# MAP 2302
# Lecture Notes

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# Preface

Welcome to my LaTex written notes for MAP-2302, or Differential Equations. I wanted to attempt to be able to write notes for mathematical classes since MAC2302 or Calculus 3 after seeing the beautiful blogs and notes by Gilles Castel, so much of what I create and design inside these notes are based on his work. Check out his website here: https://castel.dev.

# Chapter 1

## 1.1 What are differential equations?

Remember that differential equations are equations defined by equations with derivatives. One of the simplest examples, with variables x and y, are:

$$dy = dx$$

Where the initial function, through integration can be found as:

$$y = x$$

The differential equation learned from Calc 1 is the one that describes a population, where:

$$\frac{dP}{dt} = kP$$

Where the derivative, or rate of change, is dependent on the *current population*. We can also take how a bank might offer an interest rate, consider below an interest rate of 3%.

$$\frac{dy}{dx} = 0.03y$$

A solution to this equation is $y(t) = e^{0.03t}$ which we can prove by inputing $y(t)$ and it's derivative: $y'(t) = 0.03e^{0.03t}$

$$0.03e^{0.03t} = 0.03e^{0.03t}\checkmark$$

With this solution we can also find that, for any real value of $C$, the solution still proves correct ($y(t) = Ce^{0.03t}$, $y'(t) = C(0.03)e^{0.03t}$).

$$C(0.03)e^{0.03t} = Ce^{0.03t}\checkmark$$

Which means that there are *infinitly* many solutions, which we will find true for many differential equations.

Thus we must ask

1. When do we have solutions?
2. If we do, how many?
3. And how do we find these solutions for any given differential equation?

## 1.2 Lecture 2

**Example 1.1** (Excplicit Solution)

Now let's look at an explicit solution.

Verify that $\phi = 3\sin 2x + e^{-x}$ is a solution of $y'' + 4y = 5e^{-x}$.

$$\phi' = 6\sin 2x - e^{-x}$$
$$\phi'' = -12\sin 2x + e^{-x}$$

Now we plug in

$$-12\sin 2x + e^{-x} + 12\sin 2x + 4e^{-x} = 5e^{-x}$$
$$5e^{-x} = 5e^{-x}\checkmark$$

**Theorem 1.1** Existence and Uniqueness

If $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous about the point $(x_0, y_0)$ then the Initial Value Problem $y' = f(x,y)$ and $y(x_0) = y_0$ has a unique solution in a neighborhood of the point $(x_0, y_0)$.

**Example 1.2**
$y' = xy^{\frac{1}{2}} = f(x,y)$.
Clearly $f(x,y)$ is continuous about $(0,0)$ but $\frac{\partial f}{\partial y} = \frac{x}{2y^{\frac{1}{2}}}$ is not continuous at $(0,0)$. So the theorem cannot say that there is a unique solution.
Lets say $y_1 = 0$ for every $x$:

$$y_1' = 0 \qquad xy^{\frac{1}{2}} = 0$$

$$y' = xy^{\frac{1}{2}}\checkmark$$

Lets now try $y_2 = \frac{x^4}{16}$

$$y_2' = \frac{x^3}{4} xy^{\frac{1}{2}} = x(\frac{x^4}{16})^{\frac{1}{2}} = \frac{x^3}{4}$$

$$y' = xy^{\frac{1}{2}}\checkmark$$

## 1.3 Lecture 3

Let's say $y' = f(x,y)$. Try to get an idea of how the solution curves. Now, remember the interpretation of $y'$: is it the slope the tangent line. Now let's plot plenty of small tangent lines along some graph for an equation of f(x,y).

This is defined as a **Direction Field**.

**Definition 1.1: Direction Field**

A field of vectors or slopes that represent a function at any given set of points.

**Example 1.3** ($y' = x^2 = f(x,y)$)

**Example 1.4** ($y' = \frac{x}{y}$)

**Definition 1.2: Isoclines**

**Isoclines** are curves of *equal* slope. Isoclines do not intersect unless $f(x, y)$ is not defined at the point. Isoclines are used to develop vector fields or directional fields.

We can use isoclines to create a direction field. To do so we set the derivative or $y'$ to m and solve for $y$.

**Example 1.5** ($y' = \frac{x}{y} = m$ ; $y = \frac{1}{m}x$)

**Example 1.6** ($y' = -\frac{x}{y} = m$ ; $y = -\frac{1}{m}x$)

# Chapter 2

## 2.1 Introduction to the uses of the first DE's

**Example 2.1** (Gravity)

## 2.2 Separable Equation

**Definition 2.1: Separable Equations**

A differential equation is separable if $y' = f(x, y) = g(x)p(y)$

**Example 2.2** (Is $y' = e^{x+y}$ a separable equation?)

We find that $y' = f(x, y) = (e^x)(e^y)$ where $g(x) = e^x$ and $p(y) = e^y$. Therefor it is a separable equation.

**Definition 2.2: Separable Equations and Integrals**

$$\frac{dy}{dx} = f(x, y) = g(x)p(y) \tag{2.1}$$

$$\frac{1}{p(y)}\frac{dy}{dx} = g(x) \tag{2.2}$$

Let $h(y(x)) = p^{-1}(y(x))$

$$h(y(x))\frac{dy}{dx} = g(x) \tag{2.3}$$

Let $H(y(x)), G(x)$ be antiderivatives of $h(y(x)), g(x)$, respectively.

$$\frac{dH}{dy}\frac{dy}{dx} = \frac{dG}{dx} \tag{2.4}$$

$$\frac{dH}{dx} = \frac{dG}{dx} \tag{2.5}$$

## 2.3 Linear Equations

**Definition 2.3: Linear Equation Definition**

Let's first define linear functions as below

$$a_1(x)\frac{dy}{dx} + a_2(x)y = b(x)$$

or

$$\frac{dy}{dx} + p(x)y = q(x) \qquad \text{where } p(x) = \frac{a_2(x)}{a_1(x)} \quad q(x) = \frac{b(x)}{a_1(x)}$$

### 2.3.1 How do we solve linear equations?

Take from the definition above and multiply my $\mu(x)$.

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

Assume that $\frac{d\mu}{dx} = \mu(x)p(x)$. Which we can find by doing:

$$\begin{aligned}
\frac{1}{\mu(x)}\frac{d\mu}{dx} &= p(x)dx \\
\frac{d}{dx}\left[\ln(\mu(x))\right] &= p(x) \\
ln(\mu(x)) &= \int p(x)dx \\
\mu(x) &= e^{\int p(x)dx}
\end{aligned}$$

However, now with $\mu(x)$ the function can be simplified to a much easier to understand form.

$$\begin{aligned}
\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y &= \mu(x)q(x) \\
\frac{d}{dx}\left[\mu(x)y\right] &= \mu(x)q(x) \\
y &= \frac{1}{\mu(x)}\int \mu(x)q(x)dx
\end{aligned}$$

With both equations combined:

$$y = \frac{1}{e^{\int p(x)dx}}\int e^{\int p(x)dx}q(x)dx$$

## 2.4 Exact Equations

Let $F(x, y(x)) = 0$ be an implicit solution of a differential equation. Find $\frac{dy}{dx} = f$

$$\begin{aligned}
\frac{d}{dx}\left[F(x, y(x))\right] = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial y}\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} = m - \frac{F_x}{F_y}
\end{aligned}$$

**Definition 2.4**

$\frac{dy}{dx}$ is said to be exact if there exists an $F(x, y(x))$ such that $f = -\frac{F_x}{F_y}$

**Example 2.3**

$\frac{dy}{dx} = \frac{2xy}{1+y}$, is it exact?

**Theorem 2.1**

Let $f = -\frac{M}{N}$ or $\frac{dy}{dx} = -\frac{M}{N}$. $Ndy = -Mdx$ or $Mdx + Ndy = 0$. $\frac{dy}{dx} = f$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

# Chapter 3

## 3.1 Vibrations

The simplist vibration model is of a spring.
The most ideal case in a spring, where $k$ is the stiffness.

$$my'' = -ky$$

Where the solution is

$$y = cos(wt)$$

Where $w = \sqrt{\frac{k}{m}}$.

If in a viscous fluid the equation becomes more complicated where viscosity is taken into consideration.

$$my'' = -ky - by'$$