MAP 2302
Lecture Notes

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Preface

Welcome to my LaTex written notes for MAP-2302, or Differential Equations. I wanted to attempt to be able to write notes for mathematical classes since MAC2302 or Calculus 3 after seeing the beautiful blogs and notes by Gilles Castel, so much of what I create and design inside these notes are based on his work. Check out his website here: https://castel.dev.

Chapter 1

1.1 What are differential equations?

Remember that differential equations are equations defined by equations with derivatives. One of the simplest examples, with variables x and y, are:

$$dy = dx$$

Where the initial function, through integration can be found as:

$$y = x$$

The differential equation learned from Calc 1 is the one that describes a population, where:

$$\frac{dP}{dt} = kP$$

Where the derivative, or rate of change, is dependent on the *current population.* We can also take how a bank might offer an interest rate, consider below an interest rate of 3%.

$$\frac{dy}{dx} = 0.03y$$

A solution to this equation is $y(t) = e^{0.03t}$ which we can prove by inputing $y(t)$ and it's derivative: $y'(t) = 0.03e^{0.03t}$

$$0.03e^{0.03t} = 0.03e^{0.03t} \checkmark$$

With this solution we can also find that, for any real value of C, the solution still proves correct ($y(t) = Ce^{0.03t}$, $y'(t) = C(0.03)e^{0.03t}$).

$$C(0.03)e^{0.03t} = Ce^{0.03t} \checkmark$$

Which means that there are *infinitly* many solutions, which we will find true for many differential equations.

Thus we must ask

1. When do we have solutions?
2. If we do, how many?
3. And how do we find these solutions for any given differential equation?

1.1.1 Key Definitions

We define ODE's as follows:

Definition 1.1: Orinary Differential Equations

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$$

The ***order*** is the highest derivative of the function.

Well then we ask how do we solve for these ODE's?

Example 1.1

Lets first look at a simple example: $y'' - 4y' + 3y = 0$. We can actually solve this with $y_1(t) = e^t$ or $y_2(t) = C_1e^t$. Or even $y_3(t) = C_1e^{3t}$. A ***general solution***, a solution that contains all possible solutions, would be $y(t) = C_1e^t + C_2e^{3t}$. But then where did these two equations derive from?

Definition 1.2

A solution to an ODE is defined as:

$$\phi^{(n)} = f(t, \phi, \phi', \phi'', ..., \phi^{(n-1)})$$

Once we find the solution to the ODE, it can then be found the *exact* equation using initial conditions where the initial conditions are defined as follows:

Definition 1.3

$$y(0) = y_0, y'(0) = y_0', ..., y^{(n-1)}(0) = y_0^{(n-1)}$$

This will then solve for the ***Initial Value Problem***.

1.2 Separation of Variables

Lets take the **Exponential Growth** equation: $\frac{dy}{dt} = ky$. This equation can model many real world ideas like population growth, virus spread, compound interest, etc. It's a very powerful equation. The solution is as follows: $y = C_1e^{kt}$. We can find this solution by separating the variables.

$$\begin{aligned}
\frac{dy}{dt} &= ky \\
\frac{1}{y}\frac{dy}{dt} &= k \\
\int \frac{1}{y}\frac{dy}{dt}dt &= \int kdt \\
\int \frac{1}{y}dy &= \int kdt \\
ln|y| &= kt + C \\
y &= e^{kt+C} = e^Ce^{kt} = C_1e^{kt}
\end{aligned}$$

Generically we solve through separation of variables by:

$$\begin{aligned}
\frac{dy}{dt} &= g(y)f(t) \\
\frac{1}{g(y)}\frac{dy}{dt} &= f(t) \\
\int \frac{1}{g(y)}\frac{dy}{dt}dt &= \int f(t)dt \\
\int \frac{1}{g(y)}dy &= \int f(t)dt
\end{aligned}$$

Sometimes these solutions can be implicit and cannot be solved for y or x.

1.2.1 Newton's Law of Cooling

We can model cooling or heating of an object in a room with the following:

$$\frac{dT}{dt} = -k(T - A)$$

Where T is the temperature of the object, A is the ambient temperature or the temperature of the room, and k is some constant.

We can solve for T through the separation of variables.

$$\begin{aligned}\frac{dT}{dt} &= -k(T - A)\\ \frac{1}{T - A} &= -k\\ \int \frac{1}{T - A} dT &= \int -k dt\\ ln|T - A| &= -kt + C\\ T - A &= e^{-kt+C}\\ T &= C_1 e^{-kt} + A\end{aligned}$$

1.3 Geometric Meaning

Slope Fields, Integral Curves, and Isoclines

Slope fields are graphs of small slopes plotted to help visualize a differential equation. We can either plot them directly or use ***Isoclines*** to graph the DE. We can find these Isoclines by finding when the slope equals some integer (usually from -2 to 2) and plotting the curves with their integer slopes. ***Integral Curves*** are then the *solutions* that follow the lines in the slope field.

1.4 Existance and Uniqueness

Given an Initial Value Problem: $y' = f(x, y), y(x_0) = y_0$

1. Does a solution *exist*?
2. Is that solution *unique*?

Theorem 1.1

If f and $\frac{df}{dy}$ are continuous near (x_0, y_0), then there is a unique solution on an interval $\alpha < x_0 < \beta$ to the I.V.P.

$$y' = f(x, y), y(x_0) = y_0$$

If f is continuous then it guarantees existance only.

1.5 Linear Differential Equations

A linear differential equation is a ordinary differential equation where $y, y'', y''', ...$ are all linear with respect to y. Or if it follows:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + ...a_0(x)y = b(x)$$

A Linear ODE is Homogeneous if $b(x) = 0$.

Let's take a first order linear differential equation like the following $y' + p(x)y = f(x)$. How might we solve this equation? Let's first multiply a third function, $r(x)$ on both sides of the equation and we get:

$$r(x)y' = r(x)p(x)y = r(x)f(x)$$

Is it then possible to force the left hand side to be the derivative of some function of x multiplied by y? The left hand side *seems* to be comparable to the product rule. Let's see if this follows:

$$\frac{d}{dx}[r(x)y] = r(x)y' + r'(x)y$$

Now we see some similarities, the only differences actually are that of the functional coefficients of y. Well let's see what happens when we set them equal to each other.

$$r'(x) = r(x)p(x)$$

Now we can solve this as a separable equation.

$$\frac{1}{r(x)}\frac{dr}{dx} = p(x)$$
$$\int \frac{1}{r(x)}dr = \int p(x)dx$$
$$ln|r(x)| = \int p(x)dx$$
$$r(x) = e^{\int p(x)dx}$$

Now that we know that an $r(x)$ exists we can use it in our differential equation

$$\frac{d}{dx}[r(x)y] = r(x)f(x)$$

Then lastly integrating dx we find that

$$y = \frac{1}{r(x)}\int r(x)f(x)dx$$
$$r(x) = e^{\int p(x)dx}$$

Theorem 1.2 Existance and Uniqueness for Linear

If $f(x)$ and $p(x)$ are continuous on (a,b) then a solution exists and is unique on (a,b).

Note that everything after here is based on my prof's teachings and not the lecture videos.

1.6 Lecture 2

Example 1.2 (Excplicit Solution)

Now let's look at an explicit solution.
Verify that $\phi = 3\sin 2x + e^{-x}$ is a solution of $y'' + 4y = 5e^{-x}$.

$$\phi' = 6\sin 2x - e^{-x}$$
$$\phi'' = -12\sin 2x + e^{-x}$$

Now we plug in

$$-12\sin 2x + e^{-x} + 12\sin 2x + 4e^{-x} = 5e^{-x}$$
$$5e^{-x} = 5e^{-x}\checkmark$$

Theorem 1.3 Existence and Uniqueness

If $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous about the point (x_0,y_0) then the Initial Value Problem $y' = f(x,y)$

and $y(x_0) = y_0$ has a unique solution in a neighborhood of the point (x_0, y_0).

Example 1.3
$y' = xy^{\frac{1}{2}} = f(x, y)$.
Clearly $f(x, y)$ is continuous about $(0, 0)$ but $\frac{\partial f}{\partial y} = \frac{x}{2y^{\frac{1}{2}}}$ is not continuous at $(0, 0)$. So the theorem cannot say that there is a unique solution.
Lets say $y_1 = 0$ for every x:

$$y_1' = 0 \qquad xy^{\frac{1}{2}} = 0$$

$$y' = xy^{\frac{1}{2}} \checkmark$$

Lets now try $y_2 = \frac{x^4}{16}$

$$y_2' = \frac{x^3}{4} xy^{\frac{1}{2}} = x(\frac{x^4}{16})^{\frac{1}{2}} = \frac{x^3}{4}$$

$$y' = xy^{\frac{1}{2}} \checkmark$$

1.7 Lecture 3

Let's say $y' = f(x, y)$. Try to get an idea of how the solution curves. Now, remember the interpretation of y': is it the slope the tangent line. Now let's plot plenty of small tangent lines along some graph for an equation of f(x,y).

This is defined as a **Direction Field**.

Definition 1.4: Direction Field

A field of vectors or slopes that represent a function at any given set of points.

Example 1.4 $(y' = x^2 = f(x, y))$

Example 1.5 $(y' = \frac{x}{y})$

Definition 1.5: Isoclines

Isoclines are curves of *equal* slope. Isoclines do not intersect unless $f(x, y)$ is not defined at the point. Isoclines are used to develop vector fields or directional fields.

We can use isoclines to create a direction field. To do so we set the derivative or y' to m and solve for y.

Example 1.6 $(y' = \frac{x}{y} = m \ ; \ y = \frac{1}{m}x)$

Example 1.7 $(y' = -\frac{x}{y} = m \ ; \ y = -\frac{1}{m}x)$

Chapter 2

2.1 Introduction to the uses of the first DE's

Example 2.1 (Gravity)

2.2 Separable Equation

Definition 2.1: Separable Equations

A differential equation is separable if $y' = f(x, y) = g(x)p(y)$

Example 2.2 (Is $y' = e^{x+y}$ a separable equation?)

We find that $y' = f(x, y) = (e^x)(e^y)$ where $g(x) = e^x$ and $p(y) = e^y$. Therefor it is a separable equation.

Definition 2.2: Separable Equations and Integrals

$$\frac{dy}{dx} = f(x, y) = g(x)p(y) \tag{2.1}$$

$$\frac{1}{p(y)}\frac{dy}{dx} = g(x) \tag{2.2}$$

Let $h(y(x)) = p^{-1}(y(x))$

$$h(y(x))\frac{dy}{dx} = g(x) \tag{2.3}$$

Let $H(y(x)), G(x)$ be antiderivatives of $h(y(x)), g(x)$, respectively.

$$\frac{dH}{dy}\frac{dy}{dx} = \frac{dG}{dx} \tag{2.4}$$

$$\frac{dH}{dx} = \frac{dG}{dx} \tag{2.5}$$

2.3 Linear Equations

Definition 2.3: Linear Equation Definition

Let's first define linear functions as below

$$a_1(x)\frac{dy}{dx} + a_2(x)y = b(x)$$

or

$$\frac{dy}{dx} + p(x)y = q(x) \qquad \text{where } p(x) = \frac{a_2(x)}{a_1(x)} \quad q(x) = \frac{b(x)}{a_1(x)}$$

2.3.1 How do we solve linear equations?

Take from the definition above and multiply my $\mu(x)$.

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

Assume that $\frac{d\mu}{dx} = \mu(x)p(x)$. Which we can find by doing:

$$\begin{aligned}
\frac{1}{\mu(x)}\frac{d\mu}{dx} &= p(x)dx \\
\frac{d}{dx}\left[\ln(\mu(x))\right] &= p(x) \\
ln(\mu(x)) &= \int p(x)dx \\
\mu(x) &= e^{\int p(x)dx}
\end{aligned}$$

However, now with $\mu(x)$ the function can be simplified to a much easier to understand form.

$$\begin{aligned}
\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y &= \mu(x)q(x) \\
\frac{d}{dx}\left[\mu(x)y\right] &= \mu(x)q(x) \\
y &= \frac{1}{\mu(x)}\int \mu(x)q(x)dx
\end{aligned}$$

With both equations combined:

$$y = \frac{1}{e^{\int p(x)dx}}\int e^{\int p(x)dx}q(x)dx$$

2.4 Exact Equations

Let $F(x, y(x)) = 0$ be an implicit solution of a differential equation. Find $\frac{dy}{dx} = f$

$$\begin{aligned}
\frac{d}{dx}\left[F(x, y(x))\right] &= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \\
\frac{dy}{dx} &= m - \frac{F_x}{F_y}
\end{aligned}$$

Definition 2.4

$\frac{dy}{dx}$ is said to be exact if there exists an $F(x, y(x))$ such that $f = -\frac{F_x}{F_y}$

Example 2.3

$\frac{dy}{dx} = \frac{2xy}{1+y}$, is it exact?

Theorem 2.1

Let $f = -\frac{M}{N}$ or $\frac{dy}{dx} = -\frac{M}{N}$. $Ndy = -Mdx$ or $Mdx + Ndy = 0$. $\frac{dy}{dx} = f$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Chapter 3

3.1 Vibrations

The simplist vibration model is of a spring.

The most ideal case in a spring, where k is the stiffness.

$$my'' = -ky$$

Where the solution is

$$y = cos(wt)$$

Where $w = \sqrt{\frac{k}{m}}$.

If in a viscous fluid the equation becomes more complicated where viscosity is taken into consideration.

$$my'' = -ky - by'$$