2D Rigid Body Equations

WB 1418-07

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- ✓ **We said that we can have a rigid body made of a number of points masses connected by rigid links.**

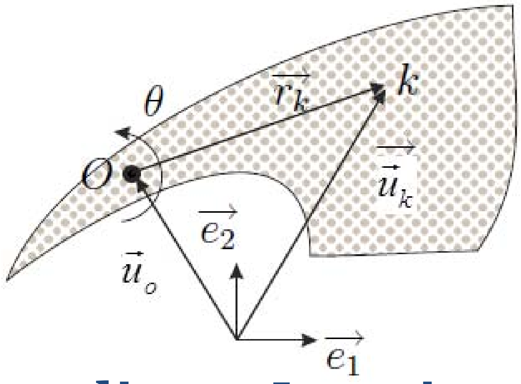
- ✓ **We showed that a rigid body will have 6 Dofs.**

- ✓ **Now let's assume we have a 2D rigid body and we want to find the equations of motion.**

Point O is an arbitrary point on the rigid body.

3 DOFs: 2 translations along e_1 and e_2+a rotation

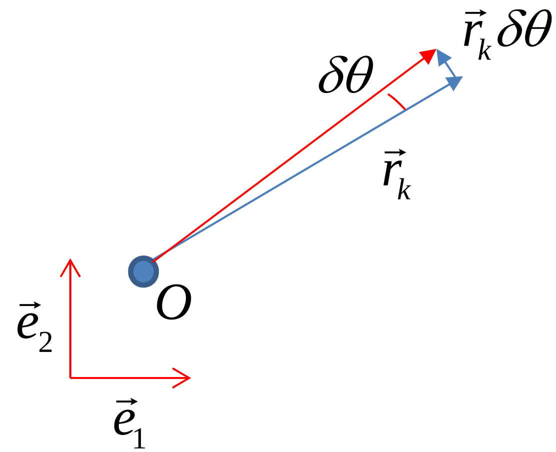
Position of any arbitrary point on the rigid body ⇒ $$\vec{u}_k = \vec{u}_O + \vec{r}_k(\theta)$$



- ✓ **Applying Newton's 2nd law would be quite tedious: Imagine finding the acceleration of each point+ internal forces.**

- ✓ **If we use D'Alambert principle all we need is to find virtual displacements**

$$\delta\vec{u}_k = \delta\vec{u}_O + \delta\vec{r}_k$$



Virtual displacement is indeed a vector so how can I show that $r_k\delta\theta$ is in the direction shown here?

$$\delta\vec{u}_k = \delta\vec{u}_O + \vec{e}_3 \times \vec{r}_k\delta\theta$$

e_3 is the out-of-plane unit vector.

D'Alambert principle states that we would need to project dynamic equilibrium in the direction of the virtual displacements. Therefore:

$$m_k\ddot{\vec{u}}_k - \vec{X}_k - \vec{R}_k = 0 \quad , \quad k = 1,\ldots,N$$

$$\sum_{k=1}^{N}\left(m_k\ddot{\vec{u}}_k - \vec{X}_k \cancel{- \vec{R}_k}\right)\cdot\delta\vec{u}_k = 0$$

$$\sum_{k=1}^{N}\left(m_k\ddot{\vec{u}}_k - \vec{X}_k\right)\cdot\left(\delta\vec{u}_O + \vec{e}_3 \times \vec{r}_k\delta\theta\right) = 0$$

$$\sum_{k=1}^{N}\left(m_k\ddot{\vec{u}}_k - \vec{X}_k\right)\cdot\left(\delta\vec{u}_O + \vec{e}_3\times\vec{r}_k\delta\theta\right) = 0$$

This relation is true for every variation. Thus:

$$\boxed{\sum_{k=1}^{N} m_k\ddot{\vec{u}}_k = \sum_{k=1}^{N}\vec{X}_k}$$

$$\sum_{k=1}^{N} m_k\ddot{\vec{u}}_k\cdot\left(\vec{e}_3\times\vec{r}_k\delta\theta\right) = \sum_{k=1}^{N}\vec{X}_k\cdot\left(\vec{e}_3\times\vec{r}_k\delta\theta\right)$$

Remark: $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right) = \vec{b}\cdot\left(\vec{c}\times\vec{a}\right)$

$$\vec{e}_3\cdot\left(\sum_{k=1}^{N} m_k\vec{r}_k\times\ddot{\vec{u}}_k\right) = \vec{e}_3\cdot\left(\sum_{k=1}^{N}\vec{r}_k\times\vec{X}_k\right)$$

$$\Downarrow$$

$$\boxed{\sum_{k=1}^{N} m_k\vec{r}_k\times\ddot{\vec{u}}_k = \sum_{k=1}^{N}\vec{r}_k\times\vec{X}_k}$$

So the rigid body equations will be

$$\sum_{k=1}^{N} m_k \ddot{\vec{u}}_k = \sum_{k=1}^{N} \vec{X}_k$$

$$\sum_{k=1}^{N} m_k \vec{r}_k \times \ddot{\vec{u}}_k = \sum_{k=1}^{N} \vec{r}_k \times \vec{X}_k$$

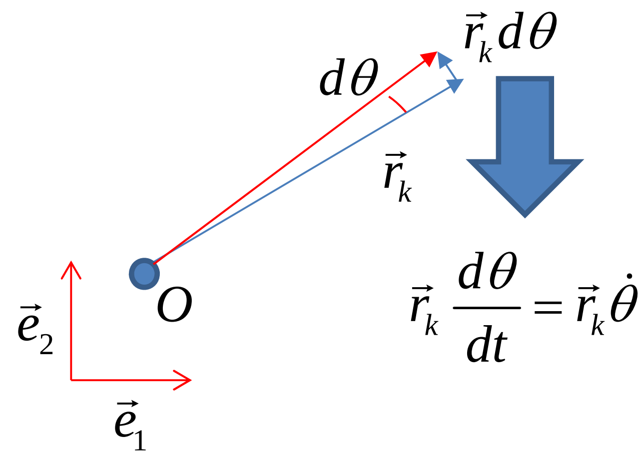
Next is to find acceleration

$$\dot{\vec{u}}_k = \dot{\vec{u}}_O + \dot{\theta}\vec{e}_3 \times \vec{r}_k$$

$$\ddot{\vec{u}}_k = \ddot{\vec{u}}_O + \ddot{\theta}\vec{e}_3 \times \vec{r}_k + \dot{\theta}\vec{e}_3 \times \dot{\vec{r}}_k$$

$$= \ddot{\vec{u}}_O + \ddot{\theta}\vec{e}_3 \times \vec{r}_k + \dot{\theta}\vec{e}_3 \times \left(\dot{\theta}\vec{e}_3 \times \vec{r}_k\right)$$

$$= \ddot{\vec{u}}_O + \ddot{\theta}\vec{e}_3 \times \vec{r}_k - \dot{\theta}^2 \vec{r}_k$$



I can still make things simpler by defining center of mass

$$\sum_{k=1}^{N} m_k \vec{r}_k = m_{tot} \vec{r}_C \quad where \quad m_{tot} = \sum_{k=1}^{N} m_k$$

So the rigid body equations will be

$$m_{tot}\ddot{\vec{u}}_O + m_{tot}\ddot{\theta}\vec{e}_3 \times \vec{r}_C - m_{tot}\dot{\theta}^2\vec{r}_C = \sum_{k=1}^{N} \vec{X}_k$$

$$\boxed{\sum_{k=1}^{N} m_k \vec{r}_k \times \left(\ddot{\vec{u}}_O + \ddot{\theta}\vec{e}_3 \times \vec{r}_k - \dot{\theta}^2\vec{r}_k\right) = \sum_{k=1}^{N} \vec{r}_k \times \vec{X}_k}$$

$$\sum_{k=1}^{N} m_k \vec{r}_k \times \left(\ddot{\vec{u}}_O\right) = m_{tot}\vec{r}_C \times \ddot{\vec{u}}_O$$

$$\sum_{k=1}^{N} m_k \vec{r}_k \times \left(-\dot{\theta}^2\vec{r}_k\right) = 0$$

$$\vec{r}_k \times \left(\vec{e}_3 \times \vec{r}_k\right) = \left\|\vec{r}_k\right\|^2 \vec{e}_3$$

$$\sum_{k=1}^{N} m_k \vec{r}_k \times \left(\ddot{\theta}\vec{e}_3 \times \vec{r}_k\right) = \sum_{k=1}^{N} m_k \left\|\vec{r}_k\right\|^2 \ddot{\theta}\vec{e}_3$$

$$= J_O \vec{e}_3 \ddot{\theta}$$

Rotational Inertia

$$\boxed{\begin{gathered} m_{tot}\ddot{\vec{u}}_O + m_{tot}\ddot{\theta}\vec{e}_3 \times \vec{r}_C - m_{tot}\dot{\theta}^2\vec{r}_C = \sum_{k=1}^{N} \vec{X}_k \\ m_{tot}\vec{r}_C \times \ddot{\vec{u}}_O + J_O\vec{e}_3\ddot{\theta} = \sum_{k=1}^{N} \vec{r}_k \times \vec{X}_k \end{gathered}}$$

So at the end we come up with two vectorial equations: one defining the translational motion and the other rotational.

$$m_{tot}\ddot{\vec{u}}_O + m_{tot}\ddot{\theta}\vec{e}_3 \times \vec{r}_C - m_{tot}\dot{\theta}^2\vec{r}_C = \sum_{k=1}^{N} \vec{X}_k$$

$$m_{tot}\vec{r}_C \times \ddot{\vec{u}}_O + J_O\vec{e}_3\ddot{\theta} = \sum_{k=1}^{N} \vec{r}_k \times \vec{X}_k$$

What would happen if I put the reference point *O* at *C*?

$$\vec{r}_C = 0$$

$$\Downarrow$$

$$m_{tot}\ddot{\vec{u}}_C = \sum_{k=1}^{N} \vec{X}_k$$

$$J_C\vec{e}_3\ddot{\theta} = \sum_{k=1}^{N} \vec{r}_k \times \vec{X}_k$$

$\Rightarrow$ Two uncoupled equations