2019 IEEE 58th Conference on Decision and Control (CDC)
Palais des Congrès et des Expositions Nice Acropolis
Nice, France, December 11-13, 2019

Nonlinear Observers for Stereo-Vision-Aided Inertial Navigation

Miaomiao Wang and Abdelhamid Tayebi

Abstract— **This paper considers the problem of attitude, position and linear velocity estimation for rigid body systems, relying on an inertial measurement unit (IMU) and a stereo vision system. We provide a generic stability result for a class of nonlinear time-varying systems evolving on $SO(3) \times \mathbb{R}^n$, which is used for the development of two nonlinear observers, guaranteeing almost global asymptotic stability and local exponential stability, for autonomous systems evolving in a 3-dimensional space. The first observer considers stereo bearing measurements of a single landmark, while the second one uses stereo bearing measurements of multiple landmarks. Two numerical examples are provided to illustrate the performance of the proposed observers.**

I. Introduction

The development of reliable state estimation algorithms is instrumental for autonomous navigation systems such as unmanned aerial vehicles. Typically, the attitude of a rigid body can be estimated from a set of body-frame measurements of known inertial vectors, obtained, for instance, via an IMU [1], [2]. The position and linear velocity of a rigid body can be obtained, for instance, from a Global Positioning System (GPS) [3]. However, this type of GPS-based navigation observers/filters is not easy to implement in GPS-denied environments (*e.g.,* indoor applications). Moreover, IMU-based attitude estimators rely on the fact that the accelerometer provides a body frame measurement of the gravity vector, which is not a valid assumption in the case where the rigid body is subject to important linear accelerations. Recently, inertial-vision systems combining IMU and onboard cameras have made their appearance in the literature [4], [5]. A class of nonlinear invariant pose (attitude and position) observers designed on the matrix Lie group $SE(3)$ using group velocity (angular and linear velocities) and vision-based landmark position measurements have been proposed in [6]–[8]. However, these observers are only shown to guarantee almost global asymptotic stability (AGAS), *i.e.*, the pose error converges to zero from almost all initial conditions except from a set of Lebesgue measure zero. To overcome this issue, hybrid nonlinear pose observers evolving on $SE(3)$ have been proposed, leading to global asymptotic stability [9] and global exponential stability [10].

In practice, obtaining the linear velocity from low-cost sensors, in either the inertial frame or the body frame, is difficult, especially in GPS-denied environments. Moreover, the dynamics of the attitude, position and linear velocity for inertial navigation are not invariant, which makes the extension of the invariant observers designed on $SE(3)$ to the estimation problem considered in this work is non-trivial. Therefore, it is of great importance to develop estimation algorithms for inertial navigation systems that provide (simultaneously) reliable estimates of the attitude, position and linear velocity. Most of the existing results in the literature are filters of the Kalman-type such as extended Kalman filters (EKF) and unscented Kalman filters (UKF) [11], [12]. Recently, nonlinear geometric observers for inertial navigation systems using landmark position measurements, have been proposed in [13]–[15]. In fact, an invariant Extended Kalman Filter (IEKF), with local stability guarantees, has been proposed in [13], and a hybrid nonlinear geometric observer, with global stability guarantees, has been proposed in [14]. In [15], a Riccati-based observer for pose, linear velocity and gravity vector estimation, with local stability guarantees, has been proposed.

In the present paper, we consider the problem of attitude, position and linear velocity estimation using IMU and stereo bearing measurements obtained from stereo vision systems. In fact, vision systems do not directly provide three-dimensional landmarks positions [16], and as such, additional algorithms are needed to obtain the three-dimensional body-frame landmark positions. To address this issue, we design two nonlinear navigation observers on $SO(3) \times \mathbb{R}^n$ using IMU and stereo bearing measurements directly (*i.e.,* without landmarks positions reconstruction). The first observer relies on the IMU (including a 3-axis magnetometer) and stereo bearing measurements of a single landmark, while the second one uses the IMU and stereo bearing measurements of multiple landmarks. One of the main contribution of our work is that both observers guarantee almost global asymptotic stability and local exponential stability. This is distinct from the classical EKF-based filters where only local stability is guaranteed.

The rest of this paper is organized as follows: Section II introduces some preliminary notions that will be used throughout this paper. Section III provides a stability result for a class of nonlinear time-varying systems on $SO(3) \times \mathbb{R}^n$ and two nonlinear observers for inertial navigation on $SO(3) \times \mathbb{R}^n$. Section IV presents two simulation results showing the performance of the proposed nonlinear geometric observers.

This work was supported by the National Sciences and Engineering Research Council of Canada (NSERC).

The authors are with the Department of Electrical and Computer Engineering, Western University, London, ON N6A 3K7, Canada. A. Tayebi is also with the Department of Electrical Engineering, Lakehead University, Thunder Bay, ON P7B 5E1, Canada. mwang448@uwo.ca, atayebi@lakeheadu.ca

II. Preliminary Material

A. Notations

The sets of real and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}^+$ respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, and denote by $\mathbb{S}^n$ the set of n-dimensional unit vectors. Given two matrices, $A, B \in \mathbb{R}^{m\times n}$, their Euclidean inner product is defined as $\langle\langle A, B\rangle\rangle = \mathrm{tr}(A^\top B)$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x^\top x}$, and the Frobenius norm of a matrix $X \in \mathbb{R}^{n\times m}$ is given by $\|X\|_F = \sqrt{\langle\langle X, X\rangle\rangle}$. The n-by-n identity matrix is denoted by I_n. For each $A \in \mathbb{R}^{n\times n}$, we define $\mathcal{E}(A)$ as the set of all unit-eigenvectors of A. The minimum and maximum eigenvalue of A are, respectively, denoted by $\lambda^A_{\min}$ and $\lambda^A_{\max}$.

The 3-dimensional Special Orthogonal group $SO(3)$ is defined as $SO(3) := \{R \in \mathbb{R}^{3\times3}, RR^\top = R^\top R = I_3, \det(R) = 1\}$. The *Lie algebra* of $SO(3)$ is given by $\mathfrak{so}(3) := \{\Omega \in \mathbb{R}^{3\times3} : \Omega = -\Omega^\top\}$. Let $\times$ be the vector cross-product on $\mathbb{R}^3$ and define the map $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ such that $x \times y = x^\times y$, for any $x, y \in \mathbb{R}^3$. Let $\mathrm{vec} : \mathfrak{so}(3) \to \mathbb{R}^3$ be the inverse isomorphism of the map $(\cdot)^\times$, such that $\mathrm{vec}(\omega^\times) = \omega$ and $(\mathrm{vec}(\Omega))^\times = \Omega$ for all $\omega \in \mathbb{R}^3$ and $\Omega \in \mathfrak{so}(3)$. For a matrix $A \in \mathbb{R}^{3\times3}$, we denote by $\mathbb{P}_a : \mathbb{R}^{3\times3} \to \mathfrak{so}(3)$ the anti-symmetric projection of A, such that $\mathbb{P}_a(A) := (A - A^\top)/2$. Define the composition map $\psi_a := \mathrm{vec}\circ\mathbb{P}_a$ such that, for a matrix $A = [a_{ij}] \in \mathbb{R}^{3\times3}$, one has $\psi_a(A) = \frac{1}{2}[a_{32} - a_{23}, a_{13} - a_{31}, a_{21} - a_{12}]^\top$. It is easy to verify that, for any $A \in \mathbb{R}^{3\times3}, x \in \mathbb{R}^3$, one has $\langle\langle A, x^\times\rangle\rangle = 2x^\top\psi_a(A)$. For any $R \in SO(3)$, we define $|R|_I \in [0, 1]$ as the normalized Euclidean distance on $SO(3)$ with respect to the identity I_3, which is given by $|R|_I^2 = \mathrm{tr}(I - R)/4$. Let the map $\mathcal{R}_a : \mathbb{R} \times \mathbb{S}^2 \to SO(3)$ represent the well-known angle-axis parametrization of the attitude, which is given by $\mathcal{R}_\alpha(\theta, u) := I_3 + \sin\theta u^\times + (1 - \cos\theta)(u^\times)^2$ with $u \in \mathbb{S}^2$ being the unit rotational axis and $\theta \in \mathbb{R}$ the rotational angle.

B. System equations and measurements

Let $\{\mathcal{I}\}$ and $\{\mathcal{B}\}$ denote the inertial and body frames, respectively. Consider the rotation matrix $R \in SO(3)$ as the attitude of the body frame $\{\mathcal{B}\}$ with respect to the inertial frame $\{\mathcal{I}\}$. Let $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ be the position and linear velocity of a rigid-body expressed in frame $\{\mathcal{I}\}$, respectively. Consider the following kinematic equations of a rigid body:

$$\dot{R} = R\omega^\times, \tag{1a}$$
$$\dot{p} = v, \tag{1b}$$
$$\dot{v} = \mathrm{g} + Ra, \tag{1c}$$

where $\mathrm{g} \in \mathbb{R}^3$ denotes the gravity vector, $\omega \in \mathbb{R}^3$ denotes the body-frame angular velocity expressed in frame $\{\mathcal{B}\}$, and $a \in \mathbb{R}^3$ denotes the body-frame "apparent acceleration" capturing all non-gravitational forces applied to the rigid body expressed in the body frame. We assume that the rigid body is equipped with an IMU, which provides the continuous measurements of ω and a.

We further assume that the vehicle is equipped with a stereo camera that provides stereo measurements of N landmarks. Let $\{\mathcal{C}_L\}$ and $\{\mathcal{C}_R\}$ denote the frames attached to the left camera and the right camera, respectively. Let p_i be the position of the i-th landmark expressed in frame $\{\mathcal{I}\}$ and $p_i^{\mathcal{B}} := R^\top(p_i - p)$ the position of the i-th landmark expressed in frame $\{\mathcal{B}\}$. Moreover, the position of the i-th landmark expressed in the left (right) camera frames is defined as $p_i^{\mathcal{C}_L} := R_{cL}^\top(p_i^{\mathcal{B}} - p_L)$ ($p_i^{\mathcal{C}_R} := R_{cR}^\top(p_i^{\mathcal{B}} - p_R)$) with (R_{cL}, p_L) and (R_{cR}, p_R) denoting the homogeneous transformation from the body frame to the left and right camera frames, respectively. Let us introduce the following bearing vectors for all $s \in \{L, R\}, i \in \{1, 2, \cdots, N\}$

$$x_i^s := \frac{p_i^{\mathcal{C}_s}}{\|p_i^{\mathcal{C}_s}\|} = \frac{R_{cs}^\top(p_i^{\mathcal{B}} - p_s)}{\|p_i^{\mathcal{B}} - p_s\|}. \tag{2}$$

III. Main results

A. Almost global asymptotic stability on $SO(3) \times \mathbb{R}^n$

Let $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and $\tilde{R} \in SO(3)$. Consider the following nonlinear time-varying system on $SO(3) \times \mathbb{R}^n$:

$$\begin{cases} \dot{\tilde{R}} &= \tilde{R}(\sigma_R)^\times \\ \dot{x} &= Ax - Ky \\ y &= Cx \end{cases} \tag{3}$$

where $A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{m\times n}$, $K = k(t)P(t)C^\top Q(t)$ with $k(t) > 1/2$ and $P \in \mathbb{R}^{n\times n}$ being the solution of the following continuous Riccati equation (CRE):

$$\dot{P} = AP + PA^\top - PC^\top Q(t)CP + V(t), \tag{4}$$

where $P(0) \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $V(t) \in \mathbb{R}^{n\times n}$ and $Q(t) \in \mathbb{R}^{m\times m}$ are uniformly positive definite matrices. The term σ_R is defined as

$$\sigma_R = -\psi_a(M\tilde{R}) + \Gamma x \tag{5}$$

where $\Gamma \in \mathbb{R}^{3\times n}$, and $M = M^\top \in \mathbb{R}^{3\times3}$ is positive semi-definite with distinct eigenvalues. Assume that Γ and its first-order derivative are uniformly bounded. Some useful properties are given in the following lemmas:

Lemma 1 ([17], [18]): Consider the CRE defined in (4). Suppose that there exist constants $\delta, \mu > 0$ such that $\forall t \geq 0$

$$\frac{1}{\delta}\int_t^{t+\delta} \Phi(\tau, t)^\top C(\tau)^\top C(\tau)\Phi(\tau, t)d\tau \geq \mu I_n, \tag{6}$$

with $\Phi(t, \tau)$ being the transition matrix associated to $A(t)$, *i.e.*, $\frac{d}{dt}\Phi(t, \tau) = A(t)\Phi(t, \tau)$. Then, the solution $P(t)$ is well defined on $\mathbb{R}^+$, and there exist positive constants $0 < p_m \leq p_M < \infty$ such that $p_m I_n \leq P(t) \leq p_M I_n$.

Lemma 2 ([19]): Consider the trajectory of $\dot{R} = R\omega^\times$ with $R(0) \in SO(3)$ and $\omega \in \mathbb{R}^3$. Define the potential function $\mathcal{L}_M(R) = \mathrm{tr}((I_3 - R)M)$ with $M = M^\top$ a positive semi-definite matrix. Then, the following properties hold:

$$4\lambda^{\bar{M}}_{\min}|R|_I^2 \leq \mathcal{L}_M(R) \leq 4\lambda^{\bar{M}}_{\max}|R|_I^2, \tag{7}$$
$$\|\psi_a(MR)\|^2 = \alpha(M, R)\mathrm{tr}((I_3 - R)\underline{M}), \tag{8}$$
$$\dot{\psi}_a(MR) = E(MR)\omega, \tag{9}$$

where $\bar{M} := \frac{1}{2}(\text{tr}(M)I - M)$, $\underline{M} := \text{tr}(\bar{M}^2)I_3 - 2\bar{M}^2$ and the map $\alpha(M, R) := (1 - |R|_I^2 \cos\langle u, \bar{M}u\rangle)$ with $u \in \mathbb{S}^2$ denoting the axis of the rotation R and $\langle\cdot,\cdot\rangle$ denoting the angle between two vectors.

Now, one can state the following result:

Theorem 1: Consider the nonlinear time-varying system (3)-(5). Assume that the pair $(A(t), C(t))$ is uniformly observable and $M = M^\top$ is positive semi-definite with three distinct eigenvalues. Then, the following statements hold:

i) The set of equilibrium points of system (3) is given by

$$\Psi := (I_3, 0) \cup \{(\tilde{R}, x) \in SO(3) \times \mathbb{R}^n : \tilde{R} = \mathcal{R}_\alpha(\pi, v), v \in \mathcal{E}(M), x = 0\}. \quad (10)$$

ii) The desired equilibrium $(I_3, 0)$ is almost globally asymptotically stable and the other three undesired equilibria are unstable.

iii) There exist a constant $\kappa^* > 0$ such that the desired equilibrium $(I_3, 0)$ is exponentially stable if

$$\text{tr}((I - \tilde{R}(0))M) + \kappa x(0)^\top P(0)^{-1} x(0) \leq \varepsilon_R,$$

where $\kappa > \kappa^*$ and $\varepsilon_R \in (0, 4\lambda_{\min}^{\bar{M}}]$.

Proof: See the proof in Appendix A. ■

B. Observer design using a single landmark

In this subsection, we assume that the IMU includes a gyroscope, an accelerometer and a magnetometer. The magnetometer provides the body frame vector measurement m_B of the (known and constant) inertial Earth magnetic field vector m_I, such that $m_B = R^\top m_I$. Let $\hat{R}, \hat{p}, \hat{v}$ and $\hat{\mathrm{g}}$ be the estimates of the attitude R, position p, linear velocity v and gravity vector g, respectively. Define the attitude estimation error $\tilde{R} := R\hat{R}^\top$, position estimation error $\tilde{p}_e = R^\top(p - p_1) - \hat{R}^\top(\hat{p} - p_1)$, velocity estimation errors $\tilde{v} := R^\top v - \hat{R}^\top \hat{v}$ and gravity vector estimation error $\tilde{\mathrm{g}} = R^\top \mathrm{g} - \hat{R}^\top \hat{\mathrm{g}}$.

Let $\pi : \mathbb{S}^2 \to \mathbb{R}^{3\times3}$ be the projection map such that $\pi_x := I_3 - xx^\top$. For each $s \in \{L, R\}$, we define the following projected vector:

$$\begin{aligned} e_1^s &:= \pi_{x_1^s} R_{cs}^\top(\hat{R}^\top(p_1 - \hat{p}) - p_s) \\ &= \pi_{x_1^s} R_{cs}^\top((\hat{R}^\top(p_1 - \hat{p}) - p_s) - (R^\top(p_1 - p) - p_s)) \\ &= \pi_{x_1^s} R_{cs}^\top \tilde{p}_e, \end{aligned} \quad (11)$$

where we made use of the fact $\pi_{x_1^s} p_1^{\mathcal{C}_s} = 0$. Let us introduce the following (virtual) output:

$$y = R_{cR} e_1^R + R_{cL} e_1^L = \Pi_1 \tilde{p}_e, \quad (12)$$

where $\Pi_1 := R_{cR}\pi_{x_1^R} R_{cR}^\top + R_{cL}\pi_{x_1^L} R_{cL}^\top$. For any $z \in \mathbb{S}^2$, one has $z^\top \Pi_1 z = 2 - \|z^\top R_{cR} x_1^R\|^2 - \|z^\top R_{cL} x_1^L\|^2$. Note that Π_1 has zero eigenvalues if and only if $R_{cR}x_1^R = R_{cL}x_1^L$. For the sake of simplicity, let $R_{cL} = R_{cR}$, which implies that Π_1 is uniformly positive definite since x_R and x_L are non-collinear.

We propose the following observer on $SO(3) \times \mathbb{R}^9$:

$$\dot{\hat{R}} = \hat{R}(\omega - \hat{R}^\top \sigma_R)^\times, \quad (13a)$$
$$\dot{\hat{p}} = \hat{v} - \sigma_R^\times(\hat{p} - p_1) + \hat{R}K_p y, \quad (13b)$$
$$\dot{\hat{v}} = \hat{\mathrm{g}} + \hat{R}a - \sigma_R^\times \hat{v} + \hat{R}K_v y, \quad (13c)$$
$$\dot{\hat{\mathrm{g}}} = -\sigma_R^\times \hat{\mathrm{g}} + \hat{R}K_{\mathrm{g}} y, \quad (13d)$$

where the gain matrices $K_p, K_v, K_{\mathrm{g}} \in \mathbb{R}^{3\times3N}$ will be designed later, and the innovation term σ_R is given by

$$\sigma_R := -\rho_1 \hat{\mathrm{g}} \times \mathrm{g} - \rho_2(\hat{R}m_B) \times m_I = -\psi_a(M\tilde{R}) + \Gamma x. \quad (14)$$

with $\rho_1, \rho_2 > 0$, $M := \rho_1 \mathrm{g}\mathrm{g}^\top + \rho_2 m_I m_I^\top$ and $\Gamma := [0_{3\times6}, -\rho_1 \mathrm{g}^\times \hat{R}] \in \mathbb{R}^{3\times9}$. In view of (1a)-(1c), (13a)-(13d) and (14), one obtains the following closed-loop system:

$$\begin{cases} \dot{\tilde{R}} &= \tilde{R}(\sigma_R)^\times, \\ \dot{\tilde{p}}_e &= -\omega^\times \tilde{p}_e + \tilde{v} - K_p y, \\ \dot{\tilde{v}} &= -\omega^\times \tilde{v} + \tilde{\mathrm{g}} - K_v y, \\ \dot{\tilde{\mathrm{g}}} &= -\omega^\times \tilde{\mathrm{g}} - K_{\mathrm{g}} y \end{cases}. \quad (15)$$

Let us introduce the new variable $x := [\tilde{p}_e^\top, \tilde{v}^\top, \tilde{\mathrm{g}}^\top]^\top \in \mathbb{R}^9$. Then, the closed-loop system (15) with σ_R defined in (14) can be rewritten in a compact form as (3) with

$$A = \begin{bmatrix} -\omega^\times & I_3 & 0 \\ 0 & -\omega^\times & I_3 \\ 0 & 0 & -\omega^\times \end{bmatrix}, C = \begin{bmatrix} \Pi_1 & 0 & 0 \end{bmatrix}, \quad (16)$$

and $\begin{bmatrix} K_p^\top & K_v^\top & K_{\mathrm{g}}^\top \end{bmatrix}^\top := K = k(t)PC^\top Q(t)$ where $P \in \mathbb{R}^{9\times9}$ is the solution to the CRE (4).

Lemma 3: Consider the pair (A, C) defined in (16). Then, there exist constants $\delta, \mu > 0$ such that the inequality (6) holds for all $t \geq 0$.

Remark 1: The proof of Lemma 3 is similar to the proof of [20, Lemma 1], which is omitted here due space limitation. Lemma 3 shows that the pair (A, C) is uniformly observable. Suppose that the matrices $Q(t)$ and $V(t)$ are uniformly positive definite. Lemma 1 guarantees the existence of positive constants p_m and p_M such that $p_m I_9 \leq P \leq p_M I_9$.

Theorem 2: Consider the closed-loop system (3) with system parameters (A, C) defined in (16) and σ_R given by (14). Suppose that the matrices $V(t)$ and $Q(t)$ for CRE (4) are uniformly positive definite. Choose $\rho_1, \rho_2 > 0, k(t) > \frac{1}{2}, \forall t \geq 0$ and $P(0)$ positive definite. Then, the results of Theorem 1 hold.

Proof: From the definition of Γ given in (14), one can easily show that $\|\Gamma\| \leq \rho_1 := c_\Gamma$ for all $t \geq 0$. Moreover, using the fact that g and m_I are non-collinear, one verifies that $M = M^\top$ is positive semi-definite and has three distinct eigenvalues through an appropriate choice of ρ_1, ρ_2. Finally, the proof of Theorem 2 is completed by using similar steps as in the proof of Theorem 1. ■

C. Observer design using multiple landmarks

In this section, we present a navigation observer using bearing measurements of $N \geq 3$ landmarks. In contrast to the observer presented in the previous section, this observer does not rely on magnetometer measurements which are known to be unreliable in practical applications.

Assumption 1: Assume that there exist at least three non-collinear landmarks among the $N \geq 3$ measurable landmarks.

Given three non-collinear landmarks, it is always possible to guarantee that the matrix M has three distinct eigenvalues through an appropriate choice of the gains $\rho_i, i = 2, \cdots, N$. It should be noted that Assumption 1 is commonly used in the problem of pose estimation on $SE(3)$ using landmark measurements [6]–[10].

Let $\hat{p}_i$ be the estimate of p_i for all $i \in \{2, 3, \cdots, N\}$. Define $r_i := p_i - p_1$, $\hat{r}_i := \hat{p}_i - p_1$ and $\tilde{r}_i = R^\top r_i - \hat{R}^\top \hat{r}_i$. For $i \in \{2, 3, \cdots, N\}$ and $s \in \{L, R\}$, we introduce the following projected vectors:

$$
\begin{aligned}
e_i^s &:= \pi_{x_i^s} R_{cs}^\top (\hat{R}^\top (\hat{p}_i - \hat{p}) - p_s) \\
&= \pi_{x_i^s} R_{cs}^\top ((\hat{R}^\top (\hat{p}_i - \hat{p}) - p_s) - (R^\top (p_i - p) - p_s)) \\
&= \pi_{x_i^s} R_{cs}^\top R^\top ((p - p_1) - \tilde{R}(\hat{p} - p_1) - r_i + \tilde{R}\hat{r}_i) \\
&= \pi_{x_i^s} R_{cs}^\top (\tilde{p}_e - \tilde{r}_i)
\end{aligned} \tag{17}
$$

where we made use of the fact $\pi_{x_i^s} p_i^{\mathcal{C}_s} = 0$ and the definitions of $\tilde{p}_e$ and $\tilde{r}_i$. Let us define the output $y = [y_1^\top, y_2^\top, \cdots, y_N^\top]^\top \in \mathbb{R}^{3N}$ with

$$
y_i = R_{cR} e_i^R + R_{cL} e_i^L = \begin{cases} \Pi_i \, \tilde{p}_e & i = 1 \\ \Pi_i (\tilde{p}_e - \tilde{r}_i) & \text{otherwise} \end{cases}, \tag{18}
$$

where $\Pi_i := R_{cR} \pi_{x_i^R} R_{cR}^\top + R_{cL}^\top \pi_{x_i^L} R_{cL}^\top$, which is uniformly positive definite.

We propose the following observer on $SO(3) \times \mathbb{R}^{6+3N}$:

$$
\begin{aligned}
\dot{\hat{R}} &= \hat{R}(\omega - \hat{R}^\top \sigma_R)^\times, && (19a) \\
\dot{\hat{p}} &= \hat{v} - \sigma_R^\times (\hat{p} - p_1) + \hat{R} K_p y, && (19b) \\
\dot{\hat{v}} &= \hat{\mathrm{g}} + \hat{R} a - \sigma_R^\times \hat{v} + \hat{R} K_v y, && (19c) \\
\dot{\hat{\mathrm{g}}} &= -\sigma_R^\times \hat{\mathrm{g}} + \hat{R} K_{\mathrm{g}} y, && (19d) \\
\dot{\hat{p}}_i &= -\sigma_R^\times (\hat{p}_i - p_1) + \hat{R} K_i y, \ \ i = 2, 3, \cdots, N, && (19e)
\end{aligned}
$$

where the gain matrices $K_p, K_v, K_{\mathrm{g}}, K_i \in \mathbb{R}^{3 \times 3N}, i \in \{2, \cdots, N\}$ will be designed later, and the innovation term σ_R is given by

$$
\sigma_R := -\sum_{i=2}^{N} \rho_i \hat{r}_i \times r_i = -\psi_a(M\tilde{R}) + \Gamma x, \tag{20}
$$

with $\rho_i > 0$ for all $i = 2, \cdots, N$, $M := \sum_{i=2}^N \rho_i r_i r_i^\top$ and $\Gamma = [0_{3\times 9}, -\rho_2 r_2^\times \hat{R}, \cdots, -\rho_N r_N^\times \hat{R}] \in \mathbb{R}^{3 \times (6+3N)}$. In view of (1a)-(1c), (19a)-(19e) and (20), one obtains the following close-loop system:

$$
\begin{cases}
\dot{\tilde{R}} &= \tilde{R}(\sigma_R)^\times, \\
\dot{\tilde{p}}_e &= -\omega^\times \tilde{p}_e + \tilde{v} - K_p y, \\
\dot{\tilde{v}} &= -\omega^\times \tilde{v} + \tilde{\mathrm{g}} - K_v y, \\
\dot{\tilde{\mathrm{g}}} &= -\omega^\times \tilde{\mathrm{g}} - K_{\mathrm{g}} y, \\
\dot{\tilde{r}}_i &= -\omega^\times \tilde{r}_i - K_i y, \quad i = 2, 3, \cdots, N
\end{cases} \tag{21}
$$

Define the new variable $x := [\tilde{p}_e^\top, \tilde{v}^\top, \tilde{\mathrm{g}}^\top, \tilde{r}_2^\top, \cdots, \tilde{r}_N^\top]^\top \in \mathbb{R}^{6+3N}$. Then, the closed-loop system (21) with σ_R defined in (20) can be rewritten in the compact form (3) with

$$
A = \begin{bmatrix}
-\omega^\times & I_3 & 0 & 0 & 0 & 0 \\
0 & -\omega^\times & I_3 & 0 & 0 & 0 \\
0 & 0 & -\omega^\times & 0 & 0 & 0 \\
0 & 0 & 0 & -\omega^\times & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & -\omega^\times
\end{bmatrix}, \tag{22}
$$

$$
C = \begin{bmatrix}
\Pi_1 & 0 & 0 & 0 & \cdots & 0 \\
\Pi_2 & 0 & 0 & -\Pi_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\Pi_N & 0 & 0 & 0 & 0 & -\Pi_N
\end{bmatrix}, \tag{23}
$$

and $\left[K_p^\top \ K_v^\top \ K_{\mathrm{g}}^\top \ K_2^\top \ \cdots K_N^\top\right]^\top := K = k(t) P C^\top Q(t)$ where $P \in \mathbb{R}^{(6+3N)\times(6+3N)}$ is the solution to the CRE (4).

Lemma 4: Consider the pair (A, C) defined in (22) and (23). Then, there exist constants $\delta, \mu > 0$ such that the inequality (6) holds for all $t \geq 0$.

Remark 2: The proof of Lemma 4 is similar to the proof of Lemma 3, which is also omitted here due to space limitations. Lemma 4 shows that the pair (A, C) is uniformly observable. Suppose that $Q(t)$ and $V(t)$ are uniformly positive definite, from Lemma 1 one guarantees the existence of positive constants p_m and p_M such that $p_m I_{6+3N} \leq P(t) \leq p_M I_{6+3N}$.

Theorem 3: Consider the closed-loop system (3) with system parameters (A, C) defined in (22) and (23) and σ_R defined by (20). Let Assumption 1 hold and $M = M^\top$ be positive semi-definite with three distinct eigenvalues. Suppose that the matrices $V(t)$ and $Q(t)$ for CRE (4) are uniformly positive definite. Choose $k(t) > \frac{1}{2}, \forall t \geq 0$ and $P(0)$ being positive definite. Then, the results of Theorem 1 hold.

Proof: From the definition of Γ in (20), one can show that $\|\Gamma\| \leq \sum_{i=2}^N \rho_i \|r_i\| := c_\Gamma$. Then, the proof of Theorem 3 is completed by using similar steps as in the proof of Theorem 1. ■

IV. Simulation Results

In this section, we consider the trajectory generated by a real flight of a quadrotor from the EuRoc dataset [21]. The sampling rate of the IMU measurements is 200Hz. The stereo bearing measurements of $N = 5$ randomly selected landmarks are generated using (2) with $p_i^{\mathcal{C}_s} = R_{cs}^\top (R_G^\top (p_i - p_G) - p_s)$ for each $s \in \{L, R\}$, with R_G, p_G being the rotation and position obtained from ground truth provided in the dataset. The sampling rate of the stereo bearing measurements is the same as the IMU for the sake of simplicity, and additional noise in the bearing measurement is considered as [22]

$$x_i^s = \frac{\text{sign}(x_{i,3}^s)}{d_{x_i^s}}(x_{i,1}^s/x_{i,3}^s + n_{i,2}, x_{i,2}^s/x_{i,3}^s + n_{i,1}, 1), \quad (24)$$

$$d_{x_i^s} = \|x_{i,1}^s/x_{i,3}^s + n_{i,1}, x_{i,2}^s/x_{i,3}^s + n_{i,2}, 1\|, \quad (25)$$

with $s \in \{L, R\}, i = 1, 2, \cdots, 5$ and $n_{i,1}, n_{i,2}$ being uncorrelated zero-mean uniformly distributed noise inputs with a maximum deviation of 0.005. The parameters of the observer (19a)-(19e) are taken as $\rho_i = 1/4, i = 2, \cdots, 4, P(0) = I_{21}, V(t) = \text{diag}(2I_6, I_3, 0.1I_{12}), Q(t) = 1000I_{15}$ and $k(t) = 1$. The proposed observer has been discretized with sampling period $dT = 1/200$ as $\hat{R}_{k+1} = \hat{R}_k \exp((dT\omega_k - \hat{R}_k^\top \sigma_R)^\times)$ for the estimated attitude, and a first-order numerical integration for the estimated state of vectors. A discrete version of the CRE (4) has been considered by assuming that the angular velocity ω is constant between two samplings. The simulation results are given in Fig. 1. As one can see, the estimates converge to the vicinity of the ground truth after a few seconds.

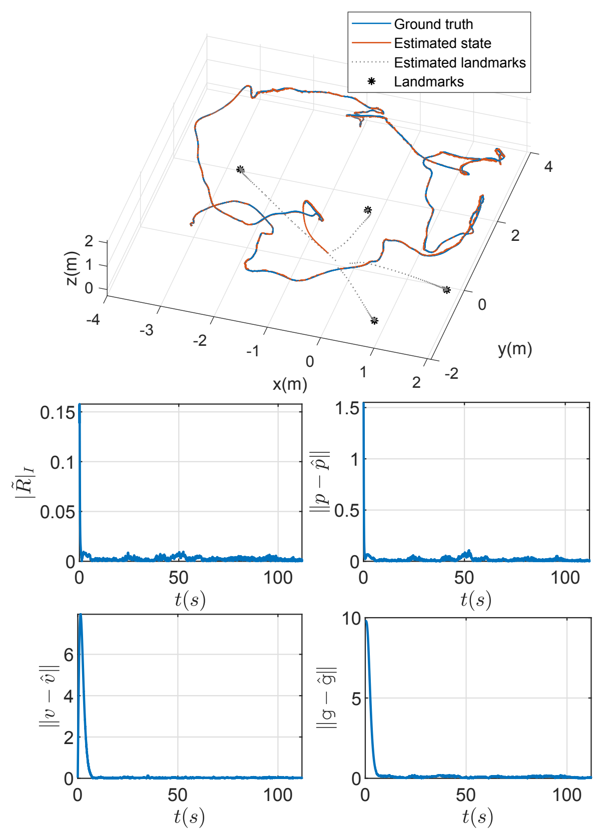

Fig. 1: Simulations using the trajectory generated by a real flight from the EuRoc dataset (Vicon Room 1 01).

V. Conclusion

In this paper, the problem of attitude, position and linear velocity estimation for inertial navigation relying on IMU and stereo bearing measurements has been addressed. First, a stability result for a generic class of nonlinear time-varying systems on $SO(3) \times \mathbb{R}^n$ has been derived. This stability result has been exploited to develop two nonlinear navigation observers, relying on IMU and landmark measurements. Both cases of a single landmark and multiple landmarks have been considered. Numerical results using simulated landmark measurements together with IMU data from the EuRoc dataset have been provided to illustrate the performance of the proposed observers.

References

[1] S. Bonnabel, P. Martin, and P. Rouchon, "Symmetry-preserving observers," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2514–2526, 2008.

[2] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the special orthogonal group," *IEEE Transactions on automatic control*, vol. 53, no. 5, pp. 1203–1218, 2008.

[3] A. Barrau and S. Bonnabel, "Invariant particle filtering with application to localization," in *Proceedings the 53rd IEEE Annual Conference on Decision and Control (CDC)*, 2014, pp. 5599–5605.

[4] H. Rehbinder and B. K. Ghosh, "Pose estimation using line-based dynamic vision and inertial sensors," *IEEE Transactions on Automatic Control*, vol. 48, no. 2, pp. 186–199, 2003.

[5] T. Hamel and C. Samson, "Position estimation from direction or range measurements," *Automatica*, vol. 82, pp. 137–144, 2017.

[6] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira, "A nonlinear position and attitude observer on SE(3) using landmark measurements," *Systems & Control Letters*, vol. 59, no. 3-4, pp. 155–166, 2010.

[7] M.-D. Hua, T. Hamel, R. Mahony, and J. Trumpf, "Gradient-like observer design on the Special Euclidean group $SE(3)$ with system outputs on the real projective space," in *Proceedings of the 54th IEEE Annual Conference on Decision and Control (CDC)*, 2015, pp. 2139–2145.

[8] A. Khosravian, J. Trumpf, R. Mahony, and C. Lageman, "Observers for invariant systems on Lie groups with biased input measurements and homogeneous outputs," *Automatica*, vol. 55, pp. 19–26, 2015.

[9] M. Wang and A. Tayebi, "Globally asymptotically stable hybrid observers design on SE(3)," in *Proceedings of the 56th IEEE Annual Conference on Decision and Control (CDC)*, 2017, pp. 3033–3038.

[10] ——, "Hybrid pose and velocity-bias estimation on se(3) using inertial and landmark measurements," *IEEE Transactions on Automatic Control*, vol. 64, no. 8, pp. 3399 – 3406, 2019.

[11] A. I. Mourikis and S. I. Roumeliotis, "A multi-state constraint Kalman filter for vision-aided inertial navigation," in *Proceedings of IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2007, pp. 3565–3572.

[12] A. I. Mourikis, N. Trawny, S. I. Roumeliotis, A. E. Johnson, A. Ansar, and L. Matthies, "Vision-aided inertial navigation for spacecraft entry, descent, and landing," *IEEE Transactions on Robotics*, vol. 25, no. 2, pp. 264–280, 2009.

[13] A. Barrau and S. Bonnabel, "The invariant extended Kalman filter as a stable observer," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1797–1812, 2017.

[14] M. Wang and A. Tayebi, "A globally exponentially stable nonlinear hybrid observer for 3D inertial navigation," in *Proceedings of the 57th IEEE Annual Conference on Decision and Control (CDC)*, 2018, pp. 1367–1372.

[15] M.-D. Hua and G. Allibert, "Riccati observer design for pose, linear velocity and gravity direction estimation using landmark position and IMU measurements," in *2018 IEEE Conference on Control Technology and Applications (CCTA)*. IEEE, 2018, pp. 1313–1318.

[16] F. Le Bras, T. Hamel, R. Mahony, and C. Samson, "Observers for position estimation using bearing and biased velocity information," in *Sensing and Control for Autonomous Vehicles*. Springer, 2017, pp. 3–23.

[17] R. S. Bucy, "Global theory of the riccati equation," *Journal of computer and system sciences*, vol. 1, no. 4, pp. 349–361, 1967.

[18] ——, "The riccati equation and its bounds," *Journal of computer and system sciences*, vol. 6, no. 4, pp. 343–353, 1972.

[19] S. Berkane, A. Abdessameud, and A. Tayebi, "Hybrid attitude and gyro-bias observer design on $SO(3)$," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 6044–6050, 2017.
[20] M. Wang and A. Tayebi, "Hybrid nonlinear observers for inertial navigation using landmark measurements," *Submitted to IEEE Transactions on Automatic Control*, pp. 1–14, 2019, arXiv:1906.04689v1.
[21] M. Burri, J. Nikolic, P. Gohl, T. Schneider, J. Rehder, S. Omari, M. W. Achtelik, and R. Siegwart, "The EuRoC micro aerial vehicle datasets," *The International Journal of Robotics Research*, vol. 35, no. 10, pp. 1157–1163, 2016.
[22] T. Hamel and C. Samson, "Riccati observers for the nonstationary PnP problem," *IEEE Transactions on Automatic Control*, vol. 63, no. 3, pp. 726–741, 2018.

APPENDIX

A. Proof of Theorem 1

From (3)-(5) and the fact that the pair (A, C) is uniformly observable (as per Lemma 1), one can show that the set of equilibria of the system is given by $\Psi = \{(\tilde{R}, x) \in SO(3) \times \mathbb{R}^n : \psi_a(M\tilde{R}) = 0, x = 0\}$. From $\psi_a(M\tilde{R}) = 0$ and using the facts that $\psi_a(M\tilde{R}) = \text{vec} \circ \mathbb{P}_a(M\tilde{R})$ and $\mathbb{P}_a(M\tilde{R}) = (M\tilde{R} - \tilde{R}^\top M)/2$, one can conclude, as in [2], that $\tilde{R} \in \{\tilde{R} \in SO(3) : \ \tilde{R} = \mathcal{R}_\alpha(\pi, v), v \in \mathcal{E}(M)\}$, which proves item (i).

Consider the following real-valued function:

$$\mathcal{L}_P(x) = x^\top P^{-1} x. \tag{26}$$

One can show that $\frac{1}{p_M}\|x\|^2 \leq \mathcal{L}_P(x) \leq \frac{1}{p_m}\|x\|^2$. In view of (3) and (4), the time-derivative of $\mathcal{L}_P$ is given by

$$\begin{aligned}\dot{\mathcal{L}}_P &= x^\top(P^{-1}A + A^\top P^{-1} - 2k(t)C^\top QC + \dot{P}^{-1})x \\ &= -(2k(t)-1)x^\top C^\top QCx - x^\top P^{-1}VP^{-1}x \\ &\leq -\frac{v_m}{p_M^2}\|x\|^2 \leq -\lambda \mathcal{L}_P, \end{aligned} \tag{27}$$

where $\lambda = v_m p_m / p_M^2$, $v_m := \inf_{t\geq 0} \lambda_{\min}^{V(t)}$, and we made use of the facts $\dot{P}^{-1} = -P^{-1}\dot{P}P^{-1} = -P^{-1}A - A^\top P^{-1} + 2C^\top QC - P^{-1}VP^{-1}$ and $2k(t) - 1 > 0$. Hence, one has $\|x(t)\| \leq \sqrt{p_M/p_m}e^{-\frac{\lambda}{2}t}\|x(0)\|$, which implies that $x, \dot{x}$ are bounded, and x converges to zero exponentially. Note that the convergence of x is independent from the dynamics of the rotation. On the other hand, consider the following real-valued function

$$\mathcal{L}_M = \text{tr}((I - \tilde{R})M), \tag{28}$$

whose time-derivative is given by

$$\begin{aligned}\dot{\mathcal{L}}_M &= \text{tr}(-M\tilde{R}(-\psi_a(M\tilde{R}) + \Gamma x)^\times) \\ &\leq -2\|\psi_R\|^2 + 2c_\Gamma\|x\|\|\psi_R\|,\end{aligned}$$

where $\psi_R := \psi_a(M\tilde{R})$, $c_\Gamma := \sup_{t\geq 0}\|\Gamma\|_F$, and we made use of the facts $\|\Gamma x\| \leq c_\Gamma\|x\|$ and $\text{tr}(A, x^\times) = 2x^\top\psi_a(A)$ for all $A \in \mathbb{R}^{3\times 3}$. Consider the following Lyapunov function candidate:

$$\mathcal{L}(\tilde{R}, x) = \mathcal{L}_M(\tilde{R}) + \kappa\mathcal{L}_P(x), \tag{29}$$

whose time-derivative is given by

$$\begin{aligned}\dot{\mathcal{L}}(\tilde{R}, x) &\leq -2\|\psi_R\|^2 + 2c_\Gamma\|x\|\|\psi_R\| - \kappa\frac{v_m}{p_M^2}\|x\|^2 \\ &= -\zeta^\top H\zeta, \quad H := \begin{bmatrix} 2 & -c_\Gamma \\ -c_\Gamma & \kappa\frac{v_m}{p_M^2}\end{bmatrix}\end{aligned} \tag{30}$$

where $\zeta := [\|\psi_R\|, \|x\|]^\top$. Choosing $\kappa > \frac{c_\Gamma^2 p_M^2}{2v_m}$, one can show that the matrix H is positive definite and $\dot{\mathcal{L}} \leq 0$. From (7) and (8) given in Lemma 2, one has $\|\psi_R\|^2 \leq \text{tr}(\underline{M}(I - R)) \leq 4\lambda_{\max}^W|\tilde{R}|_I^2 \leq 4\lambda_{\max}^W$ with $W := \frac{1}{2}(\text{tr}(\underline{M})I_3 - \underline{M})$, which implies that ψ_R is bounded. From (3), (5) and (9), one has $\dot{\psi}_R = E(M\tilde{R})(-\psi_R + \Gamma x)$ with $E(M\tilde{R}) := \frac{1}{2}(\text{tr}(M\tilde{R})I_3 - \tilde{R}^\top M)$. Using the facts that $\|E(M\tilde{R})\| \leq \|\bar{M}\|_F$ and x is bounded, one can verify that $\dot{\psi}_R$ is bounded. Thus, from the fact that $x, \dot{x}$ and $\psi_R, \dot{\psi}_R$ are bounded, it follows that $\ddot{\mathcal{L}}$ is bounded. Therefore, from Barbalat's lemma, one has $\dot{\mathcal{L}} \to 0$ as $t \to \infty$, and in turn, $\zeta \to 0$ as $t \to \infty$, *i.e.*, $(\|\psi_R\|, \|x\|) \to (0, 0)$ as $t \to \infty$. This implies that the solution $(\tilde{R}, x)$ to (3)-(5) converges to the set Ψ.

Next, we need to show that the undesired equilibria $\Psi/(I_3, 0)$ are unstable. For each $v \in \mathcal{E}(M)$, let us define $R_v^* = \mathcal{R}_\alpha(\pi, v)$ and the open set $U_v^\delta := \{(\tilde{R}, x) \in SO(3) \times \mathbb{R}^n : \tilde{R} = R_v^*\exp(\delta u^\times), u \in \mathbb{S}^2, x = 0\}$ with δ sufficiently small. For any $(\tilde{R}, x) \in U_v^\delta$, pick a sufficiently small ϵ such that $\exp(\epsilon^\times) := (R_v^*)^\top\tilde{R}$ and $\exp(\epsilon^\times) \approx I_3 + \epsilon^\times$. Consequently, one obtains the dynamics of ϵ as follows:

$$\dot{\epsilon} = -\psi_a(MR_v^*(I_3 + \epsilon^\times)) = -W_v\epsilon \tag{31}$$

where $W_v = \text{tr}(MR_v^*)I_3 - (MR_v^*)^\top = (2v^\top Mv - \text{tr}(M))I_3 - (2vv^\top M - M)$, and we made use of the facts $\psi_a(MR_v^*) = 0, \psi_a(MR_v^*\epsilon^\times) = \text{vec} \circ \mathbb{P}_a(MR_v^*\epsilon^\times)$ and $MR_v^*\epsilon^\times + \epsilon^\times(MR_v^*)^\top = (W_v\epsilon)^\times$. Using the fact that M is positive semi-definite with three distinct eigenvalues, one verifies that $-v^\top W_v v = -v^\top Mv + \text{tr}(M) > 0$, which implies that for each $v \in \mathcal{E}(M)$, the matrix $-W_v$ has at least one positive eigenvalue. It follows that the undesired equilibrium $(R_v^*, 0) \in \Psi/(I_3, 0)$ is unstable, and hence, the desired equilibrium $(I_3, 0)$ is almost globally asymptotically stable. This completes the proof of item (ii).

Now, let us prove the local exponential stability result in item (iii). From (30) with $\kappa > \frac{c_\Gamma^2 p_M^2}{2v_m}$, one has $\mathcal{L}_M(\tilde{R}(t)) \leq \mathcal{L}(\tilde{R}(t), x(t)) \leq \mathcal{L}(\tilde{R}(0), x(0)) \leq \varepsilon_R$, for all $t \geq 0$. Hence, one verifies that $|\tilde{R}(t)|_I < 1$ for all $t \geq 0$. Moreover, one has $\varrho|\tilde{R}|_I^2 \leq \|\psi_R\|^2 \leq 4\lambda_{\max}^W|\tilde{R}|_I^2$ with $\varrho := \min_{\mathcal{L}_M(\tilde{R})\leq\varepsilon_R}(1 - |\tilde{R}|_I^2\cos\langle u, \bar{M}u\rangle)4\lambda_{\min}^W > 0$ and $\tilde{R} = \mathcal{R}_a(\theta, u)$. Let $\bar{\zeta} := [|\tilde{R}|_I, \|x\|]^\top$. In view of (7), (26) and (28), one obtains that

$$\underline{\alpha}\|\bar{\zeta}\|^2 \leq \mathcal{L} \leq \bar{\alpha}\|\bar{\zeta}\|^2, \tag{32}$$

where $\underline{\alpha} := \min\{4\lambda_{\min}^{\bar{M}}, \frac{\kappa}{p_M}\}$ and $\bar{\alpha} := \max\{4\lambda_{\max}^{\bar{M}}, \frac{\kappa}{p_m}\}$. Substituting $\varrho|\tilde{R}|_I^2 \leq \|\psi_R\|^2 \leq 4\lambda_{\max}^W|\tilde{R}|_I^2$ into (30), one has

$$\begin{aligned}\dot{\mathcal{L}} &\leq -2\varrho|\tilde{R}|_I^2 + 4c_W c_\Gamma\|x\||\tilde{R}|_I - \kappa\frac{v_m}{p_M^2}\|x\|^2 \\ &\leq -\bar{\zeta}^\top\bar{H}\bar{\zeta}, \ \bar{H} := \begin{bmatrix} 2\varrho & -2c_W c_\Gamma \\ -2c_W c_\Gamma & \kappa v_m/p_M^2\end{bmatrix}.\end{aligned} \tag{33}$$

where $c_W := \sqrt{\lambda_{\max}^W}$. Choosing $\kappa > \frac{2\lambda_{\max}^W c_\Gamma^2 p_M^2}{\varrho v_m} := k^*$, it is clear that both matrices H and $\bar{H}$ are positive definite since $\varrho \leq 4\lambda_{\max}^W$. In view of (32) and (33), one concludes

$$\|\bar{\zeta}(t)\| \leq \sqrt{\bar{\alpha}/\underline{\alpha}}e^{-\frac{1}{2}\lambda_{\min}^{\bar{H}}t}\|\bar{\zeta}(0)\|, \tag{34}$$

for all $t \geq 0$, which implies that $(\tilde{R}, x)$ converges to $(I_3, 0)$ exponentially. This completes the proof.