Globally Asymptotically Stable Hybrid Observers Design on $SE(3)$

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***Abstract*— This paper deals with the design of globally asymptotically stable invariant observers on the Special Euclidian group $SE(3)$. First, we propose a generic hybrid estimation scheme (depending on a generic potential function) evolving on $SE(3) \times \mathbb{R}^6$ for pose (orientation and position) and velocity-bias estimation. Thereafter, the proposed observer is formulated explicitly in terms of inertial vectors and landmark measurements. Interestingly, the proposed globally asymptotically stable estimator, leads to decoupled translational and rotational error dynamics (in the velocity bias-free case), which is an important feature in practical applications with noisy measurements and disturbances.**

I. INTRODUCTION

The development of reliable pose (*i.e,* attitude and position) estimation algorithms is instrumental for many applications such as autonomous underwater vehicles and unmanned aerial vehicles. Since there is no sensor that directly measures the attitude, the latter is usually determined using body-frame measurements of some known inertial vectors via static determination algorithms [1] which are generally sensitive to measurement noise. Alternatively, dynamic estimation algorithm using inertial vector measurements together with the angular velocity can be used to recover the attitude while filtering measurement noise (*e.g.,* Kalman filters [2], linear complimentary filters [3], nonlinear complementary filters [4]). In low-cost applications, angular velocity and inertial vector measurements can be obtained, for instance, from an inertial measurement unit (IMU) equipped with gyroscopes, accelerometers and magnetometers. The translational position and velocity can be estimated using a Global Positioning System (GPS). However, in GPS-denied environments such as indoor applications, recovering the position and linear velocity is a challenging task. Alternatively, inertial-vision systems combing IMU and on-board camera measurements have been considered for pose estimation [5]–[7].

Recently, a class of nonlinear observers on Lie groups including $SO(3)$ and $SE(3)$ have made their appearances in the literature. Invariant observers which take into account the topological properties of the motion space were developed in [4], [8], [9]. Motivated by the work of [4] on $SO(3)$, complementary observers on $SE(3)$ were proposed in [10], [11]. In practice, measurements of group velocity (translational velocity and angular velocity) are often corrupted by an unknown bias. Pose estimation using biased velocity measurements were considered in [12]–[14]. A nice feature of [14] is the fact that the observer incorporates (naturally) both inertial vector measurements (*e.g.,* from IMU) and landmark measurements (*e.g.,* from a stereo vision system). The observers proposed in [10]–[14] are shown to guarantee almost global asymptotic stability (AGAS), *i.e.,* the pose converges to the actual one from almost any initial condition except from a set of Lebesgue measure zero. This is the strongest result one can aim at when considering continuous time-invariant state observers on $SO(3)$ or $SE(3)$.

Recently, the topological obstruction to global asymptotic stability on $SO(3)$ using continuous time-invariant controllers (observers) has been successfully addressed via hybrid techniques such as [15]–[19].

In this paper we propose an approach for hybrid observers design on $SE(3)$ leading to global asymptotic stability. To the best of our knowledge, there is no work in the literature achieving global asymptotic stability results on $SE(3)$. Interestingly, the proposed global hybrid pose and velocity-bias estimation scheme, relying on biased group-velocity measurements, body-frame vectors and landmark measurements, leads to decoupled translational and rotational error dynamics (in the velocity bias-free case), which guarantees nice robustness and performance properties.

II. BACKGROUND

A. Notations and mathematical preliminaries

The sets of real, nonnegative real and natural numbers are denoted as $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. Given two matrices, $A, B \in \mathbb{R}^{m\times n}$, their Euclidean inner product is defined as $\langle\langle A, B\rangle\rangle = \mathrm{tr}(A^\top B)$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x^\top x}$, and the Frobenius norm of a matrix $X \in \mathbb{R}^{n\times m}$ is given by $\|X\|_F = \sqrt{\langle\langle X, X\rangle\rangle}$. The n-by-n identity matrix is denoted by I_n.

For each $A \in \mathbb{R}^{n\times n}$, we define $\mathcal{E}(A)$ as the set of all eigenvectors of A. Let λ_i^A be the i-th eigenvalue of A, and $\lambda_{\min}^A$ and $\lambda_{\max}^A$ be the minimum and maximum eigenvalues of A, respectively.

B. Pose representation and useful relations

Let $\mathcal{I}$ be an inertial frame and $\mathcal{B}$ be a body-attached frame. Let $p \in \mathbb{R}^3$ denote the rigid body position expressed in the inertial frame $\mathcal{I}$, and $R \in SO(3)$ denote the rotation of the body-attached frame $\mathcal{B}$ relative to the inertial frame $\mathcal{I}$. Let $\omega \in \mathbb{R}^3$ denote the angular velocity of the body-attached frame $\mathcal{B}$ with respect to inertial frame $\mathcal{I}$, expressed in frame $\mathcal{B}$. Let $v \in \mathbb{R}^3$ be the translational velocity, expressed in frame $\mathcal{B}$.

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We consider the problem of pose estimation of the rigid body (*i.e.,* position p and rotation R). The pose of the rigid body can be represented by an element of the *Special Euclidean group* $SE(3) := SO(3) \times \mathbb{R}^3$ which is a subset of the *affine group* $GA(3) := GL(3) \times \mathbb{R}^3$, defined as

$$g := \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}, \quad R \in SO(3), p \in \mathbb{R}^3$$

This representation is known as homogeneous representation which preserves the group structure of $SE(3)$ with the $GL(4)$ operation of matrix multiplication (i.e., $g_1 g_2 \in SE(3)$, for all $g_1, g_2 \in SE(3)$). The *Lie algebra* of $SE(3)$, denoted by $\mathfrak{se}(3)$, is given by

$$\mathfrak{se}(3) := \left\{ X \in \mathbb{R}^{4\times 4} | X = \begin{bmatrix} \omega^\times & v \\ 0 & 0 \end{bmatrix}, \omega, v \in \mathbb{R}^3 \right\},$$

where, the map $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ denotes the skew-symmetric matrix related to the cross product such that $x \times y = x^\times y$, for any $x, y \in \mathbb{R}^3$. A wedge map $(\cdot)^\wedge : \mathbb{R}^6 \mapsto \mathfrak{se}(3)$ is defined as

$$\xi^\wedge = \begin{bmatrix} \omega^\times & v \\ 0 & 0 \end{bmatrix}, \quad \xi = (\omega^\top, v^\top)^\top.$$

The *tangent spaces* of the group $SE(3)$, is identified by $T_g SE(3) = \{gX | \; g \in SE(3), X \in \mathfrak{se}(3)\}$.

For all $A \in \mathbb{R}^{n\times n}$, define $\mathbb{P}_a(A)$ as the anti-symmetric projection of A, such that $\mathbb{P}_a(A) = (A - A^\top)/2$. Let $\mathbb{P} : \mathbb{R}^{4\times 4} \to \mathfrak{se}(3)$ denote the projection of $\mathbb{A}$ on the Lie algebra $\mathfrak{se}(3)$ such that for all $X \in \mathfrak{se}(3), \mathbb{A} \in \mathbb{R}^{4\times 4}$, one has $\langle\langle X, \mathbb{A}\rangle\rangle = \langle\langle X, \mathbb{P}(\mathbb{A})\rangle\rangle = \langle\langle \mathbb{P}(\mathbb{A}), X\rangle\rangle$. For all $A \in \mathbb{R}^{3\times 3}, b, c^\top \in \mathbb{R}^3$ and $d \in \mathbb{R}$, one has

$$\mathbb{P}\left(\begin{bmatrix} A & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \mathbb{P}_a(A) & b \\ 0 & 0 \end{bmatrix}.$$

For any $\mathbb{A} = [a_{ij}] \in \mathbb{R}^{4\times 4}$ and $y \in \mathbb{R}^6$, let us define the map

$$\psi(\mathbb{A}) := \frac{1}{2} \left[a_{32} - a_{23}, a_{13} - a_{31}, a_{21} - a_{12}, a_{14}, a_{24}, a_{34} \right]^\top,$$

such that $\langle\langle \mathbb{A}, y^\wedge \rangle\rangle = 2\psi(\mathbb{A})^\top y$. The adjoint action map is defined for all $g = g(R, p) \in SE(3)$ as

$$\mathrm{Ad}_g = \begin{bmatrix} R & 0 \\ p^\times R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}. \quad (1)$$

For all $g_1, g_2 \in SE(3)$ and $x \in \mathbb{R}^6$, the following properties hold:

$$g_1 x^\wedge g_1^{-1} = (\mathrm{Ad}_{g_1} x)^\wedge, \quad \mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1} \mathrm{Ad}_{g_2}.$$

For any two vectors $r, b \in \mathbb{R}^4$, let us define the following wedge product (exterior product) $\wedge$ as

$$b \wedge r := \begin{bmatrix} b_v \times r_v \\ b_s r_v - r_s b_v \end{bmatrix} \in \mathbb{R}^6, \quad (2)$$

where $r = (r_v, r_s)^\top, b = (b_v, b_s)^\top$ with $r_v, b_v \in \mathbb{R}^3$ and $r_s, b_s \in \mathbb{R}$.

It is easy to verify that $r \wedge r = 0$ and $b \wedge r = -r \wedge b$. Let us introduce some mathematical identities which can be easily verified by simple calculations

$$\psi\left((I - g)rr^\top\right) = \frac{1}{2}(gr) \wedge r, \quad (gb) \wedge (gr) = \mathrm{Ad}_{g^{-1}}^\top b \wedge r$$

where, $g \in SE(3)$ and $r \in \mathbb{R}^4$. Let $\mathcal{M}_0$ denote the submanifold of $\mathbb{R}^{4\times 4}$, defined as

$$\mathcal{M}_0 := \left\{ M \middle| M = \begin{bmatrix} M_1 & m_2 \\ 0 & 0 \end{bmatrix}, M_1 \in \mathbb{R}^{3\times 3}, m_2 \in \mathbb{R}^3 \right\}.$$

Then, for all $g \in SE(3)$, and $M, \bar{M} \in \mathcal{M}_0$, the following properties can be easily verified

$$\mathbb{P}(gM) = \mathbb{P}(g^{-\top} M) \quad \mathrm{tr}(g^\top g M \bar{M}^\top) = \mathrm{tr}(M \bar{M}^\top). \quad (3)$$

Throughout the paper, we will also make use of the following matrix decomposition:

$$\begin{bmatrix} A & b \\ b^\top & d \end{bmatrix} = \begin{bmatrix} I_3 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A - bb^\top d^{-1} & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} I_3 & 0 \\ b^\top d^{-1} & 1 \end{bmatrix}.$$

III. Problem Formulation and Preliminary Results

The pose $g \in SE(3)$ of a rigid body is governed by the following differential equation:

$$\dot{g} = g\xi^\wedge, \quad (4)$$

where, $\xi \in \mathbb{R}^6$. Note that system (4) is left invariant in the sense that it preserves the Lie group invariance properties with respect to constant translation and constant rotation of the body-attached frame $\mathcal{B}$. Assume that the group velocity ξ is bounded, continuous, and available for measurement. The measured group velocity, denoted by $\xi_y = (\omega_y^\top, v_y^\top)^\top$, can be subject to a constant or slowly varying bias $b_a = (b_\omega^\top, b_v^\top)^\top \in \mathbb{R}^6$ such that

$$\xi_y = \xi + b_a. \quad (5)$$

A family of constant n reference vectors $r_i \in \mathbb{R}^4, i = 1, 2, \cdots, n$, known in the inertial frame $\mathcal{I}$, are assumed to be measured in the body-attached frame $\mathcal{B}$ as

$$b_i := g^{-1} r_i, \quad i = 1, 2, \cdots, n \quad (6)$$

Assume that, among the n reference vectors, there are n_1 feature points (or landmarks) with the form $r_i = [(p_i^\mathcal{I})^\top, 1]^\top, i = 1, \ldots, n_1$, and $n - n_1$ inertial vectors with the form $r_i = [(v_i^\mathcal{I})^\top, 0]^\top, i = n_1+1, \cdots, n$. Then, from (6), the measurements of the landmarks and inertial vectors have the form of $b_i = [(p_i^\mathcal{B})^\top, 1]^\top, p_i^\mathcal{B} = R^\top(p_i^\mathcal{I} - p), i = 1, \cdots n_1$ and $b_i = [(v_i^\mathcal{B})^\top, 0]^\top, v_i^\mathcal{B} = R^\top v_i^\mathcal{I}, i = n_1 + 1, \cdots n$, respectively.

Define the following weighted geometric center of all the landmarks and their measurements:

$$p_c^\mathcal{I} := \sum_{i=1}^{n_1} \frac{\alpha_i}{\ell} p_i^\mathcal{I}, \; p_c^\mathcal{B} := \sum_{i=1}^{n_1} \frac{\alpha_i}{\ell} p_i^\mathcal{B} = R^\top(p_c^\mathcal{I} - p), \quad (7)$$

where, $\alpha_i > 0$ for all $i = 1, 2, \cdots, n_1$, and $\ell = \sum_{i=1}^{n_1} \alpha_i$. Let us introduce the following *modified inertial vectors*

$$v_i^\mathcal{I} = p_i^\mathcal{I} - p_c^\mathcal{I} \quad v_i^\mathcal{B} = p_i^\mathcal{B} - p_c^\mathcal{B} = R^\top v_i^\mathcal{I}, \quad (8)$$

for all $i=1,2,\cdots,n_1$. It is easy to verify that $\sum_{i=1}^{n_1}\alpha_i v_i^{\mathcal{I}}=0$. Let $V^{\mathcal{I}}=\{v_i^{\mathcal{I}}, i=1,2,\cdots,n\}$ be the set of all inertial vectors and modified inertial vectors.

Assumption 1: At least one landmark point is measured, and at least two vectors from the set $V^{\mathcal{I}}$ are non-collinear.

Remark 1: Assumption 1 implies that $n_2\geq 1$ and $n=n_1+n_2\geq 3$. This assumption is standard in estimation problems in $SE(3)$, *e.g.,* [11], [13], [14], which is satisfied in the following particular cases:

- Three different landmark points are measured such as the corresponding $v_i^{\mathcal{I}}$, $i=1,2,3$, are non-collinear.
- One landmark point and two non-collinear inertial vectors are measured.
- Two different landmark points and one inertial vector are measured such that the corresponding $v_1^{\mathcal{I}}$ and $v_3^{\mathcal{I}}$ are non-collinear.

Our objective is to design a globally asymptotically stable hybrid pose and velocity-bias estimation scheme that provides global estimates of g and b_a, respectively, using the available measurements satisfying Assumption 1.

Let us introduce the following matrix

$$\mathbb{A} := \sum_{i=1}^{n} k_i r_i r_i^\top = \begin{bmatrix} A & b \\ b^\top & d \end{bmatrix} \in \mathbb{R}^{4\times 4}, \quad (9)$$

where, $k_i>0$ for all $i=1,2,\cdots,n$, and $d:=\sum_{i=1}^{n_1} k_i\in\mathbb{R}, b:=\sum_{i=1}^{n_1} k_i p_i^{\mathcal{I}}\in\mathbb{R}^3$ and $A:=\sum_{i=1}^{n_1} k_i p_i^{\mathcal{I}}(p_i^{\mathcal{I}})^\top+\sum_{i=n_1+1}^{n} k_i v_i^{\mathcal{I}}(v_i^{\mathcal{I}})^\top\in\mathbb{R}^{3\times 3}$. Let $\alpha_i=k_i, i=1,\cdots,n_1$, one can verify that

$$p_c^{\mathcal{I}} = \frac{1}{\sum_{i=1}^{n_1} k_i}\sum_{i=1}^{n_1} k_i p_i^{\mathcal{I}} = bd^{-1}.$$

In the sequel, we will make use of $p_c^{\mathcal{I}}$ and bd^{-1} equivalently. The following lemmas (whose proofs are omitted for space limitation) will be useful in the remainder of this paper.

Lemma 1: Consider the matrix $\mathbb{A}$ defined in (9), under assumption 1 the following statements hold:

1) $d>0$.
2) Define the Schur Complement of d in matrix $\mathbb{A}$ as $Q:=A-bb^\top d^{-1}$. Then, the matrix $W_Q:=\mathrm{tr}(Q)I_3-Q$ is positive definite.

Lemma 2: Consider $\mathbb{A}$ defined in (9). Then, for any $\bar{g},g\in SE(3)$, the following statements hold:

$$\mathrm{tr}\left((I-g)\mathbb{A}(I-g)^\top\right) = \sum_{i=1}^{n} k_i\|r_i-g^{-1}r_i\|^2, \quad (10)$$

$$\psi((I-g^{-1})\mathbb{A}) = \frac{1}{2}\sum_{i=1}^{n} k_i(g^{-1}r_i)\wedge r_i. \quad (11)$$

Lemma 3: Let $Q=Q^\top$ be a positive definite matrix with three distinct eigenvalues $\lambda_1^Q>\lambda_2^Q>\lambda_3^Q>0$. Define

$$\Delta(u,v) = u^\top\left(\mathrm{tr}(Q)I_3 - Q - 2\lambda_v^Q(I_3-vv^\top)\right)u, \quad (12)$$

with $v\in\mathcal{E}(Q)$ and $u\in\mathcal{S}^2$. Then, there exists a positive Δ_Q^* given by

$$\Delta_Q^* := \min_{v\in\mathcal{E}(Q)}\max_{u\in\mathcal{E}(Q)}\Delta(u,v) = \mathrm{tr}(Q)-\lambda_{\max}^Q > 0. \quad (13)$$

IV. Main Results

A. Gradient-based Hybrid Pose and Velocity-bias Estimation Scheme

Consider a positive-valued continuously differentiable function $\mathcal{U}(g)$ on $\mathcal{D}\subseteq SE(3)$. For all $g\in SE(3)$, $\nabla_g\mathcal{U}(g)\in T_gSE(3)$ denotes the gradient of $\mathcal{U}$, with respect to g. Let $\Psi_{\mathcal{U}}\subset SE(3)$ denote the set of critical points. Let $\mathcal{A}=\{I_4\}$ be the desired critical point and $\mathcal{X}=\Psi_{\mathcal{U}}/\mathcal{A}$ be the set of undesired critical points.

Let $\hat{g}$ and $\hat{b}_a$ denote, respectively, the estimates of the rigid body pose and velocity bias. Define the pose estimation error $\tilde{g}:=g\hat{g}^{-1}$ and bias estimation error $\tilde{b}_a=\hat{b}_a-b_a$. Given a nonempty finite set *transformation set* $\mathbb{Q}$ and a positive δ, define the flow set $\mathcal{F}$ and jump set $\mathcal{J}$ as

$$\mathcal{F} = \left\{\tilde{g}:\mathcal{U}(\tilde{g})-\min\nolimits_{g_q\in\mathbb{Q}}\mathcal{U}(\tilde{g}g_q)\leq\delta\right\}, \quad (14)$$

$$\mathcal{J} = \left\{\tilde{g}:\mathcal{U}(\tilde{g})-\min\nolimits_{g_q\in\mathbb{Q}}\mathcal{U}(\tilde{g}g_q)\geq\delta\right\}, \quad (15)$$

The hysteresis gap δ and the set $\mathbb{Q}\subset SE(3)$ will be designed later. Then, we propose the following pose and velocity-bias estimation scheme:

$$\underbrace{\begin{array}{l}\dot{\hat{g}} = \hat{g}(\xi_y-\hat{b}_a+k_\beta\beta)^\wedge \\ \dot{\hat{b}}_a = \mathrm{Proj}_\Delta(\hat{b}_a,-\Gamma\sigma_b)\end{array}}_{(\tilde{g},\tilde{b}_a)\in\mathcal{F}\times\mathbb{R}^6}\ \underbrace{\begin{array}{l}\hat{g}^+ = g_q^{-1}\hat{g}, g_q=\gamma(\tilde{g}) \\ \hat{b}_a^+ = \hat{b}_a\end{array}}_{(\tilde{g},\tilde{b}_a)\in\mathcal{J}\times\mathbb{R}^6}, \quad (16)$$

$$\beta := \mathrm{Ad}_{\hat{g}^{-1}}\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g})), \quad (17)$$

$$\sigma_b := \mathrm{Ad}_{\hat{g}}^\top\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g})), \quad (18)$$

where, $\hat{g}(0)\in SE(3),\hat{b}_a(0)\in\mathbb{R}^6$ with $\|\hat{b}_a(0)\|<\Delta$, and $k_\beta\in\mathbb{R},\Gamma\in\mathbb{R}^{6\times 6}$ are strictly positive. The set-valued map $\gamma:SE(3)\to SE(3)$ is defined as $\gamma(\tilde{g})=\arg\min_{g_q\in\mathbb{Q}}\mathcal{U}(\tilde{g}g_q)$. The map Proj_Δ is a smooth projection algorithm defined as in [20]

$$\mathrm{Proj}_\Delta(\hat{b}_a,-\Gamma\sigma_b) = \begin{cases} -\Gamma\sigma_b, & \hat{b}_a\in\Pi_\Delta \text{ or } -\nabla_{\hat{b}_a}\mathcal{P}^\top\Gamma\sigma_b\leq 0 \\ -\left(I_6-\varrho(\hat{b}_a)\Gamma\frac{\nabla_{\hat{b}_a}\mathcal{P}\nabla_{\hat{b}_a}\mathcal{P}^\top}{\nabla_{\hat{b}_a}\mathcal{P}^\top\Gamma\nabla_{\hat{b}_a}\mathcal{P}}\right)\Gamma\sigma_b, & \text{otherwise}\end{cases},$$

where, $\mathcal{P}(\hat{b}_a):=\|\hat{b}_a\|-\Delta$, $\Pi_\Delta=\{\hat{b}_a|\mathcal{P}(\hat{b}_a)\leq 0\}$, $\Pi_{\Delta,\epsilon}=\{\hat{b}_a|\mathcal{P}(\hat{b}_a)\leq\epsilon\}$ and $\varrho(\hat{b}_a):=\min\{1,\mathcal{P}(\hat{b}_a)/\epsilon\}$ for some positive parameters Δ and ϵ. Given $\|\hat{b}_a(0)\|<\Delta$, one can verify that the projection map Proj_Δ satisfies the following properties

1) $\|\hat{b}_a(t,j)\|\leq\Delta+\epsilon$, for all $(t,j)\succeq(t_0,0)$;
2) $\tilde{b}_a^\top\Gamma^{-1}\mathrm{Proj}_\Delta(\hat{b}_a,-\Gamma\sigma_b)\leq-\tilde{b}_a^\top\sigma_b$.

Now, one can state one of our main results that provides a hybrid pose and velocity-bias estimation relying on a general potential function on $SE(3)$.

Theorem 1: Consider the pose kinematics (4) with the observer (14)-(18). Assume that the potential function $\mathcal{U}$ is continuously differentiable on $SE(3)$. The nonempty finite set $\mathbb{Q}$ and hysteresis gap δ are chosen such that $\mathcal{X}\subseteq\mathcal{J}$. Then, the number of discrete jumps is finite and the equilibrium set $\bar{\mathcal{A}}=\mathcal{A}\times\{0\}$ is uniformly globally asymptotically stable.

Proof: In view of (4), (5) and (16) - (18), the closed loop dynamics during the flows of $\mathcal{F}$ are given by

$$\underbrace{\begin{aligned}\dot{\tilde{g}} &= \tilde{g}(\mathrm{Ad}_{\hat{g}}(\tilde{b}_a - k_\beta \beta))^\wedge \\ \dot{\tilde{b}}_a &= \mathrm{Proj}_\Delta(\hat{b}_a, -\Gamma\sigma_b)\end{aligned}}_{(\tilde{g},\tilde{b}_a)\in\mathcal{F}\times\mathbb{R}^6} \quad \underbrace{\begin{aligned}\tilde{g}^+ &= \tilde{g}g_q, g_q = \gamma(\tilde{g}) \\ \tilde{b}_a^+ &= \tilde{b}_a\end{aligned}}_{(\tilde{g},\tilde{b}_a)\in\mathcal{J}\times\mathbb{R}^6}. \tag{19}$$

Let us consider the following real-valued function on $SE(3)$,

$$V(\tilde{g}, \tilde{b}_a) = \mathcal{U}(\tilde{g}) + \tilde{b}_a\Gamma^{-1}\tilde{b}_a, \tag{20}$$

which is positive definite with respect to $\bar{\mathcal{A}}$. Taking the time derivative of V, along the trajectories of (19), one has

$$\begin{aligned}\dot{V}(\tilde{g}, \tilde{b}_a) &= \langle \nabla_{\tilde{g}}\mathcal{U}(\tilde{g}), \tilde{g}\left(\mathrm{Ad}_{\hat{g}}(\tilde{b}_a - k_\beta\beta(\tilde{g}))\right)^\wedge\rangle_{\tilde{g}} \\ &\quad + 2\tilde{b}_a^\top\Gamma^{-1}\mathrm{Proj}_\Delta\left(\hat{b}_a, -\Gamma\sigma_b(\tilde{g})\right) \\ &\leq -2k_\beta\left\|\psi\left(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}(\tilde{g})\right)\right\|^2,\end{aligned}$$

where, we made use of the fact $\mathrm{Ad}_{\hat{g}}\mathrm{Ad}_{\hat{g}^{-1}} = I_6$, and the properties of ψ and Proj_Δ. Thus, V is non-increasing along the flows of (19). Moreover, for any $(\tilde{g}, \tilde{b}_a) \in \mathcal{J}\times\mathbb{R}^6$, one has

$$V(\tilde{g}^+, \tilde{b}_a^+) - V(\tilde{g}, \tilde{b}_a) = \min_{g_q\in\mathbb{Q}}\mathcal{U}(\tilde{g}g_q) - \mathcal{U}(\tilde{g}) \leq -\delta,$$

which implies that V is strictly decreasing over the jumps of (19). Using ([21], Theorem 23), it follows that the set $\bar{\mathcal{A}}$ is stable.

To show the asymptotic stability and finite number of jumps, one can verify that

$$0 \leq V(t,j) \leq V(t, j-1) - \delta \leq V(t_0, 0) - \delta j, \tag{21}$$

where $(t,j), (t, j-1) \in \mathrm{dom}\ (\tilde{g}, \tilde{b}_a)$ with $(t,j) \succeq (t, j-1)$. This leads to $j \leq J := \left\lceil \frac{V(t_0,0)}{\delta}\right\rceil$, where, $\lceil\cdot\rceil$ denotes the ceiling function. Hence, one concludes that the number of jumps is finite and it is linked to initial energy of the system.

In view of (21), it follows from ([22], Theorem 4.7) that $(\tilde{g}, \tilde{b}_a)$ must converge to the largest invariant subset of $(\mathcal{F}\times\mathbb{R}^6)\cap\mathcal{W}$ with

$$\mathcal{W} = \left\{(\tilde{g}, \tilde{b}_a) \in SE(3)\times\mathbb{R}^6 |\ \tilde{g}\in\Psi_{\mathcal{U}}, \tilde{b}_a \in \mathbb{R}^6\right\},$$

where, we made use of the definition of critical points. Since $\Psi_{\mathcal{U}} = \mathcal{X}\cup\mathcal{A}$, $\mathcal{X}\subseteq\mathcal{J}$ and $\mathcal{A}\subseteq\mathcal{F}$, one has $\mathcal{F}\cap\Psi_{\mathcal{U}} = \mathcal{A}$, hence the solution of $\tilde{g}$ must converge to $\mathcal{A}$. Hence, one can conclude that $\tilde{g}\to I_4$. For $\tilde{g}\equiv I_4$, it follows that $\dot{\tilde{g}}\equiv 0$. Using the fact $\beta(\tilde{g})\equiv 0$, one can conclude from (19) that $\tilde{b}_a \equiv 0$. Therefore, the solutions of $(\tilde{g}, \tilde{b}_a)$ must converge to $\bar{\mathcal{A}}$. Finally, the set $\bar{\mathcal{A}}$ is globally attractive and stable which shows that $\bar{\mathcal{A}}$ is globally asymptotically stable. ■

Remark 2: The observers proposed in [10]–[14] are shown to guarantee almost global asymptotic stability due to the topological obstruction when considering continuous time-invariant state observers on $SE(3)$. The proposed hybrid observer in Theorem 1, uses a new switching mechanism, inspired from [19], which changes directly the observer state through appropriate jumps in the direction of a decreasing potential function on $SE(3)$. The jump transitions occur when the estimation error is close to the critical points. This switching mechanism is different from the principle used, for instance in [16], [17], which consists in incorporating the jumps in the observer's correcting term.

B. Implementation Using Inertial Vector Measurements

Let us consider the following potential function

$$\mathcal{U}_1(\tilde{g}) = \frac{1}{2}\mathrm{tr}\left((I_4 - \tilde{g})\mathbb{A}(I_4 - \tilde{g})^\top\right). \tag{22}$$

with $\mathbb{A}$ defined in (9). Some useful properties of the potential function are given in the following lemma (whose proof are omitted due to space limitation).

Lemma 4: Consider $\mathbb{A}$ defined in (9) such that $d > 0$ and $Q = A - bb^\top d^{-1}$ is symmetric positive definite. Consider the potential function $\mathcal{U}_1$ defined in (22). Then, the following properties hold:

$$\nabla_g\mathcal{U}_1(g) = g\mathbb{P}((I_4 - g^{-1})\mathbb{A}), \tag{23}$$

$$\begin{aligned}\Psi_{\mathcal{U}_1} = \{I_4\}\bigcup\{g = g(R,p)|\ &R = \mathcal{R}_a(\pi, v), \\ &p = (I_3 - \mathcal{R}_a(\pi, v))bd^{-1}, v\in\mathcal{E}(Q)\},\end{aligned} \tag{24}$$

where, $\mathcal{E}(Q)$ is the set of eigenvectors of Q, and the map $\mathcal{R}_a : \mathbb{R}\times\mathcal{S}^2\to SO(3)$ is known as angle-axis parametrization of $SO(3)$ defined as $\mathcal{R}_a(\theta, u) = I_3 + \sin\theta u^\times + (1 - \cos\theta)(u^\times)^2$.

Let us introduce the following transformation set

$$\begin{aligned}\mathbb{Q} = \{g\in SE(3)|g = g(R,p), R = \mathcal{R}_a(\theta, u), \\ p = (I - R)bd^{-1}, u\in\mathcal{E}(Q), \theta\in(0,\pi]\}.\end{aligned} \tag{25}$$

Let $\mathcal{X}_{\mathcal{U}_1} = \Psi_{\mathcal{U}_1}/\mathcal{A}$ be the set of all undesired critical points. Given a positive definite matrix Q with three distinct eigenvalues, choose $0 < \delta < (1 - \cos\theta)\Delta_Q^*$. From Lemma 3, one has

$$\begin{aligned}\mathcal{U}_1(\tilde{g}) - \min_{g_q\in\mathbb{Q}}\mathcal{U}_1(\tilde{g}g_q) &= (1 - \cos\theta)\min_{u\in\mathcal{E}(Q)}\Delta(u,v) \\ &> \delta, \quad \forall\tilde{g}\in\mathcal{X}_{\mathcal{U}_1}\end{aligned}$$

which implies $\mathcal{X}_1 \subseteq \mathcal{F}$.

Given the potential function (23), the correction terms (17) and (18) can be rewritten as follows with respect to inertial vector measurements:

$$\beta = \frac{1}{2}\mathrm{Ad}_{\hat{g}^{-1}}\sum_{i=1}^n k_i(\hat{g}b_i)\wedge r_i, \tag{26}$$

$$\sigma_b = \frac{1}{2}\mathrm{Ad}_{\hat{g}}^\top\sum_{i=1}^n k_i(\hat{g}b_i)\wedge r_i, \tag{27}$$

Now, we can state the following theorem that provides a hybrid pose and velocity-bias estimation scheme relying directly on the available measurements.

Theorem 2: Consider the pose kinematics (4) and the observer (16) with (26)-(27). Let Assumption 1 hold. The flow and jump sets $\mathcal{F}$ and $\mathcal{J}$ are given in (14)-(15) with $\mathcal{U} = \mathcal{U}_1$ given in (22). Consider the transformation set $\mathbb{Q}$ given in (25) and choose $0 < \delta < (1 - \cos\theta)\Delta_Q^*$ such that $\mathcal{X}_{\mathcal{U}_1}\subseteq\mathcal{J}$. Then, the number of discrete jumps is finite and the equilibrium set $\bar{\mathcal{A}}$ is uniformly globally asymptotically stable.

Remark 3: In view of (19) and (26), the rotational and translational error dynamics, in the flow set $\mathcal{F}$, are given by

$$\dot{\tilde{R}} = \tilde{R}\left(-k_\beta(\mathbb{P}_a(A\tilde{R}) + \frac{1}{2}(b^\times \tilde{R}^\top \tilde{p})^\times) + (\hat{R}\tilde{b}_\omega)^\times\right), \quad (28)$$

$$\dot{\tilde{p}} = -\frac{1}{2}k_\beta(d\tilde{p} - (I_3 - \tilde{R})b) + \tilde{R}(\hat{p}^\times \hat{R}\tilde{b}_\omega + \hat{R}\tilde{b}_v), \quad (29)$$

This is similar to the one in [14]. It is clear that the dynamics of $\tilde{R}$ and $\tilde{p}$ are coupled if $p_c^{\mathcal{I}} = bd^{-1} \neq 0$. Therefore, it is expected that noisy or erroneous position measurements would affect the rotation estimation. This motivated us to re-design the estimation scheme in a way that leads to decoupled translational and rotational error dynamics (in the velocity bias-free case).

C. Implementation Using Modified Inertial Measurements

Consider a translation matrix $g_c = g(I_3, p_c^{\mathcal{I}}) \in SE(3)$ with $p_c^{\mathcal{I}}$ defined in (7). Let us introduce the following linear transformation

$$\bar{r}_i := g_c^{-1} r_i \in \mathbb{R}^4, i = 1, 2, \cdots, n \ ,$$

One can verify that $\bar{r}_i = r_i$ for $i = n_1 + 1, \cdots, n$, and $\bar{r}_i = [(p_i^{\mathcal{I}} - p_c^{\mathcal{I}})^\top, 1]^\top$ for $i = 1, 2, \cdots, n_1$. Interestingly, one can verify that $\sum_{i=1}^{n_1} k_i(p_i^{\mathcal{I}} - p_c^{\mathcal{I}}) = 0$, which implies that the weighted center of all translated landmarks is the origin of the inertial frame. This property is instrumental in achieving decoupled translational and rotational error dynamics (in the velocity bias-free case). Note that in [13] this property has been achieved through the choice of the parameters α_i assuming that the landmark points are linearly dependant. Our approach does not put such restrictions on the landmarks and the parameters α_i.

Define the modified homogeneous representation $\underline{g} := g_c^{-1} g$ such that $b_i = \underline{g}^{-1}\bar{r}_i$. Similarly, define the modified estimation $\underline{\hat{g}} := g_c^{-1}\hat{g}$ and the new pose error $\underline{\tilde{g}} := \underline{g}\underline{\hat{g}}^{-1}$. One verifies that $\underline{\tilde{g}} = g_c^{-1}\tilde{g}g_c = g(\tilde{R}, \tilde{p} - (I_3 - \tilde{R})p_c^{\mathcal{I}})$. Consequently, if $\underline{\tilde{g}}$ converges to I_4, so does $\tilde{g}$. Note that the modified position estimation error $\underline{\tilde{p}} = (p - p_c^{\mathcal{I}}) - \tilde{R}(\hat{p} - p_c^{\mathcal{I}})$, is different from the one $(R^\top p - \hat{R}^\top \hat{p})$ defined in [13] and the one $(p - \tilde{R}\hat{p})$ defined in [14].

Let us consider the following potential function:

$$\mathcal{U}_2(\underline{\tilde{g}}) = \frac{1}{2}\mathrm{tr}\left((I_4 - \underline{\tilde{g}})\mathbb{D}(I_4 - \underline{\tilde{g}})^\top\right), \quad (30)$$

with $\mathbb{D} = \sum_{i=1}^{n} k_i \bar{r}_i \bar{r}_i^\top = g_c^{-1}\mathbb{A}g_c^{-\top} = \mathrm{diag}(Q, d)$. Using the second property of (3), one can easily verify that $\mathcal{U}_2(\underline{\tilde{g}}) = \mathcal{U}_1(\tilde{g})$. Similarly, from Lemma 4, the gradient and the set of critical points of $\mathcal{U}_2$ are given by

$$\nabla_{\underline{\tilde{g}}}\mathcal{U}_2(\underline{\tilde{g}}) = \underline{\tilde{g}}\mathbb{P}((I_4 - \underline{\tilde{g}}^{-1})\mathbb{D}), \quad (31)$$

$$\Psi_{\mathcal{U}_2} = \{I_4\} \bigcup \{g = g(R, p) |\ R = \mathcal{R}_a(\pi, v), \ p = 0, v \in \mathcal{E}(Q)\}, \quad (32)$$

Define $\underline{g}_q = g_c^{-1} g_q g_c$ such that $g_c^{-1} g(g_q^{-1}\hat{g})^{-1} g_c = \underline{\tilde{g}}\underline{g}_q$. Consider the modified flow set $\mathcal{F}$ and jump set $\mathcal{J}$

$$\mathcal{F} = \left\{\underline{\tilde{g}} : \mathcal{U}_2(\underline{\tilde{g}}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}_2(\underline{\tilde{g}}\underline{g}_q) \leq \delta\right\}, \quad (33)$$

$$\mathcal{J} = \left\{\underline{\tilde{g}} : \mathcal{U}_2(\underline{\tilde{g}}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}_2(\underline{\tilde{g}}\underline{g}_q) \geq \delta\right\}, \quad (34)$$

Let $\mathcal{X}_2 = \Psi_{\mathcal{U}_2}/\mathcal{A}$ be the set of all undesired critical points. Given a positive definite matrix Q with three distinct eigenvalues, choose $0 < \delta < (1 - \cos\theta)\Delta_Q^*$. From Lemma 3, one has

$$\begin{aligned}\mathcal{U}_2(\underline{\tilde{g}}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}_2(\underline{\tilde{g}} g_c^{-1} g_q g_c) &= (1 - \cos\theta) \max_{u \in \mathcal{E}(Q)} \Delta(u, v) \\ &> \delta, \quad \forall \tilde{g} \in \mathcal{X}_2\end{aligned}$$

which implies $\mathcal{X}_2 \subseteq \mathcal{F}$.

Given the potential function (30), one has the following new correction terms with respect to inertial vector measurements

$$\beta = \frac{1}{2}\mathrm{Ad}_{\hat{g}^{-1}g_c} \sum_{i=1}^{n} k_i(g_c^{-1}\hat{g}b_i) \wedge (g_c^{-1}r_i), \quad (35)$$

$$\sigma_b = \frac{1}{2}\mathrm{Ad}_{\hat{g}}^\top \sum_{i=1}^{n} k_i(\hat{g}b_i) \wedge (r_i), \quad (36)$$

Now, we can state our Theorem that provides a hybrid pose and velocity-bias estimation scheme, relying directly on the available measurements, and leading to decoupled translational and rotational error dynamics (in the velocity bias-free case).

Theorem 3: Consider the pose kinematics (4) and the observer (16) with (35)-(36). Let Assumption 1 hold. The flow and jump sets $\mathcal{F}$ and $\mathcal{J}$ are given in (33)-(34). Consider the transformation set $\mathbb{Q}$ given in (25) and choose $0 < \delta < (1 - \cos\theta)\Delta_Q^*$ such that $\mathcal{X}_2 \subseteq \mathcal{J}$. Then, the number of discrete jumps is finite and the equilibrium set $\bar{\mathcal{A}}$ is uniformly globally asymptotically stable.

Remark 4: In view of (4)-(5), (16) and (35), the rotational and translational error dynamics are given by

$$\dot{\underline{\tilde{R}}} = \underline{\tilde{R}}\left(-k_\beta\mathbb{P}_a(Q\underline{\tilde{R}}) + (\underline{\hat{R}}\tilde{b}_\omega)^\times\right), \quad (37)$$

$$\dot{\underline{\tilde{p}}} = -\frac{1}{2}k_\beta d\underline{\tilde{p}} + \underline{\tilde{R}}(\underline{\hat{p}}^\times \underline{\hat{R}}\tilde{b}_\omega + \underline{\hat{R}}\tilde{b}_v). \quad (38)$$

It is clear that, in the velocity bias-free case, the dynamics of $\underline{\tilde{R}} = \tilde{R}$ and $\underline{\tilde{p}} = \tilde{p} - (I_3 - \tilde{R})p_c^{\mathcal{I}}$ are decoupled. Compared to (28) and (29), the cross term $\tilde{R}^\top\tilde{p}$ does not appear in (37), and $\underline{\tilde{p}}$ enjoys exponential stability when $\tilde{b}_a = 0$ as shown in (38).

V. Simulation

In this section, some simulation results are presented to illustrate the performance of the decoupled hybrid observer (16) with (35)-(36) compared to the non-decoupled hybrid observer (16) with (26)-(27) and the standard smooth (non-hybrid) observer (*i.e.,* considering just the flow set (16) and (26) without the switching mechanism).

The vehicle is initialized at the origin, *i.e.,* $R(0) = I_3$ and $p(0) = [0\ 0\ 0]^\top$, and driven by the following linear and angular velocities $v(t) = [10\cos(0.5t)\ 10\sin(0.5t)\ 0]^\top$, $\omega(t) = [\sin(t)\ \ \sin(t)\ 0]^\top$. The angular and linear velocity measurements are corrupted by the following constant biases: $b_{a,\omega} = [-0.02\ 0.02\ 0.1]^\top$ and $b_{a,v} = [0.2\ \ -0.1\ 0.01]^\top$. Let us consider the following known inertial elements r_i^v ($i = 1, 2, 3, 4$), $v_1 = [0\ 0\ 1]^\top$, $v_2 = [\frac{\sqrt{3}}{2}\ \frac{1}{2}\ 0]^\top$, $p_1 = [1\ 0\ 0]^\top$, $p_2 = [-\frac{1}{2}\ \frac{\sqrt{3}}{2}\ 0]^\top$. The gain parameters involved in all the observers are taken as follows: $k_1 = k_2 = k_3 =$

$k_4 = 1$, $k_\beta = 1, \Gamma = \text{diag}(0.01 I_3, I_3)$. The estimates initial conditions are taken as: $\hat{R}(0) = \mathcal{R}_a(\pi - 0.001, u)$, $\hat{p}(0) = [20\ 0\ 10]^\top$, with $u = [0\ 0\ 1]^\top \in \mathcal{E}(Q)$. For the jump set and jump map, we choose $\theta = 2\pi/3$ and $\Delta^* = 1.2$ such that $\delta = 3.6217$ .The simulation results are given in Fig. 1 - Fig. 2, from which one can clearly see the improved performance of the decoupled hybrid observer as compared to the non-decoupled hybrid observer and smooth (non-hybrid) observer.

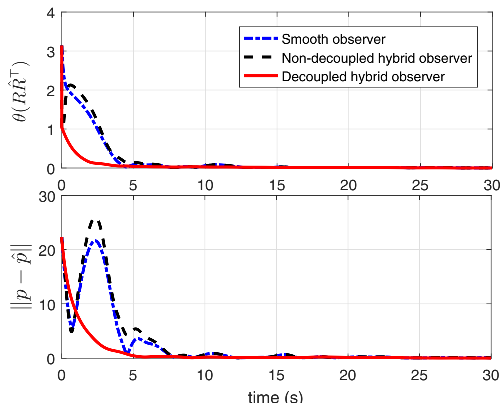

Fig. 1: The rotation angle error and the position error of three observers.

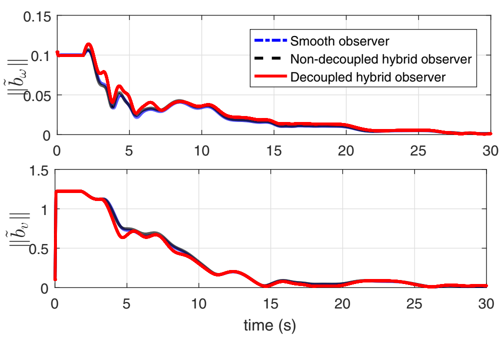

Fig. 2: Norms of the velocity-bias errors.

VI. Conclusion

A globally asymptotically stable hybrid pose and velocity-bias estimation scheme evolving on $SE(3) \times \mathbb{R}^6$ has been proposed. The proposed observer is formulated in terms of body-frame measurements of known inertial vectors and landmark points. It relies on a switching mechanism designed to avoid the undesired critical points while ensuring a decrease of the potential function in flow and jump sets. A translational transformation is introduced on the inertial measurements leading to an estimation scheme resulting in decoupled translational and rotational error dynamics (in the velocity bias-free case) which is an interesting practical feature in applications involving noisy or poor landmark measurements.

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