Vibrations & mode superposition

WB 1418-07

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- ✓ The last step after obtaining linearized equations is to understand how systems behave when vibrating freely or subjected to external excitations.

- ✓ Vibrations play an important role in system dynamics.

- ✓ Here we will be looking only at small oscillations around equilibrium. So we don't deal with non-linear vibrations.

Eigen-modes and Eigen-frequencies of undamped systems

Free linear motion (No external forces)

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0$$

If **M** and **K** are positive definite, the system is stable. We can seek a general synchronous solution (generalized coordinates have the same temporal law):

$$\mathbf{q} = \mathbf{x}\phi(t)$$

x is the eigenshape (scale factor).

Substituting

$$\ddot{\phi}(t)\mathbf{M}\,\mathbf{x}+\phi(t)\mathbf{K}\,\mathbf{x}=0\Rightarrow\ \mathbf{K}\,\mathbf{x}=-\frac{\ddot{\phi}(t)}{\phi(t)}\mathbf{M}\,\mathbf{x}$$

Therefore

$$\begin{cases}\dfrac{\ddot{\phi}(t)}{\phi(t)}=\lambda^2\\ \mathbf{K}\,\mathbf{x}=-\lambda^2\,\mathbf{M}\,\mathbf{x}\end{cases}$$

M and **K** are symmetric and positive definite! Therefore

- λ^2 is real and negative, therefore λ can be renamed as $i\omega$ with ω being real

$$\begin{cases}\dfrac{\ddot{\phi}(t)}{\phi(t)}=-\omega^2\\ \left(\mathbf{K}-\omega^2\,\mathbf{M}\right)\mathbf{x}=\mathbf{0}\end{cases}$$

$$\left(\mathbf{K}-\omega_r^2\,\mathbf{M}\right)\mathbf{x}_{(\mathbf{r})}=0$$

Given a certain eigen frequency, the matrix becomes singular such that we will have a non-trivial solution $\mathbf{x}_{(\mathbf{r})}$ (eigen mode)

Mathematically this means that:

$$\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = 0$$

Is the eigenvalue problem of degree *n* in ω^2, and posses n roots which have been shown to be positive and real . For each root ω_r^2 there exists a real solution $\mathbf{x_{(r)}}$ which is known as the eigenmode associated to that eigenvalue.

What happens if ω_r is zero?

We have a special kind of eigenmode that are displacement modes (*rigid body modes* **u**) . These modes do not generate elastic forces

$$\mathbf{K}\,\mathbf{u} = 0$$

A system of dimension *n* can have *m* rigid-body modes *u* and *n* – *m* elastic modes $\mathbf{x_{(r)}}$.

Oscillation of free vibration modes

Temporal solution

$$\ddot{\phi}_r(t) + \omega_r^2 \phi_r(t) = 0$$

If the eigenfrequency is nonzero:

$$\phi_r(t) = \alpha_r \cos \omega_r t + \beta_r \sin \omega_r t$$

ω_r = circular frequency of mode r in radians per second or frequency $\nu = \omega/2\pi$ in hertz .

NOTE1 : As discussed previously stability is a direct consequence of s.p.d of **M** and **K.**

NOTE2: α_r and β_r should be obtained from initial conditions.

If $\omega = 0$

$$\ddot{\phi}(t) = 0 \Rightarrow \phi(t) = \gamma + \delta t$$

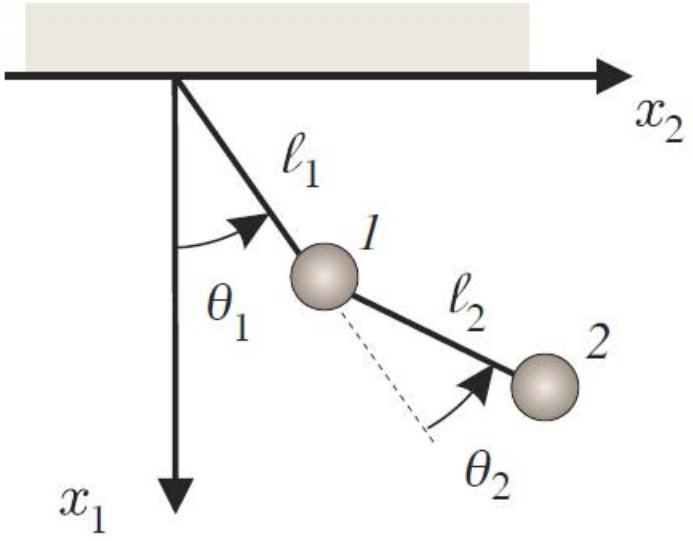
So in general we will have 2 types of solution: (i) Rigid-body modes $\mathbf{u}_{(i)}$, is number equal to the degree of singularity of the stiffness matrix, describing an overall transport motion; (ii) Eigenmodes $\mathbf{x}_{(r)}$ associated with non-zero eigenfrequencies , describing vibration about the overall motion

Double Pendulum

Linearized EoMs for $\theta_1 = \theta_2 = 0$

$$\begin{pmatrix} (m_1+m_2)l_1^2 + m_2\left(l_1^2+l_2^2\right) & m_2(l_1 l_2) \\ m_2(l_1 l_2) & m_2 l_2^2 \end{pmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} +$$

$$\begin{pmatrix} (m_1+m_2)gl_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



We showed that $\theta_1 = \theta_2 = 0$ is a stable equilibrium, suppose $m_1 = 4m$, $m_2 = m$, $l_1 = 2l$, $l_2 = l$

$$\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = 0$$

$$\left(\begin{pmatrix} 11mgl & 0 \\ 0 & mgl \end{pmatrix} - \omega^2 \begin{pmatrix} 20ml^2 & 2ml^2 \\ 2ml^2 & ml^2 \end{pmatrix} \right) \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$$

Introducing $\nu^2 = \dfrac{\omega^2 l}{g}$

$$\left(\begin{pmatrix}11 & 0\\ 0 & 1\end{pmatrix}-\nu^2\begin{pmatrix}20 & 2\\ 2 & 1\end{pmatrix}\right)\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}=0$$

A

Characteristic polynomial

$$\left(11-20\nu^2\right)\left(1-\nu^2\right)-4\nu^2=0,$$

$$\nu_1^2=\frac{\omega_1^2 l}{g}=0.4678 \quad , \quad \nu_1^2=\frac{\omega_2^2 l}{g}=1.4697$$

Next is to find the shapes at those frequencies: Eigenmodes

$$\mathbf{x}_{(1)}=\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}_{@\,\nu_1} \quad , \quad \mathbf{x}_{(2)}=\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}_{@\,\nu_2}$$

We can find a solution for the eigenmodes up to a constant. In other words, we cannot find a vector with a specific magnitude, rather it is only possible to find the ratio between the generalized coordinates.

We can arbitrary say that the magnitude of θ_1 at the frequencies is 1. The two equations of the eigenvalue problem $\mathbb{A}$ are telling the same thing so each could be used to obtain the magnitude of θ_2

$$\mathbf{x}_{(1)} = \begin{Bmatrix} 1 \\ X_1 \end{Bmatrix}, \quad \mathbf{x}_{(2)} = \begin{Bmatrix} 1 \\ X_2 \end{Bmatrix}$$

From first equation of $\mathbb{A}$

$$X_1 = \frac{11 - 20\nu_1^2}{2\nu_1^2} = \frac{11 - 20(0.4678)}{2(0.4678)} = 1.7578$$

$$X_2 = \frac{11 - 20\nu_2^2}{2\nu_2^2} = \frac{11 - 20(1.4697)}{2(1.4697)} = -6.2578$$

We can check using the second equation of $\mathbb{A}$

$$X_1 = \frac{2\nu_1^2}{1 - \nu_1^2} = \frac{2(0.4678)}{1 - (0.4678)} = 1.7578$$

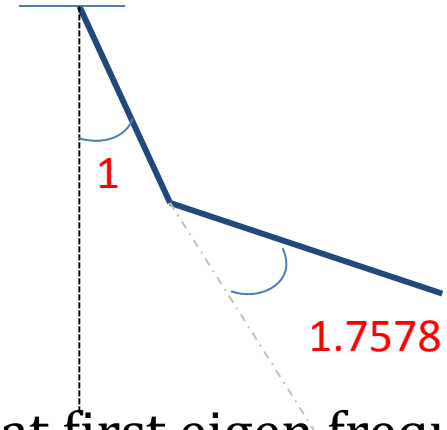
$$X_2 = \frac{2\nu_2^2}{1 - \nu_2^2} = \frac{2(1.4697)}{1 - (1.4697)} = -6.2578$$

The eigensolutions are thus

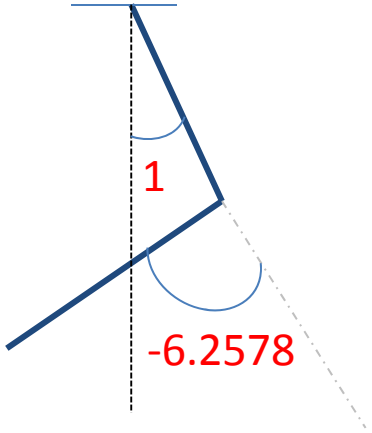
$$\omega_1^2 = 0.4678\frac{g}{l} \quad , \quad \mathbf{x}_{(1)} = \begin{Bmatrix} 1 \\ 1.7578 \end{Bmatrix}$$

$$\omega_2^2 = 1.4697\frac{g}{l} \quad , \quad \mathbf{x}_{(2)} = \begin{Bmatrix} 1 \\ -6.2578 \end{Bmatrix}$$

The two possible motions:



Shape at first eigen frequency



Shape at second eigen frequency

Remember modes are just shapes and do not depend on time

Orthogonality of eigenmodes

Let's assume two eigenmodes $\mathbf{x_{(r)}}$ and $\mathbf{x_{(s)}}$ associated to distinct eigenfrequencies ω_r and ω_s, respectively.

$$\mathbf{K}\,\mathbf{x_{(r)}} = \omega_r^2\,\mathbf{M}\,\mathbf{x_{(r)}} \quad , \quad \mathbf{K}\,\mathbf{x_{(s)}} = \omega_s^2\,\mathbf{M}\,\mathbf{x_{(s)}}$$

Let's pre multiply the two by $\mathbf{x_{(s)}}^{\mathbf{T}}$, and $\mathbf{x_{(r)}}^{\mathbf{T}}$, respectively.

$$\mathbf{x_{(s)}^T}\,\mathbf{K}\,\mathbf{x_{(r)}} = \omega_r^2\,\mathbf{x_{(s)}^T}\,\mathbf{M}\,\mathbf{x_{(r)}} \quad , \quad \mathbf{x_{(r)}^T}\,\mathbf{K}\,\mathbf{x_{(s)}} = \omega_s^2\,\mathbf{x_{(r)}^T}\,\mathbf{M}\,\mathbf{x_{(s)}}$$

Subtracting the two equations:

$$\mathbf{x_{(s)}^T}\,\mathbf{K}\,\mathbf{x_{(r)}} - \mathbf{x_{(r)}^T}\,\mathbf{K}\,\mathbf{x_{(s)}} = \omega_r^2\,\mathbf{x_{(s)}^T}\,\mathbf{M}\,\mathbf{x_{(r)}} - \omega_s^2\,\mathbf{x_{(r)}^T}\,\mathbf{M}\,\mathbf{x_{(s)}}$$

NOTE: we already know that mass and stiffness matrices are symmetric

$$\left(\omega_r^2 - \omega_s^2\right)\mathbf{x_{(s)}^T}\,\mathbf{M}\,\mathbf{x_{(r)}} = 0$$

Since eigenfrequencies are distinct

$$\mathbf{x}_{(s)}^{T}\,\mathbf{M}\,\mathbf{x}_{(r)} = 0$$

Therefore,

$$\mathbf{x}_{(s)}^{T}\mathbf{K}\mathbf{x}_{(r)} = 0$$

The above two relations are called orthogonality relations between eigenmodes with respect to the mass and stiffness matrices.

We can also interpret this as:

- ✓ The work produced by inertia forces $\mathbf{M}\mathbf{x}_{(r)}$ (of mode r)on a displacement described by mode s is zero.
- ✓ The work produced by elastic forces $\mathbf{K}\mathbf{x}_{(r)}$ (of mode r)on a displacement described by mode s is zero.

Generalized mass (modal mass) and generalized stiffness (modal stiffness):

$$\mathbf{x}_{(r)}^{T}\,\mathbf{M}\,\mathbf{x}_{(r)} = \mu_r \qquad , \qquad \mathbf{x}_{(r)}^{T}\mathbf{K}\mathbf{x}_{(r)} = \gamma_r$$

Both quantities are not unique since the amplitude of eigenmode $\mathbf{x}_{(r)}$ is undetermined. This indeterminacy may be removed by calculating their ratio: (Rayleigh's quotient)

$$\frac{\mathbf{x}_{(r)}^{T}\mathbf{K}\mathbf{x}_{(r)}}{\mathbf{x}_{(r)}^{T}\mathbf{M}\mathbf{x}_{(r)}} = \frac{\gamma_r}{\mu_r} = \omega_r^2$$

The indeterminacy in the norm of eigenmode $\mathbf{x}_{(r)}$ allows us, to choose it appropriately, the most common choices being:

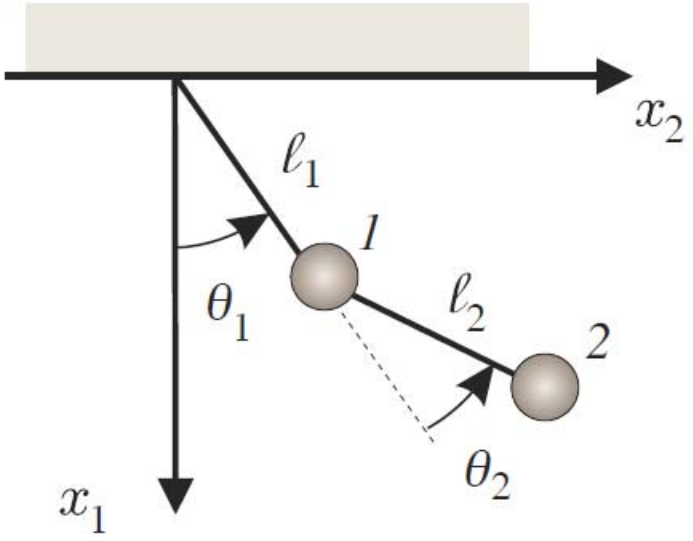
- ✓ Setting the largest component to unity.
- ✓ Setting the generalized mass to unity

When the latter is used:

$$\begin{cases} \mathbf{x}_{(s)}^{T}\,\mathbf{K}\,\mathbf{x}_{(r)} = \omega_r^2\delta_{rs} \\ \mathbf{x}_{(s)}^{T}\mathbf{M}\mathbf{x}_{(r)} = \delta_{rs} \end{cases}$$

Double Pendulum

$m_1=4m,\ m_2=m,\ l_1=2l,\ l_2=l$



$$\omega_1^2 = 0.4678\frac{g}{l} \quad , \quad \mathbf{x}_{(1)} = \begin{Bmatrix} 1 \\ 1.7578 \end{Bmatrix}$$

$$\omega_2^2 = 1.4697\frac{g}{l} \quad , \quad \mathbf{x}_{(2)} = \begin{Bmatrix} 1 \\ -6.2578 \end{Bmatrix}$$

$$\mathbf{x}_{(r)}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{x}_{(r)} = \mu_r \quad , \quad \mathbf{x}_{(r)}^{\mathrm{T}}\mathbf{K}\mathbf{x}_{(r)} = \gamma_r$$

$$\mu_1 = \mathbf{x}_{(1)}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{x}_{(1)} = \begin{bmatrix} 1 & 1.7578 \end{bmatrix}\begin{pmatrix} 20 & 2 \\ 2 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 1.7578 \end{bmatrix} = 30.121$$

$$\mu_2 = \mathbf{x}_{(2)}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{x}_{(2)} = \begin{bmatrix} 1 & -6.2578 \end{bmatrix}\begin{pmatrix} 20 & 2 \\ 2 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ -6.2578 \end{bmatrix} = 34.129$$

$$\gamma_1 = \mathbf{x}_{(1)}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{x}_{(1)} = \begin{bmatrix} 1 & 1.7578 \end{bmatrix}\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 1.7578 \end{bmatrix} = 14.09$$

$$\gamma_2 = \mathbf{x}_{(2)}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{x}_{(2)} = \begin{bmatrix} 1 & -6.2578 \end{bmatrix}\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ -6.2578 \end{bmatrix} = 50.16$$

Let's check if the modes are orthogonal!

$$\mathbf{x}_{(1)}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{x}_{(2)} = \begin{bmatrix}1 & 1.7578\end{bmatrix}\begin{pmatrix}20 & 2\\ 2 & 1\end{pmatrix}\begin{bmatrix}1\\ -6.2578\end{bmatrix} = \begin{bmatrix}23.516 & 3.7578\end{bmatrix}\begin{bmatrix}1\\ -6.2578\end{bmatrix} = 23.516 - 23.516 = 0$$

$$\mathbf{x}_{(1)}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{x}_{(2)} = \begin{bmatrix}1 & 1.7578\end{bmatrix}\begin{pmatrix}11 & 0\\ 0 & 1\end{pmatrix}\begin{bmatrix}1\\ -6.2578\end{bmatrix} = \begin{bmatrix}11 & 1.7578\end{bmatrix}\begin{bmatrix}1\\ -6.2578\end{bmatrix} = 11 - 11 = 0$$

Also it can be verified that

$$\frac{\gamma_1}{\mu_1} = \omega_1^2\frac{l}{g} = 0.4678 \quad , \quad \frac{\gamma_2}{\mu_2} = \omega_2^2\frac{l}{g} = 1.4697$$

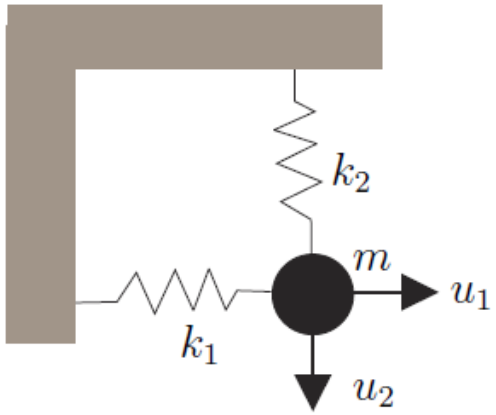
Now modes can be mass normalized

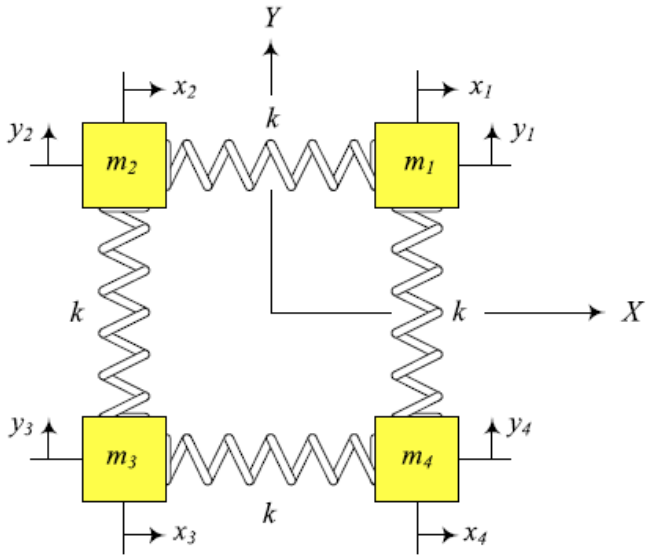
$$\tilde{\mathbf{x}}_{(1)} = \frac{\mathbf{x}_{(1)}}{\sqrt{\mu_1}} = \begin{Bmatrix}0.1822\\ 0.3203\end{Bmatrix} \quad , \quad \tilde{\mathbf{x}}_{(2)} = \frac{\mathbf{x}_{(2)}}{\sqrt{\mu_2}} = \begin{Bmatrix}0.1712\\ -1.0712\end{Bmatrix}$$

Degeneracy theorem

What happens if a multiple circular frequency is encountered?

In particular, here different modes can exist for a same eigenfrequency. In that case, some eigenvalues are multiple roots of the characteristic polynomial and we will call *m* that multiplicity.



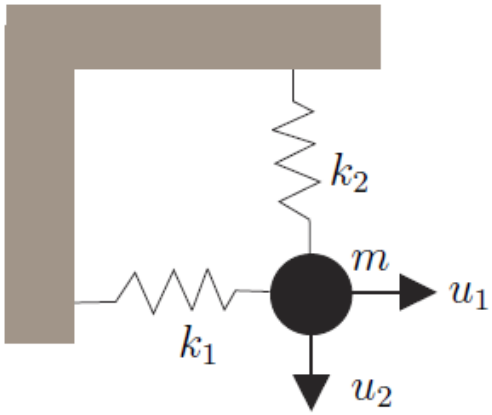


This theory states:

To a multiple root ω_p^2 of system (Kx- ω^2Mx)=0, corresponds a number of linearly independent eigenvectors equal to the root multiplicity

In other words: the m eigenvectors corresponding to an eigenvalue problem with multiplicity m are linearly independent. Moreover, any linear combination of the eigenmodes associated with ω_p^2 , is also a solution of the algebraic system (Kx- ω_p^2 Mx)=0

$$\left(\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right)\begin{Bmatrix} x \\ y \end{Bmatrix} = 0$$



The characteristic polynomial would be:

$$\left(k - \omega^2 m\right) = 0$$

$$\omega_1 = \omega_2 = \sqrt{\frac{k}{m}}$$ → Eigenvalue of multiplicity 2

Any choice $\mathbf{x}_{(s)} = \begin{Bmatrix} X \\ Y \end{Bmatrix}$ is an eigenmode.

We can choose $\mathbf{x}_{(1)} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ & $\mathbf{x}_{(2)} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ as the two satisfying K & M orthogonality.

Modal decoupling and normal equations

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t)$$

$$\textit{given} \quad \mathbf{q}(0) = \mathbf{q}_0 \ , \ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0$$

Since the normal vibration modes represent an orthogonal set of *n* vectors, any arbitrary vector can be expanded in the basis of the modes

$$\mathbf{q(t)} = \sum_{s=1}^{n} \eta_s(t)\mathbf{x}_{(s)}$$

Time dependent amplitudes — *Mode shapes*

$$\mathbf{M}\sum_{s=1}^{n} \ddot{\eta}_s(t)\,\mathbf{x}_{(s)} + \mathbf{K}\sum_{s=1}^{n} \eta_s(t)\mathbf{x}_{(s)} = \mathbf{p}(t)$$

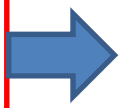
Let's now pre-multiply by the *r*th eigenmode

$$\mathbf{x}_{(r)}^{T}\,\mathbf{M}\sum_{s=1}^{n}\ddot{\eta}_s(t)\mathbf{x}_{(s)}+\mathbf{x}_{(r)}^{T}\,\mathbf{K}\sum_{s=1}^{n}\eta_s(t)\mathbf{x}_{(s)}=\mathbf{x}_{(r)}^{T}\,\mathbf{p}(t)$$

Owing to the orthogonality conditions we can say

$$\begin{cases}\mathbf{x}_{(s)}^{\mathbf{T}}\,\mathbf{K}\,\mathbf{x}_{(r)}=\omega_r^2\delta_{rs}\\ \mathbf{x}_{(s)}^{\mathbf{T}}\mathbf{M}\mathbf{x}_{(r)}=\delta_{rs}\end{cases}$$

$$\ddot{\eta}_r(t)+\omega_r^2\eta_r(t)=\varphi_r(t)$$



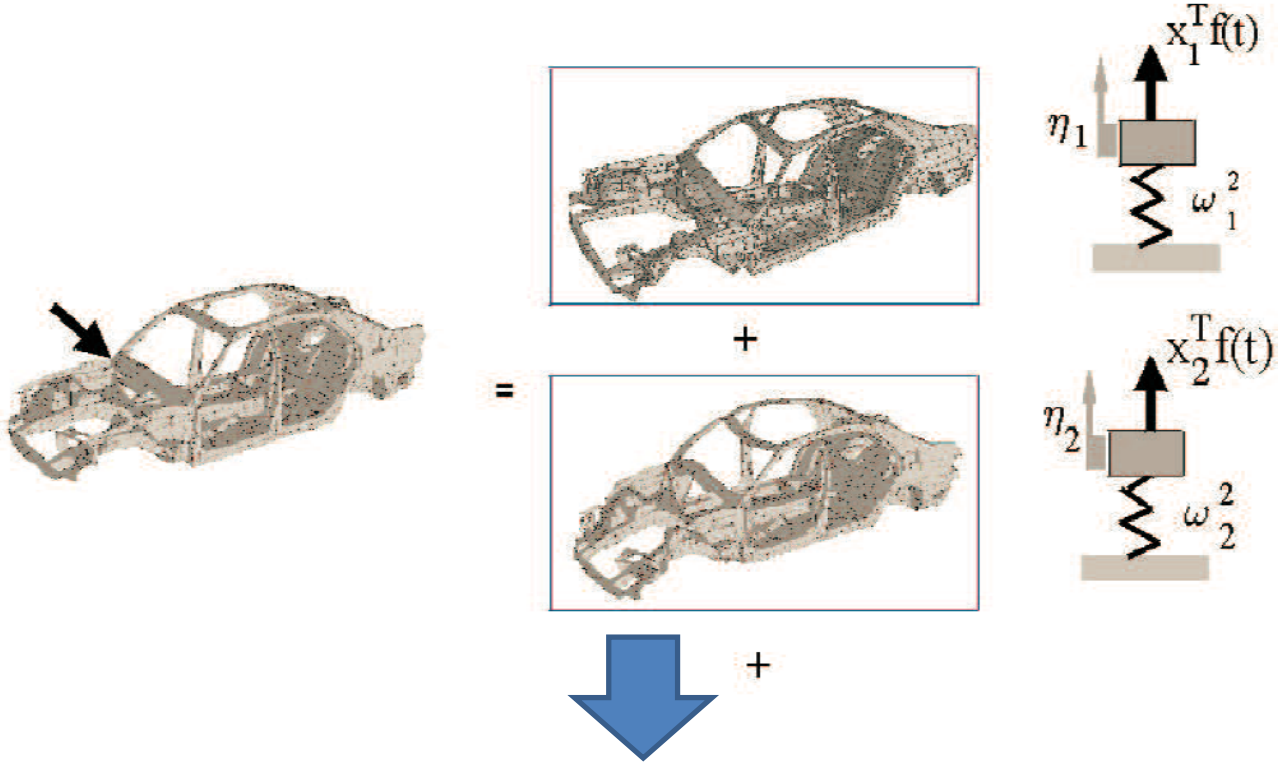
Normal equations

Where

η_r  Normal/modal coordinates

$$\varphi_r(t)=\frac{x_{(r)}^{T}\,\mathbf{p}(t)}{\mu_r}$$  Modal participation factor of mode *r* of the load(which modes will be excited during vibrations)

In the absence of damping, calculating the response of an n-degree-of freedom system reduces to the solution of n uncoupled single-degree-of-freedom systems excited by the external forces



The method transforms the fully coupled dynamical equation into a set of scalar differential equations, which by solving each separately and using $\mathbf{q(t)}=\sum_{s=1}^{n}\eta_s(t)x_{(s)}$ the final solution can be obtained!

NOTES:

- In case of rigid body modes, then the normal equations are still like before except that the corresponding frequency is zero (no elastic term).
- Initial conditions for normal equations:

$$\mathbf{q_0} = \sum_{s=1}^{n} \eta_s(0)\mathbf{x}_{(s)} \quad , \quad \dot{\mathbf{q}}_\mathbf{0} = \sum_{s=1}^{n} \dot{\eta}_s(0)\mathbf{x}_{(s)}$$

We can pre-multiply above by $\mathbf{x}_{(r)}^{\mathbf{T}}\mathbf{M}$

$$\eta_r(0) = \frac{\mathbf{x}_r^T\mathbf{M}\mathbf{q_0}}{\mu_r} \quad , \quad \dot{\eta}_r(0) = \frac{\mathbf{x}_r^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0}}{\mu_r} \quad , \quad r = 1, ..., n$$

Time integration of normal equations

Once I have obtained the uncoupled equations of motion the next step is to solve them when the system is subjected to an arbitrary excitation.

It is helpful to use Laplace transform

$$\bar{\eta}_r(s)=\mathbf{L}\left(\eta_r(t)\right)=\int_0^\infty e^{-st}\eta_r(t)\,dt \quad , \quad \bar{\phi}_r(s)=\mathbf{L}\left(\phi_r(t)\right)=\int_0^\infty e^{-st}\phi_r(t)\,dt$$

The Laplace transform of the normal equation is:

$$s^2\bar{\eta}_r(s)-s\eta_r(0)-\dot{\eta}_r(0)+\omega_r^2\bar{\eta}_r(s)=\bar{\phi}_r(s)$$

$$\bar{\eta}_r(s)=\frac{s\eta_r(0)+\dot{\eta}_r(0)+\bar{\phi}_r(s)}{s^2+\omega_r^2}$$

$$\mathbf{L}^{-1}\left(\bar{\eta}_r(s)\right)=\eta_r(t) \quad \Rightarrow \quad \mathbf{q(t)}=\sum_{r=1}^{n}\eta_r(t)x_{(r)}$$

Let's assume that there are no rigid body modes

$$\bar{\eta}_r(s) = \frac{1}{s^2+\omega_r^2}\dot{\eta}_r(0) + \frac{s}{s^2+\omega_r^2}\eta_r(0) + \frac{\bar{\varphi}_r(s)}{s^2+\omega_r^2}$$

$$\mathbf{L}^{-1}\left(\frac{1}{s^2+\omega_r^2}\right) = \frac{\sin\omega_r t}{\omega_r} = h_r(t) \qquad \mathbf{L}^{-1}\left(\frac{s}{s^2+\omega_r^2}\right) = \cos\omega_r t$$

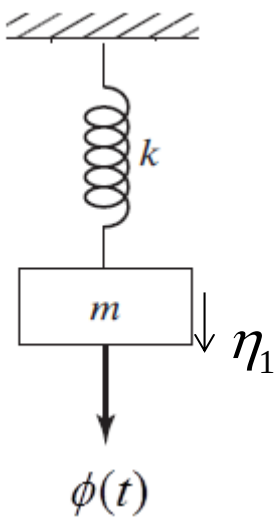
We already know are contributions of initial conditions

$$\mathbf{L}^{-1}\left(\frac{\bar{\varphi}_r(s)}{s^2+\omega_r^2}\right) = \int_0^t \varphi_r(t)h(t-\tau)d\tau$$

Duhamel's integral, representing the forced response or contribution of **p(t)**

What does the convolution product mean?

The impulse response at time zero. The impulse imparts a change in the momentum and thus change in velocity. Unit impulse is a function that has magnitude 1 at the time that is applied and is zero elsewhere



$$\tilde{\phi} = \phi(t)\Delta t = m\dot{\eta}_0$$

$$\ddot{\eta} + \omega_1^2 \eta = 0 \quad , \quad \eta(0) = 0 \quad , \quad \dot{\eta}(0) = \dot{\eta}_0 \quad , \quad \omega_1^2 = \frac{k}{m}$$

$$\eta_1 = \alpha \cos \omega_1 t + \beta \sin \omega_1 t \Rightarrow \eta_1 = \frac{\dot{\eta}_0}{\omega_1} \sin \omega_1 t \Rightarrow \eta_1 = \frac{\tilde{\phi}}{m\omega_1} \sin \omega_1 t$$

$$\eta_1 = \frac{\tilde{\phi}}{m} h(t)$$

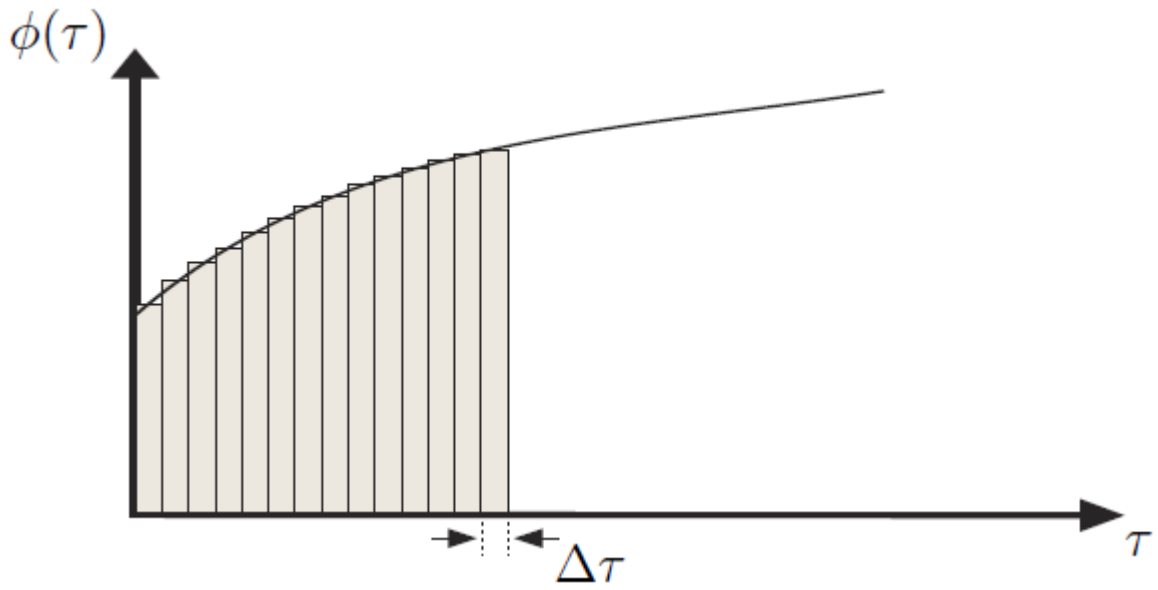
If the impulse is applied at time τ, then the system is changing velocity and for unit mass:

$$\eta_1 = \tilde{\phi} h(t - \tau)$$

h(t-τ) is called impulse response function of a system

Now if I have a force of general form, I can look at the force as a train of impulses. Therefore, the convolution product is understood as the sum of the responses to impulses at times τ between $t = 0$ and t.

$$\eta_r^{(p)} = \int_0^t \phi_r(t) h(t-\tau) d\tau$$



In the above, superscript (p) is used to show that this is indeed the particular solution. The general solution is the solution to initial conditions.

What happens if the force is harmonic?

- ✓ This happens in many engineering applications, e.g. rotor dynamics, turbine blades and etc.
- ✓ Using Fourier transform I can decompose a general periodic function to a summation of harmonics. If we can solve for one harmonic then we can solve for all.
- ✓ Harmonic excitation is also very useful in experimental dynamics.

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{s}\cos\omega t$$

ω is the excitation frequency and "s" is the magnitude.

In the absence of damping

$$\mathbf{q} = \mathbf{x}\cos\omega t$$

Which yields a system of algebraic equations:

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right)\mathbf{x} = \mathbf{s}$$

x is a generic shape that should be obtained and is not necessarily eigenmodes. What is the difference between the above equation and the previous eigenvalue problem?

We can solve the previous system but most of the times we are not just interested in the response for just one ω. Imagine solving the system for a large number of ω's and for millions of dofs! (Extremely expensive)

But we know modal analysis and can expand **x** in terms of eigenmodes and use orthogonality conditions:

$$\mathbf{x} = \sum_{i=1}^{m} \xi_i \mathbf{u}_{(i)} + \sum_{s=m+1}^{n} \eta_s \mathbf{x}_{(s)}$$

$$\xi_i = -\frac{\mathbf{u}_{(i)}^T \mathbf{s}}{\omega^2 \mu_i} \quad , \quad \eta_s = -\frac{\mathbf{x}_{(s)}^T \mathbf{s}}{\left(\omega_s^2 - \omega^2\right)\mu_s}$$

Therefore the generic shape can be obtained as

$$\mathbf{x} = \left\{ -\frac{1}{\omega^2} \sum_{i=1}^{m} \frac{\mathbf{u}_{(i)} \mathbf{u}_{(i)}^T}{\mu_i} + \sum_{s=m+1}^{n} \frac{\mathbf{x}_{(s)} \mathbf{x}_{(s)}^T}{\left(\omega_s^2 - \omega^2\right)\mu_s} \right\} \mathbf{s}$$

This formula is known as ***frequency response function*** and gives the shape of the motion. Given a excitation frequency we can simply calculate what the shape will be. Efficient!

$$\mathbf{x} = \left\{ -\frac{1}{\omega^2} \sum_{i=1}^{m} \frac{\mathbf{u}_{(i)} \mathbf{u}_{(i)}^T}{\mu_i} + \sum_{s=m+1}^{n} \frac{\mathbf{x}_{(s)} \mathbf{x}_{(s)}^T}{\left(\omega_s^2 - \omega^2\right) \mu_s} \right\} \mathbf{s}$$

NOTE1: **x** is not a mode , and therefore it has certain magnitude, it can be small or large.

NOTE2: The point at which the above expression explodes is when $\omega_s = \omega$ *(resonance)*

NOTE3: The reason that in reality you don't see infinite amplitude is because of the damping.

NOTE4: The other element important to see large amplitudes is to have nonzero projection of the force on the mode shape.

NOTE5: You can have responses in between frequencies as a combination of different eigenmodes.

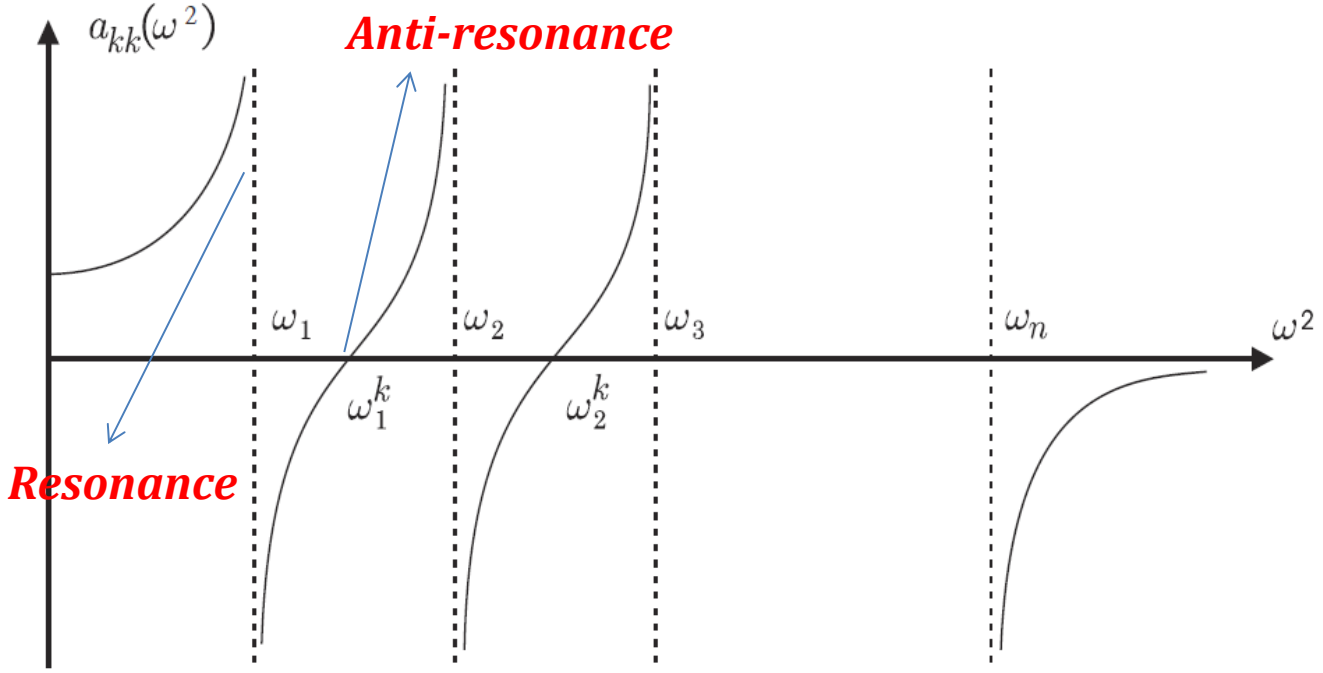
NOTE6: The previous expression is indeed a vector sometimes we are interested to see what happens at certain dof "*k*" when the load is applied at "*k*". Sometimes this is called ***frequency response function.***

$$\frac{x_k}{s_k}\left(\omega^2\right) = a_{kk}\left(\omega^2\right) = -\frac{1}{\omega^2} \sum_{i=1}^{m} \frac{u_{k(i)}^2}{\mu_i} + \sum_{s=m+1}^{n} \frac{x_{k(s)}^2}{\left(\omega_s^2 - \omega^2\right) \mu_s}$$

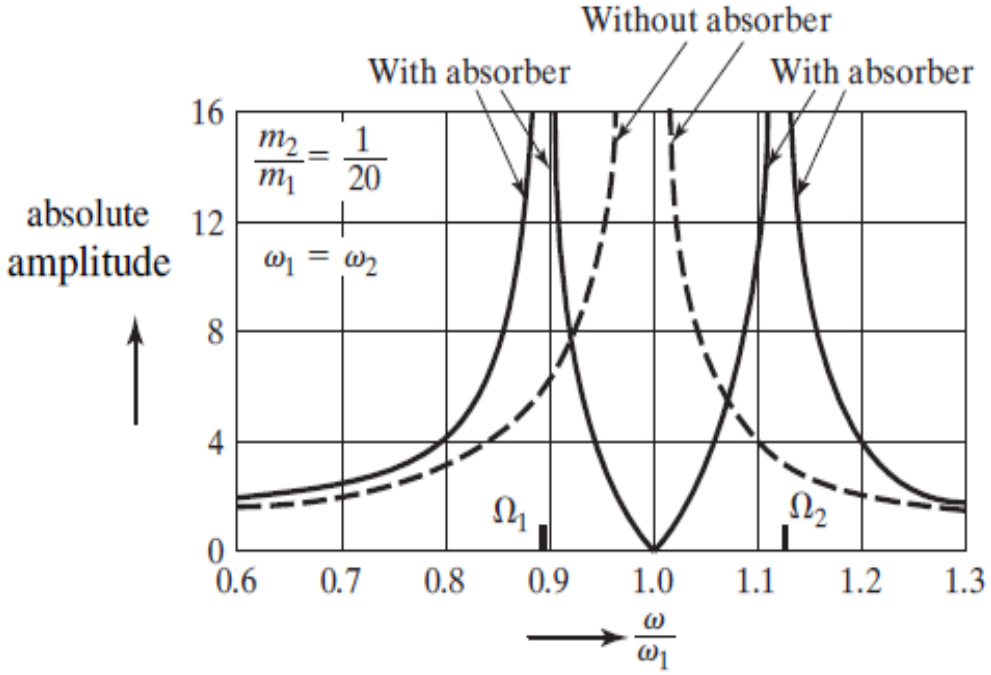
Now if we take the derivative with respect to ω^2:

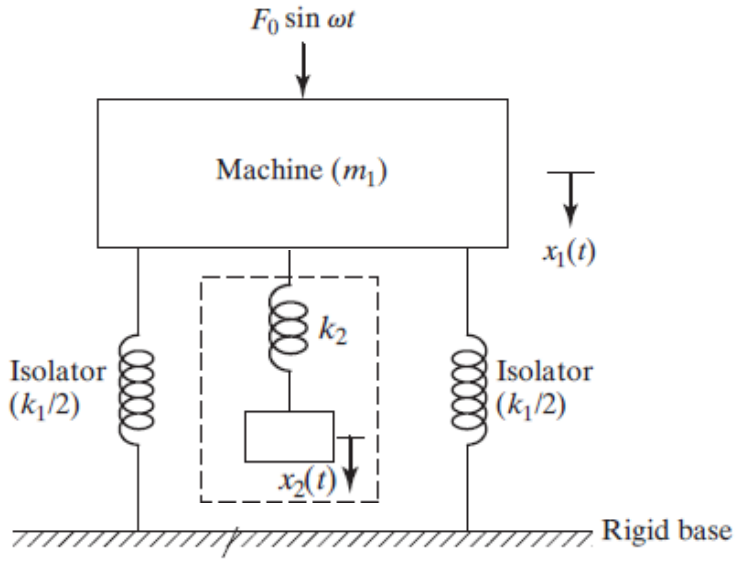
$$\frac{da_{kk}}{d\omega^2} = \frac{1}{\left(\omega^2\right)^2}\sum_{i=1}^{m}\frac{u_{k(i)}^2}{\mu_i} + \sum_{s=1}^{n-m}\frac{x_{k(s)}^2}{\left(\omega_s^2 - \omega^2\right)^2\mu_s} > 0$$

Which means that the amplitude response is always increasing with the increase in the excitation frequency. If there are no rigid body mode, plotting a_{kk} versus ω^2:



The presence of anti-resonance means that, paradoxically I can excite the system at a specific frequency that the system moves but that point that I am applying the force stands still! This is indeed the principle of passive vibration absorbers (e.g. dumbbell shape devices that are hung from transmission lines)





In many practical applications, you cannot redesign the system, rather you can tune stiffness and mass in such a way that the vibration dies out for undesired frequencies by adding certain degrees of freedom (absorbers)

Approximation by truncated modal series

Unfortunately it is not always possible to extract all eigenmodes. A practical way is to build an approximate solution by truncating the modal superposition. This is in fact the basis is what is known as ***reduced-order modelling.***

$$\mathbf{q(t)}=\sum_{s=1}^{k}\eta_s(t)\,\mathbf{x}_{(s)} \quad k<n$$

$\eta_s(t)$ being the solutions of the normal equations.

$$\mathbf{x}=\left\{-\frac{1}{\omega^2}\sum_{i=1}^{m}\frac{\mathbf{u}_{(i)}\mathbf{u}_{(i)}^T}{\mu_i}+\sum_{s=m+1}^{k}\frac{\mathbf{x}_{(s)}\mathbf{x}_{(s)}^T}{\left(\omega_s^2-\omega^2\right)\mu_s}\right\}\mathbf{s}$$

Clearly this is a good approximation as long as ω^2 is much smaller ω_s^2

Now let's suppose that instead of a harmonic force we have a generic one:

$$\mathbf{p(t)=g}\varphi(t)$$

Therefore, for a system initially at rest the response using Duhamel's integral :

$$\mathbf{q}(t)=\sum_{s=1}^{k}\frac{\mathbf{x}_{(s)}\mathbf{x}_{(s)}^{\mathbf{T}}}{\mu_{\mathbf{s}}}\mathbf{g}\frac{1}{\omega_{\mathbf{s}}}\int_{0}^{t}\mathbf{sin}\left(\omega_s(t-\tau)\right)\varphi(\tau)d\tau$$

So the convergence of solution depends on:

✓ A spatial factor (Quasi-static convergence)

if the applied load **g** admits a sufficiently accurate spatial representation in the basis of the $k < n$ retained eigenmodes

$$\frac{\mathbf{x}_{(s)}\mathbf{x}_{(s)}^{\mathbf{T}}}{\mu_{\mathbf{s}}}\mathbf{g}$$

✓ A temporal factor (spectral convergence)

Depending on both on the frequency content of the excitation and on the system eigenspectrum

$$\theta_{\mathbf{s}}(t)=\frac{1}{\omega_{\mathbf{s}}}\int_{0}^{t}\mathbf{sin}\left(\omega_s(t-\tau)\right)\varphi(\tau)d\tau$$

To clarify the second one:

- In case of step load $\varphi(t)=1$ for $t>0$

$$\theta_s(t) = \frac{1-\cos\omega_s t}{\omega_s^2}$$

It is clear that first modes of vibration will have the largest contribution.

- In case of harmonic force $\varphi(t)=\cos\omega t$

$$\theta_s(t) = \frac{\omega_s \sin\omega t - \omega \sin\omega_s t}{\omega_s\left(\omega_s^2 - \omega^2\right)}$$

Clearly modes having frequencies much higher than the excitation frequency will have little influence.

Bottom line: you should look at both spatial and temporal convergences

Modal analysis of damped systems

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t)$$

- ✓ Damping is generally a complicated non-linear phenomena
- ✓ Here we assume linear viscous damping. **C** is symmetric and non-negative.
- ✓ We already know from the associated conservative system that we have eigenfrequencies ω_r^2 and eigenmodes $\mathbf{x}_{(r)}$ which are both real! Let's try to see if we can uncouple the damped system using eigenmode properties!

$$\mathbf{q}(t) = \sum_{s=1}^{n} \eta_s(t)\,\mathbf{x}_{(s)}$$

$$\mathbf{M}\sum_{s=1}^{n} \ddot{\eta}_s(t)\,\mathbf{x}_{(s)} + \mathbf{C}\sum_{s=1}^{n} \dot{\eta}_s(t)\,\mathbf{x}_{(s)} + + \mathbf{K}\sum_{s=1}^{n} \eta_s(t)\mathbf{x}_{(s)} = \mathbf{p}(t)$$

Again projecting onto generic mode $\mathbf{x}_{(r)}$

$$\mathbf{x}_{(r)}^T \mathbf{M} \sum_{s=1}^{n} \ddot{\eta}_s(t)\mathbf{x}_{(s)} + \mathbf{x}_{(r)}^T \mathbf{C} \sum_{s=1}^{n} \dot{\eta}_s(t)\mathbf{x}_{(s)} + \mathbf{x}_{(r)}^T \mathbf{K} \sum_{s=1}^{n} \eta_s(t)\mathbf{x}_{(s)} = \mathbf{p(t)}$$

$$\mathbf{x}_{(r)}^T \mathbf{C} \mathbf{x}_{(s)} = \beta_{rs} \quad \text{(damping coefficients)}$$

Note that eigenmodes are not orthogonal to one another with respect to damping matrix and as a matter of fact the system is not decoupled!

$$\ddot{\eta}_r(t) + \sum_{s=1}^{n} \frac{\beta_{rs}}{\mu_r} \dot{\eta}_s(t) + \omega_r^2 \eta_r(t) = \varphi_r(t) \qquad r = 1, 2, ..., n$$

- ✓ The above equation still could be useful when dealing with reduced-order models, but even in that case we have coupled equations.
- ✓ Note that in this case the eigenmodes and eigenvalues are complex indicating that the free vibration solutions are non-synchronous.
- ✓ In practice, many systems are lightly damped (e.g. structures). In those cases the above system could be decoupled by introducing modal damping.

Modal damping assumption

If we are able to show that the damping matrix $[\beta_{rs}]$, could be diagonal then we are able to uncouple equations. This assumption has no physical background but we are going to show that it is consistent with the lightly damped systems

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=0 \quad , \quad \mathbf{q}=\mathbf{z}e^{\lambda t}$$

$$\left(\lambda_k^2\,\mathbf{M}+\lambda_k\mathbf{C}+\mathbf{K}\right)\mathbf{z}_{(k)}=0$$

In the associated undamped system

$$\lambda_k=\pm i\omega_k \quad , \quad \mathbf{z}_{(k)}=\mathbf{x}_{(k)}$$

$$\left(\mathbf{K}-\omega_k^2\mathbf{M}\right)\mathbf{x}_{(k)}=0$$

We begin by saying that the eigenfrequencies and modes of the damped system should not be very different from undamped ones.

$$\lambda_k = i\omega_k + \Delta\lambda \ ,$$

$$\mathbf{z}_{(k)} = \mathbf{x}_{(k)} + \Delta\mathbf{z}$$

Inserting the above relations in the damped eigenvalue problem

$$\left(\left(-\omega_k^2 + 2i\omega_k\Delta\lambda + \left(\Delta\lambda\right)^2\right)\mathbf{M} + \left(i\omega_k + \Delta\lambda\right)\mathbf{C} + \mathbf{K}\right)\left(\mathbf{x}_{(k)} + \Delta\mathbf{z}\right) = 0$$

Neglecting second order terms and (K- ω_k^2 M) $x_{(k)}$ =0

$$\left(\mathbf{K} - \omega_k^2\,\mathbf{M}\right)\Delta\mathbf{z} + \left(2i\omega_k\,\mathbf{M} + \mathbf{C}\right)\mathbf{x}_{(k)}\Delta\lambda + i\omega_k\mathbf{C}\left(\mathbf{x}_{(k)} + \Delta\mathbf{z}\right) \simeq 0$$

We are assuming lightly damped systems, therefore $C\Delta\lambda=0$, $C\Delta z=0$, this will yield the following first order approximation:

$$\left(\mathbf{K} - \omega_k^2\,\mathbf{M}\right)\Delta\mathbf{z} + i\omega_k\left(\mathbf{C} + 2\Delta\lambda\,\mathbf{M}\right)\mathbf{x}_{(k)} \simeq 0$$

Premultiplying by $x_{(k)}^T$

$$i\omega_k \mathbf{x}_{(k)}^T \left(\mathbf{C} + 2\Delta\lambda \mathbf{M}\right)\mathbf{x}_{(k)} \simeq 0 \quad , \quad \mathbf{x}_{(k)}^T \mathbf{C}\mathbf{x}_{(k)} = \beta_k \quad , \quad \mathbf{x}_{(k)}^T \mathbf{M}\mathbf{x}_{(k)} = \mu_k$$

$$i\omega_k \left(\beta_k + 2\Delta\lambda\mu_k\right) \simeq 0 \Rightarrow \quad \Delta\lambda \simeq -\frac{\beta_k}{2\mu_k}$$

Therefore

$$\lambda_k = i\omega_k - \frac{\beta_k}{2\mu_k},$$

- ✓ $\Delta\lambda$ is real & negative. Thus, it transforms each term of the fundamental solution into a damped oscillatory motion. (consequence of positiveness of **C**).
- ✓ It was seen that the first order correction involves only diagonal terms β_k . Therefore non diagonal terms are of the second order and can simply be neglected.

We can also find the correction on the mode shapes. The correction can be expanded in terms of the undamped eigenmodes

$$\Delta\mathbf{z} = \sum_{\substack{s=1 \\ s\neq k}}^{n} \alpha_s \mathbf{x}_{(s)}$$

If $\Delta\mathbf{z}$ is part of $\mathbf{x}_{(k)}$ then we are just scaling it . Therefore, *k*-th mode is not included in the above relation.

$$\left(\mathbf{K} - \omega_k^2 \mathbf{M}\right) \sum_{\substack{s=1 \\ s\neq k}}^{n} \alpha_s \mathbf{x}_{(s)} + i\omega_k \left(\mathbf{C} + 2\Delta\lambda \mathbf{M}\right) \mathbf{x}_{(k)} \simeq 0$$

Now we premultiply by $\mathrm{x}_{(\mathrm{l})}{}^{\mathrm{T}}$

$$\mathbf{x}_{(l)}^{\mathbf{T}} \left(\mathbf{K} - \omega_k^2 \mathbf{M}\right) \sum_{\substack{s=1 \\ s\neq k}}^{n} \alpha_s \mathbf{x}_{(s)} + i\omega_k \mathbf{x}_{(l)}^{\mathbf{T}} \left(\mathbf{C} + 2\Delta\lambda \mathbf{M}\right) \mathbf{x}_{(k)} \simeq 0$$

$$\mathbf{x}_{(l)}^{\mathbf{T}} \mathbf{C} \mathbf{x}_{(k)} = \beta_{kl}$$

$$\alpha_l = \frac{i\omega_k \beta_{kl}}{\mu_l \left(\omega_k^2 - \omega_l^2\right)}$$

Purely imaginary

$$\mathbf{z}_{(k)} = \mathbf{x}_{(k)} + \sum_{\substack{s=1 \\ k \neq s}}^{n} \frac{i\omega_k \beta_{ks}}{\mu_s \left(\omega_k^2 - \omega_s^2 \right)} \mathbf{x}_{(s)}$$

- ✓ The correction is purely imaginary. This is consistent with the fact that modes in damped case are complex!
- ✓ The biggest corrections come from the modes that are closer to one another.
- ✓ β_{ks} is likely to be second order as long as the modes are well separated. This means that if a structure has light damping the mode complexity is very small and thus it is still possible to use the real normal modes!

$$\beta_{ks} = 0 \quad k \neq s$$

Diagonal or modal damping assumption

$$\ddot{\eta}_r(t) + \frac{\beta_r}{\mu_r} \dot{\eta}_s(t) + \omega_r^2 \eta_r(t) = \varphi_r(t) \qquad r = 1, 2, \ldots, n$$

We can define modal damping coefficient as (damping ratio): $\varepsilon_r = \dfrac{\beta_r}{2\omega_r \mu_r}$

Therefore,

$$\ddot{\eta}_r(t) + 2\varepsilon_r \omega_r \dot{\eta}_r(t) + \omega_r^2 \eta_r(t) = \varphi_r(t) \qquad r = 1, 2, ..., n$$

- ✓ ε_r are obtained from experiments!
- ✓ If the damping is distributed in the same way as inertia and stiffness (proportional damping), ε_r can be estimated through a diagonal damping matrix

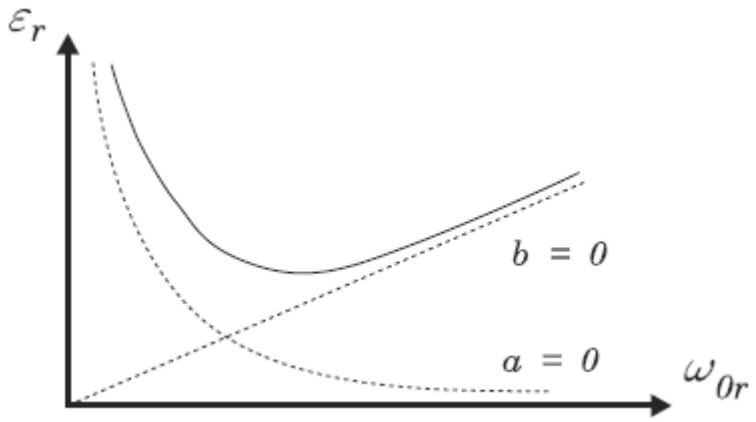
$$\mathbf{C} = a\mathbf{M} + b\mathbf{K}$$

This immediately gives

$$\beta_r = a\mu_r + b\gamma_r$$

$$\varepsilon_r = \frac{1}{2}\left(a\omega_r + \frac{b}{\omega_r}\right)$$

Hence, if the damping ratios are known from experimental vibration testing for two eigenfrequencies, the coefficients a and b for the proportional damping matrix can be determined



Nevertheless, it can be seen that the previous formula will generate a higher damping in the lower and higher frequency ranges, and a lower one in the intermediate range.

Possible solutions: use least squares technique.

Force vibrations of damped systems

$$\ddot{\eta}_r(t) + 2\varepsilon_r \omega_r \dot{\eta}_s(t) + \omega_r^2 \eta_r(t) = \varphi_r(t) \qquad r = 1, 2, ..., n$$

We recall that

$$\varphi_r(t) = \frac{x_{(r)}^T \mathbf{p}(t)}{\mu_r}$$

Applying Laplace transform and assuming zero initial conditions

$$s^2 \bar{\eta}_r(s) + 2\varepsilon_r \omega_r s \bar{\eta}_r(s) + \omega_r^2 \bar{\eta}_r(s) = \bar{\varphi}_r(s) \Rightarrow \bar{\eta}_r(s) = \frac{\bar{\varphi}_r(s)}{s^2 + 2\varepsilon_r \omega_r s + \omega_r^2}$$

$$\mathbf{L}^{-1}\left(\bar{\eta}_r(s)\right) = \int_0^t \varphi_r(\tau) h(t-\tau)\, d\tau$$

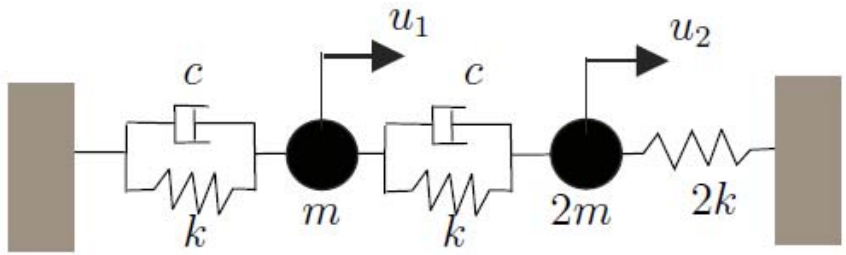
The convolution integral in this case includes the impulse response of the single dof damped system

$$h(t-\tau) = \frac{1}{\omega_{dr}} e^{-\varepsilon_r \omega_r (t-\tau)} \sin\left(\omega_{dr}(t-\tau)\right) \qquad , \omega_{dr} = \omega_r \sqrt{1-\varepsilon_r^2}$$

Modal analysis of a damped system

$$\mathbf{M}=\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix},\ \mathbf{C}=\begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix}$$

$$\mathbf{K}=\begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix}$$



$$\det\left(\begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix}-\omega^2\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\right)=0 \quad \xrightarrow{\nu^2=\omega^2\frac{m}{k}} \quad \left(2-\nu^2\right)\left(3-\nu^2\right)-1=0$$

$$\nu_1^2=1\Rightarrow\omega_1^2=\frac{k}{m}$$

$$\nu_2^2=\frac{5}{2}\Rightarrow\omega_2^2=\frac{5k}{2m}$$

$$\Rightarrow \quad \mathbf{x}_{(1)}=\begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad \mathbf{x}_{(2)}=\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$$

$$\mu_1=\mathbf{x}_{(1)}^{\mathrm{T}}\mathbf{M}\,\mathbf{x}_{(1)}=3m \quad , \quad \mu_2=\mathbf{x}_{(2)}^{\mathrm{T}}\mathbf{M}\,\mathbf{x}_{(2)}=1.5m$$

Modal damping matrix

$$\beta=\begin{bmatrix} \mathbf{x}_{(1)}^{\mathrm{T}} \\ \mathbf{x}_{(2)}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \mathbf{x}_{(1)} & \mathbf{x}_{(1)} \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}=c\begin{bmatrix} 1 & 1 \\ 1 & 13/4 \end{bmatrix}$$

The system will remained coupled. Now using the modal damping assumption and assuming $c = 0.06\sqrt{km}$

$$\varepsilon_1 = \frac{\beta_{11}}{2\omega_1\mu_1} = 0.01$$

$$\varepsilon_2 = \frac{\beta_{22}}{2\omega_2\mu_2} = 0.0411$$

Now let's check and see if modal damping is a valid assumption by finding the eigenvalues and eigenvectors of the complete system

$$\left(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}\right)\mathbf{z} = 0$$

Or alternatively

$$\left(\lambda\begin{bmatrix} 0 & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} + \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix}\right)\begin{bmatrix} \lambda\mathbf{z} \\ \mathbf{z} \end{bmatrix} = 0$$

$$\lambda_1 = \left(-0.009986 + i1.0002\right)\sqrt{\frac{k}{m}} \Rightarrow \mathbf{z}_{(1)} = \begin{bmatrix} 0.5246 + i0.4715 \\ 0.5065 + i0.4932 \end{bmatrix}$$

$$\lambda_2 = \left(-0.065014 + i1.5794\right)\sqrt{\frac{k}{m}} \Rightarrow \mathbf{z}_{(2)} = \begin{bmatrix} 0.2482 - i0.3798 \\ -0.1114 + i0.1964 \end{bmatrix}$$

With modal damping assumption

$$\lambda_1 = -\frac{\beta_{11}}{\mu_1} + i\omega_1 = \left(-0.01 + i\right)\sqrt{\frac{k}{m}}$$

$$\lambda_2 = -\frac{\beta_{22}}{\mu_2} + i\omega_2 = \left(-0.065 + i1.58\right)\sqrt{\frac{k}{m}}$$

There is almost no difference between the values predicted by modal damping assumption and those obtained from full system

To check damped eigenmodes with respect to the undamped ones

$$\mathbf{z}_{(1)} = \begin{bmatrix} 0.5246 + i0.4715 \\ 0.5065 + i0.4932 \end{bmatrix} \Big/ \left(0.5246 + i0.4715\right) \Rightarrow \mathbf{z}_{(1)} = \begin{bmatrix} 1 \\ 1.0015 + i0.03997 \end{bmatrix}$$

$$\mathbf{z}_{(2)} = \begin{bmatrix} 0.2482 - i0.3798 \\ -0.1114 + i0.1964 \end{bmatrix} \Big/ \left(0.2482 - i0.3798\right) \Rightarrow \mathbf{z}_{(2)} = \begin{bmatrix} 1 \\ -0.497 + i0.03138 \end{bmatrix}$$

Little complexity with respect to undamped modes. However, still normal modes could be used to define system response due to the fact that damping is small and frequencies are well separated