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A General Method for the Evaluation of Vehicle Manoeuvrability with Special Emphasis on Motorcycles

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A General Method for the Evaluation of Vehicle Manoeuvrability with Special Emphasis on Motorcycles

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SUMMARY

This paper presents a novel approach to the assessment of the manoeuvrability of vehicles which is not based on the simulation of open-loop manoeuvres, nor does it rely on the modelling of the driver as a control system.

Instead, the essence of the method is the solution of a two-point optimal control boundary value problem, in which a vehicle, subject to physical constraints like tyre adherence and road borders, among others, is required to go between given initial and final positions as fast as possible. The control inputs – i.e., the driver's actions – that make the vehicle move between the two states in the most efficient way are found as a part of the solution procedure and represent the actions of a sort of *ideal, perfect driver*. The resulting motion is called the *optimal manoeuvre* and, besides being the most efficient way that the given vehicle has for travelling between the two points according to the chosen optimal criterion, may be taken as a reference for meaningful comparisons with other vehicles. The value of the penalty function, used to define the optimal condition occurring at the optimal manoeuvre, may be taken as a measure of manoeuvrability or handling. With this approach the manoeuvrability properties are established as intrinsic to the vehicle, being defined with respect to an ideal perfect driver.

Some possible forms of the penalty function, which means slightly different concepts of manoeuvrability and handling, are discussed. In the end, the case of motorcycles and some examples of optimal manoeuvres are given.

1. INTRODUCTION

Traditional evaluation of vehicle handling is carried out either by simulating its open-loop response to typical inputs or by establishing a driver model and simulating a closed-loop pilot-vehicle system performance [1–4].

The former case is, for example, that of the lane-change manoeuvre for cars and trucks. In cases such as these, the use of pre-defined input may be justified because the drivers of these vehicles act mostly as path planners but rarely attempt to modify the basic dynamic behaviour of the vehicle itself. Obviously this may not always be true, as is the case, for example, of motorcycle and race car drivers who use better control: they do not only plan the path but also modify the natural

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behaviour of the vehicle. For example, motorcycle drivers stabilise the capsize mode and sometimes attempt to control instabilities that may occasionally occur, like the weave mode. Professional drivers use very effective control inputs, specially tailored for the vehicle and the manoeuvre to be done.

The latter case (i.e., simulations based on pilot models) is sometimes used for vehicles that have some unstable modes (in which case the modelling of pilots serves as a means to provide the necessary stabilisation) or for simulating a vehicle following a prescribed path. Once again this might be the case of motorcycles which cannot be simulated without the modelling of at least ‘‘a basic driver’’ to prevent the vehicle from capsizing.

Both approaches may be criticised.

The open-loop simulations cannot be used in the case of unstable vehicles and, moreover, the input used for the simulations might be far from being the most effective for that vehicle and that manoeuvre; whereas it is a fact that expert drivers always provide control actions which are very suited to the combination of vehicle and manoeuvre to be done. In addition, for different vehicles there might be the problem of defining input that is in some sense ‘‘equivalent’’. In fact, comparisons should be done among vehicles performing the same manoeuvre, not (necessarily) with the same input (intended that comparisons are made on vehicles that must perform the same manoeuvre, not necessarily that use the same input). Lastly the interpretation of the results is not straightforward, being based on the history of many state variables.

Simulations based on driver models, on the other hand, are limited by the fact that they largely depend on the model used. No satisfactory driver model exists and therefore there is the risk of drawing misleading conclusions, largely influenced by the model of driver in use and poorly related to the intrinsic properties of the vehicles (whereas it is a fact that vehicles can be better or worse regardless of the driver’s ability).

The basic idea developed in this paper to overcome the above problems is that of comparing different vehicles completing the same manoeuvre, with each vehicle being driven in *its* most efficient way, i.e., to compare the best possible performance of every vehicle. This idea comes from early inverse dynamic analyses, which were introduced with the aim of evaluating car handling on prescribed manoeuvres by a few authors in the recent past [5–9]. More recently Hendrix et al. [10] and Da Lio [11–12] independently and almost simultaneously formulated the idea of using optimal control for carrying out inverse dynamic analysis, the former authors having in mind car handling, the latter thinking of motorcycle manoeuvrability. This paper further develops these ideas, showing that the method is really worthy of attention.

2. VEHICLE MANOEUVRABILITY AND HANDLING

Concepts like manoeuvrability and handling are often used in vehicle dynamics, although they still have no uniform interpretation nor are they rigorously defined.

Therefore, before entering into the mathematical details of the method we are going to describe, we feel that an explanation of what we mean by the above two terms, as well as by the term "manoeuvre", is necessary.

Handling and Manoeuvrability We give the word *handling* the meaning of *easy to drive*. This involves the judgement of a driver who may find different vehicles easier or more difficult to drive. Conversely, we give the word *manoeuvrability* the meaning of *able to perform complex manoeuvres fast*. This means the ability of a vehicle to complete a manoeuvre as fast as possible and without exceeding existing physical limitations, like tyre adherence or road borders.

Unlike handling, manoeuvrability is thus defined regardless of a driver's skill and does not involve his/her judgement. It may therefore be regarded as a more intrinsic property of the vehicle, which is exactly what we are aiming to study.

Manoeuvre By the term *manoeuvre* we mean a generic motion between given initial and final positions or states. During the motion the vehicle must respect some trajectory constraints, which means that some states are not allowed (e.g., again it must move according to available driving forces and respect spatial constraints like lane borders). According to this definition a manoeuvre may be either very simple, such as the change of the travelling direction, or rather complex, such as when a vehicle must follow a portion of a real track.

2.1. Penalty formulation

Both manoeuvrability and handling may be measured by the integral of a penalty function (the lower the penalty, the better the manoeuvrability/handling). For example, in the form of:

$$I = \int_0^T f_0(\boldsymbol{x}, \boldsymbol{u}, t)\, dt \tag{1}$$

where T is the duration of the motion being considered and the integrand f_0 a proper function of system state $\boldsymbol{x}$, control inputs $\boldsymbol{u}$, and possibly explicit time t.

There may be many possible definitions of f_0 to express mathematically the idea of a vehicle moving in a fast and efficient way; the exact meaning of *handling* and *manoeuvrability* will indeed depend on the form actually chosen. However, in view of the above definitions, we may say that if f_0 includes terms that express a *cost* for the driver (e.g., steering torque or rate of variation of steering torque) then equation (1) should be considered as a measure of handling. On the contrary, if f_0 includes only terms related to the efficiency of the motion and to physical limitations (e.g., tyre adherence, thrust limits, spatial constraints), equation (1) should be considered as a measure of manoeuvrability.

It is thus clear that there are as many concepts of handling and manoeuvrability as the number of possible forms of f_0, and that the distinction between handling and manoeuvrability is in fact vanishing. The actual form that we used will be clarified below.

2.2. The Optimal Control Problem

In order to evaluate manoeuvrability by means of the integral above, the motion of the vehicle needs to be given, i.e., functions $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ must be known. However $\boldsymbol{u}(t)$ are inputs and there are many different possible choices, the only requirements being that the generated motion must comply with endpoint and trajectory constraints that define the manoeuvre.

In other words, there are many acceptable solutions of the equations of motion, each being produced by a different choice of control inputs $\boldsymbol{u}(t)$ and representing a different possible way of completing the same manoeuvre.

Of all these solutions however, only one minimises the penalty and represents the most efficient way to do the manoeuvre. This special solution is called the *optimal manoeuvre*. To find it we need to solve an optimal control problem which is shortly outlined below (more detail may be found in the literature [13–15]).

Statement of the optimal control problem

We need to find inputs $\boldsymbol{u}(t)$ such that the penalty given by equation (1) is minimised subject to the following constraints:

a. Equations of motion:

$$\boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{u}, t) = 0 \tag{2}$$

b. Boundary conditions:

$$\begin{aligned} \boldsymbol{\Psi}_j'(\boldsymbol{x}(0)) &= 0 \qquad j = 1, m' \\ \boldsymbol{\Psi}_j''(\boldsymbol{x}(T)) &= 0 \qquad j = 1, m'' \end{aligned} \tag{3}$$

c. Trajectory constraints:

$$\boldsymbol{\Phi}_i(\boldsymbol{x}, \boldsymbol{u}, t) < 0 \quad i = 1, m \tag{4}$$

It is also assumed here for simplicity that the final time T is fixed [1]

The major difference with the usual way of formulating optimal control problems [3–15] is the fact that the equations of motion (2) are given in the form of an *implicit* first-order system, whereas they are usually given in the explicit form. In addition, some of them may be algebraic. The reason is that they are derived from multi-body schematisation [2] which yields the form $\boldsymbol{A}(\boldsymbol{x}, t)\dot{\boldsymbol{x}} = \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{u}, t)$, where matrix $\boldsymbol{A}$ is not necessarily full rank and, in any case, the

[1] Problems with free final time could also be handled by augmenting the system dynamics with additional states.

[2] Of the various methods for deriving the equations of motion, we prefer the natural coordinates approach [1] because it produces a set of simple equations in a moderately redundant number of variables $\boldsymbol{x}$. We prefer to derive the equations in symbolic form by exploiting symbolic-algebra tools.

number of equations, i.e., the order of matrix $\boldsymbol{A}$, is large enough to discourage matrix inversion either symbolically or numerically.

Boundary conditions, as given by (3), usually take the simple form of $x_j(0) = cost'_j$ and $x_j(T) = cost''_j$, which means that state variables are assigned at the beginning and at the end. However, not the whole state vector might be given. In fact, it might be desirable to leave some of the state variables undetermined, as is the case of final velocity and final curvilinear abscissa in the examples below (for this reason the sum of numbers m' and m'' may be less than the number n of equations of motion (2) and of state variables).

Lastly, a final remark concerns trajectory constraints. In equations (4) they are given as inequality constraints. However, manipulating this kind of constraint is not simple (unless the inequality constraint involves only the inputs, in which case the Pontryagin principle can be used). We prefer, therefore not to use equations (4) but rather to modify the penalty function by adding additional terms that apply a large penalty if inequality constraints are violated. In place of (1) we thus use the following:

$$\boldsymbol{I} = \int_0^T \left\{ f_0(\boldsymbol{x},\boldsymbol{u},t) + \sum_{i=1}^{m} \mathrm{wf}_i(\boldsymbol{x},\boldsymbol{u},t) \right\} dt \tag{5}$$

where $\mathrm{wf}_i(\boldsymbol{x},\boldsymbol{u},t)$ are weighting functions that take on a large value (compared to f_0) if corresponding constraints $\boldsymbol{\Phi}_i(\boldsymbol{x},\boldsymbol{u},t)$ are violated, and are instead negligible if constraints are not violated.

The use of (5) in place of (4) is clearly a practical solution. To ensure that constraints (4) are never violated we need to impose a non-negligible penalty, even when functions $\boldsymbol{\Phi}_i(\boldsymbol{x},\boldsymbol{u},t)$ become close to zero, which will become a very large penalty when (4) are definitely violated.

Solution of the optimal control problem

The solution of the optimal control problem formed by equations (1–3) with the penalty function modified according to equation (5) may be reduced to the solution of a system of ordinary differential equations with boundary conditions, as outlined in Appendix 1. If some of the equations forming the mathematical model are algebraic, then algebraic equations are also obtained for the co-equations (the latter equations in (A1.2)). But in the case of the occurrence of algebraic equations we prefer to first reduce the differential-algebraic system to the underlying system of ordinary differential equations by means of one of the methods described in [16], for example the penalty formulation.

We derive the equations of the boundary value problem in symbolic form by means of symbolic algebra software and automatically produce the include files that encode the various terms of the expressions of Appendix 1. These are linked with a main code to produce a program specific for the given mathematical model (if the mathematical model is changed we regenerate the include files and recompile the program).

For the numerical solution of the boundary value problems we initially used the routine BVPFD of the IMSL libraries [17]. It worked fine, but requires that the equations of motion are explicit in $\dot{\boldsymbol{x}}$ which in turn means that system $\boldsymbol{A}(\boldsymbol{x},t)\dot{\boldsymbol{x}} = \boldsymbol{B}(\boldsymbol{x},\boldsymbol{u},t)$ must be solved and this has turned out to be a great computational cost whether it is done symbolically prior to the compilation of the program or numerically during the iteration (the first choice being even worse). In addition, BVPFD needs to know the jacobians of $\dot{\boldsymbol{x}}$ with respect to both the state variables $\boldsymbol{x}$ and the inputs $\boldsymbol{u}$. These can be derived from the implicit form $\boldsymbol{A}(\boldsymbol{x},t)\dot{\boldsymbol{x}} = \boldsymbol{B}(\boldsymbol{x},\boldsymbol{u},t)$ and expressed in terms of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, but the resulting expressions are cumbersome and the whole process is not computationally efficient. For these reasons we turned to the use of the routine SOLVDE of NUMERICAL RECIPES [18]. It allows the equations of motion to be used in the implicit form and this has turned out to be a great advantage.

3. MOTORCYCLE MANOEUVRABILITY

The special case of motorcycles will now be considered, but most of what will be said here may easily be adapted to other types of road vehicles.

3.1. Dynamic Model

The dynamic model used is given in Appendix 2. It is quite simple, focusing on gross motion only, and does not include steering angle or torque (for further details, see Appendix 2). However, this simplicity does not affect the demonstration of the optimal manoeuvre method, but allows the number of equations for the boundary value problem in this first stage to be kept to a reasonable number [3]. Useful information about basic manoeuvrability properties, mainly those related to mass distribution within the vehicle, may still be obtained. By simply changing the mathematical model, the optimal manoeuvre for different vehicles and/or for more sophisticated motorcycle models may be solved.

The model is composed of 7 equations (A2.1) and (A2.2), which only become 8 when reduced to the first order, because many coordinates are cyclic.

Correspondingly, there are 8 state variables: u longitudinal velocity [4], v lateral velocity, ϕ roll angle, $\phi' = \dot{\phi}$ roll rate, $\Psi' = \dot{\Psi}$ yaw rate, N_f and N_r vertical wheel loads and F_r lateral force on the rear wheel.

[3] Nevertheless, the optimal control problem that results from this model is composed of 27 ordinary differential equations with boundary conditions, plus three equations for the three control inputs (see Appendix 1).

[4] More precisely we consider the velocity of the point of the rigid system which coincides with the contact point of the rear wheel when the vehicle is unrolled.

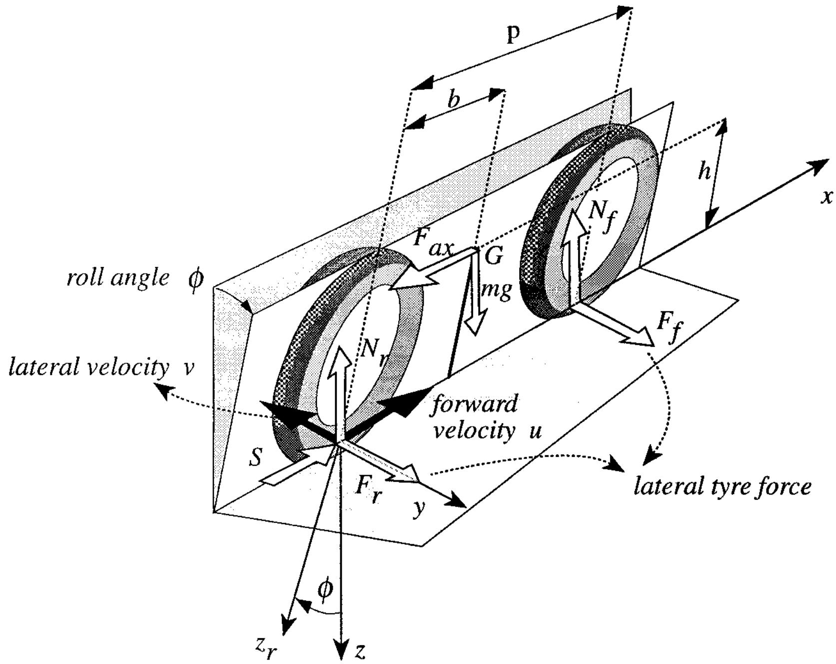


Fig. 1. Mathematical model of motorcycle. External forces acting on vehicle.

There are 3 control inputs, namely ff, s and k_S, which determine driving forces F_f and S and the distribution of S between the front and rear wheels when braking. As discussed in Appendix 2, we chose as inputs ff, s and k_S which are respectively the time derivatives of F_f and S and the fraction of the total thrust at the rear wheel. These are used in place of F_f and S because limits may thus be imposed on the rate of change of both F_f and S. However, this leads to the two additional differential equations (A2.3) that link the inputs to F_f and S.

3.2. Track and Manoeuvre Description

The position of the vehicle is given in a system of curvilinear coordinates that follow the road lane, rather than by means of absolute coordinates x, y and absolute yaw Ψ. As shown in Fig. 2, the position of the vehicle [5] is defined by curvilinear abscissas s_1, s_2, the former following the lane centreline, the latter defining the position of the vehicle perpendicular to the centreline. Similarly, the orientation of the vehicle is measured from the centreline tangent by angle α in place of using absolute yaw Ψ.

The choice of curvilinear coordinates makes it easier to formulate initial and final conditions relative to the lane and to define the trajectory constraint that

[5] As for the velocity, we consider the position of the point in the rigid system which coincides with the contact point of the rear wheel when the vehicle is unrolled.

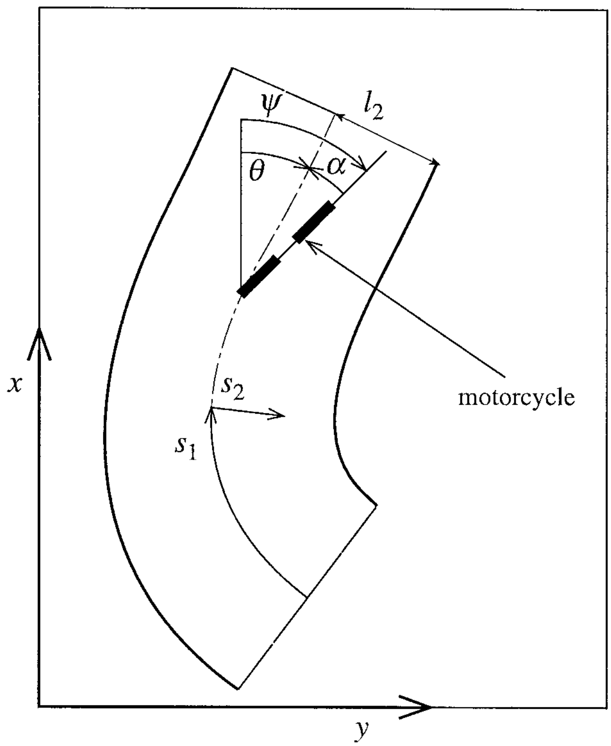


Fig. 2. System of curvilinear coordinates following road lane.

makes the ''vehicle stay within the lane''. In fact, if l_2 represents the half-width of the lane (constant or possibly a function of s_1) this constraint is written simply as:

$$-l_2 < s_2 < l_2 \tag{6}$$

The shape of the road lane is described by the given functions $xc(s_1)$, $yc(s_1)$ that define the centreline as a parametric curve. These functions are derived from double integration of centreline curvature $\Theta(s_1)$. Similarly, slope $\theta(s_1)$ of the centreline is derived by simple integration of curvature $\Theta(s_1)$. At this point the transformation from curvilinear coordinates s_1, s_2, α, and absolute coordinates x, y and Ψ is simple:

$$\begin{aligned} x &= xc(s_1) - s_2 \sin(\theta(s_1)) \\ y &= yc(s_1) + s_2 \cos(\theta(s_1)) \\ \Psi &= \theta(s_1) - \alpha \end{aligned} \tag{7}$$

Table 1 Boundary conditions used for optimal control problem.

State variable	**Boundary conditions**	
	Entrance	**Exit**
Forward velocity u	assigned (u_0)	free
Lateral velocity v	assigned (0)	assigned (0)
Roll angle ϕ	assigned (0)	assigned (0)
Roll rate ϕ'	assigned (0)	assigned (0)
Yaw rate ψ'	assigned (0)	assigned (0)
Front tyre vertical force N_f	assigned *	free
Front tyre lateral force F_f	assigned (0)	assigned (0)
Rear tyre vertical force N_r	assigned *	free
Rear tyre lateral force F_r	assigned (0)	assigned (0)
Thrust (global) S	free	free
Yaw relative to centreline direction α	assigned (0)	assigned (0)
Curvilinear abscissa s_1	assigned (0)	free
Curvilinear ordinate s_2	assigned (s_{20})	free

* Consistent with steady-state equilibrium.

The position of the vehicle is obtained by integrating the velocities in the system of curvilinear coordinates s_1, s_2, α, as described in detail in Appendix 3. The following 3 additional equations result:

$$\dot{s}_1 = \frac{1}{1 - s_2 \sin[\Theta(s_1)]}\{u\cos(\alpha) + v\sin(\alpha)\}$$

$$\dot{s}_2 = v\cos(\alpha) + u\sin(\alpha) \tag{8}$$

$$\dot{\alpha} = \Theta(s_1)\,\dot{s}_1 - \psi'$$

With these, the total number of equations becomes 13: the 8 first order equations derived from (A2.1) and (A2.2), the 2 in (A2.3) and the 3 in (8) [6]. Correspondingly, there are 13 state variables: u, v, ϕ, ϕ', ψ', N_f, N_r, F_f, F_r, S, s_1, s_2, α; the system being controlled by the 3 inputs ff, s, k_S.

Boundary conditions

To complete the description of the manoeuvre, the initial and final conditions must be given. There are many possibilities. The choices that we made are summarised in Table 1. At the beginning we assign initial position and velocity, leave the initial thrust S free and assign vertical reactions consistent with it. At the end we leave the final longitudinal position and velocity (s_1 and u) free so that the travel distance may really be maximised as is required by the penalty function below. We accept different exit positions s_2 (not necessarily the centreline) but the final

[6] The optimal control problem will therefore have 27 equations: the 13 above, plus the 13 co-equations, plus the penalty equation (which is integrated simultaneously) [13].

direction of travel must be parallel to the centreline. The conditions chosen work well when the final and initial parts of the path, as described by equations (7), are straight. This is not a problem because, in practice, all simulations will be made on parts of a track taken between straight lines.

3.3. Penalty function

The forms that can be used for the penalty function will now be considered. A major distinction should be made concerning the final time T. There are basically two possibilities: either T is free, or fixed.

In the former case there is, in theory, the possibility of solving the so-called *minimum time* problem (the penalty function in this case is a unit function, i.e., $f_0 = 1$). This would mean finding the motion of a vehicle that travels in a *minimum time* between the two endpoints and satisfies all the trajectory constraints. If this were done, manoeuvrability would be measured by the time required to complete the manoeuvre.

However, although seemingly very attractive, this approach has not been adopted here mainly for simplicity reasons: because of the additional state variables and equations that are needed for having a free final time, and, above all, because of the consideration that the additional terms introduced in equation (5) for the trajectory constraints would negatively affect the meaning of the penalty (i.e., time).

If T is considered to be fixed, as in fact was done here, the penalty may be written in a way that the average velocity and travel distance are maximised. It can be said that, instead of minimising the time for covering a fixed travel distance, we are trying to maximise the travel distance in a fixed amount of time. This of course requires that the final abscissa s_1 and velocity u be left free, as shown in Table 1.

In other words, we are solving the problem of a vehicle *travelling as far as possible* in the available time, so that manoeuvrability be measured by the travel distance, or more properly by the value of integral (1).

The form used for the penalty function is thus the following:

$$f_0 = -\frac{\dot{s}_1}{V_0} + \left(\frac{s_2}{l_2}\right)^{2n} + \left(\frac{F_r^2 + S_r^2}{(f_a N_r)^2}\right)^n + \left(\frac{F_f^2 + S_f^2}{(f_a N_f)^2}\right)^n + \left(\frac{\tau \cdot ff}{F_0}\right)^2 + \left(\frac{\tau_1 \cdot s}{S_0}\right)^2$$

$$+ \begin{cases} \left(\dfrac{S_r}{S_{\max}}\right)^{2n} & S > 0 \\ 0 & S \le 0 \end{cases} \tag{9}$$

The first term is the most important. It requires that the final abscissa s_1 be maximised [7], V_0 being a suitable velocity scale (in fact, all terms of the penalty

[7] In fact the minimisation of integral $\int_0^T -\frac{\dot{s}_1}{V_0} dt$ results in the maximisation of final distance $s_1(T)$.

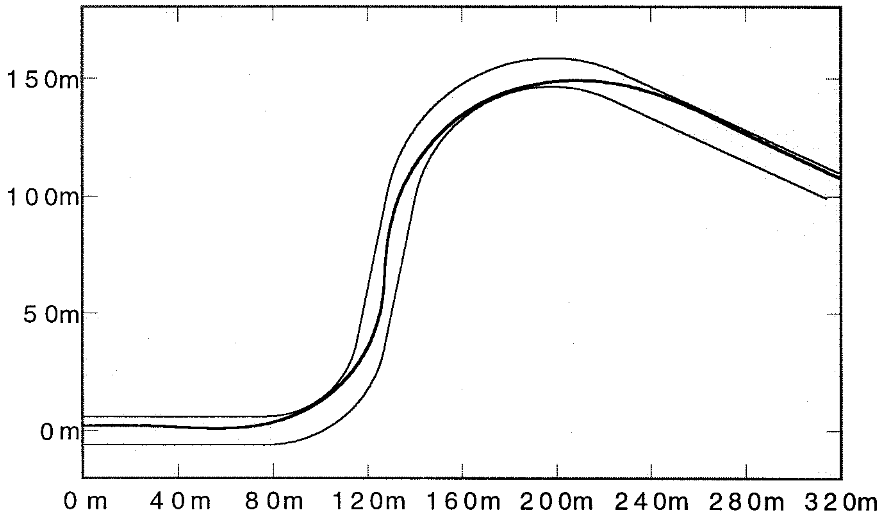


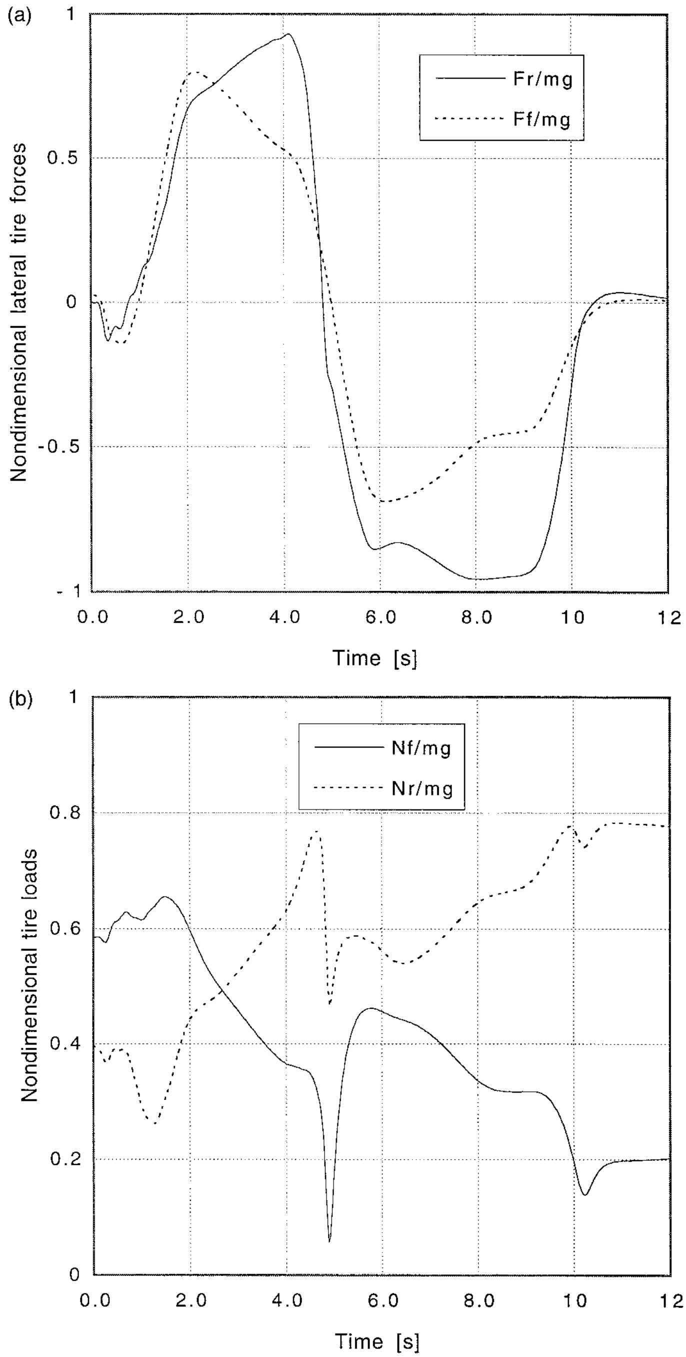
Fig. 3. Luco - Poggio Secco curves at Mugello racing track. Figure also shows optimal path computed with optimal manoeuvre method for a motorcycle whose data reflects those of a real racing motorcycle.

function are made non-dimensional, scaled in a way that they take a value greater than one when they become non-acceptable).

Table 2 Parameters of two motorcycle designs.

Parameter	**Value**	
	Motorcycle 1 (reference)	**Motorcycle 2**
Motorcycle mass * m (Kg)	172.0	169.0
Roll moment of inertia * I_x (Kg m^2)	9.0	8.5
Pitch moment of inertia * I_y (Kg m^2)	30.0	30.5
Yaw moment of inertia * I_z (Kg m^2)	20.0	20.0
Roll-Yaw product of inertia * I_{xz} (Kg m^2)	−2.5	2.5
Wheelbase p (m)	1.35	1.37
Horizontal position of mass centre b - see Fig. 1 (m)	0.63	0.60
Height of mass centre h - see Fig. 1 (m)	0.640	0.670
Parameter for wheels/flyweel gyroscopic effect I_e (Kg)	3.05	3.05
Average toroidal radius rt (m)	0.065	0.065
Rear wheel toroidal radius rt_r (m)	0.08	0.08
Front wheel toroidal radius rt_f (m)	0.05	0.05
Rear tyre sideslip stiffness C_1 (N/rad)	16000	18000
Rear tyre camber stiffness C_2 (N/rad)	640	720
Rear tyre relaxation length σ_r (m)	0.2	0.2
Coefficient for the adherence in the penalty function f_a	1.8	1.8

* driver included



(a)
Fr/mg
Ff/mg
Nondimensional lateral tire forces
1
0.5
0
-0.5
-1
0.0
2.0
4.0
6.0
8.0
10
12
Time [s]
(b)
Nf/mg
Nr/mg
Nondimensional tire loads
1
0.8
0.6
0.4
0.2
0
0.0
2.0
4.0
6.0
8.0
10
12
Time [s]

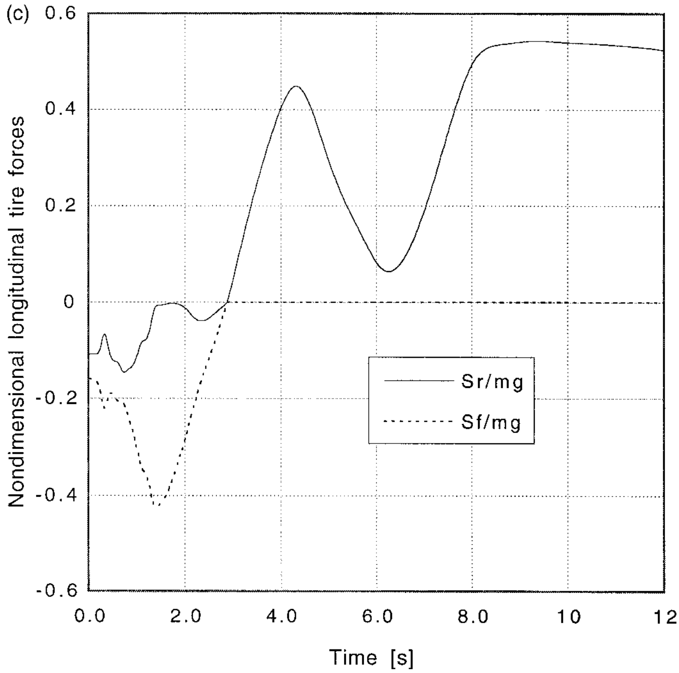


Fig. 4. Forces acting on vehicle during simulated optimal manoeuvre. *a* - lateral tyre forces. *b* - vertical tyre forces. *c* - thrust-braking forces. Of special interest are lateral force on front wheel and thrust.

The second term stands for the trajectory constraints given in (6). It depends on an integer parameter *n*. The larger *n*, the closer the term will resemble constraints given in equations (6) [8].

The third and fourth terms express the requirement that forces on the rear and front wheel must stay within the limits of tyre traction. Here numerators are the squared values of the horizontal force acting on each wheel (see Appendix 2), denominators are the squared admissible forces, taken as the vertical load times the traction coefficient f_a.

The 5th and 6th terms apply a penalty to the rate of change, respectively *ff* and *s*, of the two control forces F_f and S (A2.3). These two terms, unlike the former 4, are not to the power $2n$, but simply squared. This choice derives from the fact that there is no well-defined limit on the rate of change of the control forces, but rather there is an ever increasing cost of actuation, F_0/τ and S_0/τ_1 being the values at which this cost becomes a unit. In other words, the two latter terms do not express

[8] In practice, to avoid numerical troubles, we solve a sequence of optimisation problems, starting with $n = 1$ and increasing until it is large enough (about 10–20).

an inequality constraint, as did term 2 for equations (6), but rather a cost of actuation which we may interpret as a ‘‘handling’’ term (in the sense explained above for handling, i.e., driver-related cost).

Lastly, the 7th term expresses the constraint that the thrust on the rear wheel S_r be less than the maximum available thrust S_{max} (i.e., $S < S_r$).

4. EXAMPLES OF MOTORCYCLE MANOEUVRABILITY

Two examples are given here of the evaluation of the manoeuvrability of motorcycles, done with the method and dynamic model described above.

The track chosen is the circuit The Mugello in Italy. We focused on a group of curves named Luco – Poggio Secco which are depicted in Fig. 3. It is an S-shaped path. The motorcycle enters at the left lower corner and exits to the right. Fig. 3 shows the path – evaluated with the method of the optimal manoeuvre – followed by the ‘‘reference’’ motorcycle, whose data are given in Table 2; this reflects those of a real racing motorcycle (although the entrance point is not exactly known). Figs. 4.a, 4.b, 4.c show the tyre forces during the manoeuvre which results from the application of the method. Of special interest are the lateral forces on the front wheel (Fig. 4.a, dashed line) and the thrust (Fig. 4.c) because they sum up the effects of the three control inputs. In Fig. 4.b at time 4.8 s there is a sudden decrease on both front and rear vertical tyre forces which occurs when the vehicle rapidly changes its roll angle in the chicane. It also shows the transfer of load between the two wheels during braking (at the beginning) and acceleration (at the end). Fig. 5 is the most interesting as it shows the resulting vehicle velocity and compares it to the velocities that may be computed on the basis of the measurements of the angular velocities of the two wheel (these latter are different because the slip of the two wheels is different, especially during acceleration when only the rear wheel slips). There are two minima in the velocity, in both the function computed by the optimal manoeuvre method and the one derived from the telemetry. The minima occur at the passage of the chords of the two curves (Fig. 3).

Lastly, Fig. 6 shows the potential benefits that could derive if certain changes (given in the second column of Table 2) were applied to the design of the vehicle. As shown, the modified vehicle is predicted to be faster. The travel distance by the modified vehicle is in fact estimated to increase by 2.52 meters (from 505.64 m to 508.16 m). Since the exit velocity is about 55 m/s, this increase in distance corresponds to an advantage of about 0.05 s. Actually, the changes applied here are beyond what can be done easily and are used only to emphasise the differences in Fig. 5. But the important fact is that the application of the method *clearly shows* which vehicle is better and how much improvement can be obtained. In fact, the application of design changes similar to the above has given the expected improvements in a real modified vehicle. Fig. 7 shows a different point of view: it plots the difference between the penalty functions for the two vehicles and points

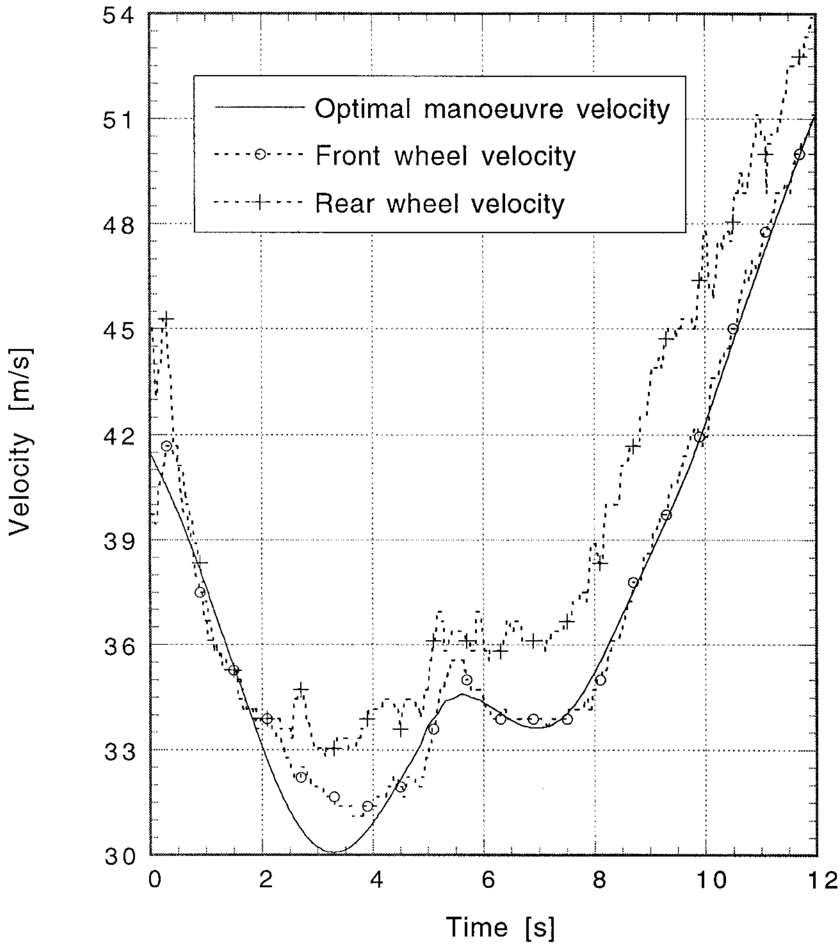


Fig. 5. Velocity of vehicle obtained from optimal manoeuvre method (solid line) and velocities derived from telemetry (rear wheel: dotted line with cross markers; front wheel: dotted line with circle markers).

out when the newer design has an advantage with respect to the older one, i.e., almost always.

CONCLUSIONS

This paper demonstrates a new approach for the evaluation of vehicle handling and manoeuvrability. It consists of finding the optimal manoeuvre that a given vehicle may execute for moving between two given points according to all holding

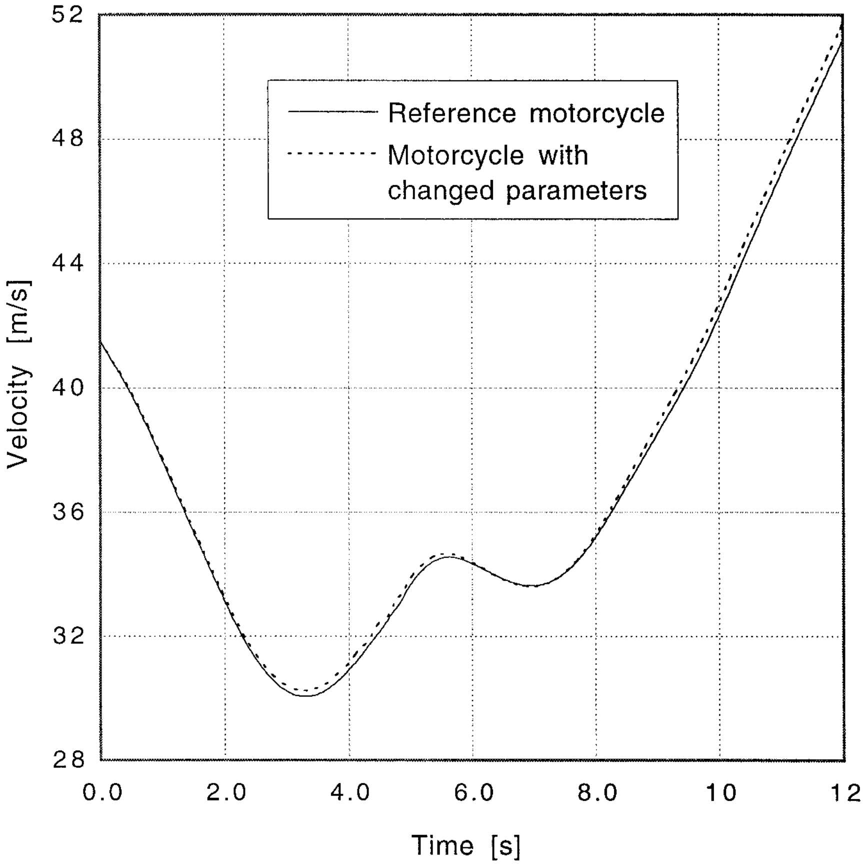


Fig. 6. Evaluation of improvements yielded by modified design. Dashed line: older design.

trajectory constraints. This is done by solving a two-point optimal control problem.

The penalty function that defines the optimal criterion gives the exact meaning of ''handling'' and ''manoeuvrability'' and provides a means for *measuring* them as a unique scalar index that is easy to compare. In our examples, we used a penalty function providing maximisation of the travel distance in a given time. In this way manoeuvrability is measured by the distance travelled.

This method has a great advantage over open loop simulations and, to a lesser extent, also over closed-loop simulations based on pilot models. The advantage is that comparisons are based on the ''best'' performance that each vehicle is able to produce. This provides a clear, unambiguous comparison even when, because the differences involved are very small, the comparison of the history of state variables (forces, velocities, etc.) is not easily interpreted.

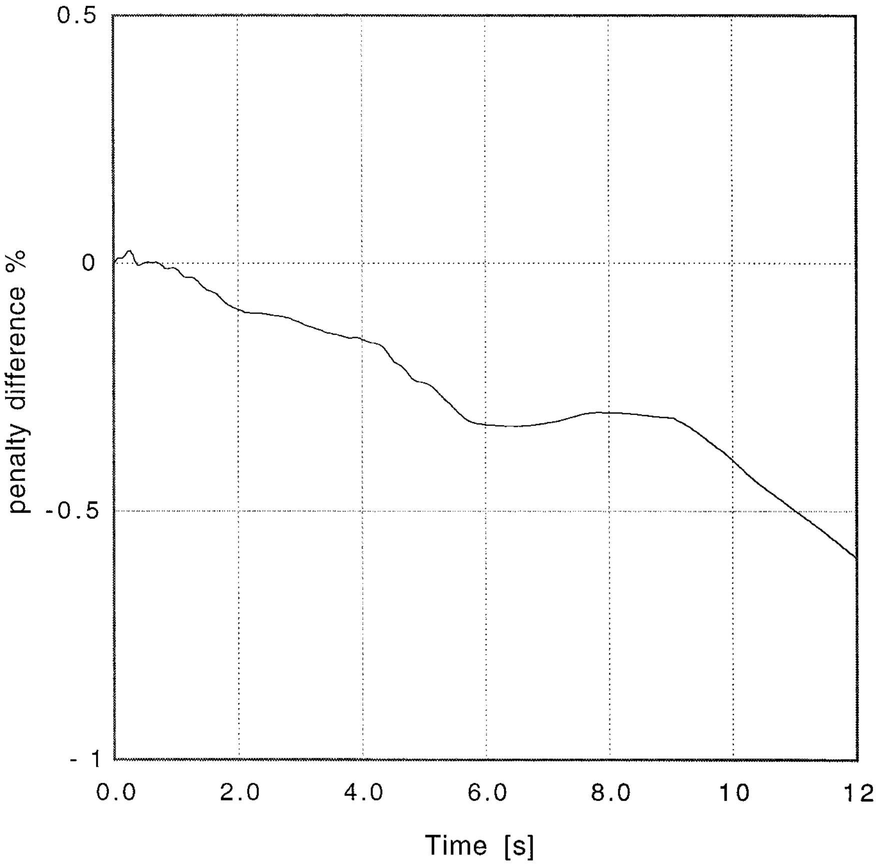


Fig. 7. Comparison between penalty function of two alternative designs reveals not only advantage of newer design, but also where the latter is better.

We presented an example of a racing motorcycle and compared the velocity resulting from the method to that derived from the telemetry, showing there is good agreement. The potential benefit of an alternative design has been shown.

APPENDIX 1 - SOLUTION OF OPTIMAL CONTROL PROBLEM

The solution of the constrained minimisation problem given by equation (5,2,3) is reduced to the unconstrained minimisation of:

$$I' = \sum_{j=1}^{m'} \nu_j' \boldsymbol{\Psi}_j'(\boldsymbol{x}(0)) + \sum_{j=1}^{m''} \nu_j'' \boldsymbol{\Psi}_j''(\boldsymbol{x}(T)) + \int_0^T \left\{ f(\boldsymbol{x},\boldsymbol{u},t) + \sum_{i=1}^{n} \lambda_i(t) F_i(\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{u},t) \right\} dt \qquad \text{(A1.1)}$$

where ν', ν'' and $\lambda(t)$ are Lagrange multipliers and $f(\boldsymbol{x},\boldsymbol{u},t)$ stands for the modified penalty function $f_0(\boldsymbol{x},\boldsymbol{u},t) + \sum_{i=1}^{m'} \mathrm{wf}_i(\boldsymbol{x},\boldsymbol{u},t)$.

By means of the calculus of variations, the following conditions are derived [13–15]:

$$\begin{aligned} &F_i(\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{u},t) && i = 1,n \\ &\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}L - \frac{\partial}{\partial x_i}L = 0 && i = 1,n \end{aligned} \tag{A1.2}$$

$$\frac{\partial}{\partial \boldsymbol{u}}L = 0 \tag{A1.3}$$

$$\begin{aligned} &\boldsymbol{\Psi}_j'(\boldsymbol{x}(0)) = 0 && j = 1,m' \\ &\boldsymbol{\Psi}_j''(\boldsymbol{x}(T)) = 0 && j = 1,m'' \\ &\sum_{j=1}^{m'} \nu_j' \frac{\partial \boldsymbol{\Psi}_j'}{\partial x_i}\bigg|_0 + \frac{\partial L}{\partial \dot{x}_i}\bigg|_0 = 0 && i = 1,n \\ &\sum_{j=1}^{m'} \nu_j'' \frac{\partial \boldsymbol{\Psi}_j''}{\partial x_i}\bigg|_T + \frac{\partial L}{\partial \dot{x}_i}\bigg|_T = 0 && i = 1,n \end{aligned} \tag{A1.4}$$

where L stands for the Lagrange function:

$$L = L(\boldsymbol{x},\dot{\boldsymbol{x}},\lambda,\boldsymbol{u},t) = f(\boldsymbol{x},\boldsymbol{u},t) + \sum_{i=1}^{n} \lambda_i(t) F_i(\boldsymbol{x},\dot{\boldsymbol{x}},\boldsymbol{u},t)$$

and subscripts *0* and T in equations (A1.4) indicate that the conditions must be evaluated for $t = 0$ and $t = T$.

The result is a system of $2n$ differential equations (A1.2) in the unknowns $\boldsymbol{x}$, $\boldsymbol{\lambda}$ with $2n$ boundary conditions, that remain from equations (A1.4) after the Lagrange multipliers ν', ν'' are eliminated. Equations (A1.3) yield the values for inputs $\boldsymbol{u}$ that at any time t produce optimal control. If inequality constraints hold for the inputs, then the inputs to be used are those that minimise L within the constraints, according to the Pontryagin principle [13–15].

APPENDIX 2 - DYNAMIC MODEL OF MOTORCYCLE

The model given here focuses on the description of the gross motion only. A motorcycle, which is regarded as a rigid body, is thought of as having only 4 degrees of freedom, namely: roll, yaw, and the two displacements in the horizontal plane. The influence of pitch, as well as that of the rotation of the front frame upon the linear and angular momenta are neglected. The gyroscopic effect of the wheels and flywheel is accounted for, except for the contribution due to the steering rate, which is definitely small compared to the other angular rates. It is considered that the lateral displacement of driver torso does not occur (e.g., the pilot is thought of as rigidly connected to the motorcycle – but this assumption can be easily eliminated, as explained below). The road is considered even since we are essentially committed to the evaluation of the ability of the vehicle in changing the travel direction. For the same reason, the suspensions are not modelled, which again means that the effect of pitch and heave due to suspension stroke is not evaluated (for now).

The whole system is subject to the forces depicted in Fig. 1, i.e., gravity $m\boldsymbol{g}$, aerodynamic drag F_{ax}, reactions normal to the road plane N_f and N_r, lateral forces on tyres F_f and F_r, and thrust S. The tyre forces are applied to the point where the wheels meet the road plane, taking into account the radii of the tyre cross sections (except for the fact that pitch and front frame rotation are neglected).

The model is made up of six differential equations for the motion of the vehicle in the unknowns longitudinal velocity [9], ν lateral velocity, ϕ roll angle, ψ yaw angle, N_f and N_r tyre reactions normal to road plane.

$$m\left\{\dot{u}-\dot{\psi}\nu-h\sin(\phi)\ddot{\psi}-b\dot{\psi}^2-2h\dot{\psi}\dot{\phi}\cos(\phi)\right\}=-F_{ax}+S$$

$$m\left\{\dot{\nu}+\dot{\psi}u+h\cos(\phi)\ddot{\phi}+b\ddot{\psi}-h\dot{\phi}^2\sin(\phi)-h\dot{\psi}^2\sin(\phi)\right\}=F_f+F_r \quad \text{(A2.1)}$$

$$m\left\{(h-rt)\dot{\phi}^2\cos(\phi)+(h-rt)\sin(\phi)\ddot{\phi}\right\}=-N_f-N_r+mg$$

$$\begin{aligned}I_x\ddot{\phi}-I_{xz}\cos(\phi)\ddot{\psi}+\left(I_z-I_y\right)\dot{\psi}^2\cos(\phi)\sin(\phi)+I_e\dot{\psi}u\cos(\phi)\\=\left(rt_f-h\right)F_f\cos(\phi)+(rt_r-h)F_r\cos(\phi)-\left(rt_f-h\right)N_f\sin(\phi)\\-(rt_r-h)N_r\sin(\phi)-rt_fF_f-rt_rF_r\end{aligned}$$

$$\begin{aligned}I_y\sin(\phi)\ddot{\psi}-I_e\dot{u}+I_{xz}\dot{\phi}^2-I_{xz}\psi^2\cos^2(\phi)+\left(I_x+I_y-I_z\right)\dot{\psi}\dot{\phi}\cos(\phi)\\=(p-b)F_f\sin(\phi)-bF_r\sin(\phi)+(p-b)N_f\cos(\phi)-bN_r\cos(\phi)\\+(rt\cos(\phi)-rt+h)S\end{aligned}$$

[9] More precisely we consider the velocity of the point of the rigid system which coincides with the contact point of the rear wheel when the vehicle is not rolled.

$$I_z \cos(\phi)\ddot{\psi} - I_{xz}\ddot{\phi} - I_e\dot{\phi}u + \left(-I_x + I_y - I_z\right)\dot{\psi}\dot{\phi}\sin(\phi)$$

$$+ I_{xz}\dot{\psi}^2\cos(\phi)\sin(\phi) = (p-b)F_f\cos(\phi) - bF_r\cos(\phi)$$

$$-(p-b)N_f\sin(\phi) + bN_r\sin(\phi) - rtS\sin(\phi)$$

A seventh equation yields the lateral force on the rear wheel F_r as a function of tyre roll and slip angles, here by means of a simple linear dynamic model.

$$\frac{\sigma_r}{u}\dot{F}_r + F_r = C_1\lambda + C_2\phi \tag{A2.2}$$

with σ_r being the relaxation length and λ the side slip angle.

The vehicle is controlled by means of the remaining forces F_f and S. These are assumed to be controllable by the driver. For example, the lateral force on the front wheel F_f will be the result of the rotation of the front frame, which is controlled by the driver. Similarly, the thrust or braking force S will be controlled by the driver by means of throttle and brakes. We do not focus here on the details of the formation of these forces, we simply assume that the desired forces F_f and S may be produced by the driver with some suitable action (the penalty function will ensure that F_f and S stand within admissible values). For example, in the case of the lateral force F_f, we do not analyse how the steering torque applied by the driver becomes a rotation of the front frame and ultimately a lateral force on the tyre. We assume that a desired force F_f will be produced by some torque history. To evaluate manoeuvrability it is not necessary to know what the driver's torque is; it would be necessary only if the cost of actuation for the driver were to be accounted for, i.e., only for the evaluation of handling (according to the above definitions).

Forces F_f and S are not considered as inputs themselves, but rather they are given by means of the following equations:

$$\begin{aligned} \dot{F}_f &= ff \\ \dot{S} &= s \end{aligned} \tag{A2.3}$$

where ff and s are the actual inputs and represent the time derivatives of F_f and S. This is done only in order to be able to impose limits on the rate of change of F_f and S. In fact, if equations (A2.3) were not used and F_f and S were considered themselves as inputs, then they would tend to have abrupt changes at certain times (like in a bang-bang control). With the assumption of equations (A2.3) and the imposition of limits (see the penalty function) on the maximum and minimum values of ff and s we obtain a smooth and more realistic behaviour for F_f and S.

In addition, force S is considered to be applied to the rear wheel only during thrust, but shared between the two wheels during braking, i.e., the longitudinal forces S_r and S_f, respectively acting on rear and front wheels, are:

$$\begin{cases} S_r = S, \quad S_f = 0 & S \geq 0 \\ S_r = k_S \cdot S, \quad S_f = (1 - k_S) \cdot S & S < 0 \end{cases} \tag{A2.4}$$

the former row holding during thrust, the latter holding during braking. In the above equations k_S is therefore the coefficient which represents the fraction of the total force applied to the rear wheel. It is treated as a third control input (which appears only in the penalty function) and found like the others in solving the optimal control problem (which means that the braking applied is always the optimal braking).

A final remark concerns the lateral displacement of the driver’s torso. As mentioned above, this is not considered here, but, if it were to be included, this could be done easily by means of an additional control input defining the lateral displacement of the system centre of mass in a way similar to equations (A2.3). It is worth noting that the solution of the optimal control problem in this case would yield the most efficient way of leaning the torso as part of the solution.

APPENDIX 3 - INTEGRATION OF VEHICLE VELOCITIES IN TRACK CURVILINEAR SYSTEM OF COORDINATES

The position of the vehicle, in the absolute coordinates x, y and ψ may be obtained by integrating the velocities as follows:

$$\begin{cases} \dot{\psi} = \psi' \\ \dot{x} = u\cos(\psi) - v\sin(\psi) \\ \dot{y} = v\cos(\psi) + u\sin(\psi) \end{cases} \tag{A3.1}$$

By substituting equations (7) in the above, equations (8) may be derived.

LIST OF SYMBOLS

u	forward (longitudinal) velocity
v	lateral velocity

ϕ	roll angle
ϕ'	roll rate
ψ'	yaw rate
N_f	front tyre load
F_f	front tyre lateral force
N_r	rear tyre load
F_r	rear tyre lateral force
S	thrust (global)
α	yaw relative to centreline direction (see Fig. 2)
s_1	curvilinear abscissa (see Fig. 2)
s_2	curvilinear ordinate (see Fig. 2)
λ	side slip angle (in Eq. A2.2)
m	motorcycle mass (driver included)
I_x	roll moment of inertia relative to centre of mass (driver included)
I_y	pitch moment of inertia relative to centre of mass (driver included)
I_z	yaw moment of inertia relative to centre of mass (driver included)
I_{xz}	roll-yaw product of inertia relative to centre of mass (driver included)
p	wheelbase
b	longitudinal distance of centre of mass from rear wheel centre (see Fig. 1)
h	height of the centre of mass (see Fig. 1)
I_e	parameter for wheels/flywheel global gyroscopic effect
rt	average toroidal radius
rt_r	rear wheel toroidal radius
rt_f	front wheel toroidal radius
C_1	rear tyre sideslip stiffness
C_2	rear tyre camber stiffness
f_a	weight for tyre traction in the penalty function
l_2	lane half-width
V_0	velocity scale in the penalty function
$F_0/\tau,\ S_0/\tau_1$	scales for control inputs in the penalty function
$S_{\max}$	maximum thrust
F_{ax}	aerodynamic drag
t	time
T	final time
$\boldsymbol{x}$	system's state vector
$\boldsymbol{u}$	control inputs
$ff,\ s,\ k_S$	the three control inputs
f_0	penalty function
f	modified penalty function

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