Asymmetric All-Pay Contests with Spillovers

Maria Betto* Matthew Wildrick Thomas†

April 17, 2021

Abstract

When opposing parties compete for a prize, the sunk effort players exert during the conflict can affect the value of the winner's reward. These *spillovers* can have substantial influence on the equilibrium behavior of participants in applications such as lobbying, warfare, labor tournaments, marketing, and R&D races. To understand this influence, we study a general class of asymmetric, two-player all-pay contests where we allow for spillovers in each player's reward. The link between participants' efforts and rewards yields novel effects. In particular, players with higher costs and lower values than their opponent sometimes extract larger payoffs.

Keywords: all-pay, contests, auctions, spillovers, war of attrition.
JEL Codes: C65, C72, D44, D62, D74

*Maria Betto: Northwestern University, Email: maria.betto@u.northwestern.edu
†Matthew Wildrick Thomas: Northwestern University, Email: mthomas@u.northwestern.edu

1 Introduction

All-pay contests model strategic interaction among players who must expend some effort in order to win a prize. They have been successfully applied in diverse settings such as labor (Rosen 1986), R&D races (Che and I. L. Gale 1998; Dasgupta 1986), and litigation (Baye et al. 2005). For tractability, the recent literature mostly assumes that players' actions affect their opponent's probability of winning, but not the value of the prize. Yet, in many settings, such *spillover effects* arise naturally.

For example, consider an all-pay version of a standard labor tournament. Division managers apply effort to some production technology in order to win a promotion that is awarded to the most productive division. If this promotion is for a partnership or involves stock options, the prize will be increasing in the efforts of all players. The ways in which this affects the strategic play between participants has not yet been addressed.

As another application, consider the setting in Che and I. L. Gale 1998, where two lobbyists compete in an all-pay auction to win an incumbent politician's favor through campaign contributions. If the politician were instead a candidate running for office, then she would only be able to provide the reward if successfully elected. In this case, it is natural to assume that total campaign contributions increase the candidate's chances of prevailing. Therefore, each lobbyist's contributions increase her opponent's value for winning the politician's political favor. This changes incentives in interesting ways: is it better to curb one's own contributions to make their opponent lose interest? Or is it preferable to ramp up the competition? These questions have been largely left unanswered.

This paper constructs and establishes the uniqueness of equilibrium strategies and payoffs in general two-player contests with spillovers. We consider games with (i) complete information, (ii) deterministic prizes, (iii) at least partially sunk investment costs, and (iv) a general dependence of each participant's value for the prize on both players' actions. This family of games includes all-pay contests (Siegel 2009, 2010), while also allowing for general spillovers to affect the winner's payoff.[1]

The addition of spillovers can have a significant impact on equilibrium behavior. First, players with strictly higher costs can have higher payoffs than those with lower

[1] We also allow the payoffs of the losers to depend linearly on the actions of other players. See Section 7.

costs, even if their value functions for the prize are identical. In fact, in some settings, players could increase their payoffs if they were allowed to commit to a schedule of costly handicaps (See Section 4). Thus, trying to favor an "underdog" participant in a contest by means of reducing their costs may have the opposite of the desired effect, and in fact decrease their welfare in equilibrium. This is also important in settings in which players can commit to increasing their costs (e.g. by selecting an inefficient technology) as they may choose to do so.

Another contribution of this paper is to provide a novel way of constructing equilibrium strategy profiles. The equilibrium strategy distributions of asymmetric all-pay contests have two distinct parts: the densities and a mass-point at zero. In the the literature on all-pay contests without spillovers, starting with Baye et al. 1996, expected payoffs are obtained independently of the equilibrium distribution. This independence is exploited to derive the probability mass at zero for the weaker player from the payoffs, which is then used to compute the densities. In the presence of spillovers, however, a player's payoffs cannot be derived without the equilibrium strategy of their opponent, and as such the same process cannot be followed. To overcome this issue we introduce an algorithm that works in exactly the opposite order: first, it solves for the density independently of the mass-point, and then uses this density to find the probability mass at zero.

Our method capitalizes on the theory of Volterra Integral Equations (VIEs), which are integral equations with a unique fixed-point that can be obtained via iteration. To the best of the authors' knowledge, these techniques have not previously been applied to the determination of equilibrium mixed-strategy profiles.[2]

The game we study is general enough to encompass many different applications in which externalities matter. In particular, investment wars, contests with winner's regret, and militaristic conflicts all fit our framework well, since spillovers are key in each of these settings. Our model also subsumes a natural extension to the war of attrition which, unlike the classical model, yields a unique equilibrium on a bounded support. We are also able to use the same framework to describe wars of attrition where rational agents face uncompromising (never-yielding) types with positive probability, as in Abreu and Gul 2000 and Kambe 2019. Our approach identifies why

[2]Few other works in Economics use VIE methods in general. We note McAfee, McMillan, et al. 1989 and McAfee and Reny 1992 as some early examples. More recently, Gomes and Sweeney 2014 also used VIEs, to compute the unique efficient equilibrium bidding functions in generalized second-price auctions.

these games admit unique equilibria when the regular war of attrition does not: the addition of an uncompromising type introduces an unavoidable cost that depends on a player's own score and, as it turns out, this single characteristic is sufficient for a unique equilibrium.

The paper is organized as follows. We introduce the model, the equilibrium concept and the assumptions in Section 2. We then construct the equilibrium and prove its uniqueness in Section 3. Section 4 presents sufficient conditions under which a player has a positive expected payoff. This includes an example where a player with higher costs and lower values receives a positive expected payoff while her opponent receives zero. In Section 5, we introduce a general perturbation of the war of attrition that ensures the equilibrium is unique. This perturbation admits the war of attrition with the possibility of an uncompromising type as a special case. Section 6 explores applications where equilibrium analysis can be simplified significantly. In Section 7, we extend our results in two ways: (1) contests which also have linearly separable spillovers on the losers' payoff and (2) contests which have more than two players, but have ranked costs. We show that when costs are ranked, only two players participate in equilibrium. In Section 8, we review related literature.

2 Model

We focus, for now, on contests with two participants. Extensions with more than two players are considered in section 7.2.

An asymmetric contest with spillovers is a family $\{I, \{\tilde{S}_i\}_{i \in I}, \{u_i\}_{i \in I}\}$, where

1. $I := \{1, 2\}$ is the set of players.

2. For each $i \in I$, $\tilde{S}_i := [0, \infty)$ is Player i's action space,[3] i.e., her set of available scores s_i. We let s_{-i} denote the action of Player $j \neq i$.

3. For each $i \in I$, $u_i : \tilde{S} \to \mathbb{R}$ is Player i's payoff, where $\tilde{S} := \prod_{i \in I} \tilde{S}_i$.

   Let $s := (s_i; s_{-i})$ denote an arbitrary element of $\tilde{S}$. Then, for each $(s_i; s_{-i})$, we further define

$$u_i(s_i; s_{-i}) := p_i(s_i; s_{-i}) v_i(s_i; s_{-i}) - c_i(s_i)$$

[3]We use a tilde because a later assumption will allow us to replace the action set with a bounded interval.

where (i) $p_i(s_i; s_{-i})$ denotes the probability that i wins the prize given the score profile $(s_i; s_{-i})$, with $p_i(s_i; s_{-i}) = 1 - p_{-i}(s_{-i}; s_i)$ and

$$\begin{aligned} p_i(s_i; s_{-i}) &= 1 && \text{if } s_i > s_{-i}, \\ p_i(s_i; s_{-i}) &= \lambda \in [0,1] && \text{if } s_i = s_{-i}, \\ p_i(s_i; s_{-i}) &= 0 && \text{if } s_i < s_{-i}; \end{aligned}$$

(ii) $v_i : \tilde{S} \to \mathbb{R}_+$ maps each score profile $(s_i; s_{-i})$ to Player i's satisfaction $v_i(s_i; s_{-i})$ from winning the prize, and (iii) $c_i : \tilde{S}_i \to \mathbb{R}_+$ outputs Player i's private cost $c_i(s_i)$ given her submitted score s_i.

Definition 1 (Two-player contest with spillovers)**.** A two-player contest is said to have *spillovers* if, for some $i \in I$ and $s_i \in \tilde{S}_i$, there exists $s_{-i}, \hat{s}_{-i} \in \prod_{j \neq i} \tilde{S}_j$ such that

$$v_i(s_i, s_{-i}) \neq v_i(s_i, \hat{s}_{-i})$$

i.e., the prize's value for at least one player and an action of that player is not constant in their opponent's action.

Accommodating spillovers is the distinguishing feature of our analysis. As is standard, we are interested in characterizing the Nash equilibrium of these general contests.

Definition 2 (Best-responses)**.** Consider a two-player contest $\{I, \{\tilde{S}_i\}_{i \in I}, \{u_i\}_{i \in I}\}$. For each $i \in I$, let $\Delta\tilde{S}_i$ denote the set of probability distributions on $\tilde{S}_i$ and let $\Delta\tilde{S} := \prod_{i \in I} \Delta\tilde{S}_i$. Player i's best response set $b_i(G_{-i})$ to $G_{-i} \in \Delta\tilde{S}_{-i}$ is given by

$$b_i(G_{-i}) := \arg\max_{s \in \tilde{S}_i} \int_{\tilde{S}_{-i}} u_i(s; s_{-i}) dG_{-i}(s_{-i})$$

Definition 3 (Nash equilibrium)**.** Consider the two-player contest $\{I, \{\tilde{S}_i\}_{i \in I}, \{u_i\}_{i \in I}\}$. A Nash equilibrium of this game is a profile $\mathbf{G}^\star := (G_i^\star)_{i \in I} \in \Pi_{i \in I} \Delta\tilde{S}_i$ where, for each $i \in I$, supp $(G_i^\star) \subseteq b_i(G_{-i}^\star)$.

2.1 Assumptions

The following assumptions are imposed throughout whenever a two-player contest is invoked.

Assumption 1 (A1, Smoothness)**.** *The function $v_i(s_i; y)$ is continuously differentiable in s_i and continuous in y for all $i \in I$, $s_i \in \tilde{S}_i$, and $y \in \tilde{S}_{-i}$ with $s_i \geq y$. The function $c_i(s_i)$ is continuously differentiable in s_i for all $i \in I$, $s_i \in \tilde{S}_i$.*

Assumption 2 (A2, Monotonicity)**.** *For all $i \in I$ and $s_i \in \tilde{S}_i$, $c_i'(s_i) > 0$ and*

$$v_i'(s_i; y) < c_i'(s_i)$$

for almost all y, where $v_i'(s; y) := \frac{\partial v_i(s_i;y)}{\partial s_i}$.

Assumption 3 (A3, Interiority)**.** *For all $i \in I$,*

$$v_i(0,0) > c_i(0) = 0 \quad \text{and} \quad \lim_{s_i \to \infty} \sup_{y \in \tilde{S}_{-i}} v_i(s_i; y) < \lim_{s_i \to \infty} c(s_i).$$

Versions of assumptions A1, A2, and A3 are adopted by all papers in the all-pay contests literature. A2 formalizes the sense in which these contests are all-pay and A3 ensures that bids are positive and bounded.

Note that, for each $i \in I$, there exist $T_i \in \tilde{S}_i$ such that Player i will never choose a score $s \geq T_i$. Thus, we can restrict the action space to $S_i := [0, T_i]$.

Assumption 4 (A4, Discontinuous at ties)**.** *For all $i \in I$ and $s \in S_i \cap S_{-i}$,*

$$v_i(s; s) > 0.$$

Assumption A4 is a novel, yet natural assumption. It states that agents would prefer to win a tie than lose one. It is satisfied if the prize is always valuable (i.e. winning is better than losing). Note that this assumption is equivalent to assuming a discontinuity in payoffs near the point of a tie[4] – a property of all-pay auctions. In contrast, Tullock contests (Buchanan et al. 1980) and Lazear and Rosen contests (Lazear and Rosen 1981) are continuous at ties.

3 Characterization of equilibrium

By standard arguments,[5] any pair of equilibrium strategies will be mixed with support on some interval $[0, \bar{s}]$ and at most one player will have an atom at zero. Players must

[4]By A1 and A3, $v_i(s; s) \neq 0$ implies A4.
[5]These arguments are in the appendix.

therefore be indifferent between all points on their interval support:

$$\bar{u}_i(G_{-i}) := \int_0^s v_i(s;y)dG_{-i}(y) - c_i(s) \qquad \text{for all } s \in [0, \bar{s}]. \tag{1}$$

Any pair of distributions (G_1, G_2) that satisfy (1) is an equilibrium. This paper's main contribution to the literature is an application of Volterra Integral Equations to characterize the solution to this system of equations and show that it is unique.

Theorem 1. *Every two-player all-pay contest has a* ***unique Nash equilibrium*** $(G_i^\star)_{i\in I} \in \Delta S$ *in mixed strategies. Furthermore,*

$$G_i^\star(s) = \int_0^s \tilde{g}_i(y)dy + \int_{\bar{s}}^{\bar{s}_i} \tilde{g}_i(y)dy, \tag{2}$$

where $\tilde{g}_i(s)$ *solves*

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s;s)} - \int_0^s \frac{v'_{-i}(s;y)}{v_{-i}(s;s)}\tilde{g}_i(y)dy, \tag{3}$$

$\bar{s}_i$ *solves* $\int_0^{\bar{s}_i} \tilde{g}_i(y)dy = 1$ *and* $\bar{s} = \min_{i\in I} \bar{s}_i$.

We outline the proof here with an emphasis on the general methodology. We show in the appendix that in any equilibrium, players choose strictly increasing, continuous mixed strategies with common support on some interval $[0, \bar{s}]$, and that at most one participant can have a mass-point at zero. From this, we conclude that any equilibrium must satisfy (3) on $[0, \bar{s}]$ for some $\bar{s}$.

The key step is recognizing that we can apply results about Volterra Integral Equations (VIE) to show that (3) has a unique solution. The relevant result is summarized in the following Lemma. For a proof, see e.g. Brunner 2017.

Lemma 1 (Volterra 1896)**.** *Let $K(s;y)$ and $f(s)$ be continuous functions. Then, the integral equation*

$$\tilde{g}(s) = f(s) - \int_0^s K(s;y)\tilde{g}(y)dy \qquad \textit{for all } s \in [0, \bar{s}] \tag{4}$$

has a unique solution almost everywhere. Moreover, (4) defines a contraction mapping, and so the solution can be found by iteration. This iteration reduces to:

$$\tilde{g}(s) = f(s) + \int_0^s R(s;y)f(y)dy,$$

where $R(s;y)$[6] *is a unique function defined by*

$$R(s;y) = \sum_{m=1}^{\infty} K_m(s;y) = \sum_{m=1}^{\infty} \int_y^s K_{m-1}(s;z)K(z;y)dz.$$

To construct the unique equilibrium, we must show that the unique solutions $(\tilde{g}_1, \tilde{g}_2)$ can both be densities, i.e., for each i there is an interval $[0, \bar{s}_i]$ where $\tilde{g}_i$ is non-negative and integrates to one.

Lemma 2. *Assume a two-player contest where* $(\tilde{g}_i)_{i \in I}$ *satisfy the indifference condition in* (3)*. Then, for each* $i \in I$*, there exists* $\bar{s}_i \in S_i$ *such that*

$$\int_0^{\bar{s}_i} \tilde{g}_i(y)dy = \tilde{G}_i(\bar{s}_i) = 1, \tag{5}$$

and $\tilde{g}_i(s)$ *is positive for* $s \leq \bar{s}_i$*.*

Lemma 2 is proven in the appendix. With the candidate densities found, all that is left is to find the mass-point at zero. The next key insight is that the mass-point can be obtained via the following algorithm:

1. Find each $\tilde{g}_i(s)$.

2. Integrate each $\tilde{g}_i(s)$ to find $\bar{s}_i$ given by equation 5.

3. Take $\bar{s} = \min_i \bar{s}_i$ and give each player an atom of size

   $$1 - \tilde{G}_i(\bar{s}),$$

   which is positive for at most one player.

The algorithm is illustrated by Figures 1, 2, and 3. Since the cumulative distribution functions are useful, we sometimes use the alternate expression in Corollary 1.1.

Corollary 1.1. *Consider a two-player all-pay contest where* $v_i(s;y)$ *is continuously differentiable in both arguments for all* $i \in I$*.*[7] *Then, we can alternatively express the*

[6] In the Volterra Integral Equation literature, this object is known as the *resolvent kernel*.

[7] A1 only imposes that $v(s;y)$ is continuous in both arguments and continuously differentiable in the first argument.

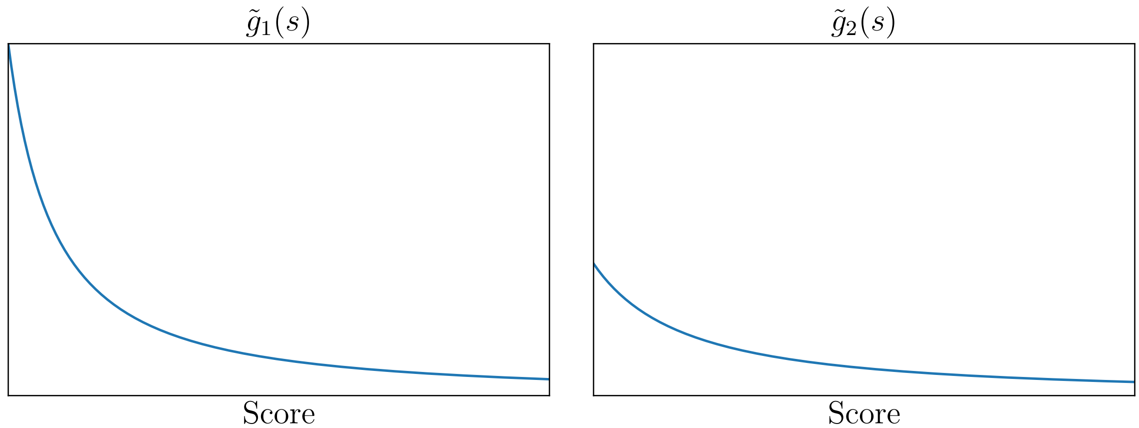

Figure 1: Lemma 1 says A1 and A4 imply there exist functions, $\tilde{g}_i$, which satisfy the indifference condition (3). The displayed densities are from the contest with $v_1(s_1; s_2) = 2 + s_1 + 2s_2$, $v_2(s_2; s_1) = 1 + s_2 + 2s_1$, $c_1(s_i) = 3s_1$, and $c_2(s_2) = 4s_2$.

unique equilibrium as

$$G_i(s) = \left[\tilde{G}_i(\bar{s}_i) - \tilde{G}_i(\bar{s})\right] + \tilde{G}_i(s),$$

where

$$\tilde{G}_i(s) = \frac{c_{-i}(s)}{v_{-i}(s; s)} + \int_0^s \frac{\partial v_{-i}(s; y)}{\partial y} \frac{\tilde{G}_i(y)}{v_{-i}(s; s)} dy. \tag{6}$$

4 Payoffs

Since payoffs are constant on the interval $[0, \bar{s}]$, each player i receives an expected payoff of $v_i(0; 0)G_{-i}(0) \geq 0$. Only one player can have an atom (at zero), so there can be at most one player – their opponent – with a positive payoff.[8]

In contests without spillovers, it is easy to identify the player with a positive payoff when normalized costs (i.e. the cost-value ratio) are ranked.

Lemma 3. *Consider a two-player contest without spillovers. If*

$$\frac{c_i(s)}{v_i(s)} < \frac{c_{-i}(s)}{v_{-i}(s)}$$

for all $s > 0$, then player i has a positive payoff.

[8]In a symmetric contest, note that both players receive an expected payoff of zero.

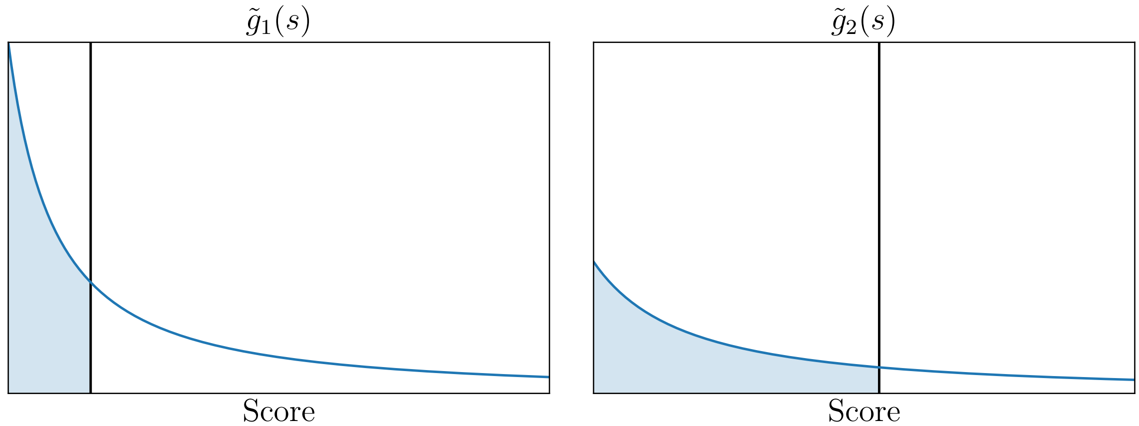

Figure 2: Lemma 2 says the functions, $\tilde{g}_i$, are positive and each integrates to one on some interval $[0, \bar{s}_i]$. As above, the two functions need not integrate to one on the same interval.

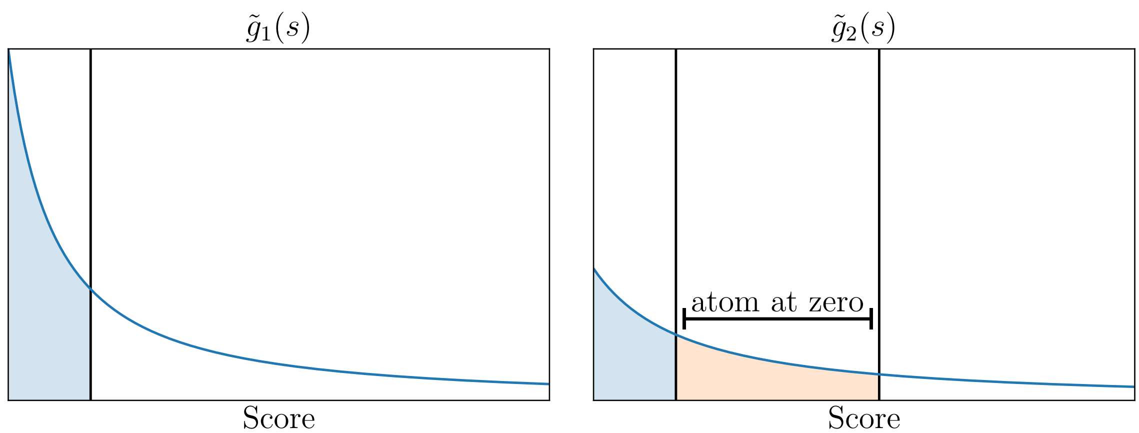

Figure 3: We construct an equilibrium on the smaller support by moving the excess density to an atom at zero.

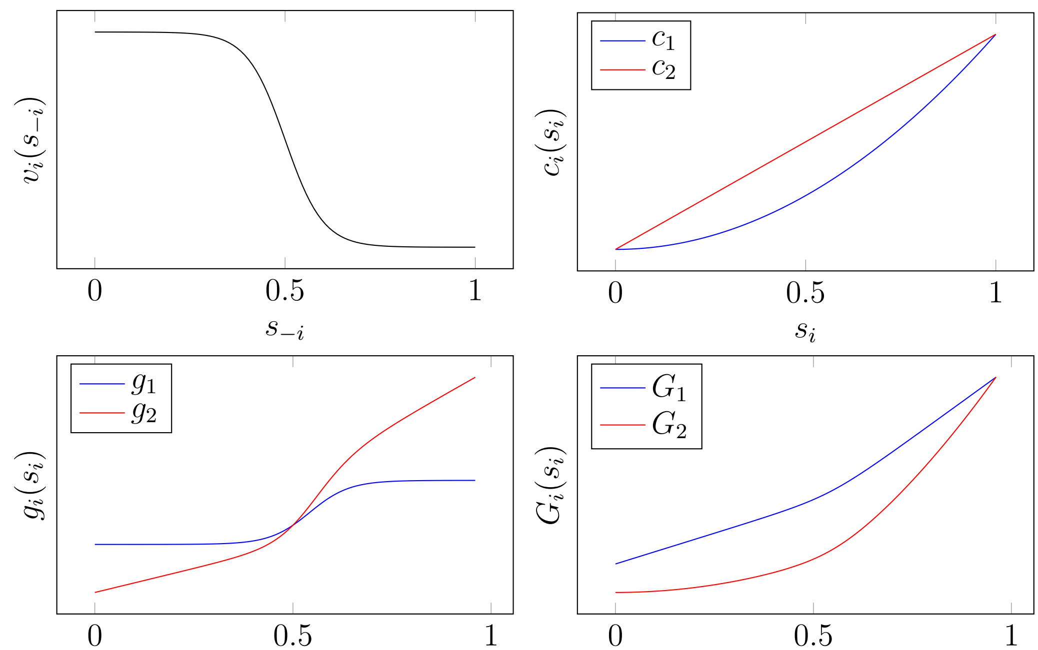

Figure 4: Player 1 has lower costs and both value the prize in the same way, yet Player 2 receives a positive expected payoff and Player 1 receives zero. This is possible because because Player 2 has a lower marginal cost for scores above $\frac{1}{2}$ (upper right) and, because of spillovers, these scores massively devalue the prize for Player 1 (upper left).

The above results from the observation that, in a two-player all-pay contest without spillovers, player i's opponent has an atom if, and only if,

$$\bar{s}_i < \bar{s}_{-i}, \qquad \text{where} \qquad \tilde{G}_i(\bar{s}_i) = \frac{c_{-i}(\bar{s}_i)}{v_{-i}(\bar{s}_i)} = 1.$$

However, a similar result does not apply to a contest with spillovers, as is illustrated by Example 1. In fact, this example shows that a player with strictly higher costs can indeed receive a positive payoff even if both players have the same value function v for the prize.

Example 1 (Higher cost player has positive payoffs)**.** Consider a two-player contest with spillovers. Let $v(s; y) := v_1(s; y) = v_2(s; y)$ be given by:

$$v(s; y) = \frac{3}{4} + \frac{\exp(10 - 20y)}{1 + \exp(10 - 20y)}.$$

Let $c_1(s) = s^2$ and $c_2(s) = s$. Then,

$$\tilde{g}_1(s) = \frac{1}{\frac{3}{4} + \frac{\exp(10-20s)}{1+\exp(10-20s)}} \qquad \tilde{g}_2(s) = \frac{2s}{\frac{3}{4} + \frac{\exp(10-20s)}{1+\exp(10-20s)}}.$$

Integrating these shows that $\int_0^1 \tilde{g}_1(s)ds < 1$ and $\int_0^1 \tilde{g}_2(s)ds > 1$. Thus, $\bar{s} < 1$ and player 1 has an atom, even though their costs are smaller than player 2's in the equilibrium strategy's support. $\triangle$

Example 1 is possible because the marginal costs are not ranked. While Player 1 has lower costs in absolute terms, Player 2 has a lower marginal cost for all scores above $\frac{1}{2}$. This causes Player 2 to place comparatively more density on these bids. As can be seen in Figure 4, spillovers make the prize sharply less valuable when the opponent bids above $\frac{1}{2}$. So, these higher bids from player 2 damage player 1's valuation enough to reduce her participation.

Example 1 highlights a potential problem when giving one side an advantage in a contest. This can be critical for several legal structures which are designed to asymmetrically reduce litigation costs for one of two parties. The example also implies that it's possible to have a contest where one or more players would prefer to ex-ante increase their own costs.[9]

Theorem 2. *Consider a two-player contest where $v_i(s; y)$ is continuously differentiable in s and y for all $i \in I$. Suppose that the following two conditions hold:*

$$\frac{c_1(s)}{v_1(s;s)} < \frac{c_2(s)}{v_2(s;s)} \tag{7}$$

$$\frac{1}{v_1(s;s)} \left| \frac{\partial v_1(s;y)}{\partial y} \right| \leq \frac{1}{v_2(s;s)} \frac{\partial v_2(s;y)}{\partial y} \tag{8}$$

for all $s \in (0, \bar{s}]$ and $y \in [0, s]$. Then, player 1 has a positive payoff bounded below by

$$u_1 \geq v_1(0;0) \left[\frac{c_2(\bar{s})}{v_2(\bar{s};\bar{s})} - \frac{c_1(\bar{s})}{v_1(\bar{s};\bar{s})} \right] > 0.$$

Theorem 2 gives an analogue of Lemma 3 for some contests with spillovers. The proof is in the Appendix. It relies on the main comparison theorem for VIEs found

[9] Suppose both players are as in Example 1 except $c_1(s) = c_2(s) = s^2$. Then, the game is symmetric. So, both players have a payoff of zero. If player i increased her cost to $c_i(s) = s$, then she would receive a positive payoff, as in the example.

in Beesack 1969.

Conditions (7) and (8) have clear interpretations: (7) imposes that Player 2 has higher normalized costs on the margin,[10] whereas (8) ensures Player 1's valuation is less dependent on her opponent's action than Player 2's is. Putting it differently, (8) states that any spillover effects must be more significant for Player 2.[11]

Example 1 violates (8) because it featured a value function $v(s;y) := v_1(s;y) = v_2(s;y)$ with $\frac{\partial v(s;y)}{\partial y} < 0$. This suggests the following Corollary.

Corollary 2.1. *Assume a two-player contest. Suppose the two players have the same value $v(s;y) := v_1(s;y) = v_2(s;y)$, $c_2(s) > c_1(s)$, and*

$$\frac{\partial v(s;y)}{\partial y} \geq 0.$$

Then, Player 1 has a positive payoff.

Whenever players' scores adversely affect their opponent's value, determining which player has a competitive edge becomes more subtle.

Proposition 1 (Ranked marginal costs)**.** *Consider a two-player contest. Suppose the players have the same value $v(s;y) := v_1(s;y) = v_2(s;y)$, $c_2'(s) > c_1'(s)$, and at least one of the following applies for all $s \in [0, \bar{s}]$ and $y \in [0, s]$:*

1. *Costs are scaled: $c_2(s) = \lambda c_1(s)$ for some $\lambda \in (0, 1)$.*

2. *The effect of a player's own score on the value is bounded above by the distance between the marginal costs: $v'(s;y) < c_2'(s) - c_1'(s)$.*

3. *The effect of one's opponent on the marginal values is bounded above by the distance between the marginal costs:*

$$\max_{y \leq s} v'(s;y) - \min_{y \leq s} v'(s;y) < c_2'(s) - c_1'(s).$$

[10] Recall that in a contest without spillovers, (7) is sufficient to attain the same result (Lemma 3). In contests with spillovers, (7) is not sufficient.

[11] Since (7) does not differentiate between spillovers and the effects of the player's own score. A player with $v_i(s;y) = 1 + s$ has a higher valuation than someone with $v_i(s;y) = 1 + y$ for all $s > y$, i.e., whenever they are the winner. Yet (7) treats these two as the same. Condition (8) corrects for this.

4. *The common value function is supermodular in the sense that*

$$\frac{\partial v(s; y)}{\partial s \partial y} \geq 0$$

almost everywhere.

Then, Player 1 has a positive payoff.

Proposition 1, which is proven in the Appendix, is useful for contests where the opponent's actions are detrimental to the prize's value. It also pins down the circumstances under which the higher cost players can win: either the marginal costs are not ranked (as in Example 1) or all of conditions 1-4 of Proposition 1 fail to hold.

5 War of attrition with costly preparation

To illustrate the wide applicability of our framework, we first explore extensions to the classical war of attrition (henceforth WoA). The WoA first appeared in theoretical biology to explain how individual selection works in favor of animal species that outlast others (Smith 1974). In Economics, WoAs have since been popularized in the study of bargaining (Abreu and Gul 2000; Kambe 2019), filibusters (Bulow and Klemperer 1999), delays in the implementation of stabilizing macroeconomic policies (Alesina and Drazen 1991) exit, competition and price wars (Fudenberg and Tirole 1986; Ghemawat and Nalebuff 1985), boycotts and activism (Egorov and Harstad 2017), among others.

The canonical war of attrition is a game between two players $i = 1, 2$. Each picks a score[12] in $[0, \infty)$ and the player i to select the largest score s_i wins an amount that depends on the loser's choice s_{-i}.[13] A player's payoff function is thus given by:

$$u(s_i; s_{-i}) = \begin{cases} f(s_{-i}) & \text{if } s_i > s_{-i} \\ \ell(s_i) & \text{if } s_i < s_{-i} \\ \alpha f(s_{-i}) + (1-\alpha)\ell(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

[12] In the classic war of attrition, in general each player will chose an exit time.

[13] Again in the classic war of attrition, this value is typically the fixed value of a prize, time-discounted for however much the losing opponent lasted in the game.

where f, ℓ are strictly decreasing, continuously differentiable functions such that $f(s) > \ell(s)$, $\lim_{s\to\infty} \ell_i(s) = -\infty$, $\ell(0) = 0$, and $\alpha \in (0,1)$.

The typical WoA is known to admit multiple equilibria and therefore cannot satisfy the assumptions in Section 2.1.[14] However, known perturbations of the WoA do induce uniqueness of equilibria, and our framework is able to provide further insights into how the multiplicity issue is resolved.

Abreu and Gul 2000 and Kambe 2019, for example, extend the WoA to let a rational player's opponent be of an uncompromising type with positive probability, where "uncompromising" describes someone who bids (or exits at) infinity. Let z_i denote the (known) probability that player i is of an uncompromising type.[15] Against such an opponent, a rational or compromising player loses with certainty; thus, it is as if their value for the prize was multiplied by a constant $(1 - z_{-i})$.

As it turns out, this game is a two-player contest with spillovers where $v_i(s_i; s_{-i}) := (1 - z_{-i})\,(f_i(s_{-i}) - \ell_i(s_i))$ and $c_i(s_i) := -\ell_i(s_i)$. This satisfies assumptions A1-4 if and only if $z_i \in (0,1)$ for all i. Therefore, the equilibrium is unique and there exists some $\bar{s}$ such that no compromising player bids above $\bar{s}$.

We can further generalize this result and extend this equilibrium characterization to games with more general payoffs. Consider another modified WoA where, instead of scaling down the prize, we assume that the winner's outcome is decreasing in a their own score – even if this dependence is minimal:

$$u(s_i; s_{-i}) = \begin{cases} f(s_{-i}) - \varepsilon(s_i) & \text{if } s_i > s_{-i} \\ \ell(s_i) - \varepsilon(s_i) & \text{if } s_i < s_{-i} \\ \alpha(f(s_{-i}) - \varepsilon(s_i)) + (1-\alpha)\ell(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

for any strictly increasing continuously differentiable function ε with $\varepsilon(0) = 0$ and $\lim_{s\to\infty} \varepsilon(s) > f_i(0)$ for both i.

We denominate this variant a *WoA with costly preparation*, as there is some small preparation cost $\varepsilon(s)$ incurred to set the maximum amount of time s one wishes to participate for. For example, a company engaged in a price war might have to build up inventory in advance or secure a costly line of credit.[16]

[14] In particular, it violates assumptions A2 and A3.

[15] Note whether or not a player i's opponent of the uncompromising type is the only unknown information to i.

[16] Alternatively, We could also call this a WoA with imperfect monitoring. The costly preparation

A WoA with costly preparation is a two-player all-pay contest with spillovers where

$$v_i(s_i; s_{-i}) := f_i(s_{-i}) - \ell_i(s_i) \text{ and}$$

$$c_i(s_i) := \varepsilon(s_i) - \ell_i(s_i),$$

which again satisfy assumptions A1-4. Therefore, this game also has a unique equilibrium, and there exists some $\bar{s}$ such that no player bids above $\bar{s}$. Theorem 1 then further allows us to fully characterize the equilibrium. Note that this is equivalent to the WoA with an uncompromising type when $\varepsilon_i(s) := -\frac{z_{-i}}{1-z_{-i}}\ell_i(s)$.

This result sheds light on the uniqueness of the equilibrium in the WoA with an uncompromising type. When we add the possibility of an uncompromising type, we introduce an unavoidable cost that depends on the player's own score. What the WoA with costly preparation says is that this single characteristic is sufficient for a unique equilibrium.

As the preparation costs become small (with $\varepsilon'(s) \to 0$ uniformly for all s), the unique equilibrium of a WoA with costly preparation approaches the mixed-strategy equilibrium of the classic WoA.[17]

One of the counter-intuitive results from a standard WoA is the fact that the player with the lowest value wins most of the time. Example 2 shows why the higher-value player can receive a positive payoff in our perturbation.[18]

Example 2 (Simple WoA)**.** Consider a WoA where $f_1(t_2) = 1 - t_2$, $f_2(t_1) = 2 - t_1$, and $\ell_i(t_i) = -t_i$ for each $i \in \{1, 2\}$. The equilibrium of this game is:

$$G_1(s) = 1 - \exp\left(-\frac{s}{2}\right) \qquad G_2(s) = 1 - \exp\left(-\frac{s}{1}\right).$$

Note that Player 1's strategy first order stochastically dominates Player 2's. In fact, Player 1 exits at half the rate of Player 2, and wins $\frac{2}{3}$ of the time – despite having the lower valuation.

In the corresponding WoA with costly preparation where $\varepsilon(s) := \delta s$, with $\delta > 0$,

is equivalent to a small probability of not detecting that the war of attrition is over. In this story, you determine an exit point in advance and exit early only if you are aware that your opponent has exited.

[17]Refer to the Appendix for a direct proof.

[18]This is not always the case in the WoA with costly preparation. In fact, Example 1 is a WoA with costly preparation where v_i is constant.

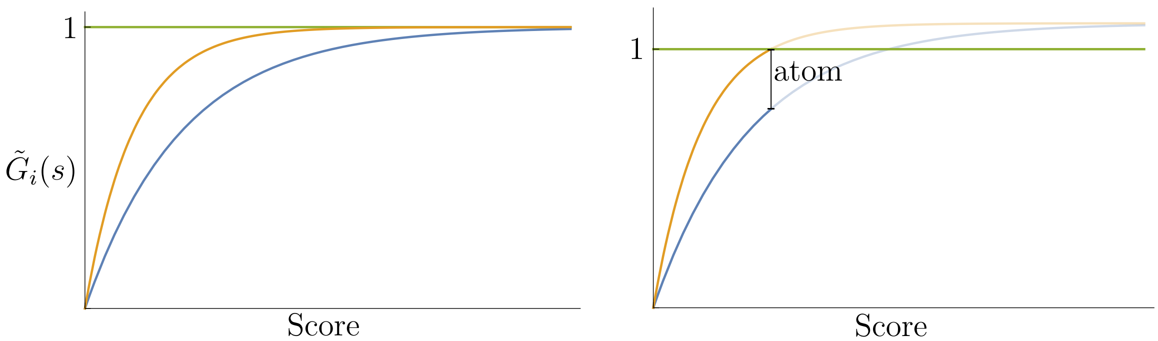

Figure 5: WoA (left) and WoA with costly preparation (right). If $\tilde{G}_1$ first-order stochastically dominates $\tilde{G}_2$, our algorithm implies that Player 1 has an atom. However, when the support is unbounded, there need not be an atom and the first-order stochastic dominance holds on the actual strategies.

the equilibrium is

$$G_1(s) = (1+\delta)\left(1 - \exp\left(-\frac{s}{2}\right)\right) + \underbrace{\sqrt{\delta^2+\delta} - \delta}_{\text{atom}}$$
$$G_2(s) = (1+\delta)\left(1 - \exp\left(-\frac{s}{1}\right)\right).$$

Because Player 1 has an atom, Player 2 receives a positive payoff. This is because the algorithm for finding the equilibrium of an all-pay contest stops at some finite $\bar{s}$. There, $\tilde{G}_2(\bar{s}) > \tilde{G}_1(\bar{s})$, implies that 2 has a positive payoff. Figure 5 provides a visualization on how the stronger player gets a positive payoff when the support is finite.

When δ is large enough ($\delta > \frac{2\sqrt{7}-5}{6} \approx 0.05$) Player 2 wins most of the time, in addition to receiving a positive payoff. $\triangle$

6 Other applications

In this section we focus on examples where closed-form solutions of the construction in Section 3 are obtainable through a variety of methods. The functional forms were chosen partially for tractability, but also to showcase useful methodologies in finding equilibrium strategies. Section 6.1 discusses a simple warfare model which captures elements of the offensive/defensive balance debate. More generally, it also exemplifies how one is able to solve for the equilibrium when spillovers are additively separable.

Sections 6.2 and 6.3 consider a price war or entry deterrence model, and a contest with winner's regret, respectively. They typify cases where spillovers induce an indifference condition, either (3) or (6), that depends only on the margin of victory.[19] This invites the use of Laplace transforms.

These are evidently not the only cases of interest. We note that when closed-form solutions are not available, there are multiple algorithms for computing the solutions numerically.[20]

6.1 Offensive/Defensive Balance

Military strategists generally agree that warfare is naturally asymmetric: the defending party can usually prevail with less expenditure of resources than the attacker (Clausewitz 1982). More generally, scholars have tried to identify which factors influence the so-called offensive/defensive balance – that is, the many elements of military technology that generate either offensive or defensive advantages, and thus affect the probability of war (Levy 1984). Our model is able to capture both the defensive advantage and the role of the prize-depleting nature of war in the offensive/defensive balance debate.

An attacker (a) invades a defender's (d) territory, which is worth V. Both combatants purchase costly scores in $[0, \infty)$, and the combatant with the higher score wins. A score of s_i costs $c_i s_i$, where $c_i > 0$ is a positive constant, for player $i \in I := \{a, d\}$. Furthermore, a's score inflicts $\delta_a s_a$ damage to the territory.[21] If the attacker wins, it internalizes all costs faced by the defender, as these costs effectively depleted the resources available from the territory. Consider the following payoff functions $u_a : [0, \infty) \to \mathbb{R}$ for the attacker:

$$u_a(s_a, s_d) = p_a(s_a, s_d)(V - \delta_a s_a) - c_a s_a,$$

[19] i.e. either $v'_{-i}(s; y)/v_{-i}(s; s)$ in (3) or $\frac{\partial v_{-i}(s;y)}{\partial y}/v_{-i}(s; s)$ for (6) are functions of $(s - y)$.

[20] One such way is to use the fact that the resolvent kernel of Lemma 1 is defined by a series that converges absolutely and uniformly. This suggests using a conceptually simple successive approximations method (see Appendix III for a more in-depth exposition). Other methods abound – for references, see Wazwaz 2011 and Brunner 2004.

[21] Assuming the defender also inflicts a cost of $\delta_d s_d$ onto the attacker does not change the analysis.

and the following payoff function $u_d : [0, \infty) \to \mathbb{R}$ for the defender:

$$u_d(s_a, s_d) = (1 - p_a(s_a, s_d))(V - \delta_a s_a) - c_d s_d,$$

where $p_a(\cdot) : [0, \infty)^2 \to [0, 1]$ denotes the probability that the attacker is victorious. Accordingly, we let $p_a(s_a, s_d) = 1$ whenever $s_a > s_d$, $p_a(s_a, s_d) = 0$ when $s_a < s_d$, and $p_a(s_a, s_d) = \lambda \in [0, 1]$ whenever $s_a = s_d$.

When we transform this model into our framework, we get $c_i(s_i) := c_i s_i$ and

$$v_a(s_a; s_d) = v_d(s_d; s_a) := V - \delta_a s_a.$$

Assume it costs weakly more to attack than to defend (i.e., $c_a \geq c_d$). The attacker does not have any spillovers while the defender is harmed by her opponent.

We are able to leverage the linearity of payoffs in this case to obtain a closed-form solution to the problem.

Proposition 2 (Additively separable spillovers)**.** *Consider a two-player contest where $v'_{-i}(s_{-i}; y)$ does not depend on y. That is, for each $i \in I$, $v_{-i}(s_{-i}; y) = v^{-i}_{-i}(s_{-i}) + v^{i}_{-i}(y)$. Then,*

$$\tilde{G}_i(s) = \frac{1}{f(s)} \int_0^s \frac{c'_{-i}(y)}{v_{-i}(y; y)} f(y) dy, \tag{9}$$

where $f(y) := \exp\left(-\int_0^y \frac{(v^{-i}_{-i})'(u)}{v_{-i}(u;u)} du\right)$. [22]

Using Proposition 2, we derive that the defender receives positive payoffs if, and only if,

$$\bar{s}_d := \frac{V}{c_a + \delta_a} < \frac{V}{\delta_a}\left[1 - \exp\left(-\frac{\delta_a}{c_d}\right)\right] =: \bar{s}_a,$$

which holds whenever $\delta_a > 0$ and $c_a \geq c_d$. In this case,

$$G_a(s) = 1 + \frac{c_d}{\delta_a} \log\left[\frac{c_a V}{(c_a + \delta_a)(V - \delta_a s)}\right] \quad \text{and} \quad G_d(s) = \frac{c_a s}{V - \delta_a s}.$$

The probability $P(s_a > s_d | \delta_a, c_a, c_d)$ that the attacker succeeds,[23] in equilibrium, is

[22] Equation (9) defines the strategy of any player with no atom. For example, this is the case in the symmetric game.

[23] Since ties happen with zero probability, the event where a wins has the same probability as the event where $s_a > s_d$.

given by

$$P(s_a > s_d|\delta_a, c_a, c_d) = \frac{c_d}{\delta_a^2}\left(\delta_a + c_a \log\left[\frac{c_a}{c_a+\delta_a}\right]\right) < \frac{c_d}{2c_a} \leq \frac{1}{2},$$

where the supremum is reached as $\delta_a \to 0$. If the war damages the territory at least as much as it costs the attacker to inflict such damage, ($\delta_a \geq c_a$), a tighter bound is obtained:

$$P(s_a > s_d|\delta_a, c_a, c_d) < 1 - \log(2) < \frac{1}{3}.$$

Even if $c_a = c_d$, the defender is more than twice as likely to win than the attacker is. In our model, the stronger position of the defensive party comes as a byproduct of the inverse relationship between the attacker's strength and the erosion of the prize's value. This provides an alternate explanation on why it is typically easier to defend than to attack, something usually attributed to the high costs of maintaining long supply lines and of keeping seized territories (Glaser and Kaufmann 1998). The defender's stronger position also suggests that any positive participation cost in a war contest imposed on the aggressor would be effective in discouraging aggression. [24]

6.2 War of Investment

Investment has long been considered as a method of committing to entry deterrence (Dixit 1980), while the war of attrition is a popular model of exit (Fudenberg and Tirole 1986). Our model can combine the two attributes into a single model of competition in continuous time, where players invest to stay in the game, but are able to recoup part of that investment if their opponent invests less.[25] Wars of investment can also be used to model Cold-War style defense spending and competition between technology companies and R&D races.

Assume two competitors, 1 and 2, invest in capital s_i at cost $c_i(s_i)$. The capital is necessary to engage in competition and depreciates at a constant rate. Competition

[24] In the more general nonlinear model, where the value of the territory after invasion is given by $v_\delta(s_a)$ and the cost of choosing score s_i to player i is given by a continuously differentiable function $c_i : [0, \infty) \to \mathbb{R}_+$ satisfying the required assumptions A1 to A4, $c_d(s) \leq c_a(s)$ is sufficient to ensure that $\tilde{G}_a(s) < \tilde{G}_d(s)$ for all $s > 0$. By our algorithm, this guarantees the defender's payoff remains positive, with $G_d(s) = \frac{c_a(s)}{v_\delta(s)}$.

[25] The combined model is similar to the war of attrition with costly preparation (Section 5). In fact, the two overlap when payoffs are linear. However, when payoffs are not linear, the two can be very different.

results in zero profits. However, the winner is able to extract monopoly profits and benefits from the remaining capital according to an increasing function $v_i(s_i - s_{-i})$. More concretely, assume payoffs are

$$u(s_i; s_{-i}) = \begin{cases} v_i(s_i - s_{-i}) - c_i(s_i) & \text{if } s_i > s_{-i} \\ -c_i(s_i) & \text{if } s_i < s_{-i} \\ \alpha_i v_i(0) - c_i(s_i) & \text{if } s_i = s_{-i} \end{cases}$$

for any $\alpha_i \in [0, 1)$.

If assumptions A1-4 are met, there is a unique equilibrium of capital investments in mixed strategies on finite support. Moreover, the equilibrium admits a closed-form solution, which can be solved using Laplace transforms.

Definition 4 (Laplace Transform)**.** [26] A function f defined on $\mathbb{R}_+$ admits a Laplace transform $F : \mathbb{C} \to \mathbb{C}$ given by

$$F(x) := \mathcal{L}\{f(s)\} = \int_0^\infty f(s)e^{-sx}ds$$

if and only if the above integral conditionally converges.[27]

To insure that the integral converges, the following regularity condition is imposed for this application.

Assumption 5 (A5, Exponential order)**.** [28] *A function F is of exponential order if and only if there exist s', $q, M \in [0, \infty)$ such that, for all $s \geq s'$,*

$$|F(s)| \leq Me^{qs}.$$

Proposition 3 (Margin of victory spillovers)**.** *Assume a two-player contest such that function $\nu_i(s - y) := \frac{v_i'(s;y)}{v_i(s;s)}$ exists and A5 holds for ν_i and $\frac{c_{-i}'(s)}{v_{-i}(s;s)}$. Then, for any $s \in (0, \overline{s}]$ and $i \in I$,*

$$\tilde{g}_i(s) = \mathcal{L}^{-1}\left\{ \frac{\mathcal{L}\left\{\frac{c_{-i}'(s)}{v_{-i}(s;s)}\right\}}{1 + \mathcal{L}\left\{\nu_{-i}(s)\right\}} \right\}$$

[26] See Churchill 1972 for an exposition on Laplace transforms.

[27] That is, it does not need to converge absolutely.

[28] Assumption A5 could be replaced by anything that ensures the Laplace transforms in Proposition 3 converge.

and

$$\tilde{G}_i(s) = \mathcal{L}^{-1}\left\{ \frac{\mathcal{L}\left\{\frac{c'_{-i}(s)}{v_{-i}(s;s)}\right\}}{x + x\mathcal{L}\left\{\nu_{-i}(s)\right\}} \right\},$$

where $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace and inverse Laplace transforms, respectively.

Example 3. Let $v_i(s;y) := e^{r_i(s-y)}\omega_i$ and $c_i(s) = e^{r_i s} - 1$, where $\omega_i, r_i \in (0,1)$ for each $i \in I := \{1,2\}$. Then, $\nu_{-i}(s) = r_{-i}e^{r_{-i}s}$ and Proposition 3 yields

$$\tilde{g}_i(s) = \mathcal{L}^{-1}\left\{ \frac{\mathcal{L}\left\{\frac{c'_{-i}(s)}{v_{-i}(s;s)}\right\}}{1 + \mathcal{L}\left\{\nu_{-i}(s)\right\}} \right\} = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{\omega_{-i}}\frac{r_{-i}}{x-r_{-i}}}{1 + \frac{r_{-i}}{x-r_{-i}}} \right\} = \frac{r_{-i}}{\omega_{-i}}$$

so the equilibrium strategies, excluding the possible mass point at zero, will be uniform with $\tilde{G}_i(s) = \left(\frac{r_{-i}}{\omega_{-i}}\right) s$.

The pair of ratios $\frac{\omega_i}{r_i}$ is therefore a sufficient statistic for the equilibrium of this game. Assume, without loss of generality that this ratio is weakly larger for Player 1. Then, the maximum duration of the game is Player 2's ratio $\bar{s} = \frac{\omega_2}{r_2}$.

The equilibrium is fully characterized by the overall *strength* of the players $\bar{s}$ and the *competitive balance* $\delta := \frac{\omega_2/r_2}{\omega_1/r_1} \in (0,1]$.

Because the strategies are uniform, Player 1's average commitment duration is half of the strength. Player 2 on the other hand has an atom of size

$$G_2(0) = 1 - \delta$$

which decreases as the competition becomes more balanced.

Overall, the conflict is expected to last for

$$\mathbb{E}[\min(s_1, s_2)] = \int_0^{\bar{s}} (1 - G_1(y))(1 - G_2(y))dy = \frac{\delta\bar{s}}{3}$$

total periods. The relationship between overall power and war duration is one to one. The duration is also increasing in the competitive balance. So, a large strength differential implies the conflict will typically be short-lived, whereas close contests can have delayed resolutions.

△

6.3 All-pay contest with winner's regret

Winner's regret is the remorse that the winner has from spending more than is necessary to win a contest or auction. This phenomenon has mostly been studied in the context of winner-pay first-price, auctions (Engelbrecht-Wiggans 1989; Filiz-Ozbay and Ozbay 2007). We instead apply our framework to model winner's regret in an all-pay contest

Let each Player $i \in I := \{1, 2\}$ choose a score in $[0, \infty)$. Suppose i values the prize at $\mu_i(s_i)[1 - h_i(s_i - s_{-i})]$, where $\mu_i(s_i)$ is the player's objective value of the prize and $h_i(s_i - s_{-i})$ is the share of the winnings that is unappreciated due to regret. Each player pays the cost $c_i(s_i)$ whether they win or lose. So payoffs are

$$u(s_i; s_{-i}) = \begin{cases} \mu_i(s_i)[1 - h_i(s_i - s_{-i})] - c_i(s_i) & \text{if } s_i > s_{-i} \\ -c_i(s_i) & \text{if } s_i < s_{-i}, \\ \alpha_i \mu_i(s_i) - c_i(s_i) & \text{if } s_i = s_{-i}, \end{cases}$$

for any $\alpha_i \in [0, 1)$. We assume all functions are continuously differentiable with $c'(s) > 0$ and $h_i'(s) \geq 0$. Moreover, $\mu(0) > h(0) = c(0) = 0$ and $c'(s) > \mu'(s)$ for each s, so that lower bids are preferable even with no regret. Intuitively, the regret function, h, should not exceed one.[29] The equilibrium strategies will admit closed-form solutions, which, as in the previous application, can be solved for using Laplace transforms.

Proposition 4 (Multiplicative margin of victory spillovers)**.** *Assume a two-player contest such that function $\psi_i(s - y) := \frac{1}{v_i(s;s)} \frac{\partial v_i(s;y)}{\partial y}$ exists and A5 holds for ψ_i and $\frac{c_i(s)}{v_i(s;s)}$. Then, for any $s \in (0, \overline{s}]$ and $i \in I$,*

$$\tilde{g}_i(s) = \mathcal{L}^{-1} \left\{ \frac{x \mathcal{L} \left\{ \frac{c_{-i}(s)}{v_{-i}(s;s)} \right\}}{1 - \mathcal{L} \left\{ \psi_{-i}(s) \right\}} \right\}$$

and

$$\tilde{G}_i(s) = \mathcal{L}^{-1} \left\{ \frac{\mathcal{L} \left\{ \frac{c_{-i}(s)}{v_{-i}(s;s)} \right\}}{1 - \mathcal{L} \left\{ \psi_{-i}(s) \right\}} \right\}$$

[29] Note that this is not a technical requirement.

where $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace and inverse Laplace transforms, respectively.

In our current application, $v_i(s_i; s_{-i}) = \mu_i(s_i)\,[1 - h_i(s_i - s_{-i})]$. So, $\psi_i(s) = h_i'(s)$ and

$$\frac{c_i(s)}{v_i(s;s)} = \frac{c_i(s)}{\mu_i(s_i)}.$$

As long as these two functions are of exponential order (A5) we can express the equilibrium in terms of the Laplace transform using Proposition 4.

Example 4. Let $\mu_i(s) := \omega_i \in \left(0, \frac{1}{2}\right)$, $h_i(s) = \frac{s^2}{2}$ and $c_i(s) := s - \frac{s^2}{2}$ for $s \in [0,1]$. Then,

$$\tilde{g}_i(s) = \mathcal{L}^{-1}\left\{\frac{x\mathcal{L}\left\{\frac{s-\frac{s^2}{2}}{\omega_{-i}}\right\}}{1 - \mathcal{L}\left\{s\right\}}\right\} = \mathcal{L}^{-1}\left\{\frac{x\left(\frac{x-1}{x^3\omega_{-i}}\right)}{1 - \left(\frac{1}{x^2}\right)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\omega_{-i}(1+x)}\right\} = \frac{e^{-s}}{\omega_{-i}}.$$

Without loss of generality, let $\omega_1 \geq \omega_2$, implying Player 1 receives non-negative payoffs. Player 1 will thus play a truncated exponential distribution with parameter 1 and support $[0, -\log(1-\omega_2)]$. Her expected score will be:

$$\mathbb{E}[s_1|\omega_1, \omega_2] = 1 + \left(\frac{1-\omega_2}{\omega_2}\right)\log(1-\omega_2).$$

which depends negatively on her opponent's payoff scaling factor ω_2. This is lower than in the same game without regret.

The player with zero expected payoffs will place a mass-point at zero of size:

$$G_2(0) = 1 - \frac{\omega_2}{\omega_1}$$

which is exactly the same size as if there were no regret. Player 2 will have expected score

$$\mathbb{E}[s_2|\omega_1, \omega_2] = \frac{\omega_2}{\omega_1}\left[1 + \left(\frac{1-\omega_2}{\omega_2}\right)\log(1-\omega_2)\right]$$

which is also less than in the same game without regret. The expected sum of the two scores score is:

$$\mathbb{E}[s_1 + s_2|\omega_1, \omega_2] = \left(1 + \frac{\omega_2}{\omega_1}\right)\left[1 + \left(\frac{1-\omega_2}{\omega_2}\right)\log(1-\omega_2)\right]$$

which is decreasing in ω_1 and increasing in ω_2. In contests such as a labor tournaments, a large productivity differential between participants in the form of a high ω_1 and low ω_2 depress aggregate effort. This is true in a contest with no spillovers, but the partial derivative of ω_1 is larger in absolute value when there is regret. So, the effect is exacerbated by the fact that the stronger player is penalized for winning by a large margin. △

7 Extensions

The first extension (7.1) allows for spillovers in the payoff of the loser. We show that this can be accommodated in our model so long as these spillovers are linearly separable. The second extension (7.2) allows for more than two players. We characterize an equilibrium with more players if the players' payoffs follow a particular ranked costs condition.

7.1 Separable spillovers on the losers payoff

There are several contexts where it makes more sense to have spillovers in the loser's payoff rather than the winner's. For example, models of litigation in English law must include the fact that the loser pays the winner's legal fees (Baye et al. 2005). While this may seem like a completely different scenario, it can easily be translated into our model if the cost is linearly separable. To see how, consider a two-player contest where player i's payoff u_i is given by

$$u_i(s_i; s_{-i}) := p_i(s_i; s_{-i})\hat{v}_i(s_i; s_{-i}) - (1 - p_i(s_i; s_{-i}))\left(c_i^{\alpha}(s_i) + c_i^{\beta}(s_{-i})\right)$$

where $c_i^{\alpha}(s_i) : \tilde{S}_i \to \mathbb{R}_+$ is the portion of i's costs that depends on their own score, and $c_i^{\beta} : \tilde{S}_{-i} \to \mathbb{R}_+$ is the portion of i's costs that depends on $-i$'s scores.

The above fits our model once we let $v_i(s_i; s_{-i}) := \hat{v}_i(s_i; s_{-i}) + c_i^{\beta}(s_{-i}) + c_i^{\alpha}(s_i)$ and $c_i(s_i) := c_i^{\alpha}(s_i)$.[30]

[30] To see this, consider Player i's expected utility in each model:

$$\int_0^{s_i} v_i(s_i; y)dG_{-i}(y) - (1 - G_{-i}(s_i))c_i^{\alpha}(s_i) - \int_{s_i}^{\bar{s}} c_i^{\beta}(y)dG_{-i}(y) \qquad \text{for the original model}$$
$$\int_0^{s_i} v_i(s_i; y)dG_{-i}(y) - (1 - G_{-i}(s_i))c_i^{\alpha}(s_i) + \int_0^{s_i} c_i^{\beta}(y)dG_{-i}(y) \qquad \text{in our framework}$$

7.2 More players

In contests with spillovers and more than two players, many of the results considered here and in the existing literature are violated. Existence still holds (see Olszewski and Siegel 2019, for example), but uniqueness doesn't.

Even without spillovers, it is difficult to construct an equilibrium of a contest where the normalized costs are not ranked. However, when the normalized costs are ranked, only two players ever participate in equilibrium, which effectively collapses the problem into a two-player contest.

A generalized version of this condition holds in our setting where $n > 2$. We still require normalized costs to be ranked in some sense, but in a way that takes the spillovers into account.

Theorem 3. *Assume i, j, $i \neq j$, are two of the $n > 2$ players in a contest satisfying assumptions A1 to A4. Suppose that Player i has a positive payoff in the two-player contest where i and j are the participants, and that the following "ranked costs" condition holds for all $s \in \tilde{S}_k$,$s_i \in \tilde{S}_i$ and $s_j \in \tilde{S}_j$, and all $k \notin \{i, j\}$*

$$\frac{c_k(s)}{v_k\left(s; \mathbf{s}_{\{i,j\}}\right)} \geq \frac{c_j(s)}{v_j\left(s; \mathbf{s}_{\{i\}}\right)} \tag{10}$$

where $\mathbf{s}_H$ is a vector of opponent scores that is zero for all players not in set H. Then, there exists an equilibrium where only Players i and j participate.

To understand condition (10), consider the candidate equilibrium where Players i and j compete using their two-player strategies and Player k does not participate. By not participating, Player k earns a payoff of zero – the same payoff as Player j. Condition (10) says that if she enters, Player k's normalized cost will be higher at every point than Player j's already is. Therefore, her payoff from participating is at most zero (Player j's payoff). So, there is no profitable deviation for any player.

Note that it is possible for multiple interval equilibria to satisfy Theorem 3 when spillovers decrease the value of the prize. If this decrease is sufficiently large, it's reasonable to have $k \succ j$ and $j \succ k$ in the sense of (10).

In the absence of spillovers, multiple equilibria also arise with three or more players. However, if payoffs are asymmetric, there can be at most one equilibrium where

These two payoffs differ only by $\int_0^{\bar{s}} c_i^{\beta}(y) dG_{-i}(y)$, which is a constant. This insight is from Xiao 2018.

the support of each player's strategy is a union of intervals. Additionally, the payoffs of each player are consistent across all equilibria. Neither of these properties hold in contests with spillovers. Indeed, payoffs can vary between different interval equilibria, as Example 5 illustrates..

Example 5. Suppose $n = 3$, and let:

$$\begin{aligned} v_i(s_i; s_{-i}) &= 1 - s_j - s_k, i = \{1, 3\}, \quad j, k \neq i \\ v_2(s_2; s_{-2}) &= \frac{3}{4} - s_1 - s_3 \end{aligned}$$

Further, assume all three players have identical cost functions $c_i(s_i) = s_i$ for all $i \in \{1, 2, 3\}$.

Consider now the following proposed equilibrium: Player 3 chooses $s_3 = 0$ with probability 1, while Players 1 and 2 submit scores as in the two-player equilibrium where only 1 and 2 participate, and thus choose

$$G_2(s) = \log\left(\frac{1}{3-4s}\right) + \log(4-e) \qquad G_1(s) = \log\left(\frac{1}{1-s}\right).$$

We can show that this profile of strategies is indeed an equilibrium; Players 1 and 2 have no incentive to deviate. Moreover, there's no $s > 0$ such that Player 3 obtains nonnegative by playing s, given 1 and 2's distributions.

Now, clearly, Players 1 and 3 are identical, and so they are interchangeable. Thus, we could also have the following equilibrium: Player 1 chooses $s_1 = 0$ with probability 1, while

$$G_2(s) = \log\left(\frac{1}{3-4s}\right) + \log(4-e) \qquad G_3(s) = \log\left(\frac{1}{1-s}\right)$$

Note that only Player 1 has a positive payoff in the first equilibrium and only Player 3 has a positive payoff in the second equilibrium. △

8 Related literature and conclusions

Throughout this paper, we characterized and established uniqueness for the equilibrium of two-player contests using techniques from the theory of integral equations.

This allowed us to derive insights on equilibrium payoffs, winners and losers, and on the importance of spillovers for applications. The fact that ranked normalized costs are not enough to establish dominance demonstrates how spillovers can favor high-cost, low-value players that nevertheless have a marginal cost advantage over their opponent when bids are high. In particular, the results in this paper suggest several potential consequences of legal structures, conflicts and competition. [31]

This paper is most closely related to two others. Baye et al. 2012, also considers spillovers in two-player contests, but focuses on linear symmetric costs and valuations. We extended the analysis to include asymmetric players and general functional forms for the prize values, which allows us to express novel results about payoffs and to characterize the equilibrium in different applications (Sections 5 and 6).

The second paper that approaches a similar question to our own is Xiao 2018. The author, however, focuses on constant prize value and separable spillovers in the cost functions, which are independent of winning or losing. This independence significantly restricts the equilibrium effects of the spillovers,[32] which is not true in our model. We applied Xiao's results to analyze contests with spillovers in the prizes and, linearly separable spillovers in the loser's payoff in Section 7.1.

This paper is also connected more broadly to the literature of spillovers in other contest frameworks. Hodler and Yektaş 2012, for example, use a linear first-price auction with spillovers to model war.[33] Hirai and Szidarovszky 2013 and Damianov et al. 2018 consider Tullock contests where the value of prize depends on the sum of the bids.

We identify several avenues for future work. The class of contests that include spillovers is very large and fits many applications. The fact that we are able to construct very different contests with the same equilibrium strategies (e.g. the all-pay contest with winner's regret in Section 6.3 has the same equilibrium as a war of attrition with costly preparation of Section 5) suggest that it might be possible for a contest designer to induce behavior more cheaply through spillovers.

Other contest design problems where spillovers are available are also of interest.

[31]This model does not require or imply that the results of a contest are known in advance. In fact, players are always uncertain of their own victory. However, this uncertainty stems from not knowing the resources that your opponent dedicated to the contest.

[32]Linearly separable spillovers on the cost have no effect on the equilibrium while multiplicatively separable spillovers scale the cost of bids by an endogenous constant.

[33]The authors refer to this as an all-pay contest, but only the winner actually pays.

Appendix II contains a brief exposition that shows that when a constrained designer that cares about aggregate effort can reward contestants with prizes that may include spillovers, no contestant will be allowed positive rents. This in particular would make computing equilibrium strategies straightforward. Under what circumstances introducing spillovers is desirable to a contest designer is however still an open question.

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9 Appendix

Appendix I Proofs

Proof of interval equilibrium

Lemma 1. *Assume a two-player contest where A1, A2, A3 and A4 hold. Then, in any equilibrium, players choose mixed strategies with a common support* $[0, \bar{s}]$*. Moreover, these strategies are continuous in* $(0, \bar{s}]$*, with at most one participant having a mass-point at zero.*

Proof. The argument is standard in the all-pay auction literature, and is presented here for completeness. Our proof is in several steps.

1. By A3, each player $i \in I$ would select scores in $S_i = [0, T_i]$, for T_i finite.

2. *The minimum score in the support of both players' strategies is zero.* Let $\underline{s}_1, \underline{s}_2$ denote the lower bounds of player 1 and player 2's strategies' supports, respectively. If $\underline{s}_1, \underline{s}_2 > 0$, then $\underline{s}_1 = \underline{s}_2$ – otherwise, player $i := \arg\min_{j \in I} \underline{s}_j$ would benefit from shifting the probability mass between $[\underline{s}_i, \underline{s}_{-i}]$ to zero. Let $\underline{s} := \underline{s}_1 = \underline{s}_2 > 0$. The two players can't both have a mass point at $\underline{s}$: either player would rather move that mass infinitesimally above $\underline{s}$. Similarly, it can't be that a single player i has a mass point at $\underline{s}$ – then, they would benefit from shifting that mass to zero, as that would reduce costs without changing their probability of winning. If neither player places a mass point at $\underline{s}$, then there exists $\varepsilon > 0$ such that player i would prefer to shift the probability mass between $\underline{s}$ and $\underline{s} + \varepsilon$ to zero.

   Now, we also can't have $\underline{s}_i > \underline{s}_{-i} = 0$, as player i could reduce the lower bound of her support, reduce costs and leave her probability of winning unchanged.

3. *Both players will have the same maximum score in their strategies' support* ($\bar{s}$)*.* Otherwise, the player with the highest upper bound to her support could reduce it and pay less costs without impacting the probability of winning.

4. *There are no gaps in the density.* If one player has a gap, then clearly both players must have the same gap. If two players have the same gap, then both players can benefit by moving some density from after the gap to just before.

5. *There are no mass points on the half open interval* $(0, \bar{s}]$*.* Otherwise, their opponent could ensure a positive payoff by shifting up density from below the mass point.

6. $G_i^\star$ *is strictly increasing in* $(0, \bar{s}]$. Suppose not, so that $G_i^\star$ is constant in some interval $(a, b) \subset (0, \bar{s}]$. Since there are no mass points in this interval, $G_i^\star(a) = G_i^\star(b) < 1$. Then, there exists $\varepsilon > 0$ such that i would benefit from shifting mass from $b + \varepsilon$ into a.

7. *At most one player will place a mass point at zero.* The two players can't both have a mass point at $\underline{s} = 0$: either player would rather move that mass infinitesimally above $\underline{s}$.

$\square$

Proof of Lemma 2

Proof. The finite definite integral cannot diverge because the function is continuous. Also note that (3) gives us $g_i(0) = \frac{c'_{-i}(0)}{v_{-i}(0;0)} > 0$.

We still need to confirm that $\tilde{g}_i(s) > 0$ on the relevant interval $\{s : \int_0^s |\tilde{g}_i(y)|dy \leq 1\}$. Suppose, by way of contradiction, that it is not. Then, by continuity, there must be an initial point $s^\star$ such that $\tilde{g}_i(s^\star) = 0$, $\int_0^{s^\star} \tilde{g}_i(y)dy \leq 1$, and $\tilde{g}_i(s) > 0$ for all $s < s^\star$. However, this is impossible because

$$\begin{aligned}
\tilde{g}_i(s^\star) &= \frac{1}{v_{-i}(s^\star; s^\star)} \left(c'_{-i}(s^\star) - \int_0^{s^\star} v'_{-i}(s^\star, y)|\tilde{g}_i(y)|dy \right) \\
&\geq \frac{1}{v_{-i}(s^\star; s^\star)} \left[c'_{-i}(s^\star) - \left| \max_{y \in [0, s^\star]} v'_{-i}(s^\star; y) \right| \underbrace{\left(\int_0^{s^\star} |\tilde{g}_i(y)|dy \right)}_{\leq 1} \right] \\
&\geq \frac{1}{v_{-i}(s^\star; s^\star)} \underbrace{\left[c'_{-i}(s^\star) - \left| \max_y v'_{-i}(s^\star; y) \right| \right]}_{>0 \text{ (A2)}} > 0.
\end{aligned}$$

We must now show that it is not possible for $\int_0^\infty |\tilde{g}_i(y)|dy \leq 1$. We can do this in one step with Holder's inequality.

$$c_{-i}(s) = \int_0^s v_{-i}(s; y) g_i(y) dy \leq \left(\int_0^s |g_i(y)|dy \right) \left(\max_{y \in [0,s]} v_{-i}(s; y) \right)$$

so $\int_0^s |g_i(y)|dy \geq \frac{c_{-i}(s)}{\max_y v_{-i}(s;y)}$ which is assumed to be greater than one as s approaches infinity (A3). By continuity, there exists an $\bar{s}_i$ such that $\int_0^{\bar{s}_i} |g_i(y)|dy = 1$ (A1). $\square$

Proof of Corollary 1.1

Proof. Equation (6) is obtained by applying integration by parts to (1). This defines a Volterra Integral Equation which has a unique solution by lemma 1. This solution coincides with the one in Theorem 1 because Equation (1) cannot have two solutions. □

Proof of Theorem 2

Proof. Consider equation (6). The main result of Beesack 1969 allows us to compare the solutions of two VIEs. In our setting, this means that conditions (7), (8) imply

$$\tilde{G}_2(s) \leq \tilde{G}_1(s) + \frac{c_1(s)}{v_1(s;s)} - \frac{c_2(s)}{v_2(s;s)} < \tilde{G}_1(s).$$

From this, it is clear that $\bar{s}_1 \leq \bar{s}_2$ which implies that player 2 has an atom. The bound comes from

$$u_1 = v_1(0;0)(1 - \tilde{G}_2(\bar{s})) \geq v_1(0;0)\left[\frac{c_2(\bar{s})}{v_2(\bar{s};\bar{s})} - \frac{c_1(\bar{s})}{v_1(\bar{s};\bar{s})}\right].$$

□

Proof of Proposition 1 Each condition is proven separately. However, the last two share the same setup.

Proof. Suppose $v(s;y) := v_1(s;y) = v_2(s;y)$ and $c_2(s) = \lambda c_1(s)$ where $\lambda > 1$. Because the two players have the same kernel, they must share the same resolvent (R). Then,

$$\begin{aligned}
\tilde{G}_1(\bar{s}) &= \frac{c_2(\bar{s})}{v(\bar{s};\bar{s})} + \int_0^{\bar{s}} R(\bar{s},y)\frac{c_2(y)}{v(y;y)}dy \\
&= \lambda\left(\frac{c_1(\bar{s})}{v(\bar{s};\bar{s})} + \int_0^{\bar{s}} R(\bar{s},y)\frac{c_1(y)}{v(y;y)}dy\right) \\
&= \lambda\tilde{G}_2(\bar{s}),
\end{aligned}$$

implying $\tilde{G}_1(\bar{s}) > \tilde{G}_2(\bar{s})$ and thus player 2 must have a point mass at zero. □

Proof. We would like to show that $\tilde{g}_1(s) - \tilde{g}_2(s) > 0$ for all $s \in [0, \bar{s}]$. This would imply that $\tilde{G}_1(\bar{s}) > \tilde{G}_2(\bar{s})$, which implies that player 2 must have the atom at zero.

By (3):

$$\tilde{g}_1(s) - \tilde{g}_2(s) = \frac{c_2'(s) - c_1'(s)}{v(s;s)} - \int_0^s \frac{v'(s;y)}{v(s;s)} \left(\tilde{g}_1(s) - \tilde{g}_2(s)\right) dy.$$

This is positive by Lemma 2. □

Proof. Suppose, by way of contradiction that Player 1 does not have a positive payoff. Then, there exists an $\bar{s}$ such that $\tilde{G}_2(\bar{s}) = 1$ and $\tilde{G}_1(\bar{s}) \leq 1$. Note that $\tilde{G}_1(0) - \tilde{G}_2(0) = 0$ and

$$\tilde{g}_1(0) - \tilde{g}_2(0) = \frac{c_2'(0) - c_1'(0)}{v(0;0)} > 0.$$

Therefore, there exists some $r \in (0, \bar{s}]$ that is the first point such that $\tilde{G}_1(r) - \tilde{G}_2(r) = 0$ and $\tilde{g}_1(r) - \tilde{g}_2(r) \leq 0$.

If we assume $c_2'(r) - c_1'(r) > \max_{y \leq r} v'(r;y) - \min_{y \leq r} v'(r;y)$, then

$$\begin{aligned}
\tilde{g}_1(r) - \tilde{g}_2(r) &= \frac{c_2'(r) - c_1'(r)}{v(r;r)} - \int_0^r \frac{v'(r;y)}{v(r;r)} [\tilde{g}_1(y) - \tilde{g}_2(y)] dy \\
&= \frac{c_2'(r) - c_1'(r)}{v(r;r)} - \int_0^r \frac{v'(r;y)}{v(r;r)} \tilde{g}_1(y) dy + \int_0^r \frac{v'(r;y)}{v(r;r)} \tilde{g}_2(y) dy \\
&\geq \frac{c_2'(r) - c_1'(r)}{v(r;r)} - \max_{y \leq r} \frac{v'(r;y)}{v(r;r)} \tilde{G}_1(r) dy + \min_{y \leq r} \frac{v'(r;y)}{v(r;r)} \tilde{G}_2(r) \\
&= \frac{c_2'(r) - c_1'(r) - [\max_{y \leq r} v'(r;y) - \min_{y \leq r} v'(r;y)] \tilde{G}_1(r)}{v(r;r)} > 0
\end{aligned}$$

If we assume $\frac{\partial v(r;y)}{\partial r \partial y} \geq 0$ a.e.

$$\begin{aligned}
\tilde{g}_1(r) - \tilde{g}_2(r) &= \frac{c_2'(r) - c_1'(r)}{v(r;r)} - \int_0^r \frac{v'(r;y)}{v(r;r)} [\tilde{g}_1(y) - \tilde{g}_2(y)] dy \\
&\geq \frac{c_2'(r) - c_1'(r)}{v(r;r)} + \frac{1}{v(r;r)} \int_0^r \frac{\partial v(r;y)}{\partial s \partial y} \left[\tilde{G}_1(y) - \tilde{G}_2(y)\right] dy > 0
\end{aligned}$$

where we apply integration by parts in the second line. □

Proof that WoA with costly preparation approximates WoA

Proof. A direct application of (3) yields the following differential equation:

$$\tilde{g}_{-i}^{\epsilon}(s) = \frac{1}{\ell_i(s) - f_i(s)} \left(\ell_i'(s) \left[1 - \tilde{G}_{-i}^{\epsilon}(s)\right] + \epsilon'(s)\right)$$

Because this is a continuous linear mapping, we can take the limit as $\epsilon'(s)$ approaches zero. This simplifies to the same differential equation used to describe the equilibrium of the WoA (e.g. in Hendricks et al. 1988):

$$\frac{\tilde{g}_{-i}(s)}{1-\tilde{G}_{-i}(s)} = \frac{\ell_i'(s)}{\ell_i(s)-f_i(s)}.$$

$\square$

Appendix II Optimal Contest Design

In this section, we consider how a designer should bias a contest to increase the scores. Several papers have analyzed this problem of assigning prizes to maximize total scores, or the average score of the winner. For example, Mealem and Nitzan 2014 consider prize redistribution in a two-player all-pay contest with fixed values and symmetric costs. They show equalizing the prize values maximizes the total scores and that the this contest yields weakly more total score than any similar Tullock-type lottery contest. Che and I. Gale 2003 investigate the optimal design of contests for innovation procurement, and find that the procurer might want to limit the maximum prize available to the most efficient firms – effectively eliminating any positive rents – in order to increase their own expected maximum surplus. The problem of optimal contest design in all-pay contests with spillovers has not been previously analyzed.

This is relevant because principals are constrained in the prizes that they can offer. Many of the tools that principals use to make prizes have spillovers. For example, if an employer chooses to construct a compensation package using a cash bonus and stock options, then the inclusion of the stock options will generate spillovers. This section analyzes the optimal prize choice when prizes can be constructed from multiple instruments.

Let $\Lambda_i \subset \mathbb{R}^{\tilde{S}_i}$ denote the set of prize functions available to the designer for player i, and let $V : \prod_{i\in I} \tilde{S}_i \times \prod_{i\in I} \Lambda_i \to \mathbb{R}$ denote the designer's payoff function, i.e., given the pair of scores $\mathbf{s} := (s_1, s_2)$ and the pair of value functions $\mathbf{v} = (v_1(\cdot;\cdot), v_2(\cdot;\cdot))$, $V(\mathbf{s}, \mathbf{v})$ denotes the designer's derived net benefit from the contest.

We make the following (mild) assumptions:

Assumption 1 (Completeness, D1)**.** *For each $i \in I$, set of prizes Λ_i, is convex and its closure contains an element with $v_i(\cdot;\cdot) \equiv 0$.*

Assumption 2 (Productive scores, D2)**.** *For each $i \in I$ and $\mathbf{v} \in \prod_{i \in I} \Lambda_i$, the designer's objective function $V(\mathbf{s}, \mathbf{v})$ is strictly increasing in s_i.*

Assumption 3 (Costly prizes, D3)**.** *For each $i \in I$, $\mathbf{s} \in \prod_{i \in I} \tilde{S}_i$ and $v_{-i} \in \Lambda_{-i}$, $V(\mathbf{s}, \mathbf{v})$ is decreasing in v_i[34].*

The primary complication with the construction in this paper is the atom is difficult to compute. Fortunately, if the mechanism designer can discriminate between the two players, an optimal mechanism will have no atoms in many specifications. This is formalized in the following proposition.

Proposition 5. *Assume a two-player contest where a fully informed principal with payoff function V chooses the prize $v_i \in \Lambda_i$ for each $i \in I$. Assume that Λ_i and V satisfy assumptions D1 to D3, and that for all i and all $v_i \in \Lambda_i$, assumptions A1 to A4 hold. Then, no contestant in equilibrium can have a positive payoff. Equivalently, no player will have a point-mass as part of their strategy.*

Proposition 5 implies that there will be no strictly dominant player in any discriminating contest design problem where the principal benefits from the efforts of participants and pays for prizes. This proposition comes from the fact that the equilibrium strategy of the dominant player is locally invariant to changes in her prize value. Intuitively, for any contest with a strictly dominant player, there exists a more competitive contest where their prize is reduced and scores are larger.

Proof. Take an optimal choice of $\mathbf{v} := (v_i)_{i \in I} \in \prod_{i \in I} \Lambda_i$. Suppose, by contradiction, that player i has a strictly positive payoff. Her strategy is defined by

$$\tilde{g}_i(s) = \frac{c'_{-i}(s)}{v_{-i}(s;s)} - \int_0^s \frac{v'_{-i}(s;y)}{v_{-i}(s;s)} \tilde{g}_i(y) dy,$$

which does not depend on v_i. Because player $-i$ has an atom, we know that $\tilde{G}_i(\bar{s}) - \tilde{G}_{-i}(\bar{s}) > 0$. Therefore, there exists a $\gamma \in (0,1)$ such that $\tilde{G}_i(\bar{s}) = \frac{1}{\gamma} \tilde{G}_{-i}(\bar{s})$.

Then, the principal could offer $(\gamma v_i, v_{-i})$ without changing the equilibrium strategy of player i. By the costly prizes Assumption D3, this is weakly preferable given a fixed distribution of s_{-i}.

[34]That is, if $v_i, \hat{v}_i \in \Lambda_i$ are such that $v_i(s;y) \leq \hat{v}_i(s;y)$ for all $(s,y) \in \tilde{S}_i \times \tilde{S}_{-i}$, then $V(\mathbf{s}, (v_i, v_{-i})) \geq V(\mathbf{s}, (\hat{v}_i, v_{-i}))$.

By construction, player 2's new equilibrium strategy is $\frac{1}{\gamma}\tilde{G}_{-i}(\bar{s})$. This first-order stochastically dominates player $-i$'s original strategy. In fact, it is the same distribution, but with the atom removed. The productive scores assumption implies that this mechanism is strictly preferred. $\square$

Proposition 5 demonstrates that the expected welfare of all agents is zero in a large class of contest design problems[35]. It also suggests the optimality, from a design perspective, of handicapping the most efficient players (as in, the players with lower costs and lower marginal costs) The idea is very much analogous to the conclusion in Che and I. Gale 2003, for example: handicapping the player that has the technological upper hand causes the less efficient player to become more aggressive, and to choose higher scores than they would otherwise.

Appendix III Numerical Approximation

Iteration method It is possible to approximate the solution by iterating numerically on this sequence:

$$\tilde{g}_{n+1}(s) = \frac{1}{v(s;s)}\left(c'(s) - \int_0^s v'(s;y)\tilde{g}_n(y)dy\right)$$

starting from $\tilde{g}_0 = 0$ to find the true $\tilde{g}$. There is a much simpler and faster way.

Matrix method (1) Consider our original equation

$$\int_0^s v_{-i}(s;y)\tilde{g}_i(y)dy = c(s)$$

and consider this 3×3 discrete approximation of this problem for $s \in [0,1]$

$$\underbrace{\frac{1}{3}\begin{bmatrix} v_{-i}({}^1\!/_3, {}^1\!/_3) & 0 & 0 \\ v_{-i}({}^2\!/_3, {}^1\!/_3) & v_{-i}({}^2\!/_3, {}^2\!/_3) & 0 \\ v_{-i}(1, {}^1\!/_3) & v_{-i}(1, {}^2\!/_3) & v_{-i}(1,1) \end{bmatrix}}_{\mathbf{V}} \cdot \underbrace{\begin{bmatrix} \tilde{g}_i({}^1\!/_3) \\ \tilde{g}_i({}^2\!/_3) \\ \tilde{g}_i(1) \end{bmatrix}}_{\mathbf{g}} \approx \underbrace{\begin{bmatrix} c_{-i}({}^1\!/_3) \\ c_{-i}({}^2\!/_3) \\ c_{-i}(1) \end{bmatrix}}_{\mathbf{c}}$$

[35] Which is not to say that there are no settings where it would not apply to. For example, the designer could wish to maximize the agents' expected welfare. In this case, the principal's objective function would violate costly prizes. It would usually also violate productive scores.

So, we can approximate $\tilde{g}_i(s)$ with

$$\mathbf{g} = 3\mathbf{V}^{-1}\mathbf{c}$$

Matrix method (2) To get a good estimate, we do the same thing with an $N \times N$ grid for N large on some interval $[0, T]$

$$\begin{bmatrix} \tilde{g}_i(1/N) \\ \tilde{g}_i(2/N) \\ \vdots \\ \tilde{g}_i(T) \end{bmatrix} \approx N \begin{bmatrix} v_{-i}(1/N, 1/N) & 0 & \cdots & 0 \\ v_{-i}(2/N, 1/N) & v_{-i}(2/N, 2/N) & \cdots & 0 \\ & & \ddots & \\ v_{-i}(T, 1/N) & v_{-i}(T, 2/N) & \cdots & v_{-i}(T; T) \end{bmatrix}^{-1} \cdot \begin{bmatrix} c_{-i}(1/N) \\ c_{-i}(2/N) \\ \vdots \\ c_{-i}(T) \end{bmatrix}$$

Getting the actual strategies Once you get $(\tilde{g}_1, \tilde{g}_2)$ you just have to:

1. take the cumulative sum and divide by N to get $(\tilde{G}_1, \tilde{G}_2)$

```
G1, G2 = cumsum(g1)/N, cumsum(g2)/N
```

2. truncate both distributions at the last value where both are ≤ 1

```
G1, G2 = G1[G1 <= 1 & G2 <= 1], G2[G1 <= 1 & G2 <= 1]
```

3. add to each CDF vector so that both end with 1 (add the atom)

```
G1, G2 = (G1 - G1[-1] + 1), (G2 - G2[-1] + 1)
```