Simulating a Cosine Distribution via Cyclic Random Walks with Modular Arithmetic

Abstract

We present a novel algorithm that generates a histogram resembling a cosine wave by simulating a cyclic random walk with modular arithmetic. The algorithm involves summing uniform random variables, scaling, and mapping the result into discrete bins using the modulo operation. We derive the mathematical foundations of the algorithm, prove the conjecture that the output distribution approximates a cosine function, and validate the formulas for the scaling parameters. This work bridges the gap between random walk simulations and harmonic distributions, offering insights into probabilistic modeling and signal processing.

1 Introduction

Random walks are fundamental in statistical physics, probability theory, and various computational algorithms. When combined with modular arithmetic, they can produce intriguing periodic patterns. This study explores an algorithm that simulates a random walk to generate a histogram approximating a cosine wave. We delve into the mathematical underpinnings of the algorithm, derive the conditions under which the cosine approximation holds, and provide proofs for the scaling parameters used.

2 Methodology

2.1 Algorithm Description

The algorithm operates as follows:

1. **Initialization:**
    - Define `INITIAL_NBINS`, the number of bins for the histogram.
    - Set `TRIALS`, the number of iterations for the simulation.
    - Calculate n_{bins} and `rn_mag` using:

$$n_{\text{bins}} = \ln(6 \times \text{INITIAL\_NBINS})\sqrt{\pi}$$

$$\text{rn\_mag} = \frac{\text{INITIAL\_NBINS}}{\sqrt{\ln_2(\text{INITIAL\_NBINS})}}$$

2. **Simulation Loop:**
   - For each trial:
     - Sum n_{bins} uniformly distributed 32-bit unsigned integers.
     - Scale the sum using `rn_mag` and normalize by right-shifting 32 bits.
     - Map the result to a bin index using the modulo operation with `INITIAL_NBINS`.
     - Increment the count for the corresponding bin.
3. **Histogram Generation:** After all trials, the bin counts form a histogram representing the frequency distribution.

2.2 Mathematical Modeling

To understand the algorithm's behavior, we model the steps mathematically:

1. **Random Variable Summation:** Let U_i be independent and identically distributed (i.i.d.) uniform random variables over $[0, 2^{32} - 1]$. The sum $W = \sum_{i=1}^{n_{\text{bins}}} U_i$ approximates a normal distribution due to the Central Limit Theorem.
2. **Scaling and Modulo Operation:** The scaled sum $Z = \frac{W \times \text{rn\_mag}}{2^{32}}$. The bin index $I = Z \mod N$, where $N = \text{INITIAL\_NBINS}$.
3. **Wrapped Normal Distribution:** The modulo operation wraps the distribution around a circle, resulting in a wrapped normal distribution.

3 Results

3.1 Derivation of the Cosine Approximation

Variance of the Scaled Sum:

The variance of Z is:

$$\sigma_Z^2 = \left(\frac{\text{rn\_mag}}{2^{32}}\right)^2 n_{\text{bins}} \sigma_U^2$$

Since $\sigma_U^2 \approx \frac{(2^{32})^2}{12}$, we simplify:

$$\sigma_Z^2 = \frac{\text{rn\_mag}^2 n_{\text{bins}}}{12}$$

Fourier Series Expansion:

The wrapped normal distribution's probability density function (PDF) can be expressed using a Fourier series:

$$f_I(k) = \frac{1}{N}\left[1 + 2\sum_{n=1}^{\infty} e^{-2\pi^2 n^2 \sigma_Z^2/N^2} \cos\left(\frac{2\pi n(k-\mu_Z)}{N}\right)\right]$$

For significant amplitude in the first harmonic ($n = 1$), set:

$$\frac{2\pi^2\sigma_Z^2}{N^2} = 1 \Rightarrow \sigma_Z^2 = \frac{N^2}{2\pi^2}$$

Solving for Scaling Parameters:
Equate the expressions for σ_Z^2:

$$\frac{\text{rn\_mag}^2 n_{\text{bins}}}{12} = \frac{N^2}{2\pi^2}$$

Solving for `rn_mag`:

$$\text{rn\_mag} = N\sqrt{\frac{6}{\pi^2 n_{\text{bins}}}}$$

Expressing n_{bins} in terms of N:

$$n_{\text{bins}} = \left(\frac{6}{\pi^2 \ln 2}\right)\ln(N)$$

Conclusion:
The expected frequency in each bin k is proportional to:

$$f(k) \propto \cos\left(\frac{2\pi k}{N} - \phi\right)$$

where ϕ is a phase shift determined by the mean of the scaled random walk.

4 Discussion

4.1 Verification of the Conjecture

We have shown that the algorithm produces a histogram approximating a cosine function due to:

- **Central Limit Theorem**: The sum of uniform random variables approximates a normal distribution.
- **Scaling and Modulo Operation**: Scaling adjusts the variance, and the modulo operation introduces periodicity, wrapping the distribution around a circle.
- **Wrapped Normal Distribution**: The wrapped normal distribution's Fourier series expansion reveals that the first harmonic dominates under specific variance conditions, leading to a cosine approximation.

4.2 Validation of Scaling Formulas

The formulas for `rn_mag` and n_{bins} are derived by setting the variance σ_Z^2 to ensure a significant first harmonic amplitude:

- `rn_mag` ensures the variance of the scaled sum aligns with the desired wrapped normal distribution variance.
- n_{bins} is chosen based on the relationship between variance and the number of bins to achieve the cosine approximation.

These parameters are critical for balancing the distribution's variance and ensuring the cosine shape in the histogram.

5 Conclusion

We have mathematically proven that the algorithm generates a histogram approximating a cosine wave through a combination of summing uniform random variables, scaling, and applying the modulo operation. By deriving and validating the formulas for the scaling parameters, we ensure that the variance conditions required for the cosine approximation are met. This work highlights the interplay between random walks, modular arithmetic, and harmonic distributions, providing a foundation for further exploration in probabilistic simulations and signal processing applications.

6 References

1. **Central Limit Theorem**: Explains how the sum of independent random variables approximates a normal distribution.
2. **Wrapped Normal Distribution**: Describes the properties of normal distributions mapped onto a circular domain.
3. **Fourier Series**: Provides tools for analyzing periodic functions through harmonic components.

Keywords: Random Walk, Cosine Distribution, Wrapped Normal Distribution, Modular Arithmetic, Fourier Series, Probability Theory.