A Globally Exponentially Stable Nonlinear Hybrid Observer for 3D Inertial Navigation

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***Abstract*— This paper considers the problem of orientation, position and linear velocity estimation for a rigid body navigating in 3D space. We propose a globally exponentially stable (GES) nonlinear hybrid observer, designed on the matrix Lie group $SE_2(3)$, relying on an inertial measurement unit (IMU) and landmark measurements. A rigorous stability analysis has been provided based on the framework of hybrid dynamical systems. Simulation results are presented to illustrate the performance of the proposed hybrid observer.**

I. INTRODUCTION

The development of reliable orientation (or attitude), position and linear velocity estimation algorithms for full 3D inertial navigation is instrumental in many applications, such as autonomous underwater vehicles and unmanned aerial vehicles. It is well known that the attitude of a rigid body can be estimated using body-frame observations of some known inertial vectors, *e.g.*, IMU equipped with a gyroscope, an accelerometer and a magnetometer [1]–[4], and the position and linear velocity can be obtained, for instance, from a Global Positioning system (GPS) [5], [6]. In fact, IMU-based nonlinear attitude observers rely on the fact that the accelerometer provides a measurement of the gravity vector in the body-fixed frame, which is true only in the case of negligible linear accelerations. In applications involving accelerated rigid body systems, a typical solution consists in using linear velocity measurements together with IMU measurements with the so-called velocity-aided attitude observers [7]–[11]. However, these observers are not easy to implement in GPS-denied environments (*e.g,* indoor applications), where it is challenging to obtain the linear velocity.

On the other hand, the pose (position and orientation) can be obtained using estimators relying on an inertial-vision system combining an IMU and a camera attached to the rigid body [12], [13]. Most of the existing pose estimators are typically filters of the Kalman-type. Recently, nonlinear geometric observers on Lie groups have made their appearance in the literature [14], [15]. Nonlinear pose observers designed on $SE(3)$ using landmark and group velocity measurements have been considered in [16]–[19]. However, these observers are shown to guarantee almost global asymptotic stability (AGAS), *i.e.*, the pose error converges to zero from almost all initial conditions except from a set of Lebesgue measure zero. Using the framework of hybrid dynamical systems, nonlinear hybrid observers design on $SE(3)$ have been addressed with global asymptotic stability in [20] and global exponential stability in [21].

The design of estimation schemes for the orientation, position and linear-velocity of a rigid body using IMU and landmark measurements is of great importance in practical applications where GPS measurements are not available. In this context, an invariant Extended Kalman Filter (iEKF) using IMU and landmark measurements has been proposed in [22]. In contrast to the classical EKF, the invariant EKF provides provable convergence properties. However, due to the well known topological obstruction on $SO(3)$, it is not possible to achieve global stability results for the pose and linear-velocity estimation problem.

Motivated by the prior work in [20], [21], a nonlinear hybrid observer on matrix Lie group $SE_2(3)$, using IMU and landmark measurements, is proposed for the pose and linear-velocity estimation. The proposed observer achieves global exponential stability, and guarantees global decoupling of the rotational error dynamics from the translational error dynamics. To the best of our knowledge, the proposed nonlinear hybrid observer for 3D inertial navigation, endowed with strong stability properties, has never been proposed in the literature.

The rest of this paper is organized as follows: Section II introduces some preliminary notions that will be used throughout this paper. Section III is devoted to the design of the nonlinear hybrid observer. Global exponential stability of the proposed observer is shown in Section IV. Section V presents some simulation results showing the performance of the proposed nonlinear hybrid observer.

II. PRELIMINARY MATERIAL

A. Notations

The sets of real, non-negative real and natural numbers are denoted as $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. Given two matrices, $A, B \in \mathbb{R}^{m\times n}$, their Euclidean inner product is defined as $\langle\langle A, B\rangle\rangle = \mathrm{tr}(A^\top B)$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x^\top x}$, and the Frobenius norm of a matrix $X \in \mathbb{R}^{n\times m}$ is given by $\|X\|_F = \sqrt{\langle\langle X, X\rangle\rangle}$. The n-by-n identity matrix is denoted by I_n. Let $\mathbf{0}_n$ denote the n-dimensional vector with all entries equal to zero. For each $A \in \mathbb{R}^{n\times n}$, we define $\mathcal{E}(A)$ as the set of all eigenvectors of A. Let λ_i^A be the i-th eigenvalue of A, and λ_m^A and λ_M^A be the minimum and maximum eigenvalue of A, respectively.

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Let $\{\mathcal{I}\}$ be an inertial frame and $\{\mathcal{B}\}$ be a frame attached to a rigid body moving in 3-dimensional space. Let $R \in SO(3)$ denote the rotation of frame $\{\mathcal{B}\}$ with respect to frame $\{\mathcal{I}\}$. Let $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ be the position and linear velocity of the rigid-body expressed in the inertial frame $\{\mathcal{I}\}$, respectively. Let us introduce the following extended Special Euclidean group of order 3: $SE_2(3) := SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \subset \mathbb{R}^{5\times 5}$, which is defined as

$$SE_2(3) = \{X = \mathcal{T}(R, v, p) | R \in SO(3), p, v \in \mathbb{R}^3\}, \quad (1)$$

with the map $\mathcal{T} : SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to SE_2(3)$ defined by [22]

$$\mathcal{T}(R, v, p) = \begin{bmatrix} R & v & p \\ \mathbf{0}_3^\top & 1 & 0 \\ \mathbf{0}_3^\top & 0 & 1 \end{bmatrix}.$$

For every $X = \mathcal{T}(R, v, p)$, one has $X^{-1} = \mathcal{T}(R^\top, -R^\top v, -R^\top p)$. The *Lie algebra* of $SE_2(3)$, denoted by $\mathfrak{se}_2(3)$, is given by

$$\mathfrak{se}_2(3) := \left\{ U = \begin{bmatrix} \Omega & \alpha & v \\ \mathbf{0}_{2\times 3} & \mathbf{0}_{2\times 3} & \mathbf{0}_{2\times 3} \end{bmatrix} \in \mathbb{R}^{5\times 5} \middle| \Omega \in \mathfrak{so}(3), v, \alpha \in \mathbb{R}^3 \right\}.$$

For any matrix $A_1 \in \mathbb{R}^{3\times 3}$, define $\mathbb{P}_a(A_1)$ as the anti-symmetric projection of A_1, such that $\mathbb{P}_a(A_1) = (A_1 - A_1^\top)/2$. Let $\mathbb{P} : \mathbb{R}^{5\times 5} \to \mathfrak{se}_2(3)$ denote the projection of A on the Lie algebra $\mathfrak{se}_2(3)$, such that, for all $U \in \mathfrak{se}_2(3)$, $A \in \mathbb{R}^{5\times 5}$

$$\langle\langle A, U\rangle\rangle = \langle\langle U, \mathbb{P}(A)\rangle\rangle = \langle\langle \mathbb{P}(A), U\rangle\rangle.$$

For all $A_1 \in \mathbb{R}^{3\times 3}, a_2, \cdots, a_5 \in \mathbb{R}^3$ and $a_6, \cdots, a_9 \in \mathbb{R}$, one has

$$\mathbb{P}\left(\begin{bmatrix} A_1 & a_2 & a_3 \\ a_4^\top & a_6 & a_7 \\ a_5^\top & a_8 & a_9 \end{bmatrix}\right) = \begin{bmatrix} \mathbb{P}_a(A_1) & a_2 & a_3 \\ \mathbf{0}_3^\top & 0 & 0 \\ \mathbf{0}_3^\top & 0 & 0 \end{bmatrix}. \quad (2)$$

Given a rigid body with configuration $X \in SE_2(3)$, for all $X \in SE_2(3), U \in \mathfrak{se}_2(3)$, the *adjoint map* $\mathrm{Ad} : SE_2(3) \times \mathfrak{se}_2(3) \to \mathfrak{se}_2(3)$ is given by

$$\mathrm{Ad}_X U := XUX^{-1}. \quad (3)$$

For all $X_1, X_2 \in SE_2(3), U \in \mathfrak{se}_2(3)$, one can verify that $\mathrm{Ad}_{X_1}\mathrm{Ad}_{X_2}U = \mathrm{Ad}_{X_1X_2}U$.

B. Kinematics and Measurements

Consider the following kinematics of a rigid body navigating in 3D space:

$$\dot{R} = R\omega^\times, \quad (4)$$
$$\dot{p} = v, \quad (5)$$
$$\dot{v} = \mathrm{g}e_3 + Ra, \quad (6)$$

where g is the gravity constant, $\omega \in \mathbb{R}^3$ is the measured body-frame angular velocity, and $a \in \mathbb{R}^3$ the measured body-frame acceleration. In this paper, we consider the configuration of the rigid body represented by an element of the matrix Lie group $X = \mathcal{T}(R, v, p) \in SE_2(3)$. Let us introduce the nonlinear map $f : SE_2(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^{5\times 5}$, such that the kinematics (4)-(6) can be rewritten in the following compact form

$$\dot{X} = f(X, \omega, a) := \begin{bmatrix} R\omega^\times & \mathrm{g}e_3 + Ra & v \\ \mathbf{0}_3^\top & 0 & 0 \\ \mathbf{0}_3^\top & 0 & 0 \end{bmatrix}. \quad (7)$$

Let $X_1, X_2 \in SE_2(3)$ be two distinct trajectories. The dynamics of the right-invariant error $\eta := X_1X_2^{-1}$ are given by

$$\dot{\eta} = f(\eta, \omega, a) - \eta f(I, \omega, a), \quad (8)$$

which implies that the right-invariant error η has a state-trajectory independent propagation [22].

Consider a family of n landmarks available for measurements, and let $p_i \in \mathbb{R}^3$ be the position of the i-th landmark expressed in the inertial frame $\{\mathcal{I}\}$. The landmark measurements expressed in the body frame $\{\mathcal{B}\}$ are denoted as

$$y_i := R^\top(p_i - p), \quad i = 1, 2, \cdots, n. \quad (9)$$

Practically, vision systems do not provide the 3D landmark measurements directly. However, one can, for instance, construct the 3D landmark measurements from a stereo vision system, see Eq. (26) in [23].

Let $r_i := [p_i^\top \ 0 \ 1]^\top \in \mathbb{R}^5$ for all $i = 1, \cdots, n$ be the new inertial reference vectors with respect to the inertial frame $\{\mathcal{I}\}$, and $b_i := [y_i^\top \ 0 \ 1]^\top \in \mathbb{R}^5$ be their measurements expressed in the body frame $\{\mathcal{B}\}$. From (9), one has

$$b_i = h(X, r_i) := X^{-1}r_i, \quad i = 1, 2, \cdots, n. \quad (10)$$

It is noted that, the Lie group action $h : SE_2(3) \times \mathbb{R}^5 \to \mathbb{R}^5$ is a *right group action* in the sense that for all $X_1, X_2 \in SE_2(3)$ and $r \in \mathbb{R}^5$, one has $h(X_2, h(X_1, r)) = h(X_1X_2, r)$. For later use, we define $r := [r_1 \ r_2 \ \cdots \ r_n] \in \mathbb{R}^{5\times n}$ and $b := [b_1 \ b_2 \ \cdots \ b_n] \in \mathbb{R}^{5\times n}$.

Our objective is to design a globally exponential stable hybrid pose and linear velocity estimation scheme using the above mentioned available measurements.

III. HYBRID OBSERVER DESIGN

A. Continuous observer and undesired equilibra

Let $\hat{X} := \mathcal{T}(\hat{R}, \hat{v}, \hat{p}) \in SE_2(3)$ be the estimate of X. Define the right-invariant estimation error as $\tilde{X} := X\hat{X}^{-1} = \mathcal{T}(\tilde{R}, \tilde{v}, \tilde{p})$ with $\tilde{R} := R\hat{R}, \tilde{v} := v - \tilde{R}\hat{v}$ and $\tilde{p} := p - \tilde{R}\hat{p}$. Given a set of weighting parameters $k_i > 0$ for all $i = 1, 2, \cdots, n$, let us introduce the following weighted geometric center of the landmarks and their measurements:

$$p_c := \frac{1}{\sum_{i=1}^n k_i}\sum_{i=1}^n k_ip_i \quad y_c := \frac{1}{\sum_{i=1}^n k_i}\sum_{i=1}^n k_iy_i. \quad (11)$$

One can easily verify that $y_c = R^\top(p_c - p)$.

Consider the following time-invariant smooth observer

$$\dot{\hat{X}} = f(\hat{X}, \omega, a) - \Delta\hat{X}, \quad (12)$$
$$\Delta := -\mathrm{Ad}_{X_c}\left(\mathbb{P}(X_c^{-1}(r - \hat{X}b)K_nr^\top KX_c^{-\top})\right), \quad (13)$$

where $\hat{X}(0) \in SE_2(3)$ and $X_c := \mathcal{T}(I_3, 0, p_c)$. The gain parameters are given by

$$K_n = \text{diag}(k_1, k_2, \cdots, k_n), K = \begin{bmatrix} k_R I_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3^\top & 0 & 0 \\ \mathbf{0}_3^\top & k_v & k_p \end{bmatrix}. \quad (14)$$

with $k_R, k_p, k_v, k_i > 0, i = 1, 2, \cdots, n$. It is noted that this observer is proposed on matrix Lie group $SE_2(3)$ directly. The term X_c is introduced to decouple the rotational error dynamics from the translational error dynamics, which will be discussed in a later section. A similar technique has been considered in [20], [21].

Define the matrix $Q := \sum_{i=1}^n k_i(p_i - p_c)(p_i - p_c)^\top$ with $k_i > 0$ for all $i = 1, 2, \cdots, n$. Let $A_1 := \sum_{i=1}^n k_i p_i p_i^\top$ and $k := \sum_{i=1}^n k_i$. Then, one has $Q = A_1 - k p_c p_c^\top$.

Lemma 1: ([20], [21]) Let $Q = Q^\top$ be a positive definite matrix with three distinct eigenvalues. Define the following map

$$\Delta(u, v) := u^\top(\text{tr}(M_v)I_3 - M_v)u, \quad (15)$$

where $M_v := Q(I_3 - 2vv^\top)$ and $v \in \mathcal{E}(Q)$. Then, there exists a strictly positive constant Δ_Q^* given by

$$\Delta_Q^* := \min_{v\in\mathcal{E}(Q)} \max_{u\in\mathcal{E}(Q)} \Delta(u, v) = \text{tr}(Q) - \lambda_{\max}^Q > 0. \quad (16)$$

Lemma 2: The expression (13) of Δ can be rewritten as

$$\Delta = -\begin{bmatrix} \mathbb{P}_a(\underline{\Omega}) & \Lambda & V - \mathbb{P}_a(\underline{\Omega})p_c \\ \mathbf{0}_3^\top & 0 & 0 \\ \mathbf{0}_3^\top & 0 & 0 \end{bmatrix}, \quad (17)$$

with

$$\begin{aligned} \underline{\Omega} &= k_R(I_3 - \tilde{R})^\top Q, \\ V &= k k_p \tilde{R}^\top(\tilde{p} - (I_3 - \tilde{R})p_c), \\ \Lambda &= k k_v \tilde{R}^\top(\tilde{p} - (I_3 - \tilde{R})p_c). \end{aligned}$$

See Appendix VI-A for the proof. From (8), the derivative of the right-invariant error $\tilde{X}$ is given by

$$\dot{\tilde{X}} = f(\tilde{X}, \omega, a) - \tilde{X}f(I, \omega, a) + \tilde{X}\Delta. \quad (18)$$

For the sake of simplicity, let us define the new position error $\tilde{p}_e := \tilde{p} - (I - \tilde{R})p_c$. Substituting (17) in (18), one has the following closed-loop system:

$$\begin{cases} \dot{\tilde{R}} &= -\tilde{R}(k_R\mathbb{P}_a(Q\tilde{R})) \\ \dot{\tilde{p}}_e &= -kk_p\tilde{p}_e + \tilde{v} \\ \dot{\tilde{v}} &= -kk_v\tilde{p}_e + (I_3 - \tilde{R})\mathrm{g}e_3 \end{cases} \quad (19)$$

It is clear that the dynamics of $\tilde{R}$ are decoupled from the dynamics of $\tilde{p}$ and $\tilde{v}$. Thanks to this property, one can analyze the convergence of the rotational estimation error $\tilde{R}$ independently. It is noted that, the equilibrium $\mathcal{T}(\tilde{R}, \tilde{v}, \tilde{p}) = I_5$ if and only if $\mathcal{T}(\tilde{R}, \tilde{v}, \tilde{p}_e) = I_5$. Let Ψ be the set of undesired equilibrium points (*i.e.*, all the equilibrium points except I_5) of the dynamics (19) as

$$\Psi := \Big\{(\tilde{R}, \tilde{v}, \tilde{p}_e) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 | \tilde{R} = \mathcal{R}_a(\pi, u), \\ u \in \mathcal{E}(Q), \tilde{p}_e = \frac{\mathrm{g}}{kk_v}(I - \tilde{R})e_3, \tilde{v} = kk_p\tilde{p}_e\Big\}. \quad (20)$$

It is important to mention that, due to the topology of the Lie group $SO(3)$, it impossible to achieve robust and global stability results with smooth (or even discontinuous) state observers [24], [25]. This motivates the consideration of hybrid observers leading to robust and global stability results as shown in the next section.

B. Hybrid observer with resetting mechanism

In this section, we make use of the framework of hybrid dynamical systems [26], [27], to propose a globally exponentially stable observer. Define the following real-valued cost function $\Phi : SO(3) \times \mathbb{R}^{5\times n} \times \mathbb{R}^{5\times n} \to \mathbb{R}^+$

$$\Phi(\hat{R}, r, b) := \frac{1}{2}\sum_{i=1}^n k_i\|(p_i - p_c) - \hat{R}(y_i - y_c)\|^2. \quad (21)$$

Given a non-empty and finite *transformation set* $\mathbb{Q} \subset SE_2(3)$, let us define a real-valued function $\mu : SO(3) \times \mathbb{R}^{5\times n} \times \mathbb{R}^{5\times n} \to \mathbb{R}$ as

$$\mu(\hat{R}, r, b) := \Phi(\hat{R}, r, b) - \min_{\mathcal{T}(R_q, v_q, p_q)\in\mathbb{Q}} \Phi(R_q^\top\hat{R}, r, b). \quad (22)$$

The flow set $\mathcal{F}_o$ and jump set $\mathcal{J}_o$ are defined as follows:

$$\mathcal{F}_o := \{\hat{X} = \mathcal{T}(\hat{R}, \hat{v}, \hat{p}) \in SE_2(3) \mid \mu(\hat{R}, r, b) \leq \delta\}, \quad (23)$$
$$\mathcal{J}_o := \{\hat{X} = \mathcal{T}(\hat{R}, \hat{v}, \hat{p}) \in SE_2(3) \mid \mu(\hat{R}, r, b) \geq \delta\}, \quad (24)$$

where $\delta > 0$. The *hysteresis gap* δ and the set $\mathbb{Q}$ will be given later. The sets $\mathcal{F}_o$ and $\mathcal{J}_o$ are closed, and $\mathcal{F}_o \cup \mathcal{J}_o = SE_2(3)$. We propose the following hybrid observer:

$$\underbrace{\dot{\hat{X}} = f(\hat{X}, \omega, a) - \Delta\hat{X}}_{\hat{X}\in\mathcal{F}_o} \quad \underbrace{\hat{X}^+ = X_q^{-1}\hat{X}, X_q \in \gamma(\hat{X})}_{\hat{X}\in\mathcal{J}_o}, \quad (25)$$

$$\Delta := -\text{Ad}_{X_c}\left(\mathbb{P}(X_c^{-1}(r - \hat{X}b)K_n r^\top K X_c^{-\top})\right), \quad (26)$$

where $\hat{X}(0) \in SE_2(3)$ and the map $\gamma : SE_2(3) \to SE_2(3)$ is defined by

$$\gamma(\hat{X}) := \left\{\min_{X_q=\mathcal{T}(R_q, v_q, p_q)\in\mathbb{Q}} \Phi_Q(R_q^\top\hat{R}, r, b)\right\}. \quad (27)$$

We define the extended space and state as $\mathcal{S} := SE_2(3) \times SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$ and $x := (\hat{X}, \tilde{R}, \tilde{v}, \tilde{p}_e, t)$. In view of (19), (25) and (29), one obtains the following hybrid closed-loop system:

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in \mathcal{F}_c := \{x \in \mathcal{S}, \hat{X} \in \mathcal{F}_o\} \\ x^+ \in G(x) & x \in \mathcal{J}_c := \{x \in \mathcal{S}, \hat{X} \in \mathcal{J}_o\} \end{cases} \quad (28)$$

with

$$F(x) = \begin{bmatrix} f(\hat{X}, \omega, a) - \Delta\hat{X} \\ -\tilde{R}(k_R\mathbb{P}_a(Q\tilde{R})) \\ -kk_v\tilde{p}_e + (I - \tilde{R})\mathrm{g}e_3 \\ -kk_p\tilde{p}_e + \tilde{v} \\ 1 \end{bmatrix}, G(x) = \begin{bmatrix} X_q^{-1}\hat{X} \\ \tilde{R}R_q \\ \tilde{v} \\ \tilde{p}_e \\ t \end{bmatrix}.$$

Note that the sets $\mathcal{F}_c, \mathcal{J}_c$ are closed, and $\mathcal{F}_c \cup \mathcal{J}_c = \mathcal{S}$. Note also that the closed-loop system (28) satisfies the hybrid basic conditions of [26] and is autonomous by taking ω and a as functions of time.

The main idea behind our hybrid observer is that a resetting mechanism is introduced to avoid all the undesired equilibrium points of the closed-loop system (28) in the flow set $\mathcal{F}_c$, *i.e.*, all the undesired equilibrium points of the closed-loop system are in the jump set $\mathcal{J}_c$. The innovation term Δ and the transformation set $\mathbb{Q}$ are designed to ensure a decrease in both flow set $\mathcal{F}_c$ and jump set $\mathcal{J}_c$.

Proposition 1: Consider the hybrid dynamics $\mathcal{H}$ defined in (28). Choose the following *transformation set* $\mathbb{Q} \subset SE_2(3)$

$$\begin{aligned}\mathbb{Q} := \{X = \mathcal{T}(R, v, p) | R = \mathcal{R}_a(\theta, u), \theta \in (0, \pi] \\ u \in \mathbb{U} \supseteq \mathcal{E}(Q), p = (I_3 - R)p_c, v = 0\}.\end{aligned} \quad (29)$$

Then, there exists a constant $\Delta_Q^* > 0$ defined in Lemma 1 such that for all $\delta < (1 - \cos\theta)\Delta_Q^*$, one has $SE_2(3) \times \Psi \times \mathbb{R}^+ \subseteq \mathcal{J}_c$.

See Appendix VI-B for the proof. Proposition 1 provides design example of the set $\mathbb{U}$ and the hysteresis gap δ, ensuring that the set of undesired equilibrium points of the flow dynamics of (28) is a subset of the jump set $\mathcal{J}_c$. In the following section, we provide the rigorous stability analysis of the closed-loop system (28).

IV. Stability Analysis

Let us introduce the following assumption:

Assumption 1: Assume that there exist at least three non-collinear landmarks among the $n \geq 3$ measurable landmarks.

Given three non-collinear landmarks, it is always possible to guarantee that the matrix Q has three distinct eigenvalues through an appropriate choice of the gains k_i. It should be noted that Assumption 1 is common in pose estimation on $SE(3)$ using landmark measurements [16]–[21].

Let us define the compact set $\mathcal{A} := \{x = (\hat{X}, \tilde{R}, \tilde{v}, \tilde{p}_e, t) \in \mathcal{S} : \hat{X} \in SE_2(3), \mathcal{T}(\tilde{R}, \tilde{v}, \tilde{p}_e) = I_5, t \in \mathbb{R}^+\}$ and let $|x|_{\mathcal{A}} \geq 0$ denote the distance to the set $\mathcal{A}$ such that

$$\begin{aligned}|x|_{\mathcal{A}}^2 &:= \inf_{\bar{x}=(\bar{X}, I_3, 0, 0, \bar{t}) \in \mathcal{A}} (\|\bar{X} - \hat{X}\|_F^2 + \tfrac{1}{8}\|I_3 - \tilde{R}\|_F^2 \\ &\quad + \|\tilde{v}\|^2 + \|\tilde{p}_e\|^2 + \|\bar{t} - t\|^2) \\ &= \tfrac{1}{8}\|I_3 - \tilde{R}\|_F^2 + \|\tilde{p}_e\|^2 + \|\tilde{v}\|^2.\end{aligned}$$

Now, one can state our main result.

Theorem 1: Consider the hybrid closed-loop system (28). Assume that Assumption 1 holds. The gains $k_i > 0, i = 1, 2, \cdots, n$ are chosen such that the matrix Q has three distinct eigenvalues. Choose $\mathcal{E}(Q) \subseteq \mathbb{U}$ and $\delta < (1 - \cos\theta)\Delta_Q^*$ with Δ_Q^* defined in Lemma 1. Then, the number of discrete jumps is finite and the set $\mathcal{A}$ is uniformly globally exponentially stable.

Proof: Consider the following real-valued function

$$\mathcal{L}_R = \mathrm{tr}((I - \tilde{R})Q). \quad (30)$$

Let $\bar{Q} := \frac{1}{2}(\mathrm{tr}(Q)I_3 - Q)$, $\underline{Q} := \mathrm{tr}(\bar{Q}^2)I - 2\bar{Q}^2$ and $\underline{\bar{Q}} := \frac{1}{2}(\mathrm{tr}(\underline{Q})I_3 - \underline{Q})$. Define $e_1 := \frac{1}{2\sqrt{2}}\|I_3 - \tilde{R}\|_F$. From Lemma 2 [4], one obtains

$$4\lambda_m^{\bar{Q}} e_1^2 \leq \mathcal{L}_R \leq 4\lambda_M^{\bar{Q}} e_1^2, \quad (31)$$

$$\dot{\mathcal{L}}_R \leq -\eta_1 \mathcal{L}_R. \quad (32)$$

where $\eta_1 := k_R \varrho_Q \lambda_m^{\bar{Q}} / \lambda_M^{\bar{Q}}$, and $\varrho_Q := \min_{\tilde{X} \in \mathcal{F}} \varrho(Q, \tilde{R})$ with $\varrho(Q, \tilde{R}) := (1 - \|e_1\|^2 \cos^2(u, \bar{Q}u))$ and $u \in \mathbb{S}^2$ denoting the axis of rotation $\tilde{R}$. Moreover, one can verify that for all $x \in \mathcal{F}_c$ one has $\varrho(Q, \tilde{R}) > 0$ which implies $\eta_1 > 0$.

On the other hand, consider the following real-valued function

$$\mathcal{L}_p = \frac{1}{2}\|\tilde{p}_e\|^2 + \frac{1}{2kk_v}\|\tilde{v}\|^2 - \mu \tilde{p}_e^\top \tilde{v}. \quad (33)$$

Let $e_2 := [\|\tilde{p}_e\| \ \|\tilde{v}\|]^\top$. One verifies that

$$e_2^\top \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{\mu}{2} \\ -\frac{\mu}{2} & \frac{1}{2kk_v} \end{bmatrix}}_{P_1} e_2 \leq \mathcal{L}_p \leq e_2^\top \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{\mu}{2} \\ \frac{\mu}{2} & \frac{1}{2kk_v} \end{bmatrix}}_{P_2} e_2. \quad (34)$$

The time derivative of $\mathcal{L}_p$ along the flows of (28) is given by

$$\begin{aligned}\dot{\mathcal{L}}_p &= \tilde{p}_e^\top(-kk_p\tilde{p}_e + \tilde{v}) + \frac{1}{kk_v}\tilde{v}^\top(-kk_v\tilde{p}_e + (I - \tilde{R})\mathrm{g}e_3) \\ &\quad - \mu(-kk_p\tilde{p}_e + \tilde{v})^\top\tilde{v} - \mu\tilde{p}_e^\top(-kk_v\tilde{p}_e + (I_3 - \tilde{R})\mathrm{g}e_3) \\ &\leq -(1 - \mu k_v)k\|\tilde{p}_e\|^2 - \mu\|\tilde{v}\| + \mu kk_p\|\tilde{p}_e\|\|\tilde{v}\| \\ &\quad + \frac{\mathrm{g}}{kk_v}\|\tilde{v}\|\|I - \tilde{R}\|_F + \mu\mathrm{g}\|\tilde{p}_e\|\|I_3 - \tilde{R}\|_F.\end{aligned}$$

Let $c_1 := \max\{\frac{\mathrm{g}}{kk_v}, \mu\mathrm{g}\}$, one can further deduce that

$$\begin{aligned}\dot{\mathcal{L}}_p &\leq -e_2^\top \underbrace{\begin{bmatrix} (1 - \mu k_v)k & \frac{\mu kk_p}{2} \\ \frac{\mu kk_p}{2} & \mu \end{bmatrix}}_{P_3} e_2 + 2\sqrt{2}c_1 e_1(\|\tilde{v}\| + \|\tilde{p}_e\|) \\ &\leq -e_2^\top P_3 e_2 + 4c_1 e_1\|e_2\|,\end{aligned}$$

where we made use of the facts: $\|I_3 - \tilde{R}\|_F = 2\sqrt{2}e_1$ and $(\|\tilde{v}\| + \|\tilde{p}_e\|) \leq \sqrt{2(\|\tilde{v}\|^2 + \|\tilde{p}_e\|)^2} = \sqrt{2}\|e_2\|$. To guarantee that the matrices P_1, P_2 and P_3 are positive definite, it is sufficient to pick μ as

$$0 < \mu < \min\left\{\frac{1}{k_v}, \frac{1}{\sqrt{kk_v}}, \frac{4}{4k_v + kk_p^2}\right\}.$$

Hence, one has

$$\dot{\mathcal{L}}_p \leq -\eta_2\mathcal{L}_p + \eta_3\sqrt{\mathcal{L}_R}\sqrt{\mathcal{L}_p}, \quad (35)$$

where $\eta_2 := \lambda_m^{P_3}/\lambda_M^{P_2}$ and $\eta_3 := 2c_1/\sqrt{\lambda_m^{\bar{Q}}\lambda_m^{P_1}}$, and we made use of the facts: $4\lambda_m^{\bar{Q}}e_1^2 \leq \mathcal{L}_R$ and $\lambda_m^{P_1}\|e_2\|^2 \leq \mathcal{L}_p$. Define the new state $\zeta := [\zeta_1 \ \zeta_2]^\top$ with $\zeta_1 := \sqrt{\mathcal{L}_R}$ and $\zeta_2 := \sqrt{\mathcal{L}_p}$. In view of (32) and (35), one obtains

$$\dot{\zeta} \leq -P_4\zeta, \quad P_4 := \begin{bmatrix} \frac{\eta_1}{2} & 0 \\ -\frac{\eta_3}{2} & \frac{\eta_2}{2} \end{bmatrix}. \quad (36)$$

Since η_1, η_2, η_3 are strictly positive, one can easily verify that matrix P_4 is positive definite. In view of the jumps of (22)-(24), (28) and (30) , one shows

$$\|\zeta^+\|^2 - \|\zeta\|^2 = \mathcal{L}_R^+ - \mathcal{L}_R = -\mu(\hat{R}, r, b) \leq -\delta. \quad (37)$$

In view of (24), (22), (30) and (32), for each $(t, j) \in \mathrm{dom}\ x$, one has

$$\mathcal{L}_R(t, j) \leq \mathcal{L}_R(t_j, j) < \mathcal{L}_R(t_j, j - 1) - \delta$$

$$< \cdots < \mathcal{L}_R(t_0, 0) - j\delta,$$

where $(t, j) \succeq (t_j, j) \succeq (t_j, j-1) \succeq \cdots \succeq (t_0, 0)$. Since $0 \leq \mathcal{L}_R(t, j) \leq 4\lambda_M^{\bar{Q}}$ for all $(t, j) \in \text{dom}\, x$, one verifies $j \leq J := \left\lceil 4\lambda_M^{\bar{Q}}/\delta \right\rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. Hence, one can conclude that the number of jumps is finite. It is noted that the number of jumps is independent from the initial conditions, which is different from [20], [21].

To show exponential stability, let us consider the following Lyapunov function candidate:

$$\mathcal{L}(x) := \mathcal{L}_R(\tilde{R}) + \mathcal{L}_p(\tilde{p}_e, \tilde{v}). \tag{38}$$

From (31)-(32), (34)-(35) and (36)-(37) one has

$$\underline{\alpha}|x|_{\mathcal{A}} \leq \mathcal{L}(x) \leq \bar{\alpha}|x|_{\mathcal{A}}, \tag{39}$$

$$\dot{\mathcal{L}}(x) \leq -\zeta^\top P_4 \zeta \leq -\lambda_F \mathcal{L}(x), \tag{40}$$

$$\mathcal{L}(x^+) \leq \mathcal{L}(x) - \delta \leq \exp(-\lambda_J)\mathcal{L}(x), \tag{41}$$

where $\underline{\alpha} := \min\{4\lambda_m^{\bar{Q}}, \lambda_m^{P_1}\}, \bar{\alpha} := \max\{4\lambda_M^{\bar{Q}}, \lambda_M^{P_2}\}$, $\lambda_F := \lambda_{\min}^{P_4}$, $\lambda_J := -\ln(1 - \delta/4\lambda_M^{\bar{Q}})$. From (40)-(41), one obtains

$$\mathcal{L}(t, j) \leq \exp(-2\lambda(t + j))\mathcal{L}(0, 0), \tag{42}$$

where $\lambda := \frac{1}{2}\min\{\lambda_F, \lambda_J\}$. From (39) and (42), one concludes that for each $(t, j) \in \text{dom}\, x$,

$$|x(t, j)|_{\mathcal{A}} \leq k \exp(-\lambda(t + j))|x(0, 0)|_{\mathcal{A}}, \tag{43}$$

where $k := \sqrt{\bar{\alpha}/\underline{\alpha}}$. This completes the proof. ■

V. Simulation

In this section, simulation results are presented to illustrate the performance of the proposed hybrid observer.

We consider a vehicle moving on a 10-meter diameter circle (yellow dashed line) at 10-meter height, with speed of 10 m/s and circular rate of 0.3 rad/s. The angular velocity is given by $\omega(t) = [0\ 0\ 0.1]^\top$. Moreover, 4 landmarks are randomly located on the ground such that Assumption 1 holds. The estimated state variables are initialized as $\hat{R}(0) = \mathcal{R}_a(\pi, u)$ with $u = [0\ 0\ 1]^\top$, $\hat{p}(0) = [0\ 0\ 0]^\top$ and $\hat{v} = [0\ 0\ 0]^\top$. The gain parameters are chosen as $k_i = 0.25$ for all $i = 1, 2, \cdots, 4$ and $k_R, k_v, k_p = 1$. For the hybrid design, we choose $\mathbb{U} = \mathcal{E}(Q)$, $\theta = 0.8\pi$ and $\delta = 0.6(1 - \cos\theta)\Delta_Q^*$.

The hybrid observer is given by (23)-(27). The non-hybrid observer considers just the flows of (25) and (13) without the resetting mechanism. The simulation results are given in Fig. 1 and Fig. 2. As one can see, the hybrid observer leads to higher convergence rate, due to the resetting mechanism. In this simulation, the hybrid observer is only activated at the initialization step. In this case, the algorithm proposed in [28] can be applied without the robustness guarantees of the proposed hybrid observer as pointed out in [25].

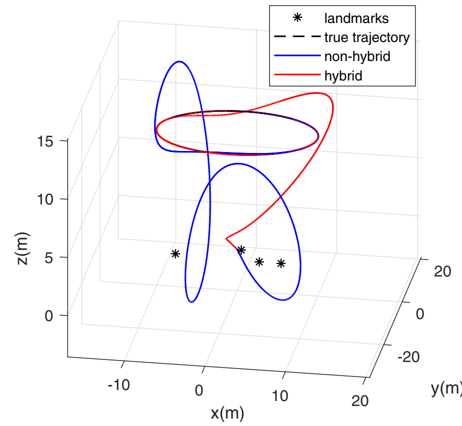

Fig. 1: 3D trajectories

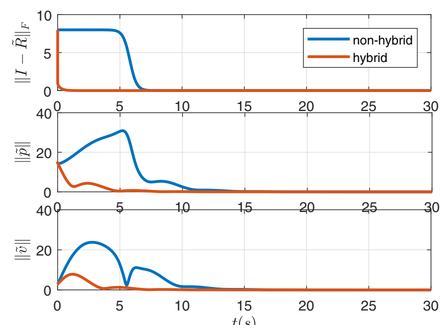

Fig. 2: Estimation errors

VI. Conclusions

A globally exponentially stable hybrid observer for 3-D inertial navigation has been proposed. The observer is designed on the matrix Lie group $SE_2(3)$ using IMU and landmark measurements. It relies on a resetting mechanism designed to avoid the undesired equilibrium points of the closed-loop system in the flows and to ensure a decrease of estimation error over the flows and jumps. Simulation results, illustrating the performance of the proposed hybrid observer, have been provided.

References

[1] S. Bonnabel, P. Martin, and P. Rouchon, "Symmetry-preserving observers," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2514–2526, 2008.

[2] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the special orthogonal group," *IEEE Transactions on automatic control*, vol. 53, no. 5, pp. 1203–1218, 2008.

[3] M.-D. Hua, G. Ducard, T. Hamel, R. Mahony, and K. Rudin, "Implementation of a nonlinear attitude estimator for aerial robotic vehicles," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 1, pp. 201–213, 2014.

[4] S. Berkane, A. Abdessameud, and A. Tayebi, "Hybrid attitude and gyro-bias observer design on $SO(3)$," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 6044–6050, 2017.

[5] A. Barrau and S. Bonnabel, "Invariant particle filtering with application to localization," in *Proceedings the 53rd IEEE Annual Conference on Decision and Control (CDC)*, 2014, pp. 5599–5605.

[6] A. H. Brodtkorb, S. A. Værnø, A. R. Teel, A. J. Sørensen, and R. Skjetne, "Hybrid controller concept for dynamic positioning of marine vessels with experimental results," *Automatica*, vol. 93, pp. 489–497, 2018.

[7] S. Bonnabel, P. Martin, and E. Salaün, “Invariant Extended Kalman filter: theory and application to a velocity-aided attitude estimation problem,” in *Proceedings of the 48th IEEE Conference on Decision and Control and the 28th Chinese Control Conference (CDC/CCC)*, 2009, pp. 1297–1304.

[8] A. Roberts and A. Tayebi, “On the attitude estimation of accelerating rigid-bodies using GPS and IMU measurements,” in *Proceedings of the IEEE CDC-ECC, Orlando, FL, USA, December 12-15*, 2011, pp. 8088–8093.

[9] M.-D. Hua, P. Martin, and T. Hamel, “Stability analysis of velocity-aided attitude observers for accelerated vehicles,” *Automatica*, vol. 63, pp. 11–15, 2016.

[10] M.-D. Hua, T. Hamel, and C. Samson, “Riccati nonlinear observer for velocity-aided attitude estimation of accelerated vehicles using coupled velocity measurements,” in *Proceedings of the 56th IEEE Annual Conference on Decision and Control (CDC)*, 2017, pp. 2428–2433.

[11] S. Berkane and A. Tayebi, “Attitude and gyro bias estimation using GPS and IMU measurements,” in *Proceedings of the 56th IEEE Annual Conference on Decision and Control (CDC), Melbourne, Australia*, 2017, pp. 2402–2407.

[12] H. Rehbinder and B. K. Ghosh, “Pose estimation using line-based dynamic vision and inertial sensors,” *IEEE Transactions on Automatic Control*, vol. 48, no. 2, pp. 186–199, 2003.

[13] G. Baldwin, “Inertial vision pose estimation using non-linear observers,” Ph.D. dissertation, The Australian National University, 2009.

[14] S. Bonnabel, P. Martin, and P. Rouchon, “Non-linear symmetry-preserving observers on Lie groups,” *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1709–1713, 2009.

[15] C. Lageman, J. Trumpf, and R. Mahony, “Gradient-like observers for invariant dynamics on a Lie group,” *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 367–377, 2010.

[16] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira, “A nonlinear position and attitude observer on SE(3) using landmark measurements,” *Systems & Control Letters*, vol. 59, no. 3-4, pp. 155–166, 2010.

[17] M.-D. Hua, M. Zamani, J. Trumpf, R. Mahony, and T. Hamel, “Observer design on the Special Euclidean Group $SE(3)$,” in *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, 2011, pp. 8169–8175.

[18] M.-D. Hua, T. Hamel, R. Mahony, and J. Trumpf, “Gradient-like observer design on the Special Euclidean group $SE(3)$ with system outputs on the real projective space,” in *Proceedings of the 54th IEEE Annual Conference on Decision and Control (CDC)*, 2015, pp. 2139–2145.

[19] A. Khosravian, J. Trumpf, R. Mahony, and C. Lageman, “Observers for invariant systems on Lie groups with biased input measurements and homogeneous outputs,” *Automatica*, vol. 55, pp. 19–26, 2015.

[20] M. Wang and A. Tayebi, “Globally asymptotically stable hybrid observers design on SE(3),” in *Proceedings of the 56th IEEE Annual Conference on Decision and Control (CDC)*, 2017, pp. 3033–3038.

[21] ——, “On the design of hybrid pose and velocity-bias observers on SE(3),” *Internal technical report, arXiv preprint arXiv:1805.00897*, 2018.

[22] A. Barrau and S. Bonnabel, “The invariant extended Kalman filter as a stable observer,” *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1797–1812, 2017.

[23] T. Hamel and C. Samson, “Riccati observers for the nonstationary PnP problem,” *IEEE Transactions on Automatic Control*, vol. 63, no. 3, pp. 726–741, 2018.

[24] C. G. Mayhew and A. R. Teel, “Synergistic potential functions for hybrid control of rigid-body attitude,” in *Proceedings of American Control Conference*, 2011, pp. 875–880.

[25] ——, “Synergistic hybrid feedback for global rigid-body attitude tracking on $SO(3)$,” *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2730–2742, 2013.

[26] R. Goebel, R. G. Sanfelice, and A. R. Teel, “Hybrid dynamical systems,” *IEEE Control Systems*, vol. 29, no. 2, pp. 28–93, 2009.

[27] ——, *Hybrid Dynamical Systems: modeling, stability, and robustness*. Princeton University Press, 2012.

[28] H. F. Grip, T. I. Fossen, T. A. Johansen, and A. Saberi, “Attitude estimation using biased gyro and vector measurements with time-varying reference vectors,” *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1332–1338, 2012.

APPENDIX

A. Proof of Lemma 2

In view of (13), one has

$$\begin{aligned}(r-\hat{X}b)K_n r^\top K &= \sum_{i=1}^n k_i(r_i-\hat{X}b_i)r_i^\top K \\ &= \begin{bmatrix}\Omega & \Lambda & V\\ \mathbf{0}_3^\top & 0 & 0\\ \mathbf{0}_3^\top & 0 & 0\end{bmatrix},\end{aligned}$$

where $\Omega := k_R((I_3-\tilde{R}^\top)\sum_{i=1}^n k_i p_i p_i^\top + k\tilde{R}^\top \tilde{p}p_c)$. One can further show that

$$X_c^{-1}(r-\hat{X}b)K_n r^\top K X_c^{-\top} = \begin{bmatrix}\underline{\Omega} & \Lambda & V\\ \mathbf{0}_3^\top & 0 & 0\\ \mathbf{0}_3^\top & 0 & 0\end{bmatrix},$$

where $\underline{\Omega} = \Omega - Vp_c^\top = (I_3-\tilde{R})^\top Q$. From the definitions of the Adjoint map (3) and the projection map (2), and X_c, one has

$$\Delta = -\begin{bmatrix}\mathbb{P}_a(\underline{\Omega}) & \Lambda & V-\mathbb{P}_a(\underline{\Omega})p_c\\ \mathbf{0}_3^\top & 0 & 0\\ \mathbf{0}_3^\top & 0 & 0\end{bmatrix},$$

which gives (17). This completes the proof.

B. Proof of Proposition 1

For each $x \in SE_2(3)\times\Psi\times\mathbb{R}^+$, let us rewrite $\tilde{R} = \mathcal{R}_a(\pi, v)$ with $v \in \mathcal{E}(Q)$, and $R_q = \mathcal{R}_a(\theta, u_q)$ with $\theta \in (0,\pi]$ and $u_q \in \mathbb{U}$. In view of (21)-(22), one can show that

$$\begin{aligned}\mu(\hat{R},r,b) &= \mathrm{tr}((I_3-\tilde{R})Q) - \min_{\mathcal{T}(R_q,p_q,v_q)\in\mathbb{Q}} \mathrm{tr}((I_3-\tilde{R}R_q)Q)\\ &= -\max_{\mathcal{T}(R_q,p_q,v_q)\in\mathbb{Q}} \mathrm{tr}(\tilde{R}(I_3-R_q)Q)\\ &= (1-\cos\theta)\max_{u_q\in\mathbb{U}}\Delta(u_q,v)\\ &\geq (1-\cos\theta)\max_{u_q\in\mathcal{E}(Q)}\Delta(u_q,v),\end{aligned}$$

where we made use of the definition (15) and the fact that $\max_{u_q\in\mathbb{U}}\Delta(u_q,v) \geq \max_{u_q\in\mathcal{E}(Q)}\Delta(u_q,v)$ for any $v\in\mathbb{R}^3$. From Lemma 1, there exists a positive constant Δ_Q^* such that for any $x\in SE_2(3)\times\Psi\times\mathbb{R}^+$, one has

$$\begin{aligned}\mu(\hat{R},r,b) &\geq (1-\cos\theta)\min_{v\in\mathcal{E}(Q)}\max_{u_q\in\mathcal{E}(Q)}\Delta(u,v)\\ &\geq (1-\cos\theta)\Delta_Q^* > \delta,\end{aligned}$$

which gives $SE_2(3)\times\Psi\times\mathbb{R}^+ \subseteq \mathcal{J}_c$ from (24) and (28). This completes the proof.