Hamilton’s principle and least action

WB 1418-07

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- ✓ **D'Alambert principle was a tool to get rid of reaction forces if one describes kinematically admissible displacements.**
- ✓ **Lagrange equations provided a systematic tool for obtaining the equations without any need for performing vectorial calculations (Physics could be lost).**
- ✓ **Lagrange's equations is suitable if we are dealing with individual masses and stiffness elements. What if the relative position of 2 points within a body is not kinematically constraint or in other words the body's displacement is a function of location within the body?**

Hamilton's principle → A general technique for obtaining EoMs of discrete and continuous media

Based on the concept of Energy

D'Alambert principle for a system of *N* particles states that:

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\ddot{u}_{ik}-X_{ik}\right)\delta u_{ik}=0$$

Let's assume

$$\sum_{k=1}^{N}\sum_{i=1}^{3}X_{ik}\,\delta u_{ik}=\delta\overline{W}$$

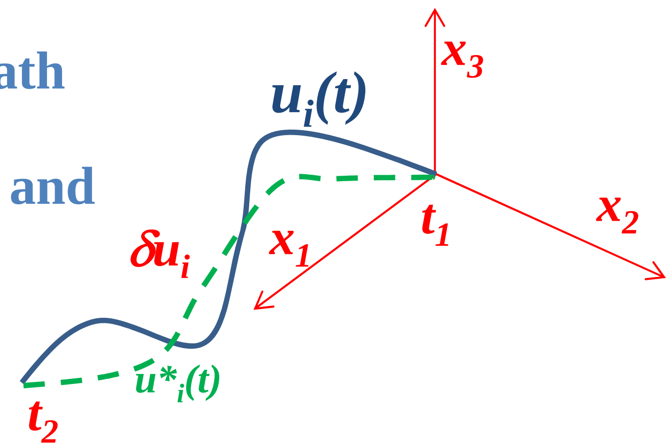
We can rewrite

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\ddot{u}_{ik}\right)\delta u_{ik}=\frac{d}{dt}\left(\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\dot{u}_{ik}\right)\delta u_{ik}\right)-\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\dot{u}_{ik}\right)\delta\dot{u}_{ik}$$

$$\delta\sum_{k=1}^{N}\sum_{i=1}^{3}\left(\frac{1}{2}m_k\dot{u}_{ik}^2\right)\Rightarrow\delta T$$

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\ddot{u}_{ik}-X_{ik}\right)\delta u_{ik}=0 \;\Longrightarrow\; \frac{d}{dt}\left(\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\dot{u}_{ik}\right)\delta u_{ik}\right)=\delta T+\delta\overline{W}$$

As time unfolds the dynamic system traces the true path $u_i(t)$ in 3N space. Let's assume that a varied path also exists ($u^*_i(t)$) that coincides with true path at times t_1 and t_2. Integrating above equation between t_1 and t_2:



$$\int_{t_1}^{t_2}\left(\delta T+\delta\overline{W}\right)dt=\int_{t_1}^{t_2}\frac{d}{dt}\left(\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\dot{u}_{ik}\right)\delta u_{ik}\right)dt$$

$$\sum_{k=1}^{N}\sum_{i=1}^{3}\left(m_k\dot{u}_{ik}\right)\delta u_{ik}\Big|_{t_1}^{t_2}=0$$

Therefore:

$$\int_{t_1}^{t_2}\left(\delta T+\delta\overline{W}\right)dt=0 \quad , \quad \delta u_{ik}(t_1)=\delta u_{ik}(t_2)=0 \;\; i=1,2,3 \;\;\&\; k=1,\ldots,N$$

In terms of generalized coordinates

$$\int_{t_1}^{t_2}\left(\delta T+\delta\overline{W}\right)dt=0 \quad , \quad \delta q_k(t_1)=\delta q_k(t_2)=0 \ \ k=1,....,n$$

Also the virtual work includes contributions of conservative and non-conservative forces

$$\delta\overline{W}=\delta W^{cons}+\delta W^{ncons}=-\delta V+\delta W^{ncons}$$

Therefore:

$$\int_{t_1}^{t_2}\left(\delta T-\delta V+\delta W^{ncons}\right)dt=0 \quad , \quad \delta q_k(t_1)=\delta q_k(t_2)=0 \ \ k=1,....,n$$

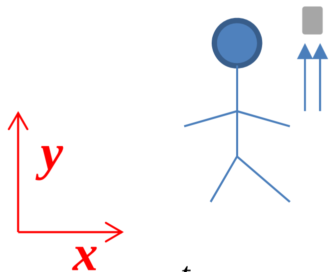
If there are no nonconservative forces, then:

$$\delta\int_{t_1}^{t_2}\underbrace{\left(T-V\right)}_{L}dt=0 \quad , \quad \delta q_k(t_1)=\delta q_k(t_2)=0 \ \ k=1,....,n$$

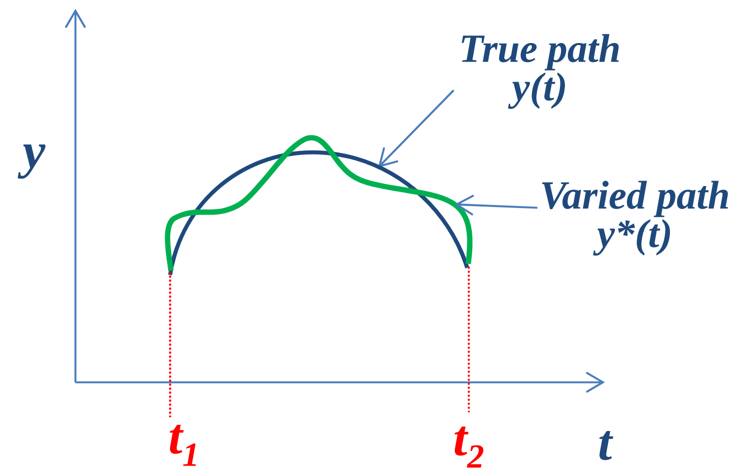
Variational operator

I=Action Integral

What does action integral mean and how Hamilton's principle could be interpreted?



$$y^*(t) - y(t) = \delta y$$



$$I_{true} = \int_{t_1}^{t_2} (T - V)\,dt = \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{y}^2 - mgy \right) dt$$

$$I_{\mathbf{var}} = \int_{t_1}^{t_2} \left(\frac{1}{2} m\left(\dot{y} + \delta\dot{y}\right)^2 - mg\left(y + \delta y\right) \right) dt$$

If we expand I_{var}

Very small=0

$$I_{\mathbf{var}} = \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{y}^2 + m\dot{y}\delta\dot{y} + \frac{1}{2} m\left(\delta\dot{y}\right)^2 - mgy - mg\delta y \right) dt$$

$$I_{\mathbf{var}} = I_{true} + \int_{t_1}^{t_2} \left(m\dot{y}\delta\dot{y} - mg\delta y \right) dt$$

$$I_{\mathbf{var}} = I_{true} + \int_{t_1}^{t_2} \left(m\dot{y}\delta\dot{y} - mg\delta y \right) dt$$

$$I_{\mathbf{var}} = I_{true} + \underbrace{\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{y}^2 - mgy \right) dt}_{I}$$

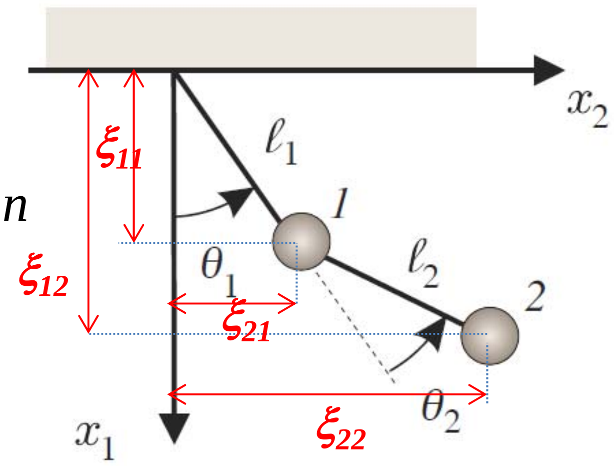
If δ*I*=0 the true path renders the value of the action integral stationary with respect to all arbitrary variations of the path between two instants t_1 and t_2 provided that the path variations vanish at these two instants. Therefore:

$$\int_{t_1}^{t_2} \left(m\dot{y}\delta\dot{y} - mg\delta y \right) dt = \underbrace{m\dot{y}\delta y \Big|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \left(m\ddot{y} + mg \right) \delta y \, dt$$

So the only way that δ*I*=0 is:

$$m\ddot{y} + mg = 0$$

The double pendulum



$$\delta\int_{t_1}^{t_2}\left(T - V + W^{ncons}\right)dt = 0 \quad , \quad \delta q_k(t_1) = \delta q_k(t_2) = 0 \;\; k = 1,\ldots,n$$

$$k = 2 \quad , \quad q_1 = \theta_1 \quad , \quad q_2 = \theta_2$$

$$Q_1^{ncons} = Q_1^{ncons} = 0$$

$$\xi_{11} = u_{11} = l_1 \cos\theta_1$$

$$\xi_{21} = u_{21} = l_1 \sin\theta_1$$

$$\xi_{12} = u_{12} = \xi_{11} + l_2 \cos(\theta_1 + \theta_2) = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$

$$\xi_{22} = u_{22} = \xi_{21} + l_2 \sin(\theta_1 + \theta_2) = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

$$\dot{u}_{11} = -l_1\dot{\theta}_1 \sin\theta_1$$

$$\dot{u}_{21} = l_1\dot{\theta}_1 \cos\theta_1$$

$$\dot{u}_{12} = -l_1\dot{\theta}_1 \sin\theta_1 - l_2\dot{\theta}_1 \sin(\theta_1 + \theta_2) - l_2\dot{\theta}_2 \sin(\theta_1 + \theta_2)$$

$$\dot{u}_{22} = l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_1 \cos(\theta_1 + \theta_2) + l_2\dot{\theta}_2 \cos(\theta_1 + \theta_2)$$

$$T = \frac{1}{2}m\dot{u}_{11}^2 + \frac{1}{2}m\dot{u}_{21}^2 + \frac{1}{2}m\dot{u}_{12}^2 + \frac{1}{2}m\dot{u}_{22}^2$$

$$= \frac{1}{2}ml_1^2\dot{\theta}_1^2 + \frac{1}{2}ml_1^2\dot{\theta}_1^2 + \frac{1}{2}ml_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + ml_1l_2\dot{\theta}_1^2\,\mathbf{cos}\,\theta_2 + ml_1l_2\dot{\theta}_1\dot{\theta}_2\,\mathbf{cos}\,\theta_2$$

$$V = -mgu_{11} - mgu_{12}$$

$$= mgl_1\left(1-\mathbf{cos}\,\theta_1\right) + mgl_1\left(1-\mathbf{cos}\,\theta_1\right) + mgl_2\left(1-\mathbf{cos}\left(\theta_1 + \theta_2\right)\right)$$

$$\delta\int_{t_1}^{t_2} Tdt = \int_{t_1}^{t_2}\Big[ml_1^2\dot{\theta}_1\delta\dot{\theta}_1 + ml_1^2\dot{\theta}_1\delta\dot{\theta}_1 + ml_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\left(\delta\dot{\theta}_1 + \delta\dot{\theta}_2\right)$$

$$+2ml_1l_2\dot{\theta}_1\,\mathbf{cos}\,\theta_2\delta\dot{\theta}_1 - ml_1l_2\dot{\theta}_1^2\,\mathbf{sin}\,\theta_2\delta\theta_2 + ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\delta\dot{\theta}_1$$

$$+ml_1l_2\dot{\theta}_1\,\mathbf{cos}\,\theta_2\delta\dot{\theta}_2 - ml_1l_2\dot{\theta}_1\dot{\theta}_2\,\mathbf{sin}\,\theta_2\delta\theta_2\Big]dt$$

$$\delta\int_{t_1}^{t_2} V\,dt = \int_{t_1}^{t_2}\Big[mgl_1\,\mathbf{sin}\,\theta_1\delta\theta_1 + mgl_1\,\mathbf{sin}\,\theta_1\delta\theta_1$$

$$+mgl_2\,\mathbf{sin}\left(\theta_1 + \theta_2\right)\delta\theta_1 + mgl_2\,\mathbf{sin}\left(\theta_1 + \theta_2\right)\delta\theta_2\Big]dt$$

Integration by parts

$$\delta\int_{t_1}^{t_2} T\,dt =$$

$$\boxed{\Big[ml_1^2\dot{\theta}_1\delta\theta_1 + ml_1^2\dot{\theta}_1\delta\theta_1 + ml_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)\delta\theta_1 + ml_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)\delta\theta_2 + 2ml_1l_2\dot{\theta}_1\,\mathbf{cos}\,\theta_2\delta\theta_1 + ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\delta\theta_1 + ml_1l_2\dot{\theta}_1\,\mathbf{cos}\,\theta_2\delta\theta_2\Big]_{t_1}^{t_2} -}$$

Zero

$$\int_{t_1}^{t_2}\Big[ml_1^2\ddot{\theta}_1\delta\theta_1 + ml_1^2\ddot{\theta}_1\delta\theta_1 + ml_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)\left(\delta\theta_1+\delta\theta_2\right)$$

$$+\frac{d}{dt}\left(2ml_1l_2\dot{\theta}_1\,\mathbf{cos}\,\theta_2\right)\delta\theta_1 - ml_1l_2\dot{\theta}_1^2\,\mathbf{sin}\,\theta_2\delta\theta_2 + \frac{d}{dt}\left(ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\right)\delta\theta_1$$

$$+\frac{d}{dt}\left(ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\right)\delta\theta_2 - ml_1l_2\dot{\theta}_1\dot{\theta}_2\,\mathbf{sin}\,\theta_2\delta\theta_2\Big]dt$$

$$\delta\int_{t_1}^{t_2} T\,dt =$$

$$-\int_{t_1}^{t_2}\left[ml_1^2\ddot{\theta}_1 + ml_1^2\ddot{\theta}_1 + ml_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + \frac{d}{dt}\left(2ml_1l_2\dot{\theta}_1\,\mathbf{cos}\,\theta_2\right) + \frac{d}{dt}\left(ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\right)\right]\delta\theta_1 dt$$

$$-\int_{t_1}^{t_2}\left[ml_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) - ml_1l_2\dot{\theta}_1^2\,\mathbf{sin}\,\theta_2 + \frac{d}{dt}\left(ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\right) - ml_1l_2\dot{\theta}_1\dot{\theta}_2\,\mathbf{sin}\,\theta_2\right]\delta\theta_2 dt$$

Finally

$$\delta\int_{t_1}^{t_2}\left(T-V\right)dt=$$

$$-\int_{t_1}^{t_2}\left[ml_1^2\ddot{\theta}_1+ml_1^2\ddot{\theta}_1+ml_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)+\frac{d}{dt}\left(2ml_1l_2\dot{\theta}_1\,\mathbf{cos}\,\theta_2\right)+\frac{d}{dt}\left(ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\right)+2mgl_1\,\mathbf{sin}\,\theta_1+mgl_2\,\mathbf{sin}\left(\theta_1+\theta_2\right)\right]\delta\theta_1 dt$$

$$-\int_{t_1}^{t_2}\left[ml_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)-ml_1l_2\dot{\theta}_1^2\,\mathbf{sin}\,\theta_2+\frac{d}{dt}\left(ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\right)-ml_1l_2\dot{\theta}_1\dot{\theta}_2\,\mathbf{sin}\,\theta_2+mgl_2\,\mathbf{sin}\left(\theta_1+\theta_2\right)\right]\delta\theta_2 dt$$

First EoM

$$2ml_1^2\ddot{\theta}_1+ml_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)+\frac{d}{dt}\left(2ml_1l_2\dot{\theta}_1\,\mathbf{cos}\,\theta_2\right)+\frac{d}{dt}\left(ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\right)+2mgl_1\,\mathbf{sin}\,\theta_1+mgl_2\,\mathbf{sin}\left(\theta_1+\theta_2\right)=0$$

Second EoM

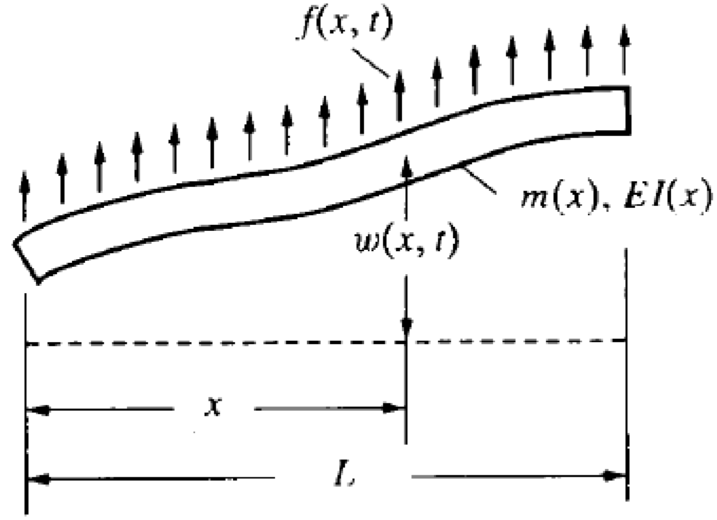
$$ml_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)-ml_1l_2\dot{\theta}_1^2\,\mathbf{sin}\,\theta_2+\frac{d}{dt}\left(ml_1l_2\dot{\theta}_2\,\mathbf{cos}\,\theta_2\right)-ml_1l_2\dot{\theta}_1\dot{\theta}_2\,\mathbf{sin}\,\theta_2+mgl_2\,\mathbf{sin}\left(\theta_1+\theta_2\right)=0$$

Important property of the variational operator

$$\delta\left[\frac{\partial w(x,t)}{\partial t}\right]=\frac{\partial}{\partial t}\left[\delta w(x,t)\right]=\delta\dot{w}\ ,\quad \delta\left[\frac{\partial w(x,t)}{\partial x}\right]=\frac{\partial}{\partial x}\left[\delta w(x,t)\right]=\delta w'$$

Beam in bending

$$\delta\int_{t_1}^{t_2}\left(T-V+W^{ncons}\right)dt=0\ ,$$



$$T=\frac{1}{2}\int_0^L m(x)\left[\frac{\partial w(x,t)}{\partial t}\right]^2 dx$$

$$V=\frac{1}{2}\int_0^L EI(x)\left[\frac{\partial^2 w(x,t)}{\partial x^2}\right]^2 dx$$

$$W^{ncons}=\int_0^L f(x,t)\,w(x,t)\,dx$$

$$\delta\int_{t_1}^{t_2}\int_0^L\left(\frac{1}{2}m(x)\left[\frac{\partial w(x,t)}{\partial t}\right]^2-\frac{1}{2}EI(x)\left[\frac{\partial^2 w(x,t)}{\partial x^2}\right]^2+f(x,t)\,w(x,t)\right)dx\,dt$$

$$\int_{t_1}^{t_2}\int_0^L \left(\underbrace{m(x)\left[\frac{\partial w(x,t)}{\partial t}\right]\delta\left[\frac{\partial w(x,t)}{\partial t}\right]}_{\text{A}} \underbrace{- E(x)I(x)\left[\frac{\partial^2 w(x,t)}{\partial x^2}\right]\delta\left[\frac{\partial w^2(x,t)}{\partial x^2}\right]}_{\text{B}} \underbrace{+ f(x,t)\,\delta w(x,t)}_{\text{C}}\right) dx\,dt$$

A:

$$\int_{t_1}^{t_2}\int_0^L \left(m(x)\,\dot{w}\,\delta\dot{w}\right)dx\,dt = \int_0^L \underbrace{m(x)\,\dot{w}\,\delta w\Big|_{t_1}^{t_2}}_{=0} dx - \int_{t_1}^{t_2}\int_0^L \left(m(x)\,\ddot{w}\,\delta w\right)dx\,dt = -\int_{t_1}^{t_2}\int_0^L \left(m(x)\,\ddot{w}\,\delta w\right)dx\,dt$$

B:

$$\int_{t_1}^{t_2}\int_0^L \left(E(x)I(x)w''\delta w''\right)dx\,dt = \int_{t_1}^{t_2}\left(E(x)I(x)w''\right)\delta w'\Big|_0^L dt - \int_{t_1}^{t_2}\int_0^L \left(\left(E(x)I(x)w''\right)'\delta w'\right)dx\,dt$$

$$= \int_{t_1}^{t_2}\left(E(x)I(x)w''\right)\delta w'\Big|_0^L dt - \int_{t_1}^{t_2}\left(E(x)I(x)w''\right)'\delta w\Big|_0^L dt + \int_{t_1}^{t_2}\int_0^L \left(\left(E(x)I(x)w''\right)''\delta w\right)dx\,dt$$

C:

$$\int_{t_1}^{t_2}\int_0^L f\,\delta w\,dx\,dt$$

Combining previous expressions

$$\int_{t_1}^{t_2}\left(\delta T-\delta V+\delta W^{ncons}\right)dt$$

$$=\int_{t_1}^{t_2}\int_{0}^{L}\left(m(x)\ddot{w}+\left(\left(E(x)I(x)w''\right)''\right)-f\right)\delta w\,dx\,dt+\int_{t_1}^{t_2}\left[\left(E(x)I(x)w''\right)\delta w'\Big|_0^L-\left(E(x)I(x)w''\right)'\delta w\Big|_0^L\right]dt$$

EOM

$$m(x)\ddot{w}+\left(\left(E(x)I(x)w''\right)''\right)-f=0$$

Boundary conditions

$$Either\quad \underbrace{E(x)I(x)w''}_{bending\ moment}=0\quad or\quad \underbrace{w'}_{slope}=0\quad at\quad x=0,L$$

$$Either\quad \underbrace{\left(E(x)I(x)w''\right)'}_{shearing\ \ force}=0\quad or\quad \underbrace{w}_{deflection}=0\quad at\quad x=0,L$$