Notes on ODEs from Simmons 1974

January 18, 2018

1 The Nature of Differential Equations

- DEFN: An equation involving one dependent variable and its derivatives with respect to one or more independent variables.
- Recall that for an equation $y = f(x)$, its derivative dy/dx can be interpreted as the rate of change of y wrt x.
- A differential equation is an *ordinary differential equation* if there is only one independent variable, so that all derivatives occurring in it are ordinary derivatives.
- The *order* of a differential equation is the order of the highest derivative present.
- A differential equation is a *partial differential equation* if it involves more than one independent variables, such that the derivatives occurring are partial derivatives.
  - For example, if we have some function $w = f(x, y, z, t)$, then the following are partial differential equations of the second order:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$
$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial w}{\partial t}$$
$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial^2 w}{\partial t^2}$$

  - As an aside, these are the *Laplace's*, *heat*, and *wave* equations, respectively.

1.1 General remarks on solutions

- The general ordinary differential equation of the nth order is:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{d^2x}, \ldots, \frac{d^ny}{d^nx}\right) = 0$$

  or, using the prime notation,

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

  - A general first order equation is taking the case of $n = 1$,

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

  - let us assume that this can be solved for dy/dx,

$$\frac{dy}{dx} = f(x, y)$$

- we also assume that $f(x, y)$ is a continuous function throughout some rectable R in the xy plane.
- The geometric meaning of this solution can be thought as such:
  * if $P_0 = (x_0, y_0)$ is a point in R, then
    $$\left(\frac{dy}{dx}\right)_{P_0} = f(x_0, y_0)$$
    determines a direction at P_0
  * now define a new point $P_1 = (x_1, y_1)$, such that
    $$\left(\frac{dy}{dx}\right)_{P_1} = f(x_1, y_1)$$
    determines a new direction at P_1
  * if we continue this process, defining new points $P_i = (x_i, y_i)$ and their directions, we obtain a broken line with points scattered along it like beads
  * as we decrease the distance between these beads, the line collapses into a smooth curve through the initial point P_0, and this curve is the solution $y = y(x)$ of eq. 1.1

- this is merely meant to provide plausibility to the following theorem:

Theorem 1.1 (Picard's Theorem) *If $f(x, y)$ and $\partial f/\partial y$ are continuous functions on a closed rectangle R, then through each point (x_0, y_0) in the interior of R there passes a unique integral curve of the equation $dy/dx = f(x, y)$*