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A New Hybrid Control Strategy for the Global Attitude Tracking Problem

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***Abstract*— This paper presents a new hybrid feedback design strategy for the attitude tracking problem on $SO(3)$ guaranteeing global asymptotic stability. The proposed hybrid controller is derived from a modified potential function on $SO(3)\times\mathbb{R}$ involving a virtual state variable with hybrid flow and jump dynamics. We propose a new resetting mechanism that keeps the state away from the undesired critical points while, at the same time, guaranteeing a decrease of the potential function. A systematic design of the instrumental potential function is provided. Numerical results are presented to illustrate the performance of the proposed hybrid controller.**

I. Introduction

Recently, geometric control design on Lie groups such as $SO(3)$ and $SE(3)$, has generated a great deal of research work in the literature [1]–[4]. It is well known that achieving global stability results with feedback control schemes designed on Lie groups such as $SO(3)$ and $SE(3)$, is a fundamentally difficult task due to the topological obstruction of the motion space induced by the fact that these manifolds are not homeomorphic to $\mathbb{R}^n$ and that there is no smooth vector field that can have a global attractor [1]. To achieve almost global asymptotic stability (AGAS), a class of *suitable* "navigation functions" has been introduced in [1]. In [2], the Riemannian structure of the configuration manifold for a class of mechanical systems, is used to derive a local exponential control law, while in [4], an almost global tracking controller has been proposed for a general class of Lie groups via intrinsic globally-defined error dynamics. Thus, in general, when considering continuous time-invariant state-feedback control laws, it is impossible to achieve global stability results on Lie groups, and as such, almost global stability is the strongest result one can aim at in this case. In particular, for any smooth potential function on $SO(3)$, there exist at least four critical points where its gradient vanishes as per Morse theory [5]. Motivated by the framework of hybrid systems [6], [7], the recent work in [8] proposed a hybrid strategy that uses a "synergistic" family of potential functions to overcome the topological obstruction on $SO(3)$ and achieve global asymptotic stability results. This family of potential functions is called "centrally" synergistic, if the identity is the common critical point of all the potential functions in the family. This synergistic hybrid approach was successfully applied to the rigid body attitude control problem in [9], where a hysteresis-based switching mechanism was introduced to avoid all the undesired critical points and ensure some robustness to measurement noise. However, only numerical examples were provided to construct such a synergistic family of potential functions via angular warping on $SO(3)$. Motivated by the work in [8], [9], several hybrid controllers and observers on $SO(3)$ and $SE(3)$ have been proposed in the literature [10]–[18]. The work in [10], provides a systematic and comprehensive procedure for the construction of synergistic potential functions, which is then used for the design of a velocity-free hybrid attitude stabilization scheme. In [15], a hybrid attitude control scheme on $SO(3)$ with global exponential stability has been proposed using the concept of "exp-synergistic" potential functions. These results are further extended to solve the problem of hybrid output attitude tracking in [11]. On the other hand, non-central synergistic potential functions have been considered in [12], [13] to relax the centrality assumption. Another technique inspired from the hybrid synergistic approach appeared in [19] using local coordinate charts to design a controller that stabilizes a set-point through gradient-based feedback during the flows, and appropriately designed jumps near the undesired critical points.

In this paper, we aim at developing a new type of hybrid feedback control laws, relying on a modified potential function on $SO(3)\times\mathbb{R}$, for the attitude tracking problem leading to global asymptotic stability results. In contrast to the hybrid controllers designed from a synergistic family of potential functions on $SO(3)$, where at least two synergistic potential functions are required, the proposed hybrid controller is designed using a single potential function. A min-resetting mechanism, that resets the virtual scalar state, near the undesired critical points, to the one leading to the minimum value of the potential function, is designed to drive the state in $SO(3)\times\mathbb{R}$ away from the undesired critical points. In contrast to the synergistic approaches which cannot be extended in a straightforward manner to other Lie groups (*e.g.,* $SE(3)$), the proposed transformation map in this paper does not require the "diffeomorphism" condition (required in the synergistic approaches), which makes the proposed approach a good candidate for hybrid control design on general matrix Lie groups.

The remainder of this paper is organized as follows: Section II provides the preliminary background related to the problem at hand. In Section III, we formulate the attitude tracking problem. Section IV is devoted to the design of the hybrid feedback on $SO(3)$ leading to global asymptotic stability. Simulation results are presented in Section V.

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Section VI wraps up the paper with a conclusion.

II. Background

A. Notations and mathematical preliminaries

The sets of real and nonnegative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and $\mathbb{S}^n$ the set of $n+1$-dimensional unit vectors. Given two matrices, $A, B \in \mathbb{R}^{m\times n}$, their Euclidean inner product is defined as $\langle\langle A, B\rangle\rangle = \mathrm{tr}(A^\top B)$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x^\top x}$. Let I_n denote the n-dimensional identity matrix and $A \in \mathbb{R}^{n\times n}$ denote a square $n \times n$ matrix with real entries $[a_{ij}], i, j = 1, 2\cdots, n$. For each $A \in \mathbb{R}^{n\times n}$, we define $\mathcal{E}(A)$ as the set of all unit-eigenvectors of A. Let λ_i^A and v_i^A be the i-th eigenvalue and eigenvector of A. Let $e_1, \cdots, e_n$ be the standard basis vectors of $\mathbb{R}^n$, that is, e_i has all entries equal to zero except for the ith entry which equals to 1.

B. Representation on $SO(3)$

Consider the 3-dimensional *Special Orthogonal group* denoted by $SO(3)$ as $SO(3) := \{R \in \mathbb{R}^{3\times3} | R^\top R = I_3, \det(R) = 1\}$. The *Lie algebra* of $SO(3)$, denoted by $\mathfrak{so}(3)$ is given by $\mathfrak{so}(3) := \left\{\Omega \in \mathbb{R}^{3\times3} | \Omega^\top = -\Omega\right\}$. The *tangent spaces* of the group $SO(3)$ is identified by $T_R SO(3) := \{R\Omega |\ R \in SO(3), \Omega \in \mathfrak{so}(3)\}$. Define the skew map $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ projecting the 3-dimensional vector on $\mathfrak{so}(3)$. Let $\times$ be the vector cross-product on $\mathbb{R}^3$, then one has $x \times y = x^\times y$ with any $x, y \in \mathbb{R}^3$. Let vec : $\mathfrak{so}(3) \to \mathbb{R}^3$ be the inverse isomorphism of the map $(\cdot)^\times$, such that $\mathrm{vec}(\omega^\times) = \omega$ and $(\mathrm{vec}(\Omega))^\times = \Omega$ for all $\omega \in \mathbb{R}^3$ and $\Omega \in \mathfrak{so}(3)$. For a matrix $A \in \mathbb{R}^{3\times3}$, we denote by $\mathbb{P}_a : \mathbb{R}^{3\times3} \to \mathfrak{so}(3)$ the anti-symmetric projection of A, such that $\mathbb{P}_a(A) := (A - A^\top)/2$. Define the composition map $\psi := \mathrm{vec} \circ \mathbb{P}_a$ such that, for a matrix $A = [a_{ij}] \in \mathbb{R}^{3\times3}$, one has $\psi(A) := \mathrm{vec}(\mathbb{P}_a(A)) = \frac{1}{2}\left[a_{32} - a_{23}, a_{13} - a_{31}, a_{21} - a_{12}\right]^\top$. For any $A \in \mathbb{R}^{3\times3}, x \in \mathbb{R}^3$, one can verify that $\langle\langle A, x^\times\rangle\rangle = 2x^\top \psi(A)$. The adjoint action map for $SO(3)$ is given as $\mathrm{Ad}_R = R$, and for all $R \in SO(3)$ and $x \in \mathbb{R}^3$, the following property holds: $Rx^\times R^\top = (Rx)^\times$. For any $R \in SO(3)$, we define $|R|_I \in [0, 1]$ as the normalized Euclidean distance on $SO(3)$ with respect to the identity I_3, which is given by $|R|_I^2 = \mathrm{tr}(I_3 - R)/4$. Let the map $\mathcal{R}_a : \mathbb{R} \times \mathbb{S}^2 \to SO(3)$ represent the well-known angle-axis parametrization of the attitude, which is given by $\mathcal{R}_a(\theta, u) := \exp(\theta u^\times) = I_3 + \sin\theta u^\times + (1 - \cos\theta)(u^\times)^2$.

Consider a continuously differentiable function $V : SO(3) \to \mathbb{R}_{\geq 0}$ as a potential function on $SO(3)$ with respect to the set $\{I_3\}$. The gradient of V at R, denoted by $\nabla_R V : R \to T_R SO(3)$. A point $R \in SO(3)$ is a critical point of V if $\nabla_R V(R) = 0$. The set of all the critical points of V is defined as $\Psi_V = \{R \in SO(3) | \nabla_R V(R) = 0\} \subset SO(3)$, and the set of undesired critical points is defined as $\Psi_V / \{I_3\}$. From the Lusternik-Schnirelmann theorem [20] and Morse theory [5], there exist at least four critical points of any smooth function on $SO(3)$.

III. Problem Statement

The dynamic equations of motion of a rigid body on $SO(3)$ is given by

$$\dot{R} = R\omega^\times \tag{1}$$
$$J\dot{\omega} = -\omega^\times J\omega + \tau \tag{2}$$

where R denotes the attitude of the rigid body, $\omega \in \mathbb{R}^3$ is the body-frame angular velocity, $J = J^\top \in \mathbb{R}^{3\times3}$ is the inertia matrix, $\tau \in \mathbb{R}$ is the control torque.

Let $M > 0$ and $\mathbb{B} := \{x \in \mathbb{R}^n, \|x\| < 1\}$ be the closed unit ball in $\mathbb{R}^m$. The desired reference trajectories are generated by the system [12]

$$\left.\begin{aligned} \dot{R}_r &= R_r\omega_r^\times \\ \dot{\omega}_r &= z \\ z\ &\in M\mathbb{B} \end{aligned}\right\} (R_r, \omega_r) \in \mathcal{W}_d. \tag{3}$$

As pointed out in [12], $\mathcal{W}_d$ is always viable since $0 \in M\mathbb{B}$, and every maximal solution to (3) is complete (has unbounded domain [7, Proposition 2.10]) since $\mathcal{W}_d$ is compact. Also, any possible solution component ω_r of (3) is Lipschitz continuous with Lipschitz constant M, but not necessarily differentiable. We assume that (R_r, ω_r, z) of the reference trajectory are measurable at any instant of time.

Let us define the right invariant attitude tracking error $R_e := R_r^\top R$ and the geometric angular velocity error as $\omega_e := \omega - R_e^\top \omega_r$. In view of (1)-(3), one obtains the following error dynamics [12]:

$$\dot{R}_e = R_e\omega_e^\times \tag{4}$$
$$J\dot{\omega}_e = \Sigma(R_e, \omega_e, \omega_r)\omega_e - \Upsilon(R_e, \omega_r, z) + \tau, \tag{5}$$

where the feedforward torque $\Upsilon : SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\Upsilon(R_e, \omega_r, z) = JR_e^\top z + (R_e^\top \omega_r)^\times JR_e^\top \omega_r$, and the function $\Sigma : SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathfrak{so}(3)$ is given by $\Sigma(R_e, \omega_e, \omega_r) = (J\omega_e)^\times + (JR_e\omega_r)^\times - ((R_e^\top \omega_r)^\times J + J(R_e^\top \omega_r)^\times)$.

Define the extended state space and state as $\mathcal{X} := SO(3) \times \mathbb{R}^3 \times \mathcal{W}_d$ and $\zeta := (R_e, J\omega_e, R_r, \omega_r) \in \mathcal{X}$, respectively. In view of (3)-(5), one obtains the following system:

$$\left\{\begin{aligned} \dot{R}_e &= R_e\omega_e^\times \\ J\dot{\omega}_e &= \Sigma(R_e, \omega_e, \omega_r)\omega_e - \Upsilon(R_e, \omega_r, z) + \tau \\ \dot{R}_r &= R_r\omega_r^\times \\ \dot{\omega}_r &= z \\ z\ &\in M\mathbb{B} \end{aligned}\right. . \tag{6}$$

The objective in this paper, consists in designing a hybrid feedback control input τ such that $(R_e = I_3, \omega_e = 0)$ is globally asymptotically stable.

IV. Main Results

A. Hybrid Feedback Using Min-Resetting Mechanism

In this paper, we make use of the framework of hybrid dynamical systems presented in [6], [7]. As mentioned before, the obstruction to global asymptotic stability on $SO(3)$, is due to the fact that any smooth potential function on $SO(3)$ admits at least four critical points, and that there is no a global attractor for smooth vector fields on $SO(3)$.

To overcome this issue, a family of synergistic potential functions relying on the angular warping techniques and a hysteresis "min-switching mechanism", have been proposed in [8]–[12]. In this subsection, we propose a new hybrid feedback strategy based on a modified potential function on $SO(3) \times \mathbb{R}$ and a hysteresis "min-resetting mechanism".

Let us introduce the virtual state $\theta \in \mathbb{R}$. Define the new compact set $\mathcal{A}_o := \{I_3\} \times \{0\} \subset SO(3) \times \mathbb{R}$. Let $U : SO(3) \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ be the potential function on $SO(3) \times \mathbb{R}$ with respect to the set $\mathcal{A}_o$. Consider the trajectory $\dot{R} = R\omega^\times$ and $\dot{\theta} = v$ with $R(0) \in SO(3), \omega \in \mathbb{R}^3$ and $\theta(0), v \in \mathbb{R}$. One can show that

$$\begin{aligned}\dot{U}(R,\theta) &= \langle\langle \nabla_R U(R,\theta), R\omega^\times \rangle\rangle + \langle\langle \nabla_\theta U(R,\theta), v \rangle\rangle \\ &= 2\omega^\top \psi(R^\top \nabla_R U(R,\theta)) + v\nabla_\theta U(R,\theta) \end{aligned} \tag{7}$$

where we made use of property $\langle\langle A, x^\times \rangle\rangle = 2x^\top \psi(A)$. Thus, no matter what the values of ω, v are, when $\psi(R^\top \nabla_R U(R,\theta)) = \nabla_\theta U(R,\theta) = 0$, there is no infinitesimal change in U. We define the set of critical points of U, denoted by Ψ_U, as follows:

$$\begin{aligned}\Psi_U := &\{(R,\theta) \in SO(3) \times \mathbb{R} : \\ &\psi(R^\top \nabla_R U(R,\theta)) = \nabla_\theta U(R,\theta) = 0\}. \end{aligned} \tag{8}$$

Given a nonempty and finite real set $\Theta \subset \mathbb{R}$, define the map $\mu_U : SO(3) \times \mathbb{R} \to \mathbb{R}$ as

$$\mu_U(R,\theta) := U(R,\theta) - \min_{\bar{\theta} \in \Theta} U(R,\bar{\theta}). \tag{9}$$

Assumption 1: Suppose that there exists a potential function U on $SO(3) \times \mathbb{R}$ with respect to $\mathcal{A}_o$, and let $\Theta \subset \mathbb{R}$ be a nonempty finite set such that $\mathcal{A}_o \subset \Psi_U$, and $\mu_U(R,\theta) > \delta, \forall (R,\theta) \in \Psi_U/\mathcal{A}_o$ with some gap $\delta > 0$.

Remark 1: Assumption 1 implies that for any point $(R,\theta) \in \Psi_U/\mathcal{A}_o$, there exists another state $(R,\bar{\theta})$ such that $U(R,\bar{\theta})$ is lower than $U(R,\theta)$ by a constant gap δ. Note that the modified potential function $U(R,\theta)$ is similar to the family of synergistic potential functions $U(R,q)$ defined in [10], [12] with q being a index variable. The main difference is that the variable θ considered in this paper is real-valued and composed of continuous flows and discrete jumps. In other worlds, an additional hybrid scalar variable θ is used instead of using a family of synergistic potential functions as [10], [12].

We consider the following hybrid dynamics for θ :

$$\begin{cases} \dot{\theta} \;\; = f(R_e,\theta) & (R_e,\theta) \in \mathcal{F} \\ \theta^+ \in g(R_e,\theta) & (R_e,\theta) \in \mathcal{J} \end{cases} \tag{10}$$

where $\theta(0) \in \mathbb{R}$, and the flow map $f : SO(3) \times \mathbb{R} \to \mathbb{R}$ and the jump map $g : SO(3) \times \mathbb{R} \rightrightarrows \Theta$ are defined as

$$f(R_e,\theta) := -k_\theta \nabla_\theta U(R_e,\theta) \tag{11}$$

$$g(R_e,\theta) := \{\theta \in \Theta : \mu_U(R_e,\theta) = 0\} \tag{12}$$

with $k_\theta > 0$, $\Theta \subset \mathbb{R}$ denoting a finite and nonempty set, and the flow and jump sets are defined as

$$\mathcal{F} = \{(R_e,\theta) \in SO(3) \times \mathbb{R} : \mu_U(R_e,\theta) \leq \delta\} \tag{13}$$

$$\mathcal{J} = \{(R_e,\theta) \in SO(3) \times \mathbb{R} : \mu_U(R_e,\theta) \geq \delta\}. \tag{14}$$

For the sake of simplicity, define the new state space $\mathcal{S} := \mathcal{X} \times \mathbb{R}$ and the new state $x := (\zeta, \theta) \in \mathcal{S}$. We propose the following hybrid controller:

$$\underbrace{\begin{aligned}\tau &= \Upsilon(R_e,\omega_r,z) - \kappa(R_e,\theta,\omega_e) \\ \dot{\theta} &= f(R_e,\theta)\end{aligned}}_{x \in \mathcal{F}_c} \quad \underbrace{\theta^+ \in g(R_e,\theta)}_{x \in \mathcal{J}_c} \tag{15}$$

where the map $\kappa : SO(3) \times \mathbb{R} \times \mathbb{R}^3$ is given by

$$\kappa(R_e,\theta,\omega_e) := 2k_R \psi(R_e^\top \nabla_{R_e} U(R_e,\theta)) + k_\omega \omega_e, \tag{16}$$

with $k_R, k_\omega > 0$, and the flow and jump sets are given by $\mathcal{F}_c := \{x \in \mathcal{S} : (R_e,\theta) \in \mathcal{F}\}$ and $\mathcal{J}_c := \{x \in \mathcal{S} : (R_e,\theta) \in \mathcal{J}\}$, respectively. It is easy to verify that $\mathcal{F}_c$ and $\mathcal{J}_c$ are closed, and $\mathcal{F}_c \cup \mathcal{J}_c = \mathcal{S}$. Letting $\mathcal{U} := (\tau,\kappa,f,g)$ and $\mathcal{K} := (\Theta, \delta, k_\theta, k_R, k_\omega)$, the closed-loop system can be written as

$$\mathcal{H}(U,\mathcal{K}) : \begin{cases} \dot{x} \;\; \in F(x,\mathcal{U},\mathcal{K}) & x \in \mathcal{F}_c \\ x^+ \in G(x,\mathcal{U},\mathcal{K}) & x \in \mathcal{J}_c \end{cases} \tag{17}$$

where the flow and jump maps are given by

$$F = \begin{bmatrix} R_e\omega_e^\times \\ \Sigma(R_e,\omega_e,\omega_r)\omega_e - \kappa(R_e,\theta,\omega_e) \\ R_r\omega_r^\times \\ M\mathbb{B} \\ f(R_e,\theta) \end{bmatrix}, G = \begin{bmatrix} R_e \\ J\omega_e \\ R_r \\ \omega_r \\ g(R_e,\theta) \end{bmatrix}$$

where the maps f, g and κ are defined in (11), (12) and (16) respectively. Note that the closed-loop system (17) satisfies the hybrid basic conditions of [6], [7]. Define the following compact set $\mathcal{A} := \{x \in \mathcal{S} : (R_e,\theta) \in \mathcal{A}_o, \omega_e = 0\}$. Then, one can state the following result:

Theorem 1: Let $\Theta \subset \mathbb{R}$ be a finite nonempty set and let U be a potential function on $SO(3) \times \mathbb{R}$ with respect to $\mathcal{A}_o$. Consider the input vectors $\mathcal{U}$ and let $\mathcal{K}$ satisfy Assumption 1 and $k_\theta, k_R, k_\omega > 0$. Then, the set $\mathcal{A}$ is globally asymptotically stable for the closed-loop system $\mathcal{H}$ (17) and the number of jumps is finite.

Proof: See the proof in Appendix A. ■

B. Construction of the potential function on $SO(3) \times \mathbb{R}$

In the previous subsection, it was shown that once a potential function U with respect to $\mathcal{A}_o$, satisfying Assumption 1, is obtained, a hybrid feedback controller (15), leading to global asymptotic stability, immediately follows. In this subsection, we will provide a systematic procedure for the construction of the potential function U using the angular warping techniques inspired by [8].

Consider the following new transformation map $\mathcal{T} : SO(3) \times \mathbb{R} \to SO(3)$:

$$\mathcal{T}(R,\theta) = R\exp(\theta u^\times) \tag{18}$$

where $u \in \mathbb{S}^2$ denotes a constant unit vector and $\theta \in \mathbb{R}$ is a real-valued variable whose dynamics are given by (10). $\mathcal{T}$ applies a rotation in the amount of θ to R about the unit vector u. Note that the main difference compared to the transformation maps considered in [8]–[10], is that the angular warping angle θ considered in this paper is a real-valued variable with hybrid flow and jump dynamics. Note also that one has exactly the same transformation map as in [10] if θ is chosen as a function of R, *i.e.,* $\theta = \vartheta(R)$ with $\vartheta : SO(3) \to \mathbb{R}$.

Consider the modified trace function $V(R) = \text{tr}(A(I - R))$ with $A = A^\top$ being positive semidefinite, which is commonly used as a potential function on $SO(3)$. Some useful properties of the potential function V are given in [10, Lemma 2]. Let us introduce the following modified potential function on $SO(3) \times \mathbb{R}$

$$U(R, \theta) = \text{tr}(A(I - \mathcal{T}(R, \theta))) + \frac{\gamma}{2}\theta^2 \tag{19}$$

where $\gamma > 0$. From (18), one can easily verify that $U(R, \theta) \geq 0$ for all $(R, \theta) \in SO(3) \times \mathbb{R}$, and $U(R, \theta) = 0$ if and only if $(R, \theta) \in \mathcal{A}_o$. The following Theorem provides useful properties of the potential function U.

Theorem 2: Let $\gamma > 0$, $u \in \mathbb{S}^2$ and $A = A^\top$ such that $W = \text{tr}(A)I - A$ is positive definite. Consider the potential function $V(R) = \text{tr}(A(I - R))$ with Ψ_V being the set of critical points of V. Consider the transformation map $\mathcal{T}$ defined in (18), the potential function U defined in (19), and the trajectory $\dot{R} = R\omega^\times$ and $\dot{\theta} = v$. Then, for all $(R, \theta) \in SO(3) \times \mathbb{R}$ the following statements hold:

$$\dot{\mathcal{T}}(R, \theta) = \mathcal{T}(R, \theta)(\mathcal{R}_a(\theta, u)^\top \omega + vu)^\times \tag{20}$$

$$\psi(R^\top \nabla_R U(R, \theta)) = \mathcal{R}_a(\theta, u)\psi(A\mathcal{T}(R, \theta)) \tag{21}$$

$$\nabla_\theta U(R, \theta) = \gamma\theta + 2u^\top \psi(A\mathcal{T}(R, \theta)) \tag{22}$$

$$\Psi_U = \Psi_V \times \{0\} \tag{23}$$

$$\mathcal{A}_o \subset \Psi_U. \tag{24}$$

Moreover, for any $v \in \mathcal{E}(A)$ and $\theta \in \mathbb{R}$, one has

$$U(\mathcal{R}_a(\pi, v), 0) = 2\lambda_v^W \tag{25}$$

$$U(\mathcal{R}_a(\pi, v), \theta) = 2\lambda_v^W - 2\sin^2\left(\tfrac{\theta}{2}\right)\Delta(v, u) + \tfrac{\gamma}{2}\theta^2 \tag{26}$$

where the map $\Delta(u, v)$ is defined as $\Delta(u, v) := u^\top\left(\text{tr}(A)I - A - 2v^\top Av(I_3 - vv^\top)\right)u$, and λ_v^W denotes the eigenvalue of W associated to the eigenvector $v \in \mathcal{E}(W) \equiv \mathcal{E}(A)$.

Proof: See the proof in Appendix B. ■

Remark 2: In view of (21) and (22), the maps f defined in (11) and κ defined in (16) can be rewritten as

$$\kappa(R_e, \theta, \omega_e) = 2k_R\mathcal{R}_a(\theta, u)\psi(A\mathcal{T}(R_e, \theta)) + k_\omega\omega_e \tag{27}$$

$$f(R_e, \theta) = -k_\theta(\gamma\theta + 2u^\top\psi(A\mathcal{T}(R_e, \theta))). \tag{28}$$

Note that (21) is different from the results given in [8, Theorem 6]. It turns out that no diffeomorphism condition is required on the transformation map $\mathcal{T}$ defined in (18).

Define $\mathcal{P}_U := \{\Theta, A, u, \gamma, \delta\}$ as a set of parameters associated to the design of the potential function U. Let (λ_i^A, v_i^A) be the i-the pair of eigenvalue-eigenvector of the matrix A. Consider the set $\mathcal{P}_U$ given as follows:

$$\mathcal{P}_U : \begin{cases} \Theta = \{|\theta_i| \in (0, \pi], i = 1, \cdots, m\} \\ A : 0 < \lambda_1^A \leq \lambda_2^A < \lambda_3^A \\ u = \alpha_1 v_1^A + \alpha_2 v_2^A + \alpha_3 v_3^A \\ \gamma < \frac{4\Delta^*}{\pi^2} \\ \delta < (\frac{4\Delta^*}{\pi^2} - \gamma)\frac{\theta_M^2}{2}, \ \theta_M := \max\limits_{\theta \in \Theta} |\theta| \end{cases} \tag{29}$$

where $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $\Delta^* > 0$ are given as follows:

1) if $0 < \lambda_1^A = \lambda_2^A < \lambda_3^A$, one has $\alpha_3^2 = 1 - \frac{\lambda_2^A}{\lambda_3^A}, \Delta^* = \lambda_1^A(1 - \frac{\lambda_2^A}{\lambda_3^A})$.
2) if $0 < \lambda_1^A < \lambda_2^A < \lambda_3^A$ and $\lambda_2^A > \frac{\lambda_1^A\lambda_3^A}{\lambda_3^A - \lambda_1^A}$, one has $\alpha_2^2 = \frac{\lambda_2^A}{\lambda_2^A + \lambda_3^A}, \alpha_3^2 = \frac{\lambda_3^A}{\lambda_2^A + \lambda_3^A}, \Delta^* = \lambda_1^A$.
3) if $0 < \lambda_1^A < \lambda_2^A < \lambda_3^A$ and $\lambda_2^A \leq \frac{\lambda_1^A\lambda_3^A}{\lambda_3^A - \lambda_1^A}$, one has $\alpha_i^2 = 1 - 4\prod_{j \neq i}\lambda_j^A/(\sum_{l=1}^3\sum_{k \neq l}^3\lambda_l^A\lambda_k^A), i \in \{1, 2, 3\}$, and $\Delta^* = 4\prod_j\lambda_j^A/\sum_{l=1}^3\sum_{k \neq l}^3\lambda_l^A\lambda_k^A$.

From [10, Proposition 2], the design of the vector u is optimal in terms of $\Delta^* = \max_{u \in \mathbb{S}^2}(\min_{v \in \mathcal{E}(A)}\Delta(v, u))$.

Theorem 3: Consider the map μ_U defined in (9) with the potential function U defined in (19). Then, given the set $\mathcal{P}_U$ designed in (29), the condition given in Assumption 1 holds.

Proof: See the proof in Appendix C ■

Remark 3: Theorem 3 provides a design option for the potential function U through the choice of the set $\mathcal{P}_U$ given in (29). The vector u is carefully chosen using the eigenvalues and eigenvectors of the matrix A (see [10, Proposition 2] for more details). It is important to point out that a decrease in the value of γ results in an increase of the gap δ (strengthening the robustness to measurement noise), which may slow down the convergence of θ (in view of (11)) leading to lower convergence rates for the overall closed-loop system. Hence, the parameter γ needs to be carefully chosen via a trade-off between the robustness to measurement noise and the convergence rates of the closed loop system.

V. SIMULATION

In this section, two numerical simulations are presented to illustrate the performance of the proposed hybrid feedback controller. We consider a quadrotor UAV whose inertia matrix is given in [21] as $J = \text{diag}([0.0159, 0.0150, 0.0297])$. The reference rotation and angular velocity are generated by (3) with $R_r(0) = I_3$ and $\omega_r(0) = 0$ and $z(t) = [\sin(0.1t), -\cos(0.3t), 0.1]^\top$. Letting $A = \text{diag}([2, 4, 6])$, one has $0 < \lambda_1^A < \lambda_2^A < \lambda_3^A$ and $\lambda_2^A > \lambda_1^A\lambda_3^A/\lambda_3^A - \lambda_1^A$. Hence, from (29) one chooses $u = \sqrt{2/5}e_2 + \sqrt{3/5}e_3$ and $\Delta^* = \lambda_1^A = 2$, $\Theta = \{0.3\}, \gamma = 0.9(4\Delta^*/\pi^2) = 0.7295$ and $\delta = 0.003$. The gain parameters are simply chosen as $k_R = 0.4, k_\omega = 0.1$ and $k_\theta = 10$.

We consider the hybrid controller (15), referred to as 'hybrid', with κ specified as (27), and the dynamics of θ specified as (12) and (28). For comparison purposes, we also consider the following smooth controller, referred to as 'smooth':

$$\tau = \Upsilon(R_e, \omega_r, z) - 2k_R\psi(AR_e) - k_\omega\omega_e \tag{30}$$

which is modified from (15) by taking $\theta \equiv 0$. Note that the term $\psi(AR_e)$ is generated from the commonly used modified trace potential function $V = \mathrm{tr}(A(I - R_e))$. The same gain parameters are considered for this smooth controller.

The simulation results are given in Fig. 1. In the left plot of Fig. 1, the initial conditions do not coincide with the critical point, $R(0) = \mathcal{R}_a(0.2\pi e_3^\times), \omega(0) = 0, \theta(0) = 0$. As one can see, the rotation tracking error $|R_e|_I$ and the angular velocity error $\|\omega_e\|$ for both control schemes converge to zero asymptotically. One can also see that the solution θ of (10) is continuous and no resetting happens in this case. In the right plot of Fig. 1, the initial condition is chosen close to one of the critical points, $R = \mathcal{R}_a(\pi, e_3), \omega = 0, \theta = 0$. In this case, the solution θ, immediately resets from 0 to 0.3 at $t = 0$, which forces the state (R_e, θ) to leave the critical point of the potential function U. Finally, the solution θ converges to zero as $t \to \infty$. One can see that the proposed hybrid controller improves the convergence rates as compared to the smooth controller.

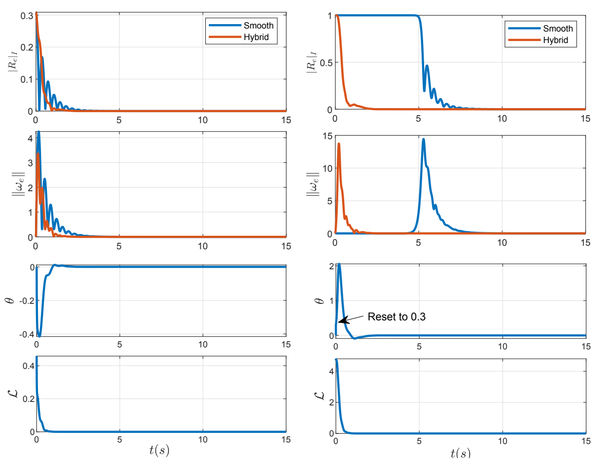

Fig. 1: Simulation results with $R(0) = \mathcal{R}(0.2\pi, e_3)$ and zero angular velocity are shown in the left plot. Simulation results with $R(0) = \mathcal{R}(\pi, e_3)$ and zero angular velocity.

VI. Conclusion

We proposed a new hybrid feedback control scheme for the attitude tracking problem on $SO(3)$, leading to global asymptotic stability. Instrumental to the proposed approach, the introduction of a new potential function on $SO(3) \times \mathbb{R}$ involving a virtual state θ with flow and jump dynamics. The virtual state flows away from the undesired critical points and jumps in the neighborhood of the undesired critical points. A min-resetting mechanism has been designed to reset the extended state in $SO(3) \times \mathbb{R}$ away from the undesired critical points while guaranteeing a decrease of the potential function. The proposed hybrid strategy, involving a single potential function, is simpler than the hybrid synergistic approach involving a family of potential function, and as such exhibits a great potential for other applications involving more complex non-compact matrix Lie groups such as $SE(3)$.

Appendix

A. Proof of Theorem 1

Consider the Lyapunov function candidate

$$\mathcal{L}(x) = k_R U(R_e, \theta) + \frac{1}{2}\omega_e^\top J \omega_e. \tag{31}$$

Since $U(R_e, \theta)$ is a potential function on $SO(3) \times \mathbb{R}$ with respect to $\mathcal{A}_o$, and J is positive definite, one verifies that $\mathcal{L}$ is positive definite on $\mathcal{S}$ with respect to $\mathcal{A}$. The time-derivative of $\mathcal{V}$ along the flows of (17) is given by

$$\dot{\mathcal{L}}(x) = k_R \dot{U}(R_e, \theta) - \omega_e^\top \kappa(R_e, \theta, \omega_e) \tag{32}$$

where we made use of the fact $\omega_e^\top \Sigma(R_e, \omega_e, \omega_r)\omega_e = 0$ since the matrix $\Sigma(R_e, \omega_e, \omega_r)$ is skew symmetric. From (4), (7) and (10), one obtains

$$\begin{aligned}\dot{U}(R_e, \theta) = 2\omega_e^\top \psi(R_e^\top \nabla_{R_e} U(R_e, \theta)) \\ + \nabla_\theta U(R_e, \theta) f(R_e, \theta).\end{aligned} \tag{33}$$

Substituting (11), (16) and (33) into (32), one can further show that

$$\begin{aligned}\dot{\mathcal{L}}(x) &= 2k_R\omega_e^\top \psi(R_e^\top \nabla_{R_e} U(R_e, \theta)) - k_R k_\theta \|\nabla_\theta U(R_e, \theta)\|^2 \\ &\quad - \omega_e^\top (2k_R \psi(R_e^\top \nabla_{R_e} U(R_e, \theta)) + k_\omega \omega_e) \\ &= -k_\omega \|\omega_e\|^2 - k_R k_\theta \|\nabla_\theta U(R_e, \theta))\|^2 \leq 0.\end{aligned} \tag{34}$$

Thus, $\mathcal{L}$ is non-increasing along the flows of (17) on $\mathcal{F}_c$. Moreover, in view of (17) and (31), for any $x^+ = (R_e, \omega_e, R_r, \omega_r, \theta^+) \in \mathcal{J}$ with $\theta^+ \in g(R_e, \theta)$, one obtains

$$\begin{aligned}\mathcal{L}(x) - \mathcal{L}(x^+) &= k_R U(R_e, \theta) - k_R \min_{\bar{\theta} \in \Theta} U(R_e, \bar{\theta}) \\ &= k_R \mu_U(R_e, \theta) \geq k_R \delta.\end{aligned} \tag{35}$$

Thus, $\mathcal{L}$ is strictly decreasing over the jumps of (17) on $\mathcal{J}_c$. From (34) and (35), one concludes that the set $\mathcal{A}$ is stable as per [6, Theorem 23]. Moreover, from (34) and (35) one obtains $\mathcal{L}(x(t, j)) \leq \mathcal{L}(x(t_j, j))$ and $\mathcal{L}(x(t_j, j)) \leq \mathcal{L}(x(t_j, j-1)) - k_R\delta$ with $(t, j), (t_j, j), (t_j, j-1) \in \mathrm{dom}\, x$ and $(t, j) \succeq (t_j, j) \succeq (t_j, j-1)$. Hence, it is clear that $0 \leq \mathcal{L}(x(t, j)) \leq \mathcal{L}(x(0, 0)) - jk_R\delta$, for all $j \geq 1$ and $t \geq t_j$. This leads to $j \leq \left\lceil \frac{\mathcal{L}(x(0,0))}{k_R \delta} \right\rceil := J_M$, where $\lceil \cdot \rceil$ denotes the ceiling function. This shows that the number of jumps is finite and depends on the initial conditions.

Next, we will show the global attractivity of $\mathcal{A}$. Since $\dot{\mathcal{L}}(x) \leq 0$ for all $x \in \mathcal{F}_c$, from (34) one verifies that $\dot{\mathcal{L}} = 0$ if and only if $\omega_e = 0$ and $\nabla_\theta U(R_e, \theta) = 0$. In view of (34) and (35) and applying the invariance principle for hybrid systems given in [22, Theorem 4.7], one concludes that any solution x to (17) must converge to the largest invariant set contained in $\mathcal{W} = \{x \in \mathcal{F}_c \mid \nabla_\theta U(R_e, \theta) = 0, \omega_e = 0\}$. If $\omega_e \equiv 0$, it follows that $\dot{\omega}_e = 0$. From (5), (15) and (16), $\omega_e = \dot{\omega}_e = 0$ implies that $\psi(R_e^\top \nabla_{R_e} U(R_e, \theta)) = 0$. Then, $\psi(R_e^\top \nabla_{R_e} U(R_e, \theta)) = \nabla_\theta U(R_e, \theta) = 0$ implies that $(R_e, \theta) \in \Psi_U$ with Ψ_U defined in (8). Thus, any solution x to (17) must converge to $\bar{\mathcal{W}} := \{x \in \mathcal{F}_c \mid \omega_e = 0, (R_e, \theta) \in \mathcal{F} \cap \Psi_U\}$.

Next, we will show that $\mathcal{F} \cap \Psi_U = \mathcal{A}_o$ (*i.e.*, $\bar{\mathcal{W}} = \mathcal{A}$). Since U is positive definite on $SO(3) \times \mathbb{R}$ with respect to

$\mathcal{A}_o$, one can verify that $\mu_U(R_e,\theta) = -\min_{\bar\theta\in\Theta} U(R_e,\bar\theta) \leq 0$, which implies that $\mathcal{A}_o \subset \mathcal{F}$ and $\mathcal{A}_o \cap \mathcal{J} = \emptyset$. From Assumption 1, one has $\mathcal{A}_o \subset \Psi_U$ and in turn $\mathcal{A}_o \subset \mathcal{F}\cap\Psi_U$. On the other hand, using the facts $\Psi_U \subset (\Psi_U/\mathcal{A}_o) \cup \mathcal{A}_o$ and $\mathcal{F}\cap\mathcal{A}_o = \mathcal{A}_o$, and using set-theoretic arguments, it follows that $\mathcal{F}\cap\Psi_U \subset (\mathcal{F}\cap(\Psi_U/\mathcal{A}_o)) \cup (\mathcal{F}\cap\mathcal{A}_o) = (\mathcal{F}\cap(\Psi_U/\mathcal{A}_o))\cup\mathcal{A}_o$. From Assumption 1 and the definition of $\mathcal{F}$ given in (13), one concludes that $\mathcal{F}\cap(\Psi_U/\mathcal{A}_o) = \emptyset$, and it follows that $\mathcal{F}\cap\Psi_U \subset \mathcal{A}_o$. From $\mathcal{F}\cap\Psi_U \subset \mathcal{A}_o$ and $\mathcal{A}_o \subset \mathcal{F}\cap\Psi_U$, one verifies that $\mathcal{F}\cap\Psi_U = \mathcal{A}_o$, or equivalently $\bar{\mathcal{W}} = \mathcal{A}$. Therefore, the set $\mathcal{A}$ is globally stable and attractive, *i.e.*, the set $\mathcal{A}$ is globally asymptotically stable. This completes the proof.

B. Proof of Theorem 2

From (18), the time-derivative of the transformation map $\mathcal{T}$ along the trajectories of $\dot R = R\omega^\times$ and $\dot\theta = v$ is given by $\dot{\mathcal{T}}(R,\theta) = R\omega^\times\mathcal{R}_a(\theta,u) + vR\mathcal{R}_a(\theta,u)u^\times = \mathcal{T}(R,\theta)(\mathcal{R}_a(\theta,u)^\top\omega + vu)^\times$, where we made use of the fact $\mathcal{R}_a(\theta,u) = \exp(\theta u^\times)$. From (19) and (20) the time-derivative of U can be written as

$$\begin{aligned}\dot U(R,\theta) &= \langle\langle A\mathcal{T}(R,\theta), (\mathcal{R}_a(\theta,u)^\top\omega + vu)^\times\rangle\rangle + \gamma\theta v\\ &= \langle\langle \mathbb{P}_a(A\mathcal{T}(R,\theta)), (\mathcal{R}_a(\theta,u)^\top\omega + vu)^\times\rangle\rangle + \gamma\theta v\\ &= 2\omega^\top\mathcal{R}_a(\theta,u)\psi(A\mathcal{T}(R,\theta))\\ &\quad + v\left(2u^\top\psi(A\mathcal{T}(R,\theta)) + \gamma\theta\right)\end{aligned} \tag{36}$$

where we made use of the facts $\langle\langle A, x^\times\rangle\rangle = 2x^\top\psi(A)$ and $\psi(\mathbb{P}_a(A)) = \psi(A)$ for all $x\in\mathbb{R}^3, A\in\mathbb{R}^{3\times3}$. In view of (7) and (36), one can easily obtain (21) and (22).

Moreover, from (21) and (22), $\psi(R^\top\nabla_R U(R,\theta)) = \nabla_\theta U(R,\theta) = 0$ implies that $\psi(A\mathcal{T}(R,\theta)) = \theta = 0$. From the definition of $\mathcal{T}(R,\theta)$ given in (18), it follows that $\psi(A\mathcal{T}(R,\theta)) = \psi(AR) = 0$. Since $\psi(AR) = 0$, one obtains $\mathbb{P}_a(AR) = 0$, which implies that $AR = R^\top A$. Applying [8, Lemma 2], one concludes that $R \in \Psi_V = \{I_3\} \cup \mathcal{R}_a(\pi, \mathcal{E}(A)) \subset SO(3)$, which gives (23). Since $I_3 \in \Psi_V$, It is clear that $(I_3, 0) \in \Psi_U$, *i.e.*, $\mathcal{A}_o \subset \Psi_U$.

For any vector $v \in \mathcal{E}(A)$ and scalar $\theta\in\mathbb{R}$, one can show that $U(\mathcal{R}_a(\pi,v),0) = V(\mathcal{R}_a(\pi,v)) = 2v^\top Wv$ and $U(\mathcal{R}_a(\pi,v),\theta) = V(\mathcal{R}_a(\pi,v)\mathcal{R}_a(\theta,u)) + \frac{\gamma}{2}\theta^2 = V(\mathcal{R}_a(\pi,v)) + \frac{\gamma}{2}\theta^2 + \mathrm{tr}(A\mathcal{R}_a(\pi,v)(I-\mathcal{R}_a(\theta,u)))$. Using the facts $A\mathcal{R}_a(\pi,v) = A(I + 2(v^\times)^2) = 2Avv^\top - A$, $\mathrm{tr}(A\mathcal{R}_a(\pi,v)(I-\mathcal{R}_a(\theta,u))) = -(1-\cos(\theta))\Delta(u,v)$ and $(1-\cos(\theta)) = 2\sin^2(\frac{\theta}{2})$, one obtains (25) and (26). This completes the proof.

C. Proof Theorem 3

From (23), $(R,\theta)\in\Psi_U/\mathcal{A}_o$, one has $R = \mathcal{R}_a(\pi,v)$ with $v\in\mathcal{E}(A)$ and $\theta = 0$. In view of (9), (25) and (26), for any $(R,\theta)\in\Psi_U/\mathcal{A}_o$, one has $\mu_U(R,\theta) = U(\mathcal{R}_a(\pi,v),0) - \min_{\bar\theta\in\Theta} U(\mathcal{R}_a(\pi,v),\bar\theta) = \max_{\bar\theta\in\Theta}(2\sin^2(\frac{\bar\theta}{2})\Delta(v,u) - \frac{\gamma}{2}\bar\theta^2)$. Applying the facts that $\frac{4\Delta^*}{\pi^2} - \gamma > 0$ and $|\sin(\theta)| \geq \frac{|\theta|}{\pi}$ for all $|\theta|\in[0,\pi]$, one obtains $\mu_U(R,\theta) \geq \max_{\bar\theta\in\Theta}(\frac{2\bar\theta^2}{\pi^2}\Delta^* - \frac{\gamma}{2}\bar\theta^2) \geq (\frac{4\Delta^*}{\pi^2} - \gamma)\frac{\theta_M^2}{2} > \delta$. This completes the proof.

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