The Four Color Theorem

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Recall the following formula:

Euler's formula for polyhedra

For any simple planar graph $V - E + F = 2$

Now, we will introduce (but not fully prove) an important corollary and theorem before we begin the proof.

Recall the following formula:

Corollary

From Euler's theorem, we can show that $E \leq 3V - 6$

This is not difficult to prove. Note each face is bounded by at least 3 edges, and 2 faces share an edge, we have: $3F \leq 2E$. Now substituting and rearranging, we can get the above corollary.

Required Lemmas

Lets briefly skim through these theorems, the second one will be needed for our proof of the five and six color theorems.

Theorem

First Theorem of Graph Theory: If G is a graph with m vertices, then $\sum_{V \in V(G)} |V| = 2m$

Theorem

For any simple planar graph G, the minimum degree is less than equal to 5.

Six Color Theorem

Now we can begin the six color theorem!

Theorem

Every planar graph is 6 colorable.

Proof.

We prove this by induction.

- Base case: $V = 1$ is the simplest planar graph. Color this vertex and we are done.
- For $k > 1$, assume that all planer graphs with $v \leq k$ vertices can be colored with six colors (each vertex differently from its neighbors). Consider a graph with $v = k + 1$. There exists a vertex in this graph with degree ≤ 5. Remove that vertex, and this graph has order k, meaning it is six-colorable by our inductive hypothesis. This vertex is connected to at most five colors, meaning the vertex has a remaining color. The graph with order $|k + 1|$ is six colorable.

Sketch of this proof

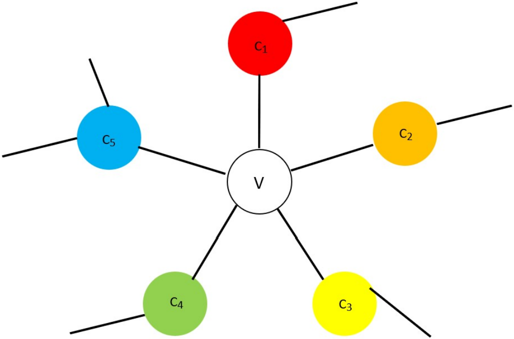

Figure: The vertex of degree 5

Five Color Theorem

The five color map theorem involves more steps, but is not too complicated.

Theorem

Every planar graph is 5 colorable.

Proof.

We can follow the same procedures as in the 6 color map theorem.

- Use the same base case and induction idea
- Now we have a problem! Based on Lemma 1.1, there exists a vertex $|V| \leq 5$, but we only have 5 colours. What if the five neighboring vertices are each different colours? We now need to show that if $|V| = 5$, we can always rearrange the colors of the surrounding vertices to open a new color for V. (You can check that this is the only scenario that requires more work as all the others have at least one colour left for V.)

Issue

- Step 1: Label each color with a subscript number; for example, C_1, C_2, ...

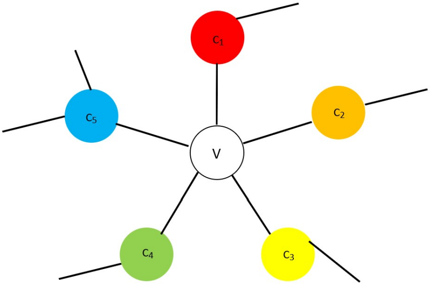

Figure: What about the middle color?

Solution

- Step 2: Pick any two non-adjacent vertices and look at the line connecting them. If this line is not exactly alternating with $C_a, C_b, C_a, C_b...$, we can find the end of the alternating line and simply switch the colors by $C_a = C_b$ and $C_b = C_a$ and we have an extra color. Else if the line actually connects the two vertices, nothing can be done but we keep the line.

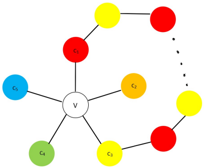

Figure: Find an alternating chain

Five Color Theorem

- Step 3: Now pick another pair of vertices, one directly in between C_a and C_b, the other one as one of the remaining 2. However, now C in between C_a and C_b is enclosed and can never reach the other color in a line with only alternating the two colors. here, we can simply swap the colors on one of the vertices touching our main V, along with the rest of the line. Now we are done since this three step procedure solves all cases of planer graphs.

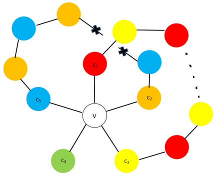

Figure: Even if the last one was a chain, this one can't be

Five Color Theorem

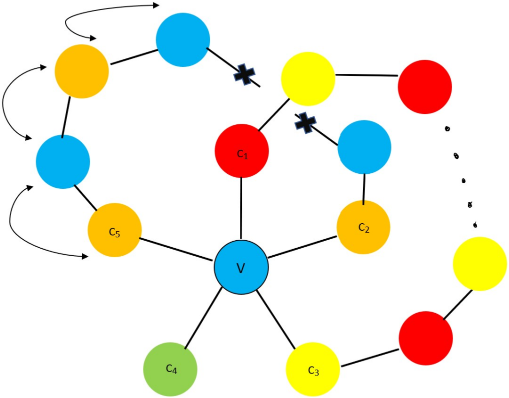

Figure: swapping the colors on one end to free up a color

Something to think about

These two proofs were not so bad! However, think about a few things:

- How many cases would the four color theorem require if we followed the same rout? we would need to eliminate 2 colors for some cases!
- This Kempe chain proof was very easy in the five color theorem. But what if we allowed these chains to go off the plane? (see figure below)

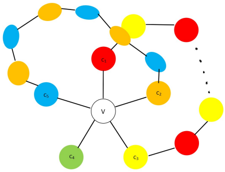

Figure: What if the orange blue chain could connect above the plane?

These questions will be answered soon!

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Four Color Theorem (4CT)

The Four Color Theorem states that no more than four colors are required to color the regions of any map so that no two adjacent regions have the same color.

More concretely, 4CT can be phrased in terms of Graph Theory:

4CT

Every simple, loopless, planar graph is four-colorable.

Examples

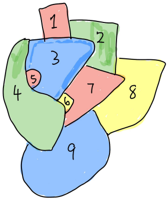

Figure: Example four-coloring of a map.

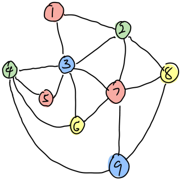

Figure: Equivalence four-coloring of a planar graph.

Kempe's (false!) Proof

Definition

In graph theory, a planar graph is said to be *triangulated* if each region has exactly three edges on its boundary.

Observation

Observation: We only need to consider the graphs that are triangulated.

Kempe's Proof

Theorem

For any triangulation G, there exists a vertex v, with $d(v) \leq 5$.

Proof.

Since G is triangulated, we must have $2e = 3f$. Euler's formula states that $v + f - e = 2$, or $v + \frac{3}{2}e - e = 2$, or $6v - 2e = 12$. If we define v_i as the number of vertices with degree i, and D as the maximum degree that a vertex has, we obtain $6\sum_{i=1}^{D} v_i - \sum_{i=1}^{D} iv_i = 12$, or equivalently $\sum_{i=1}^{D}(6-i)v_i = 12$. As a result, there must be at least one vertex with degree 5 or less, in order for the left-hand-side to be positive. □

Then, Kempe continued by using induction on the number of vertices of the graph. Base case is trivial, so now suppose every triangulated graph with n vertices is four-colorable, consider a graph with $n + 1$ vertices.

Case 1: $d(v) = 4$

By the previous theorem, we know that there exists a vertex v, with $d(v) \leq 5$. If $d(v) \leq 3$ or the vertices connected to v don't take up all four colors, we can simply reduce v and $G\backslash v$ is four-colorable by induction hypothesis.

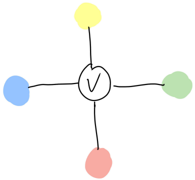

Figure: Vertex v with degree 4.

Kempe's Chain

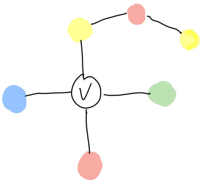

Figure: If the alternating yellow-red chain is not connected.

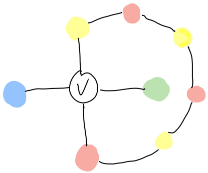

Figure: If the alternating yellow-red chain *is* connected.

Kempe adopted a similar argument as the previous case.

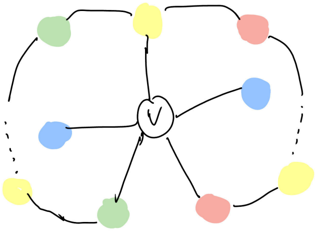

Figure: Kempe was almost right!

Why is it Wrong?

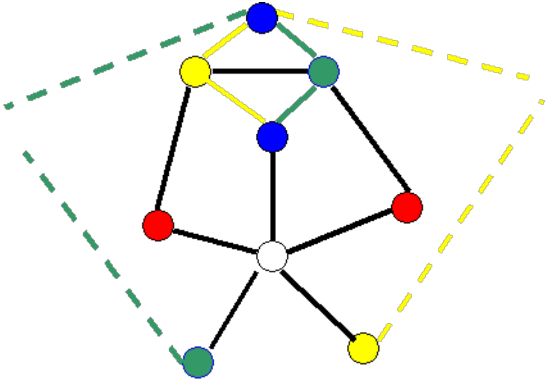

Figure: The fatal flaw.

Correct Proof

- The Four Color Theorem was finally proven by Kenneth Appel and Wolfgang Haken.
- The proof required the assistance of computers and was later improved by different researchers.
- The main idea of the proof involves looking at reducible configurations and generating an unavoidable set of reducible configurations.

Configurations & Unavoidable Set

Definition

A configuration is a connected subgraph of G, where each vertex is given a value that's equal to the vertex's degree in G. A configuration is *reducible* if the configuration is removed and the remaining graph four-colored, then the coloring can be modified in such a way that when the configuration is re-added, the four-coloring can be extended to it as well.

Example

A single vertex with degree 4 is a configuration, and so is a single vertex with degree 5. The former is shown to be a reducible configuration, but the latter isn't.

Definition

A set of configurations is called unavoidable if at least one of the configurations must appear in a graph G.

Outline of Proof

Therefore, it suffices to generate an unavoidable set of reducible configurations. The two critical steps are (1) enumerating all distinct four-colorings of a ringed configuration, and (2) using the method of discharging to discover an unavoidable set.

Coloring Theorem for Sphere

Next, we explore coloring theorems on higher dimensional surfaces. To begin, consider the sphere. We can show that the four coloring theorem holds on a sphere, as we can deform the sphere into a plane.

Coloring Theorem for Torus

Continuing, we consider the torus. The four color theorem no longer holds on the torus, and in fact we need at most seven colors to properly color any map on the torus.

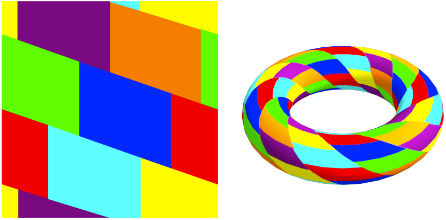

Figure: Seven colored torus, id space

Heawood's Conjecture... I need more than four colors

Recall that we call a g-holed torus a surface of genus g. For example, a sphere has genus 0, whereas a torus has genus 1 and a two holed torus has genus 2.

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Heawood conjectured that the minimum number of colors necessary to color all graphs drawn on an orientable surface of genus g is given by

$$\gamma(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor$$

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Indeed, when $\gamma(0) = 4$ and $\gamma(1) = 7$, consistent with the sphere and torus, respectively.

Heawood's Conjecture Proof

Unlike the proof of the four color theorem, Heawood's conjecture has a much simpler proof.