Stability

WB 1418-07

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What have we learnt so far?

- **Constraints: Rheonomic, Scleronomic, non-holonomic**

- **Finding EoMs**

D'Alambert	Lagrange	Hamilton
$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\ddot{u}_{ik}-X_{ik}\right)\delta u_{ik}=0$	$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}+\frac{\partial V}{\partial q_s}-Q_s^{ncons}=0 \quad s=1,\ldots n$	$\int_{t_1}^{t_2}\left(\delta T-\delta V+\delta W^{ncons}\right)dt=0$, $\delta q_s(t_1)=\delta q_s(t_2)=0 \quad s=1,\ldots.,n$

- **Finding Equilibrium**

Scleronomic	Rheonomic
$\left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}}=0$	$\left(\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{\mathbf{q}}}\right)\right)_{\substack{\dot{\mathbf{q}}=0,\\ \mathbf{q}=\mathbf{q}_{eq}}}-\left(\frac{\partial T_0}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}}+\left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}}=0$

- **Finding Linearized equations**

Scleronomic	Rheonomic
$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=0$	$\mathbf{M}\ddot{\mathbf{q}}+\left(\mathbf{G}+\mathbf{C}\right)\dot{\mathbf{q}}+\mathbf{K}^*\mathbf{q}=0$

Next step: understanding the concept of stability

We look the stability of the system once linearized.

A system is linearly stable around an equilibrium position when a perturbation results in dynamic displacements that remain small and thus bounded in time.

There are systems that are designed to be unstable!

Scleronomic systems with no damping

We will look at linearized equations of motion

Kinetic energy is a quadratic function of generalized velocities

$$2T = \sum_{s=1}^{n} \dot{q}_s \frac{\partial T}{\partial \dot{q}_s}$$

Taking the total derivative in time

$$2\frac{dT}{dt} = \sum_{s=1}^{n} \ddot{q}_s \frac{\partial T}{\partial \dot{q}_s} + \sum_{s=1}^{n} \dot{q}_s \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)$$

On the other hand we know $T(\dot{\mathbf{q}}, \mathbf{q})$

$$\frac{dT}{dt} = \sum_{s=1}^{n} \ddot{q}_s \frac{\partial T}{\partial \dot{q}_s} + \sum_{s=1}^{n} \dot{q}_s \frac{\partial T}{\partial q_s}$$

If we subtract the two previous equations

$$\frac{dT}{dt}=\sum_{s=1}^{n}\dot{q}_s\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}\right)=\sum_{s=1}^{n}\dot{q}_s\left(-\frac{\partial V}{\partial q_s}\right)$$

And we know that $V(\mathbf{q})$, so

$$\frac{dV}{dt}=\sum_{s=1}^{n}\dot{q}_s\left(\frac{\partial V}{\partial q_s}\right)$$

Thus

$$\frac{dT}{dt}+\frac{dV}{dt}=0 \Rightarrow T+V=E \Rightarrow$$ **Conservation of Energy**

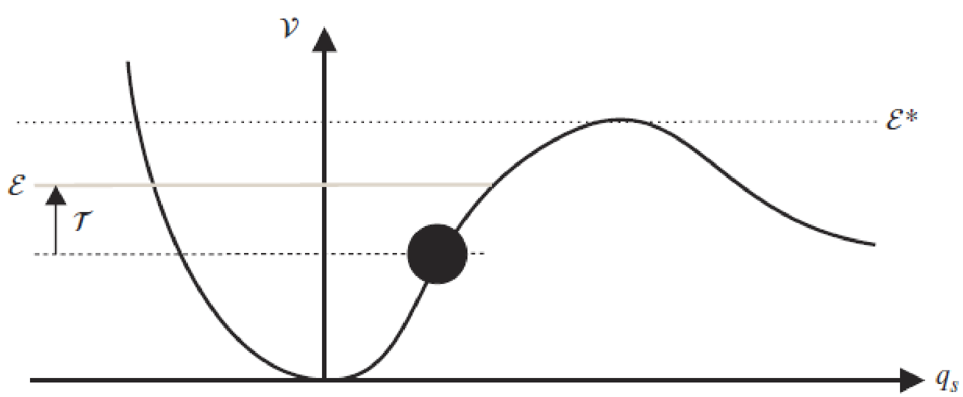
This is true for large displacements, but considering the displacements to be small and in the vicinity of an equilibrium position, the energy conservation allows to easily analyze the stability of a system.

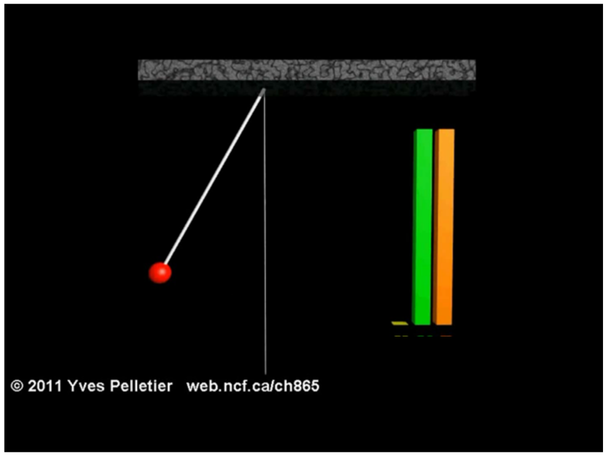
For simplicity assume that we move the origin of generalized coordinates so that, at equilibrium q_s=0, and thus at equilibrium $V(q_s=0)$ =0.

If an initial energy E is given to the system at t = 0, energy conservation at any later instant implies:

$$T + V = E$$

Therefore, the equilibrium position (q_s=0) is stable when an energy bound E^* exists such that for any energy $E < E^*$ given to the system , $T \leq E$ at any later instant, and equality occurring only at the equilibrium.





As a consequence, in the vicinity of a stable equilibrium

$$V \geq 0$$

Stable equilibrium $\Rightarrow$ Relative minimum of potential energy

Recalling

$$V(\tilde{\mathbf{q}}) = \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} k_{sr}\tilde{q}_s\tilde{q}_r = \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}\, \tilde{\mathbf{q}} > 0$$

A conservative system is stable if and only if **K** is positive definite!

NOTE: A positive definite matrix is a symmetric matrix with positive eigenvalues!

There is another way of checking the stability too:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0$$

We can look at the free motion. The above system has a general solution as:

$$\mathbf{q}(t) = \mathbf{x}\,e^{\lambda t} = \left(\mathit{Shape\ of\ the\ motion}\right)\cdot\left(\mathit{Time\ dependent\ component}\right)$$

$\mathbf{x}$ and λ are complex numbers. Hence we have an eigenvalue problem

$$\left(\lambda^2\,\mathbf{M} + \mathbf{K}\right)\mathbf{x} = 0$$

If $\mathbf{x}$=0: we are at equilibrium

If λ^2M+K =0:we have motion

$$\lambda^2 = -\frac{\mathbf{K}\left(\mathbf{Re}(\mathbf{x}) + i\,\mathbf{Im}(\mathbf{x})\right)}{\mathbf{M}\left(\mathbf{Re}(\mathbf{x}) + i\,\mathbf{Im}(\mathbf{x})\right)}$$

Let's pre-multiply numerator and denominator by complex conjugate of **x**:

$$\lambda^2 = -\frac{\left(\mathbf{Re(x)} - i\,\mathbf{Im(x)}\right)^T \mathbf{K}\left(\mathbf{Re(x)} + i\,\mathbf{Im(x)}\right)}{\left(\mathbf{Re(x)} - i\,\mathbf{Im(x)}\right)^T \mathbf{M}\left(\mathbf{Re(x)} + i\,\mathbf{Im(x)}\right)}$$

Since **K** and **M** are symmetric

$$\lambda^2 = -\boxed{\frac{\left(\mathbf{Re(x)}\right)^T \mathbf{K}\left(\mathbf{Re(x)}\right) + \left(\mathbf{Im(x)}\right)^T \mathbf{K}\left(\mathbf{Im(x)}\right)}{\left(\mathbf{Re(x)}\right)^T \mathbf{M}\left(\mathbf{Re(x)}\right) + \left(\mathbf{Im(x)}\right)^T \mathbf{M}\left(\mathbf{Im(x)}\right)}}$$

Always real and always positive because **M** and **K** are positive definite! Therefore $\lambda^2 < 0$

- **x** always real (synchronous motion)
- We can define a real ω such that $\lambda = \pm i\omega$
- This will give purely oscillatory motion and thus stable

Damped Scleronomic systems- General approach

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0$$

Again we can look at the free motion

$$\mathbf{q}(t) = \mathbf{x}\, e^{\lambda t} = \left(\mathbf{Re}(\mathbf{x}) + i\,\mathbf{Im}(\mathbf{x})\right) e^{(\mathbf{Re}(\lambda) + i\,\mathbf{Im}(\lambda))t}$$

- Different phase shifts between dofs depends on real and imaginary parts of **q**, and gives possibly non-synchronous motion
- λ is complex: response oscillates harmonically with exponentially increasing or decreasing amplitude

$$\left(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}\right)\mathbf{x} = 0$$

Free motion exists only if

$$\mathbf{det}\left(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}\right) = 0$$ (Eigenvalue problem)

if there exists an eigenvalue with a strictly positive real part, then there exist free motion displacements that are not bounded in time and the system is thus unstable.

Damped Scleronomic systems: State-space approach

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0$$

The state space representation of a system replaces an *n*th order differential equation with a single first order matrix differential equation. This converts the problem to a symmetric eigenvalue one.

$$\mathbf{z} = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{bmatrix}$$

Therefore, the equations of motion could be written as

$$\mathbf{A}\dot{\mathbf{z}} + \mathbf{B}\mathbf{z} = \mathbf{0}$$

Where

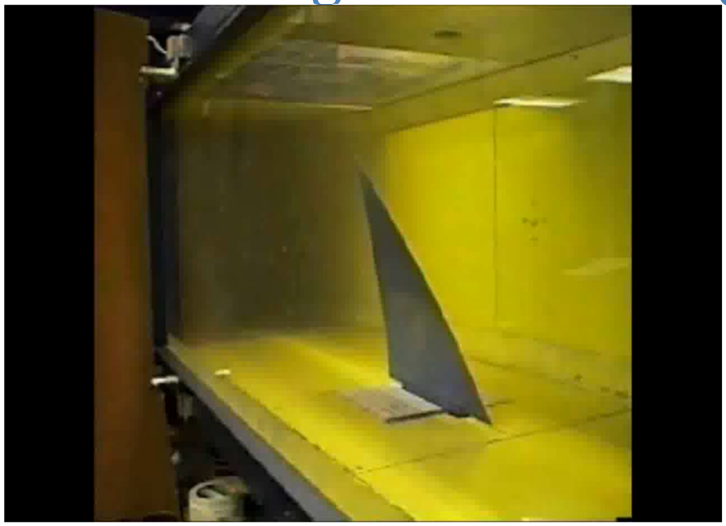
$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{pmatrix} \quad , \quad \mathbf{B} = \begin{pmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix}$$

Since $\mathbf{z} = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \lambda \mathbf{x} \\ \mathbf{x} \end{bmatrix} e^{\lambda t}$

Stability is analyzed by monitoring the sign of the real part of λ from the following eigenvalue problem

$$\mathbf{det}\left(\lambda\, \mathbf{A} + \mathbf{B}\right) = 0$$

- ✓ The system is stable if C is positive definite (The system dissipates energy)
- ✓ Free modes **x** are generally complex and eigenvalues are either complex conjugate with negative real parts (small damping) pure real negative in case of high (super-critical) damping.
- ✓ If C is negative definite (aero coupling), the system can be



Flutter instability

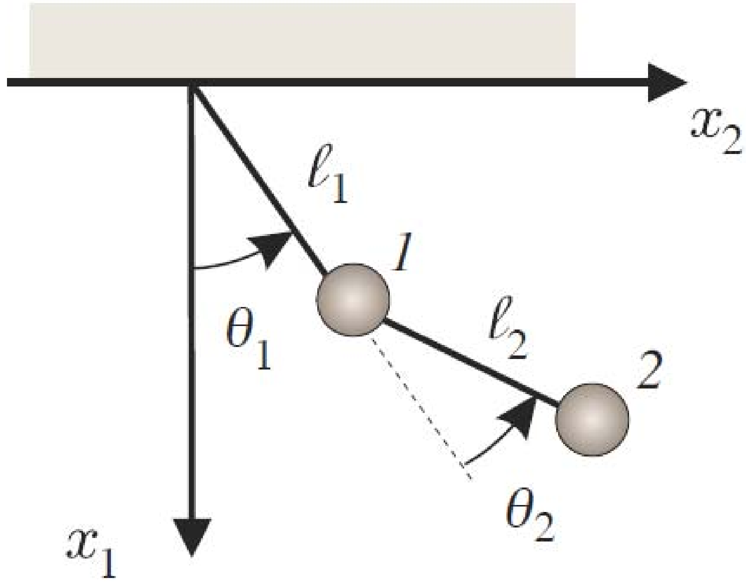
Double Pendulum

Linearized EoMs for

$\theta_1 = \theta_2 = 0$

$$\begin{pmatrix} m_1 l_1^2 + m_2\left(l_1^2 + l_2^2\right) & m_2\left(l_2^2 + l_1 l_2\right) \\ m_2\left(l_2^2 + l_1 l_2\right) & m_2 l_2^2 \end{pmatrix} \begin{Bmatrix} \ddot{\tilde{\theta}}_1 \\ \ddot{\tilde{\theta}}_2 \end{Bmatrix} +$$

$$\begin{pmatrix} \left(m_1 + m_2\right) g l_1 + m_2 g l_2 & m_2 g l_2 \\ m_2 g l_2 & m_2 g l_2 \end{pmatrix} \begin{Bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



The stability can be checked either by checking the positive definiteness of the stiffness matrix or eigenvalues of the free motion.

Here we check the positive definiteness of the K matrix

$$\det(\mathbf{K} - \lambda \mathbf{I}) = 0 \Rightarrow$$ Should have positive eigenvalues

$$\textbf{det}\left|\begin{pmatrix}(m_1+m_2)gl_1+m_2gl_2 & m_2gl_2\\ m_2gl_2 & m_2gl_2\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=0$$

Taking the determinant gives the following characteristic polynomial

$$\left((m_1+m_2)gl_1+m_2gl_2-\lambda\right)(m_2gl_2-\lambda)-(m_2gl_2)^2=0$$

$$or$$

$$\lambda^2-\left((m_1+m_2)gl_1+2m_2gl_2\right)\lambda+(m_1+m_2)m_2g^2l_1l_2=0$$

Assuming $\alpha=\left((m_1+m_2)gl_1+2m_2gl_2\right)$

$$\lambda=\frac{\alpha\pm\sqrt{\alpha^2-4(m_1+m_2)m_2g^2l_1l_2}}{2}$$

Which are positive and therefore the system is stable around equilibrium.

Equilibrium position corresponding to $\theta_1=\pi,\ \theta_2=0$

$$\mathbf{K}=\begin{pmatrix} -(m_1+m_2)gl_1+m_2gl_2 & -m_2gl_2 \\ -m_2gl_2 & -m_2gl_2 \end{pmatrix}$$

Stiffness matrix=- stiffness of previous case

So we will have two negative eigenvalues and thus the system is unstable around $\theta_1=\pi,\ \theta_2=0$ *(degree of instability equal to 2)*

Equilibrium position corresponding to $\theta_1=\pi,\ \theta_2=\pi$

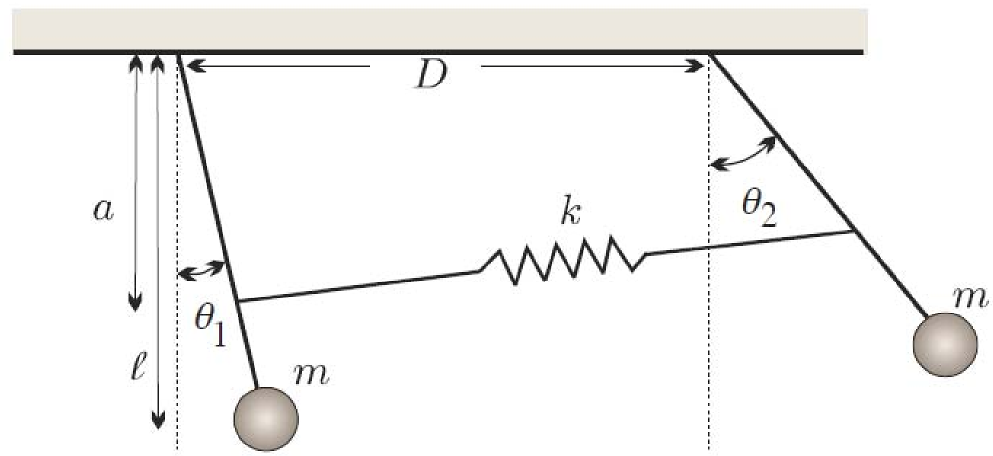
$$\mathbf{K}=\begin{pmatrix} -(m_1+m_2)gl_1+m_2gl_2 & m_2gl_2 \\ m_2gl_2 & m_2gl_2 \end{pmatrix}$$

$$\lambda^2+\underbrace{\left((m_1+m_2)gl_1-2m_2gl_2\right)}_{\beta}\lambda-(m_1+m_2)m_2g^2l_1l_2=0$$

$$\lambda=\frac{-\beta\pm\sqrt{\beta^2+4(m_1+m_2)m_2g^2l_1l_2}}{2}$$

One positive and one negative eigenvalues; degree of instability is 1.

The coupled pendulum



Linearized EoMs for

$\theta_1 = \theta_2 = 0$

$$\mathbf{M} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} mgl + ka^2 & -ka^2 \\ -ka^2 & mgl + ka^2 \end{pmatrix}$$

$$\det\left(\lambda^2 \mathbf{M} + \mathbf{K}\right) = 0 \xrightarrow{\alpha^2 = \frac{ka^2}{ml^2}} \left(\lambda^2 + \frac{g}{l} + \alpha^2\right)^2 - \alpha^4 = 0$$

$$\left(\left(\lambda^2 + \frac{g}{l} + \alpha^2\right) - \alpha^2\right)\left(\left(\lambda^2 + \frac{g}{l} + \alpha^2\right) + \alpha^2\right) = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{g}{l}},\ \lambda = \pm i\sqrt{\frac{g}{l} + 2\alpha^2}$$

Hence the free motions are oscillations of frequencies equal to the frequency of the single uncoupled pendulum or of frequency equal to $\sqrt{\frac{g}{l} + 2\alpha^2}$

Rheonomic systems

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} + \mathbf{G})\dot{\mathbf{q}} + \mathbf{K}^{*}\mathbf{q} = 0$$

K* is the modified potential and includes transport kinetic energy, and G is the gyroscopic matrix. Similar to the damped scleronomic systems we can write:

$$\left(\lambda^{2}\mathbf{M} + \lambda(\mathbf{C}+\mathbf{G}) + \mathbf{K}^{*}\right)\mathbf{x} = 0$$

Or in state space form

$$\mathbf{z} = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{bmatrix}$$

$$\tilde{\mathbf{A}}\dot{\mathbf{z}} + \tilde{\mathbf{B}}\mathbf{z} = \mathbf{0}$$

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{*} \end{pmatrix} \quad , \quad \tilde{\mathbf{B}} = \begin{pmatrix} \mathbf{G}+\mathbf{C} & \mathbf{K}^{*} \\ -\mathbf{K}^{*} & \mathbf{0} \end{pmatrix} \quad ,$$

$$\tilde{\mathbf{A}} = symmetric \;,\; \tilde{\mathbf{B}} = skew\;\; symmetric\;\; if\;\; \mathbf{C} = 0$$

And the eigenvalue problem will convert to:

$$\left(\lambda\,\tilde{\mathbf{A}}+\tilde{\mathbf{B}}\right)\mathbf{x}=0$$

As usual if **Re(λ)>0** then there exists an exponential motion that grows in time and thus the system is unstable.

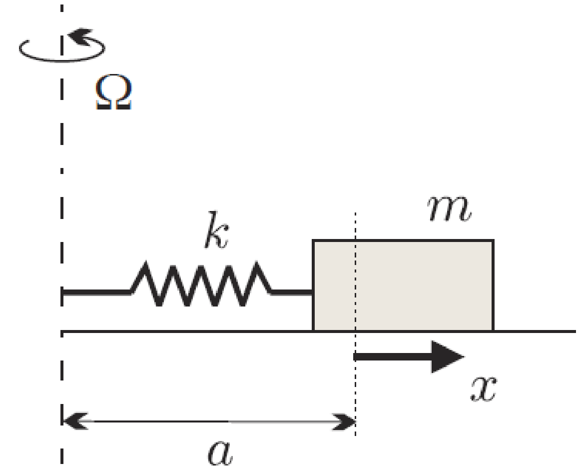
Major difference with scleronomic systems

Due to the presence of transport kinetic energy there might be a possibility that K* becomes negative definite. However, the gyroscopic terms ($G\dot{x}$) can stabilize the system even if K* is not positive definite.

Also it can be shown that the presence of damping matrix can stabilize or destablize the system.

Rotating mass

Equilibrium was obtained at the point were centrifugal and elastic forces balance



Linearized equation:

$$m\ddot{\tilde{x}} + \left(k - m\Omega^2\right)\tilde{x} = 0$$

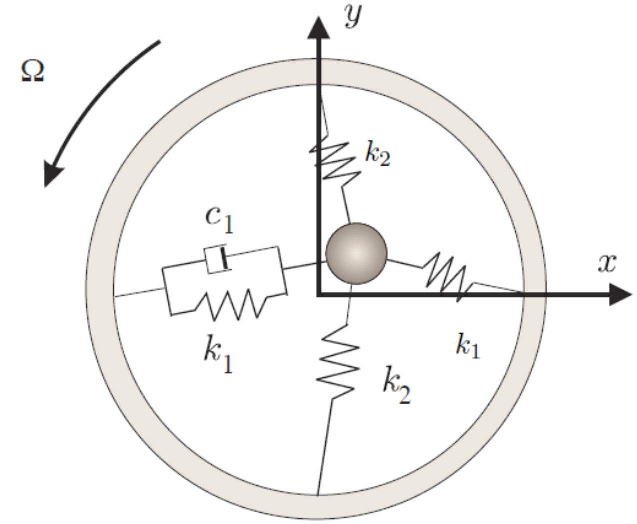
There are no gyroscopic forces, since there is only 1 dof.

$$\lambda^2 m + \underbrace{\left(k - m\Omega^2\right)}_{K^*} = 0 \Rightarrow \lambda^2 = -\frac{k - m\Omega^2}{m}$$

- ✓ As long as $\Omega^2 < k/m$, K^* is definite and the system is stable since λ will be purely imaginary. Once $\Omega^2 > k/m$, K^* is negative, λ will be positive , and the system becomes unstable.

A two-dimensional rotating system

We found linearized matrices before for equilibrium at (0,0):



$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \quad ,$$

$$\mathbf{G} = \begin{bmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{bmatrix} \quad , \quad \mathbf{K^*} = \begin{bmatrix} -m\Omega^2 + 2k_1 & 0 \\ 0 & -m\Omega^2 + 2k_2 \end{bmatrix}$$

We should solve the following eigenvalue problem

$$\mathbf{det}\left(\lambda\,\tilde{\mathbf{A}} + \tilde{\mathbf{B}}\right) = 0$$

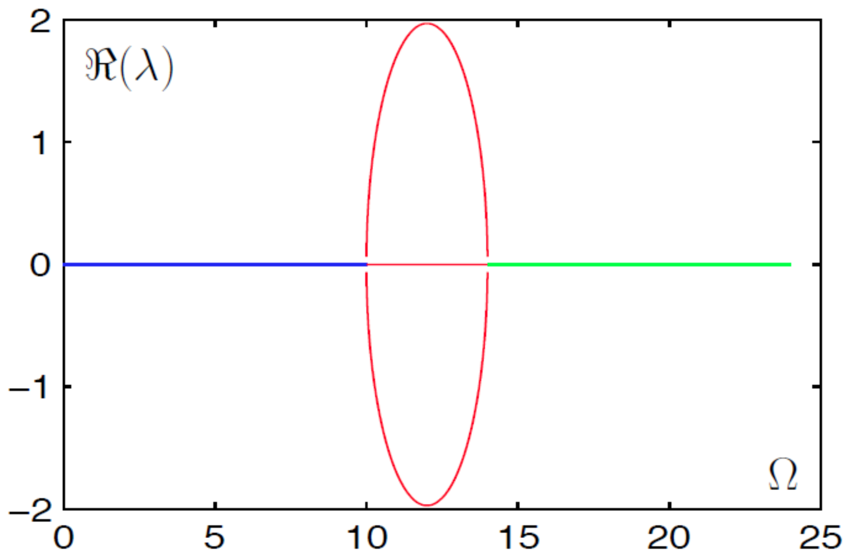
$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K^*} \end{pmatrix} \quad , \quad \tilde{\mathbf{B}} = \begin{pmatrix} \mathbf{G+C} & \mathbf{K^*} \\ \mathbf{-K^*} & \mathbf{0} \end{pmatrix} \quad ,$$

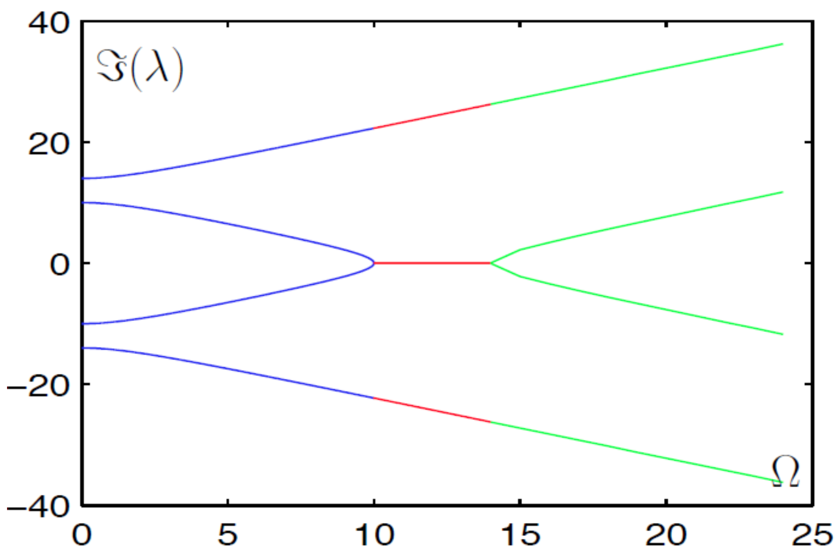
Lets consider the following values: m=1, k_1=50, k_2=98 and c_1=0 (See the Matlab example)

For this case:

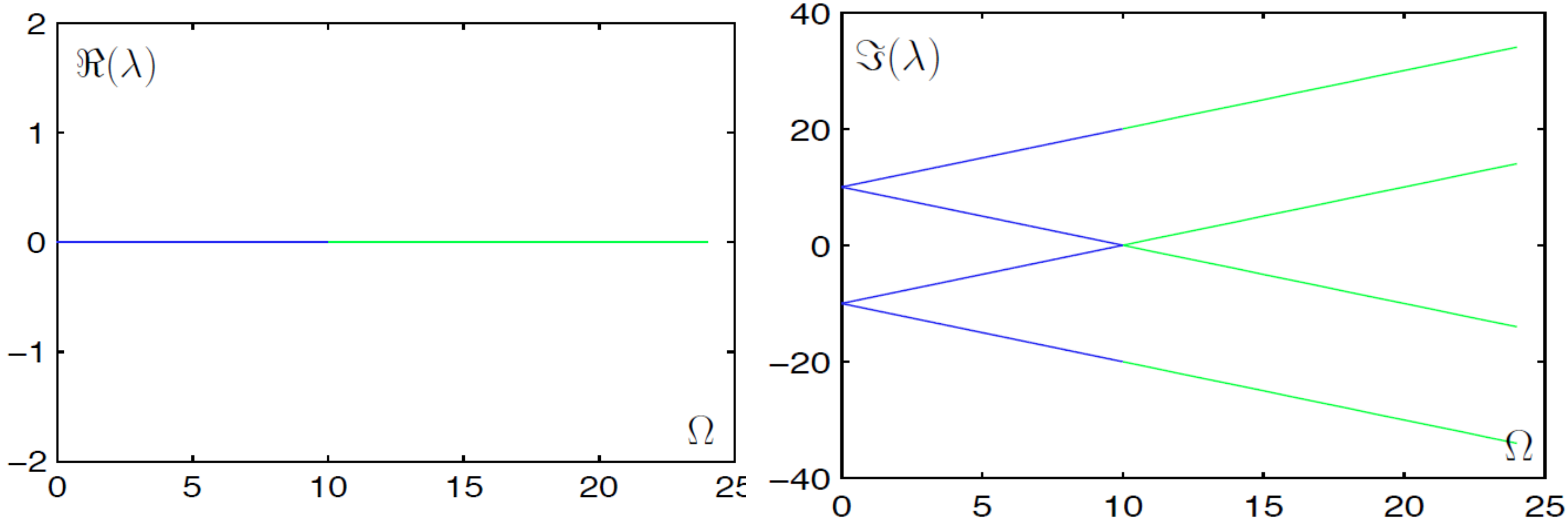
$$\omega_1 = \sqrt{\frac{2k_1}{m}}\ : in\ x\ direction \quad , \quad \omega_2 = \sqrt{\frac{2k_2}{m}}\ : in\ y\ direction$$

- ✓ If $0<\Omega<\omega_1$, K* is positive definite.
- ✓ If $\omega_1<\Omega<\omega_2$, K* has one strictly negative eigenvalue.
- ✓ If $\Omega>\omega_2$, K* has two negative eigenvalues and thus is negative definite.



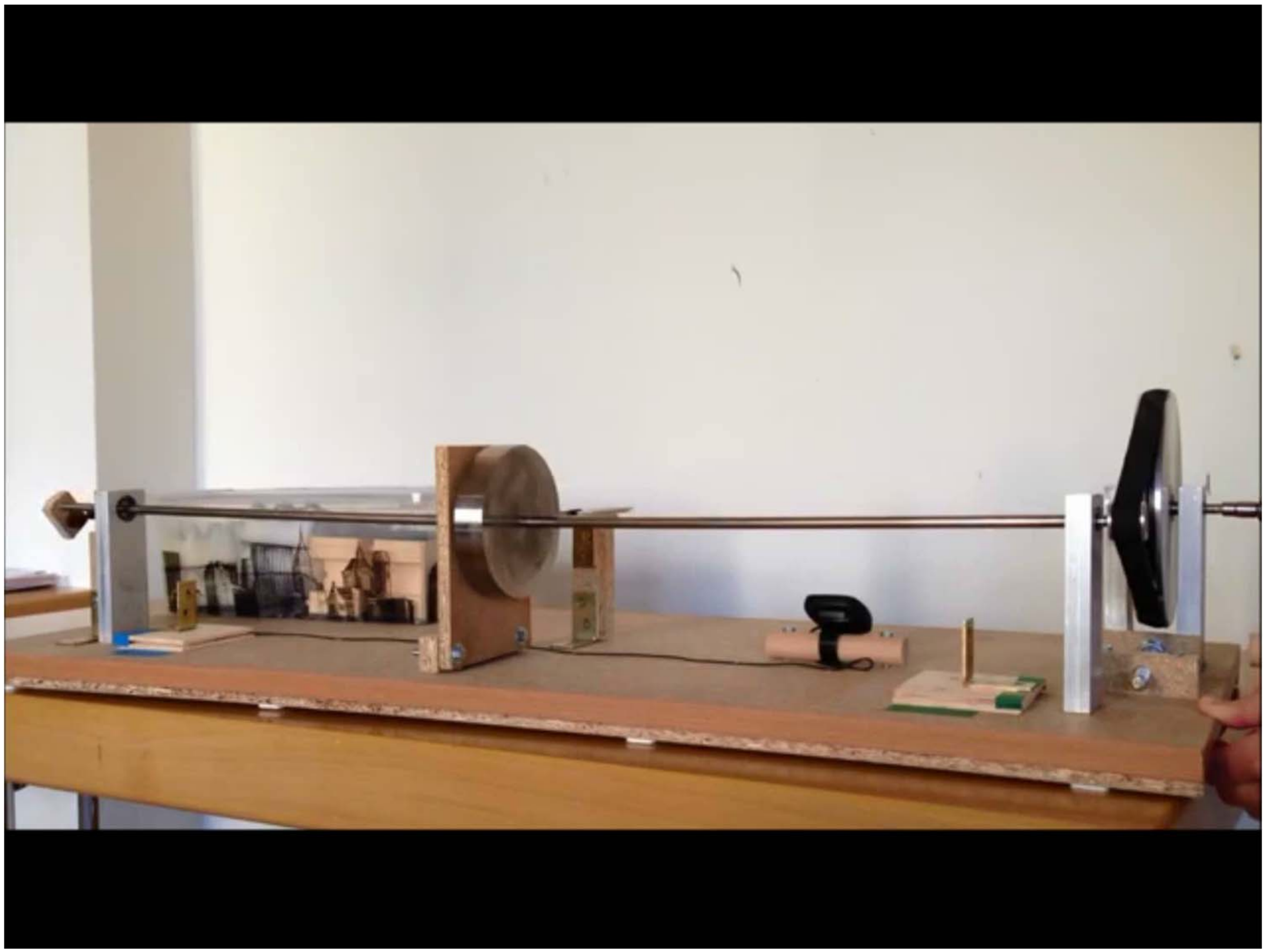


Now suppose: m=1, k_1=50, k_2=50



The system is stable for every rotation speed.

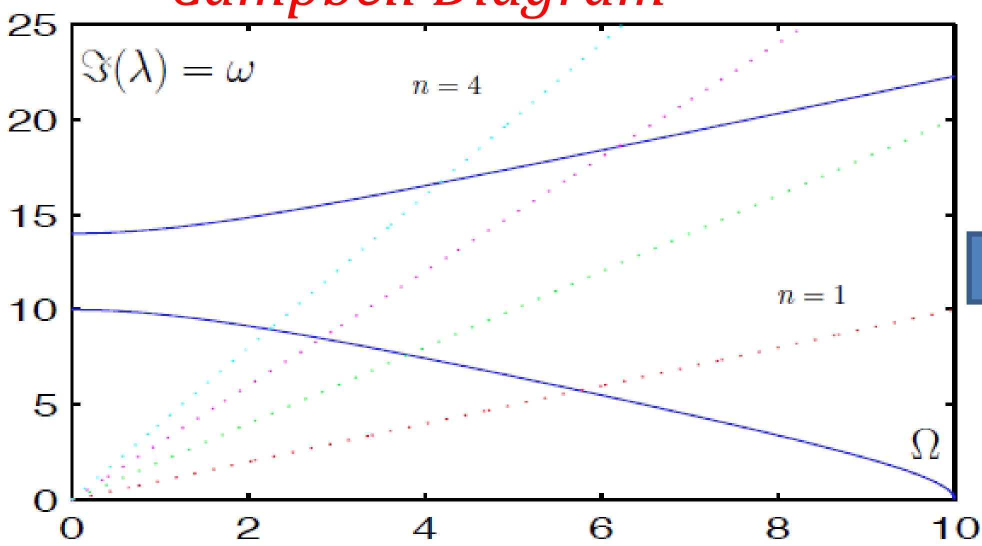
Similar dynamic behavior could be seen in rotating machinery mounted on a shaft, where slight imbalances could cause undesirable motions. In this case at certain speeds, the presence of gyroscopic forces could balance the motion.



Also, in nearly every practical application, rotation systems are subjected to external excitations that have a frequency equal to n times the rotation speed (n is sometimes called the engine order).

✓ if M bearing elements are placed around the rotating system, the corresponding excitation support forces will generate harmonic forces in the rotating frame with frequency $M\Omega$. These frequencies should be avoided in order not to have instability.

Campbell Diagram



Points where engine order lines intersect with frequency curves are resonance!