Introductory Macroeconomics for Engineers

Martin A. Valdez

IE 1

TD Macro 1

Exercise 1

Express the following equations as log-linear functions, i.e., take logs and simplify:

(a) $Y = zK^{\alpha}N^{1-\alpha}$.

(b) $Z = ce^{rt^{\beta}K}$.

Exercise 2

Calculate the first and second derivatives of the following functions:

(a) $f(c) = \ln(c)$.

(b) $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$.

(c) $h(w) = (-6w^3 + 17w - 4)^{\beta} - \ln(\theta w^{\beta})$.

Exercise 3

Calculate all the first, second, and cross derivatives of the following functions:

(a) $F(K, N) = \theta K^{\alpha}N^{1-\alpha}$.

(b) $F(K, N) = \ln\theta + \alpha\ln K + (1-\alpha)\ln N$.

Exercise 4

Solve the following constrained maximization problem.

$$\max_{x,w} U = \alpha\ln(x) + \beta\ln(w)$$

subject to

$$p_x x + p_w w \leq y$$
$$\alpha + \beta = 1.$$

Exercise 5

Consider the function $f(x) = \ln(1 + x)$. Calculate the first-order Taylor expansion of $f(x)$ around the point $x = 0$. Show that a growth rate can be approximated by the first-order Taylor expansion of the logarithm function around the point 1.

Note on First Order Taylor Expansion

The first-order Taylor expansion of a function $f(x)$ around a point a provides a linear approximation of $f(x)$ near a. It is given by:

$$f(x)\Big|_{x=a} \approx f(a) + f'(a)(x - a),$$

where $f'(a)$ is the derivative of f at a.

Exercise 6

Suppose an economy produces steel, wheat, and oil. Here are the economic activities of each industry:

- The steel industry produces \$100,000 in revenue, spends \$4,000 on oil, \$10,000 on wheat, and pays workers \$80,000.
- The wheat industry produces \$150,000 in revenue, spends \$20,000 on oil, \$10,000 on steel, and pays workers \$90,000.
- The oil industry produces \$200,000 in revenue, spends \$40,000 on wheat, \$30,000 on steel, and pays workers \$100,000.

There is no government, and there are neither exports nor imports. None of the industries accumulate or deaccumulate inventories.

1. Calculate the GDP of this economy using the production method.
2. Calculate the GDP using the income method.

Exercise 7

Excel exercise: Download file data_td_1.xlsx from the moodle.

1. Generate a series for the natural logarithm of realGDP and realGDP per capita for both countries.
2. Use the series for the natural logarithm of gdp to calculate the growth rate of real gdp and real gdp per capita for both countries.
3. Compute the average growth rate of real gdp and real gdp per capita for both countries.
4. Plot the evolution of realGDP, realGDP per capita, and the growth rate of realGDP for both countries. How does the French economy compare to the economy of the United States?

Exercise 8

Consider the following Cobb-Douglas production function:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha},$$

with total factor productivity represented by A_t. Derive the expression for growth rates using the logarithmic form of the production function. Show that the growth rate of output (g_{Y_t}) can be decomposed into the weighted sum of the growth rates of capital (g_{K_t}), labor (g_{L_t}), and total factor productivity (g_{A_t}):

$$g_{Y_t} = \alpha g_{K_t} + (1-\alpha) g_{L_t} + g_{A_t}.$$

Exercise 9

Maximize the profits of a representative competitive firm that produces the economy's total output using constant returns to scale Cobb-Douglas technology, paying wages w_t and renting capital at rate r_t. Demonstrate that the weights for capital and labor growth rates, from the previous exercise, are the capital's and labor's shares of income. Hint: The firm's profit maximization problem is:

$$\max_{K_t, L_t} \Pi_t = A_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - r_t K_t.$$

Exercise 10

Download the data_td_2.xlsx file from Moodle. Follow these steps to compute the labor income share, capital income share, and the Solow residual, which is defined as the growth rate of productivity.

1. Calculate labor and capital income shares as the ratio of total labor income and total capital income to total output, respectively.

2. Calculate the growth rates of output, labor, and capital using the definition of the growth rate.

3. Compute the Solow residual as the difference between the growth rate of output and the weighted sum of the growth rates of labor and capital.

4. Now do the same using logarithms to compute the growth rates.

5. Plot both solow residuals.

Exercise 11

Consider the Solow model we studied in class, in terms of per-worker variables. The model's dynamics are given by the capital accumulation equation:

$$k_{t+1} = sAk_t^{\alpha} + (1-\delta)k_t$$

That defines the level of capital per worker at time $t+1$ as a function of the level of capital per worker at time t. Define $h(k_t) = sAk_t^{\alpha} - (1-\delta)k_t$.

1. Show that $h'(k_t) \to \infty$ as $k_t \to 0$ from the right and $h'(k_t) \to (1-\delta)$ as $k_t \to \infty$.

   We say that a function $f(x)$ satisfies the Inada conditions if:

   $$\lim_{x\to 0} f'(x) = \infty, \quad \lim_{x\to\infty} f'(x) = 0.$$

   We just proved that sAk_t^{α} satisfies the Inada conditions.

2. Graph the curve $k_{t+1} = k_t$ on a plot with k_t on the horizontal axis and k_{t+1} on the vertical axis. What is the slope of this very simple curve?

3. Graph the curve $k_{t+1} = h(k_t)$ on the same plot. What is the slope of this curve as $k_t \to 0$ from the right? What is the slope of this curve as $k_t \to \infty$?

4. Solve for points where the two curves intersect. What do these points represent in terms of the Solow model? Which one is the non-trivial steady state?

Exercise 12

The steady-state, or the equilibrium, of the model is characterized by the the following equations:

$$
\begin{aligned}
k^* &= \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}. \\
y^* &= Ak^{*\alpha}, \\
c^* &= (1-s)Ak^{*\alpha}, \\
i^* &= sAk^{*\alpha}, \\
R^* &= \alpha Ak^{*\alpha-1}, \\
w^* &= (1-\alpha)Ak^{*\alpha}.
\end{aligned}
$$

1. Express the equilibrium values of y^*, R^*, and w^* in terms of the parameters of the model.

2. Suppose that the economy is initially in a steady state, so that $k_0 = k^*$. Show that the growth rate of output and the growth rate of capital are zero. Hint: Use the log approximation of growth rates.

3. Suppose that the economy is initially in a steady state, so that $k_0 = k^*$. During this period, the economy experiences a shock to the total factor productivity, $A \to A' > A$. Graph the new curve $k_{t+1} = h(k_t)$ on the same plot as the previous exercise, in a new color. What happens to the equilibrium level of capital per worker? What happens to the growth rates of output and capital per worker? Do they at some point return to zero?

4. Suppose the economy is initially in a steady state, so that $k_0 = k^*$. Recall that the steady state level of consumption is given by $c^* = (1-s)Ak^{*\alpha}$, which can also be written as $c^* = (1-s)y^*$. Use calculus to state the conditions for the level of savings that maximizes steady-state consumption. Interpret the conditions - which are the competing forces that determine the optimal level of savings?

Exercise 13

We now add technological progress and population growth to the Solow model. Suppose that the production function is given by:

$$Y_t = AF(K_t, N_t) = AK_t^{\alpha}(Z_tN_t)^{1-\alpha}$$

where Z_t is labor-augmenting technological progress and N_t is the population. We continue to consider A as a constant. Recall that the Solow model is given by the following equations:

$$
\begin{aligned}
K_{t+1} &= I_t + (1-\delta)K_t, \\
Y_t &= C_t + I_t, \\
C_t &= (1-s)Y_t, \\
I_t &= sY_t, \\
R_t &= F_K(K_t, N_t), \\
w_t &= F_N(K_t, N_t)
\end{aligned}
$$

Suppose that the population grows at a constant rate n, so that $N_{t+1} = (1+n)N_t$, and that technological progress grows at a constant rate z, so that $Z_{t+1} = (1+z)Z_t$. Define the per-worker variables in terms of *effective labor*:

$$\hat{k}_t = \frac{K_t}{Z_t N_t}$$

1. Show that $\hat{y}_t = \frac{Y_t}{Z_t N_t} = A\hat{k_t}^{\alpha}$
2. Show that the capital accumulation equation can be written as:

$$\hat{k}_{t+1} = \frac{1}{(1+z)(1+n)}[sA\hat{k}_t^{\alpha} + (1-\delta)\hat{k}_t]$$

3. Plot the curve $\hat{k}_{t+1} = \hat{k}_t$ on a plot with $\hat{k}_t$ on the horizontal axis and $\hat{k}_{t+1}$ on the vertical axis, as well as the curve $\hat{k}_{t+1}\frac{1}{(1+z)(1+n)}[sA\hat{k}_t^{\alpha} + (1-\delta)\hat{k}_t]$ on the same plot.
4. What is the growth rate of capital and capital per worker in the new model? Use the steady state to find this growth rate, as well as the definition of growth rate. Hint: recall that $\hat{k}_{t+1} = \hat{k}_t$ in the steady state, and that $\frac{K_{t+1}}{K_t} = 1 + g_K$.

Exercise 14

Let us work through the two-period consumption-savings model we discussed in class. In this problem, we have a representative agent who lives for two periods. Each period the agent receives an endowment of Y_1 and Y_2, respectively. The agent can consume C_1 and C_2 in each period, respectively. The agent has the opportunity to save in the first period. These savings are invested, which generates a return of r in the second period. The agent is impatient, so the agent discounts future consumption with a discount factor $\beta \in (0, 1)$.

The agent's problem is to maximize the utility function:

$$\max_{C_1, C_2} U = u(C_1) + \beta u(C_2)$$

Subject to each period's budget constraint:

$$C_1 + S = Y_1$$
$$C_2 = Y_2 + (1 + r)S.$$

At the end of the second period, after consuming C_2, the agent dies.

1. Combine the two budget constraints to form a single, *inter-temporal* budget constraint.

2. Substitute the inter-temporal budget constraint into the utility function to simplify the problem.

3. Calculate the first-order conditions for the problem. This is the **Euler equation**.

4. Suppose that the utility function is given by $u(C) = \ln(C)$. Substitute this utility function into the Euler equation.

5. Use this equation to solve for the optimal level of present consumption, C_1.

6. What happens to the optimal level of present consumption if the interest rate increases? Why?

7. What happens to the optimal level of present consumption if the agent becomes more impatient (i.e., β gets smaller)? What if the agent becomes more patient? Why?

8. Suppose that the agent's endowment in the first period increases. What happens to the optimal level of present consumption? Why? What happens to the optimal level of future consumption? Why? What happens if the endowment in the second period increases?

9. Solve for the optimal level of savings, S. What happens to savings if the interest rate increases? Why?

Let us now make the link with the Solow model. Suppose that we are in the steady state of the Solow model, so that $c_1 = c_2 = c^*$, and that the economy is in the steady state, so that $y_1 = y_2 = y^*$.

10. Show that the Euler equation can be written as:

$$\frac{1}{c^*} = \beta(1 + r)\frac{1}{c^*} \implies 1 = \beta(1 + r).$$

We call β a *preference parameter*, which describes an *intrinsic* property of the agent's preferences. On the other hand, r can be thought of as an *equilibrium quantity*, which is determined by the market. If the previous equation is not satisfied, the agent will adjust their consumption to satisfy it by saving or dis-saving.