D'Alambert Principle & Kinematic Constraints

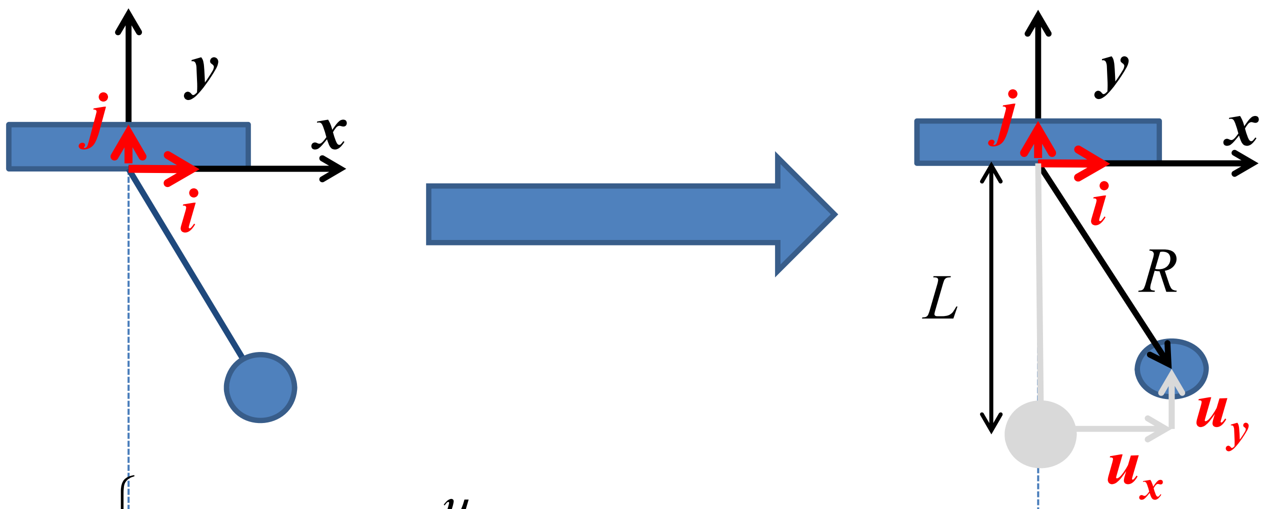
An Introduction to Analytical Dynamics

WB 1418-07

Farbod Alijani

We found that if we directly apply Newton's 2nd law to a system we may find a complicated system of equations

$$\mathbf{X} = m\,\mathbf{a}$$



$$\begin{cases} m\ddot{u}_x = -\dfrac{u_x}{L}T \\ m\ddot{u}_y = -\dfrac{u_y - L}{L}T - mg \\ L = \sqrt{(u_y - L)^2 + u_x^2} \end{cases}$$

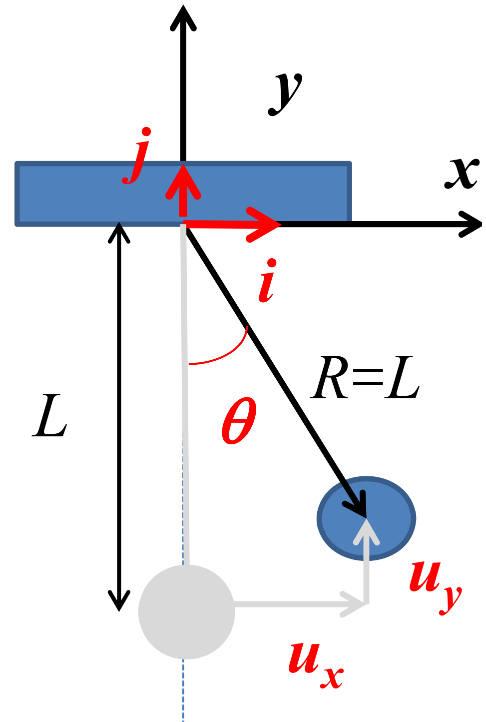
Constraint

3 equations, 3 unknowns

How can we get rid of constraint and its associated force?

- ✓ We introduced generalized coordinate θ.
- ✓ We defined a function **U** in such a way that it gives kinematically admissible displacements

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} L\sin(\theta) \\ L - L\cos(\theta) \end{bmatrix} \Rightarrow L = \sqrt{(L - L\cos(\theta) - L)^2 + (L\sin(\theta))^2}$$

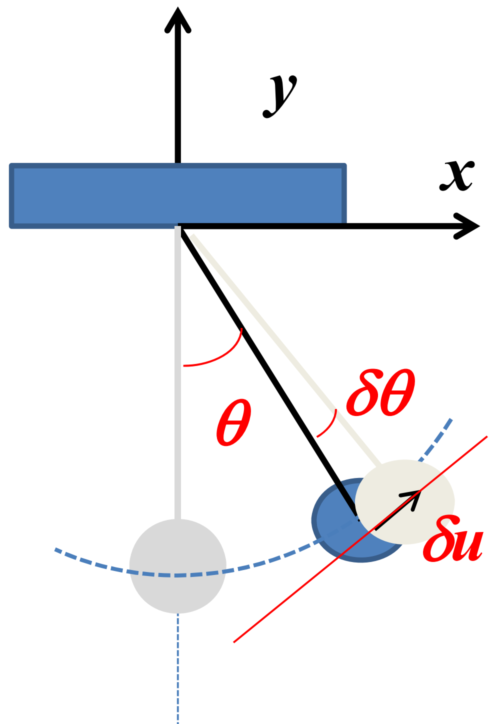


- ✓ We found the dynamic equilibrium in terms of the new coordinate.

$$\mathbf{F} = \begin{cases} mL\ddot{\theta}\cos(\theta) - mL\dot{\theta}^2\sin(\theta) + T\sin(\theta) = 0 \\ mL\ddot{\theta}\sin(\theta) + mL\dot{\theta}^2\cos(\theta) - T\cos(\theta) + mg = 0 \end{cases}$$

2 equations, 2 unknowns

So what if I kick the mass and then look at the admissible direction?



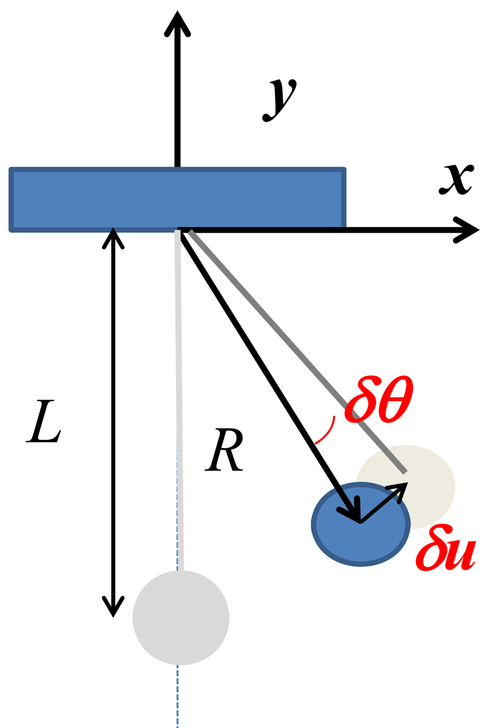
$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} L\,\mathbf{sin}(\theta) \\ L - L\,\mathbf{cos}(\theta) \end{bmatrix} \Rightarrow \begin{bmatrix} \delta u_x \\ \delta u_y \end{bmatrix} = \begin{bmatrix} L\,\mathbf{cos}(\theta)\delta\theta \\ L\,\mathbf{sin}(\theta)\delta\theta \end{bmatrix}$$

- ✓ **So it makes sense that the mass moves only in tangential direction which is indeed the admissible direction.**
- ✓ **Since admissible direction is orthogonal to position it might be possible to get rid of reaction forces**

$$\mathbf{R}\,.\,\delta\,\mathbf{u} = 0$$

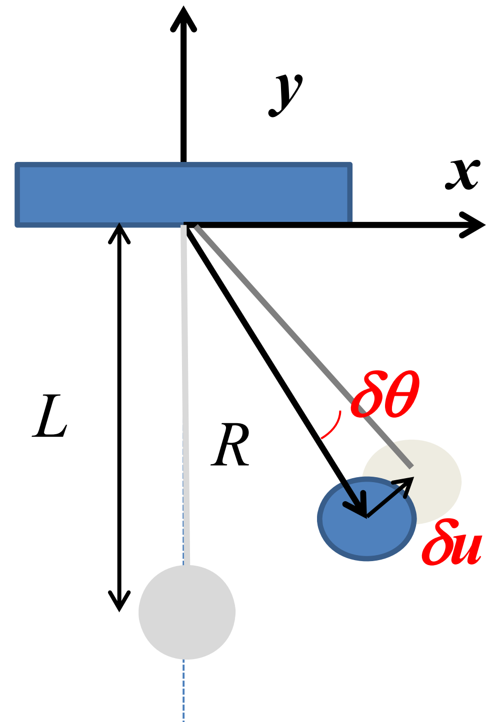
By projecting dynamic equilibrium onto the admissible direction

$$\mathbf{F}\,.\,\delta\,\mathbf{u} = 0.\delta\,\mathbf{u} = 0$$



$$\mathbf{F}.\delta\mathbf{u} = 0.\delta\mathbf{u} = 0$$

$$\mathbf{F} = \begin{bmatrix} mL\ddot{\theta}\,\mathbf{cos}(\theta) - mL\dot{\theta}^2\,\mathbf{sin}(\theta) + T\,\mathbf{sin}(\theta) \\ mL\ddot{\theta}\,\mathbf{sin}(\theta) + mL\dot{\theta}^2\,\mathbf{cos}(\theta) - T\,\mathbf{cos}(\theta) + mg \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$\delta\mathbf{u} = \begin{bmatrix} L\,\mathbf{cos}(\theta)\delta\theta \\ L\,\mathbf{sin}(\theta)\delta\theta \end{bmatrix}$$

$$\left(mL\ddot{\theta}\,\mathbf{cos}(\theta) - mL\dot{\theta}^2\,\mathbf{sin}(\theta) + T\,\mathbf{sin}(\theta)\right).\left(L\,\mathbf{cos}(\theta)\delta\theta\right) +$$
$$\left(mL\ddot{\theta}\,\mathbf{sin}(\theta) + mL\dot{\theta}^2\,\mathbf{cos}(\theta) - T\,\mathbf{cos}(\theta) + mg\right).\left(L\,\mathbf{sin}(\theta)\delta\theta\right) = 0$$

$$\Downarrow$$

$$\left(mL^2\ddot{\theta} + mgL\,\mathbf{sin}(\theta)\right)\delta\theta = 0 \;\; \forall\,\delta\theta \rightarrow mL^2\ddot{\theta} + mgL\,\mathbf{sin}(\theta) = 0 \;\; \mathbf{or} \;\; \ddot{\theta} + \frac{g}{L}\mathbf{sin}(\theta) = 0$$

The reaction force is disappeared ! We are left with 1 equation and 1 unknown. This is the consequence of projection onto the direction perpendicular to constraint

Analytical Dynamics

- ✓ D'Alambert Principle is the basis of analytical dynamics.

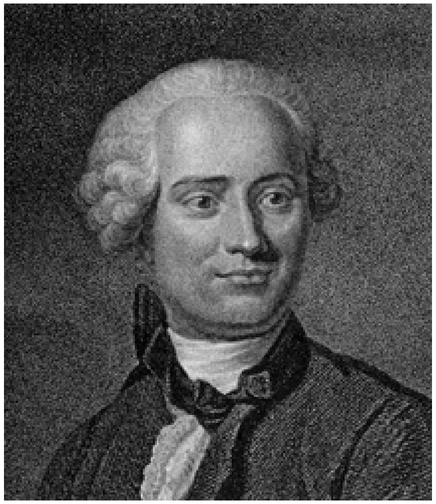
- ✓ The whole idea is to come up with the formulation that does not require vector analysis.

- ✓ Applying Newton's second law to complex problems is pretty complicated since it requires obtaining the acceleration. Analytical dynamics provides an automated formulation that greatly simplifies the way of finding EoMs.

- ✓ It gives rise to approximate techniques for the solution of discrete or continuous systems in a natural manner.

D'Alambert Principle

Would it be possible to reformulate Newton's 2nd law in such a way that we could apply virtual work principle?

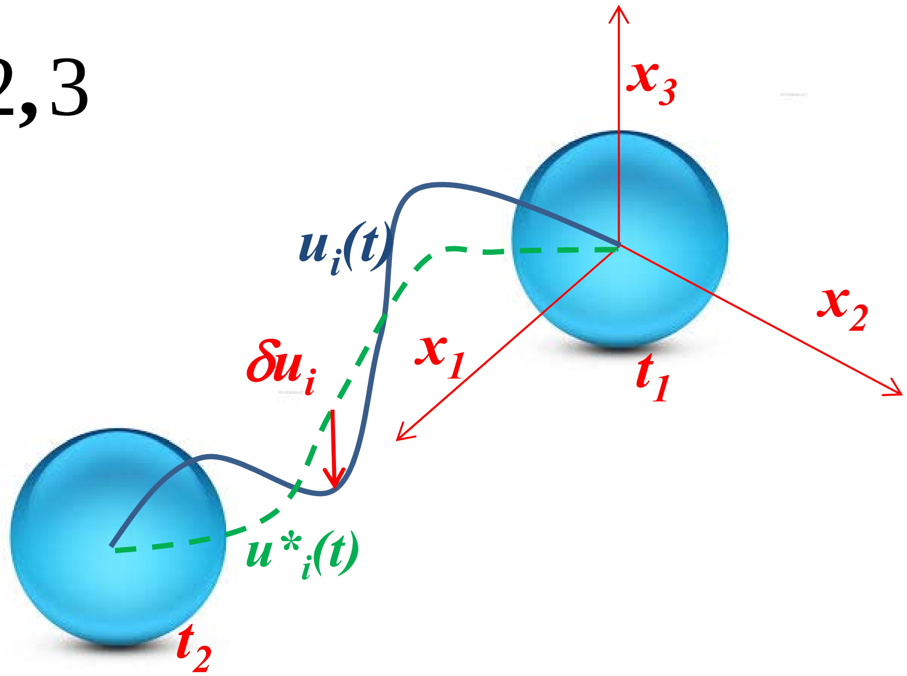


Suppose that we have a single particle in space in equilibrium

$$m\ddot{u}_i - X_i = 0 \quad , \quad i = 1,2,3$$

Virtual displacement δu_i

$$\delta u_i = u_i^* - u_i$$



Let’s multiply dynamic equilibrium by the virtual displacement

$$\sum_{i=1}^{3}\left(m\ddot{u}_i - X_i\right)\delta u_i = 0$$

How many equations is the above expression if expanded?

The above equation shows the projection of equilibrium along δu_i . If the above equation is satisfied for all variations δu_i, then the trajectory u_i(t) satisfies the dynamic equilibrium in all directions.

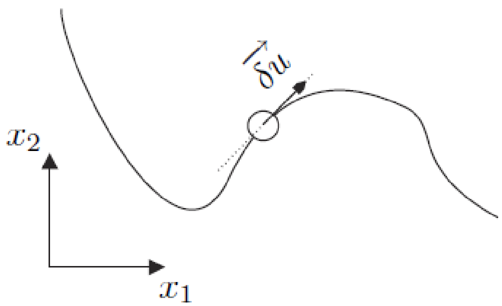
$$\delta W = 0$$

The virtual work done by effective forces acting on an unconstraint particle through infinitesimal virtual displacements is zero

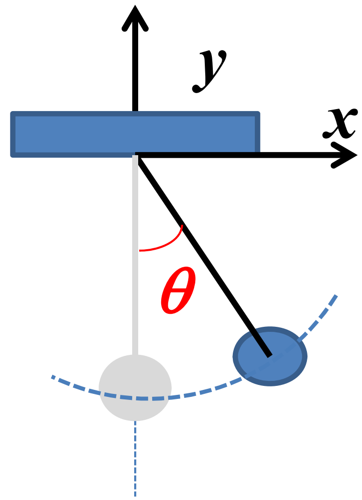
This seems to be somehow trivial !

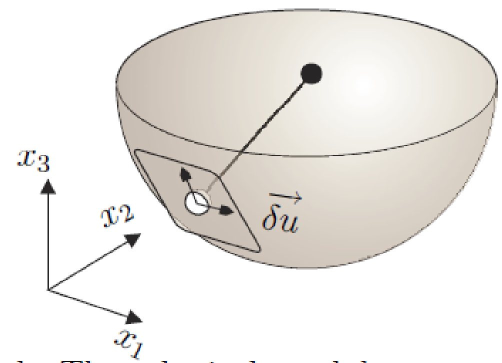
What if there are Kinematic constraints?

The presence of constraints leads to unknown reaction forces.



a. A particle on a plane curve





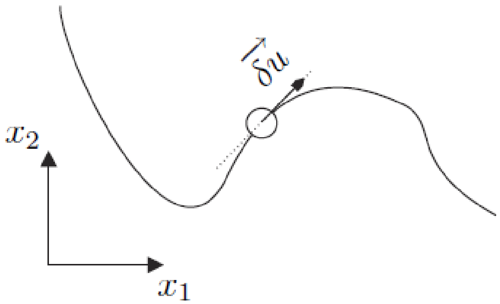
b. The spherical pendulum

- ✓ On one hand, reaction forces are unknown; There must be a force to keep the mass on the trajectory.
- ✓ On the other hand, position in constraint direction is known

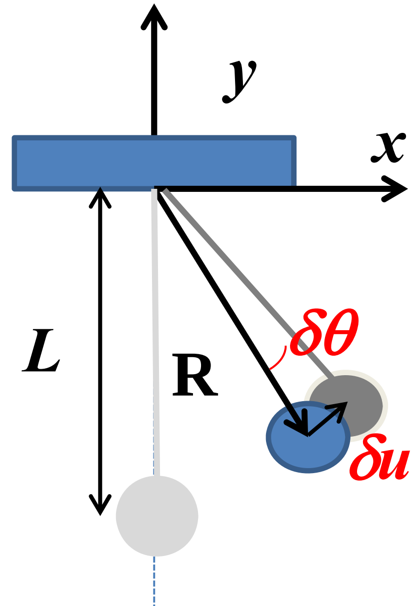
So if I consider virtual displacements compatible with the constraint (kinematically admissible) then I can get rid of reaction forces since they are always perpendicular to the motion and in the direction of the constraint.

Looking at the motion
in the admissible direction

We can get rid of reaction forces



a. A particle on a plane curve



Dynamic principle of virtual work states that virtual work performed by the **external and inertial forces** through **infinitesimal virtual displacements compatible** with the system **constraints** is zero

$$\delta W = 0$$

Now let’s suppose we have a system of *N* particles

The dynamic equilibrium for a system of particles is

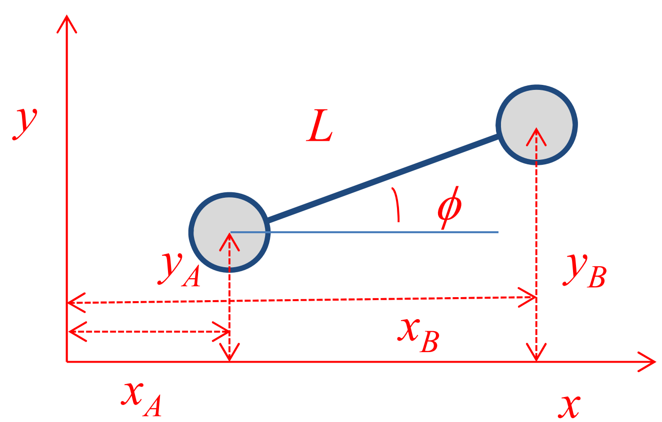
$$m_k \ddot{u}_{ik} - X_{ik} - R_{ik} = 0 \quad , \; i = 1,2,3 \; , \; k = 1,\ldots,N$$

X_{ik} are the external forces and R_{ik} are unknown reaction forces from the constraints

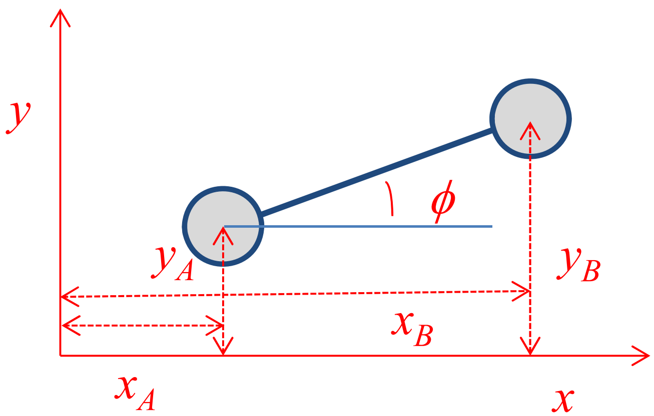
Again for each and every particle we define:

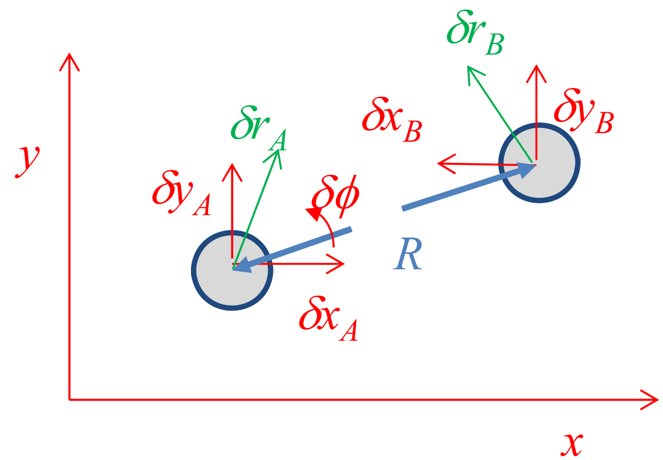
$$\delta u_{ik} = u_{ik}^{*} - u_{ik} \quad , \; i = 1,2,3$$

Which verifies the kinematic constraint e.g. every two particles are constrained via a rigid link



Let’s assume (x_A, y_A, ϕ) as the generalized coordinates





$$\mathbf{r_A} = x_A i + y_A j \quad , \quad \mathbf{r_B} = x_B i + y_B j$$

$$x_B = x_A + L\cos\phi \quad , \quad y_B = y_A + L\sin\phi$$

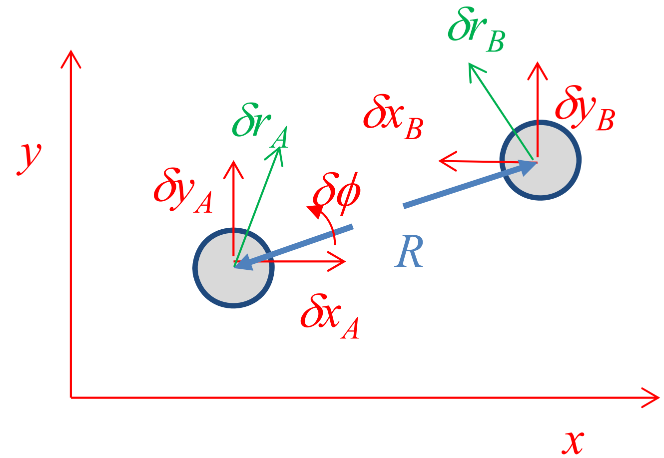
The virtual displacements

$$\delta\,\mathbf{r_A} = x_A i + y_A j \quad ,$$

$$\delta\mathbf{r_B} = \left(\delta x_A - L\sin\phi\,\delta\phi\right)i + \left(\delta y_A + L\cos\phi\,\delta\phi\right)j$$

How can I calculate the virtual work done by force R?

$$
\begin{aligned}
\delta W = &\left(-R\cos\phi i - R\sin\phi j\right)\cdot\left(\delta x_A i + \delta y_A j\right)\\
&+\left(R\cos\phi i + R\sin\phi j\right)\cdot\\
&\quad\left(\left(\delta x_A - L\sin\phi\,\delta\phi\right) i + \left(\delta y_A + L\cos\phi\,\delta\phi\right) j\right)
\end{aligned}
$$



$$
\begin{aligned}
\delta W = &-R\delta x_A\cos\phi - R\delta y_A\sin\phi + R\delta x_A\cos\phi + R\delta y_A\sin\phi\\
&- RL\cos\phi\sin\phi\ \delta\phi + RL\cos\phi\sin\phi\ \delta\phi = 0
\end{aligned}
$$

As expected, the constraint force does not work

What if I change the generalized coordinates and use (x_A, y_A, L, ϕ)

$$\delta \mathbf{r}_B = \left(\delta x_A - L\,\mathbf{sin}\,\phi\,\delta\phi + \mathbf{cos}\,\phi\delta L\right)i + \left(\delta y_A + L\,\mathbf{cos}\,\phi\,\delta\phi + \mathbf{sin}\,\phi\delta L\right)j$$

$$\begin{aligned}\delta W &= \left(-R\,\mathbf{cos}\,\phi i - R\,\mathbf{sin}\,\phi\, j\right)\cdot\left(\delta x_A i + \delta y_A j\right) \\ &\quad + \left(R\,\mathbf{cos}\,\phi i + R\,\mathbf{sin}\,\phi\, j\right)\cdot \\ &\quad \left(\left(\delta x_A - L\,\mathbf{sin}\,\phi\,\delta\phi\right)i + \left(\delta y_A + L\,\mathbf{cos}\,\phi\,\delta\phi\right)j\right) \\ &\quad + \left(R\,\mathbf{cos}\,\phi i + R\,\mathbf{sin}\,\phi\, j\right)\cdot\left(\mathbf{cos}\,\phi\delta L i + \mathbf{sin}\,\phi\delta L j\right)\end{aligned}$$

$$\delta W = R\delta L$$

So if the constraint condition is violated then *R* will do work

The same scenario could be repeated for *N* particles. Therefore:

$$m_k \ddot{u}_{ik} - X_{ik} - R_{ik} = 0 \quad , \; i = 1,2,3 \;\; , \; k = 1,...,N$$

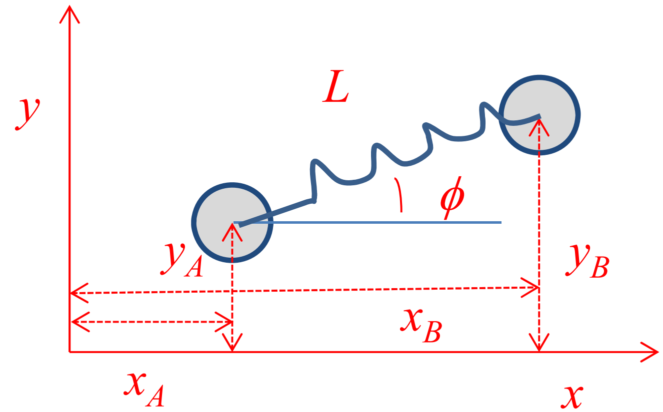
$$\sum_{k=1}^{N} \sum_{i=1}^{3} \left(m_k \ddot{u}_{ik} - X_{ik} - R_{ik} \right) \delta u_{ik} = 0$$

$$\boxed{\sum_{k=1}^{N} \sum_{i=1}^{3} R_{ik} \delta u_{ik} = 0 \quad , \quad \sum_{k=1}^{N} \sum_{i=1}^{3} \left(m_k \ddot{u}_{ik} - X_{ik} \right) \delta u_{ik} = 0}$$

If the virtual work equation is satisfied for any virtual displacement compatible with the kinematic constraints, the system is in dynamic equilibrium

What if we have a spring instead of rigid link?

There would be no constraints on the motion of the system in this case so the system will indeed have four degrees of freedom i.e (x_A, y_A, L, ϕ)



$$R = k\Delta = k(L - L_0)$$

So the virtual work done by the spring force would be:

$$\delta W = k(L - L_0)\ \delta L$$

What is the nature of constraints?

If the kinematic constraint is an implicit relation of the form

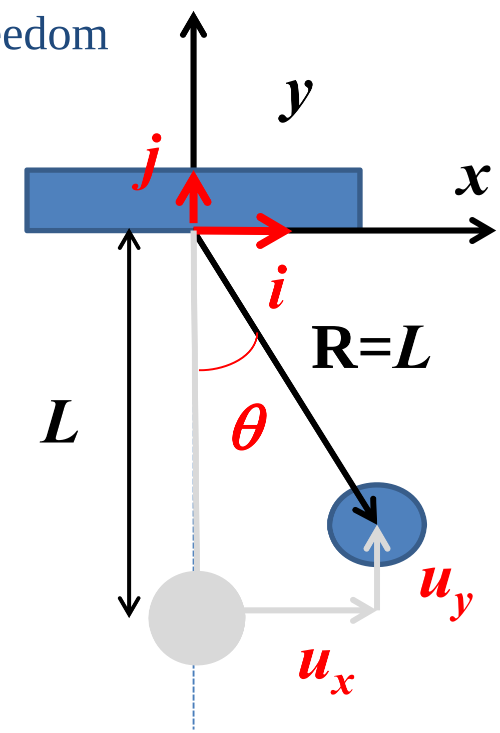
$$f\left(\xi_{ik},t\right)=0$$

The constraint is said to be holonomic (integrable)

- ✓ A holonomic constraint is said to be **scleronomic** if time does not explicitly appear in the constraint equation.
- ✓ A holonomic constraint is said to be **rheonomic** if time does explicitly appear in the constraint equation.
- ✓ A holonomic constraints reduces the number of degrees of freedom

Is the constraint in case of pendulum holonomic?

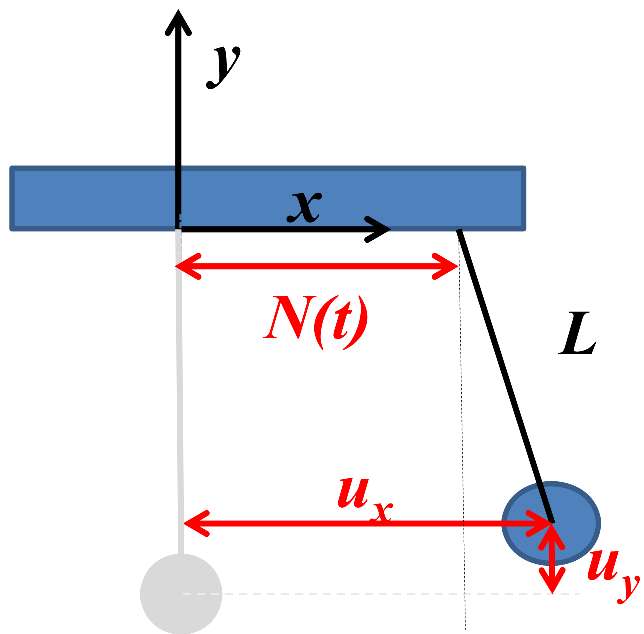
$$\left(u_x\right)^2+\left(u_y-L\right)^2-L^2=0$$



Is the above constraint rheonomic, if not how can I make it rheonomic?

Suppose that the point of suspension is moved smoothly in a prescribed direction

$$\left(u_x - N(t)\right)^2 + \left(u_y - L\right)^2 - L^2 = 0$$



What about a YOYO?

$$\left(u_x\right)^2 + \left(u_y - L_0 + \alpha t\right)^2 - \left(L_0 - \alpha t\right)^2 = 0$$

In both cases we have a time varying component that is explicitly appearing in the constraint so the constraint is **rheonomic.**

What about a rigid link is it rheonomic or scleronomic?

What is the role of constraints on degrees of freedom of a rigid body?

Pfaffian form of a constraint equation

The *Pfaffian* form of a constraint equation is a differential form as follows

$$f_i\left(q_j,t\right)=0 \qquad j=1,\ldots,M$$

$$df_i=\sum_{j=1}^{M}\frac{\partial f_i}{\partial q_j}dq_j+\frac{\partial f_i}{\partial t}dt$$

$$\Rightarrow$$

$$\boxed{\sum_{j=1}^{M}a_{ij}\left(q_j,t\right)dq_j+b_i\left(q_j,t\right)dt=0}$$

i is the number of constraints and j number of coordinates considered to define the motion.

For instance *Pfaffian* form of the constraint in case of the pendulum is:

$$u_x du_x+\left(u_y-L\right)du_y+0\cdot dt=0$$

$$a_{11}=u_x \quad , \quad a_{12}=\left(u_y-L\right) \quad , \quad b_1=0\ .$$

You will see that the above equation can be easily integrated and lead to the displacement constraint but is that always the case?

Non-holonomic constraint

Non-holonomic constraints are generally in form of differential equations, and their *pfaffian* form is neither integrable nor an integrating factor could be found to make the differential equation integrable

IMPORTANT: A non-holonomic constraint **does not reduce** the number of degrees of freedom.

Remark: A differential equation is said to be **Exact** if

$$\frac{\partial}{\partial q_k}\left(g_i a_{ij}\right) = \frac{\partial}{\partial q_j}\left(g_i a_{ik}\right), \quad j,k = 1,2,\ldots\ldots,M \;\&\; j \neq k$$

$$\frac{\partial}{\partial q_k}\left(g_i b_i\right) = \frac{\partial}{\partial t}\left(g_i a_{ik}\right)$$

Where g_i is an integrating factor

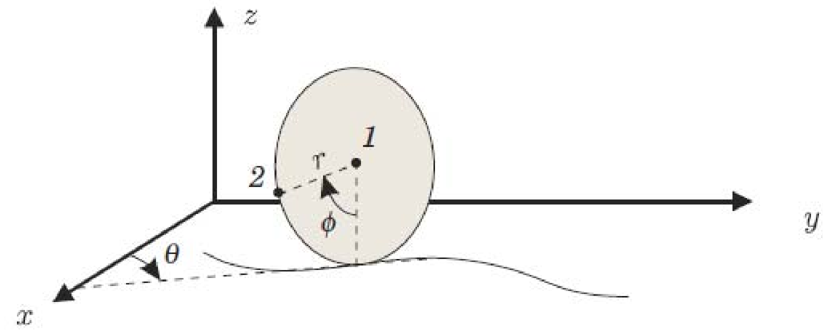
In case of pendulum

$$a_{11} = u_x \;,\; a_{12} = \left(u_y - L\right) \;,\; b_1 = 0 \;,\; g_1 = 1.$$

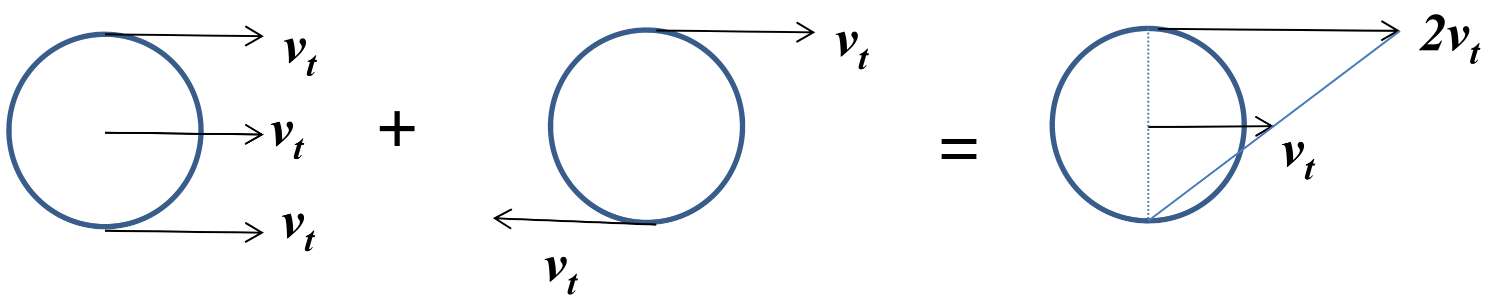
$$\frac{\partial}{\partial u_x}\left(a_{12}\right) = \frac{\partial}{\partial u_y}\left(a_{11}\right) = 0 \;, \quad \frac{\partial}{\partial t}\left(a_{11}\right) = \frac{\partial}{\partial u_x}\left(b_1\right) = 0 \;,\; \frac{\partial}{\partial t}\left(a_{12}\right) = \frac{\partial}{\partial u_y}\left(b_1\right) = 0 \;.$$

The rolling coin/wheel: example of non-holonomic system

What is rolling without slipping?

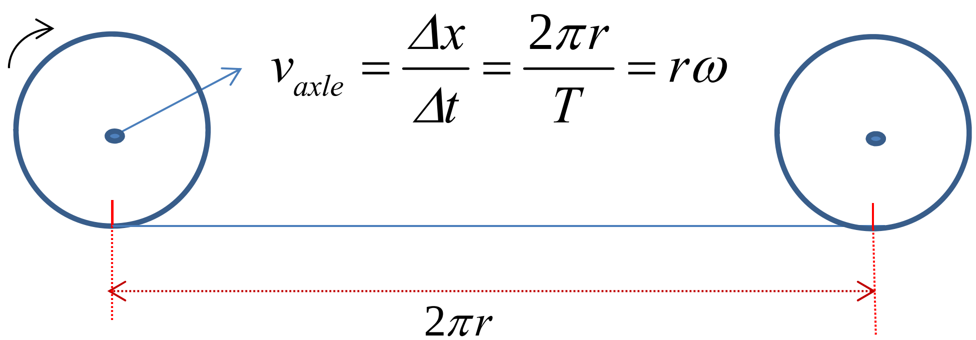


Translation **rotation**

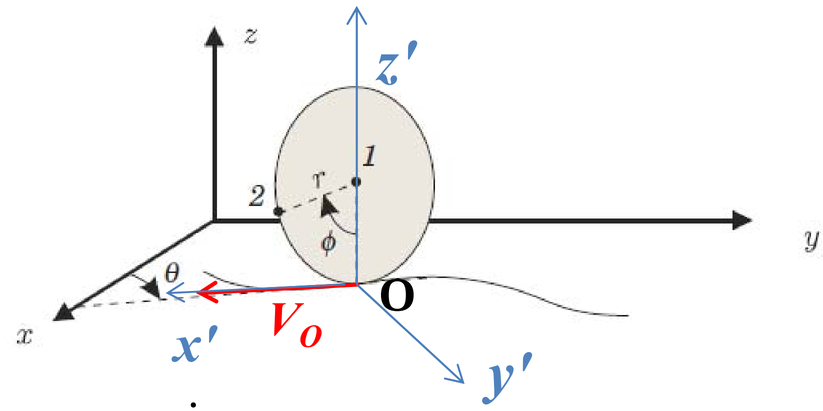


So the bottom of the coin is always at rest, meaning that the wheel is always in contact with the ground.

$$v_{axle} = \frac{\Delta x}{\Delta t} = \frac{2\pi r}{T} = r\omega$$



The rolling coin/wheel: example of non-holonomic system



$$V_O = V_1 + \omega \times r = \dot{x}_1 i + \dot{y}_1 j + (-\dot{\phi} j') \times (-r k')$$
$$= \dot{x}_1 i + \dot{y}_1 j + r\dot{\phi} i'$$
$$i' = \cos\theta i - \sin\theta j$$

$$V_O = \left(\dot{x}_1 + r\dot{\phi}\cos\theta\right) i + \left(\dot{y}_1 - r\dot{\phi}\sin\theta\right) j$$

$$\dot{x}_1 + r\dot{\phi}\cos\theta = 0$$
$$\dot{y}_1 - r\dot{\phi}\sin\theta = 0$$

$$dx_1 + 0\cdot dy_1 + r\cos\theta d\phi + 0\cdot d\theta + 0\cdot dt = 0$$
$$0\cdot dx_1 + dy_1 - r\sin\theta d\phi + 0\cdot d\theta + 0\cdot dt = 0$$

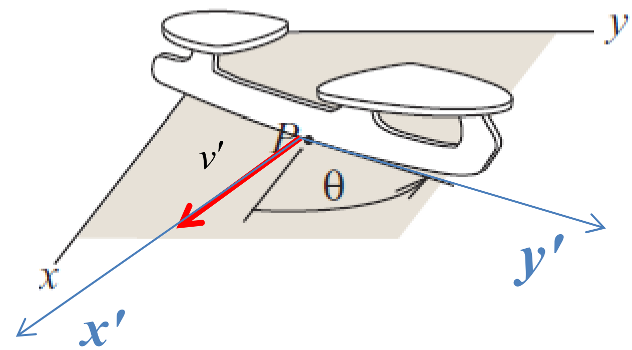
$$a_{11} = 1,\ a_{12} = 0,\ a_{13} = r\cos\theta,\ a_{14} = 0,\ b_1 = 0$$
$$a_{21} = 0,\ a_{22} = 1,\ a_{23} = -r\sin\theta,\ a_{24} = 0, b_2 = 0$$

$$\frac{\partial}{\partial\theta}(a_{13}) \neq \frac{\partial}{\partial\phi}(a_{14})$$
$$\frac{\partial}{\partial\theta}(a_{23}) \neq \frac{\partial}{\partial\phi}(a_{24})$$

The constraint is not integrable so it is non-holonomic and thus the system has 4 degrees of freedom

Non-holonomic constraint

Consider the action of the edge of an ice skate blade. The blade's position is set by the coordinates x_p and y_P of point P and the angle θ from the x axis along which the blade is aligned. The velocity of the point of contact must be aligned with the plate. In other words the blade can not have a velocity component orthogonal to the direction of motion.



$$\vec{v}_p = \dot{x}_p i + \dot{y}_p j$$

$$\vec{v}_p = v_p j'$$

$$\vec{v}' = v' i'$$

$$i = \sin\theta i' + \cos\theta j'$$

$$j = -\cos\theta i' + \sin\theta j'$$

$$\vec{v}_p = \dot{x}_p\left(\sin\theta i' + \cos\theta j'\right) + \dot{y}_p\left(-\cos\theta i' + \sin\theta j'\right)$$

$$v' = \dot{x}_p \sin\theta - \dot{y}_p \cos\theta = 0$$

$$\sin\theta dx_p - \cos\theta dy_p + 0\cdot d\theta + 0\cdot dt = 0$$

$$\frac{\partial}{\partial\theta}(a_{11}) \neq \frac{\partial}{\partial x_p}(a_{13}) \quad , \quad \frac{\partial}{\partial\theta}(a_{12}) \neq \frac{\partial}{\partial y_p}(a_{13})$$

So the constraint is non-holonomic

So now that we realized kinematically admissible displacements can reduce the number of degrees of freedom, and reaction forces do not work given a virtual displacement, let's find a unified formulation for a system of particles:

Without kinematic constraints we have 3*N* degrees of freedom

$$\xi_{ik}(t) = x_{ik} + u_{ik}(x_{jk}, t) \quad i, j = 1,2,3 \;\; \& \;\; k = 1, \ldots, N$$

Initial position displacements

If there exists *R* holonomic constraints , it is then necessary to define

$$n = 3N - R = generlized\;\; coordinates\; \left(q_1, \ldots q_n\right)\; such\;\; that$$

$$u_{ik}(x_{jk}, t) = U_{ik}(q_1, \ldots, q_n, t)$$

Such that a mapping like below exists

$$\delta u_{ik} = \sum_{s=1}^{n} \frac{\partial U_{ik}(q_s, t)}{\partial q_s} \delta q_s$$

Remember that U_{ik} should not violate the constraints

At the end

$$\sum_{s=1}^{n}\left[\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\ddot{u}_{ik}-X_{ik}\right)\frac{\partial U_{ik}(q_s,t)}{\partial q_s}\right]\delta q_s=0$$

And the *n* equations of motion become

$$\boxed{\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\frac{d^2U_{ik}\left(q_1,\ldots,q_n,t\right)}{dt^2}-X_{ik}\right)\frac{\partial U_{ik}(q_s,t)}{\partial q_s}=0\qquad s=1,\ldots,n}$$

As mentioned before these will be the dynamic equilibrium equations projected onto the admissible direction.

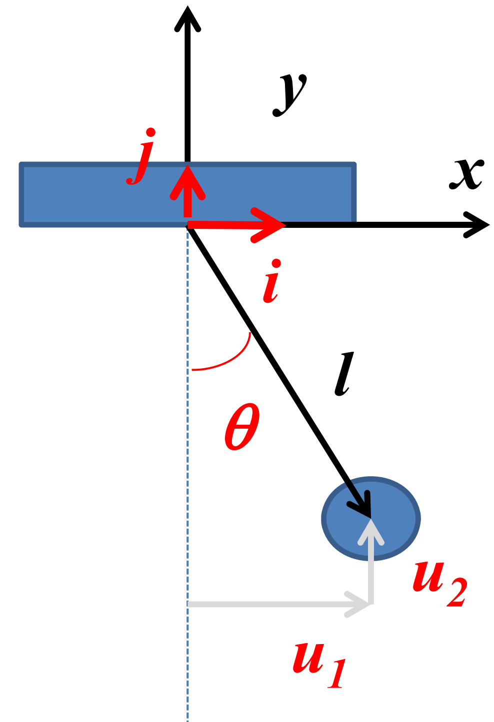
The term below is called generalized force, this name is considered simply because generalized coordinates could be of any type

$$Q_s=\sum_{k=1}^{N}\sum_{i=1}^{3}X_{ik}\frac{\partial U_{ik}(q_s,t)}{\partial q_s}$$

The pendulum problem

$$u_1 = l\,\mathbf{sin}\,\theta$$
$$u_2 = l - l\,\mathbf{cos}\,\theta$$
$$l^2 = (u_2 - l)^2 + u_1^2$$
$$n = 2 - 1 = 1$$

$$\mathbf{U} = \begin{bmatrix} l\,\mathbf{sin}\,\theta \\ l - l\,\mathbf{cos}\,\theta \end{bmatrix}$$
$$\frac{\partial \mathbf{U}}{\partial \theta} = \begin{bmatrix} l\,\mathbf{cos}\,\theta \\ l\,\mathbf{sin}\,\theta \end{bmatrix}$$



External force

$$\mathbf{X} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} = -mgj$$

Dynamic equilibrium

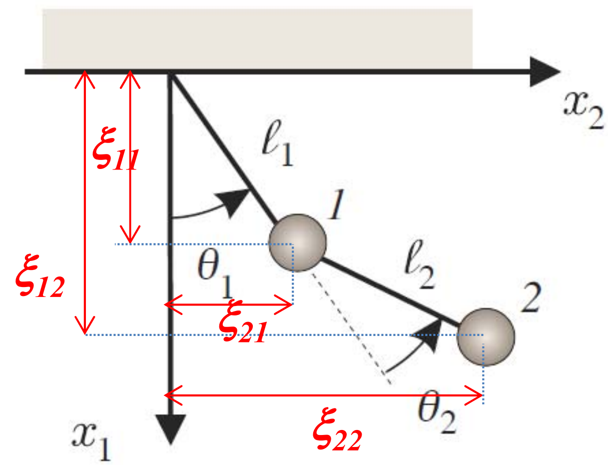
$$\begin{bmatrix} m\ddot{u}_1 \\ m\ddot{u}_2 + mg \end{bmatrix} = m\ddot{u}_1 i + \left(m\ddot{u}_2 + mg\right) j$$

Virtual work

$$m\ddot{u}_1 l\,\mathbf{cos}\,\theta + \left(m\ddot{u}_2 + mg\right) l\,\mathbf{sin}\,\theta = 0$$
$$\ddot{u}_1 = l\ddot{\theta}\,\mathbf{cos}\,\theta - l\dot{\theta}^2\,\mathbf{sin}\,\theta$$
$$\ddot{u}_2 = l\ddot{\theta}\,\mathbf{sin}\,\theta + l\dot{\theta}^2\,\mathbf{cos}\,\theta$$

$$\ddot{\theta} + \frac{g}{l}\,\mathbf{sin}\,\theta = 0$$

The double pendulum



$$\xi_{ik}(t) = x_{ik} + u_{ik}(x_{jk}, t) \quad i, j = 1,2,3 \;\; \& \;\; k = 1, \ldots, N$$

$$\xi_{11} = u_{11} = l_1 \cos\theta_1$$

$$\xi_{21} = u_{21} = l_1 \sin\theta_1$$

$$\xi_{12} = u_{12} = \xi_{11} + l_2 \cos(\theta_1 + \theta_2) = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$

$$\xi_{22} = u_{22} = \xi_{21} + l_2 \sin(\theta_1 + \theta_2) = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

$$n = 4 - 2 = 2$$

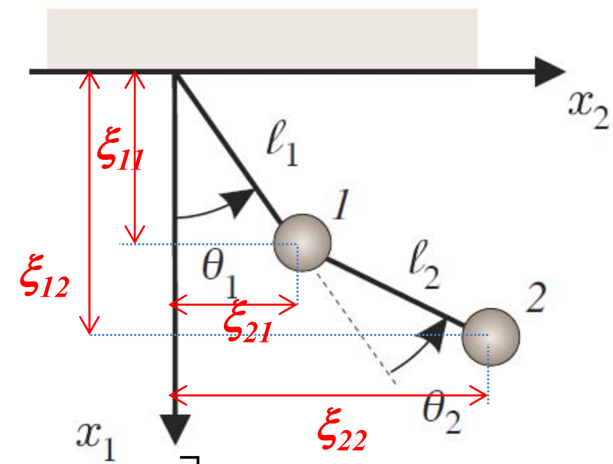
$$q_1 = \theta_1 \;,\; q_2 = \theta_2$$

The mapping $\Rightarrow$

$$\mathbf{U} = \begin{bmatrix} l_1 \cos\theta_1 & l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 & l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

$$\frac{\partial U_{ik}}{\partial q_s} \Rightarrow \begin{cases} \dfrac{\partial U_{11}}{\partial q_1} = -l_1 \sin\theta_1 \;,\; \dfrac{\partial U_{11}}{\partial q_2} = 0 \;,\; \dfrac{\partial U_{21}}{\partial q_1} = l_1 \cos\theta_1 \;,\; \dfrac{\partial U_{21}}{\partial q_2} = 0 \\ \dfrac{\partial U_{12}}{\partial q_1} = -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \;,\; \dfrac{\partial U_{12}}{\partial q_2} = -l_2 \sin(\theta_1 + \theta_2) \;, \\ \dfrac{\partial U_{22}}{\partial q_1} = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \;,\; \dfrac{\partial U_{22}}{\partial q_2} = l_2 \cos(\theta_1 + \theta_2) \end{cases}$$

Double pendulum continued



External force ➡ $X_{i1} = \begin{bmatrix} mg \\ 0 \end{bmatrix}$, $X_{i2} = \begin{bmatrix} mg \\ 0 \end{bmatrix}$

Dynamic equilibrium

$$for\ mass\,1 = \begin{bmatrix} m\ddot{u}_{11} - mg \\ m\ddot{u}_{21} \end{bmatrix} \quad , \quad for\ mass\,2 = \begin{bmatrix} m\ddot{u}_{12} - mg \\ m\ddot{u}_{22} \end{bmatrix}$$

$$\sum_{k=1}^{2}\sum_{i=1}^{2}(m_k\ddot{u}_{ik} - X_{ik})\frac{\partial U_{ik}(q_s,t)}{\partial q_s} = 0 \qquad s = 1,2$$

$$(m\ddot{u}_{11} - mg)\frac{\partial U_{11}}{\partial q_1} + (m\ddot{u}_{21})\frac{\partial U_{21}}{\partial q_1} + (m\ddot{u}_{12} - mg)\frac{\partial U_{12}}{\partial q_1} + (m\ddot{u}_{22})\frac{\partial U_{22}}{\partial q_1} = 0$$

$$(m\ddot{u}_{11} - mg)\frac{\partial U_{11}}{\partial q_2} + (m\ddot{u}_{21})\frac{\partial U_{21}}{\partial q_2} + (m\ddot{u}_{12} - mg)\frac{\partial U_{12}}{\partial q_2} + (m\ddot{u}_{22})\frac{\partial U_{22}}{\partial q_2} = 0$$

Double pendulum continued

First Equation of Motion:

$$\left(m\ddot{u}_{11}-mg\right)\frac{\partial U_{11}}{\partial q_1}+\left(m\ddot{u}_{21}\right)\frac{\partial U_{21}}{\partial q_1}+\left(m\ddot{u}_{12}-mg\right)\frac{\partial U_{12}}{\partial q_1}+\left(m\ddot{u}_{22}\right)\frac{\partial U_{22}}{\partial q_1}=0$$

$$\Downarrow$$

$$\left(m\ddot{u}_{11}-mg\right)\left(-l_1\,\mathbf{sin}\,\theta_1\right)+\left(m\ddot{u}_{21}\right)\left(l_1\,\mathbf{cos}\,\theta_1\right)$$
$$+\left(m\ddot{u}_{12}-mg\right)\left(l_1\,\mathbf{cos}\,\theta_1+l_2\,\mathbf{cos(}\theta_1+\theta_2\mathbf{)}\right)+\left(m\ddot{u}_{22}\right)\left(l_1\,\mathbf{cos}\,\theta_1+l_2\,\mathbf{cos(}\theta_1+\theta_2\mathbf{)}\right)=0$$

Second Equation of Motion:

$$\left(m\ddot{u}_{11}-mg\right)\frac{\partial U_{11}}{\partial q_2}+\left(m\ddot{u}_{21}\right)\frac{\partial U_{21}}{\partial q_2}+\left(m\ddot{u}_{12}-mg\right)\frac{\partial U_{12}}{\partial q_2}+\left(m\ddot{u}_{22}\right)\frac{\partial U_{22}}{\partial q_2}=0$$

$$\Downarrow$$

$$\left(m\ddot{u}_{12}-mg\right)\left(-l_2\,\mathbf{sin(}\theta_1+\theta_2\mathbf{)}\right)+\left(m\ddot{u}_{22}\right)\left(l_2\,\mathbf{cos(}\theta_1+\theta_2\mathbf{)}\right)=0$$

Still the accelerations should be calculated! This makes D'Alambert approach still tedious

For more examples concerning constraints and obtaining the generalized forces, refer to the blackboard!