Lagrange Equations

WB 1418-07

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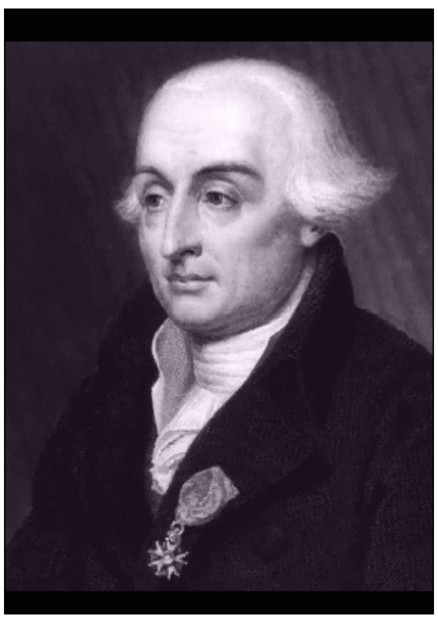
- ✓ D’Alambert principle facilitates dynamics of systems of particles and rigid bodies since it neglects reaction forces if virtual displacements are kinematically admissible.

- ✓ However, there is still the need to find accelerations and still we would need to perform vectorial operations.

- ✓ Lagrange posed the following questions:

Would it be possible not to perform vectorial calculations & instead define everything in terms of scalar functions?

Would it be possible to get rid of reaction forces in a similar fashion to D”Alambert principle and do not calculate accelerations?



Lagrange Equations

An automated technique to find EoMs
(Physical interpretation would be lost)

Smart way of finding equations with just few mathematical tricks

Suppose that we have a system of particles that could be connected in such a way that you can write your kinematically admissible relations

The kinetic energy is then obtained as:

$$T = \frac{1}{2}\sum_{k=1}^{N}\sum_{i=1}^{3} m_{ik}\dot{u}_{ik}^{2}(q_1,....q_n,t)$$

First trick: I can find the inertia forces as follows

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_{ik}}\right) = m_{ik}\ddot{u}_{ik}$$

If I recall the D'Alambert equations:

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\ddot{u}_{ik} - X_{ik}\right)\frac{\partial u_{ik}}{\partial q_s} = 0 \quad , \quad s = 1,....,n$$

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_{ik}}\right)-X_{ik}\right)\frac{\partial u_{ik}}{\partial q_s}=0 \quad , \quad s=1,....,n$$

Second trick: How can I re-write the above equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_{ik}}\frac{\partial u_{ik}}{\partial q_s}\right)=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_{ik}}\right)\frac{\partial u_{ik}}{\partial q_s}+\frac{\partial T}{\partial \dot{u}_{ik}}\frac{d}{dt}\left(\frac{\partial u_{ik}}{\partial q_s}\right)$$

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_{ik}}\frac{\partial u_{ik}}{\partial q_s}\right)-\frac{\partial T}{\partial \dot{u}_{ik}}\frac{d}{dt}\left(\frac{\partial u_{ik}}{\partial q_s}\right)-X_{ik}\frac{\partial u_{ik}}{\partial q_s}\right)=0 \quad , \quad s=1,....,n$$

Still inertia forces projected on to the virtual displacements

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_{ik}}\frac{\partial u_{ik}}{\partial q_s}\right)-\frac{\partial T}{\partial \dot{u}_{ik}}\frac{d}{dt}\left(\frac{\partial u_{ik}}{\partial q_s}\right)-X_{ik}\frac{\partial u_{ik}}{\partial q_s}\right)=0 \quad , \quad s=1,....,n$$

Third trick: Total derivative in time of u_{ik}

$$\dot{u}_{ik}=\frac{\partial u_{ik}}{\partial t}+\sum_{s=1}^{n}\frac{\partial u_{ik}}{\partial q_s}\dot{q}_s \quad , \quad s=1,....,n$$

$$\Downarrow$$

$$\frac{\partial \dot{u}_{ik}}{\partial \dot{q}_s}=\frac{\partial u_{ik}}{\partial q_s}$$

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(\frac{d}{dt}\left(\frac{\partial T}{\cancel{\partial \dot{u}_{ik}}}\frac{\cancel{\partial \dot{u}_{ik}}}{\partial \dot{q}_s}\right)-\frac{\partial T}{\cancel{\partial \dot{u}_{ik}}}\frac{\cancel{\partial \dot{u}_{ik}}}{\partial q_s}-X_{ik}\frac{\partial u_{ik}}{\partial q_s}\right)=0 \quad , \quad s=1,....,n$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}=\sum_{k=1}^{N}\sum_{i=1}^{3}X_{ik}\frac{\partial u_{ik}}{\partial q_s} \quad , \quad s=1,....,n}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} = \sum_{k=1}^{N}\sum_{i=1}^{3} X_{ik}\frac{\partial u_{ik}}{\partial q_s} \quad , \quad s = 1, ..., n$$

Two more important points:

1. We defined generalized forces associated with generalized coordinates

$$Q_s = \sum_{k=1}^{N}\sum_{i=1}^{3} X_{ik}\frac{\partial u_{ik}}{\partial q_s}$$

2. If we have conservative forces then a potential energy can be derived such that

$$X_{ik} = -\frac{\partial V}{\partial u_{ik}}$$

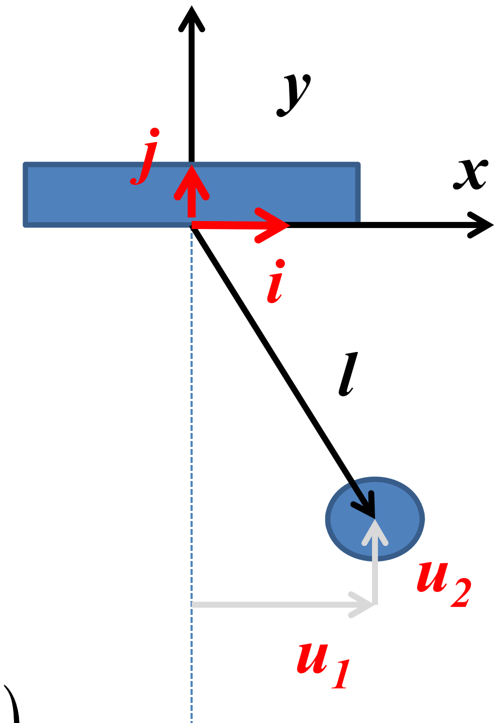
$$Q_s^{cons} = -\sum_{k=1}^{N}\sum_{i=1}^{3}\frac{\partial V}{\partial u_{ik}}\frac{\partial u_{ik}}{\partial q_s} = -\frac{\partial V}{\partial q_s}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = Q_s^{ncons} \quad , \quad s = 1, ..., n}$$

Far more automatic and easier than D'Alambert principle and Newton's second law

The pendulum

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}+\frac{\partial V}{\partial q_s}=Q_s^{ncons} \quad , \quad s=1,....,n$$



$$s=1 \quad , \quad q_1=\theta$$

$$Q_1^{ncons}=0$$

$$u_1=l\,\mathbf{sin}\,\theta \quad , \quad u_2=l-l\,\mathbf{cos}\,\theta$$

$$T=\frac{1}{2}m\dot{u}_1^2+\frac{1}{2}m\dot{u}_2^2=\frac{1}{2}ml^2\left(\dot{\theta}^2\,\mathbf{cos}^2\,\theta\right)+\frac{1}{2}ml^2\left(\dot{\theta}^2\,\mathbf{sin}^2\,\theta\right)$$

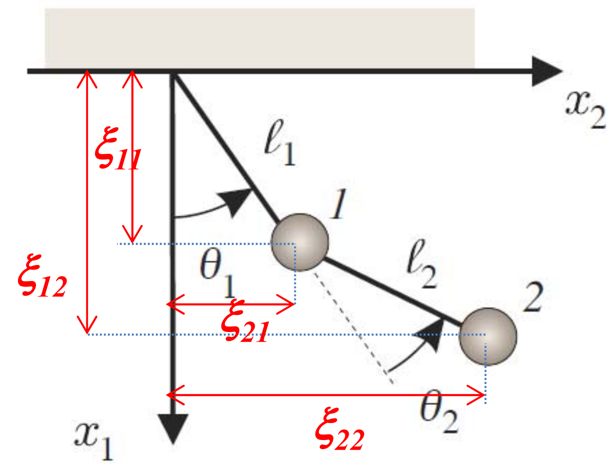
$$=\frac{1}{2}ml^2\dot{\theta}^2$$

$$V=mgu_2=mgl\left(1-\mathbf{cos}\,\theta\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)=ml^2\ddot{\theta} \quad , \quad \frac{\partial V}{\partial \theta}=mgl\,\mathbf{sin}\,\theta \quad \Rightarrow \quad ml^2\ddot{\theta}+mgl\,\mathbf{sin}\,\theta=0$$

The double pendulum

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}+\frac{\partial V}{\partial q_s}=Q_s^{ncons} \quad , \quad s=1,....,n$$



$$s=2 \quad , \quad q_1=\theta_1 \quad , \quad q_2=\theta_2$$

$$Q_1^{ncons}=Q_1^{ncons}=0$$

$$\xi_{11}=u_{11}=l_1\cos\theta_1$$

$$\xi_{21}=u_{21}=l_1\sin\theta_1$$

$$\xi_{12}=u_{12}=\xi_{11}+l_2\cos(\theta_1+\theta_2)=l_1\cos\theta_1+l_2\cos(\theta_1+\theta_2)$$

$$\xi_{22}=u_{22}=\xi_{21}+l_2\sin(\theta_1+\theta_2)=l_1\sin\theta_1+l_2\sin(\theta_1+\theta_2)$$

$$\dot{u}_{11}=-l_1\dot{\theta}_1\sin\theta_1$$

$$\dot{u}_{21}=l_1\dot{\theta}_1\cos\theta_1$$

$$\dot{u}_{12}=-l_1\dot{\theta}_1\sin\theta_1-l_2\dot{\theta}_1\sin(\theta_1+\theta_2)-l_2\dot{\theta}_2\sin(\theta_1+\theta_2)$$

$$\dot{u}_{22}=l_1\dot{\theta}_1\cos\theta_1+l_2\dot{\theta}_1\cos(\theta_1+\theta_2)+l_2\dot{\theta}_2\cos(\theta_1+\theta_2)$$

$$\dot{u}_{11} = -l_1\dot{\theta}_1 \sin\theta_1$$

$$\dot{u}_{21} = l_1\dot{\theta}_1 \cos\theta_1$$

$$\dot{u}_{12} = -l_1\dot{\theta}_1 \sin\theta_1 - l_2\dot{\theta}_1 \sin(\theta_1+\theta_2) - l_2\dot{\theta}_2 \sin(\theta_1+\theta_2)$$

$$\dot{u}_{22} = l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_1 \cos(\theta_1+\theta_2) + l_2\dot{\theta}_2 \cos(\theta_1+\theta_2)$$

$$\begin{aligned} T &= \frac{1}{2}m\dot{u}_{11}^2 + \frac{1}{2}m\dot{u}_{21}^2 + \frac{1}{2}m\dot{u}_{12}^2 + \frac{1}{2}m\dot{u}_{22}^2 \\ &= \frac{1}{2}ml_1^2\dot{\theta}_1^2 + \frac{1}{2}ml_1^2\dot{\theta}_1^2 + \frac{1}{2}ml_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + \frac{1}{2}m\left(2l_1l_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)\cos\theta_2\right) \end{aligned}$$

$$\begin{aligned} V &= -mgu_{11} - mgu_{12} \\ &= mgl_1\left(1-\cos\theta_1\right) + mgl_1\left(1-\cos\theta_1\right) + mgl_2\left(1-\cos\left(\theta_1+\theta_2\right)\right) \end{aligned}$$

Terms to be evaluated:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right), \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right), \quad \frac{\partial T}{\partial \theta_1}, \quad \frac{\partial T}{\partial \theta_2}, \quad \frac{\partial V}{\partial \theta_1}, \quad \frac{\partial V}{\partial \theta_2}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)=\frac{d}{dt}\left(ml_1^2\dot{\theta}_1+ml_1^2\dot{\theta}_1+ml_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)+m\left(l_1l_2\left(2\dot{\theta}_1+\dot{\theta}_2\right)\mathbf{cos}\,\theta_2\right)\right)$$

$$=m\left(2l_1^2\ddot{\theta}_1+l_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)+l_1l_2\left(2\ddot{\theta}_1+\ddot{\theta}_2\right)\mathbf{cos}\,\theta_2-l_1l_2\left(2\dot{\theta}_1+\dot{\theta}_2\right)\dot{\theta}_2\,\mathbf{sin}\,\theta_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right)=\frac{d}{dt}\left(ml_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)+ml_1l_2\dot{\theta}\,\mathbf{cos}\,\theta_2\right)$$

$$=m\left(l_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)+l_1l_2\ddot{\theta}_1\,\mathbf{cos}\,\theta_2-l_1l_2\dot{\theta}_1\dot{\theta}_2\,\mathbf{sin}\,\theta_2\right)$$

$$\frac{\partial T}{\partial \theta_1}=0$$

$$\frac{\partial T}{\partial \theta_2}=-ml_1l_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)\mathbf{sin}\,\theta_2$$

$$\frac{\partial V}{\partial \theta_1} = mgl_1 \sin\theta_1 + mg\left(l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)\right)$$

$$\frac{\partial V}{\partial \theta_2} = mgl_2 \sin(\theta_1 + \theta_2)$$

Therefore final set of equations will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

How Lagrange equations can be obtained using a symbolic toolbox?

Refer to Matlab and Mathematica examples on the blackboard

Lagrange Multipliers

So far we were stressing that if we use kinematically admissible displacements we can neglect reaction forces (Because their virtual work is zero). What if we introduce the constraint explicitly and then want to see the reaction forces?

The question here is if we do as such: What is the direction of reaction forces?

Suppose we have a holonomic constraint as:

$$f\left(u_{ik},t\right)=0$$

$$\delta f\left(u_{ik},t\right)=\sum_{k=1}^{N}\sum_{i=1}^{3}\frac{\partial f}{\partial u_{ik}}\delta u_{ik}=0$$

The above equation is equal to the scalar product of two orthogonal vectors. Since δu_{ik} are virtual displacements so the other vector would be the direction of the constraint!

Knowing that the reaction forces are in the direction of the constraints:

$$R_{ik} = \lambda \frac{\partial f}{\partial u_{ik}}$$

Magnitude
(Lagrange Multiplier)

So if we choose coordinates q_s such a way that do not satisfy the constraints:

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k \frac{d^2 U_{ik}\left(q_1,\ldots,q_n,t\right)}{dt^2} - X_{ik} - \underbrace{\lambda \frac{\partial f}{\partial u_{ik}}}_{R_{ik}} \right)\frac{\partial U_{ik}(q_s,t)}{\partial q_s} = 0 \qquad s = 1,\ldots n$$

$$f\left(u_{ik},t\right) = f\left(x_{ik} + U_{ik}(q_s,t),t\right)$$

So the Lagrange equations will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}+\frac{\partial V}{\partial q_s}-\lambda\frac{\partial f}{\partial q_s}=Q_s^{ncons} \quad , \quad s=1,....,n$$

$$f\left(u_{ik},t\right)=f\left(x_{ik}+U_{ik}(q_s,t),t\right)$$

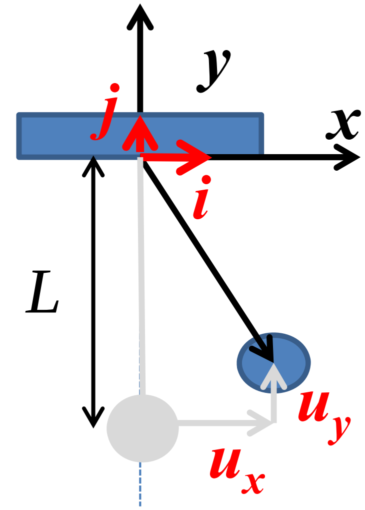
This is the case where we have only one constraint. For each constraint a new λ is required.

NOTE: It can be shown that if we can again introduce kinematically admissible generalized coordinates we can have the previous set of Lagrange equations (PART 1.4.2 of your notes):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}+\frac{\partial V}{\partial q_s}=Q_s^{ncons} \quad , \quad s=1,....,n$$

The same pendulum problem

$$f = \sqrt{(u_y - L)^2 + u_x^2} - L = 0$$



First finding the direction of the reaction force

$$\frac{\partial f}{\partial u_x} = \frac{u_x}{\sqrt{(u_y - L)^2 + u_x^2}} = \frac{u_x}{L}$$

$$\frac{\partial f}{\partial u_y} = \frac{u_y - L}{\sqrt{(u_y - L)^2 + u_x^2}} = \frac{u_y - L}{L}$$

$$\Rightarrow \quad \vec{\mathbf{R}} = \lambda \begin{bmatrix} \dfrac{u_x}{L} \\ \dfrac{u_y - L}{L} \end{bmatrix}$$

$$T = \frac{1}{2} m \dot{u}_x^2 + \frac{1}{2} m \dot{u}_y^2 \quad , \quad V = m g u_y$$

So the equations will be

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_x}\right)-\frac{\partial T}{\partial u_x}+\frac{\partial V}{\partial u_x}-\lambda\frac{\partial f}{\partial u_x}=0 \quad\Rightarrow\quad \boxed{m\ddot{u}_x=\lambda\frac{u_x}{L}}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_y}\right)-\frac{\partial T}{\partial u_y}+\frac{\partial V}{\partial u_y}-\lambda\frac{\partial f}{\partial u_y}=0 \quad\Rightarrow\quad \boxed{m\ddot{u}_y=\lambda\frac{u_y-L}{L}-mg}$$

Constraint $\quad\Rightarrow\quad \boxed{f=\sqrt{(u_y-L)^2+u_x^2}-L=0}$

You will see that if

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix}=\begin{bmatrix} L\sin\theta \\ L-L\cos\theta \end{bmatrix} \quad , \quad \begin{bmatrix} \delta u_x \\ \delta u_y \end{bmatrix}=\begin{bmatrix} L\cos\theta \\ L\sin\theta \end{bmatrix}\delta\theta$$

$$\vec{\mathbf{R}}\cdot\delta\mathbf{u}=\lambda\begin{bmatrix} \dfrac{u_x}{L} \\ \dfrac{u_y-L}{L} \end{bmatrix}\cdot\begin{bmatrix} L\cos\theta \\ L\sin\theta \end{bmatrix}\delta\theta=\lambda\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}\cdot\begin{bmatrix} L\cos\theta \\ L\sin\theta \end{bmatrix}\delta\theta=\lambda L\cos\theta\sin\theta-\lambda L\cos\theta\sin\theta=0$$

Case of nonholonomic constraints

In this case we will have *M* generalized coordinates and *R* constraint equations

Moreover, the constraint will not be integrable and we will have reaction forces associated with each constraint that do virtual work. Therefore:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k}-R_k=Q_k^{ncons} \quad , \quad k=1,....,M$$

$$f_j\left(\dot{q}_k, q_k, t\right)=0 \qquad\qquad j=1,....,R$$

Constraint in Pfaffian form in terms of generalized coordinates:

$$\sum_{k=1}^{M} a_{jk}\, dq_i + b_j\, dt = 0 \; , \quad j=1,....,R$$

$$\frac{\partial f_j}{\partial q_k} \qquad\qquad \frac{\partial f_j}{\partial t}$$

Direction of reaction forces

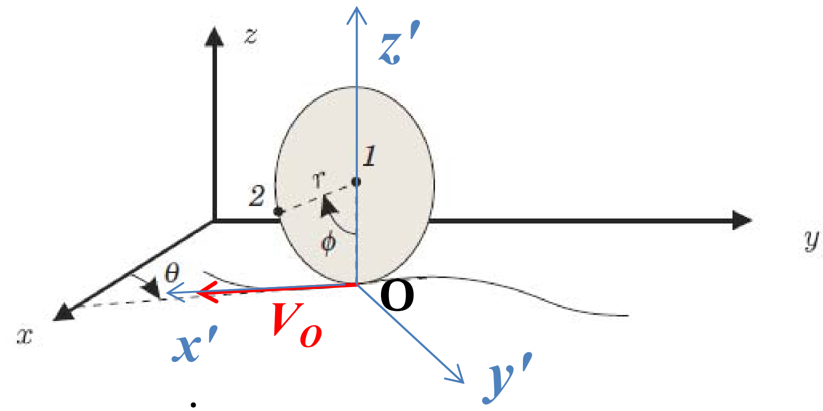
For each reaction force direction there has to be a Lagrange multiplier. Therefore:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} - \sum_{j=1}^{R} \lambda_j a_{jk}\left(q_k, t\right) = Q_k^{ncons} \quad , \quad k = 1, \ldots, M$$

$$f_j\left(\dot{q}_k, q_k, t\right) = 0 \qquad j = 1, \ldots, R$$

The rolling wheel

$$V_O = V_1 + \omega \times r = \dot{x}_1 i + \dot{y}_1 j + (-\dot{\phi} j') \times (-r k')$$
$$= \dot{x}_1 i + \dot{y}_1 j + r\dot{\phi} i'$$
$$i' = \cos\theta i - \sin\theta j$$



$$V_O = \left(\dot{x}_1 + r\dot{\phi}\cos\theta\right) i + \left(\dot{y}_1 - r\dot{\phi}\sin\theta\right) j$$

$$\dot{x}_1 + r\dot{\phi}\cos\theta = 0$$
$$\dot{y}_1 - r\dot{\phi}\sin\theta = 0$$

$$dx_1 + 0\cdot dy_1 + r\cos\theta d\phi + 0\cdot d\theta + 0\cdot dt = 0$$
$$0\cdot dx_1 + dy_1 - r\sin\theta d\phi + 0\cdot d\theta + 0\cdot dt = 0$$

$$a_{11} = 1,\ a_{12} = 0,\ a_{13} = r\cos\theta,\ a_{14} = 0,\ b_1 = 0$$
$$a_{21} = 0,\ a_{22} = 1,\ a_{23} = -r\sin\theta,\ a_{24} = 0, b_2 = 0$$

$$\frac{\partial}{\partial\theta}(a_{13}) \neq \frac{\partial}{\partial\phi}(a_{14})$$
$$\frac{\partial}{\partial\theta}(a_{23}) \neq \frac{\partial}{\partial\phi}(a_{24})$$

The constraint is not integrable so it is non-holonomic and thus the system has 4 degrees of freedom

The rolling wheel

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{y}_1^2 + \frac{1}{2}\left(\frac{1}{2} mR^2\right)\dot{\phi}^2 + \frac{1}{2}\left(\frac{1}{4} mR^2\right)\dot{\theta}^2 \quad , \quad V = mgR$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} - \lambda_1 a_{11} - \lambda_2 a_{21} = 0 \quad \Longrightarrow \quad m\ddot{x}_1 = \lambda_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) - \frac{\partial T}{\partial y_1} + \frac{\partial V}{\partial y_1} - \lambda_1 a_{12} - \lambda_2 a_{22} = 0 \quad \Longrightarrow \quad m\ddot{y}_1 = \lambda_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} - \lambda_1 a_{13} - \lambda_2 a_{23} = 0 \quad \Longrightarrow \quad \frac{1}{2} mR^2\ddot{\phi} = \lambda_1 r \mathbf{cos}\,\theta - \lambda_2 r \mathbf{sin}\,\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} - \lambda_1 a_{14} - \lambda_2 a_{24} = 0 \quad \Longrightarrow \quad \frac{1}{4} mR^2\ddot{\theta} = 0$$

$$\dot{x}_1 + r\dot{\phi}\,\mathbf{cos}\,\theta = 0 \quad \Longrightarrow \quad \dot{x}_1 + r\dot{\phi}\,\mathbf{cos}\,\theta = 0$$

$$\dot{y}_1 - r\dot{\phi}\,\mathbf{sin}\,\theta = 0 \quad \Longrightarrow \quad \dot{y}_1 - r\dot{\phi}\,\mathbf{sin}\,\theta = 0$$

$$a_{11} = 1\,, \quad a_{12} = 0, \quad a_{13} = r\,\mathbf{cos}\,\theta \;, \; a_{14} = 0, \quad b_1 = 0$$

$$a_{21} = 0\,, \quad a_{22} = 1, \quad a_{23} = -r\,\mathbf{sin}\,\theta \;, \; a_{24} = 0, b_2 = 0$$

Additional examples concerning Lagrange equations can be found on the blackboard!