# Geometric Nonlinear Observer Design for SLAM on a Matrix Lie Group

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***Abstract*— This paper considers the Simultaneous Localization and Mapping (SLAM) problem for a rigid body evolving in 3D space. We propose a nonlinear geometric observer, designed on the matrix Lie group $SE_{1+n}(3)$, using group velocity and landmark measurements. The proposed observer is also extended to handle unknown biases in the linear and angular velocity. Simulation results are presented to illustrate the effectiveness of the proposed observer.**

## I. Introduction

The development of reliable navigation algorithms for autonomous vehicles evolving in unknown environments is instrumental in many applications, such as autonomous underwater vehicles and unmanned aerial vehicles. This is particularly important when the absolute positioning systems (*e.g.,* Global Positioning Systems (GPS)) are not available (*e.g.,* indoor applications). The problem of building a map of an unknown environment while keeping track of the vehicle's location is refereed to as Simultaneous Localization and Mapping (SLAM) problem. The SLAM problem has been widely studied in robotics over the past three decades [1], [2]. One of the most widely used techniques dealing with the SLAM problem is the extended Kalman filter (EKF). However, It is well known that this type of EKF-based SLAM techniques suffers from the consistency problem and the lack of global stability results [3], [4]. Considerable efforts have been made to improve the consistency and convergence of the EKF-based SLAM [4]–[6]. In this context, a sensor-based SLAM, with global asymptotic stability results, has been proposed in [7]. An extension of the sensor-based SLAM filter to the motion in 3D space has been presented in [8].

Recently, more and more researchers recognize the importance of Lie group representations of orientation and/or pose, which take advantage of the symmetry structure of the motion space [9], [10]. In recent years, research on symmetry-preserving observers design on Lie groups has become very popular [11]–[13]. One of the first examples of invariant EKF-based SLAM (IEFK-based SLAM), combining the symmetry-preserving approach and the EKF techniques, has been presented in [14]. An extension of the IEKF-based SLAM has been proposed in [15] to solve the inconsistency problem of classical EKF-based SLAM. The convergence and consistency properties of IEKF-based SLAM have been analyzed in [16]. However, only local convergence is guaranteed with this approaches due to the linearization of the dynamics of the intrinsic error around the group identity. To the authors' best knowledge, very few publications are available in the literature that address the full SLAM problem from a Lie group perspective with strong convergence guarantees. More recently, an interesting approach, which models the landmarks and the pose of a rigid body in a single geometric structure, has been considered in [17]. Under this symmetric structure, a novel geometric nonlinear observer for SLAM has been derived by applying the techniques presented in [13]. Another gradient-based geometric approach for SLAM was proposed in [18], which generates the gradient-based innovation terms for the estimated pose and map separately.

Motivated by [17], a geometric nonlinear observer for SLAM in terms of group velocity and landmark measurements has been proposed in this paper. A rigorous analysis of the global convergence has been provided. The proposed observer is also extended to handle linear and angular velocity-biases as in [18]. The presented paper differs from [17], [18] in many ways. First, the SLAM dynamics are formulated in the framework of a specific matrix Lie group denoted by $SE_{1+n}(3)$. Second, the innovation terms are generated directly on the Lie group $SE_{1+n}(3)$ using geometric techniques on matrix Lie groups. Finally, the problem of time-varying landmarks has been considered in the present paper.

The remainder of this paper is organized as follows: Section II provides some preliminary notations and definitions that will be used throughout this paper. In Section III, we present a geometric nonlinear observer for SLAM on a matrix Lie group along with a rigorous stability analysis. In Section IV, the proposed observer is extended to the case where the measurements of the rotational and translational velocities are corrupted by unknown constant biases. Section V presents some simulation results showing the performance of our proposed approach.

## II. Preliminary Material

### A. Notations

The sets of real, nonnegative real and natural numbers are denoted as $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively. We denote by $\mathbb{R}^n$ the $n$-dimensional Euclidean space. Given two matrices, $A, B \in \mathbb{R}^{m\times n}$, their Euclidean inner product is defined as $\langle\langle A, B\rangle\rangle := \mathrm{tr}(A^\top B)$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| := \sqrt{x^\top x}$, and the Frobenius norm of a

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matrix $X \in \mathbb{R}^{n\times m}$ is given by $\|X\|_F := \sqrt{\langle\langle X, X\rangle\rangle}$. The $n$-by-$n$ identity matrix is denoted by $I_n$ and the $n$-dimensional vector of zero elements is denoted by $\mathbf{0}_n$. Let $e_1, \cdots, e_n$ be the standard basis vectors of $\mathbb{R}^n$, that is, $e_i$ has all entries equal to zero except for the $i$-th entry which is equal to 1.

Consider the matrix Lie group $SE_{1+n}(3) := SO(3) \times \mathbb{R}^3 \times \mathbb{R}^{3\times n} \subset \mathbb{R}^{(4+n)\times(4+n)}$ defined as follows [15]:

$$SE_{1+n}(3) = \{X = \mathcal{T}(R, p, \mathsf{p}) | \; R \in SO(3), p \in \mathbb{R}^3, \mathsf{p} \in \mathbb{R}^{3\times n}\}, \quad (1)$$

with the map $\mathcal{T} : SO(3) \times \mathbb{R}^3 \times \mathbb{R}^{3\times n} \to \mathbb{R}^{(4+n)\times(4+n)}$ defined as

$$\mathcal{T}(R, p, \mathsf{p}) = \left[\begin{array}{cc|c} R & p & \mathsf{p} \\ \mathbf{0}_{1\times 3} & 1 & \mathbf{0}_{1\times n} \\ \hline \mathbf{0}_{n\times 3} & \mathbf{0}_n & I_n \end{array}\right].$$

The inverse of $X$ is given by $X^{-1} = \mathcal{T}(R^\top, -R^\top p, -R^\top \mathsf{p}) \in SE_{1+n}(3)$. It is also clear that, for any $X_1, X_2 \in SE_{1+n}(3)$, one has $X_1X_1^{-1} = X_1^{-1}X_1 = I_{n+4}$ and $X_1X_2 \in SE_{1+n}(3)$. The *Lie algebra* associated to the Lie group $SE_{1+n}(3)$, denoted by $\mathfrak{se}_{1+n}(3) := \mathfrak{so}(3)\times\mathbb{R}^3\times\mathbb{R}^{3\times n} \subset \mathbb{R}^{(4+n)\times(4+n)}$, is given by

$$\mathfrak{se}_{1+n}(3) = \left\{ U = \left[\begin{array}{cc|c} \Omega & v & \mathsf{v} \\ \hline \mathbf{0}_{(n+1)\times 3} & \mathbf{0}_{n+1} & \mathbf{0}_{(n+1)\times n} \end{array}\right] \middle| \; \Omega \in \mathfrak{so}(3), v \in \mathbb{R}^3, \mathsf{v} \in \mathbb{R}^{3\times n}\right\}. \quad (2)$$

The *tangent space* of the group $SE_{1+n}(3)$ is defined as $T_X SE_{1+n}(3) := \{XU | X \in SE_{1+n}(3), U \in \mathfrak{se}_{1+n}(3)\}$. Let $\langle\cdot,\cdot\rangle_X : T_X SE_{1+n}(3) \times T_X SE_{1+n}(3) \to \mathbb{R}$ be a *Riemannian metric* on $SE_{1+n}(3)$, such that for all $X \in SE_{1+n}(3), U_1, U_2 \in \mathfrak{se}_{1+n}(3)$ one has

$$\langle XU_1, XU_2\rangle_X := \langle\langle U_1, U_2\rangle\rangle. \quad (3)$$

Given a differentiable smooth function $f : SE_{1+n}(3) \to \mathbb{R}$, the *gradient* of $f$, denoted by $\nabla_X f \in T_X SE_{1+n}(3)$, relative to the Riemannian metric $\langle\cdot,\cdot\rangle_X$ is uniquely defined by

$$df \cdot XU = \langle \nabla_X f, XU\rangle_X = \langle\langle X^{-1}\nabla_X f, U\rangle\rangle, \quad (4)$$

Let $\times$ be the vector cross-product on $\mathbb{R}^3$ and define the map $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ such that $x \times y = x^\times y$, for all $x, y \in \mathbb{R}^3$. For any matrix $A_1 \in \mathbb{R}^{3\times 3}$, define $\mathbb{P}_a(A_1)$ as the anti-symmetric projection of $A_1$, such that $\mathbb{P}_a(A_1) = (A_1 - A_1^\top)/2$. Let $\mathbb{P} : \mathbb{R}^{(4+n)\times(4+n)} \to \mathfrak{se}_{1+n}(3)$ denote the unique orthogonal projection of $\mathbb{R}^{(4+n)\times(4+n)}$ onto $\mathfrak{se}_{1+n}(3)$ with respect to the matrix inner product, that is, for all $U \in \mathfrak{se}_{1+n}(3), A \in \mathbb{R}^{(4+n)\times(4+n)}$, one has

$$\langle\langle U, A\rangle\rangle = \langle\langle U, \mathbb{P}(A)\rangle\rangle = \langle\langle \mathbb{P}(A), U\rangle\rangle, \quad (5)$$

and for all $A_1 \in \mathbb{R}^{3\times 3}, A_2, A_3 \in \mathbb{R}^{3\times(n+1)}, A_4 \in \mathbb{R}^{(n+1)\times(n+1)}$,

$$\mathbb{P}\left(\begin{bmatrix} A_1 & A_2 \\ A_3^\top & A_4 \end{bmatrix}\right) = \begin{bmatrix} \mathbb{P}_a(A_1) & A_2 \\ \mathbf{0}_{(n+1)\times 3} & \mathbf{0}_{(n+1)\times(n+1)} \end{bmatrix}. \quad (6)$$

Let $\mathcal{M}_0$ denote the sub-manifold of $\mathbb{R}^{(4+n)\times(4+n)}$ such that

$$\mathcal{M}_0 := \left\{ M = \begin{bmatrix} M_1 & M_2 \\ \mathbf{0}_{(n+1)\times 3} & \mathbf{0}_{n+1} \end{bmatrix} \middle| \; M_1 \in \mathbb{R}^{3\times 3}, M_2 \in \mathbb{R}^{3\times(n+1)} \right\}.$$

Then, for all $M, \bar{M} \in \mathcal{M}_0$ and $X \in SE_{1+n}(3)$, one has

$$\mathbb{P}(XM) = \mathbb{P}(X^{-\top}M), \quad (7)$$
$$\mathrm{tr}(X^\top XM\bar{M}^\top) = \mathrm{tr}(M\bar{M}^\top). \quad (8)$$

For any $X \in SE_{1+n}(3), U \in \mathfrak{se}_{1+n}(3)$, the Adjoint action map $\mathrm{Ad}_X : SE_{1+n}(3) \times \mathfrak{se}_{1+n}(3) \to \mathfrak{se}_{1+n}(3)$ is defined as

$$\mathrm{Ad}_X U := XUX^{-1}. \quad (9)$$

One can verify that $\mathrm{Ad}_{X_1}\mathrm{Ad}_{X_2}U = \mathrm{Ad}_{X_1X_2}U$, for all $X_1, X_2 \in SE_{1+n}(3), U \in \mathfrak{se}_{1+n}(3)$.

### *B. SLAM Kinematics and Measurements*

Let $\{\mathcal{I}\}$ be an inertial frame and $\{\mathcal{B}\}$ be a body-attached frame. Let $R \in SO(3)$ and $p \in \mathbb{R}^3$ be the rotation of frame $\{\mathcal{B}\}$ with respect to the inertial frame $\{\mathcal{I}\}$, and position of the rigid-body expressed in the inertial frame $\{\mathcal{I}\}$, respectively. Consider a family of $n$ landmarks $p_i \in \mathbb{R}^3$, $i = 1, \cdots, n$, expressed in the inertial frame $\{\mathcal{I}\}$.

Define $\Omega \in \mathfrak{so}(3)$ and $v \in \mathbb{R}^3$ as the angular velocity and linear velocity, respectively, expressed in frame $\{\mathcal{B}\}$. Consider the following motion kinematics of a rigid-body and a family of $n$ landmarks

$$\dot{R} = R\Omega, \quad (10)$$
$$\dot{p} = Rv, \quad (11)$$
$$\dot{p}_i = Rv_i, \quad i = 1, 2, \cdots, n, \quad (12)$$

where $v_i$ is the linear velocity of the $i$-th landmark expressed in the body frame $\{\mathcal{B}\}$. In this work, we formulate the SLAM problem using the kinematics of an element of the matrix Lie group $X = \mathcal{T}(R, p, \mathsf{p}) \in SE_{1+n}(3)$ with $\mathsf{p} := [p_1, \cdots, p_n] \in \mathbb{R}^{3\times n}$ . From (1), the overall kinematics (10)-(12) can be rewritten in the following compact left invariant form

$$\dot{X} = XU. \quad (13)$$

where $U \in \mathfrak{se}_{1+n}(3)$ is defined in (2) with $\mathsf{v} = [v_1, \cdots, v_n]$.

Assume that the group velocity $U$ is continuous, bounded, and available for measurement. Assume also that the landmarks are measured in the body frame $\{\mathcal{B}\}$, and are given by

$$y_i = R^\top(p_i - p), \quad i = 1, 2, \cdots, n. \quad (14)$$

Let us introduce the following new vectors: $r_i := [\mathbf{0}_3^\top \; 1 \; -e_i^\top]^\top, b_i := [y_i^\top \; 1 \; -e_i^\top]^\top \in \mathbb{R}^{n+4}$ for all $i = 1, \cdots, n$. From (14), one has the following compact equations:

$$b_i = h(X, r_i) := X^{-1}r_i, \quad \forall i = 1, 2, \cdots, n. \quad (15)$$

Note that the Lie group action $h : SE_{1+n}(3) \times \mathbb{R}^{n+4} \to \mathbb{R}^{n+4}$ is a *right group action* in the sense that for all $X_1, X_2 \in SE_{1+n}(3)$ and $r \in \mathbb{R}^{n+4}$, one has $h(X_2, h(X_1, r)) = h(X_1X_2, r)$.

## III. Gradient-like observer design without velocity-bias

Let $\hat{R} \in SO(3)$ and $\hat{p} \in \mathbb{R}^3$ denote the estimates of the attitude $R$ and the position $p$ of the rigid body, respectively. Let $\hat{p}_i \in \mathbb{R}^3$ be the estimated position of the $i$-th landmark $p_i$. Let us define the estimate of the state $X$ as $\hat{X} := \mathcal{T}(\hat{R}, \hat{p}, \hat{\mathfrak{p}}) \in SE_{1+n}(3)$ with $\hat{\mathfrak{p}} := [\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_n]$. Then, one has the estimation error $\tilde{X} := X\hat{X}^{-1} = \mathcal{T}(\tilde{R}, \tilde{p}, \tilde{\mathfrak{p}}) \in SE_{1+n}(3)$ with $\tilde{R} := R\hat{R}^\top, \tilde{p} := p - \tilde{R}\hat{p}$, $\tilde{\mathfrak{p}} := [\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n]$ and $\tilde{p}_i := p_i - \tilde{R}\hat{p}_i$ for all $i = 1, \cdots, n$.

The following Lemmas (whose proofs are given in the Appendix) will be useful in the remainder of this paper:

*Lemma 1:* Consider the following smooth real-valued function:

$$\mathcal{U}(X) = \frac{1}{2}\mathrm{tr}((I - X)A(I - X)^\top), \tag{16}$$

where $A = A^\top \in \mathbb{R}^{(4+n)\times(n+4)}$. Then, for all $X \in SE_{1+n}(3)$, the gradient of $\mathcal{U}$ with respect to $X$ along the trajectories of $\dot{X} = XU$ is given by:

$$\nabla_X \mathcal{U} = X\mathbb{P}((I - X^{-1})A). \tag{17}$$

*Lemma 2:* Consider the definitions of $r_i, b_i, i = 1, \cdots, n$. Let $A := \sum_{i=1}^n k_i r_i r_i^\top$. From (6) and (15), one has

$$\textstyle\sum_{i=1}^n k_i\|r_i - \hat{X}b_i\|^2 = \mathrm{tr}\left(((I - \tilde{X}))A(I - \tilde{X})^\top\right), \tag{18}$$

$$\mathbb{P}\left(\textstyle\sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top\right) = \mathbb{P}((I - \tilde{X}^{-1})A). \tag{19}$$

Moreover, letting $\epsilon_i := \hat{p}_i - \hat{p} - \hat{R}y_i$ for all $i = 1, \cdots, n$, one has the following identities:

$$\textstyle\sum_{i=1}^n k_i\|r_i - \hat{X}b_i\|^2 = \sum_{i=1}^n k_i\|\epsilon_i\|^2, \tag{20}$$

$$\mathbb{P}\left(\textstyle\sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top\right) = \begin{bmatrix} \mathbf{0}_{3\times3} & \sum_{i=1}^n k_i\epsilon_i & \epsilon \\ \mathbf{0}_{(n+1)\times3} & \mathbf{0}_{n+1} & \mathbf{0}_{(n+1)\times n} \end{bmatrix}, \tag{21}$$

$$\mathbb{P}\left(\hat{X}^\top \textstyle\sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top \hat{X}^{-\top}\right) = \begin{bmatrix} \Theta_a & \sum_{i=1}^n k_i\hat{R}^\top\epsilon_i & \mathbf{0}_{3\times n} \\ \mathbf{0}_{(n+1)\times3} & \mathbf{0}_{n+1} & \mathbf{0}_{(n+1)\times n} \end{bmatrix}. \tag{22}$$

where $\epsilon := -[k_1\epsilon_1 \ \cdots \ k_n\epsilon_n] \in \mathbb{R}^{3\times n}$ and $\Theta_a := \frac{1}{2}\sum_{i=1}^n k_i(\hat{R}^\top\epsilon_i^\times \hat{R}y_i)^\times \in \mathbb{R}^{3\times3}$.

We consider the following nonlinear observer:

$$\dot{\hat{X}} = \hat{X}(U - \Delta), \tag{23}$$

where $\Delta \in \mathfrak{se}_{1+n}(3)$ is an innovation term to be designed. Using the fact that $\hat{X}^{-1}\hat{X} = I$, one has $\dot{\hat{X}}^{-1} = -\hat{X}^{-1}\dot{\hat{X}}\hat{X}^{-1}$, which leads to the following error dynamics:

$$\begin{aligned}\dot{\tilde{X}} &= XU\hat{X}^{-1} - X\hat{X}^{-1}\hat{X}(U - \Delta)\hat{X}^{-1} \\ &= \tilde{X}(\mathrm{Ad}_{\hat{X}}\Delta).\end{aligned} \tag{24}$$

Let us define the landmark estimation errors as $\eta_i := r_i - \hat{X}b_i = [\epsilon_i^\top \ \mathbf{0}_{1\times n}]^\top$, $i = 1, \cdots, n$. One can verify that $\|\eta_i\| = \|\epsilon_i\|$, and consequently, the convergence of $\epsilon_i$ to zero implies the convergence of $\eta_i$ to zero, which in turn implies that $\hat{X}^{-1}r_i$ converges to $b_i$. Let us introduce the following compact set:

$$\mathcal{A} := \{(\eta_1, \cdots, \eta_n) \in \mathbb{R}^{n+4} \times \cdots \times \mathbb{R}^{n+4} | \ \eta_i = 0, \forall i = 1, \cdots, n\}. \tag{25}$$

Now, one can state our first result.

*Theorem 1:* Consider the SLAM kinematics (13) with the system outputs (15), and the observer (23) with the following innovation term:

$$\Delta := -\mathrm{Ad}_{\hat{X}^{-1}}\mathbb{P}\left(\textstyle\sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top\right), \tag{26}$$

with $k_i > 0$ for all $i = 1, \cdots, n$. Then, the set $\mathcal{A}$ is globally exponentially stable. Moreover, there exist a constant matrix $R^* \in SO(3)$ and a constant vector $p^* \in \mathbb{R}^3$ such that $\lim_{t\to\infty}\tilde{R}(t) = R^*$ and $\lim_{t\to\infty}\tilde{p}(t) = p^*$.

*Proof:* In view of (9) and (19), the closed-loop system (24) can be rewritten as

$$\dot{\tilde{X}} = -\tilde{X}\mathbb{P}\left((I - \tilde{X}^{-1})A\right). \tag{27}$$

Consider the following Lyapunov function candidate:

$$\mathcal{U}(\tilde{X}) = \frac{1}{2}\sum_{i=1}^n k_i\|r_i - \hat{X}b_i\|^2 = \frac{1}{2}\sum_{i=1}^n k_i\|\epsilon_i\|^2, \tag{28}$$

which is positive definite with respect to $\mathcal{A}$. In view of (5) and (17), the time-derivative of $\mathcal{U}$ along the trajectories of (27) is given by

$$\begin{aligned}\dot{\mathcal{U}} &= \langle \nabla_{\tilde{X}}\mathcal{U}, -\tilde{X}\mathbb{P}((I - \tilde{X}^{-1})A)\rangle_{\tilde{X}} \\ &= \langle\langle \tilde{X}^{-1}\nabla_{\tilde{X}}\mathcal{U}, -\mathbb{P}((I - \tilde{X}^{-1})A)\rangle\rangle \\ &= -\langle\langle \mathbb{P}((I - \tilde{X}^{-1})A), \mathbb{P}((I - \tilde{X}^{-1})A)\rangle\rangle \\ &= -\|\mathbb{P}((I - \tilde{X}^{-1})A)\|_F^2.\end{aligned}$$

In view of (21) in Lemma 2, one can further show that

$$\dot{\mathcal{U}} = -\left\|\textstyle\sum_{i=1}^n k_i\epsilon_i\right\|^2 - \sum_{i=1}^n k_i^2\|\epsilon_i\|^2 \le -\lambda\mathcal{U} \tag{29}$$

where $\lambda := 2\min\{k_1, k_2, \cdots, k_n\}$. From (29), one has $\mathcal{U}(t) \le e^{-\lambda t}\mathcal{U}(0)$. It follows that $\epsilon_i$ converges to zero exponentially for all $i = 1, \cdots, n$. Therefore, one can conclude that the set $\mathcal{A}$ is globally exponentially stable. Next, we are going to show the convergence of $(\tilde{R}, \tilde{p})$. The closed-loop system (27) can be expressed as follows:

$$\begin{cases}\dot{\tilde{R}} &= 0 \\ \dot{\tilde{p}} &= -\sum_{i=1}^n k_i\tilde{R}\epsilon_i \\ \dot{\tilde{p}}_i &= -k_i\tilde{R}\epsilon_i\end{cases}. \tag{30}$$

Since $\epsilon_i = \hat{p}_i - \hat{p} - \hat{R}y_i = \tilde{R}^\top(\tilde{p} - \tilde{p}_i)$, it is clear that the closed-loop dynamics (30) are autonomous. Consequently, LaSalle's theorem can be applied. From $\dot{\mathcal{U}} \equiv 0$, it follows that $\epsilon_i \equiv 0, i = 1, \cdots, n$, and consequently $\dot{\tilde{p}} \equiv 0$ and $\dot{\tilde{p}}_i \equiv 0$. In view of $\dot{\tilde{R}} \equiv 0$ and $\dot{\tilde{p}} \equiv 0$, it is clear that there exist constants $R^* \in SO(3)$ and $p^* \in \mathbb{R}^3$ such that $(\tilde{R}(t), \tilde{p}(t)) \to (R^*, p^*)$ as $t \to \infty$. This completes the proof. ■

*Remark 1:* It is important to note that the SLAM problem is not *observable* [19]. The best one can achieve is that there exist some constants $(R^*, p^*)$ such that $(\tilde{R}(t), \tilde{p}(t)) \to (R^*, p^*)$ and $\hat{p}_i(t) \to (R^*)^\top(p^* - p_i(t))$ as $t \to \infty$.

## IV. OBSERVER DESIGN WITH VELOCITY BIAS COMPENSATION

In this section, the previous observer will be extended to the case where the measurements $\Omega_y \in \mathfrak{so}(3)$ and $v_y \in \mathbb{R}^3$ of the angular velocity $\Omega$ and linear velocity $v$ contain unknown constant biases denoted as $b_\Omega \in \mathfrak{so}(3)$ and $b_v \in \mathbb{R}^3$, respectively. That is $\Omega_y = \Omega + b_\Omega$ and $v_y = v + b_v$.

Let $\mathcal{M}_b$ denote the sub-manifold of $\mathbb{R}^{(n+4)\times(n+4)}$ defined as

$$\mathcal{M}_b := \{b_U = \mathcal{P}_b(b_\Omega, b_v) | \ b_\Omega \in \mathfrak{so}(3), b_v \in \mathbb{R}^3\}. \quad (31)$$

with the map $\mathcal{P}_b : \mathfrak{so}(3) \times \mathbb{R}^3 \to \mathbb{R}^{(n+4)\times(n+4)}$ defined as

$$\mathcal{P}_b(b_\Omega, b_v) := \left[\begin{array}{cc|c} b_\Omega & b_v & \mathbf{0}_{3\times n} \\ \hline \mathbf{0}_{(n+1)\times 3} & \mathbf{0}_{n+1} & \mathbf{0}_{(n+1)\times n} \end{array}\right].$$

Note that $\mathcal{M}_b$ is also a subset of $\mathfrak{se}_{1+n}(3)$. Define the diagonal matrix $\mathbb{I} = \mathrm{diag}([1,1,1,1,\mathbf{0}_{1\times n}])$, such that for all $b_U \in \mathcal{M}_b$, $A \in \mathbb{R}^{(n+4)\times(n+4)}$, one has $\mathbb{P}(A)\mathbb{I} \in \mathcal{M}_b$ and

$$\langle\langle A, b_U\rangle\rangle = \langle\langle \mathbb{P}(A), b_U\rangle\rangle = \langle\langle \mathbb{P}(A)\mathbb{I}, b_U\rangle\rangle. \quad (32)$$

The group velocity measurement is given by $U_y := U + b_U$ with $b_U := \mathcal{P}_b(b_\Omega, b_v) \in \mathcal{M}_b$. Let $\hat{b}_\Omega$ and $\hat{b}_v$ be the estimates of $b_\Omega$ and $b_v$, respectively. The estimate of the velocity-bias $b_U$ is defined as $\hat{b}_U := \mathcal{P}_b(\hat{b}_\Omega, \hat{b}_v) \in \mathcal{M}_b$. Define the group velocity-bias error as

$$\tilde{b}_U := b_U - \hat{b}_U = \mathcal{P}_b(\tilde{b}_\Omega, \tilde{b}_v) \quad (33)$$

where $\tilde{b}_\Omega := b_\Omega - \hat{b}_\Omega$ and $\tilde{b}_v := b_v - \hat{b}_v$. Define the following compact set:

$$\bar{\mathcal{A}} = \mathcal{A} \times \{(\tilde{b}_\Omega, \tilde{b}_v) \in \mathfrak{so}(3) \times \mathbb{R}^3 \mid \tilde{b}_\Omega = 0, \tilde{b}_v = 0\}. \quad (34)$$

*Assumption 1:* Assume that among the $n \geq 3$ measured landmarks, at least three landmarks define a plane.

*Remark 2:* This assumption is needed to show the convergence of the bias estimation errors to zero. A similar assumption has been used in [8].

Now, one can state our second result.

*Theorem 2:* Consider the SLAM kinematics (13) with the system outputs (15), and the following observer

$$\begin{cases} \dot{\hat{X}} &= \hat{X}(U_y - \hat{b}_U - \Delta) \\ \dot{\hat{b}}_U &= -\mathbb{P}\left(\hat{X}^\top \sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top \hat{X}^{-\top}\right) K \\ \Delta &:= -\mathrm{Ad}_{\hat{X}^{-1}}\mathbb{P}\left(\sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top\right) \end{cases}, \quad (35)$$

where $\hat{X}(0) \in SE_{1+n}(3)$, $\hat{b}_U(0) \in \mathcal{M}_b$, $k_i > 0, \forall i = 1, \cdots, n$, and $K := \mathrm{diag}([k_\omega, k_\omega, k_\omega, k_v, \mathbf{0}_{1\times n}])$ with $k_\omega, k_v > 0$. Assume that $X$ and $U$ are bounded for all times. Then, the set $\bar{\mathcal{A}}$ is globally asymptotically stable. Moreover, there exist a constant matrix $R^* \in SO(3)$ and a constant vector $p^* \in \mathbb{R}^3$ such that $\lim_{t\to\infty} \tilde{R}(t) = R^*$ and $\lim_{t\to\infty} \tilde{p}(t) = p^*$.

*Proof:* Using the fact $U_y - \hat{b}_U = U + \tilde{b}_U$, it follows that $\dot{\tilde{b}}_U = -\dot{\hat{b}}_U$ and

$$\dot{\tilde{X}} = \tilde{X}(\mathrm{Ad}_{\hat{X}}(\Delta - \tilde{b}_U)). \quad (36)$$

From (19) and (35), one has the following closed-loop system:

$$\begin{cases} \dot{\tilde{X}} &= \tilde{X}(-\mathbb{P}((I - \tilde{X}^{-1})A) - \mathrm{Ad}_{\hat{X}}\tilde{b}_U) \\ \dot{\tilde{b}}_U &= \mathbb{P}\left(\hat{X}^\top (I - \tilde{X}^{-1})A\hat{X}^{-\top}\right) K \end{cases}. \quad (37)$$

Consider the following Lyapunov function candidate:

$$\mathcal{V}(\tilde{X}, \tilde{b}_U) = \mathcal{U}(\tilde{X}) + \frac{1}{2}\mathrm{tr}(\tilde{b}_U K' \tilde{b}_U^\top), \quad (38)$$

with $K' := \mathrm{diag}([1/k_\omega, 1/k_\omega, 1/k_\omega, 1/k_v, \mathbf{0}_{1\times n}])$ such that $K'K = KK' = \mathbb{I}$. In view of (33), $\mathcal{V}$ can be rewritten as

$$\mathcal{V}(\tilde{X}, \tilde{b}_U) = \mathcal{U}(\tilde{X}) + \frac{1}{2k_\omega}\|\tilde{b}_\Omega\|_F^2 + \frac{1}{2k_v}\|\tilde{b}_v\|^2,$$

which is positive definite with respect to $\bar{\mathcal{A}}$. In view of (5) and (17), the time-derivative of $\mathcal{V}$ along the trajectories of (37) is given by

$$\begin{aligned} \dot{\mathcal{V}} &= \langle \nabla_{\tilde{X}}\mathcal{U}, \tilde{X}(-\mathbb{P}((I - \tilde{X}^{-1})A) - \mathrm{Ad}_{\hat{X}}\tilde{b}_U)\rangle_{\tilde{X}} \\ &\quad + \langle\langle \mathbb{P}\left(\hat{X}^\top (I - \tilde{X}^{-1})A\hat{X}^{-\top}\right) KK', \tilde{b}_U\rangle\rangle \\ &= \langle\langle \mathbb{P}((I - \tilde{X}^{-1})A), (-\mathbb{P}((I - \tilde{X}^{-1})A) - \mathrm{Ad}_{\hat{X}}\tilde{b}_U)\rangle\rangle \\ &\quad + \langle\langle \mathbb{P}\left(\hat{X}^\top (I - \tilde{X}^{-1})A\hat{X}^{-\top}\right) \mathbb{I}, \tilde{b}_U\rangle\rangle, \end{aligned}$$

where we made use of the fact $KK' = \mathbb{I}$. In view of (32), one deduces

$$\begin{aligned} \dot{\mathcal{V}} &= -\|\mathbb{P}((I - \tilde{X}^{-1})A)\|_F^2 - \langle\langle (I - \tilde{X}^{-1})A, \hat{X}\tilde{b}_U\hat{X}^{-1}\rangle\rangle \\ &\quad + \langle\langle \mathbb{P}\left(\hat{X}^\top (I - \tilde{X}^{-1})A\hat{X}^{-\top}\right) \mathbb{I}, \tilde{b}_U\rangle\rangle \\ &= -\|\mathbb{P}((I - \tilde{X}^{-1})A)\|_F^2 - \langle\langle \hat{X}^\top (I - \tilde{X}^{-1})A\hat{X}^{-\top}, \tilde{b}_U\rangle\rangle \\ &\quad + \langle\langle \mathbb{P}\left(\hat{X}^\top (I - \tilde{X}^{-1})A\hat{X}^{-\top}\right) \mathbb{I}, \tilde{b}_U\rangle\rangle. \end{aligned}$$

From (21) and (32), one can easily show that

$$\begin{aligned} \dot{\mathcal{V}} &= -\|\mathbb{P}((I - \tilde{X}^{-1})A)\|_F^2 \\ &= -\left\|\sum_{i=1}^n k_i\epsilon_i\right\|^2 - \sum_{i=1}^n k_i^2\|\epsilon_i\|^2. \end{aligned} \quad (39)$$

Since $\dot{\mathcal{V}} \leq 0$, it is clear that $\mathcal{V}$ is non-increasing along the trajectories of (37), which implies that $\tilde{b}_\Omega, \tilde{b}_v$ and $\epsilon_i, i = 1, \cdots, n$ are bounded. Since the dynamics (37) are not autonomous, we will use Barbalat's lemma to show that $\dot{\mathcal{V}}$ converges to zero. In view of (9), (22) and (33), closed-loop system (37) can be rewritten as

$$\dot{\tilde{R}} = -\tilde{R}(\hat{R}\tilde{b}_\Omega\hat{R}^\top), \quad (40)$$

$$\dot{\tilde{p}} = \tilde{R}(\hat{R}\tilde{b}_\Omega\hat{R}^\top\hat{p} - \hat{R}\tilde{b}_v - \textstyle\sum_{i=1}^n k_i\epsilon_i), \quad (41)$$

$$\dot{\tilde{p}}_i = \tilde{R}(\hat{R}\tilde{b}_\Omega\hat{R}^\top\hat{p}_i - k_i\epsilon_i), \forall i = 1, 2, \cdots, n, \quad (42)$$

$$\dot{\tilde{b}}_\Omega = \tfrac{1}{2}k_\omega \textstyle\sum_{i=1}^n k_i\hat{R}^\top(-(\tilde{R}^\top(p - p_i))^\times\epsilon_i)^\times\hat{R}, \quad (43)$$

$$\dot{\tilde{b}}_v = k_v \textstyle\sum_{i=1}^n k_i\hat{R}^\top\epsilon_i, \quad (44)$$

where we made use of the fact $\hat{R}y_i = \tilde{R}^\top(p_i - p)$. In view of (40)-(42), one has

$$\begin{aligned} \dot{\epsilon}_i &= \dot{\tilde{R}}^\top(\tilde{p} - \tilde{p}_i) + \tilde{R}^\top(\dot{\tilde{p}} - \dot{\tilde{p}}_i) \\ &= \hat{R}\tilde{b}_\Omega\hat{R}^\top\epsilon_i + \hat{R}\tilde{b}_\Omega\hat{R}^\top(\hat{p} - \hat{p}_i) - \hat{R}\tilde{b}_v - \textstyle\sum_{j=1,j\neq i}^n k_j\epsilon_j \end{aligned}$$

$$= \tilde{R}^\top R(\tilde{b}_\Omega R^\top (p-p_i) - \tilde{b}_v) - \textstyle\sum_{j=1,j\neq i}^n k_j\epsilon_j, \qquad (45)$$

where we made use of the fact $\hat{R}^\top(\hat{p}-\hat{p}_i) = -\hat{R}^\top\epsilon_i + R^\top(p-p_i)$. Since $X$ and $U$ are bounded by assumption, it follows that $R$, $p-p_i, \forall i = 1,\cdots,n$ and their derivatives are bounded, which in turn implies the boundedness of $\dot{\epsilon}_i$. From (39), $\ddot{\mathcal{V}}$ is bounded. By virtue of Barbalat's lemma, one concludes that $\dot{\mathcal{V}}$ is uniformly continuous, and $\dot{\mathcal{V}}$ converges to zero, *i.e.*, $\lim_{t\to\infty}\epsilon_i(t) = 0$. In view of (40), (43) and (44), it follows that $\tilde{R}, \tilde{b}_\Omega$ and $\tilde{b}_v$ are bounded, and consequently $\ddot{\epsilon}_i$ is bounded according to (45). Therefore, since $\ddot{\epsilon}_i$ is bounded and $\epsilon_i$ converges to zero, it follows that $\lim_{t\to\infty}\dot{\epsilon}_i(t) = 0$. Since $\lim_{t\to\infty}\dot{\epsilon}_i(t) = 0$ and $\lim_{t\to\infty}\epsilon_i(t) = 0$, from (45), one has $\lim_{t\to\infty}\tilde{b}_\Omega R^\top(p-p_i) + \tilde{b}_v = 0$ for all $i = 1,\cdots,n$. Without loss of generality, let $p_1, p_2$ and $p_3$ be three non-collinear landmarks, *i.e.,* they define a plane. Then, one has

$$\begin{cases} \lim_{t\to\infty}\tilde{b}_\Omega R^\top(p_1-p_3) = 0, \\ \lim_{t\to\infty}\tilde{b}_\Omega R^\top(p_2-p_3) = 0. \end{cases}$$

Let $\tilde{b}_\omega \in \mathbb{R}^3$ be the vector such that $\tilde{b}_\omega^\times = \tilde{b}_\Omega$. Let $x_1 := p_1 - p_3$, $x_2 := p_2 - p_3$ and $x_3 := x_1 \times x_2$. Since $x_1$ and $x_2$ are non-collinear, it is clear that $\text{rank}([x_1, x_2, x_3]) = 3$. Using the facts $R^\top(x_1\times x_2) = (R^\top x_1)\times(R^\top x_2)$ and $\tilde{b}_\Omega R^\top x_3 = \tilde{b}_\omega \times R^\top(x_1\times x_2) = (\tilde{b}_\omega \times R^\top x_1)\times(\tilde{b}_\omega\times R^\top x_2)$, one has $\lim_{t\to\infty}\tilde{b}_\Omega x_3 = 0$, consequently, $\lim_{t\to\infty}\tilde{b}_\Omega[x_1\ x_2\ x_3] = 0$, which leads to $\lim_{t\to\infty}\tilde{b}_\Omega(t) = 0$ and $\lim_{t\to\infty}\tilde{b}_v(t) = 0$. In view of (40) and (41), one concludes that there exist a constant matrix $R^* \in SO(3)$ and a constant vector $p^* \in \mathbb{R}^3$ such that $(\tilde{R}, \tilde{p})$ converges to $(R^*, p^*)$ asymptotically. This completes the proof. ■

*Remark 3:* From (21) and (22), the observer in (35) can be rewritten, in terms of the measurements, as follows:

$$\begin{cases} \dot{\hat{R}} = \hat{R}(\Omega_y - \hat{b}_\Omega) \\ \dot{\hat{p}} = \hat{R}(v_y - \hat{b}_v) + \sum_{i=1}^n k_i(\hat{p}_i - \hat{p} - \hat{R}y_i) \\ \dot{\hat{p}}_i = \hat{R}v_i - k_i(\hat{p}_i - \hat{p} - \hat{R}y_i) \\ \dot{\hat{b}}_\Omega = -\frac{1}{2}k_\omega\sum_{i=1}^n k_i\hat{R}^\top((\hat{p}_i-\hat{p})^\times\hat{R}y_i)^\times\hat{R} \\ \dot{\hat{b}}_v = -k_v\hat{R}^\top\sum_{i=1}^n k_i(\hat{p}_i-\hat{p}-\hat{R}y_i) \end{cases}, \qquad (46)$$

where we made use of the fact $(\hat{R}y_i)^\times\epsilon_i = (\hat{R}y_i)^\times(\hat{p}_i - \hat{p} - \hat{R}y_i) = (\hat{R}y_i)^\times(\hat{p}_i - \hat{p}) = -(\hat{p}_i-\hat{p})^\times\hat{R}y_i$. Due to the integral action, the terms $\hat{b}_\Omega$ and $\hat{b}_v$ may grow arbitrarily large in the presence of measurement noise. To cope with this problem, in practical applications, one can use a projection mechanism as follows:

$$\dot{\hat{b}}_\omega = \text{Proj}_\delta\left(\hat{b}_\omega, -\tfrac{k_\omega}{2}\textstyle\sum_{i=1}^n k_i\hat{R}^\top((\hat{p}_i-\hat{p})^\times\hat{R}y_i)\right), \qquad (47)$$

$$\dot{\hat{b}}_v = \text{Proj}_\delta\left(\hat{b}_v, -k_v\hat{R}^\top\textstyle\sum_{i=1}^n k_i(\hat{p}_i-\hat{p}-\hat{R}y_i)\right), \qquad (48)$$

where $\hat{b}_\omega$ is the vector representation of $\hat{b}_\Omega$, and the projection map is defined as [20]

$$\text{Proj}_\delta(\hat{b}, \sigma_b) = \begin{cases} \sigma_b, & \hat{b}\in\Pi_\delta \text{ or } \nabla_{\hat{b}}\mathcal{P}^\top\sigma_b \geq 0 \\ \left(I - \varrho(\hat{b})\frac{\nabla_{\hat{b}}\mathcal{P}\nabla_{\hat{b}}\mathcal{P}^\top}{\nabla_{\hat{b}}\mathcal{P}^\top\nabla_{\hat{b}}\mathcal{P}}\right)\sigma_b, & \text{otherwise} \end{cases},$$

where $\mathcal{P}(\hat{b}) := \|\hat{b}\| - \delta$, $\Pi_\delta = \{\hat{b}|\mathcal{P}(\hat{b}) \leq 0\}$, $\Pi_{\delta,\varepsilon} = \{\hat{b}|\mathcal{P}(\hat{b}) \leq \varepsilon\}$ and $\varrho(\hat{b}) := \min\{1, \mathcal{P}(\hat{b})/\varepsilon\}$ for some positive parameters $\delta$ and $\varepsilon$.

## V. Simulation

In this section, the performance of the proposed observer (35) with (47) and (48) is illustrated by a numerical simulation. We consider a vehicle moving on a 10-meter diameter circle at 10-meter height, along the trajectory $p(t) = 10[\cos(0.2t)\ \sin(0.2t)\ 1]^\top$. The rotation matrix is initialized as $R(0) = I_3$ and the angular velocity is given by $\omega(t) = [0\ 0\ 1]^\top$. In addition, a set of landmarks located on the ground are available for measurements. Moreover, the rotational velocity and translational velocity measurements are corrupted by the following constant biases: $b_\omega = [-0.02\ 0.02\ 0.01]^\top, b_v = [0.2\ \ -0.1\ 0.1]^\top$. The estimates are initialized as $\hat{R}(0) = \exp(0.2\pi u^\times)$ with $u = [0\ 0\ 1]^\top$, $\hat{p}(0) = 0$, $\hat{b}_\omega = 0$ and $\hat{b}_v = 0$. The initial positions of the landmarks are randomly selected. The gains involved in the proposed observer are chosen as $k_\omega = 0.02, k_v = 1, \delta = 0.5, \varepsilon = 0.1$ and $k_i = 5/22$ for all $i = 1,\cdots,22$.

The simulation results are given in Fig.1, showing the convergence to zero of landmark and biases estimation errors, and the convergence of $\|I - \tilde{R}\|_F$ and $\|\tilde{p}\|$ to some constant values as discussed earlier.

## VI. Conclusions

A geometric nonlinear observer for SLAM in terms of group velocity and landmark measurements has been derived directly on the matrix Lie group $SE_{1+n}(3)$. An extension to the case of biased linear and angular velocity measurements has also been proposed. As illustrated in the simulation results, the proposed approach is capable of mapping an unknown environment and providing a relative localization of the vehicle based on the measured landmarks and linear and angular velocities.

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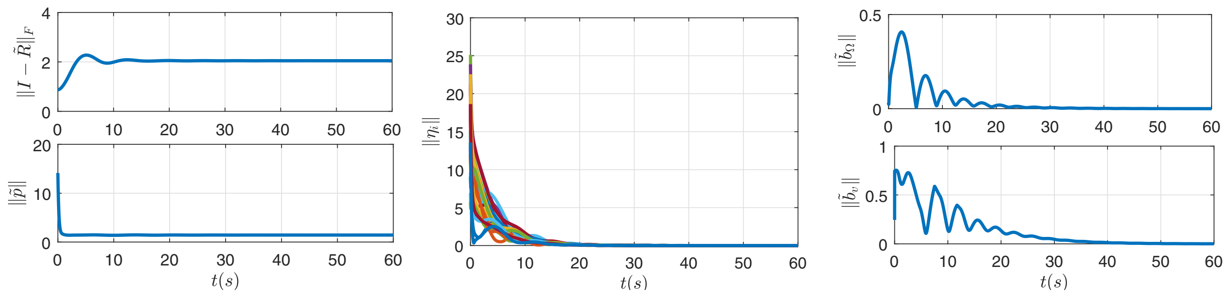


Fig. 1: Estimation errors of rotation, position and landmarks with respect to time.


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## APPENDIX

### A. Proof of Lemma 1

The gradient $\nabla_X\mathcal{U}$ can be computed using the differential of $\mathcal{U}$ in an arbitrary tangential direction $XU \in T_X SE_{1+n}(3)$ with some $U \in \mathfrak{se}_{1+n}(3)$

$$\begin{aligned} d\mathcal{U}\cdot XU &= \langle \nabla_X\mathcal{U}, XU\rangle_X \\ &= \langle XX^{-1}\nabla_X\mathcal{U}, XU\rangle_X \\ &= \langle\langle X^{-1}\nabla_X\mathcal{U}, U\rangle\rangle. \end{aligned} \tag{49}$$

On the other hand, from the definition of the tangent map, one has

$$\begin{aligned} d\mathcal{U}\cdot XU &= \mathrm{tr}\left(-XUA(I_4-X)^\top\right) \\ &= \langle\langle \mathbb{P}(X^\top(X-I)A), U\rangle\rangle \\ &= \langle\langle \mathbb{P}(X^{-1}(X-I)A), U\rangle\rangle \\ &= \langle\langle \mathbb{P}((I-X^{-1})A), U\rangle\rangle, \end{aligned} \tag{50}$$

where the fact $(I-X^{-1})A \in \mathcal{M}_0$ and property (7) has been used. Hence, in view of (49) and (50), one has (17). This completes the proof.

### B. Proof of Lemma 2

From the definition of $r_i, b_i$ and (15), one has

$$\begin{aligned} \textstyle\sum_{i=1}^n k_i\|r_i - \hat{X}b_i\|^2 &= \textstyle\sum_{i=1}^n k_i\|(I-\tilde{X}^{-1})r_i\|^2 \\ &= \mathrm{tr}((I-\tilde{X}^{-1})A(I-\tilde{X}^{-1})^\top) \\ &= \mathrm{tr}(\tilde{X}^\top\tilde{X}(I-\tilde{X}^{-1})A(I-\tilde{X}^{-1})^\top) \\ &= \mathrm{tr}((I-\tilde{X})A(I-\tilde{X})^\top), \end{aligned}$$

where we made use of the fact $\|u\|^2 = \mathrm{tr}(uu^\top)$ for all $u \in \mathbb{R}^{n+4}$ and property (8). Similarly, one verifies that

$$\mathbb{P}\left(\textstyle\sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top\right) = \mathbb{P}((I-\tilde{X}^{-1})A).$$

On the other hand, using the fact $r_i - \hat{X}b_i = [\epsilon_i^\top\ \mathbf{0}_{1\times(n+1)}]^\top$, one can easily verify (20). Moreover, one can further show that

$$\begin{aligned} &\textstyle\sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top \\ &= \begin{bmatrix} \mathbf{0}_{3\times3} & \sum_{i=1}^n k_i\epsilon_i & -k_1\epsilon_1 & \cdots & -k_n\epsilon_n \\ \mathbf{0}_{(n+1)\times3} & \mathbf{0}_{n+1} & \mathbf{0}_{n+1} & \cdots & \mathbf{0}_{n+1} \end{bmatrix}, \\ &\hat{X}^\top \textstyle\sum_{i=1}^n k_i(r_i - \hat{X}b_i)r_i^\top\hat{X}^{-\top} \\ &= \begin{bmatrix} \hat{R}^\top\Theta\hat{R} & \sum_{i=1}^n k_i\hat{R}^\top\epsilon_i & -k_1\hat{R}^\top\epsilon_1 & \cdots & -k_n\hat{R}^\top\epsilon_n \\ * & * & * & \cdots & * \end{bmatrix}, \end{aligned}$$

where $*$ denotes some irrelevant terms, and $\Theta := \left(\sum_{i=1}^n k_i\epsilon_i\hat{p}^\top - \sum_{i=1}^n k_i\epsilon_i\hat{p}_i^\top\right) = \sum_{i=1}^n k_i\epsilon_i(\hat{p}-\hat{p}_i)^\top$. Let $\Theta_a := \mathbb{P}_a(\hat{R}^\top\Theta\hat{R})$. From the definition of the map $\mathbb{P}_a(\cdot)$ and $\Theta$, one has

$$\begin{aligned} \Theta_a &= \tfrac{1}{2}\textstyle\sum_{i=1}^n k_i\hat{R}^\top\left((\hat{p}-\hat{p}_i)^\times\epsilon_i\right)^\times\hat{R} \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^n k_i(\hat{R}^\top\epsilon_i^\times\hat{R}y_i)^\times, \end{aligned}$$

where we made use of the facts: $\hat{p}-\hat{p}_i = -\hat{R}y_i - \epsilon_i$ and $yx^\top - xy^\top = (x^\times y)^\times, R^\top x^\times R = (R^\top x)^\times, x^\times y = -y^\times x, x^\times x = 0$ for all $x, y \in \mathbb{R}^3$ and $R \in SO(3)$. From (6), one obtains (21) and (22). This completes the proof.