Equilibrium & Linearization

WB 1418-07

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- ✓ We saw that even the simplest systems are non-linear in nature.
- ✓ In many practical applications there might not be a need to understand the global dynamics of the system; instead we would be interested in small motions around certain points that we call equilibrium.
- ✓ What is an equilibrium point?
- ✓ We can linearize the non-linear equations of motion having a priori knowledge of equilibrium points and thus facilitate dynamic analyses.
- ✓ The concept of equilibrium automatically leads to the concept of stability .
- ✓ Branch of dynamics that deals with motions around equilibrium is known as vibrations

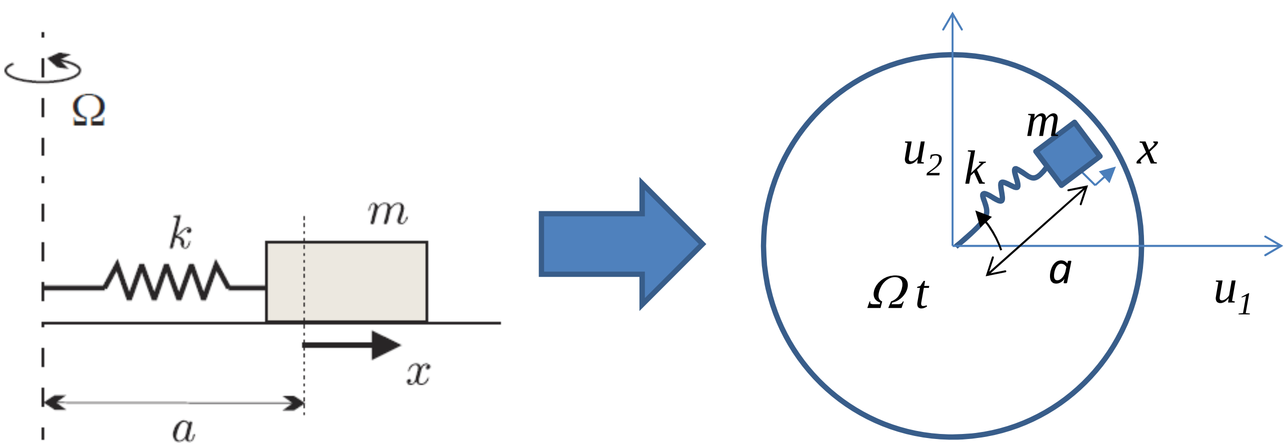
What is equilibrium in scleronomic and rheonomic systems ?

In scleronomic systems equilibrium is when the system is at rest

$$\mathbf{q}(t)=\mathbf{q}(t=0)=\mathbf{q}_{eq} \quad \textit{and thus} \quad \dot{\mathbf{q}}(t)=\ddot{\mathbf{q}}(t)=0 \ \ \forall t$$

$$\textit{where} \ \ \mathbf{q}(t)=\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix}$$

In rheonomic systems the story is different. These system can also have equilibrium(certain point at which centrifugal and elastic forces balance one another)



$$u_1-(x+a)\cos\Omega t=0$$

$$u_2-(x+a)\sin\Omega t=0$$

Finding Equilibrium from Lagrange equations

$$f_s = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} - Q_s^{ncons} = 0 \quad , \quad s = 1, \ldots, n$$

In matrix notation (Compact way of writing)

$$\mathbf{f} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} - Q_s^{ncons} = 0$$

In which

$$\frac{\partial}{\partial \dot{\mathbf{q}}} = \begin{Bmatrix} \dfrac{\partial}{\partial \dot{q}_1} \\ \dfrac{\partial}{\partial \dot{q}_2} \\ \vdots \\ \dfrac{\partial}{\partial \dot{q}_n} \end{Bmatrix} \quad , \quad \frac{\partial}{\partial \mathbf{q}} = \begin{Bmatrix} \dfrac{\partial}{\partial q_1} \\ \dfrac{\partial}{\partial q_2} \\ \vdots \\ \dfrac{\partial}{\partial q_n} \end{Bmatrix}$$

At equilibrium

$$\mathbf{f}\left(\ddot{\mathbf{q}}=0,\dot{\mathbf{q}}=0,\mathbf{q}=\mathbf{q}_{eq}\right)=0$$

Equilibrium equations are generally non-linear and thus typically have many solutions. Some would be logical and some not. A nice numerical algorithm for solving equilibrium equations is Newton-Raphson method.

Now suppose that we are able to solve equilibrium equations and want to look at dynamics around equilibrium

$$\mathbf{q}=\mathbf{q}_{eq}+\tilde{\mathbf{q}}\ ,\quad \dot{\mathbf{q}}=\dot{\tilde{\mathbf{q}}},\quad \ddot{\mathbf{q}}=\ddot{\tilde{\mathbf{q}}},$$

So we can linearize equations as follows using Taylor expansion

$$\mathbf{f}\left(\ddot{\mathbf{q}}=\ddot{\tilde{\mathbf{q}}},\dot{\mathbf{q}}=\dot{\tilde{\mathbf{q}}},\mathbf{q}=\mathbf{q}_{eq}+\tilde{\mathbf{q}}\right)\simeq\mathbf{f}\left(\ddot{\mathbf{q}}=0,\dot{\mathbf{q}}=0,\mathbf{q}=\mathbf{q}_{eq}\right)$$

$$+\left.\frac{\partial\mathbf{f}}{\partial\ddot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}\ddot{\tilde{\mathbf{q}}}+\left.\frac{\partial\mathbf{f}}{\partial\dot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}\dot{\tilde{\mathbf{q}}}+\left.\frac{\partial\mathbf{f}}{\partial\mathbf{q}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}\tilde{\mathbf{q}}=0$$

We can write previous equations in the following form

$$\mathbf{M}\ddot{\tilde{\mathbf{q}}}+\mathbf{C}\dot{\tilde{\mathbf{q}}}+\mathbf{K}\tilde{\mathbf{q}}=0$$

Mass matrix *Damping matrix* *Stiffness matrix*

Where

$$\mathbf{M}=\left.\frac{\partial\,\mathbf{f}}{\partial\,\ddot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}} \quad , \quad \mathbf{C}=\left.\frac{\partial\,\mathbf{f}}{\partial\,\dot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}} \quad \mathbf{K}=\left.\frac{\partial\,\mathbf{f}}{\partial\,\mathbf{q}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}$$

Or alternatively

$$m_{sr}=\left.\frac{\partial f_s}{\partial \ddot{q}_r}\right|_{\substack{\ddot{q}=0\\ \dot{q}=0\\ q=q_{eq}}} \quad , \quad c_{sr}=\left.\frac{\partial f_s}{\partial \dot{q}_r}\right|_{\substack{\ddot{q}=0\\ \dot{q}=0\\ q=q_{eq}}} \quad , \quad k_{sr}=\left.\frac{\partial f_s}{\partial q_r}\right|_{\substack{\ddot{q}=0\\ \dot{q}=0\\ q=q_{eq}}}$$

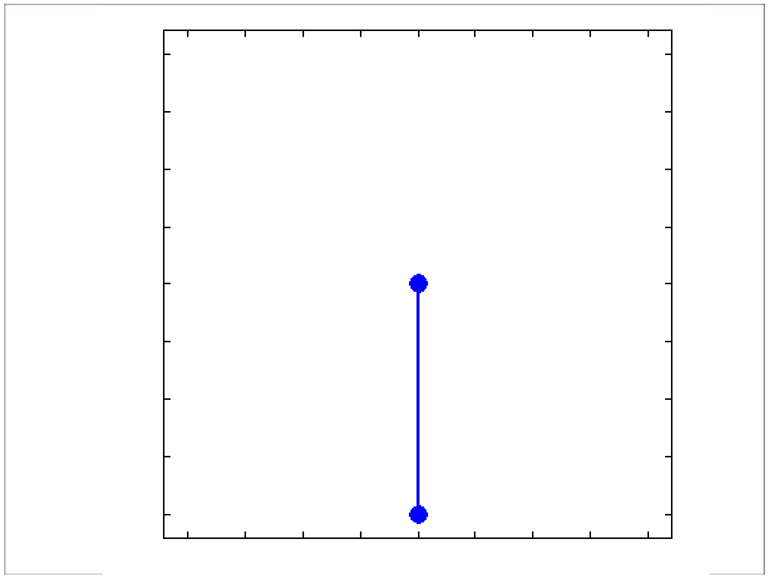
In the pendulum case

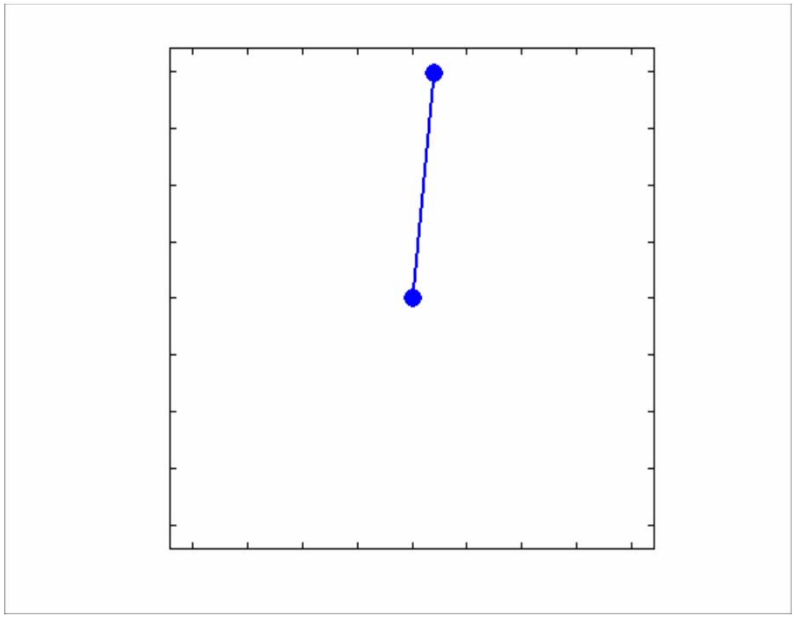
$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \quad ,$$

$$\mathbf{f}\left(\ddot{\mathbf{q}} = 0, \dot{\mathbf{q}} = 0, \mathbf{q} = \mathbf{q}_{eq}\right) = 0 \qquad \sin\theta_{eq} = 0 \Rightarrow \begin{cases} \theta_{eq} = 0 \\ \theta_{eq} = \pi \end{cases}$$

$$\ddot{\tilde{\theta}} + \frac{g}{l}\cos\theta\Big|_{\theta=0}\tilde{\theta} = 0 \Rightarrow \ddot{\tilde{\theta}} + \frac{g}{l}\tilde{\theta} = 0$$

$$\ddot{\tilde{\theta}} + \frac{g}{l}\cos\theta\Big|_{\theta=\pi}\tilde{\theta} = 0 \Rightarrow \ddot{\tilde{\theta}} - \frac{g}{l}\tilde{\theta} = 0$$





The equilibrium can also be directly obtained from energies

$$\mathbf{f}_T\left(\ddot{\mathbf{q}},\dot{\mathbf{q}},\mathbf{q}\right)=\frac{d}{dt}\left(\frac{\partial T\left(\mathbf{q},\dot{\mathbf{q}}\right)}{\partial\,\dot{\mathbf{q}}}\right)-\frac{\partial T\left(\mathbf{q},\dot{\mathbf{q}}\right)}{\partial\,\mathbf{q}} \qquad \mathbf{f}_V\left(\mathbf{q}\right)=\frac{\partial V}{\partial\,\mathbf{q}}$$

Lagrange equations

$$\mathbf{f}_T\left(\ddot{\mathbf{q}},\dot{\mathbf{q}},\mathbf{q}\right)+\mathbf{f}_V\left(\mathbf{q}\right)+\mathbf{Q}^{n\cos}\left(\dot{\mathbf{q}},\mathbf{q}\right)=0$$

$$\mathbf{f}_T\left(\ddot{\mathbf{q}}=0,\dot{\mathbf{q}}=0,\mathbf{q}=\mathbf{q}_{eq}\right)+\mathbf{f}_V\left(\mathbf{q}=\mathbf{q}_{eq}\right)+\mathbf{Q}^{n\cos}\left(\dot{\mathbf{q}}=0,\mathbf{q}=\mathbf{q}_{eq}\right)=0$$

$$\mathbf{M}=\left.\frac{\partial\,\mathbf{f}_T}{\partial\,\ddot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}},$$

$$\mathbf{C}=\left.\frac{\partial\left(\mathbf{f}_T+\mathbf{Q}^{ncons}\right)}{\partial\,\dot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}},$$

$$\mathbf{K}=\left.\frac{\partial\left(\mathbf{f}_T+\mathbf{f}_V+\mathbf{Q}^{ncons}\right)}{\partial\,\mathbf{q}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}$$

Scleronomic systems

1. Equilibrium

Note that the mapping U_{ik} does not explicitly depend on time

$$u_{ik} = U_{ik}\left(q_1, \ldots, q_n\right)$$

$$\dot{u}_{ik} = \sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial q_s} \dot{q}_s$$

Or in matrix form

$$\dot{\mathbf{u}}_k = \frac{\partial \mathbf{U}_k}{\partial \mathbf{q}} \dot{\mathbf{q}} \quad , \quad \dot{\mathbf{u}}_k = \begin{Bmatrix} \dot{u}_{1k} \\ \dot{u}_{2k} \\ \dot{u}_{3k} \end{Bmatrix} \qquad \text{(}\mathit{Virtual\ velocities}\text{)}$$

So the kinetic energy could be written as follows:

$$T = \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \dot{u}_{ik}^2 = \frac{1}{2} \sum_{k=1}^{N} m_k \, \dot{\mathbf{u}}_k^T \cdot \dot{\mathbf{u}}_k = \frac{1}{2} \sum_{k=1}^{N} m_k \, \dot{\mathbf{q}}^T \left(\frac{\partial \mathbf{U}_k}{\partial \mathbf{q}} \right)^T \cdot \frac{\partial \mathbf{U}_k}{\partial \mathbf{q}} \dot{\mathbf{q}}$$

The kinetic energy is always a quadratic function of $\dot{\mathbf{q}}$

$$\mathbf{f}_T\Big|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}} = \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}}\right)_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}} = 0$$

Therefore:

$$\mathbf{f}_V\Big|_{\mathbf{q}=\mathbf{q}_{eq}} = \left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} = 0$$

So in order to find the equilibrium equations in case of scleronomic constraints one would need to take the derivative of the potential energy with respect to the generalized coordinates only and set it equal to zero.

Scleronomic systems

1. Linearized equations

The stiffness matrix can be obtained as

$$\mathbf{K} = \cancel{\left.\frac{\partial(\mathbf{f}_T)}{\partial \mathbf{q}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}} + \left.\frac{\partial(\mathbf{f}_V)}{\partial \mathbf{q}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}$$

Because f_T is quadratic in $\dot{q}$

Recalling $\mathbf{f}_V(\mathbf{q}) = \dfrac{\partial V}{\partial \mathbf{q}}$

$$\boxed{\mathbf{K} = \left.\frac{\partial^2 V}{\partial \mathbf{q}\,\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}_{eq}} \quad , \quad k_{rs} = k_{sr} = \left(\frac{\partial^2 V}{\partial q_s \partial q_r}\right)}$$

The stiffness matrix is symmetric since you can sweep the order of derivatives.

Moreover, around certain equilibrium the potential can be approximated using stiffness matrix:

$$V(\tilde{\mathbf{q}}) = \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} k_{sr}\tilde{q}_s\tilde{q}_r = \frac{1}{2}\tilde{\mathbf{q}}^T\,\mathbf{K}\,\tilde{\mathbf{q}}$$

Since kinetic energy is quadratic in generalized velocities, the damping matrix is

$$\boxed{\mathbf{C}=\left.\frac{\partial(\mathbf{f}_T)}{\partial\dot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}=0}$$

Therefore, no damping matrix exists for scleronomic systems if there are no non-conservative forces acting on the system.

And for the mass matrix

$$\mathbf{f}_T=\left(\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\mathbf{q}}}\right)-\frac{\partial T}{\partial\mathbf{q}}\right)=\sum_{k=1}^{N}m_k\frac{d}{dt}\left(\left(\frac{\partial\mathbf{U}_k}{\partial\mathbf{q}}\right)^T\frac{\partial\mathbf{U}_k}{\partial\mathbf{q}}\right)\dot{\mathbf{q}}$$

$$+\sum_{k=1}^{N}m_k\left(\frac{\partial\mathbf{U}_k}{\partial\mathbf{q}}\right)^T\frac{\partial\mathbf{U}_k}{\partial\mathbf{q}}\ddot{\mathbf{q}}$$

$$-\frac{\partial}{\partial\mathbf{q}}\left(\frac{1}{2}\sum_{k=1}^{N}m_k\dot{\mathbf{q}}^T\left(\frac{\partial\mathbf{U}_k}{\partial\mathbf{q}}\right)^T\frac{\partial\mathbf{U}_k}{\partial\mathbf{q}}\dot{\mathbf{q}}\right)$$

Since $\mathbf{M} = \left.\dfrac{\partial(\mathbf{f}_T)}{\partial \ddot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}} = \left.\sum_{k=1}^{N} m_k \left(\dfrac{\partial \mathbf{U}_k}{\partial \mathbf{q}}\right)^T \dfrac{\partial \mathbf{U}_k}{\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}_{eq}}$

By observing that the above is the coefficients of the quadratic form of the kinetic energy, we can write:

$$\boxed{\mathbf{M} = \left.\frac{\partial^2 T}{\partial \dot{\mathbf{q}}\,\partial \dot{\mathbf{q}}}\right|_{\mathbf{q}=\mathbf{q}_{eq}}}$$

Or component wise

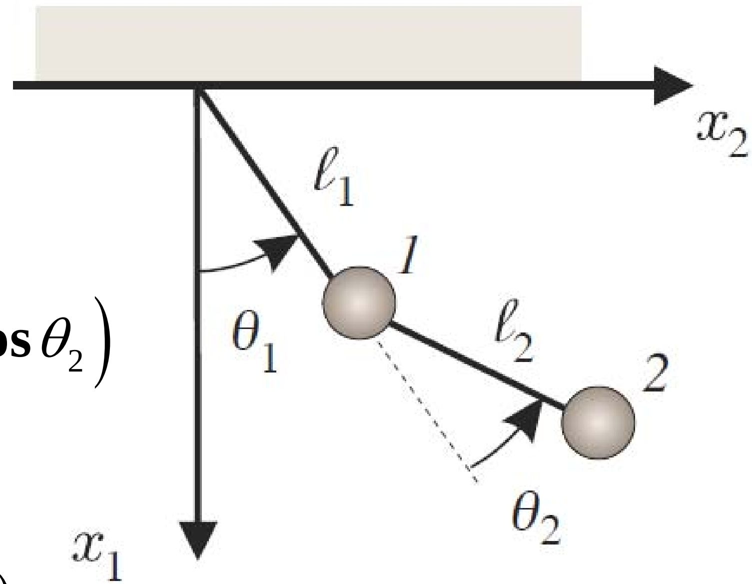
$$\boxed{m_{sr} = m_{rs} = \left(\frac{\partial^2 T}{\partial \dot{q}_s \partial \dot{q}_r}\right)_{q=q_{eq}} = \sum_{k=1}^{N}\sum_{i=1}^{3} m_k \left(\frac{\partial U_{ik}}{\partial q_s}\right)_{q=q_{eq}} \left(\frac{\partial U_{ik}}{\partial q_r}\right)_{q=q_{eq}}}$$

The kinetic energy around certain equilibrium can be approximated using mass matrix as:

$$T(\dot{\tilde{\mathbf{q}}}) = \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} m_{sr}\dot{\tilde{q}}_s \dot{\tilde{q}}_r = \frac{1}{2}\dot{\tilde{\mathbf{q}}}^T \mathbf{M}\,\dot{\tilde{\mathbf{q}}} > 0 \quad for \;\; \dot{\tilde{\mathbf{q}}} \neq 0$$

Where the mass matrix is symmetric and positive definite.

Double Pendulum



$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2$$

$$+ \frac{1}{2} m_2 l_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2} m_2 \left(2 l_1 l_2 \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2\right) \cos\theta_2\right)$$

$$V = m_1 g l_1 \left(1 - \cos\theta_1\right)$$

$$+ m_2 g l_1 \left(1 - \cos\theta_1\right) + m_2 g l_2 \left(1 - \cos\left(\theta_1 + \theta_2\right)\right)$$

To find equilibrium we don't have to bother looking at kinetic energy since the system is scleronomic.

$$Equilibrium: \frac{\partial V}{\partial q_s} = 0 \ , \quad s = 1, 2$$

$$\frac{\partial V}{\partial \theta_1} = 0 \qquad\qquad \frac{\partial V}{\partial \theta_2} = 0$$

As a result

$$\frac{\partial V}{\partial \theta_1} = \left(m_1 g l_1 + m_2 g l_1\right)\mathbf{sin}\,\theta_1 + m_2 g l_2\,\mathbf{sin}\left(\theta_1 + \theta_2\right) = 0$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g l_2\,\mathbf{sin}\left(\theta_1 + \theta_2\right) = 0$$

How many equilibrium positions we may have:

$$\theta_1 = 0\ \mathbf{,}\,\pi \quad \mathbf{,}\ \theta_2 = 0\ \mathbf{,}\,\pi$$

Now for each set of equilibrium we can have a linearized system.
Suppose for $\theta_1 = \theta_2 = 0$

$$\mathbf{K} = \left.\frac{\partial^2 V}{\partial \mathbf{q}\,\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}_{eq}} = \begin{pmatrix} \dfrac{\partial^2 V}{\partial q_1 \partial q_1} & \dfrac{\partial^2 V}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & \dfrac{\partial^2 V}{\partial q_2 \partial q_2} \end{pmatrix}_{q_1=q_2=0}$$

$$= \begin{pmatrix} \left(m_1 g l_1 + m_2 g l_1\right)\mathbf{cos}\,\theta_1 + m_2 g l_2\,\mathbf{cos}\left(\theta_1+\theta_2\right) & m_2 g l_2\,\mathbf{cos}\left(\theta_1+\theta_2\right) \\ m_2 g l_2\,\mathbf{cos}\left(\theta_1+\theta_2\right) & m_2 g l_2\,\mathbf{cos}\left(\theta_1+\theta_2\right) \end{pmatrix}_{\theta_1=\theta_2=0}$$

Stiffness matrix $\Rightarrow$ $$\mathbf{K}=\begin{pmatrix}(m_1+m_2)gl_1+m_2gl_2 & m_2gl_2\\ m_2gl_2 & m_2gl_2\end{pmatrix}$$

Mass matrix $\Rightarrow$ $$\mathbf{M}=\left.\frac{\partial^2 T}{\partial\dot{\mathbf{q}}\partial\dot{\mathbf{q}}}\right|_{\mathbf{q}=\mathbf{q}_{eq}}=\begin{pmatrix}\dfrac{\partial^2 T}{\partial\dot{q}_1\partial\dot{q}_1} & \dfrac{\partial^2 T}{\partial\dot{q}_1\partial\dot{q}_2}\\ \dfrac{\partial^2 T}{\partial\dot{q}_1\partial\dot{q}_2} & \dfrac{\partial^2 T}{\partial\dot{q}_2\partial\dot{q}_2}\end{pmatrix}_{q_1=q_2=0}$$

$$=\begin{pmatrix}m_1l_1^2+m_2\left(l_1^2+l_2^2\right) & m_2\left(l_2^2+l_1l_2\right)\\ m_2\left(l_2^2+l_1l_2\right) & m_2l_2^2\end{pmatrix}$$

Linearized EoM for $\theta_1=\theta_2=0$ $\Rightarrow$ $$\begin{pmatrix}m_1l_1^2+m_2\left(l_1^2+l_2^2\right) & m_2\left(l_2^2+l_1l_2\right)\\ m_2\left(l_2^2+l_1l_2\right) & m_2l_2^2\end{pmatrix}\begin{Bmatrix}\ddot{\tilde{\theta}}_1\\ \ddot{\tilde{\theta}}_2\end{Bmatrix}+$$

$$\begin{pmatrix}(m_1+m_2)gl_1+m_2gl_2 & m_2gl_2\\ m_2gl_2 & m_2gl_2\end{pmatrix}\begin{Bmatrix}\tilde{\theta}_1\\ \tilde{\theta}_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Linearized EoM for $\theta_1 = \pi$, $\theta_2 = 0$

$$\begin{pmatrix} m_1 l_1^2 + m_2\left(l_1^2 + l_2^2\right) & m_2\left(l_2^2 + l_1 l_2\right) \\ m_2\left(l_2^2 + l_1 l_2\right) & m_2 l_2^2 \end{pmatrix} \begin{Bmatrix} \ddot{\tilde{\theta}}_1 \\ \ddot{\tilde{\theta}}_2 \end{Bmatrix} + \begin{pmatrix} -\left(m_1 + m_2\right) g l_1 + m_2 g l_2 & -m_2 g l_2 \\ -m_2 g l_2 & -m_2 g l_2 \end{pmatrix} \begin{Bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

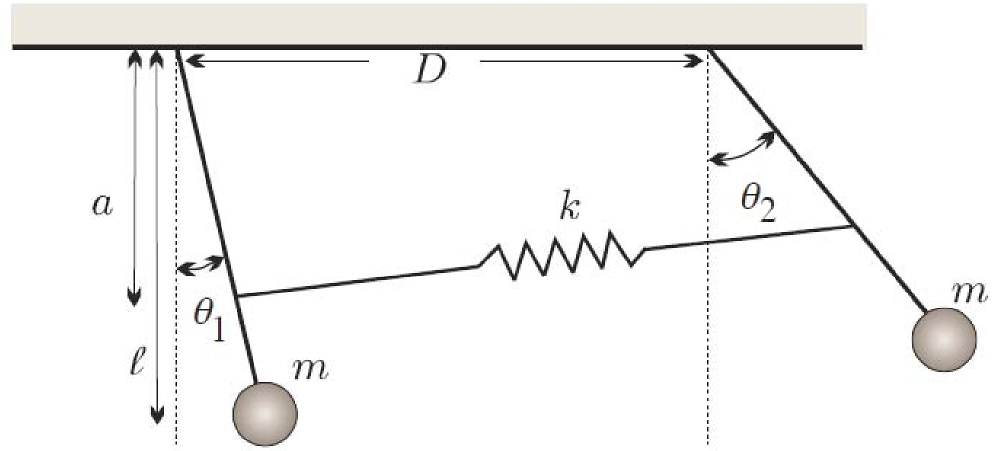
Linearized EoM for $\theta_1 = \pi$, $\theta_2 = \pi$

$$\begin{pmatrix} m_1 l_1^2 + m_2\left(l_1^2 - l_2^2\right) & m_2\left(l_2^2 - l_1 l_2\right) \\ m_2\left(l_2^2 - l_1 l_2\right) & m_2 l_2^2 \end{pmatrix} \begin{Bmatrix} \ddot{\tilde{\theta}}_1 \\ \ddot{\tilde{\theta}}_2 \end{Bmatrix} + \begin{pmatrix} -\left(m_1 + m_2\right) g l_1 + m_2 g l_2 & m_2 g l_2 \\ m_2 g l_2 & m_2 g l_2 \end{pmatrix} \begin{Bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The coupled pendulum

$$T = \frac{1}{2} m l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 \right)$$

$$V = mgl\left(1 - \cos\theta_1\right)$$
$$+ mgl\left(1 - \cos\theta_2\right) + \frac{1}{2} k \zeta^2$$



Where

$$\zeta\left(\theta_1, \theta_2\right) = L - D$$

$$L\left(\theta_1, \theta_2\right) = \sqrt{\left(a\cos\theta_2 - a\cos\theta_1\right)^2 + \left(a\sin\theta_2 - a\sin\theta_1 + D\right)^2}$$

Equilibrium will be

$$\frac{\partial V}{\partial q_s} = 0 \quad , \quad s = 1, 2$$

$$mgl\sin\theta_1 + k\zeta \frac{\partial L}{\partial \theta_1} = 0$$

$$mgl\sin\theta_2 + k\zeta \frac{\partial L}{\partial \theta_2} = 0$$

Solving the equilibrium in this case will be more complicated. However, it can be shown that $\theta_1 = \theta_2 = 0$ is an equilibrium.

$$\mathbf{K} = \left.\frac{\partial^2 V}{\partial \mathbf{q}\, \partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}_{eq}} = \begin{pmatrix} \dfrac{\partial^2 V}{\partial q_1 \partial q_1} & \dfrac{\partial^2 V}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & \dfrac{\partial^2 V}{\partial q_2 \partial q_2} \end{pmatrix}_{q_1=q_2=0}$$

$$= \begin{pmatrix} mgl + k\left(\dfrac{\partial L}{\partial \theta_1}\right)\left(\dfrac{\partial L}{\partial \theta_1}\right) + k\zeta\left(\dfrac{\partial^2 L}{\partial \theta_1^2}\right) & k\left(\dfrac{\partial L}{\partial \theta_1}\right)\left(\dfrac{\partial L}{\partial \theta_2}\right) + k\zeta\left(\dfrac{\partial^2 L}{\partial \theta_1 \theta_2}\right) \\ k\left(\dfrac{\partial L}{\partial \theta_1}\right)\left(\dfrac{\partial L}{\partial \theta_2}\right) + k\zeta\left(\dfrac{\partial^2 L}{\partial \theta_1 \theta_2}\right) & mgl + k\left(\dfrac{\partial L}{\partial \theta_2}\right)\left(\dfrac{\partial L}{\partial \theta_2}\right) + k\zeta\left(\dfrac{\partial^2 L}{\partial \theta_2^2}\right) \end{pmatrix}_{\theta_1=\theta_2=0}$$

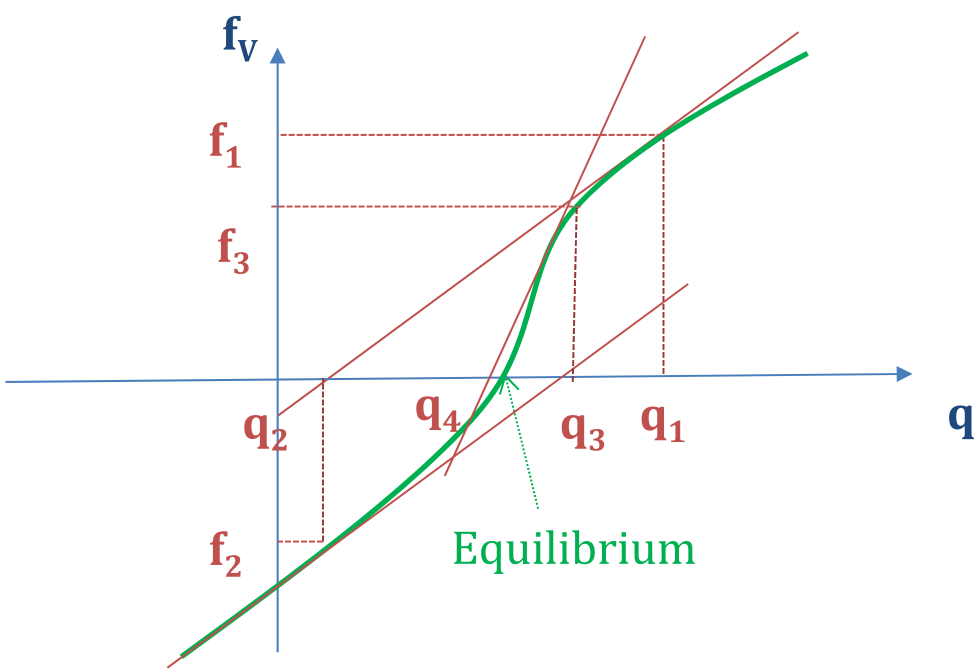
Finally

$$\mathbf{M} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} mgl + ka^2 & -ka^2 \\ -ka^2 & mgl + ka^2 \end{pmatrix}$$

As it could be seen almost always the equations that should give equilibrium points are non-linear. Most of times it is not easy to find solutions that's when we can use Newton-Raphson algorithm.

$$\mathbf{f}_V\Big|_{\mathbf{q}=\mathbf{q}_{eq}} = \left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} = 0$$

Suppose we have a 1D problem and a non-linear $\mathbf{f_V}$



The same story could be repeated in case of matrices

$$\mathbf{f}_V(\mathbf{q}_0) = \mathbf{r}_0$$

$\mathbf{q_0}$ being the initial guess and $\mathbf{r_0}$ the residual. We want the residual to become zero. Or in other words we would like to correct the guess in such a way that the residual is minimized. So we correct $\mathbf{q_0}$ by adding $\Delta\mathbf{q}$

$$\mathbf{r}_0(\mathbf{q}_0 + \Delta\mathbf{q}) = \mathbf{f}_V(\mathbf{q}_0 + \Delta\mathbf{q}) \approx \mathbf{f}_V(\mathbf{q}_0) + \left.\frac{\partial \mathbf{f}_V}{\partial \mathbf{q}}\right|_{\mathbf{q}_0} \Delta\mathbf{q} + \mathbf{H.O.T=0}$$

Stiffness matrix evaluated at $\mathbf{q_0}$
(Tangent matrix)

So

$$\Delta\mathbf{q} = -\mathbf{K}^{-1}\mathbf{r}_0$$

Then

$$\mathbf{q}_1 = \mathbf{q}_0 + \Delta\mathbf{q}$$

The scenario continues till residual becomes zero.

Rheonomic systems

1. Equilibrium

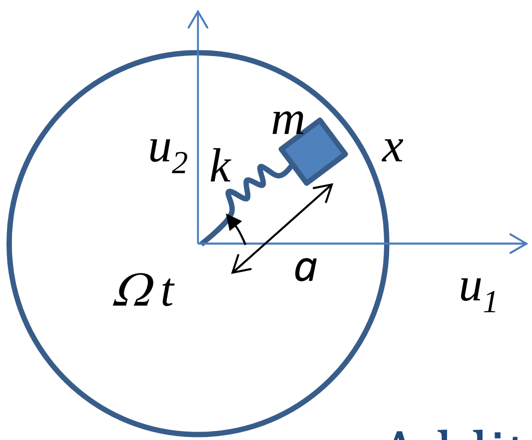
Note that the mapping U_{ik} will explicitly depend on time

$$u_{ik} = U_{ik}\left(q_1, \ldots, q_n, t\right)$$

$$\dot{u}_{ik} = \frac{\partial U_{ik}}{\partial t} + \sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial q_s}\dot{q}_s$$

Or in matrix form

$$\dot{\mathbf{u}}_k = \frac{\partial \mathbf{U}_k}{\partial t} + \frac{\partial \mathbf{U}_k}{\partial \mathbf{q}}\dot{\mathbf{q}} \quad , \quad \dot{\mathbf{u}}_k = \begin{Bmatrix} \dot{u}_{1k} \\ \dot{u}_{2k} \\ \dot{u}_{3k} \end{Bmatrix}$$



$$u_1 - \left(x + a\right)\cos \Omega t = 0$$

$$u_2 - \left(x + a\right)\sin \Omega t = 0$$

$$\dot{u}_1 = -\left(x + a\right)\sin \Omega t + \dot{x}\cos \Omega t$$

$$\dot{u}_2 = \left(x + a\right)\cos \Omega t - \dot{x}\sin \Omega t$$

Additional terms should exist in kinetic energy

Kinetic energy

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k\, \dot{\mathbf{u}}_k^T\, \dot{\mathbf{u}}_k = \frac{1}{2}\sum_{k=1}^{N} m_k \left(\frac{\partial \mathbf{U}_k}{\partial t} + \frac{\partial \mathbf{U}_k}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)^T \left(\frac{\partial \mathbf{U}_k}{\partial t} + \frac{\partial \mathbf{U}_k}{\partial \mathbf{q}}\dot{\mathbf{q}}\right) \quad ,$$

$$T_0 = \frac{1}{2}\sum_{k=1}^{N} m_k \left(\frac{\partial \mathbf{U}_k}{\partial t}\right)^T \left(\frac{\partial \mathbf{U}_k}{\partial t}\right)$$

Transport kinetic energy

Caused by applied motion of the system

$$T_1 = \sum_{k=1}^{N} m_k \left(\frac{\partial \mathbf{U}_k}{\partial t}\right)^T \left(\frac{\partial \mathbf{U}_k}{\partial \mathbf{q}}\dot{\mathbf{q}}\right) \quad ,$$

Coupling kinetic energy:

Coupling between the relative motion and the applied one

$$T_2 = \frac{1}{2}\sum_{k=1}^{N} m_k\, \dot{\mathbf{q}}^T \left(\frac{\partial \mathbf{U}_k}{\partial \mathbf{q}}\right)^T \left(\frac{\partial \mathbf{U}_k}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)$$

Relative kinetic energy:

Energy belonging to the relative motion with respect to the applied

Exactly the same as scleronomic systems

T_0, T_1, T_2, are homogenous forms of degree 0, 1 and 2 in generalized velocities

From Lagrange equations

$$\mathbf{f}_T(\ddot{\mathbf{q}},\dot{\mathbf{q}},\mathbf{q}) = \frac{d}{dt}\left(\frac{\partial T(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}} \qquad \mathbf{f}_V(\mathbf{q}) = \frac{\partial V}{\partial \mathbf{q}}$$

At equilibrium

$$\mathbf{f}_T(\ddot{\mathbf{q}}=0,\dot{\mathbf{q}}=0,\mathbf{q}=\mathbf{q}_{eq}) = \left(\frac{d}{dt}\left(\frac{\partial T(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}}\right)_{\substack{\ddot{\mathbf{q}}=0,\\ \dot{\mathbf{q}}=0,\\ \mathbf{q}=\mathbf{q}_{eq}}}$$

$$\left(\frac{d}{dt}\left(\frac{\partial (T_2+T_1)}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial (T_2+T_1+T_0)}{\partial \mathbf{q}}\right)_{\substack{\ddot{\mathbf{q}}=0,\\ \dot{\mathbf{q}}=0,\\ \mathbf{q}=\mathbf{q}_{eq}}}$$

Will include generalized velocities and thus are zero at equilibrium

$$\mathbf{f}_T(\ddot{\mathbf{q}}=0,\dot{\mathbf{q}}=0,\mathbf{q}=\mathbf{q}_{eq}) = \left(\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{\mathbf{q}}}\right)\right)_{\substack{\dot{\mathbf{q}}=0,\\ \mathbf{q}=\mathbf{q}_{eq}}} - \left(\frac{\partial T_0}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}}$$

In addition at equilibrium from potential energy we have

$$\mathbf{f}_V\Big|_{\mathbf{q}=\mathbf{q}_{eq}} = \left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} = 0$$

Therefore,

$$\boxed{\left(\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{\mathbf{q}}}\right)\right)_{\substack{\dot{\mathbf{q}}=0,\\ \mathbf{q}=\mathbf{q}_{eq}}} - \left(\frac{\partial T_0}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} + \left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} = 0}$$

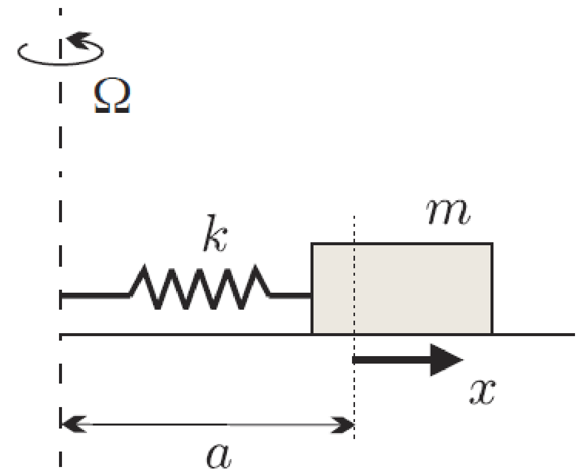
If $V^*=V-T_0$

$$\left(\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{\mathbf{q}}}\right)\right)_{\substack{\dot{\mathbf{q}}=0,\\ \mathbf{q}=\mathbf{q}_{eq}}} - \left(\frac{\partial V^*}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} = 0$$

Equilibrium will exist only if there are no time dependent components here. If such condition prevails then an equilibrium position is not possible (e.g. a pendulum that sinusoidally moves back and fort at the base)

In case of uniformly accelerate systems or systems rotating with constant angular velocity equilibrium could be found for the rheonomic system.

Consider the rotating mass problem



$$u_1 = (a+x)\cos\Omega t \quad , \quad u_2 = (a+x)\sin\Omega t$$

$$\dot{u}_1 = \dot{x}\cos\Omega t - (a+x)\Omega\sin\Omega t$$

$$\dot{u}_2 = \dot{x}\sin\Omega t + (a+x)\Omega\cos\Omega t$$

$$T = \frac{1}{2}m\dot{u}_1^2 + \frac{1}{2}m\dot{u}_2^2 \Rightarrow T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(a+x)^2\Omega^2$$

$$V = \frac{1}{2}kx^2$$

T_2 $\qquad$ T_0

The equilibrium can then be obtained as

$$\left(\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{\mathbf{q}}}\right)\right)_{\substack{\dot{\mathbf{q}}=0,\\ \mathbf{q}=\mathbf{q}_{eq}}} - \left(\frac{\partial T_0}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} + \left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} = 0$$

If angular velocity is constant

$$kx_{eq} - m\left(a + x_{eq}\right)\Omega^2 = 0 \quad \Rightarrow \quad x_{eq} = \frac{\Omega^2 ma}{k - \Omega^2 m}$$

- ✓ Equilibrium between spring force and centrifugal force (apparent force)
- ✓ As Ω increases the numerator grows and the denominators decreases.

Rheonomic systems

2. Linearized equations

Stiffness matrix

$$\mathbf{K} = \left.\frac{\partial\left(\mathbf{f}_T + \mathbf{f}_V\right)}{\partial \mathbf{q}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}} = \frac{\partial}{\partial \mathbf{q}}\left.\left(\frac{d}{dt}\frac{\partial\left(T_2+T_1\right)}{\partial \dot{\mathbf{q}}} - \frac{\partial\left(T_2+T_1+T_0\right)}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}}\right)\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}$$

$$= \left(-\frac{\partial^2 T_0}{\partial \mathbf{q}\,\partial \mathbf{q}} + \frac{\partial^2 V}{\partial \mathbf{q}\,\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}}$$

Therefore

$$\boxed{\mathbf{K}^* = \left(\frac{\partial^2 V^*}{\partial \mathbf{q}\,\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{eq}}}$$

Again by construction it is symmetric.

Damping matrix

$$\mathbf{C}=\left.\frac{\partial(\mathbf{f}_T)}{\partial\dot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}=\left.\frac{\partial}{\partial\dot{\mathbf{q}}}\left(\frac{d}{dt}\frac{\partial(T_2+T_1)}{\partial\dot{\mathbf{q}}}-\frac{\partial(T_2+T_1+T_0)}{\partial\mathbf{q}}\right)\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}$$

$$=\left.\frac{\partial}{\partial\dot{\mathbf{q}}}\left(\frac{d}{dt}\frac{\partial T_1}{\partial\dot{\mathbf{q}}}-\frac{\partial T_1}{\partial\mathbf{q}}\right)\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}=\left.\frac{\partial}{\partial\dot{\mathbf{q}}}\left(\frac{\partial}{\partial t}\frac{\partial T_1}{\partial\dot{\mathbf{q}}}+\frac{\partial^2T_1}{\partial\mathbf{q}\,\partial\dot{\mathbf{q}}}\dot{\mathbf{q}}-\frac{\partial T_1}{\partial\mathbf{q}}\right)\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}$$

$$=\left(\frac{\partial^2T_1}{\partial\mathbf{q}\,\partial\dot{\mathbf{q}}}-\frac{\partial^2T_1}{\partial\dot{\mathbf{q}}\,\partial\mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{\mathbf{eq}}}$$

Note that C is not zero, according to our notation

$$\boxed{c_{rs}=\left(\frac{\partial^2T_1}{\partial q_r\partial\dot{q}_s}-\frac{\partial^2T_1}{\partial\dot{q}_r\partial q_s}\right)_{q=q_{eq}}=-c_{sr}}$$

Skew symmetric matrix (matrix with zero diagonal terms and equal but different sign non-diagonal ones). This matrix is called *Gyroscopic matrix (G)* since it is originated from inertial couplings.

This instantaneous power related to Gyroscopic force $\mathbf{G}\,\dot{\mathbf{q}}$

$$\dot{\mathbf{q}}^T\,\mathbf{G}\,\dot{\mathbf{q}} = 0$$

This means that unlike damping forces, Gyroscopic forces do not dissipate any power from the system, this is fundamentally different from damping that takes energy from the system.

$$\boxed{\mathbf{G} = \left(\frac{\partial^2 T_1}{\partial \mathbf{q}\,\partial \dot{\mathbf{q}}} - \frac{\partial^2 T_1}{\partial \dot{\mathbf{q}}\,\partial \mathbf{q}} \right)_{\mathbf{q}=\mathbf{q}_{\mathbf{eq}}}}$$

Mass matrix

$$\mathbf{M} = \left.\frac{\partial \mathbf{f}_T}{\partial \ddot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}} = \left.\frac{\partial}{\partial \ddot{\mathbf{q}}}\left(\frac{d}{dt}\frac{\partial (T_2 + T_1)}{\partial \dot{\mathbf{q}}} - \frac{\partial (T_2 + T_1 + T_0)}{\partial \mathbf{q}} \right)\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}} = \left.\frac{\partial}{\partial \ddot{\mathbf{q}}}\left(\frac{d}{dt}\frac{\partial (T_2)}{\partial \dot{\mathbf{q}}} \right)\right|_{\mathbf{q}=\mathbf{q}_{eq}}$$

$$\boxed{\mathbf{M} = \left.\frac{\partial^2 T_2}{\partial \dot{\mathbf{q}}\,\partial \dot{\mathbf{q}}}\right|_{\mathbf{q}=\mathbf{q}_{eq}}}$$

Similar to the scleronomic case

$$\mathbf{M} = \left.\frac{\partial^2 T_2}{\partial \dot{\mathbf{q}} \partial \dot{\mathbf{q}}}\right|_{\mathbf{q}=\mathbf{q}_{eq}}$$

So the linearized equations in case of rheonomic systems

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}^* \mathbf{q} = 0$$

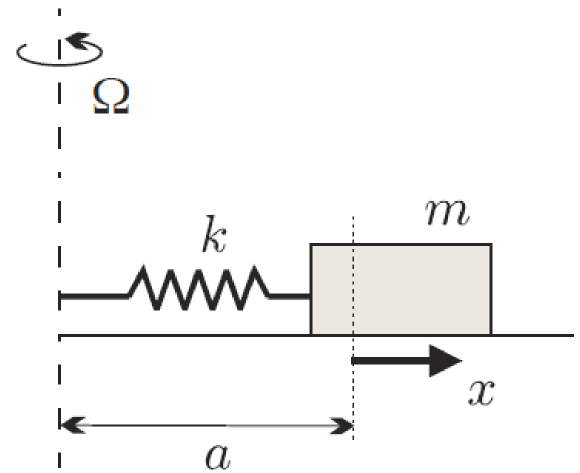
$$\mathbf{M} = \left(\frac{\partial^2 T_2}{\partial \dot{\mathbf{q}} \partial \dot{\mathbf{q}}}\right)_{\mathbf{q}=\mathbf{q}_{eq}} , \ \mathbf{G} = \left(\frac{\partial^2 T_1}{\partial \mathbf{q} \partial \dot{\mathbf{q}}} - \frac{\partial^2 T_1}{\partial \dot{\mathbf{q}} \partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{\mathbf{eq}}} , \ \mathbf{K}^* = \left(-\frac{\partial^2 T_0}{\partial \mathbf{q} \partial \mathbf{q}} + \frac{\partial^2 V}{\partial \mathbf{q} \partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{\mathbf{eq}}}$$

So for the rotating mass

$$x_{eq} = \frac{\Omega^2 ma}{k - \Omega^2 m} \qquad V = \frac{1}{2} k x^2$$

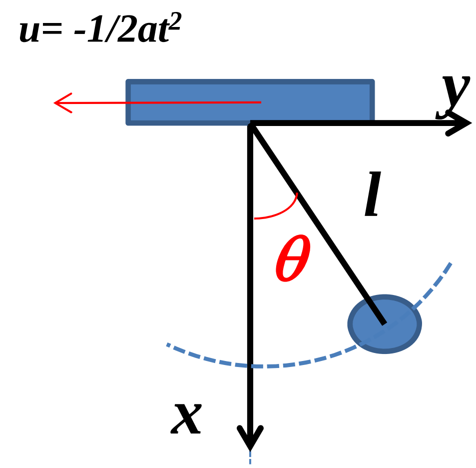
$$T_2 = \frac{1}{2} m \dot{x}^2 \quad , \quad T_0 = \frac{1}{2} m (a + x)^2 \, \Omega^2$$

$$m\ddot{\tilde{x}} + \underbrace{G}_{=0} \dot{\tilde{x}} + \left(k - m\Omega^2\right) \tilde{x} = 0$$



Pendulum with base motion

STEP 1



$$x = l\cos\theta \quad , \quad y = l\sin\theta - \frac{1}{2}at^2$$

$$\dot{x} = -l\dot{\theta}\sin\theta \quad , \quad \dot{y} = l\dot{\theta}\cos\theta - at$$

$$T = \frac{1}{2}m\left(-l\dot{\theta}\sin\theta\right)^2 + \frac{1}{2}m\left(l\dot{\theta}\cos\theta - at\right)^2$$

$$= \underbrace{\frac{1}{2}ml^2\dot{\theta}^2}_{T_2} \underbrace{-mlat\dot{\theta}\cos\theta}_{T_1} + \underbrace{\frac{1}{2}ma^2t^2}_{T_0}$$

$$V = -mgl\cos\theta$$

STEP 2

$$\left(\frac{d}{dt}\left(\frac{\partial T_1}{\partial\dot{\theta}}\right)\right)_{\substack{\dot{\theta}=0,\\ \theta=\theta_{eq}}} - \left(\frac{\partial T_0}{\partial\theta}\right)_{\theta=\theta_{eq}} + \left(\frac{\partial V}{\partial\theta}\right)_{\theta=\theta_{eq}} = 0$$

$$\left(\frac{d}{dt}\left(-malt\cos\theta\right)\right)_{\substack{\dot{\theta}=0,\\ \theta=\theta_{eq}}} - 0 + \left(mgl\sin\theta\right)_{\theta=\theta_{eq}} = -mal\cos\theta_{eq} + \underbrace{mal\dot{\theta}\sin\theta_{eq}}_{=0} + mgl\sin\theta_{eq} = 0$$

Therefore, equilibrium equation will be

$$-mal\,\mathbf{cos}\,\theta_{eq} + mgl\,\mathbf{sin}\,\theta_{eq} = 0 \quad \Rightarrow \quad \mathbf{tan}\,\theta_{eq} = \frac{a}{g}$$

STEP 3

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}^{*}\mathbf{q} = 0$$

$$\mathbf{M} = \left(\frac{\partial^2 T_2}{\partial\dot{\mathbf{q}}\,\partial\dot{\mathbf{q}}}\right)_{\mathbf{q}=\mathbf{q}_{eq}}, \quad \mathbf{G} = \left(\frac{\partial^2 T_1}{\partial\mathbf{q}\,\partial\dot{\mathbf{q}}} - \frac{\partial^2 T_1}{\partial\dot{\mathbf{q}}\,\partial\mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{\mathbf{eq}}}, \quad \mathbf{K}^{*} = \left(-\frac{\partial^2 T_0}{\partial\mathbf{q}\,\partial\mathbf{q}} + \frac{\partial^2 V}{\partial\mathbf{q}\,\partial\mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_{\mathbf{eq}}}$$

$$\Downarrow$$

$$\mathbf{M} = \left(\frac{\partial^2 T_2}{\partial\dot{\theta}\partial\dot{\theta}}\right)_{\theta=\theta_{eq}}, \quad \mathbf{G} = \left(\frac{\partial^2 T_1}{\partial\theta\partial\dot{\theta}} - \frac{\partial^2 T_1}{\partial\dot{\theta}\partial\theta}\right)_{\theta=\theta_{\mathbf{eq}}}, \quad \mathbf{K}^{*} = \left(-\frac{\partial^2 T_0}{\partial\theta\partial\theta} + \frac{\partial^2 V}{\partial\theta\partial\theta}\right)_{\theta=\theta_{\mathbf{eq}}}$$

$$\Downarrow$$

$$ml^2\ddot{\tilde{\theta}} + mgl\,\mathbf{cos}\,\theta\big|_{\theta_{eq}}\tilde{\theta} = 0 \;, \quad \ddot{\tilde{\theta}} + \frac{g}{l}\mathbf{cos}\left(\mathbf{tan}^{-1}\frac{a}{g}\right)\tilde{\theta} = 0$$

A two-dimensional rotating system

Finding linearized equations around (0,0)

$$u_1 = x\cos\Omega t - y\sin\Omega t \ ,$$
$$u_2 = x\sin\Omega t + y\cos\Omega t$$

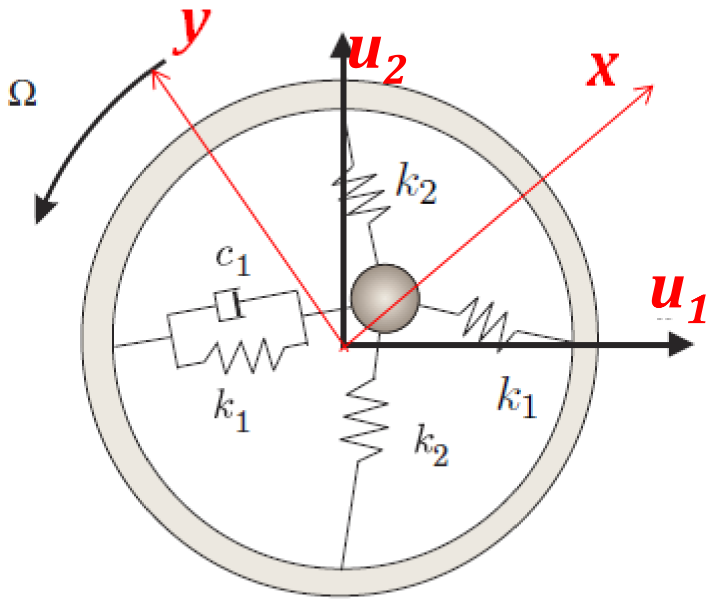
$$\dot{u}_1 = \dot{x}\cos\Omega t - x\Omega\sin\Omega t - \dot{y}\sin\Omega t - y\Omega\cos\Omega t \ ,$$
$$\dot{u}_2 = \dot{x}\sin\Omega t + x\Omega\cos\Omega t + \dot{y}\cos\Omega t - y\Omega\sin\Omega t$$

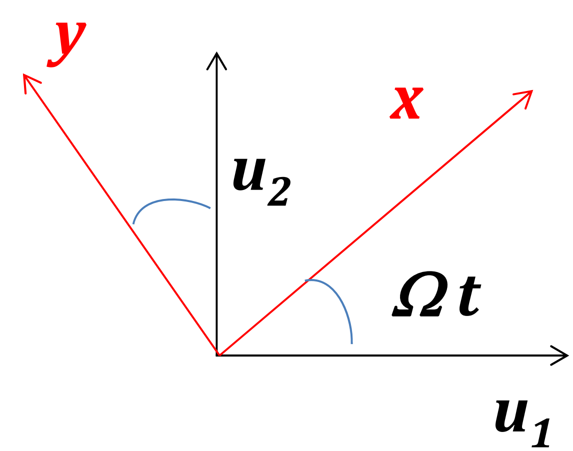
$$T = \frac{1}{2}m(\dot{u}_1)^2 + \frac{1}{2}m(\dot{u}_2)^2$$

$$= \underbrace{\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)}_{T_2} + \underbrace{m\Omega\left(-\dot{x}y+\dot{y}x\right)}_{T_1} + \underbrace{\frac{1}{2}m\Omega^2\left(x^2+y^2\right)}_{T_0}$$

$$m_{11} = \frac{\partial}{\partial\dot{x}}\left(\frac{\partial T_2}{\partial\dot{x}}\right)\Bigg|_{\substack{x=0\\y=0}} = m \quad , \quad m_{12} = \frac{\partial}{\partial\dot{x}}\left(\frac{\partial T_2}{\partial\dot{y}}\right)\Bigg|_{\substack{x=0\\y=0}} = 0$$

$$m_{22} = \frac{\partial}{\partial\dot{y}}\left(\frac{\partial T_2}{\partial\dot{y}}\right)\Bigg|_{\substack{x=0\\y=0}} = m$$





$$V=\frac{1}{2}k_1\left(a_{left}-R\right)^2+\frac{1}{2}k_1\left(a_{right}-R\right)^2+\frac{1}{2}k_2\left(a_{top}-R\right)^2+\frac{1}{2}k_2\left(a_{bottom}-R\right)^2$$

$$a_{left}=\sqrt{(R+x)^2+y^2}\ ,\ a_{right}=\sqrt{(R-x)^2+y^2}$$

$$a_{bottom}=\sqrt{(R+y)^2+x^2}\ ,\ a_{top}=\sqrt{(R-y)^2+x^2}$$

Assuming small motion around equilibrium we can approximate the potential energy in the following quadratic form

$$V\simeq\frac{1}{2}k_1x^2+\frac{1}{2}k_1x^2+\frac{1}{2}k_2y^2+\frac{1}{2}k_2y^2=k_1x^2+k_2y^2$$

The same scenario could be repeated for the damping term: $\mathbf{C}=\left.\frac{\partial\left(\mathbf{f}_T+\mathbf{Q}^{ncons}\right)}{\partial\dot{\mathbf{q}}}\right|_{\substack{\ddot{\mathbf{q}}=0\\ \dot{\mathbf{q}}=0\\ \mathbf{q}=\mathbf{q}_{eq}}}$,

$$c_{11}=\frac{\partial}{\partial\dot{x}}\left(c_1\frac{d}{dt}a_{left}\right)\Bigg|_{\substack{\dot{x}=\dot{y}=0\\ x=0\\ y=0}}=\frac{\partial}{\partial\dot{x}}\left(c_1\frac{(R+x)\dot{x}+y\dot{y}}{\sqrt{(R+x)^2+y^2}}\right)\Bigg|_{\substack{\dot{x}=\dot{y}=0\\ x=0\\ y=0}}=c_1\frac{(R+x)}{\sqrt{(R+x)^2+y^2}}\Bigg|_{\substack{\dot{x}=\dot{y}=0\\ x=0\\ y=0}}=c_1$$

$$c_{22}=\frac{\partial}{\partial\dot{y}}\left(c_1\frac{d}{dt}a_{left}\right)\Bigg|_{\substack{\dot{x}=\dot{y}=0\\ x=0\\ y=0}}=\frac{\partial}{\partial\dot{y}}\left(c_1\frac{(R+x)\dot{x}+y\dot{y}}{\sqrt{(R+x)^2+y^2}}\right)\Bigg|_{\substack{\dot{x}=\dot{y}=0\\ x=0\\ y=0}}=c_1\frac{y}{\sqrt{(R+x)^2+y^2}}\Bigg|_{\substack{\dot{x}=\dot{y}=0\\ x=0\\ y=0}}=0$$

$$\mathbf{K}^* = \left(-\frac{\partial^2 T_0}{\partial \mathbf{q} \partial \mathbf{q}} + \frac{\partial^2 V}{\partial \mathbf{q} \partial \mathbf{q}} \right)_{\mathbf{q}=\mathbf{q}_{\mathbf{eq}}}$$

$$\mathbf{G} = \left(\frac{\partial^2 T_1}{\partial \mathbf{q} \partial \dot{\mathbf{q}}} - \frac{\partial^2 T_1}{\partial \dot{\mathbf{q}} \partial \mathbf{q}} \right)_{\mathbf{q}=\mathbf{q}_{\mathbf{eq}}}$$

$$k_{11}{}^* = \left(-\frac{\partial^2 T_0}{\partial x \partial x} + \frac{\partial^2 V}{\partial x \partial x} \right)_{x=y=0} = -m\Omega^2 + 2k_1$$

$$k_{12}{}^* = \left(-\frac{\partial^2 T_0}{\partial x \partial y} + \frac{\partial^2 V}{\partial x \partial y} \right)_{x=y=0} = 0$$

$$k_{22}{}^* = \left(-\frac{\partial^2 T_0}{\partial y \partial y} + \frac{\partial^2 V}{\partial y \partial y} \right)_{x=y=0} = -m\Omega^2 + 2k_2$$

$$g_{11} = \left(\frac{\partial^2 T_1}{\partial x \partial \dot{x}} - \frac{\partial^2 T_1}{\partial \dot{x} \partial x} \right)_{x=y=0} = 0$$

$$g_{12} = -g_{21} = \left(\frac{\partial^2 T_1}{\partial \dot{x} \partial y} - \frac{\partial^2 T_1}{\partial x \partial \dot{y}} \right)_{x=y=0} = -2m\Omega$$

$$g_{22} = \left(\frac{\partial^2 T_1}{\partial y \partial \dot{y}} - \frac{\partial^2 T_1}{\partial \dot{y} \partial y} \right)_{x=y=0} = 0$$

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \;,\; \mathbf{C} = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \;,\; \mathbf{G} = \begin{bmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{bmatrix} \;,\; \mathbf{K}^* = \begin{bmatrix} -m\Omega^2 + 2k_1 & 0 \\ 0 & -m\Omega^2 + 2k_2 \end{bmatrix}$$