Global Average Temperature as a Simple Energy Balance Model ODE

This project analyzes a first-order Ordinary Differential Equation (ODE) that models Earth's global mean surface temperature (GMST). The equation links temperature change to the difference between incoming solar radiation and outgoing longwave radiation, providing a simplified baseline for understanding the global mean temperature in equilibrium.

We will be estimating the ODE :

$$\frac{dT}{dt} = \frac{Q(1-\alpha)}{R} - \frac{\epsilon\sigma T^4}{R}$$

With variables

- t – Time, measured in years since 2015, i.e., t = 0 is the year 2015
- T – Temperature, measured in °K, a function of time t.

And the following constants

- R = 2.912 $W \times yr/(m^2 \times K)$. Heat capacity of the atmosphere. The energy required to raise the temperature of 1 m^2 of surface by 1 °K.
- Q = 342 W / m^2. Global mean incoming radiation per m^2 of surface.
- α = 0.30 (unitless). Planetary albedo. The proportion of solar radiation that Earth reflects.
- $\sigma = 5.67 \times 10^{-8}$ (unitless). The Stefan–Boltzmann constant of proportionality.
- $\epsilon \approx 0.610$ (unitless)(computed in the next paragraphs). Effective emissivity. The proportion of longwave radiation the Earth emits to space.

Thus, according to the previous definitions, the first term of the right-hand side of our ODE represents the rate of temperature increase due to incoming solar radiation. In contrast, the second term represents the rate of temperature decrease due to outgoing longwave radiation. Hence, the difference between these two terms gives the net rate of GMST variation.

We assume variation of the GMST is initially in equilibrium, i.e., incoming solar radiation equals outgoing longwave radiation in the year 2015. So $\frac{dT}{dt} = 0$ at t = 0.

We calculate the resulting effective emissivity (the constant ε) of the Earth based on the initial GMST, which is 288.4K (Walsh et al., 2015). We solve for ε as follows.

$$\frac{dT}{dt} = 0 \Rightarrow 0 = \frac{Q(1-\alpha)}{R} - \frac{\epsilon\sigma T^4}{R} \Rightarrow 0 = \frac{342(1-0.30)}{2.912} - \frac{\epsilon(5.67 \times 10^{-8})(288.4)^4}{2.912} \Rightarrow \epsilon \approx 0.610$$

Computation

1. Euler's Method

We used Euler's method to approximate the ODE solution via the Python program `Euler_Approx.py`. Relevant scripts and data are on [GitHub].

Step Size: We chose a step size of 12 steps per year (approx. 0.0833) to represent monthly updates and minuscule changes in slope that can reflect the differing behavior between the changing seasons. This algorithm runs efficiently, which permits a small step size and finer resolution, allowing Euler's method to approximate the solution to the ODE precisely.

Iterations: Iterations were limited to look roughly 5 years into the 'future', i.e., $t \in [0, 5]$ the observed range, corresponding to the years 2015 through 2020. This decision allows for comparison and back-testing against reliable temperature data from sources such as NASA. It is also crucial to operate in a range not far from the year used to calibrate $\epsilon = 0.610$, since extrapolating far beyond the assumed equilibrium point increases the probability of inaccurate forecasting.

Initial Conditions: In 2015 ($t = 0$), the Global Mean Temperature was 288.4 K (Walsh et al., 2015). The year 2015 is also the equilibrium year of energy balance, as per our previous assumption; hence, near this point, the calibration is reliable. A definite solution to the ODE must go through the point (0, 288.4).

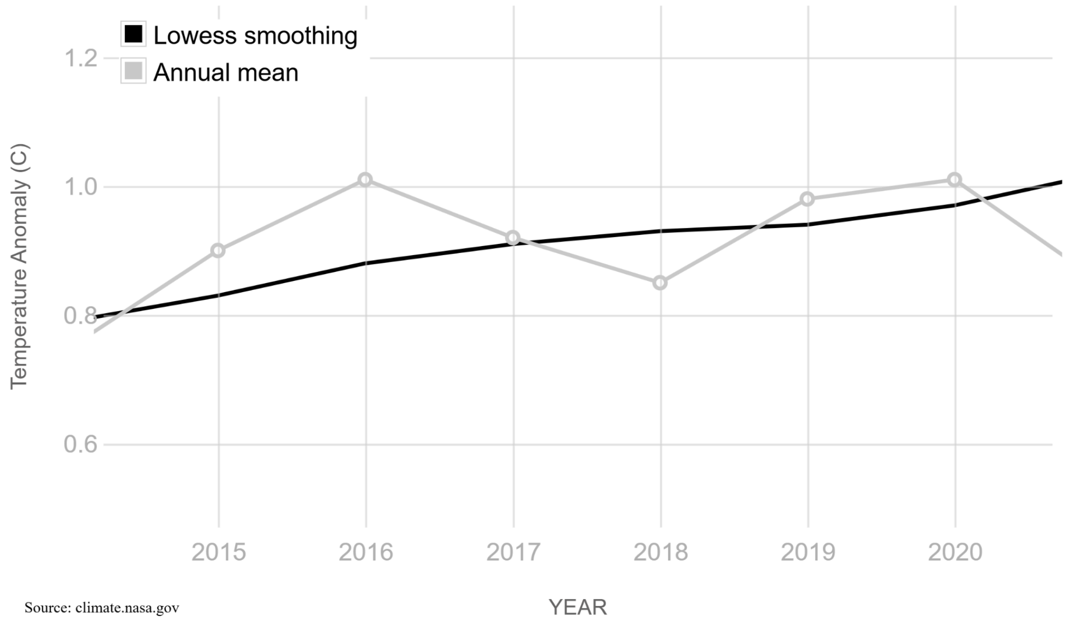
Additional Content: We displayed the steps taken by our algorithm at each iteration of Euler's method in the file `euler_steps.txt`

2. Diffrax ODE Solver

To compare the effectiveness of Euler's Method, a 'control' solution was also created, `ODE_Solver.py`. Specifically, this ODE was solved using a Diffrax, a high-performance ODE solver (Kidger, n.d.).

3. NASA GISS Global Land-Ocean Temperature Index

NASA provides the highest-accuracy empirical data on the annual Global Mean Surface Temperature, including for the period 2015–2020 (Change, n.d.-b). Plotting these data points gives a clear metric for comparing the ODE and Euler approximation. Each year, NASA reports an annual mean and a 'lowest smoothing' point; for comparison with state-of-the-art regression, we used the 'lowest smoothing' points.

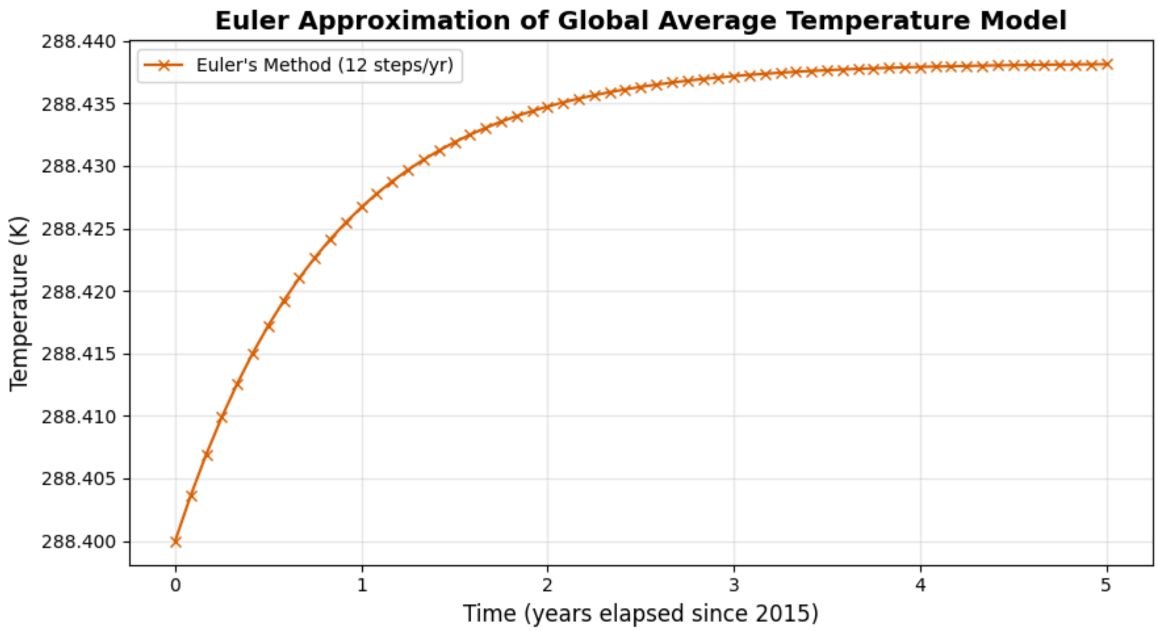


In the NASA graph, the Temperature Anomaly (y-axis) is the ± deviation from 287.56 K, which is the approximate mean global temperature from the 1950s to the 1990s. Hence, a deviation of 0.83 K (like in 2015) represents a temperature of 288.39 K.

We compared the lowest smoothing curve of the NASA graph to the Euler's method curve and the Diffrax curve in the file `Euler_Approx_vs_ODE_Solver_vs_NASA_Data.py.`

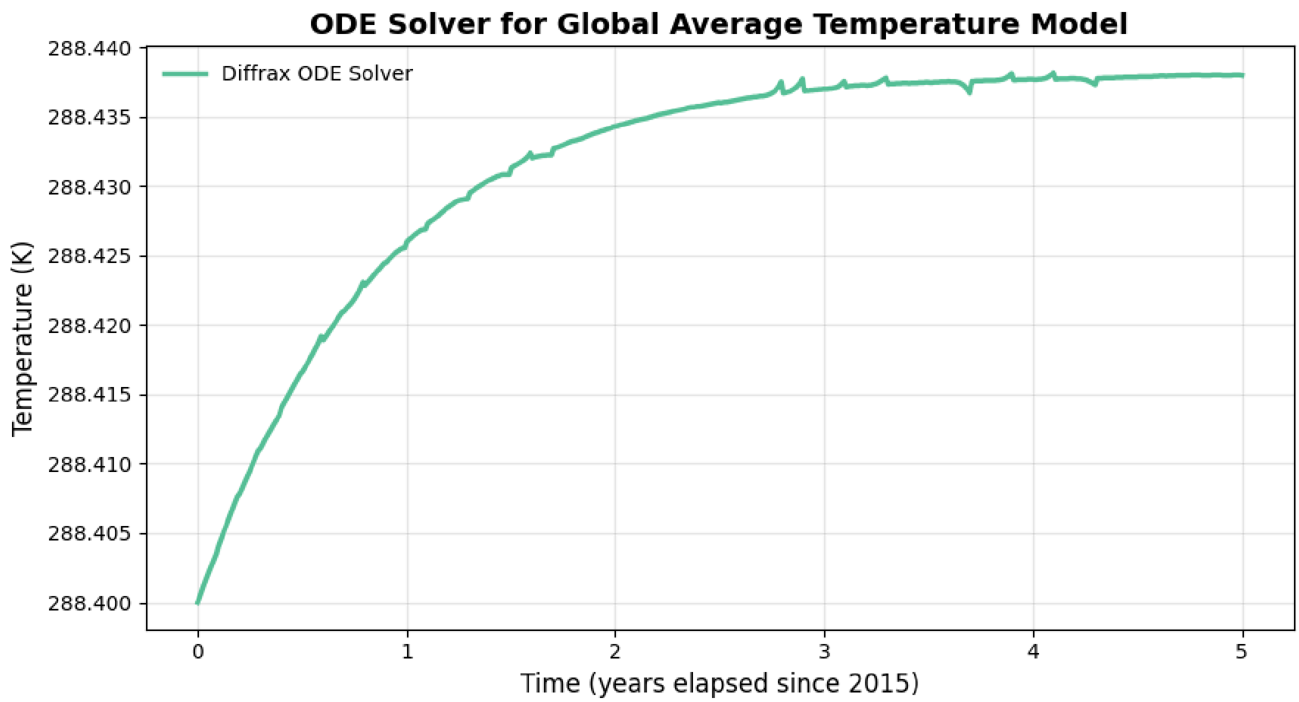
Results

The first graph shows the global mean temperature T(t) approximated using Euler's method with a step size of 0.0833 years (monthly updates), over the time interval $t \in [0, 5]$.



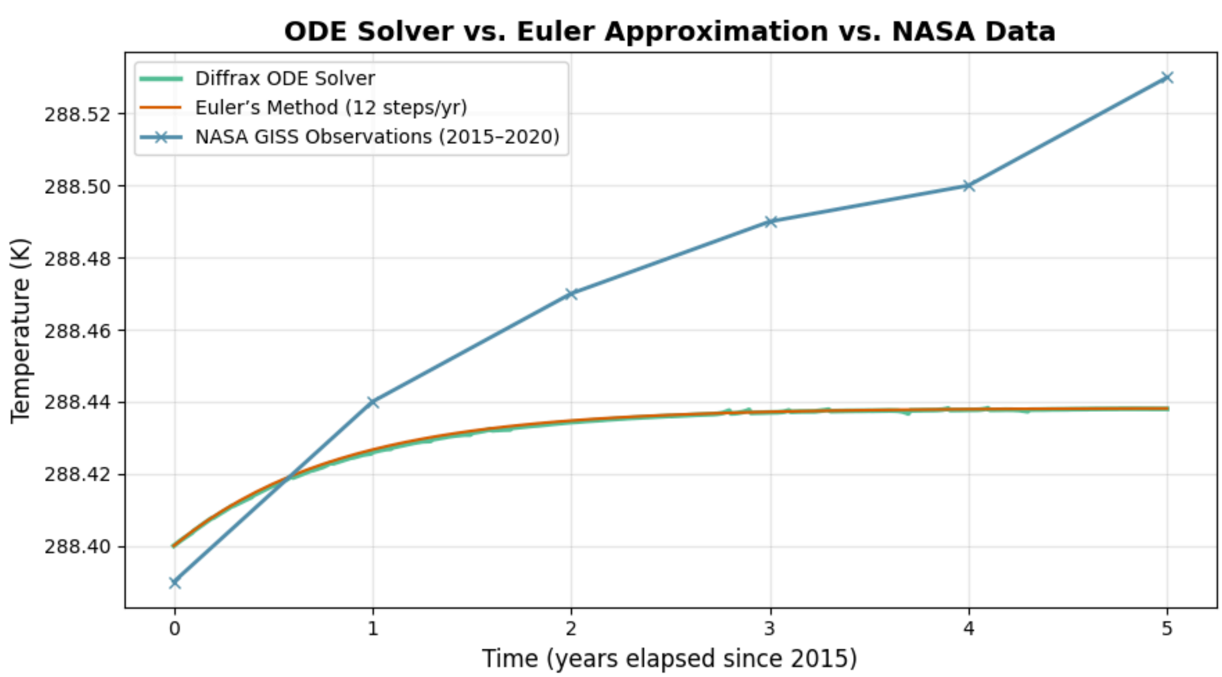
The initial condition at t = 0 is T = 288.4 K, corresponding to the initial assumption of 2015 being an equilibrium year. The function converges to an equilibrium value as time elapses. The curve is smooth, indicating that the chosen step size is small enough to capture the minuscule changes in the ODE solution.

The following graph shows the global mean temperature T(t) calculated using the Diffrax high-performance ODE solver over the same time interval $t \in [0, 5]$.



The initial condition at t = 0 is T = 288.4 K, matching Euler's method approximation.

The last graph compares the Euler approximation, the Diffrax ODE solver approximation, and the NASA empirical data estimation of the global mean temperature over the years 2015 to 2020.



The graph created using Euler’s Method closely overlaps with the Diffrax solution graph, indicating near-identical numerical solutions over the 2015–2020 period. Both numerical solutions start at 288.4 K and gradually increase to approximately 288.44 K by 2020. The NASA empirical data points over the same interval show slightly higher annual temperatures, with an upward trend relative to the model solutions.

Conclusion

This project demonstrates how a simple ODE can be used to model Earth’s energy balance and equilibrium temperature. Euler’s method, with a step size of a month, produces results that are nearly identical to those of a higher-order solver, showing that Euler’s method is effective at approximating proper functions.

However, comparing the model to NASA data highlights the limitations of simplified ODEs: real-world temperatures are rising faster than the model predicts. Although the model incorporates basic mechanisms of incoming and outgoing radiation, it fails to capture the warming trends already underway before 2015. It also overlooks well-established ecological phenomena, such as the impact of rising greenhouse gas concentrations on Earth’s effective emissivity (Galashev & Rakhmanova, 2013).

Nonetheless, this combination of ODE analysis, numerical computing, and real-world data gives both mathematical insight and an appreciation for the complexity of modern climate models, which go beyond first-order relationships to capture the ongoing trajectory of climate change.

References

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Walsh, J., Oberlin College, COMAP, Inc., & Garfunkel, S. (2015). Climate modeling in differential equations. *The UMAP Journal*, *36–36*(4), 325–363. https://www2.oberlin.edu/math/faculty/walsh/UMAPclimate.pdf