MAP 2302
Lecture Notes

Mark Williams

Contents

Preface

Welcome to my LaTex written notes for MAP-2302, or Differential Equations. I wanted to attempt to be able to write notes for mathematical classes since MAC2302 or Calculus 3 after seeing the beautiful blogs and notes by Gilles Castel, so much of what I create and design inside these notes are based on his work. Check out his website here: https://castel.dev.

Unit 1

1.1 Lecture 1: What are differential equations?

Remember that differential equations are equations defined by equations with derivatives. One of the simplest examples, with variables x and y, are:

$$dy = dx$$

Where the initial function, through integration can be found as:

$$y = x$$

The differential equation learned from Calc 1 is the one that describes a population, where:

$$\frac{dP}{dt} = kP$$

Where the derivative, or rate of change, is dependent on the *current population*.

1.2 Lecture 2

Example 1.1 (Excplicit Solution)

Now let's look at an explicit solution.
Verify that $\phi = 3\sin 2x + e^{-x}$ is a solution of $y'' + 4y = 5e^{-x}$.

$$\begin{aligned} \phi' &= 6\sin 2x - e^{-x} \\ \phi'' &= -12\sin 2x + e^{-x} \end{aligned}$$

Now we plug in

$$\begin{aligned} -12\sin 2x + e^{-x} + 12\sin 2x + 4e^{-x} &= 5e^{-x} \\ 5e^{-x} &= 5e^{-x} \checkmark \end{aligned}$$

Theorem 1.1 Existence and Uniqueness

If $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous about the point (x_0, y_0) then the Initial Value Problem $y' = f(x, y)$ and $y(x_0) = y_0$ has a unique solution in a neighborhood of the point (x_0, y_0).

Example 1.2

$y' = xy^{\frac{1}{2}} = f(x, y)$.
Clearly $f(x, y)$ is continuous about $(0, 0)$ but $\frac{\partial f}{\partial y} = \frac{x}{2y^{\frac{1}{2}}}$ is not continuous at $(0, 0)$. So the theorem cannot say that there is a unique solution.

Lets say $y_1 = 0$ for every x:

$$y_1' = 0 \qquad xy^{\frac{1}{2}} = 0$$

$$y' = xy^{\frac{1}{2}} \checkmark$$

Lets now try $y_2 = \frac{x^4}{16}$

$$y_2' = \frac{x^3}{4} xy^{\frac{1}{2}} = x(\frac{x^4}{16})^{\frac{1}{2}} = \frac{x^3}{4}$$

$$y' = xy^{\frac{1}{2}} \checkmark$$

1.3 Lecture 3

Let's say $y' = f(x, y)$. Try to get an idea of how the solution curves. Now, remember the interpretation of y': is it the slope the tangent line. Now let's plot plenty of small tangent lines along some graph for an equation of f(x,y).

This is defined as a **Direction Field**.

Definition 1.1: Direction Field

A field of vectors or slopes that represent a function at any given set of points.

Example 1.3 ($y' = x^2 = f(x, y)$)

Example 1.4 ($y' = \frac{x}{y}$)

Definition 1.2: Isoclines

Isoclines are curves of *equal* slope. Isoclines do not intersect unless $f(x, y)$ is not defined at the point. Isoclines are used to develop vector fields or directional fields.

We can use isoclines to create a direction field. To do so we set the derivative or y' to m and solve for y.

Example 1.5 ($y' = \frac{x}{y} = m$; $y = \frac{1}{m}x$)

Example 1.6 ($y' = -\frac{x}{y} = m$; $y = -\frac{1}{m}x$)

Unit 2

2.1 Introduction to the uses of the first DE's

Example 2.1 (Gravity)

2.2 Separable Equation

Definition 2.1: Separable Equations

A differential equation is separable if $y' = f(x, y) = g(x)p(y)$

Example 2.2 (Is $y' = e^{x+y}$ a separable equation?)
We find that $y' = f(x, y) = (e^x)(e^y)$ where $g(x) = e^x$ and $p(y) = e^y$. Therefor it is a separable equation.

Definition 2.2: Separable Equations and Integrals

$$\frac{dy}{dx} = f(x, y) = g(x)p(y) \tag{2.1}$$

$$\frac{1}{p(y)}\frac{dy}{dx} = g(x) \tag{2.2}$$

Let $h(y(x)) = p^{-1}(y(x))$

$$h(y(x))\frac{dy}{dx} = g(x) \tag{2.3}$$

Let $H(y(x)), G(x)$ be antiderivatives of $h(y(x)), g(x)$, respectively.

$$\frac{dH}{dy}\frac{dy}{dx} = \frac{dG}{dx} \tag{2.4}$$

$$\frac{dH}{dx} = \frac{dG}{dx} \tag{2.5}$$

2.3 Linear Equations

Definition 2.3: Linear Equation Definition

Let's first define linear functions as below

$$a_1(x)\frac{dy}{dx} + a_2(x)y = b(x)$$

or

$$\frac{dy}{dx} + p(x)y = q(x) \qquad \text{where } p(x) = \frac{a_2(x)}{a_1(x)} \quad q(x) = \frac{b(x)}{a_1(x)}$$

2.3.1 How do we solve linear equations?

Take from the definition above and multiply my $\mu(x)$.

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

Assume that $\frac{d\mu}{dx} = \mu(x)p(x)$. Which we can find by doing:

$$\begin{aligned}\frac{1}{\mu(x)}\frac{d\mu}{dx} &= p(x)dx \\ \frac{d}{dx}\left[\ln(\mu(x))\right] &= p(x) \\ ln(\mu(x)) &= \int p(x)dx \\ \mu(x) &= e^{\int p(x)dx}\end{aligned}$$

However, now with $\mu(x)$ the function can be simplified to a much easier to understand form.

$$\begin{aligned}\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y &= \mu(x)q(x) \\ \frac{d}{dx}\left[\mu(x)y\right] &= \mu(x)q(x) \\ y &= \frac{1}{\mu(x)}\int \mu(x)q(x)dx\end{aligned}$$

With both equations combined:

$$y = \frac{1}{e^{\int p(x)dx}}\int e^{\int p(x)dx}q(x)dx$$

2.4 Exact Equations

Let $F(x, y(x)) = 0$ be an implicit solution of a differential equation. Find $\frac{dy}{dx} = f$

$$\begin{aligned}\frac{d}{dx}\left[F(x, y(x))\right] &= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \\ \frac{dy}{dx} &= m - \frac{F_x}{F_y}\end{aligned}$$

Definition 2.4

$\frac{dy}{dx}$ is said to be exact if there exists an $F(x, y(x))$ such that $f = -\frac{F_x}{F_y}$

Example 2.3

$\frac{dy}{dx} = \frac{2xy}{1+y}$, is it exact?

Theorem 2.1

Let $f = -\frac{M}{N}$ or $\frac{dy}{dx} = -\frac{M}{N}$. $Ndy = -Mdx$ or $Mdx + Ndy = 0$. $\frac{dy}{dx} = f$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Unit 3

3.1 Vibrations

The simplist vibration model is of a spring.

The most ideal case in a spring, where k is the stiffness.

$$my'' = -ky$$

Where the solution is

$$y = cos(wt)$$

Where $w = \sqrt{\frac{k}{m}}$.

If in a viscous fluid the equation becomes more complicated where viscosity is taken into consideration.

$$my'' = -ky - by'$$