Hybrid Pose and Velocity-Bias Estimation on $SE(3)$ Using Inertial and Landmark Measurements

Miaomiao Wang and Abdelhamid Tayebi

Abstract—**This paper deals with the design of globally exponentially stable invariant observers on the Special Euclidian group $SE(3)$. First, we propose a generic hybrid estimation scheme (depending on a generic potential function) evolving on $SE(3) \times \mathbb{R}^6$ for pose (orientation and position) and velocity-bias estimation. Thereafter, the proposed estimation scheme is formulated explicitly in terms of inertial vectors and landmark measurements. Interestingly, the proposed estimation scheme leads to a decoupled rotational error dynamics from the translational dynamics, which is an interesting feature in practical applications with noisy measurements and disturbances.**

Index Terms—**Hybrid dynamics, inertial-vision systems, nonlinear observer, Special Euclidean group $SE(3)$.**

I. INTRODUCTION

The development of reliable pose (i.e., attitude and position) estimation algorithms is instrumental in many applications such as autonomous underwater vehicles and unmanned aerial vehicles. Since there is no sensor that directly measures the attitude, the latter is usually determined using body-frame measurements of some known inertial vectors via static determination algorithms [2], which are generally sensitive to measurement noise. Alternatively, dynamic estimation algorithms using inertial vector measurements together with the angular velocity can be used to recover the attitude while filtering measurement noise (e.g., Kalman filters [3], linear complementary filters [4], and nonlinear complementary filters [5]). In low-cost applications (nonaccelerated motions), angular velocity and inertial vector measurements can be obtained, for instance, from an inertial measurement unit (IMU) equipped with gyroscopes, accelerometers, and magnetometers. The translational position and velocity can be estimated using a global positioning system (GPS). However, in GPS-denied environments such as indoor applications, recovering the position and linear velocity is a challenging task. Alternatively, inertial-vision systems combining IMU and on-board camera measurements have been considered for pose estimation [6]–[8]. In [7], local Riccati-based pose observers have been proposed relying on the system's linear and angular velocities and the

bearing measurements of some known landmark points. Another solution with global asymptotic stability (GAS) has been proposed in [8], which considers a nongeometric pose estimation problem using biased body-frame measurements of the system's linear and angular velocities as well as body-frame measurements of landmarks. The achieved GAS results are due to the fact that the estimates are not confined to live in $SE(3)$ for all times.

Recently, a class of nonlinear observers on Lie groups including $SO(3)$ and $SE(3)$ have made their appearances in the literature. Invariant observers, which take into account the topological properties of the motion space, were developed in [5], [9], and [10]. Motivated by the work of [5] on $SO(3)$, complementary observers on $SE(3)$ were proposed in [11] and [12]. In practice, measurements of group velocity (translational and rotational velocities) are often corrupted by an unknown bias. Pose estimation using biased velocity measurements were considered in [13]–[15]. A nice feature of [15] is the fact that the observer incorporates (naturally) both inertial vector measurements (e.g., from IMU) and landmark measurements (e.g., from a vision system). The observers proposed in [11]–[15] are shown to guarantee almost GAS (AGAS), i.e., the estimated pose converges to the actual one from almost all initial conditions except from a set of Lebesgue measure zero. This is the strongest result one can aim at when considering continuous time-invariant state observers on $SO(3)$ or $SE(3)$.

Recently, the topological obstruction to GAS on $SO(3)$ using continuous time-invariant controllers (observers) has been successfully addressed via hybrid techniques such as [16]–[20]. To this end, a synergistic hybrid technique was introduced in [16]. Motivated by this approach, globally asymptotically stable hybrid attitude observers on $SO(3)$ have been proposed in [17] and globally exponentially stable hybrid attitude observers on $SO(3)$ have been proposed in [20].

In this paper, we propose a generic approach for hybrid estimation on $SE(3)$ leading to global exponential stability. To the best of our knowledge, there is no work in the literature achieving such results on $SE(3)$. Moreover, we propose some explicit and practically implementable versions of the proposed scheme, using biased group-velocity measurements (constant bias), inertial vectors, and landmark measurements. Interestingly, the proposed estimation scheme leads to decoupled rotational error dynamics from translational error dynamics, which guarantees nice robustness and performance properties.

The remainder of this paper is organized as follows. Section II introduces some preliminary notions that will be used throughout out the paper. In Section III, the problem of pose estimation on $SE(3)$ is formulated. In Section IV, a hybrid approach for pose and velocity-bias estimation is proposed. An estimation scheme leading to decoupled rotational error dynamics from the translational error dynamics is provided in Section V. Section VI presents some simulation results showing the performance of the proposed estimation schemes.

M. Wang is with the Department of Electrical and Computer Engineering, University of Western Ontario, London, ON N6A 3K7, Canada (e-mail: mwang448@uwo.ca).

A. Tayebi is with the Department of Electrical and Computer Engineering, University of Western Ontario, London, ON N6A 3K7, Canada, and also with the Department of Electrical Engineering, Lakehead University, Thunder Bay, ON P7B 5E1, Canada (e-mail: atayebi@lakeheadu.ca).

II. BACKGROUND AND PRELIMINARIES

The sets of real, non-negative real, and natural number are denoted as $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{N}$, respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and $\mathbb{S}^n$ the set of n-dimensional unit vectors. Given

two matrices, $A, B \in \mathbb{R}^{m \times n}$, their Euclidean inner product is defined as $\langle\langle A, B \rangle\rangle := \operatorname{tr}(A^\top B)$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| := \sqrt{x^\top x}$, and the Frobenius norm of a matrix $X \in \mathbb{R}^{n \times m}$ is given by $\|X\|_F := \sqrt{\langle\langle X, X \rangle\rangle}$. The n-by-n identity matrix is denoted by I_n. For a matrix $A \in \mathbb{R}^{n \times n}$, we define $\mathcal{E}(A)$ as the set of all eigenvectors of A and $\mathbb{E}(A) \subseteq \mathcal{E}(A)$ as the eigenbasis set of A. Let λ_i^A be the ith eigenvalue of A, and $\lambda_{\min}^A$ and $\lambda_{\max}^A$ be the minimum and maximum eigenvalue of A, respectively.

Consider the three-dimensional *Special Euclidean group*, defined as $SE(3) := \{g = \mathcal{T}(R, p) \in \mathbb{R}^{4 \times 4} | R \in SO(3), p \in \mathbb{R}^3\}$, with the map $\mathcal{T} : SO(3) \times \mathbb{R}^3 \to SE(3)$ given by $\mathcal{T}(R, p) := \left[\begin{smallmatrix} R & p \\ 0 & 1 \end{smallmatrix}\right]$. The *Lie algebra* of $SE(3)$, denoted by $\mathfrak{se}(3)$, is given by $\mathfrak{se}(3) := \big\{ X \in \mathbb{R}^{4 \times 4} | X = \left[\begin{smallmatrix} \Omega & v \\ 0 & 0 \end{smallmatrix}\right], \Omega \in \mathfrak{so}(3), v \in \mathbb{R}^3 \big\}$. Let $\times$ be the vector cross-product on $\mathbb{R}^3$ and define the map $(\cdot)^\times : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ such that $x \times y = x^\times y$, for any $x, y \in \mathbb{R}^3$. A wedge map $(\cdot)^\wedge : \mathbb{R}^6 \mapsto \mathfrak{se}(3)$ is defined as $\xi^\wedge := \left[\begin{smallmatrix} \omega^\times & v \\ 0 & 0 \end{smallmatrix}\right]$ with $\xi := (\omega^\top, v^\top)^\top$. Given a differentiable smooth function $f : SE(3) \to \mathbb{R}$, the *gradient* of f, denoted $\nabla_g f$, relative to the Riemannian metric on $SE(3)$ is uniquely defined by $df \cdot gX = \langle\langle g^{-1} \nabla_g f, X \rangle\rangle$, for all $g \in SE(3), X \in \mathfrak{se}(3)$. A point $g \in SE(3)$ is called *critical point* of f if the gradient of f at g is zero. The set of all critical points of f on $SE(3)$ is denoted by $\Psi_f := \{g \in SE(3) | \nabla_g f = 0\}$. For any $g \in SE(3)$, we define $|g|_I$ as the distance with respect to I_4, which is given by $|g|_I := \|I_4 - g\|_F$. The map $\mathcal{R}_a : \mathbb{R} \times \mathbb{S}^2 \to SO(3)$ known as the angle-axis parametrization of $SO(3)$ is defined as $\mathcal{R}_a(\theta, u) := I_3 + \sin\theta u^\times + (1 - \cos\theta)(u^\times)^2$.

For a matrix $A \in \mathbb{R}^{3 \times 3}$, we denote by $\mathbb{P}_a(A)$ the antisymmetric projection of A, such that $\mathbb{P}_a(A) = (A - A^\top)/2$. Let $\mathbb{P} : \mathbb{R}^{4 \times 4} \to \mathfrak{se}(3)$ denote the projection of $\mathbb{A}$ on the Lie algebra $\mathfrak{se}(3)$, such that, for all $X \in \mathfrak{se}(3)$, $\mathbb{A} \in \mathbb{R}^{4 \times 4}$ one has $\langle\langle \mathbb{A}, X \rangle\rangle = \langle\langle X, \mathbb{P}(\mathbb{A}) \rangle\rangle = \langle\langle \mathbb{P}(\mathbb{A}), X \rangle\rangle$. For all $A \in \mathbb{R}^{3 \times 3}, b, c^\top \in \mathbb{R}^3$ and $d \in \mathbb{R}$, one has $\mathbb{P}\big(\left[\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right]\big) := \left[\begin{smallmatrix} \mathbb{P}_a(A) & b \\ 0_{1 \times 3} & 0 \end{smallmatrix}\right]$. Let $\operatorname{vex} : \mathfrak{se}(3) \to \mathbb{R}^6$ denote the inverse isomorphism of the map $(\cdot)^\wedge$, such that $\operatorname{vex}(\xi^\wedge) = \xi$ and $(\operatorname{vex}(X))^\wedge = X$, for all $\xi \in \mathbb{R}^6$ and $X \in \mathfrak{se}(3)$. For a matrix $\mathbb{A} = \left[\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right]$, we also define the following map:

$$\psi_b(\mathbb{A}) := \operatorname{vex}(\mathbb{P}(\mathbb{A})) = \begin{bmatrix} \psi_a(A) \\ b \end{bmatrix} \tag{1}$$

where $\psi_a(A) := \frac{1}{2}\begin{bmatrix} a_{32} - a_{23} & a_{13} - a_{31} & a_{21} - a_{12} \end{bmatrix}^\top$ with $A = [a_{ij}]$. It is verified that for all $\mathbb{A} \in \mathbb{R}^{4 \times 4}, y \in \mathbb{R}^6$, $\langle\langle \mathbb{A}, y^\wedge \rangle\rangle = 2\psi_b(\mathbb{A})^\top \Theta y$ with $\Theta := \operatorname{diag}(I_3, \frac{1}{2}I_3)$.

Given a rigid body with configuration $g \in SE(3)$, the *adjoint map* $\operatorname{Ad}_g(\cdot) : SE(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$ is given by $\operatorname{Ad}_g(X) := gXg^{-1}$ with $g \in SE(3), X \in \mathfrak{se}(3)$. The matrix representation of the adjoint map on $\mathfrak{se}(3)$ is defined as $\operatorname{Ad}_g := \left[\begin{smallmatrix} R & 0 \\ p^\times R & R \end{smallmatrix}\right] \in \mathbb{R}^{6 \times 6}$ such that $gx^\wedge g^{-1} = (\operatorname{Ad}_g x)^\wedge$, for all $g \in SE(3), x \in \mathbb{R}^6$. One verifies that $\operatorname{Ad}_{g_1} \operatorname{Ad}_{g_2} = \operatorname{Ad}_{g_1 g_2}$, for all $g_1, g_2 \in SE(3)$. Define $\operatorname{Ad}_g^*(\cdot)$ as the Hermitian adjoint of $\operatorname{Ad}_g(\cdot)$ with respect to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathfrak{se}(3)$ associated with the right-invariant Riemannian metric, such that $\langle\langle Y, \operatorname{Ad}_g(X) \rangle\rangle = \langle\langle \operatorname{Ad}_g^*(Y), X \rangle\rangle$, for all $X, Y \in \mathfrak{se}(3), g \in SE(3)$. The matrix representation of the map $\operatorname{Ad}_g^*(\cdot)$ is given by $\operatorname{Ad}_g^* := \left[\begin{smallmatrix} R^\top & -R^\top p^\times \\ 0 & R^\top \end{smallmatrix}\right]$. For all $X \in \mathfrak{se}(3)$ and $g \in SE(3)$, one has $\psi_b(g^\top X g^{-\top}) = \Theta^{-1} \operatorname{Ad}_g^* \Theta \psi_b(X)$. For the sake of simplicity, let us define the map $\psi : \mathbb{R}^{4 \times 4} \to \mathbb{R}^6$ such that $\psi(\mathbb{A}) = \Theta \psi_b(\mathbb{A})$, for all $\mathbb{A} \in \mathbb{R}^{4 \times 4}$. The following identities are used throughout this paper:

$$\langle\langle \mathbb{A}, y^\wedge \rangle\rangle = 2\psi(\mathbb{A})^\top y, \quad \psi(g^\top X g^{-\top}) = \operatorname{Ad}_g^* \psi(X). \tag{2}$$

For any two vectors $r, b \in \mathbb{R}^4$, we define the following wedge product (exterior product) $\wedge$:

$$b \wedge r := \begin{bmatrix} b_v \times r_v \\ b_s r_v - r_s b_v \end{bmatrix} \in \mathbb{R}^6 \tag{3}$$

where $r = (r_v^\top, r_s)^\top, b = (b_v^\top, b_s)^\top$ with $r_v, b_v \in \mathbb{R}^3$ and $r_s, b_s \in \mathbb{R}$. For all $r, b \in \mathbb{R}^4, g \in SE(3)$, one can easily verify that $r \wedge r = 0, b \wedge r = -r \wedge b, \psi((I_4 - g)rr^\top) = \frac{1}{2}(gr) \wedge r$ and $((gb) \wedge (gr))^\wedge = (\operatorname{Ad}_{g^{-1}}(b \wedge r))^\wedge$.

Let $\mathcal{M}_0$ denote a submanifold of $\mathbb{R}^{4 \times 4}$, defined as $\mathcal{M}_0 := \{M | M = \left[\begin{smallmatrix} M_1 & m_2 \\ 0 & \end{smallmatrix}\right], M_1 \in \mathbb{R}^{3 \times 3}, m_2 \in \mathbb{R}^3\}$. Then, for all $g \in SE(3), M, \bar{M} \in \mathcal{M}_0$, one verifies that

$$\mathbb{P}(gM) = \mathbb{P}(g^{-\top}M), \quad \operatorname{tr}(g^\top g M \bar{M}^\top) = \operatorname{tr}(M \bar{M}^\top). \tag{4}$$

III. PROBLEM FORMULATION AND PRELIMINARY RESULTS

Let $\mathcal{I}$ be an inertial frame and $\mathcal{B}$ be a body-fixed frame. Let $p \in \mathbb{R}^3$ denote the rigid body position expressed in the inertial frame $\mathcal{I}$, and $R \in SO(3)$ the rigid body attitude describing the rotation of frame $\mathcal{B}$ with respect to frame $\mathcal{I}$. We consider the problem of pose estimation of the rigid body, i.e., position p and attitude R. The pose of the rigid body can be represented by $g = \mathcal{T}(R, p) \in SE(3)$. This representation is commonly known as the homogeneous representation. Let $\omega \in \mathbb{R}^3$ denote the angular velocity of the body-fixed frame $\mathcal{B}$ with respect to the inertial frame $\mathcal{I}$, expressed in frame $\mathcal{B}$. Let $v \in \mathbb{R}^3$ be the translational velocity, expressed in frame $\mathcal{B}$. The pose is governed by the following dynamics:

$$\dot{g} = g\xi^\wedge \tag{5}$$

where $\xi := (\omega^\top, v^\top)^\top \in \mathbb{R}^6$. Note that system (8) is left invariant in the sense that it preserves the Lie group invariance properties with respect to constant translation and constant rotation of the body-fixed frame $\mathcal{B}$. Let the group velocity be piecewise-continuous, and consider the following biased group velocity measurement:

$$\xi_y := \xi + b_a \tag{6}$$

where $b_a := (b_\omega^\top, b_v^\top)^\top \in \mathbb{R}^6$ with $b_\omega, b_v \in \mathbb{R}^3$ denotes the unknown constant velocity biases. Moreover, a family of n constant homogeneous vectors $r_i \in \mathbb{R}^4, i = 1, 2, \ldots, n$, known in the inertial frame $\mathcal{I}$, are assumed to be measured in the frame $\mathcal{B}$ as

$$b_i = h(g, r_i) := g^{-1} r_i, \quad i = 1, 2, \ldots, n. \tag{7}$$

Assume that, among the n inertial elements, there are n_1 feature points (or landmarks) $p_i^\mathcal{I}, i = 1, 2, \ldots, n_1$, and $n_2 = n - n_1$ inertial vectors $v_i^\mathcal{I}, i = 1, 2, \ldots, n_2$, i.e.,

$$r_i = \begin{bmatrix} p_i^\mathcal{I} \\ 1 \end{bmatrix}, i = 1, \ldots, n_1 \quad r_{j+n_1} = \begin{bmatrix} v_j^\mathcal{I} \\ 0 \end{bmatrix}, j = 1, \ldots, n_2. \tag{8}$$

Define the following modified inertial vectors and weighted geometric landmark center: $\bar{v}_i^\mathcal{I} := p_i^\mathcal{I} - p_c^\mathcal{I}, \quad p_c^\mathcal{I} := \sum_{i=1}^{n_1} \alpha_i p_i^\mathcal{I}$ with $\alpha_i > 0$ for all $i = 1, 2, \ldots, n_1$ and $\sum_{i=1}^{n_1} \alpha_i = 1$.

Assumption 1: Among the n measurements, at least one landmark point is measured, and at least two vectors from the set $V^\mathcal{I} := \{\bar{v}_1^\mathcal{I}, \ldots, \bar{v}_{n_1}^\mathcal{I}, v_1^\mathcal{I}, \ldots, v_{n-n_1}^\mathcal{I}\}$ are noncollinear.

Remark 1: From Assumption 1, one verifies that $n_1 \geq 1$ and $n \geq 3$. This assumption is standard in estimation problems in $SE(3)$, e.g., [12], [13], and [15], which is satisfied in the following particular cases.

1) Three different landmark points are measured such that the corresponding $\bar{v}_i^\mathcal{I}, i = 1, 2, 3$, are noncollinear.
2) One landmark point and two noncollinear inertial vectors are measured.
3) Two different landmark points and one inertial vector are measured such that the corresponding $v_1^\mathcal{I}$ and $\bar{v}_i^\mathcal{I}, i = 1, 2$ are noncollinear.

Assumption 2: The pose g and group velocity ξ of the rigid body are uniformly bounded.

Our objective is to design a globally exponentially stable hybrid pose and velocity-bias estimation scheme that provides estimates $\hat{g}$ and $\hat{b}_a$ of g and b_a, respectively, using the available measurements satisfying Assumptions 1 and 2.

IV. GRADIENT-BASED HYBRID OBSERVER DESIGN

In this paper, we make use of the framework of hybrid dynamical systems presented in [21] and [22]. Consider a positive-valued continuously differentiable function $\mathcal{U} : SE(3) \to \mathbb{R}_{\geq 0}$. The function $\mathcal{U}(g)$ is said to be a potential function on $SE(3)$ if $\mathcal{U}(g) \geq 0$ for all $g \in SE(3)$ and $\mathcal{U}(g) = 0$ if and only if $g = I_4$. For all $g \in SE(3)$, $\nabla_g \mathcal{U}(g)$ denotes the gradient of $\mathcal{U}$ with respect to g. Let $\Psi_{\mathcal{U}}$ denote the set of critical points and $\mathcal{X}_{\mathcal{U}} := \Psi_{\mathcal{U}}/\{I_4\}$ be the set of undesired critical points.[1]

A. Generic Hybrid Pose and Velocity-Bias Estimation Filter

Let $\hat{g} := \mathcal{T}(\hat{R}, \hat{p})$ and $\hat{b}_a$ denote, respectively, the estimates of the rigid body pose and velocity bias. Define the pose estimation error $\tilde{g} := \mathcal{T}(\tilde{R}, \tilde{p}) = g\hat{g}^{-1}$ and bias estimation error $\tilde{b}_a := \hat{b}_a - b_a$. Given a nonempty finite set $\mathbb{Q} \subset SE(3)$, we propose the following generic hybrid pose and velocity-bias estimation scheme relying on a generic potential function on $SE(3)$

$$\underbrace{\begin{aligned} \dot{\hat{g}} &= \hat{g}(\xi_y - \hat{b}_a + k_\beta \beta)^\wedge \\ \dot{\hat{b}}_a &= -\Gamma \sigma_b \end{aligned}}_{(\hat{g}, \hat{b}_a) \in \mathcal{F}_o} \quad \underbrace{\begin{aligned} \hat{g}^+ &= g_q^{-1} \hat{g}, g_q \in \gamma(\hat{g}) \\ \hat{b}_a^+ &= \hat{b}_a \end{aligned}}_{(\hat{g}, \hat{b}_a) \in \mathcal{J}_o} \quad (9)$$

$$\beta := \mathrm{Ad}_{\hat{g}^{-1}} \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g})) \quad (10)$$

$$\sigma_b := \mathrm{Ad}_{\hat{g}}^* \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g},)) \quad (11)$$

where $\hat{g}(0) \in SE(3), \hat{b}_a(0) \in \mathbb{R}^6$, $\Gamma := \mathrm{diag}(k_\omega I_3, k_v I_3) \in \mathbb{R}^{6 \times 6}$ and $k_\omega, k_v, k_\beta > 0$. The set-valued map $\gamma : SE(3) \rightrightarrows \mathbb{Q}$ is defined as $\gamma(\hat{g}) := \{g_q \in \mathbb{Q} : g_q = \arg \min_{g_q \in \mathbb{Q}} \mathcal{U}(\tilde{g} g_q)\}$. The flow set $\mathcal{F}_o$ and jump set $\mathcal{J}_o$ are defined by

$$\mathcal{F}_o := \{(\hat{g}, \hat{b}_a) \in SE(3) \times \mathbb{R}^6 : \mathcal{U}(\tilde{g}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}(\tilde{g} g_q) \leq \delta\} \quad (12)$$

$$\mathcal{J}_o := \{(\hat{g}, \hat{b}_a) \in SE(3) \times \mathbb{R}^6 : \mathcal{U}(\tilde{g}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}(\tilde{g} g_q) \geq \delta\} \quad (13)$$

for some $\delta > 0$. The potential function $\mathcal{U}$, and the parameters δ and $g_q \in \mathbb{Q} \subset SE(3)$ will be designed later. Note that the vector ξ_y involved in (9) is a known bounded function of time. Note also that $\mathcal{U}(\tilde{g})$ and $\mathcal{U}(\tilde{g} g_q)$ involved in (12) and (13), and $\psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}))$ involved in (10) and (11) can be rewritten in terms of $\hat{g}$ and the available measurements as it is going to be shown later. We define the extended space and state as $\mathcal{S} := SE(3) \times \mathbb{R}^6 \times SE(3) \times \mathbb{R}^6 \times \mathbb{R}_{\geq 0}$ and $x := (\tilde{g}, \tilde{b}_a, \hat{g}, \hat{b}_a, t) \in \mathcal{S}$. In view of (5), (6), (9)–(11), one has the following hybrid closed-loop system:

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in \mathcal{F}_c := \{x \in \mathcal{S} : (\hat{g}, \hat{b}_a) \in \mathcal{F}_o\} \\ x^+ \in G(x) & x \in \mathcal{J}_c := \{x \in \mathcal{S} : (\hat{g}, \hat{b}_a) \in \mathcal{J}_o\} \end{cases} \quad (14)$$

with

$$F(x) = \begin{bmatrix} \tilde{g}(\mathrm{Ad}_{\tilde{g}} \tilde{b}_a - k_\beta \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g})))^\wedge \\ -\Gamma \mathrm{Ad}_{\hat{g}}^* \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g})) \\ \hat{g}(\xi_y - \hat{b}_a + k_\beta \mathrm{Ad}_{\hat{g}^{-1}} \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g})))^\wedge \\ -\Gamma \mathrm{Ad}_{\hat{g}}^* \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g})) \\ 1 \end{bmatrix}, G(x) = \begin{bmatrix} \tilde{g} g_q \\ \tilde{b}_a \\ g_q^{-1} \hat{g} \\ \hat{b}_a \\ t \end{bmatrix}.$$

Note that the closed-loop system (14) satisfies the hybrid basic conditions of [21] and is autonomous. Define the closed set $\bar{\mathcal{A}} := \{x \in \mathcal{S} : \tilde{g} = I_4, \tilde{b}_a = 0\}$ and let $|x|_{\bar{\mathcal{A}}}$ denote the distance to the set $\bar{\mathcal{A}}$ such that $|x|_{\bar{\mathcal{A}}}^2 := \inf_{y=(I_4, 0, \hat{g}, \hat{b}_a, \bar{t}) \in \bar{\mathcal{A}}} (\|I_4 - \tilde{g}\|_F^2 + \|\tilde{b}_a\|^2 + \|\hat{g} - \hat{g}\|_F^2 + \|\bar{b}_a - \hat{b}_a\|^2 + \|\bar{t} - t\|^2) = |\tilde{g}|_I^2 + \|\tilde{b}_a\|^2$. Now, one can state one of our main results.

---

[1]As shown in [23], no smooth vector field on Lie groups, which are not homeomorphic to $\mathbb{R}^n$, can have a global attractor. Therefore, any smooth potential function on $SE(3)$(or $SO(3)$), has at least four critical points.

Theorem 1: Consider system (14) with a continuously differentiable potential function $\mathcal{U}$ on $SE(3)$, and choose the nonempty finite $\mathbb{Q}$ and the gap $\delta > 0$ such that

$$\alpha_1 |\tilde{g}|_I^2 \leq \mathcal{U}(\tilde{g}) \leq \alpha_2 |\tilde{g}|_I^2, \qquad x \in \mathcal{S} \quad (15)$$

$$\alpha_3 |\tilde{g}|_I^2 \leq \|\psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}))\|^2 \leq \alpha_4 |\tilde{g}|_I^2, \quad x \in \mathcal{F}_c \quad (16)$$

$$\|\mathrm{Ad}_{\hat{g}^{-1}}^* \psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}))\|^2 \leq \alpha_5 |\tilde{g}|_I^2, \quad x \in \mathcal{F}_c \quad (17)$$

where $\alpha_1, \ldots, \alpha_5$ are strictly positive scalars. Let Assumptions 1 and 2 hold. Then, the number of jumps is finite and for any initial condition $x(0, 0) \in \mathcal{S}$, the solution $x(t, j)$ is complete and there exist $k > 0$ and $\lambda > 0$ such that

$$|x(t, j)|_{\bar{\mathcal{A}}} \leq k \exp(-\lambda(t + j))|x(0, 0)|_{\bar{\mathcal{A}}} \quad (18)$$

for all $(t, j) \in \mathrm{dom}\, x$.

Proof: See Appendix A. ∎

Remark 2: Theorem 1 provides exponential stability results for the generic estimation scheme (9)–(13) relying on a generic potential function $\mathcal{U}$. The flow and jump sets $\mathcal{F}_o$ and $\mathcal{J}_o$, given in (12) and (13), depend on some parameters δ and $\mathbb{Q}$ that have to be designed together with the potential function $\mathcal{U}$ such that conditions (15)–(17) are fulfilled. It is worth pointing out that condition (16) implies that the undesired critical points belong to the jump set $\mathcal{J}_o$. In the next section, we will design $\mathcal{U}$, δ, and $\mathbb{Q}$ such that (15)–(17) are fulfilled.

Remark 3: The filters proposed in [11]–[15] are shown to guarantee AGAS due to the topological obstruction when considering continuous time-invariant state observers on $SE(3)$. The proposed hybrid estimation scheme uses a new observer-state jump mechanism, inspired from [19], which changes directly the observer state through appropriate jumps in the direction of a decreasing potential function on $SE(3)$. The jump transitions occur when the estimation error is close to the critical points. This observer-state jump mechanism is different from the principle used, for instance in [16], [17], and [20], which consists in incorporating the jumps in the observer's correcting term derived from a family of synergistic potential functions.

B. Explicit Hybrid Observers Design Using the Available Measurements

In this subsection, we provide an explicit expression for the proposed hybrid observers in terms of available measurements. Before proceeding with the design, some useful properties are given in the following lemmas (the reader is referred to [24] for additional details).

Lemma 1: Consider a family of n elements of homogeneous space $r_i \in \mathbb{R}^4, i = 1, 2, \ldots, n$ defined in (8). Given $k_i \geq 0$ for all $i = 1, 2, \ldots, n$, define the following matrix:

$$\mathbb{A} := \sum_{i=1}^{n} k_i r_i r_i^\top = \begin{bmatrix} A & b \\ b^\top & d \end{bmatrix} \in \mathbb{R}^{4 \times 4} \quad (19)$$

where $A := \sum_{i=1}^{n_1} k_i p_i^{\mathcal{I}} (p_i^{\mathcal{I}})^\top + \sum_{j=1}^{n_2} k_{j+n_1} v_j^{\mathcal{I}} (v_j^{\mathcal{I}})^\top, b := \sum_{i=1}^{n_1} k_i p_i^{\mathcal{I}}$, and $d := \sum_{i=1}^{n_1} k_i$. Then, under Assumption 1 the following statements hold:

1) $d > 0$.
2) Matrix $Q := A - bb^\top d^{-1}$ is a positive semidefinite.
3) Matrix $\bar{Q} := \frac{1}{2}(\mathrm{tr}(Q)I_3 - Q)$ is a positive definite.

Lemma 2: Let $Q = Q^\top$ be a positive semidefinite matrix. Consider the map $\Delta_Q : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ defined as

$$\Delta_Q(u, v) = u^\top ((\mathrm{tr}(Q) - 2v^\top Q v)I_3 - Q + 2Q vv^\top)u. \quad (20)$$

Let $\mathbb{U} \subset \mathbb{S}^2$ be a finite and nonempty set. Define the constant scalar $\Delta_Q^* := \min_{v \in \mathcal{E}(Q)} \max_{u \in \mathbb{U}} \Delta_Q(u, v)$. Then, the following results hold.

1) Let $\mathbb{U}$ be a superset of $\mathbb{E}(Q)$ (*i.e.*, $\mathbb{U} \supseteq \mathbb{E}(Q)$), then the following inequality holds:

$$\Delta_Q^* \geq \begin{cases} \frac{2}{3}\lambda_1^Q & \text{if } \lambda_1^Q = \lambda_2^Q = \lambda_3^Q > 0 \\ \min\{\lambda_1^Q + \lambda_2^Q, \lambda_3^Q\} & \text{if } \lambda_1^Q = \lambda_2^Q \neq \lambda_3^Q > 0 \\ \text{tr}(Q) - \lambda_{\max}^Q & \text{if } \lambda_i^Q \neq \lambda_j^Q \geq 0, i \neq j \end{cases} \quad (21)$$

2) Let Q be a matrix such that $\text{tr}(Q) - 2\lambda_{\max}^Q > 0$, and let $\mathbb{U}$ be a set that contains any three orthogonal unit vectors in $\mathbb{R}^3$, then the following inequality holds:

$$\Delta_Q^* \geq \frac{2}{3}(\text{tr}(Q) - 2\lambda_{\max}^Q). \quad (22)$$

Lemma 3: Let $\mathbb{A} = \sum_{i=1}^n k_i r_i r_i^\top$ with $k_i > 0$ and $r_i \in \mathbb{R}^4, i = 1, \ldots, n$. Then, for all $g, \bar{g} \in SE(3)$, the following identities hold:

$$\text{tr}((I_4 - g)\mathbb{A}(I_4 - g)^\top) = \sum_{i=1}^n k_i \|r_i - g^{-1}r_i\|^2 \quad (23)$$

$$\psi(\mathbb{P}((I_4 - g^{-1})\mathbb{A})) = \frac{1}{2}\sum_{i=1}^n k_i (g^{-1}r_i) \wedge r_i \quad (24)$$

$$\text{Ad}_{\bar{g}}^* \sum_{i=1}^n k_i (\bar{g}g^{-1}r_i) \wedge r_i = \sum_{i=1}^n k_i (g^{-1}r_i) \wedge (\bar{g}^{-1}r_i). \quad (25)$$

Lemma 4: Let Assumption 1 hold and consider the following smooth potential function on $SE(3)$:

$$\mathcal{U}(g) = \frac{1}{2}\text{tr}((I_4 - g)\mathbb{A}(I_4 - g)^\top) \quad (26)$$

where the matrix $\mathbb{A}$ is defined in (19). For any $g = \mathcal{T}(R, p) \in SE(3)$, one has

$$\nabla_g \mathcal{U}(g) := g\mathbb{P}((I_4 - g^{-1})\mathbb{A}), \quad (27)$$

$$\Psi_{\mathcal{U}}(g) := \{I_4\} \bigcup \{g = \mathcal{T}(R, p) : R = \mathcal{R}_a(\pi, v),$$
$$p = (I_3 - \mathcal{R}_a(\pi, v))bd^{-1}, v \in \mathcal{E}(Q)\}. \quad (28)$$

Lemma 5: Consider the potential function (26) under Assumption 1. Define the following set

$$\mathbb{Q} := \{\mathcal{T}(R, p) \in SE(3) | R = \mathcal{R}_a(\theta^*, u), \theta^* \in (0, \pi],$$
$$p = (I_3 - \mathcal{R}_a(\theta^*, u))bd^{-1}, u \in \mathbb{U}\}. \quad (29)$$

where $\mathbb{U} = \{u_q | u_q \in \mathbb{S}^2, q = 1, \ldots, m\}$. There exist strictly positive scalars $\alpha_1, \alpha_2, \alpha_3$, and α_4, such that the following inequalities hold:

$$\alpha_1 |g|_I^2 \leq \mathcal{U}(g) \leq \alpha_2 |g|_I^2, \qquad g \in SE(3) \quad (30)$$
$$\alpha_3 |g|_I^2 \leq \|\psi(g^{-1}\nabla_g \mathcal{U}(g))\|^2 \leq \alpha_4 |g|_I^2, \quad g \in \Upsilon \quad (31)$$
$$\|\text{Ad}_{g^{-1}}^* \psi(g^{-1}\nabla_g \mathcal{U}(g))\|^2 \leq \alpha_4 |g|_I^2, \quad g \in \Upsilon \quad (32)$$

where $\Upsilon := \{g \in SE(3) : \mathcal{U}(g) - \min_{g_q \in \mathbb{Q}} \mathcal{U}(gg_q) \leq \delta\}$ with $\delta < (1 - \cos\theta^*)\Delta_Q^*$, $\mathbb{U}$ and Δ_Q^* designed as per Lemma 2.

In view of (7), (23), and (26), let us introduce the following potential function, which can be written in terms of the homogeneous output measurements:

$$\mathcal{U}_1(\tilde{g}) := \frac{1}{2}\text{tr}((I_4 - \tilde{g})\mathbb{A}(I_4 - \tilde{g})^\top) = \frac{1}{2}\sum_{i=1}^n k_i \|r_i - \hat{g}b_i\|^2 \quad (33)$$

where the matrix $\mathbb{A}$ is given in (19). Making use of (24), (25) in Lemma 3, and (27) in Lemma 4, one has the following identities:

$$\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}_1(\tilde{g})) = \frac{1}{2}\sum_{i=1}^n k_i (\hat{g}b_i) \wedge r_i \quad (34)$$

$$\text{Ad}_{\hat{g}}^* \psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}_1(\tilde{g})) = \frac{1}{2}\sum_{i=1}^n k_i b_i \wedge (\hat{g}^{-1}r_i). \quad (35)$$

Proposition 1: Consider the following hybrid state observer:

$$\underbrace{\begin{aligned} \dot{\hat{g}} &= \hat{g}(\xi_y - \hat{b}_a + k_\beta\beta)^\wedge \\ \dot{\hat{b}}_a &= -\Gamma\sigma_b \end{aligned}}_{(\hat{g}, \hat{b}_a) \in \mathcal{F}_o} \quad \underbrace{\begin{aligned} \hat{g}^+ &= g_q^{-1}\hat{g}, g_q \in \gamma(\hat{g}) \\ \hat{b}_a^+ &= \hat{b}_a \end{aligned}}_{(\hat{g}, \hat{b}_a) \in \mathcal{J}_o} \quad (36)$$

$$\beta = \frac{1}{2}\text{Ad}_{\hat{g}^{-1}} \sum_{i=1}^n k_i (\hat{g}b_i) \wedge (r_i) \quad (37)$$

$$\sigma_b = \frac{1}{2}\sum_{i=1}^n k_i b_i \wedge (\hat{g}^{-1}r_i). \quad (38)$$

Choose the set $\mathbb{Q}$ as per Lemma 5. Let Assumption 1 and Assumption 2 hold. Then, the results of Theorem 1 hold.

Proof: See Appendix B ∎

Remark 4: In view of (5), (6), and (36)–(38), the rotational and translational error dynamics in the flow $\mathcal{F}_c$ are given by

$$\dot{\tilde{R}} = \tilde{R}(-k_\beta(\psi_a(Q\tilde{R}) + \frac{1}{2}b^\times \tilde{R}^\top \tilde{p}_e) + (\hat{R}\tilde{b}_\omega))^\times \quad (39)$$

$$\dot{\tilde{p}} = -\frac{1}{2}k_\beta d(\tilde{p} - (I_3 - \tilde{R})bd^{-1}) + \tilde{R}(\hat{p}^\times \hat{R}\tilde{b}_\omega + \hat{R}\tilde{b}_v). \quad (40)$$

The error dynamics (39) and (40) have the same form as [12, Eq. (23)], in the velocity-bias-free case. Note that the dynamics of $\tilde{R}$ and $\tilde{p}$ are coupled as long as $b = dp_c^\mathcal{I} \neq 0$. Therefore, it is expected that noisy or erroneous position measurements would affect the attitude estimation. This motivated us to redesign the estimation scheme in a way that leads to a decoupled rotational error dynamics from the translational error dynamics.

V. DECOUPLING THE ROTATIONAL ERROR DYNAMICS FROM THE TRANSLATIONAL ERROR DYNAMICS

Define an auxiliary configuration $g_c := \mathcal{T}(I_3, p_c^\mathcal{I})$ with $p_c^\mathcal{I} = \alpha_i p_i^\mathcal{I}$ and $\alpha_i := k_i / \sum_{i=1}^{n_1} k_i$. Consider the modified inertial elements of the homogeneous space $\bar{r}_i$, defined as $\bar{r}_i := h(g_c, r_i) = g_c^{-1} r_i, i = 1, \ldots, n$. Define the modified inertial landmarks as $\bar{p}_i^\mathcal{I} := p_i^\mathcal{I} - p_c$. It is clear that $\sum_{i=1}^{n_1} \alpha_i \bar{p}_i^\mathcal{I} = 0$, which implies that the centroid of the weighted modified landmarks coincides with the origin. This property is instrumental in achieving decoupled rotational error dynamics from the translational error dynamics. Note that in [13] this property has been achieved through the choice of the parameters α_i assuming that the landmark points are linearly dependent. Our approach does not put such restrictions on the landmarks and the parameters α_i.

Define the modified pose and pose estimate as $g := \mathcal{T}(\underline{R}, \underline{p}) = g_c^{-1} g$ and $\underline{\hat{g}} := \mathcal{T}(\hat{\underline{R}}, \hat{\underline{p}}) = g_c^{-1} \hat{g}$. One verifies that $b_i = h(g, \bar{r}_i) = g^{-1}\bar{r}_i$. Define the new pose estimation error $\tilde{g} := \mathcal{T}(\tilde{\underline{R}}, \tilde{\underline{p}}) = g\underline{\hat{g}}^{-1} = g_c^{-1} g\hat{g}^{-1} g_c$. It is clear that $\tilde{g}$ tends to I_4 if $\tilde{g}$ tends to I_4. Let us introduce the following potential function:

$$\mathcal{U}_2(\tilde{g}) := \frac{1}{2}\text{tr}((I_4 - \tilde{g})\bar{\mathbb{A}}(I_4 - \tilde{g})^\top) = \frac{1}{2}\sum_{i=1}^n k_i \|\bar{r}_i - \underline{\hat{g}}b_i\|^2 \quad (41)$$

where $\bar{\mathbb{A}} := \sum_{i=1} k_i \bar{r}_i \bar{r}_i^\top = \text{diag}(Q, d)$. In view of (4), (33), and (41), one can show that, for any $\tilde{g} \in SE(3), \mathcal{U}_2(\tilde{g}) = \mathcal{U}_2(g_c^{-1}\tilde{g}g_c) = \mathcal{U}_1(\tilde{g})$. In the sequel, we will make use of $\mathcal{U}_2(\tilde{g})$ and $\mathcal{U}_1(\tilde{g})$ equivalently. Making use of (24) in Lemma 3 and (27) in Lemma 4, one can also

show that

$$\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}_2(\tilde{g})) = \frac{1}{2}\sum_{i=1}^{n}k_i(g_c^{-1}\hat{g}b_i)\wedge(g_c^{-1}r_i). \tag{42}$$

Define the extended state $x' := (\tilde{g}, \tilde{b}, \hat{g}, \hat{b}_a, t) \in \mathcal{S}$ and the closed set $\bar{\mathcal{A}}' := \{x' \in \mathcal{S} : \tilde{g} = I_4, \tilde{b}_a = 0\}$. Let $|x'|_{\bar{\mathcal{A}}'}$ denote the distance to the set $\bar{\mathcal{A}}'$ such that $|x'|_{\bar{\mathcal{A}}'}^2 := \inf_{y=(I_4,0,\hat{g},\hat{b}_a,\bar{t})\in\bar{\mathcal{A}}'}(\|I_4 - \tilde{g}\|_F^2 + \|\tilde{b}_a\|^2 + \|\bar{g} - \hat{g}\|_F^2 + \|\bar{b}_a - \hat{b}_a\|^2 + \|\bar{t} - t\|^2) = |\tilde{g}|_I^2 + \|\tilde{b}_a\|^2$.

Let us consider the following modified estimation scheme:

$$\underbrace{\begin{aligned}\dot{\hat{g}} &= \hat{g}(\xi_y - \hat{b}_a + k_\beta\beta)^\wedge \\ \dot{\hat{b}}_a &= -\Gamma\sigma_b\end{aligned}}_{(\hat{g},\hat{b}_a)\in\mathcal{F}_o} \quad \underbrace{\begin{aligned}\hat{g}^+ &= g_q^{-1}\hat{g}, \; g_q\in\gamma(\hat{g}) \\ \hat{b}_a^+ &= \hat{b}_a\end{aligned}}_{(\hat{g},\hat{b}_a\in\mathcal{J}_o)} \tag{43}$$

$$\beta = \frac{1}{2}\mathrm{Ad}_{\hat{g}^{-1}g_c}\sum_{i=1}^{n}k_i(g_c^{-1}\hat{g}b_i)\wedge(g_c^{-1}r_i) \tag{44}$$

$$\sigma_b = \frac{1}{2}\Lambda^\top\sum_{i=1}^{n}k_i(g_c^{-1}\hat{g}b_i)\wedge(g_c^{-1}r_i) \tag{45}$$

where $\Lambda := \mathrm{diag}(\hat{R},\hat{R})$. In view of (5), (6), and (43)–(45), one can write the closed-loop system as an autonomous hybrid system

$$\mathcal{H}' : \begin{cases} \dot{x}' \in F(x') & x' \in \mathcal{F}'_c := \{x'\in\mathcal{S}:(\hat{g},\hat{b}_a)\in\mathcal{F}_o\} \\ x'^+ \in G(x') & x' \in \mathcal{J}'_c := \{x'\in\mathcal{S}:(\hat{g},\hat{b}_a)\in\mathcal{J}_o\} \end{cases} \tag{46}$$

with

$$F(x') = \begin{bmatrix} \tilde{g}(\mathrm{Ad}_{\hat{g}}\tilde{b}_a - k_\beta\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}_2(\tilde{g})))^\wedge \\ -\Gamma\Lambda^\top\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}_2(\tilde{g})) \\ \hat{g}(\xi_y - \hat{b}_a + k_\beta\mathrm{Ad}_{\hat{g}^{-1}}\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}_2(\tilde{g})))^\wedge \\ -\Gamma\Lambda^\top\psi(\tilde{g}^{-1}\nabla_{\tilde{g}}\mathcal{U}_2(\tilde{g})) \\ 1 \end{bmatrix}, G(x') = \begin{bmatrix} \tilde{g}\bar{g}_q \\ \tilde{b}_a \\ g_q^{-1}\hat{g} \\ \hat{b}_a \\ t \end{bmatrix}$$

where $\bar{g}_q := g_c^{-1}g_q g_c$. Note that the closed-loop system (46) also satisfies the hybrid basic conditions of [21]. Now, one can state the following theorem.

Theorem 2: Consider the closed-loop system (46) with potential function $\mathcal{U}_2$ in (41). Choose the set $\mathbb{Q}$ as per Lemma 5. Let Assumption 1 and Assumption 2 hold. Then, the number of jumps is finite and for any initial condition $x'(0,0) \in \mathcal{S}$, the solution $x'(t,j)$ is complete and there exist $k' > 0$ and $\lambda' > 0$ such that

$$|x'(t,j)|_{\bar{\mathcal{A}}'} \leq k'\exp(-\lambda'(t+j))|x'(0,0)|_{\bar{\mathcal{A}}'} \tag{47}$$

for all $(t,j) \in \mathrm{dom}\,x'$.

Proof: See Appendix C. ∎

Remark 5: From (25), (44), and (45), one can show that

$$\beta = \frac{1}{2}\mathrm{Ad}_{\hat{g}^{-1}}\mathrm{Ad}_{g_c}\mathrm{Ad}_{g_c}^*\sum_{i=1}^{n}k_i(\hat{g}b_i)\wedge(r_i) \tag{48}$$

$$\sigma_b = \frac{1}{2}\Lambda^\top\mathrm{Ad}_{g_c}^*\sum_{i=1}^{n}k_i(\hat{g}b_i)\wedge(r_i). \tag{49}$$

In view of (36)–(38), (43) and (48)–(49), one can notice that the difference between the observer in Theorem 2 and the observer in Proposition 1 is related to the terms β and σ_b. It is worth pointing out that the observer in Theorem 2, without "hybridation" (i.e., in the flow set), is not gradient-based as in [10], [12], [14], and [15].
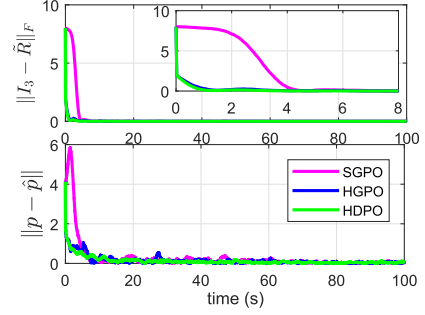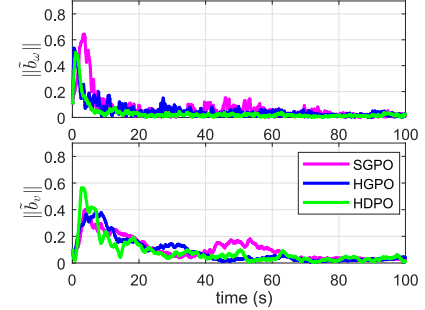
Fig. 1. Simulation results with additive white Gaussian noise of variance 0.1 in the output measurements and time-varying velocity-bias $b_a(t) = \cos(0.02t)[-0.02 \; 0.02 \; 0.10.2 \; -0.1 \; 0.01]^\top$. (a) Rotation and position estimation errors. (b) Velocity-bias estimation errors.

Remark 6: In view of (25), (27), and (46), one has the following error dynamics in the flows:

$$\dot{\tilde{R}} = \tilde{R}(-k_\beta\psi_a(Q\tilde{R}) + \hat{R}\tilde{b}_\omega)^\times \tag{50}$$

$$\dot{\tilde{p}} = -\frac{1}{2}k_\beta d\tilde{p} + \tilde{R}(\hat{p}^\times\hat{R}\tilde{b}_\omega + \hat{R}\tilde{b}_v) \tag{51}$$

$$\dot{\tilde{b}}_\omega = -k_\omega\hat{R}^\top\psi_a(Q\tilde{R}) \tag{52}$$

$$\dot{\tilde{b}}_v = -\frac{1}{2}k_v d\hat{R}^\top\tilde{R}^\top\tilde{p}. \tag{53}$$

Using the facts $\tilde{R} = \bar{\tilde{R}}$, $\tilde{p} = \bar{\tilde{p}} - (I_3 - \tilde{R})p_c^\mathcal{I}$, $\hat{R} = \bar{\hat{R}}$, and $\hat{p} = \bar{\hat{p}} - p_c^\mathcal{I}$, one can notice that the rotational error dynamics (50) together with (52) do not depend on the translational estimation, which guarantees the aimed at decoupling property.

VI. SIMULATION

In this section, simulation results are presented to illustrate the performance of the hybrid gradient-based pose observer in Proposition 1 and the hybrid decoupled pose observer in Theorem 2, referred to, respectively, as HGPO and HDPO. We refer to the smooth gradient-based pose observer (i.e., the HGPO in Proposition 1 without the jumps) as SGPO.

As commonly used in practical applications, to avoid the bias estimation drift, in the presence of measurement noise, we introduce the following projection mechanism [25]:

$$\mathrm{Proj}_\Delta(\hat{b},\Gamma\sigma) :=$$

$$\begin{cases} \Gamma\sigma, & \text{if } \hat{b}\in\Pi_\Delta \text{ or } \nabla_{\hat{b}}\mathcal{P}^\top\Gamma\sigma \leq 0 \\ \left(I - \varrho(\hat{b})\Gamma\frac{\nabla_{\hat{b}}\mathcal{P}\nabla_{\hat{b}}\mathcal{P}^\top}{\nabla_{\hat{b}}\mathcal{P}^\top\Gamma\nabla_{\hat{b}}\mathcal{P}}\right)\Gamma\sigma, & \text{otherwise} \end{cases}$$

where $\hat{b}, \sigma \in \mathbb{R}^n$, $\Gamma \in \mathbb{R}^{n \times n}$, $\mathcal{P}(\hat{b}) := \|\hat{b}\| - \Delta$, $\Pi_\Delta := \{\hat{b}|\mathcal{P}(\hat{b}) \leq 0\}$, $\Pi_{\Delta,\epsilon} := \{\hat{b}|\mathcal{P}(\hat{b}) \leq \epsilon\}$ and $\varrho(\hat{b}) := \min\{1, \mathcal{P}(\hat{b})/\epsilon\}$ for some positive parameters Δ and ϵ. Given $\|\hat{b}(0)\| < \Delta$, one can verify that the projection map Proj_Δ satisfies the following properties: 1) $\|\hat{b}(t)\| \leq \Delta + \epsilon$, for all $t \geq 0$; and 2) $\tilde{b}^\top \Gamma^{-1} \text{Proj}_\Delta(\hat{b}, \Gamma\sigma) \leq \tilde{b}^\top \sigma$ and $\|\text{Proj}_\Delta(\hat{b}, \Gamma\sigma)\| \leq \|\Gamma\sigma\|$.

Consider the three inertial vectors $v_1^{\mathcal{I}} = [0\ 0\ 1]^\top$, $v_2^{\mathcal{I}} = [\frac{\sqrt{3}}{2}\ \frac{1}{2}\ 0]^\top$, $v_3^{\mathcal{I}} = [-\frac{1}{2}\ \frac{\sqrt{3}}{2}\ 0]^\top$ and one landmark $p^{\mathcal{I}} = [\frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}\ 2]^\top$ are available. The initial pose for all the observers is taken as the identity, i.e., $\hat{g}(0) = I_4$. The system's initial conditions are taken as follows: $R(0) = \mathcal{R}_a(\pi, v)$ with $v = [1\ 0\ 0]^\top$ and $p(0) = [0\ 1\ 4]^\top$. The system is driven by the following linear and angular velocities: $v(t) = 2[\cos(t)\ \sin(t)\ 0]^\top$, $\omega(t) = [-\sin(t)\ \cos(t)\ 0]^\top$. For the hybrid design, we choose $\theta^* = 2\pi/3$, $\delta = 1$, and $\mathbb{U} = \mathbb{E}(Q)$. The gain parameters involved in all the observers are taken as follows: $k_i = 1, i = 1, \ldots, 4$, $k_\beta = 1, k_\omega = 1, k_v = 1$.

The simulation results are given in Fig. 1, from which one can clearly see the improved performance of the decoupled hybrid observer as compared to the nondecoupled hybrid observer and nonhybrid observer.

VII. CONCLUSION

A globally exponentially stable hybrid pose and velocity-bias estimation scheme evolving on $SE(3) \times \mathbb{R}^6$ has been proposed. The proposed observer is formulated in terms of homogeneous output measurements of known inertial vectors and landmark points. It relies on an observer-state jump mechanism designed to avoid the undesired critical point while ensuring a decrease of the potential function in the flow and jump sets. Moreover, an auxiliary coordinate transformation is introduced on the landmark measurements, and a modified observer, leading to decoupled rotational error dynamics from translational error dynamics, is proposed.

APPENDIX

A. Proof of Theorem 1

Consider the following real-valued function:

$$V(x) = \mathcal{U}(\tilde{g}) + \frac{1}{2}\tilde{b}_a \Gamma^{-1} \tilde{b}_a. \tag{54}$$

Taking the time derivative of V along the flow trajectories of (14), one can show that

$$\dot{V} = \left\langle \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}), \tilde{g}(\text{Ad}_{\hat{g}}(\tilde{b}_a - k_\beta \beta))^\wedge \right\rangle_{\tilde{g}} - \tilde{b}_a^\top \Gamma^{-1} \Gamma \sigma_b$$

$$= \langle \langle \tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}), (\text{Ad}_{\hat{g}}(\tilde{b}_a - k_\beta \beta))^\wedge \rangle \rangle - \tilde{b}_a^\top \sigma_b$$

$$= -k_\beta \|\psi(\tilde{g}^{-1} \nabla_{\tilde{g}} \mathcal{U}(\tilde{g}))\|^2 \tag{55}$$

where we made use of (2) and $\text{Ad}_{\hat{g}} \text{Ad}_{\hat{g}^{-1}} = I_6$. Then, for any $x \in \mathcal{F}_c$, V is nonincreasing along the flows of (14). Using the fact $\mathcal{F}_c \cup \mathcal{J}_c = \mathcal{S}$ and the definitions of $\mathcal{X}_{\mathcal{U}}$ and $\mathcal{J}_c$, from (16), one has $\mathcal{X}_{\mathcal{U}} \times \mathbb{R}^6 \times SE(3) \times \mathbb{R}^6 \times \mathbb{R}_{\geq 0} \subseteq \mathcal{J}_c$.

For any $x \in \mathcal{J}_c$, one can show that

$$V(x^+) - V(x) = \mathcal{U}(\tilde{g}^+) - \mathcal{U}(\tilde{g})$$

$$\leq \min_{g_q \in \mathbb{Q}} \mathcal{U}(\tilde{g}g_q) - \mathcal{U}(\tilde{g}) \leq -\delta \tag{56}$$

which implies that V is strictly decreasing over the jumps of (14). Thus, in view of (55) and (56), one can easily show that $0 < V(x(t,j)) \leq V(x(0,0)) - j\delta$, which leads to $0 \leq j \leq J_{\max} := \lceil V(x(0,0))/\delta \rceil$ with $\lceil \cdot \rceil$ denoting the ceiling function. This implies that the number of jumps is finite.

To show exponential stability, let us consider the following Lyapunov function candidate:

$$\mathcal{L}(x) = V(x) - z^\top U \tilde{b}_a \tag{57}$$

where $U := \text{diag}(\mu_1 I_3, \mu_2 I_3)$ with $\mu_1, \mu_2 > 0$ and $z := [\psi_a(\tilde{R})^\top \hat{R}, \tilde{p}^\top R]^\top$. Let $e := [\|\tilde{g}\|_I, \|\tilde{b}_\omega\|, \|\tilde{b}_v\|]^\top$. From (15), one obtains

$$e^\top \underbrace{\begin{bmatrix} \alpha_1 & -\frac{\sqrt{2}}{4}\mu_1 & -\frac{1}{2}\mu_2 \\ -\frac{\sqrt{2}}{4}\mu_1 & \frac{1}{k_\omega} & 0 \\ -\frac{1}{2}\mu_2 & 0 & \frac{1}{k_v} \end{bmatrix}}_{P_1} e \leq \mathcal{L}(x) \leq e^\top \underbrace{\begin{bmatrix} \alpha_2 & \frac{\sqrt{2}}{4}\mu_1 & \frac{1}{2}\mu_2 \\ \frac{\sqrt{2}}{4}\mu_1 & \frac{1}{k_\omega} & 0 \\ \frac{1}{2}\mu_2 & 0 & \frac{1}{k_v} \end{bmatrix}}_{P_2} e.$$

Using the fact $|x|_{\mathcal{A}}^2 = |\tilde{g}|_I^2 + \|\tilde{b}_a\|^2 = \|e\|^2$, one obtains the following inequalities:

$$\lambda_{\min}^{P_1} |x|_{\mathcal{A}}^2 \leq \mathcal{L}(x) \leq \lambda_{\max}^{P_2} |x|_{\mathcal{A}}^2. \tag{58}$$

Let $\psi_{\tilde{R}} := \psi_a(\tilde{R})$. One verifies that $\|\psi_{\tilde{R}}\|^2 \leq \frac{1}{2}\|I_3 - \tilde{R}\|_F^2 \leq 4$ and $\|\psi_{\tilde{R}}\|^2 \leq \frac{1}{2}|\tilde{g}|_I^2$. From (14), one has

$$\dot{\tilde{R}} = \tilde{R}(-k_\beta \psi_\omega + \hat{R}\tilde{b}_\omega)^\times \tag{59}$$

$$\dot{\tilde{p}} = \tilde{R}(-k_\beta \psi_v + \hat{p}^\times \hat{R}\tilde{b}_\omega + \hat{R}\tilde{b}_v) \tag{60}$$

$$\dot{\psi}_{\tilde{R}} = E(\tilde{R})(-k_\beta \psi_\omega + \hat{R}\tilde{b}_\omega) \tag{61}$$

where $E(\tilde{R}) := \frac{1}{2}(\text{tr}(\tilde{R}) - \tilde{R}^\top)$. The arguments of ψ have been omitted for simplicity, and ψ_ω, ψ_v are given by $\psi := [\psi_\omega^\top, \psi_v^\top]^\top$. From [20, Lemma 2], one has $\|E(\tilde{R})\|_F \leq \sqrt{3}$ and $v^\top(I_3 - E(\tilde{R}))v \leq \frac{1}{4}\|I_3 - \tilde{R}\|_F^2 \|v\|^2 \leq \frac{\sqrt{2}}{2}\|I_3 - \tilde{R}\|_F \|v\|^2 \leq \frac{\sqrt{2}}{2}|\tilde{g}|_I \|v\|^2$ for all $v \in \mathbb{R}^3$. Define the constants $c_\omega := \sup_{t \geq 0} \|\omega(t)\|$ and $c_p := \sup_{t \geq 0} \|p(t)\|$. Since $\tilde{b}_\omega$ is bounded both in the flow and jump sets, there exists a constant $c_{b_\omega} := \sup_{(t,j) \succeq (0,0)} \|\tilde{b}_\omega(t,j)\|$. In view of (59)–(61), the time-derivative of the cross term $\mathfrak{X} := -z^\top U \tilde{b}_a = -\mu_1 \tilde{b}_\omega^\top \hat{R}^\top \psi_{\tilde{R}} - \mu_2 \tilde{p}^\top R \tilde{b}_v$ is obtained as

$$\dot{\mathfrak{X}} = \mu_1 \tilde{b}_\omega^\top (\omega - \tilde{b}_\omega + k_\beta \psi_\omega)^\times \hat{R}^\top \psi_{\tilde{R}}$$

$$\quad - \mu_1 \tilde{b}_\omega^\top \hat{R}^\top E(\tilde{R})(-k_\beta \psi_\omega + \hat{R}\tilde{b}_\omega) + \mu_2 \tilde{b}_v^\top (\omega)^\times R^\top \tilde{p}$$

$$\quad - \mu_2 \tilde{b}_v^\top \hat{R}^\top (-k_\beta \psi_v + \hat{p}^\times \hat{R}\tilde{b}_\omega + \hat{R}\tilde{b}_v) + z^\top U \Gamma \text{Ad}_{\hat{g}}^* \text{Ad}_{\hat{g}^{-1}}^* \psi$$

$$\quad \leq -\mu_1 \|\tilde{b}_\omega\|^2 - \mu_2 \|\tilde{b}_v\|^2 + k_\Gamma c_g \sqrt{\alpha_5}(u_1 + u_2)|\tilde{g}|_I^2$$

$$\quad + \mu_1 \left(\frac{\sqrt{2}}{2}c_\omega + 2\sqrt{\alpha_4}k_\beta + \frac{\sqrt{2}}{2}c_{b_\omega} + \sqrt{3\alpha_4}k_\beta\right)|\tilde{g}|_I \|\tilde{b}_\omega\|$$

$$\quad + \mu_2(c_\omega + \sqrt{\alpha_4}k_\beta + c_{b_\omega})\|\tilde{b}_v\|\|\tilde{g}\|_I + \mu_2 c_p \|\tilde{b}_v\|\|\tilde{b}_\omega\|$$

where $k_\Gamma := \|\Gamma\|_F$, $c_g := \|\text{Ad}_g^*\|_F = \sqrt{6 + 2c_p^2}$, and the following facts have been used: $|\tilde{g}|_I^2 = \|I_3 - \tilde{R}\|^2 + \|\tilde{p}\|^2$, $\|\hat{p}\| = \|\tilde{R}^\top(p - \tilde{p})\| \leq c_p + \|\tilde{p}\|$, $\|\psi_\omega\|^2 + \|\psi_v\|^2 \leq \alpha_4|\tilde{g}|_I^2$, $\|z\| = \|\psi_{\tilde{R}}\| + \|\tilde{p}\| \leq |\tilde{g}|_I$, $\|U\|_2 \leq (\mu_1 + \mu_2)$ and (17). Let $c_1 := \frac{\sqrt{2}}{2}c_\omega + 2k_\beta\sqrt{\alpha_4} + \frac{\sqrt{2}c_{b_\omega}}{2} + k_\beta\sqrt{3\alpha_4}$, $c_2 := k_\beta\sqrt{\alpha_4} + c_{b_\omega} + c_\omega$, $c_3 := k_\Gamma\sqrt{\alpha_5}c_g$. Then, the time-derivative of $\mathfrak{X}$ satisfies

$$\dot{\mathfrak{X}} \leq -\mu_1 \|\tilde{b}_\omega\|^2 - \mu_2 \|\tilde{b}_v\|^2 + \mu_1 c_1 \|\tilde{b}_\omega\|\|\tilde{g}\|_I + \mu_2 c_2 \|\tilde{b}_v\|\|\tilde{g}\|_I$$

$$\quad + (\mu_1 c_3 + \mu_2 c_3)|\tilde{g}|_I^2 + \mu_2 c_p \|\tilde{b}_v\|\|\tilde{b}_\omega\|. \tag{62}$$

Consequently, in view of (55) and (62), one obtains

$$\dot{\mathcal{L}} \leq -2k_\beta \alpha_3 e_1^2 + (\mu_1 c_3 + \mu_2 c_3)e_1^2 - \mu_1 \|\tilde{b}_\omega\|^2 - \mu_2 \|\tilde{b}_v\|^2$$

$$\quad + \mu_1 c_1 \|\tilde{b}_\omega\|e_1 + \mu_2 c_2 \|\tilde{b}_v\|e_1 + \mu_2 c_p \|\tilde{b}_v\|\|\tilde{b}_\omega\|$$

$$\quad = -e_{12}^\top P_{31} e_{12} - e_{13}^\top P_{32} e_{13} - e_{23}^\top P_{33} e_{23}$$

where $e_{ij} := [e_i, e_j]^\top, i, j \in \{1, 2, 3\}$ with e_i denoting the ith elements of e, and

$$P_{31} := \begin{bmatrix} k_\beta \alpha_3 - \mu_1 c_3 & -\frac{1}{2}\mu_1 c_1 \\ -\frac{1}{2}\mu_1 c_1 & \frac{1}{2}\mu_1 \end{bmatrix}$$

$$P_{32} := \begin{bmatrix} k_\beta \alpha_3 - \mu_2 c_3 & -\frac{1}{2}\mu_2 c_2 \\ -\frac{1}{2}\mu_2 c_2 & \frac{1}{2}\mu_2 \end{bmatrix}, \quad P_{33} := \begin{bmatrix} \frac{1}{2}\mu_1 & -\frac{1}{2}\mu_2 c_p \\ -\frac{1}{2}\mu_2 c_p & \frac{1}{2}\mu_2 \end{bmatrix}.$$

To guarantee that the matrices P_1, P_2, P_{31}, P_{32}, and P_{33} are positive definite, the parameters μ_1 and μ_2 are chosen such that $0 < \mu_1 < \min\left\{\frac{2\sqrt{\alpha_1}}{\sqrt{k_\omega}}, \frac{k_\beta \alpha_3}{c_3 + \frac{1}{4}c_1^2}\right\}, 0 < \mu_2 < \min\left\{\frac{\sqrt{2\alpha_1}}{\sqrt{k_v}}, \frac{k_\beta \alpha_3}{c_3 + \frac{1}{4}c_2^2}, \frac{1}{c_p^2}\mu_1\right\}$. Then, one concludes that

$$\dot{\mathcal{L}}(x) \le -\lambda_F \mathcal{L}(x), \quad x \in \mathcal{F}_c \tag{63}$$

with $\lambda_F := \min\{\lambda_{\min}^{P_{31}}, \lambda_{\min}^{P_{32}}, \lambda_{\min}^{P_{33}}\}/\lambda_{\max}^{P_2}$. On the other hand, from (55) and (56), it is clear that b_a is bounded in the flow and jump sets. Hence, there exists a constant $c_{b_a} := \sup_{(t,j) \succeq (0,0)} \|\tilde{b}_a(t, j)\|$. Let $g_q = \mathcal{T}(R_q, p_q) \in \mathbb{Q}$. Using the facts: $\tilde{R}^+ = \tilde{R}R_q$ and $\tilde{p}^+ = \tilde{p} + \tilde{R}p_q$, one has $\|z^+ - z\| \le \|R_q \psi_a(\tilde{R}R_q) - \psi_a(\tilde{R})\| + \|\tilde{p}^+ - \tilde{p}\| \le 4 + \|p_q\|$, which implies

$$\mathcal{L}(x^+) - \mathcal{L}(x) = V(x^+) - V(x) + (z^+)^\top U\tilde{b}_a - z^\top U\tilde{b}_a$$
$$\le -\delta + (\mu_1 + \mu_2)c_4$$

where $c_4 := (4 + \|p_q\|)c_{b_a}$. Choosing μ_1 and μ_2 such that $0 < \mu_1 < \min\left\{\frac{2\sqrt{\alpha_1}}{\sqrt{k_\omega}}, \frac{k_\beta \alpha_3}{c_3 + \frac{1}{4}c_1^2}, \frac{\delta}{2c_4}\right\}, 0 < \mu_2 < \min\left\{\frac{\sqrt{2\alpha_1}}{\sqrt{k_v}}, \frac{k_\beta \alpha_3}{c_3 + \frac{1}{4}c_2^2}, \frac{1}{c_p^2}\mu_1, \mu_1\right\}$, there exists a constant $\delta^* := \delta - 2\mu_1 c_4 > 0$ such that

$$\mathcal{L}(x^+) \le \mathcal{L}(x) - \delta^* \quad \forall x \in \mathcal{J}_c. \tag{64}$$

In view of (63) and (64), it follows that $\mathcal{L}(t, j) \le \exp(-\lambda_F t)\mathcal{L}(0, 0)$ for each $(t, j) \in \text{dom}\,x$. One verifies that $\mathcal{L}(x^+) \le \exp(-\lambda_J)\mathcal{L}(x)$, with $\lambda_J := -\ln(1 - \delta^*/\mathcal{L}(0, 0))$. Consequently, one obtains

$$\mathcal{L}(t, j) \le \exp(-2\lambda(t + j))\mathcal{L}(0, 0) \tag{65}$$

where $\lambda := \frac{1}{2}\min\{\lambda_F, \lambda_J\}$. From (58) and (65), for each $(t, j) \in \text{dom}\,x$ one obtains (18) with $k := \sqrt{\lambda_{\max}^{P_2}/\lambda_{\min}^{P_1}}$. Since $\mathcal{F}_c \cup \mathcal{J}_c = \mathcal{S}$, the number of jumps if finite and there is no finite escape-time, one concludes that the solution to the hybrid system $\mathcal{H}$ is complete as per [22, Proposition 2.10]. This completes the proof.

B. Proof of Proposition 1

Let $\tilde{g} = \mathcal{T}(\tilde{R}, \tilde{p})$ with $\tilde{R} = \mathcal{R}_a(\theta, v), \theta \in \mathbb{R}$ and $v \in \mathbb{R}^3$. In view of (19) and (33), one obtains

$$\mathcal{U}_1(\tilde{g}) = 2(1 - \cos\theta)v^\top \bar{Q}v + \frac{1}{2}d\|\tilde{p} - (I_3 - \tilde{R})bd^{-1}\|^2 \tag{66}$$

where matrix $\bar{Q}$ is defined in Lemma 1. Under Assumption 1, one has $d > 0$ and $\bar{Q}$ is a positive definite. Consequently, $\mathcal{U}_1(\tilde{g})$ has a unique global minimum at $\tilde{g} = I_4$, i.e., $\mathcal{U}_1$ is a potential function on $SE(3)$. From (66), for any $\tilde{g} = \mathcal{T}(\tilde{R}, \tilde{p}) \in \mathcal{X}_{\mathcal{U}_1}$ and $g_q = \mathcal{T}(R_q, p_q) \in \mathbb{Q}$, one has

$$\mathcal{U}_1(\tilde{g}g_q) = \text{tr}((I_3 - \tilde{R}R_q)Q) + \frac{1}{2}d\left\|p_q - (I_3 - R_q)bd^{-1}\right\|^2$$
$$= \text{tr}((I_3 - \tilde{R})Q) + \text{tr}(\tilde{R}(I_3 - R_q)Q)$$
$$= \mathcal{U}_1(\tilde{g}) - (1 - \cos\theta^*)\Delta_Q(u_q, v) \tag{67}$$

where $u_q \in \mathbb{U}, v \in \mathcal{E}(Q), \Delta_Q(\cdot)$ is defined in (20). We also made use of the following facts: $p_q = (I_3 - R_q)bd^{-1}, \tilde{R} = \mathcal{R}_a(\pi, v) = 2vv^\top - I_3, R_q = \mathcal{R}_a(\theta^*, u_q)$. From (67), one can obtain

$$\mathcal{U}_1(\tilde{g}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}_1(\tilde{g}g_q) = (1 - \cos\theta^*) \max_{u_q \in \mathbb{U}} \Delta_Q(u_q, v) \quad \forall v \in \mathcal{E}(Q)$$
$$\ge (1 - \cos\theta^*) \min_{v \in \mathcal{E}(Q)} \max_{u_q \in \mathbb{U}} \Delta_Q(u_q, v)$$
$$= (1 - \cos\theta^*)\Delta_Q^* > \delta$$

which implies that $\mathcal{X}_{\mathcal{U}_1} \times \mathbb{R}^6 \times SE(3) \times \mathbb{R}^6 \times \mathbb{R}_{\ge 0} \subseteq \mathcal{J}_c$ from (13). In view of (5), (6) and (34)–(38), one can write the hybrid closed-loop system as in (14). The proof is completed by using Lemma 5, Theorem 1 and the fact that $\mathcal{X}_{\mathcal{U}_1} \times \mathbb{R}^6 \times SE(3) \times \mathbb{R}^6 \times \mathbb{R}_{\ge 0} \subseteq \mathcal{J}_c$.

C. Proof Theorem 2

As shown in Proposition 1, for any $\tilde{g} \in \mathcal{X}_{\mathcal{U}_2}$, one verifies that

$$\mathcal{U}_2(\tilde{g}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}_2(g_c^{-1}\tilde{g}g_q g_c) = \mathcal{U}_2(\tilde{g}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}_2(\tilde{g}g_c^{-1}g_q g_c)$$
$$= \mathcal{U}_1(\tilde{g}) - \min_{g_q \in \mathbb{Q}} \mathcal{U}_1(\tilde{g}g_q) > \delta$$

which implies that $\mathcal{X}_{\mathcal{U}_2} \times \mathbb{R}^6 \times SE(3) \times \mathbb{R}^6 \times \mathbb{R}_{\ge 0} \subseteq \mathcal{J}_c'$. Let us consider the following real-valued function:

$$\mathcal{L}_R = \text{tr}(Q(I - \hat{R})) + \frac{1}{2k_\omega}\tilde{b}_\omega^\top \tilde{b}_\omega - \bar{\mu}_1 \psi_a(\hat{R})^\top \hat{R}\tilde{b}_\omega \tag{68}$$

where $\bar{\mu}_1 > 0$. Let $\bar{e}_1 := [\|I_3 - \hat{R}\|_F, \|\tilde{b}_\omega\|]^\top$. Following similar steps as in the proof of Theorem 1, it is clear that there, exists a constant $c_{b_\omega} := \sup_{(t,j) \succeq (0,0)} \|\tilde{b}_\omega(t, j)\|$, and a constant $\bar{\mu}_1^*$ such that for all $\bar{\mu}_1 < \bar{\mu}_1^*$

$$\bar{c}_1 \|\bar{e}_1\|^2 \le \mathcal{L}_R \le \bar{c}_2 \|\bar{e}_1\|^2 \tag{69}$$
$$\dot{\mathcal{L}}_R \le -\bar{c}_3 \mathcal{L}_R, \quad x' \in \mathcal{F}_c' \tag{70}$$

for some positive constants $\bar{c}_1, \bar{c}_2$, and $\bar{c}_3$. Let us consider the following real-valued function:

$$\mathcal{L}_p = \frac{d}{2}\tilde{p}^\top \tilde{p} + \frac{1}{k_v}\tilde{b}_v^\top \tilde{b}_v - \bar{\mu}_2 \tilde{p}^\top \tilde{R}\tilde{b}_v. \tag{71}$$

Let $\bar{e}_2 := [\|\tilde{p}\|, \|\tilde{b}_v\|]^\top$. It is straightforward to show that

$$\bar{e}_2^\top \underbrace{\begin{bmatrix} \frac{d}{2} & -\frac{\bar{\mu}_2}{2} \\ -\frac{\bar{\mu}_2}{2} & \frac{1}{k_v} \end{bmatrix}}_{H_1} \bar{e}_2 \le \mathcal{L}_p \le \bar{e}_2^\top \underbrace{\begin{bmatrix} \frac{d}{2} & \frac{\bar{\mu}_2}{2} \\ \frac{\bar{\mu}_2}{2} & \frac{1}{k_v} \end{bmatrix}}_{H_2} \bar{e}_2. \tag{72}$$

The time-derivative of $\mathcal{L}_p$ along the trajectories of (51) and (53) is obtained as

$$\dot{\mathcal{L}}_p = -\frac{d^2}{2}k_\beta \tilde{p}^\top \tilde{p} - \bar{\mu}_2 \tilde{b}_v^\top \tilde{b}_v + d\tilde{p}^\top \tilde{R}\hat{p}^\times \hat{R}\tilde{b}_\omega + \bar{\mu}_2 k_\beta \frac{d}{2}\tilde{p}^\top R\tilde{b}_v$$
$$+ \bar{\mu}_2 \tilde{b}_\omega^\top \hat{R}^\top \hat{p}^\times \hat{R}^\top R\tilde{b}_v + \frac{d}{2}k_v \bar{\mu}_2 \tilde{p}^\top R\hat{R}^\top \tilde{R}^\top \tilde{p} - \bar{\mu}_2 \tilde{p}^\top R\omega^\times \tilde{b}_v$$
$$\le -\frac{d^2}{2}k_\beta \|\tilde{p}\|^2 - \bar{\mu}_2 \|\tilde{b}_v\|^2 + d\bar{c}_p \|\tilde{p}\|\|\tilde{b}_\omega\| + \bar{\mu}_2 k_\beta \frac{d}{2}\|\tilde{p}\|\|\tilde{b}_v\|$$
$$+ \bar{\mu}_2 \bar{c}_p \|\tilde{b}_\omega\|\|\tilde{b}_v\| + \bar{\mu}_2 c_{b_\omega} \|\tilde{p}\|\|\tilde{b}_v\| + \bar{\mu}_2 \frac{d}{2}k_v \|\tilde{p}\|^2$$
$$+ \bar{\mu}_2 c_\omega \|\tilde{p}\|\|\tilde{b}_v\|$$

where $\bar{c}_p := \|\underline{p}\| = \|p - bd^{-1}\|$, and the following facts have been used: $\hat{\underline{R}} = \hat{R}$, $\underline{R} = R$ and $\hat{\underline{p}} = \tilde{\underline{R}}^\top(p - \tilde{p})$. Let $\bar{c}_4 := \max\{d, \bar{\mu}_2\}$ and $\bar{c}_5 := k_\beta \frac{d}{2} + c_\omega + c_{b_\omega}$. Then, one has

$$\dot{\mathcal{L}}_p \leq -\bar{e}_2^\top \underbrace{\begin{bmatrix} \frac{d^2}{2}k_\beta - \bar{\mu}_2\frac{d}{2}k_v & -\frac{1}{2}\bar{\mu}_2\bar{c}_5 \\[2mm] -\frac{1}{2}\bar{\mu}_2\bar{c}_5 & \bar{\mu}_2 \end{bmatrix}}_{H_3} \bar{e}_2 + \sqrt{2}\bar{c}_4\bar{c}_p\|\tilde{b}_\omega\|\|\bar{e}_2\|.$$

To guarantee that the matrices H_1, H_2, and H_3 are positive definite, it is sufficient to pick $\bar{\mu}_2$ such that $0 < \bar{\mu}_2 < \min\{\frac{\sqrt{2d}}{\sqrt{k_v}}, \frac{k_\beta d^2}{dk_v + \frac{1}{2}\bar{c}_5^2}\}$. Hence, one has

$$\dot{\mathcal{L}}_p \leq -\lambda_{\min}^{H_3}\|\bar{e}_2\|^2 + \sqrt{2}\bar{c}_4\bar{c}_p\|\bar{e}_1\|\|\bar{e}_2\|$$
$$\leq -\eta_2\mathcal{L}_p + \eta_3\sqrt{\mathcal{L}_R}\sqrt{\mathcal{L}_p}, \quad x' \in \mathcal{F}_c' \tag{73}$$

where $\eta_2 := \lambda_{\min}^{H_3}/\lambda_{\max}^{H_2} > 0$, $\eta_3 := \sqrt{2}\bar{c}_4\bar{c}_p/\sqrt{\lambda_{\min}^{H_1}\bar{c}_1} > 0$, and we made use of the fact $\|\tilde{b}_\omega\| \leq \|\bar{e}_1\| \leq \frac{1}{\sqrt{\bar{c}_1}}\sqrt{\mathcal{L}_R}$. Let $\zeta_1 := \sqrt{\mathcal{L}_R}$, $\zeta_2 := \sqrt{\mathcal{L}_p}$ and $\zeta := [\zeta_1, \zeta_2]^\top$. From (70) and (73), one obtains

$$\dot{\zeta} \leq -H_4\zeta, \quad H_4 := \begin{bmatrix} \bar{c}_3 & 0 \\ -\frac{\eta_3}{2} & \frac{\eta_2}{2} \end{bmatrix}. \tag{74}$$

One can easily verify that H_4 is a positive definite. Let us consider the following Lyapunov function candidate:

$$\mathcal{L}'(x') = \mathcal{L}_R + \mathcal{L}_p = \|\zeta\|^2. \tag{75}$$

Using the fact $|x'|_{\bar{\mathcal{A}}'}^2 = |\tilde{g}|_I^2 + \|\tilde{b}_a\|^2 = \|\bar{e}_1\|^2 + \|\bar{e}_2\|^2$, from (69), (70), (72), and (73), one has

$$\underline{\alpha}|x'|_{\bar{\mathcal{A}}'}^2 \leq \mathcal{L}'(x') \leq \bar{\alpha}|x'|_{\bar{\mathcal{A}}'}^2 \tag{76}$$
$$\dot{\mathcal{L}}'(x') \leq -\zeta^\top H_4\zeta \leq -\lambda_F'\mathcal{L}'(x'), \quad x' \in \mathcal{F}_c' \tag{77}$$

where $\lambda_F' := \lambda_{\min}^{H_4}$, $\underline{\alpha} := \min\{\bar{c}_1, \lambda_{\min}^{H_1}\}$ and $\bar{\alpha} := \max\{\bar{c}_2, \lambda_{\max}^{H_2}\}$. On the other hand, using the facts $\tilde{\underline{R}}^+ = \tilde{\underline{R}}R_q$, $\tilde{b}_\omega^+ = \tilde{b}_\omega$, $\tilde{b}_v^+ = \tilde{b}_v$ and $\tilde{\underline{p}}^+ = \tilde{\underline{p}}$, one has

$$\mathcal{L}(x'^+) - \mathcal{L}(x') = \operatorname{tr}(Q(I - \tilde{\underline{R}}^+)) - \operatorname{tr}(Q(I - \tilde{\underline{R}}))$$
$$- \bar{\mu}_1\psi_a(\tilde{\underline{R}})^\top\hat{\underline{R}}\tilde{b}_\omega + \bar{\mu}_1\psi_a(\tilde{\underline{R}}^+)^\top\hat{\underline{R}}^+\tilde{b}_\omega^+ \leq -\delta + 4\bar{\mu}_1c_{b_\omega}.$$

Choosing $\bar{\mu}_1 < \min\{\bar{\mu}_1^*, \delta/4c_{b_\omega}\}$ such that

$$\mathcal{L}(x'^+) \leq \mathcal{L}(x') - \bar{\delta}^*, \quad x' \in \mathcal{J}_c' \tag{78}$$

where $\bar{\delta}^* := -\delta + 4\bar{\mu}_1c_{b_\omega} > 0$. In view of (77) and (78), one has $\mathcal{L}'(t, j) \leq \mathcal{L}'(0, 0)\exp(-\lambda_F't)$ and $j \leq J_{\max}' := \lceil\mathcal{L}'(0, 0)/\bar{\delta}^*\rceil$. One can show that $\mathcal{L}'(x'^+) \leq \exp(-\lambda_J')\mathcal{L}'(x')$ with $\lambda_J' := -\ln(1 - \bar{\delta}^*/\mathcal{L}'(0, 0))$. Consequently, for each $(t, j) \in \operatorname{dom}x'$, one obtains

$$\mathcal{L}'(t, j) \leq \exp(-2\lambda'(t + j))\mathcal{L}'(0, 0) \tag{79}$$

where $\lambda' := \frac{1}{2}\min\{\lambda_F', \lambda_J'\}$. From (76) and (79), for each $(t, j) \in \operatorname{dom}x'$, one obtains the solution of $x'(t, j)$ as (47) wtih $k' := \sqrt{\bar{\alpha}/\underline{\alpha}}$. Using the same arguments as in the proof of Theorem 1, one can conclude that the solution to the hybrid system $\mathcal{H}'$ is complete. This completes the proof.

REFERENCES

[1] M. Wang and A. Tayebi, "Globally asymptotically stable hybrid observers design on $SE(3)$," in *Proc. 56th IEEE Ann. Conf. Decision Control*, 2017, pp. 3033–3038.

[2] M. D. Shuster and S. D. Oh, "Three-axis attitude determination from vector observations," *J. Guid. Control*, vol. 4, pp. 70–77, 1981.

[3] J. L. Crassidis, F. L. Markley, and Y. Cheng, "Survey of nonlinear attitude estimation methods," *J. Guid., Control, Dyn.*, vol. 30, no. 1, pp. 12–28, 2007.

[4] A. Tayebi and S. McGilvray, "Attitude stabilization of a VTOL quadrotor aircraft," *IEEE Trans. Control Syst. Technol.*, vol. 14, no. 3, pp. 562–571, May 2006.

[5] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the Special Orthogonal group," *IEEE Trans. Autom. Control*, vol. 53, no. 5, pp. 1203–1218, Jun. 2008.

[6] H. Rehbinder and B. K. Ghosh, "Pose estimation using line-based dynamic vision and inertial sensors," *IEEE Trans. Autom. Control*, vol. 48, no. 2, pp. 186–199, Feb. 2003.

[7] T. Hamel and C. Samson, "Riccati observers for the non-stationary PnP problem," *IEEE Trans. Autom. Control*, vol. 63, no. 3, pp. 726–741, Mar. 2018.

[8] A. Moeini and M. Namvar, "Global attitude/position estimation using landmark and biased velocity measurements," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 52, no. 2, pp. 852–862, Apr. 2016.

[9] S. Bonnabel, P. Martin, and P. Rouchon, "Non-linear symmetry-preserving observers on Lie groups," *IEEE Trans. Autom. Control*, vol. 54, no. 7, pp. 1709–1713, Jul. 2009.

[10] C. Lageman, J. Trumpf, and R. Mahony, "Gradient-like observers for invariant dynamics on a Lie group," *IEEE Trans. Autom. Control*, vol. 55, no. 2, pp. 367–377, Feb. 2010.

[11] G. Baldwin, R. Mahony, J. Trumpf, T. Hamel, and T. Cheviron, "Complementary filter design on the special euclidean group $SE(3)$," in *Proc. Eur. Control Conf.*, 2007, pp. 3763–3770.

[12] M.-D. Hua, M. Zamani, J. Trumpf, R. Mahony, and T. Hamel, "Observer design on the Special Euclidean group $SE(3)$," in *Proc. 50th IEEE Conf. Decision Control Eur. Control Conf.*, 2011, pp. 8169–8175.

[13] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira, "A nonlinear position and attitude observer on $SE(3)$ using landmark measurements," *Syst. Control Lett.*, vol. 59, no. 3, pp. 155–166, 2010.

[14] A. Khosravian, J. Trumpf, R. Mahony, and C. Lageman, "Observers for invariant systems on Lie groups with biased input measurements and homogeneous outputs," *Automatica*, vol. 55, pp. 19–26, 2015.

[15] M.-D. Hua, T. Hamel, R. Mahony, and J. Trumpf, "Gradient-like observer design on the special euclidean group $SE(3)$ with system outputs on the real projective space," in *Proc. 54th IEEE Ann. Conf. Decision Control*, 2015, pp. 2139–2145.

[16] C. G. Mayhew and A. R. Teel, "Hybrid control of rigid-body attitude with synergistic potential functions," in *Proc. Am. Control Conf.*, 2011, pp. 287–292.

[17] T.-H. Wu, E. Kaufman, and T. Lee, "Globally asymptotically stable attitude observer on $SO(3)$," in *Proc. 54th IEEE Conf. Decision Control*, 2015, pp. 2164–2168.

[18] S. Berkane and A. Tayebi, "Construction of synergistic potential functions on $SO(3)$ with application to velocity-free hybrid attitude stabilization," *IEEE Trans. Autom. Control*, vol. 62, no. 1, pp. 495–501, Jan. 2017.

[19] S. Berkane and A. Tayebi, "Attitude and gyro bias estimation using GPS and IMU measurements," in *Proc. 56th IEEE Conf. Decision Control*, 2017, pp. 2402–2407.

[20] S. Berkane, A. Abdessameud, and A. Tayebi, "Hybrid attitude and gyro-bias observer design on $SO(3)$," *IEEE Trans. Autom. Control*, vol. 62, no. 11, pp. 6044–6050, Nov. 2017.

[21] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Syst.*, vol. 29, no. 2, pp. 28–93, Apr. 2009.

[22] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness.* Princeton, NJ, USA: Princeton Univ. Press, 2012.

[23] D. E. Koditschek, "The application of total energy as a Lyapunov function for mechanical control systems," in *Dynamics and Control of Multibody Systems*, (Contemporary Mathematics), vol. 97, J. E. Marsden, P. S. Krishnaprasad, and J. C. Simo, Eds. Providence, RI, USA: Amer. Math. Soc., 1989, pp. 131–157.

[24] M. Wang and A. Tayebi, "On the design of hybrid pose and velocity-bias observers on *SE*(3)," Internal Tech. Rep., 2018, arXiv:1805.00897.

[25] P. A. Ioannou and J. Sun, *Robust Adaptive Control.* Englewood Cliffs, NJ, USA: Prentice-Hall, 1995.