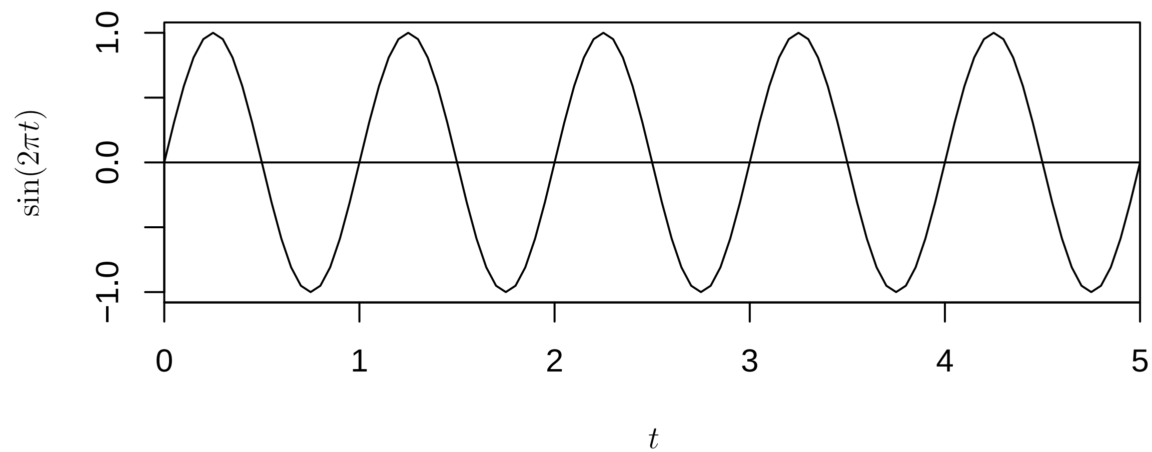
Sine Waves and Frequency

Think of pure tone

- Wave form is a sine (or cosine) wave



- Equations in time are

$$y(t) = \sin(2\pi f t) = \sin(2\pi t/L)$$
$$y(t) = \cos(2\pi f t) = \cos(2\pi t/L)$$

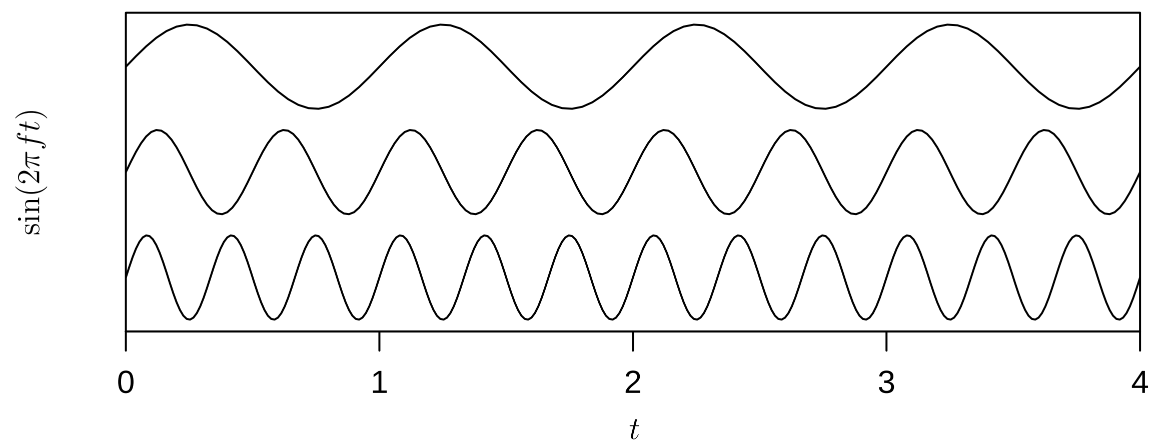
  where f is the *frequency* and $L = 1/f$ the *wavelength*
  - The argument of the sine or cosine is measured in radians
    * The wave goes through one complete cycle in $360°$ or 2π radians

$$1° = 0.01745 \text{ radian} \qquad \text{and} \qquad 1 \text{ radian} = 57.30°$$

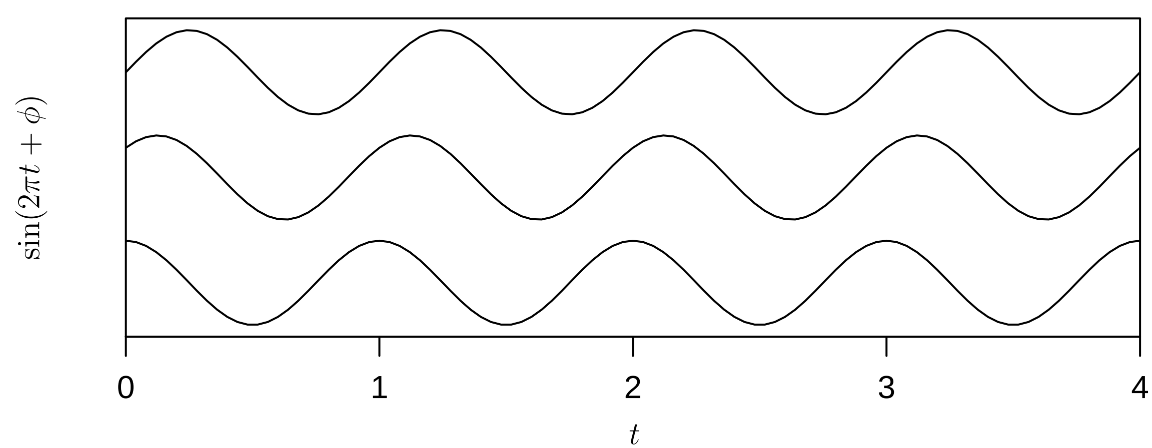
  - The cosine is often the function chosen
    * The cosine is an *even function*, symmetric about zero: $y(t) = y(-t)$
    * The sine is an *odd function*, antisymmetric about zero: $y(t) = -y(-t)$

Frequency and Phase

- Sine waves can have different frequencies



- Sine waves can have different displacements or *phase*



  – The cosine is a displacement of the sine by $\pi/2$: $\cos(t) = \sin(t + \pi/2)$

Three Representations of Phase

1. As an increment to the argument of a sine or cosine

$$y(t) = \sin(2\pi f t + \phi)$$

2. As the sum of a sine and a cosine term

$$y(t) = a\cos(2\pi f t) + b\sin(2\pi f t),$$
$$a = \sin(\phi) \qquad \text{and} \qquad b = \cos(\phi)$$
$$\phi = \arctan(a/b)$$

3. By coding as a complex exponential with a complex coefficient

$$y(t) = z e^{i2\pi f t} + \overline{z} e^{i2\pi(-f)t} \quad \text{where} \quad z = \tfrac{1}{2}(a - ib), \quad \overline{z} = \tfrac{1}{2}(a + ib) \quad \text{and} \quad i = \sqrt{-1}$$

- Use of imaginary numbers and negative frequencies is just a computational convenience
- Expect to find it in computer output

The exponential of an imaginary number ix is a point on the unit circle at a distance x along the circle from $(1, 0)$

$$e^{x} = \cos(x) + i\sin x$$

Thus, for the representation of phase

$$\begin{aligned} y(t) &= z e^{i2\pi f t} + \overline{z} e^{-i2\pi f t} \\ &= \tfrac{1}{2}(a - ib)] \cos(2\pi f t) + i\sin(2\pi f t)] + \tfrac{1}{2}(a + ib)[\cos(-2\pi f t) + i\sin(-2\pi f t)] \\ &= a\cos(2\pi f t) - i^2 b\sin(2\pi f t) \\ &= a\cos(2\pi f t) + b\sin(2\pi f t) \end{aligned}$$

Nonsinusoidal Waveforms

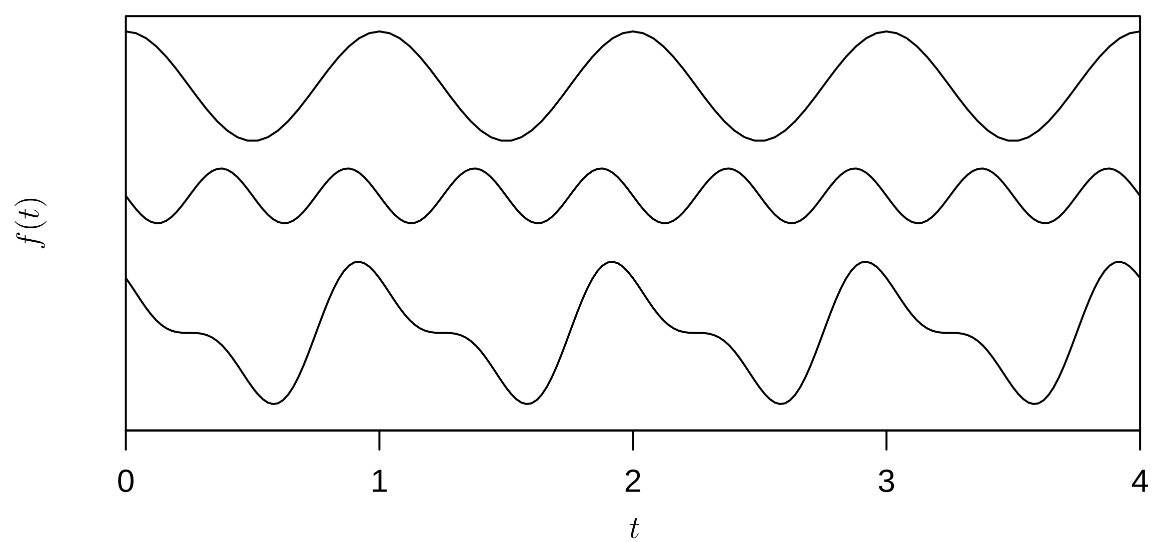
A combination of sinusoidal waves of different frequencies is periodic but not sinusoidal

$$f(t) = a_1 \cos(2\pi f_1 t + \phi_1) + a_2 \cos(2\pi f_2 t + \phi_2) + a_3 \cos(2\pi f_3 t + \phi_3) + \cdots$$

The fundamental (longest) wavelength must be an integer multiple of the wavelength of each shorter component

$$L_1 = kL_j \qquad \text{or} \qquad f_j = f_1/k$$

- Changing the phase or amplitude of the combination changes the shape of the sum
- $f(t) = \cos(2\pi t) - \frac{1}{2}\sin(4\pi t)$



Fourier Series

Fundamental Result: Any periodic function can be expressed as the sum of a series of sine and cosine waves

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f t) + b_n \sin(2\pi n f t)$$

$$a_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\,dt$$

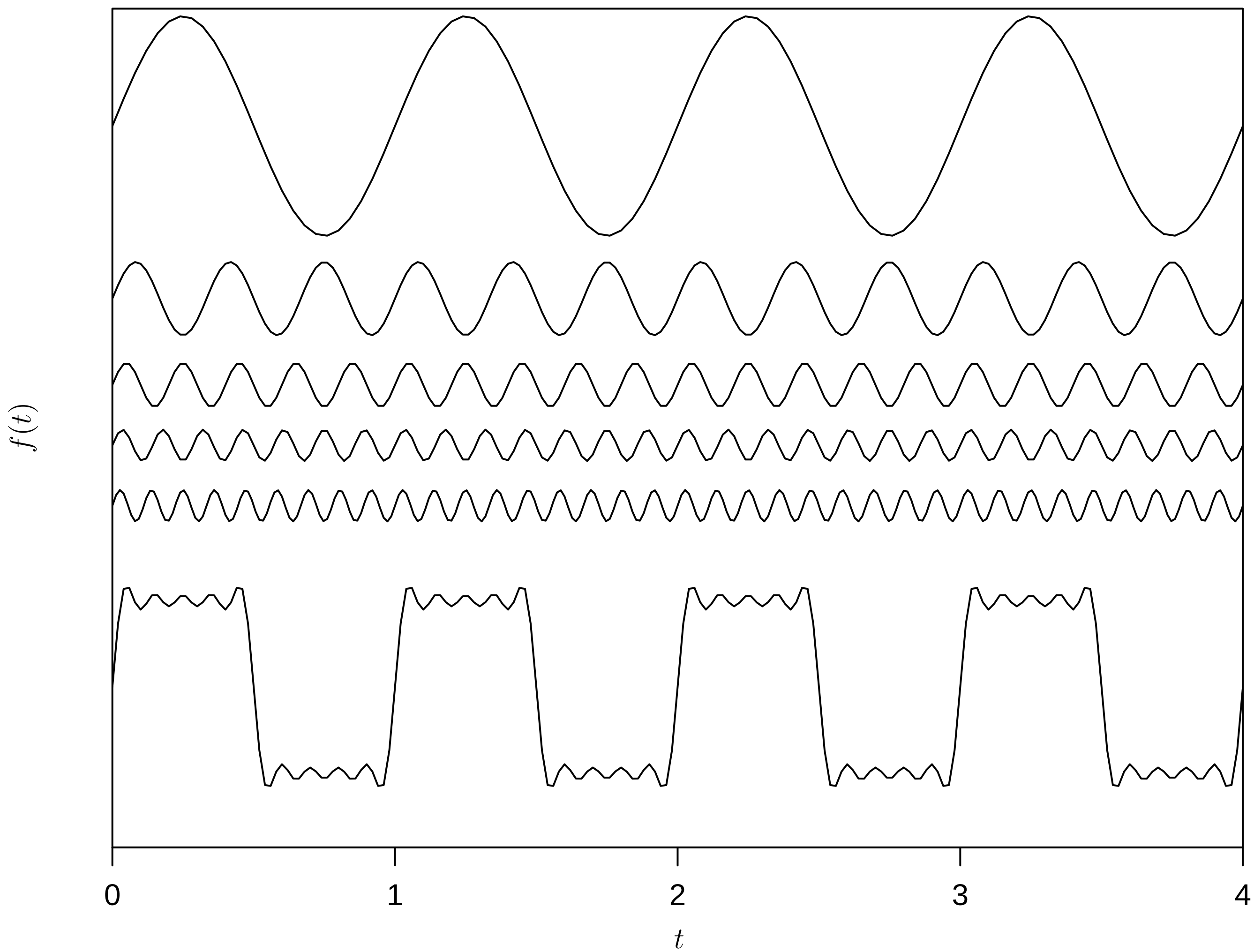
$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} \cos(2\pi n f t) f(t)\,dt \qquad \text{and} \qquad b_n = \frac{2}{L}\int_{-L/2}^{L/2} \sin(2\pi n f t) f(t)\,dt$$

- The function $f(t)$ is fully determined by the coefficients a_0, a_1, a_2 $\ldots$, and b_1, b_2, $\ldots$
  - By analogy with visual stimuli, call this sequence the *spectrum* of $f(t)$
- The component functions—$\cos(2\pi n f t)$ and $\sin(2\pi n f t)$—are mutually orthogonal
  - Each term picks up a different aspect of the function $f(t)$
    - a_0 is an average over one cycle
    - a_n and b_n are like the correlations of $f(t)$ with the cosine or sine
  - They are the *basis functions* or *eigenfunctions* for the representation
    - Sines and cosines are appropriate for *linear systems*
- Can also use a single series of complex coefficients

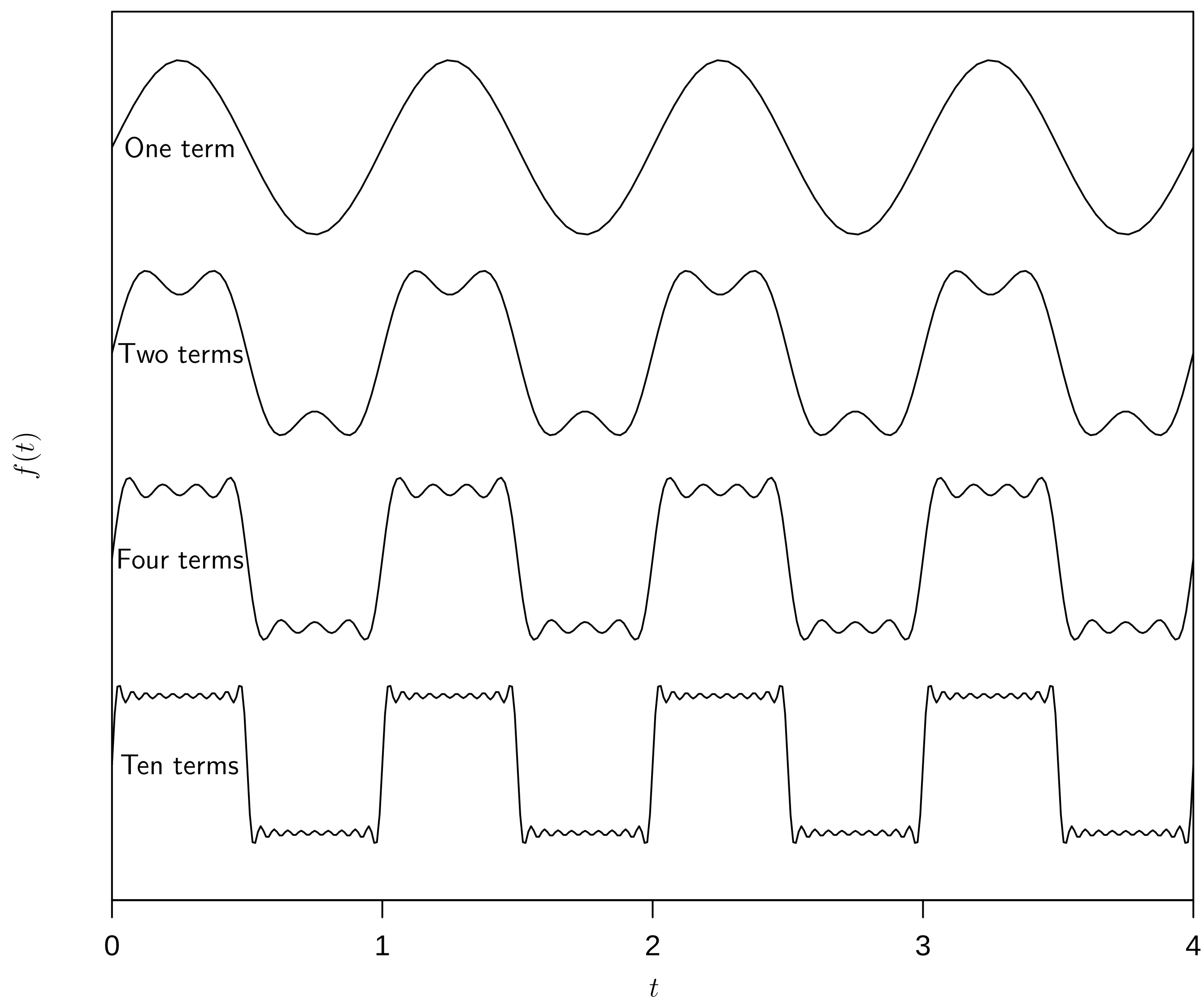
$$f(t) = \sum_{n=-\infty}^{\infty} z_n e^{-int} \qquad \text{with} \qquad z_{-n} = \overline{z}_n$$

Fourier Series for a Square Wave

$$f(t) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin \frac{n\pi t}{L}$$



Approximations to a Square Wave



- Each approximation has a 9% transient overshoot

The Two Domains

- A function can be represented by either its waveform or its spectrum

  $$\boxed{\text{Waveform } f(t)} \Longleftrightarrow \boxed{\text{Spectrum } F(f)}$$

  The Fourier transform moves between these representations
- The two representations describe complementary domains
  - Problems can be solved in either domain
  - Many problems are easier to solve in one domain than the other
  - Sensory systems take advantage of this fact
    - ∗ Auditory: Receive a waveform but perceive frequencies
    - ∗ Visual
      - · Color
      - · Spatial arrays
- The two domains arise in many areas where linear processes appear
  - Signal processing
    - ∗ Audio and video
  - Temporal evolution of a system
    - ∗ Markov chains and processes
  - Statistics
    - ∗ Distributions and moments
      - · Moment generating functions and characteristic functions
    - ∗ Multivariate variation and eigenvectors

Nonperiodic Functions—the Fourier Transform

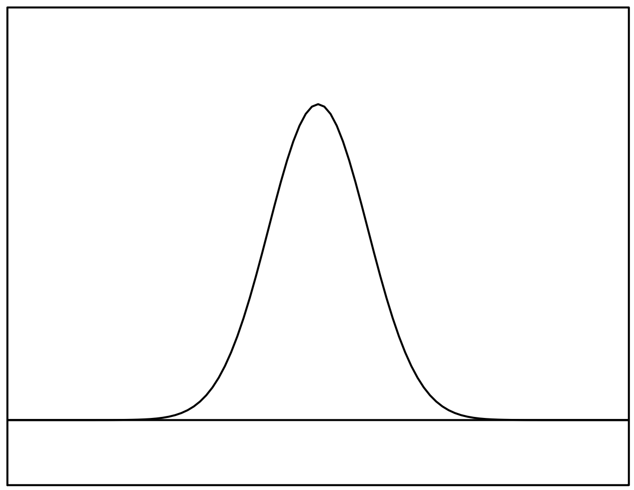
- A nonperiodic function is like one that has an infinite wavelength
  - $L \to \infty$ implies $f_1 \to {}^1\!/_L = 0$
  - The spectrum is continuous
- The Fourier transform and its inverse are complementary operations

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t s} dt$$
$$f(t) = \int_{-\infty}^{\infty} F(s) e^{2\pi i t s} ds$$

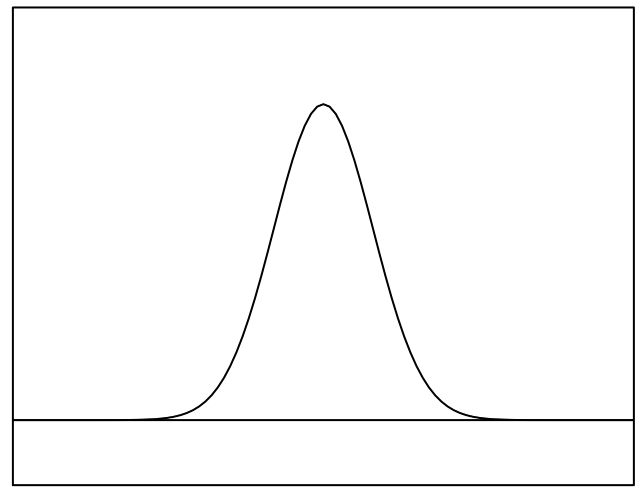
  - These equations are equivalent to those that use cosine and sine or the phase
  - The complex representation emphasizes the similarity of the two transforms
  - The Fourier transform is a *functional*, which maps one function to another

Example of Fourier Transforms

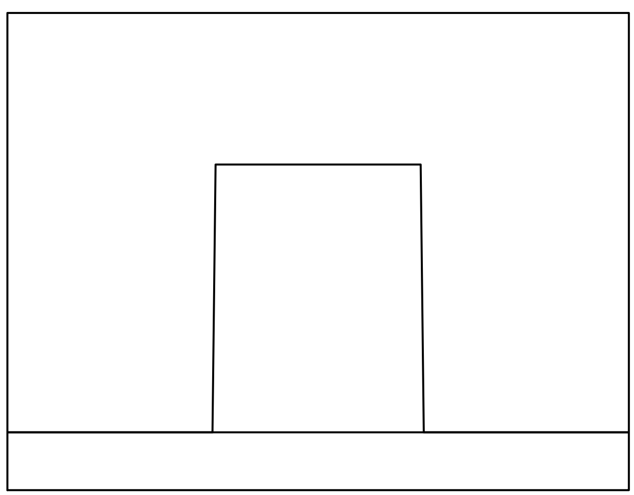
Gaussian waveform



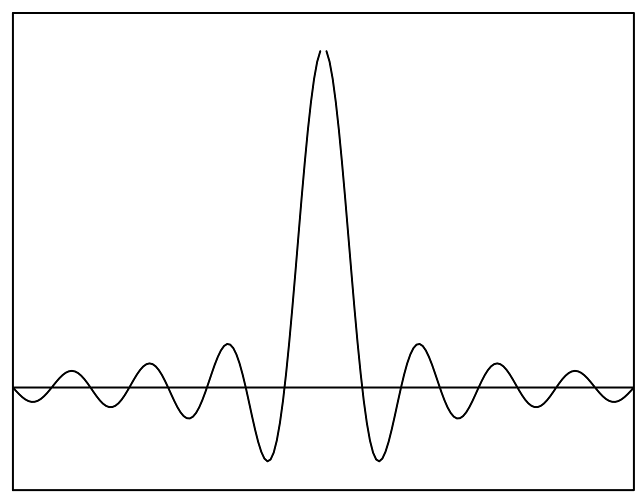
Fourier transform of Gaussian waveform



Square waveform



Fourier transform of square waveform



Tradeoffs between Domains

A number of results relate the width of a function to the width (or *bandwidth*) of its spectrum

- Narrow functions $f(t)$ have wide spectra and visa versa
  - The spectrum of the constant $f(t) = c$ is a spike at zero
  - The transform of a single spike contains all frequencies
- The *uncertainty principle*: The product of the variance of $f(t)$ and $F(f)$ cannot be too small
  - The precision of a signal in time is measured by the standard deviation Δt of the function $f(t)$
  - The precision of the signal in frequency is measured by the standard deviation Δf of the spectrum $F(f)$
  - These quantities must obey the inequality $(\Delta t)(\Delta f) \geq \frac{1}{4\pi}$
- The *sampling theorem*: A continuous signal with limited bandwidth (i.e., $F(f) = 0$ for $f > f_{\max}$) can be complete reconstructed from a set of discrete observations of spaced no more than $1/(2f_{\max})$ apart
  - The *Nyquist limit*: From a set of observations spaced at intervals D, one can recover the spectrum up to, but not above $f_c = 1/(2D)$
  - If the signal contains higher frequencies, they will be *aliased* to components of the calculated spectrum

Fourier Transforms in Two Dimensions

A two-dimensional function $f(x, y)$ has a Fourier transform $F(u, v)$

- The basic elements are one-dimensional waves
  - Each wave has an frequency and orientation
- The Fourier transformation determines an amplitude and phase for these wave
  - Alternatively, a complex value and its conjugate
- It is often convenient to represent the transform in polar coordinates
  - Angle is orientation of wave
  - Radial position is frequency
  - Amplitude is indicated at each point
- The two-dimensional transform is important in image processing
- Seemingly used by the visual system

Local Representations

- A Fourier transform describes the complete waveform
  - The only location information is the origin of $f(t)$ or $f(x,y)$
- In analyzing visual processing, it is useful to center the Fourier transform at particular points
  - Doing so amounts to shifting the origin of the space of $f(x,y)$
  - The *shift theorem* (in complex form states that

$$\text{If } f(x,y) \Longleftrightarrow F(u,v), \text{ then } f(x-a,y-b) \Longleftrightarrow e^{-2\pi i(au+bv)}F(u,v)$$

Transformation of Waveforms and Spectra

- Consider a process that transforms a waveform $f(t)$ to a waveform $g(t)$
  - Auditory signal changed by the ear
  - Light changed by passing into the eye
  - Neural signal transmitted forward
- The transformation of $f(t)$ to $g(t)$ induces a complementary transformation of the spectrum

$$\boxed{f(t) \rightarrow g(t)} \quad \Longleftrightarrow \quad \boxed{F(f) \rightarrow G(f)}$$

- Likewise, a transformation of a spectrum $F(f)$ induces a complementary transformation of the waveform,

$$\boxed{F(f) \rightarrow G(f)} \quad \Longleftrightarrow \quad \boxed{f(t) \rightarrow g(t)}$$

Modulation or Windowing

A simple transformation of a waveform increases or decreases its magnitude at each time according to a transmission function $w(t)$

$$g(t) = w(t)f(t)$$

- When $w(t)$ falls to zero as $|t| \to \infty$, it acts as a *window* on the function
  - For a function observed only between t_1 and t_2

$$w(t) = \begin{cases} 1, & t_1 < t < t_2 \\ 0, & \text{otherwise} \end{cases}$$

  - For a two-dimensional function observed in a Gaussian neighborhood of the origin

$$w(t) = \frac{1}{\pi} e^{-x^2+y^2}$$

- A series of modulations or windows can be described by multiplying their transmission functions:

$$g_1(t) = w_1(t)f(t) \qquad \text{and} \qquad g(t) = w_2(t)g_1(t)$$

  is equivalent to

$$g(t) = w(t)f(t) \qquad \text{where} \qquad w(t) = w_2(t)w_1(t)$$

- Not all processes can be described by modulations or windows

Filters

A *filter* is a transformation whose action is to multiple the spectrum by a frequency-specific *(modulation) transfer function* $A(f)$

$$G(f) = A(f)F(f)$$

- A *low-pass filter* blocks high frequencies and lets low frequencies pass
- A *high-pass filter* block low frequencies and lets high frequencies pass
- A series of filters can be analyzed by multiplying their transfer functions:

  $$G_1(f) = A_1(f)F(f) \qquad \text{and} \qquad G(f) = A_2(f)G_1(f)$$

  is equivalent to

  $$G(f) = A(f)F(f) \qquad \text{where} \qquad A(f) = A_2(f)A_1(f)$$

- Described this way, a filter is
  - Linear
  - Time invariant
- Not all processes can be described by filters

Convolutions

A *convolution* is a transformation of a function $f(t)$ created by combining its values according to a sliding linear transformation

$$g(t) = h_2 f(t-2) + h_1 f(t-1) + h_0 f(0) + h_{-1} f(t+1) + h_{-2} f(t+2)$$

The convolution of a discrete process is

$$g(t) = \sum f(t-u)h(u)$$

The convolution of a continuous process is

$$g(t) = \int_{-\infty}^{\infty} f(t-u)h(u)du$$

- Convolutions are denoted by an asterisk $g(t) = f(t) * h(t)$
- Convolutions are often easy to implement
  - Physical: blurring, running averages
  - Neural: cells in one layer combine the activation of cells in an earlier layer
  - Probability: distributions of sums of random variables

Convolutions and Fourier Transforms

- When a convolution is applied to a pure sinusoidal signal, it can change the amplitude and phase, but it does not change the frequency

$$f(t) = \cos(2\pi f t) \qquad g(t) = f(t) * h(t) = A\cos(2\pi f t + \phi)$$

- The effect of a convolution can be described by a transfer function

$$\boxed{g(t) = f(t) * h(t)} \Longleftrightarrow \boxed{G(f) = F(f)H(f)}$$

  - Convolutions of waveforms and filters are descriptions of the same process

- The effect of a modulation or window can be described by a convolution of the spectra

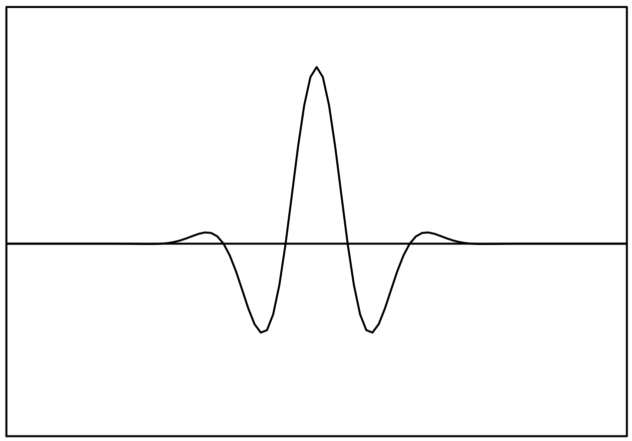
$$\boxed{g(t) = w(t)f(t)} \Longleftrightarrow \boxed{G(f) = W(f) * F(f)}$$

  - Windows and convolutions of spectra are descriptions of the same process

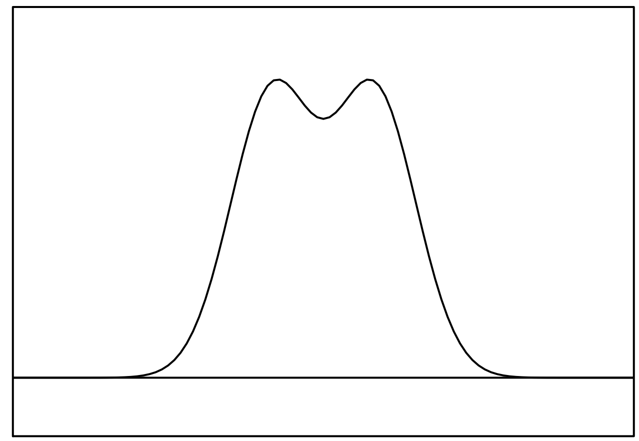
Gabor functions

A *Gabor function* is a cosine or sine wave modulated by a Gaussian window

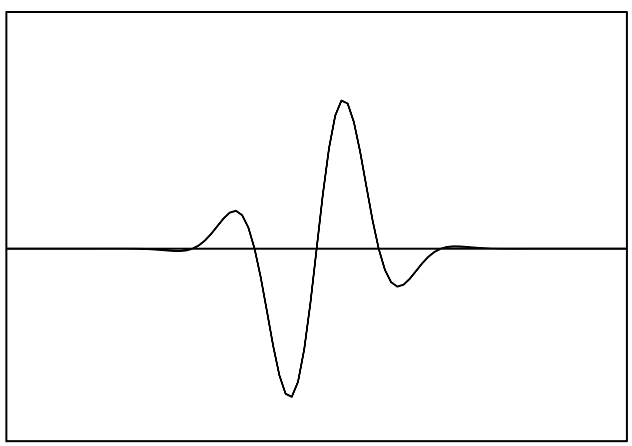
Even (cosine) Gabor function



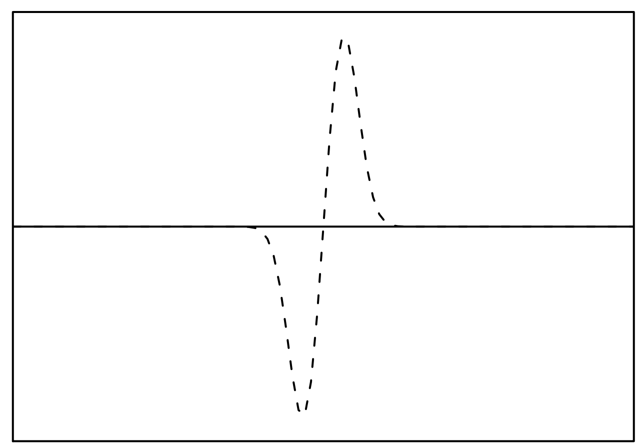
Fourier transform of even Gabor function



Odd (sine) Gabor function



Fourier transform of odd Gabor function (complex)



Impulse Response Functions

- An *impulse* is a spike of negligible width
- An impulse at time t is denoted by the "function" $\delta(t)$
  - $\delta(t)$ is not really a function
    - It has infinitesimal width and finite area
  - $\delta(t)$ is defined mathematically by limits
  - In practice the pulse must simply be short in comparison to the behavior of the process
- The Fourier transform of an impulse function is flat
  - It contains all frequencies equally
- The response of system to $\delta(0)$ is known as the *impulse response function.*
- The Fourier Transform of the impulse response function is the modulation transfer function of the system

The Fast Fourier Transform (FFT)

- Fourier transforms are computed numerically using an algorithm known as the Fast Fourier Transform (FFT)
  - Computation is much more efficient than calculating the Fourier sums of integrals
- The FFT is built into many computer languages (`Matlab`, Mathematica, `R`)
  - Usually the output is given in complex form
- One- and two-dimensional forms of the FFT exist