$f(x) - g(x) = \left(\frac{x^3}{3} - x\right) - \frac{x}{3} = \frac{1}{3}(x^3 - 4x)$$

$$\therefore A_1 = \int_{-2}^{0} [f(x) - g(x)]\, dx = \frac{1}{3}\int_{-2}^{0} \left(x^3 - 4x\right) dx$$

$$= \frac{1}{3}\left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{-2}^{0} = \left[0 - \frac{1}{3}(4-8)\right] = \frac{4}{3}$$

A_2: For the sketch lower limit is 0 and we find the upper limit by solving the equations $y = \frac{x^3}{3} - x$ and $y = \frac{x}{3}$ simultaneously for x.

$$\frac{x^3}{3} - x = \frac{x}{3} \Rightarrow \frac{x^3}{3} - \frac{4}{3}x = 0 \Rightarrow \frac{x}{3}(x-2)(x+2) = 0$$

$\Rightarrow x = -2, 0, 2 \Rightarrow$ Upper limit is 2

Upper curve $f(x) = \frac{x}{3}$ and lower curve $g(x) = \frac{x^3}{3} - x$

$$f(x) - g(x) = \frac{x}{3} - \left(\frac{x^3}{3} - x\right) = -\frac{1}{3}(x^3 - 4x)$$

$$\therefore A_2 = \int_{0}^{2} [f(x) - g(x)]\, dx = -\frac{1}{3}\int_{0}^{2} \left(x^3 - 4x\right) dx$$

$$= -\frac{1}{3}\left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{0}^{2} = -\left[\frac{1}{3}(4-8) - 0\right] = \frac{4}{3}$$

$A_3: 2 \le x \le 3$, upper curve $f(x) = \frac{x^3}{3} - x$ and lower curve $g(x) = \frac{x}{3}$,

$$f(x) - g(x) = \left(\frac{x^3}{3} - x\right) - \frac{x}{3} = \frac{1}{3}(x^3 - 4x)$$

$$\therefore A_3 = \int_{2}^{3} [f(x) - g(x)]\, dx = \frac{1}{3}\int_{2}^{3} \left(x^3 - 4x\right) dx$$

$$= \frac{1}{3}\left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{2}^{3} = \left[\left(\frac{81}{4} - 18\right) - \left(\frac{16}{4} - 8\right)\right] = \frac{25}{12}$$

Therefore, the total area of the shaded region is

$$A = |A_1| + |A_2| + |A_3| = \left|\frac{4}{3}\right| + \left|\frac{4}{3}\right| + \left|\frac{25}{12}\right| = \frac{19}{4}$$

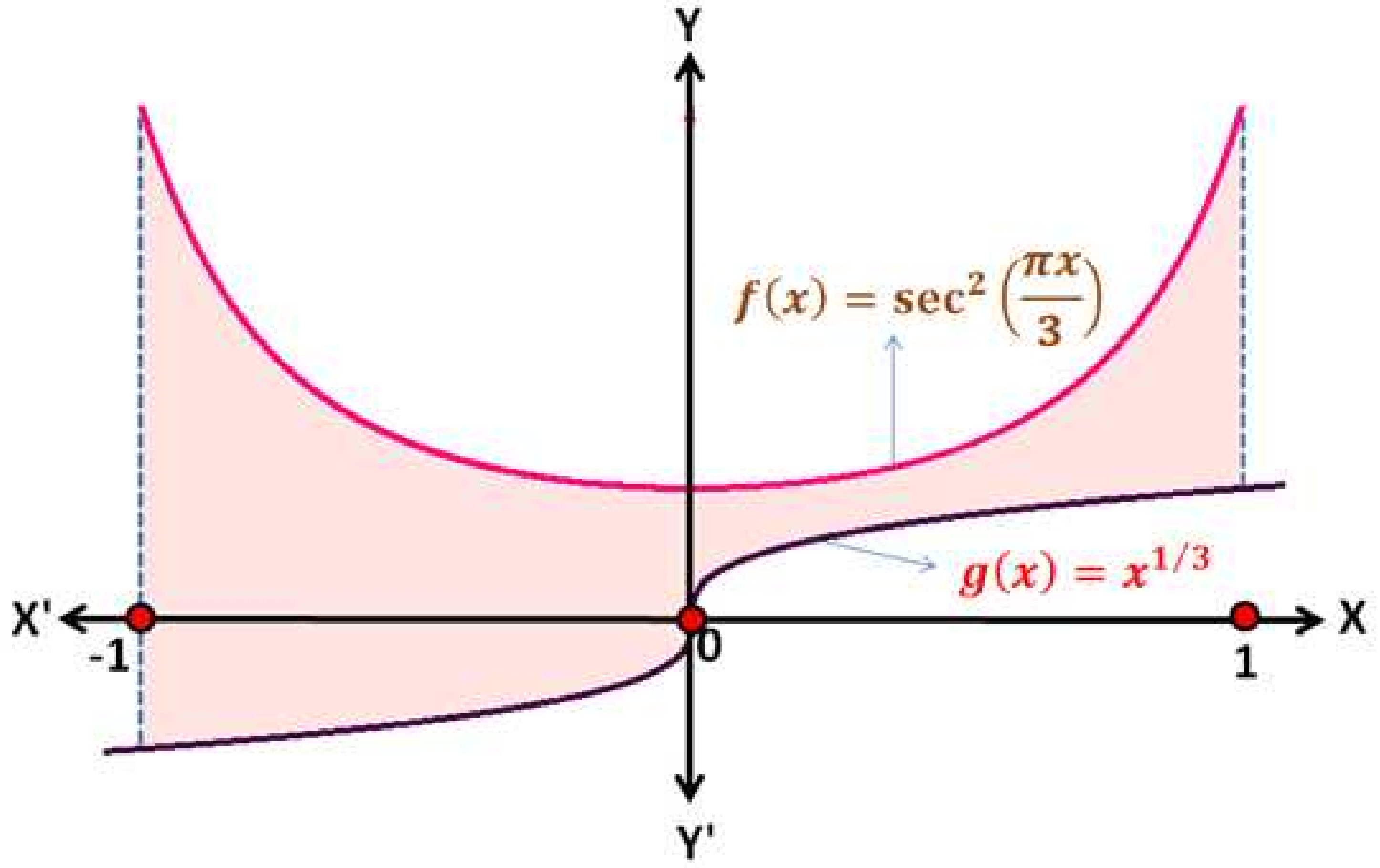
IP2:

Find the area of the region enclosed by the curves $y = x^{1/3}$ and $y = \sec^2\left(\frac{\pi x}{3}\right)$, $-1 \le x \le 1$.

Solution:

Given curves are $y = x^{1/3}$ and $y = \sec^2\left(\frac{\pi x}{3}\right)$ where $-1 \le x \le 1$.

The graph is plotted between the curves $y = x^{1/3}$ and $y = \sec^2\left(\frac{\pi x}{3}\right)$ over the interval $[-1, 1]$



From the graph, upper curve is $f(x) = \sec^2\left(\frac{\pi x}{3}\right)$ and lower curve $g(x) = x^{1/3}$

Here $a = -1$ and $b = 1$

$\therefore$ Area A of the shaded region in the given graph is

$$A = \int_{-1}^{1} [f(x) - g(x)]\, dx$$

$$= \int_{-1}^{1} \left[\sec^2\left(\frac{\pi x}{3}\right) - x^{1/3}\right] dx$$

$$= \int_{-1}^{1} \sec^2\left(\frac{\pi x}{3}\right) dx - \int_{-1}^{1} x^{1/3}\, dx$$

$$= \left[\frac{3}{\pi}\tan\left(\frac{\pi x}{3}\right)\right]_{-1}^{1} - \left[\frac{x^{1/3+1}}{1/3+1}\right]_{-1}^{1}$$

$$= \frac{3}{\pi}\left[\sqrt{3} - \left(-\sqrt{3}\right)\right] - \frac{4}{3}[1-1] = \frac{6\sqrt{3}}{\pi}$$

IP3:

Find the area of the region bounded between the curves
$\boldsymbol{y = 7 - 2x^2}$ and $\boldsymbol{y = x^2 + 4}$

Solution:

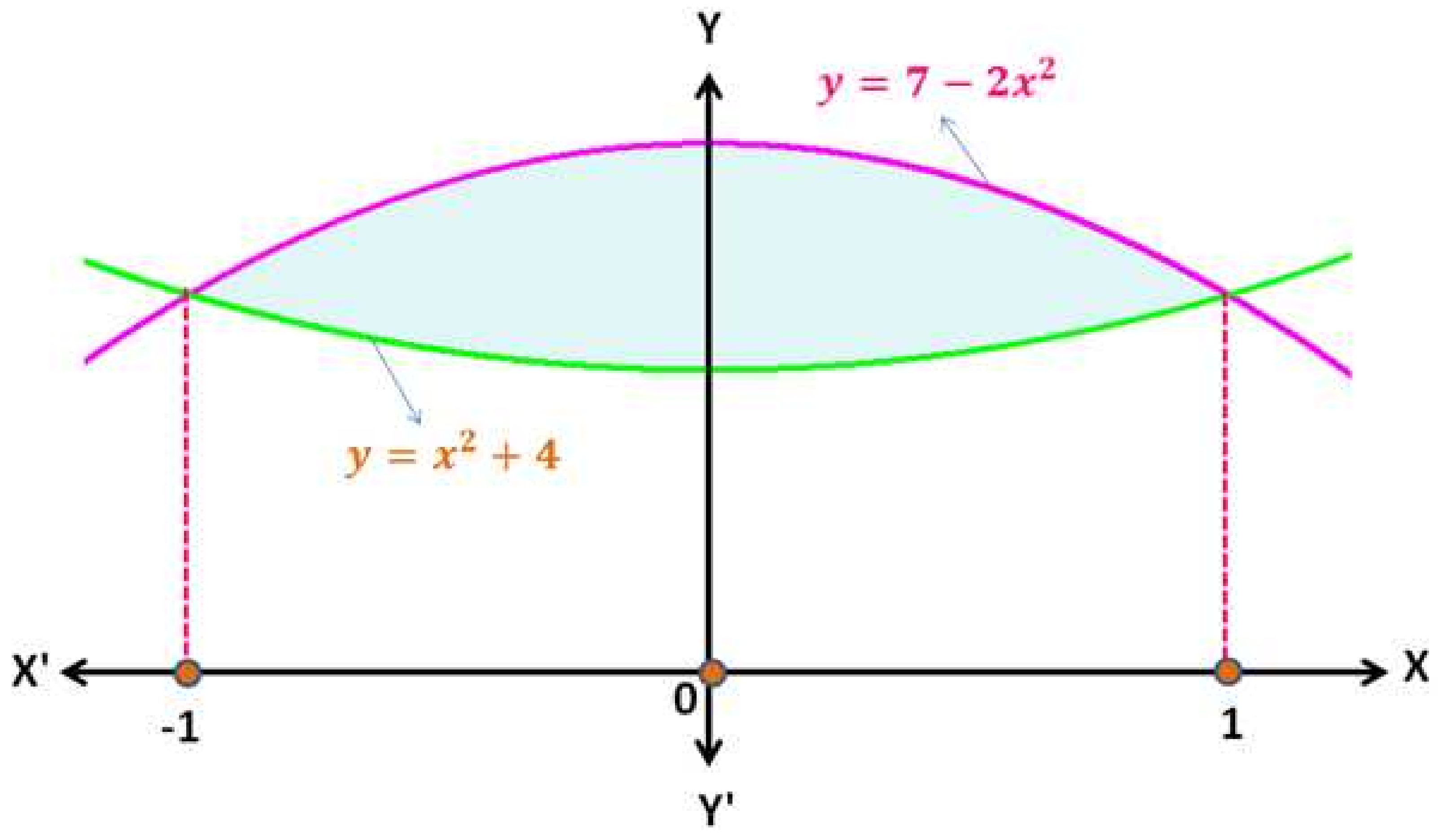
Given curves are $y = 7 - 2x^2$ and $y = x^2 + 4$

Now, we have to calculate the limits of integration.

$$7 - 2x^2 = x^2 + 4 \Rightarrow 3x^2 - 3 = 0 \Rightarrow 3(x-1)(x+1) = 0$$

$$\Rightarrow y = -1, 1 \Rightarrow a = -1, \ \ b = 1$$

The graph is plotted between the curves $y = 7 - 2x^2$ and
$y = x^2 + 4$ over the interval $[-1\ , 1]$



From the graph, upper curve is $f(x) = 7 - 2x^2$ and lower curve
$g(x) = x^2 + 4$

$\therefore$ Area A of the shaded region in the given graph is

$$A = \int_{-1}^{1} [f(x) - g(x)]\, dx$$

$$= \int_{-1}^{2} \left[\left(7 - 2x^2\right) - \left(x^2 + 4\right)\right] dx = \int_{-1}^{2} \left(3 - 3x^2\right) dx$$

$$= 3\int_{-1}^{1} dx - 3\int_{-1}^{1} x^2 dx = 3[x]_{-1}^{1} - 3\left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= 3[1 - (-1)] - 3\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] = 4$$

IP4:

Find the area of the propeller-shaped region enclosed by the curves $\boldsymbol{x - y^{1/3} = 0}$ and $\boldsymbol{x - y^{1/5} = 0}$

Solution:

Given curves are $x - y^{1/3} = 0 \Rightarrow y = x^3$ and

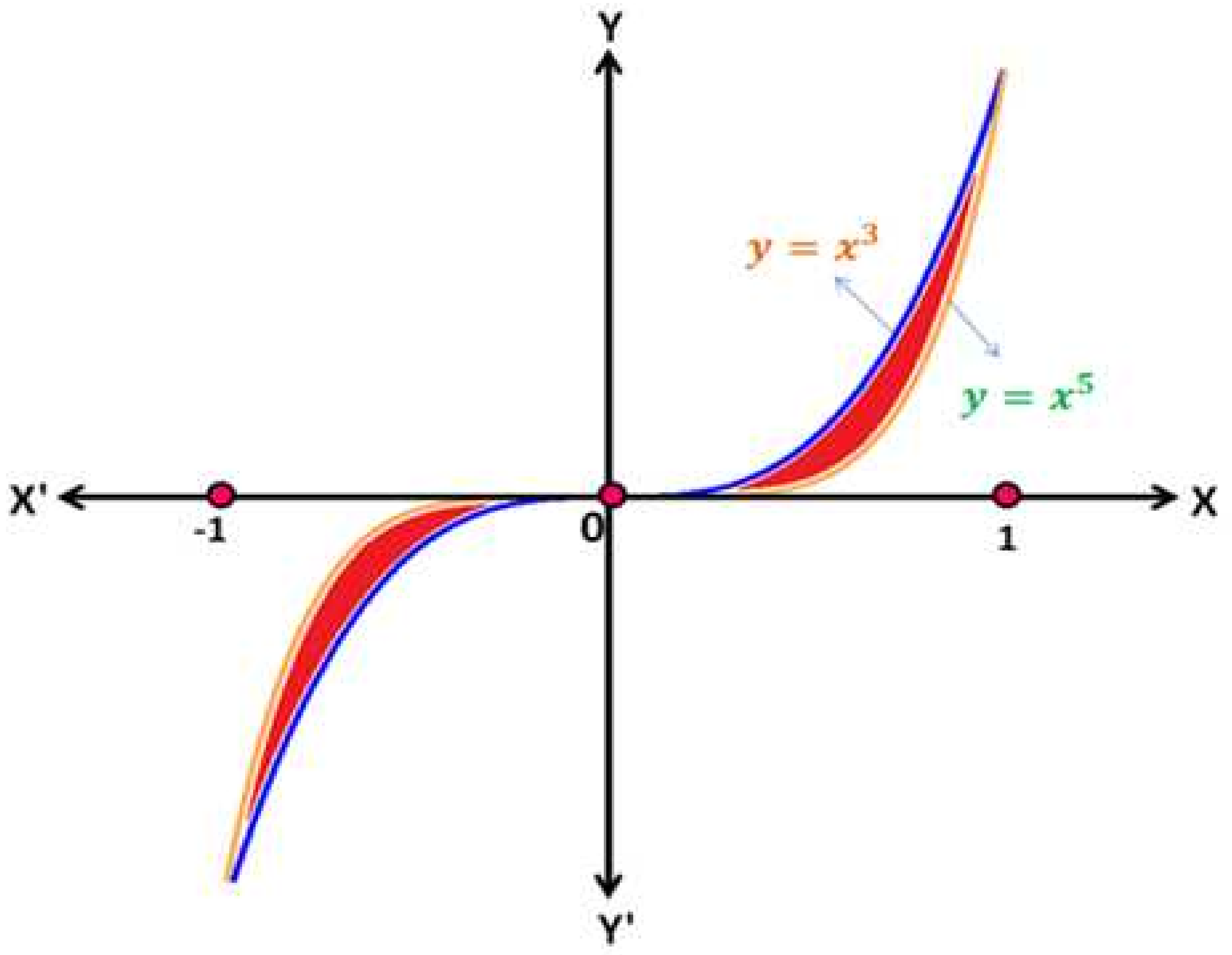
$x - y^{1/5} = 0 \Rightarrow y = x^5$

Now, we have to calculate the limits of integration.

$x^3 = x^5 \Rightarrow x^5 - x^3 = 0 \Rightarrow x^3(x^2 - 1) = 0$

$\Rightarrow x = -1, 0, 1 \Rightarrow a = -1$ and $b = 1$

The graph is plotted between the curves $y = x^3$ and $y = x^5$ over the interval $[-1, 1]$, which is a propelled- shaped region.



To obtain the total area A of the shaded region in the graph, we have to calculate the areas of the two shaded regions A_1, A_2 and add the absolute the values of the areas, where

$A_1: -1 \leq x \leq 0, f_1(x) - g_1(x) = x^5 - x^3$ and

$A_2: 0 \leq x \leq 1, f_2(x) - g_2(x) = x^3 - x^5$

By observing the graph, it is symmetry about the origin.

$$\therefore A = A_1 + A_2 = 2A_2$$

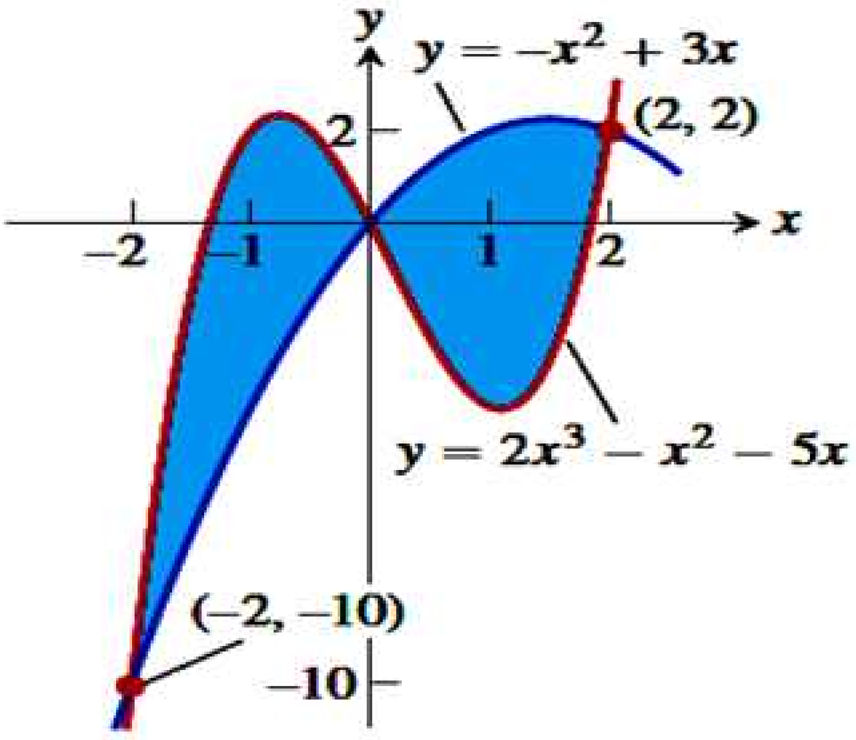
$$= 2\int_0^1 \left(x^3 - x^5\right) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = 2\left[\left(\frac{1}{4} - \frac{1}{6}\right) - 0\right] = \frac{1}{6}$$

Hence, the total area of the shaded region on $[-1, 1]$ is $A = \frac{1}{6}$

P1:

Find the total area of the shaded region of the graph



Solution:

To obtain the total area A of the shaded region in the given graph, we have to calculate the areas of the two shaded regions A_1, A_2 and add the absolute the values of the areas, where

$A_1: -2 \leq x \leq 0$, upper curve $f(x) = 2x^3 - x^2 - 5x$ and lower curve $g(x) = -x^3 + 3x$

$$\therefore A_1 = \int_{-2}^{0} [f(x) - g(x)]\, dx$$

$$= \int_{-2}^{0} \left[\left(2x^3 - x^2 - 5x\right) - \left(-x^2 + 3x\right)\right] dx$$

$$= \int_{-2}^{0} \left(2x^3 - 8x\right) dx = \left[\frac{x^4}{2} - 4x^2\right]_{-2}^{0} = [0 - (8 - 16)] = 8$$

$A_2: 0 \leq x \leq 2$, upper curve $f(x) = -x^3 + 3x$ and lower curve $g(x) = 2x^3 - x^2 - 5x$

$$\therefore A_2 = \int_{0}^{2} [f(x) - g(x)]\, dx$$

$$= \int_{0}^{2} \left[\left(-x^2 + 3x\right) - \left(2x^3 - x^2 - 5x\right)\right] dx$$

$$= \int_{0}^{2} \left(8x - 2x^3\right) dx = \left[\frac{8x^2}{2} - \frac{2x^4}{4}\right]_{0}^{2} = [(16 - 8) - 0] = 8$$

Therefore, the total area of the shaded region is

$$A = |A_1| + |A_2| = |8| + |8| = 16$$

P2:

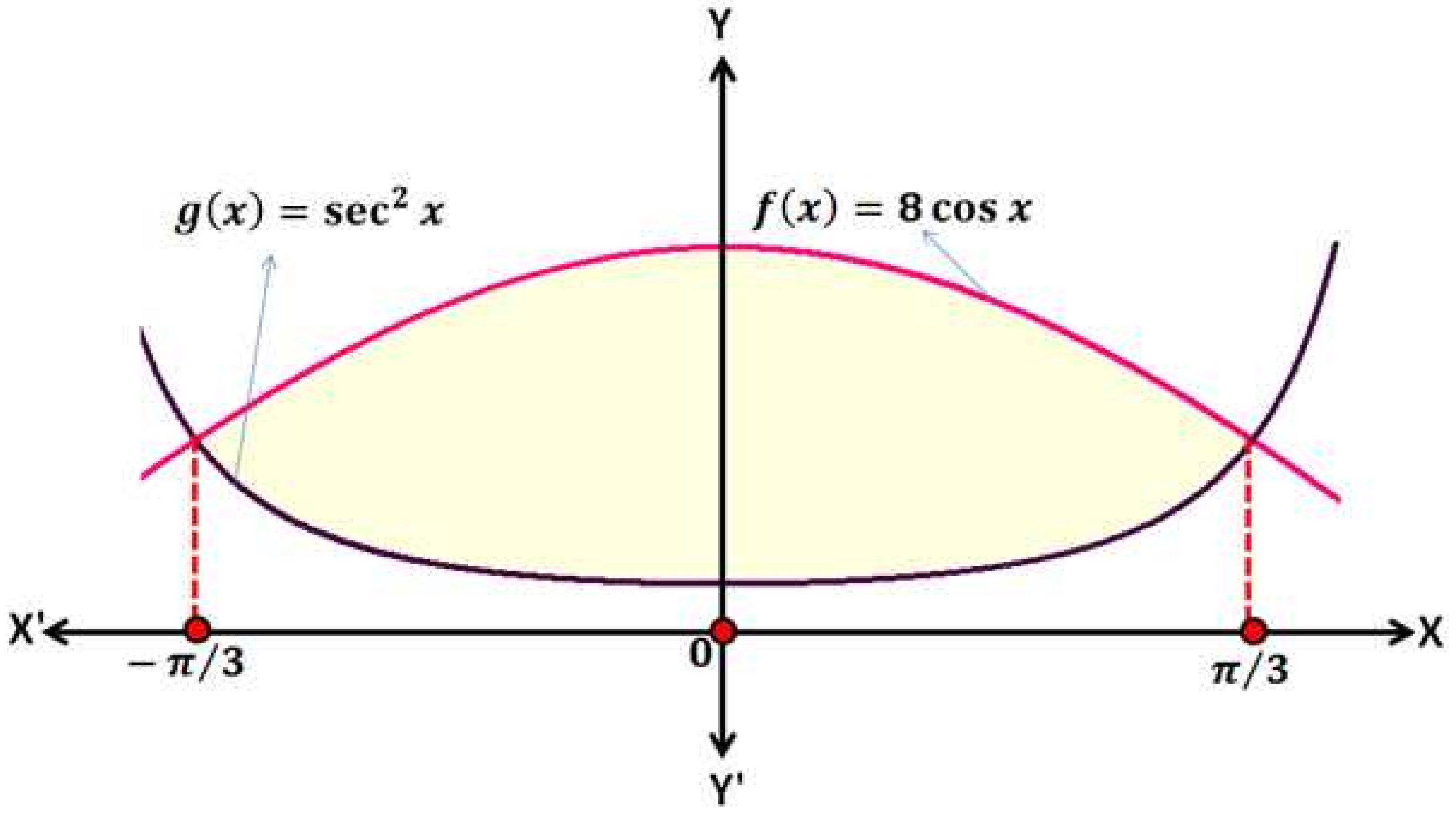
Find the area of the region enclosed by the curves $y = 8\cos x$ and $y = \sec^2 x$, $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$.

Solution:

Given curves are $y = 8\cos x$ and $y = \sec^2 x$ where $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$.

The graph is plotted between the curves $y = 8\cos x$ and

$y = \sec^2 x$ over the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$



From the graph, upper curve is $f(x) = 8\cos x$ and lower curve $g(x) = \sec^2 x$

Here $a = -\frac{\pi}{3}$ and $b = \frac{\pi}{3}$

$\therefore$ Area A of the shaded region in the given graph is

$$A = \int_{-\pi/3}^{\pi/3} [f(x) - g(x)]\, dx$$

$$= \int_{-\pi/3}^{\pi/3} \left[8\cos x - \sec^2 x\right] dx$$

$$= [8\sin x - \tan x]_{-\pi/3}^{\pi/3}$$

$$= \left[\left(8.\frac{\sqrt{3}}{2} - \sqrt{3}\right) - \left(-8.\frac{\sqrt{3}}{2} + \sqrt{3}\right)\right] = 6\sqrt{3}$$

P3:

Find the area of the region bounded between the curves $x = y^2$ and $x = y + 2$.

Solution:

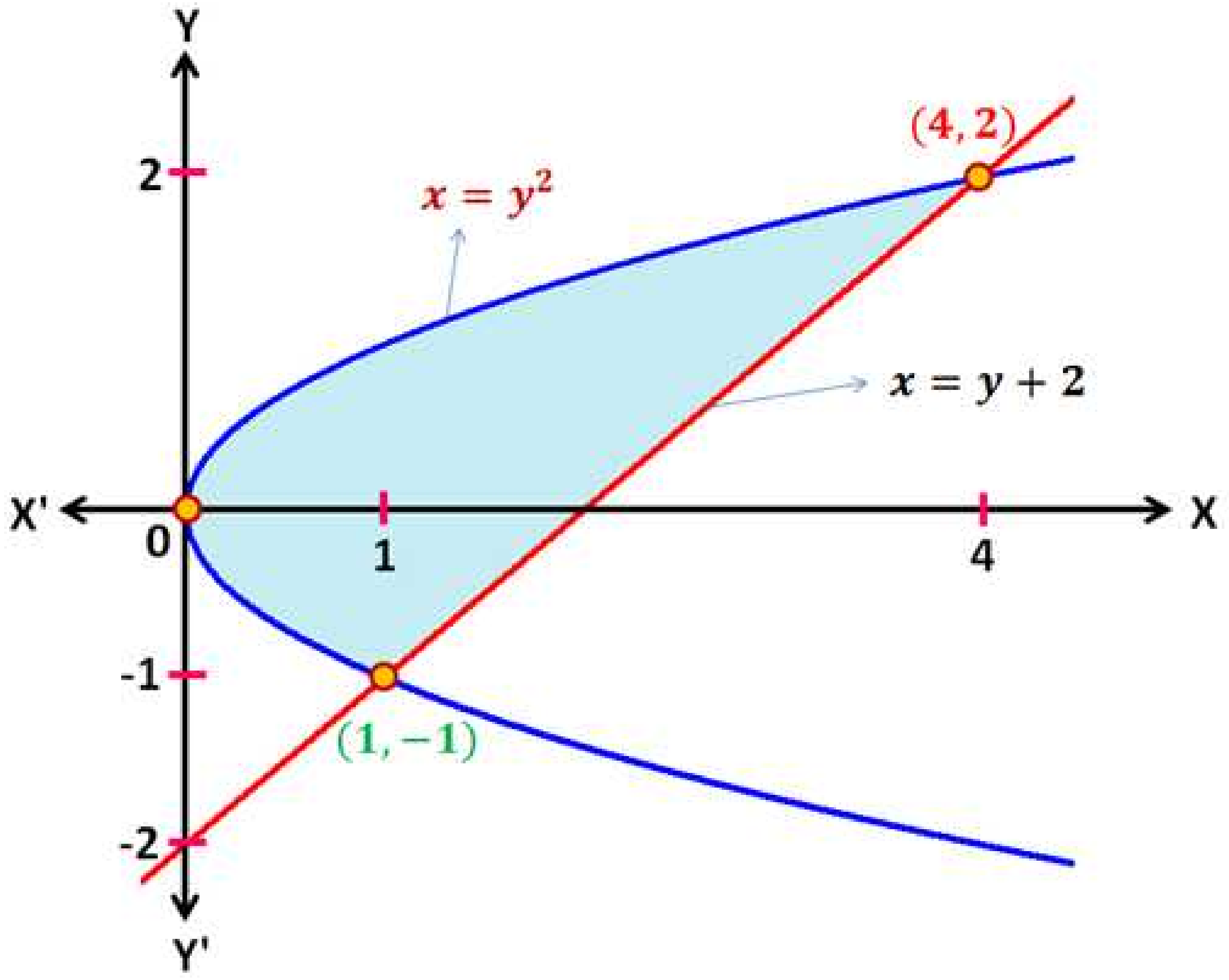
Given curves are $x = y^2$ and $x = y + 2$

Now, we have to calculate the limits of integration.

$$y^2 = y + 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0$$

$$\Rightarrow y = -1, 2 \Rightarrow c = -1, \quad d = 2$$

The graph is plotted between the curves $x = y^2$ and $x = y + 2$ over the interval $[-1, 2]$



From the graph, upper curve is $f(y) = y + 2$ and lower curve $g(x) = y^2$

$\therefore$ Area A of the shaded region in the given graph is

$$A = \int_{-1}^{2} [f(y) - g(y)]\, dy$$

$$= \int_{-1}^{2} \left[y + 2 - y^2\right] dy = \int_{-1}^{2} y\, dy + 2\int_{-1}^{2} dy - \int_{-1}^{2} y^2\, dy$$

$$= \left[\frac{y^2}{2}\right]_{-1}^{2} + 2[y]_{-1}^{2} - \left[\frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{4}{2} - \frac{1}{2}\right] + 2[2 - (-1)] - \left[\frac{8}{3} - \left(-\frac{1}{3}\right)\right] = \frac{9}{2}$$

P4:

Find the area of the region in the first quadrant bounded on the left by the y −axis, below by the curve $x = 2\sqrt{y}$, above left the curve $x = (y-1)^2$ and above right by the line $x = 3 - y$.

Solution:

Given curves are $x = 2\sqrt{y}, x = (y-1)^2$ and lines are y −axis that is $x = 0,\ x = 3 - y$.

Now, we have to calculate the limits of integration.

$(y-1)^2 = 3 - y \Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0 \Rightarrow y = 2$ and $y = -1$ is not a solution because $y > 0$

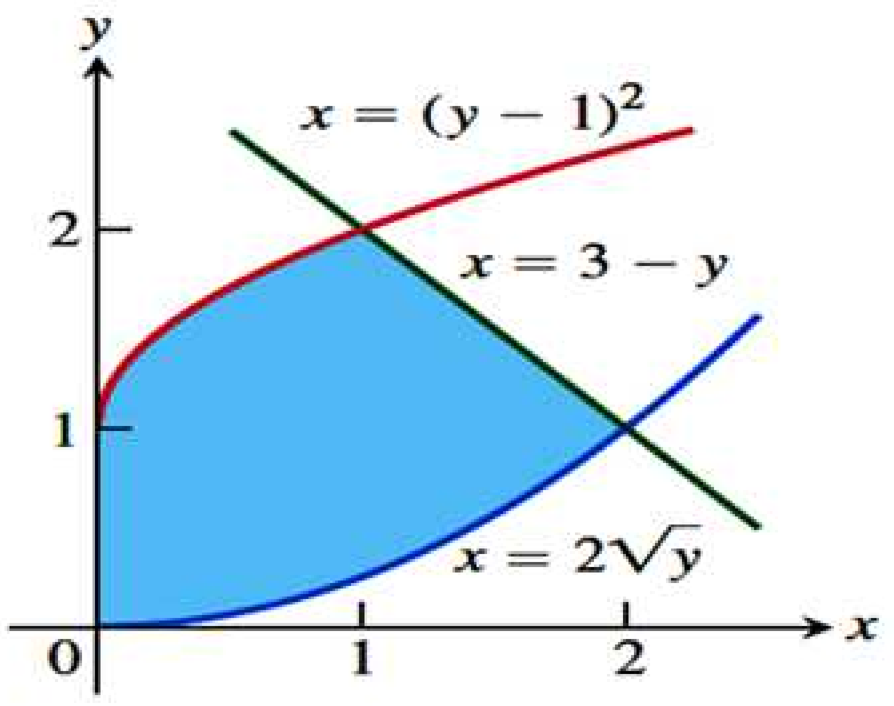
Again, $2\sqrt{y} = 3 - y \Rightarrow 4y = 9 - 6y + y^2$

$\Rightarrow y^2 - 10y + 9 = 0 \Rightarrow (y-9)(y-1) = 0$

$\Rightarrow y = 1$ and $y = 9$ is not a solution because it does not satisfy the equation.

Hence, $a = 0$ and $b = 2$

The graph is plotted between the curves $x = 2\sqrt{y}, x = (y-1)^2$ and lines are y −axis that is $x = 0,\ x = 3 - y$ over $[0, 2]$



To obtain the total area A of the shaded region in the graph, we have to calculate the areas A_1, A_2 and add the absolute the values of the areas, where

$A_1: 0 \le y \le 1, f_1(y) - g_1(y) = 2\sqrt{y} - 0 = 2\sqrt{y}$

$$\therefore A_1 = \int_0^1 2\sqrt{y}\ dy = 2\int_0^1 y^{1/2}\ dy$$

$$= 2\left[\frac{y^{1/2+1}}{1/2+1}\right]_0^1 = \frac{4}{3}[1-0] = \frac{4}{3}$$

$A_2: 1 \le y \le 2, f_2(y) - g_2(y) = (3-y) - (y-1)^2$

$$\therefore A_2 = \int_1^2 \left[3 - y - (y-1)^2\right] dy$$

$$= \left[3y - \frac{y^2}{2} - \frac{(y-1)^3}{3}\right]_1^2$$

$$= \left[\left(6 - 2 - \frac{1}{3}\right) - \left(3 - \frac{1}{2} - 0\right)\right] = \frac{7}{6}$$

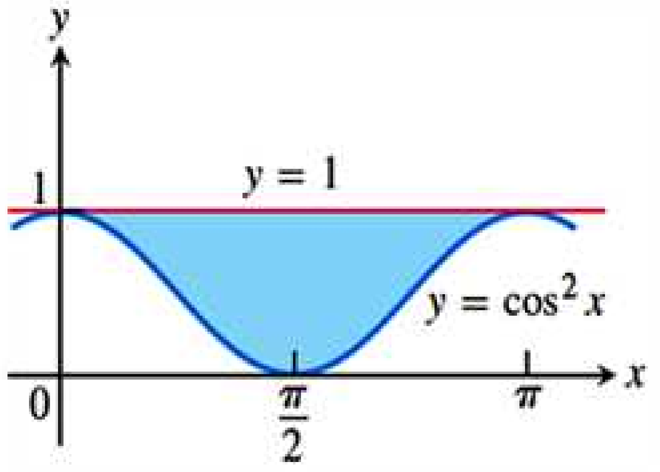
Therefore, the total area of the shaded region on $[0, 2]$ is

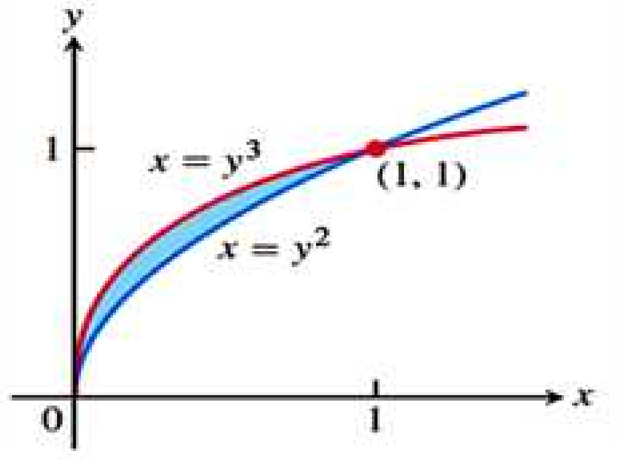
$$A = |A_1| + |A_2| = \left|\frac{4}{3}\right| + \left|\frac{7}{6}\right| = \frac{5}{2}$$

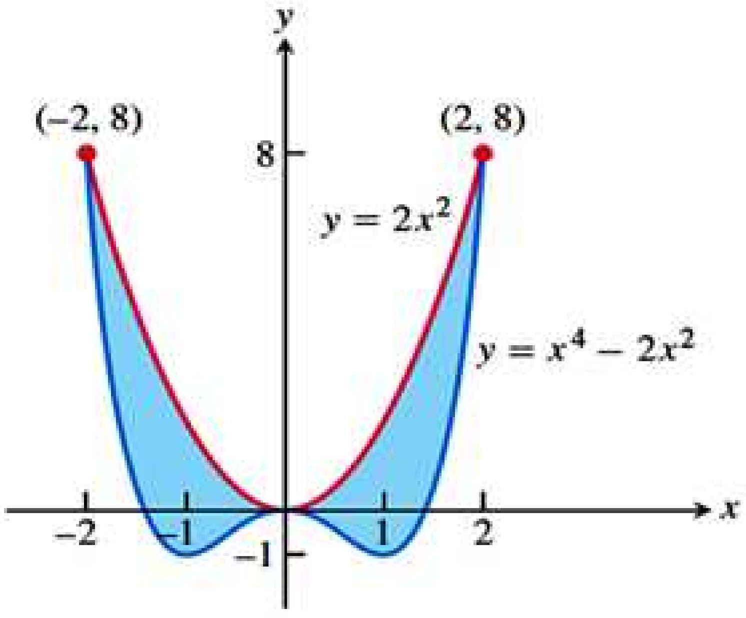
3.1. Areas between Curves

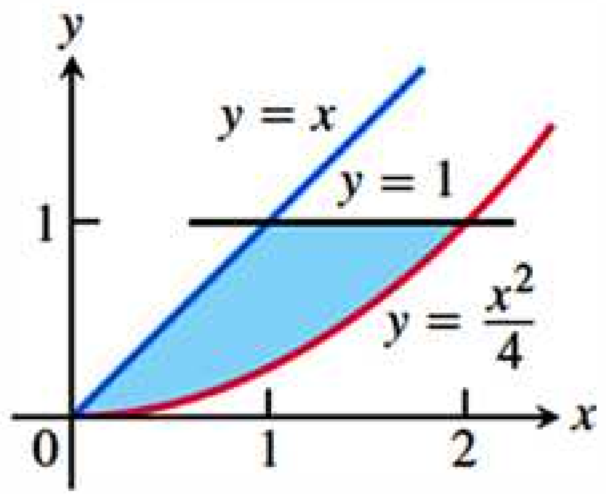
Exercise:

1. **Find the area of the shaded regions in following graphs**









2. **Find the areas of the regions enclosed by the lines and curves**

   a) $y = x^2 - 2$ and $y = 2$

   b) $y = x^4$ and $y = 8x$

   c) $x = 2y^2$, $x = 0, y = 3$

   d) $y^2 - 4x = 4, 4x - y = 16$

3. **Find the areas of the regions enclosed by the curves**

   a. $4x^2 + y = 4, x^4 - y = 1$

   b. $x + 4y^2 = 4, \quad x + y^4 = 1, \textit{ for } x \geq 0$

   c. $x + y^2 = 0, \quad x + 3y^2 = 2$

   d. $y = x^2$ and $y = -x^2 + 4x$

4. **Find the areas of the regions enclosed by the curves**

   a. $y = 2\sin x \quad$ and $\; y = \sin 2x, \quad 0 \leq x \leq \pi$

   b. $y = \cos(\pi x / 2) \quad$ and $\; y = 1 - x^2$

   c. $y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\pi / 4, \quad x = \pi / 4$

5. Find the area of the propeller–shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$

6. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = \frac{1}{x^2}$, and the $x - axis$.