Bachelor of Science in Computer Science
University of Colombo School of Computing

SCS 1211 – Mathematical Methods I (Linear Algebra)

Topic -2: System of Linear Equations

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Introduction

The fundamental problem of linear algebra is to solve m linear equations in n unknowns $x_1, x_2, \cdots, x_n$:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m;
\end{aligned}
$$

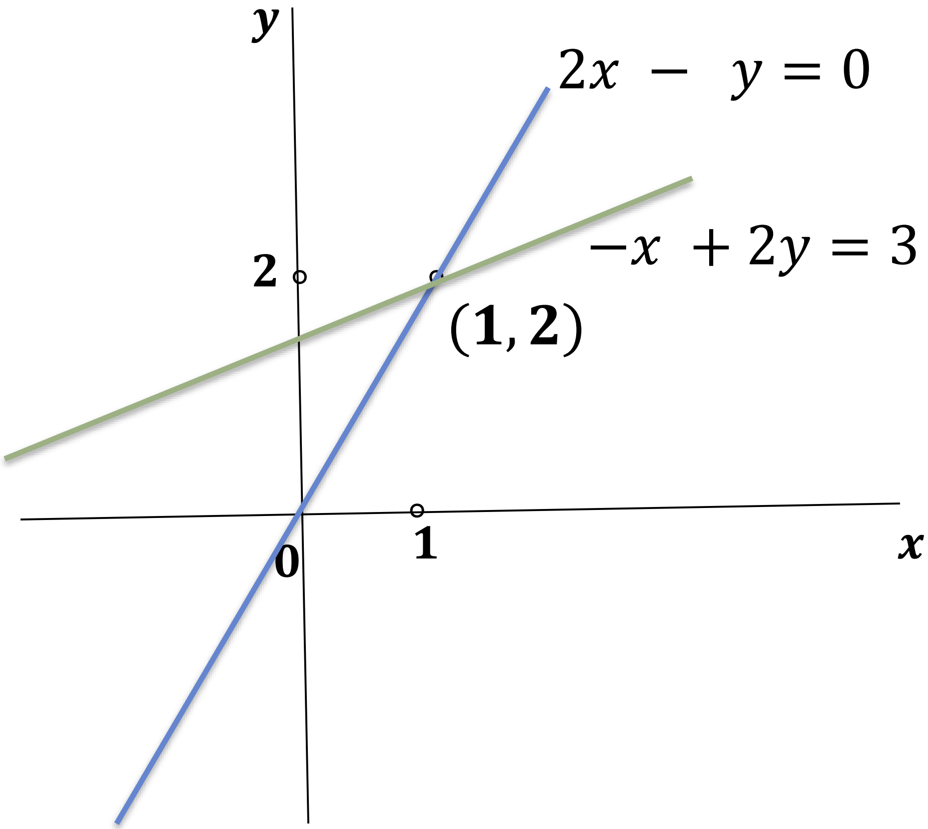
where a_{ij} and b_i are real numbers (scalars). The numbers a_{ij} are called the coefficients and the numbers b_i are called the constant terms of the system of equations.

In this lecture, we view this problem in three different ways.

Geometric view of systems of equations (Raw Picture)

Example 2.1: Consider the system of linear equations:

$$2x - y = 0$$
$$-x + 2y = 3.$$



The lines $2x - y = 0$ and $-x + 2y = 3$ intersect at the point (1, 2).

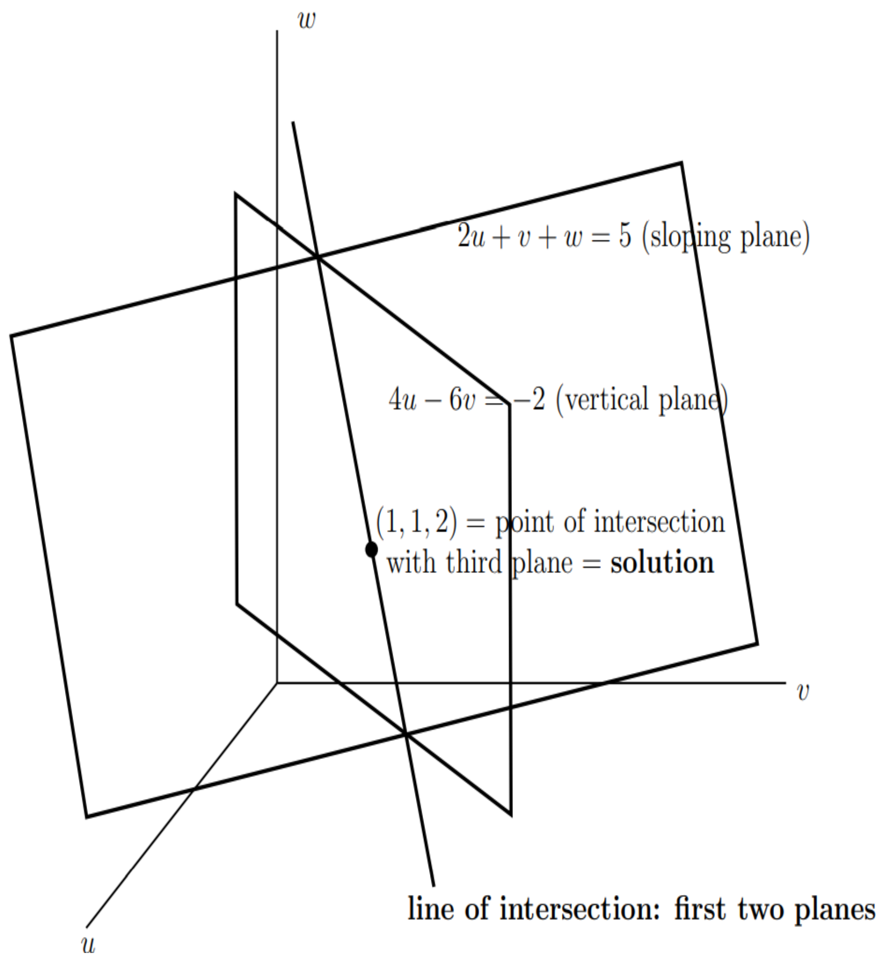
Hence $x = 1$, and $y = 2$ is the only solution to the above system of linear equations.

Geometric view of systems of equations (Raw Picture)

Example 2.2: Consider the system of linear equations:

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v &= -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

- Each equation describes a plane in three dimensions.
- The second plane is $4u - 6v = -2$. It is drawn vertically, because w can take any value.
- The figure shows the intersection of the second plane with the first.
- Finally the third plane intersects this line in a point (1,1,2).



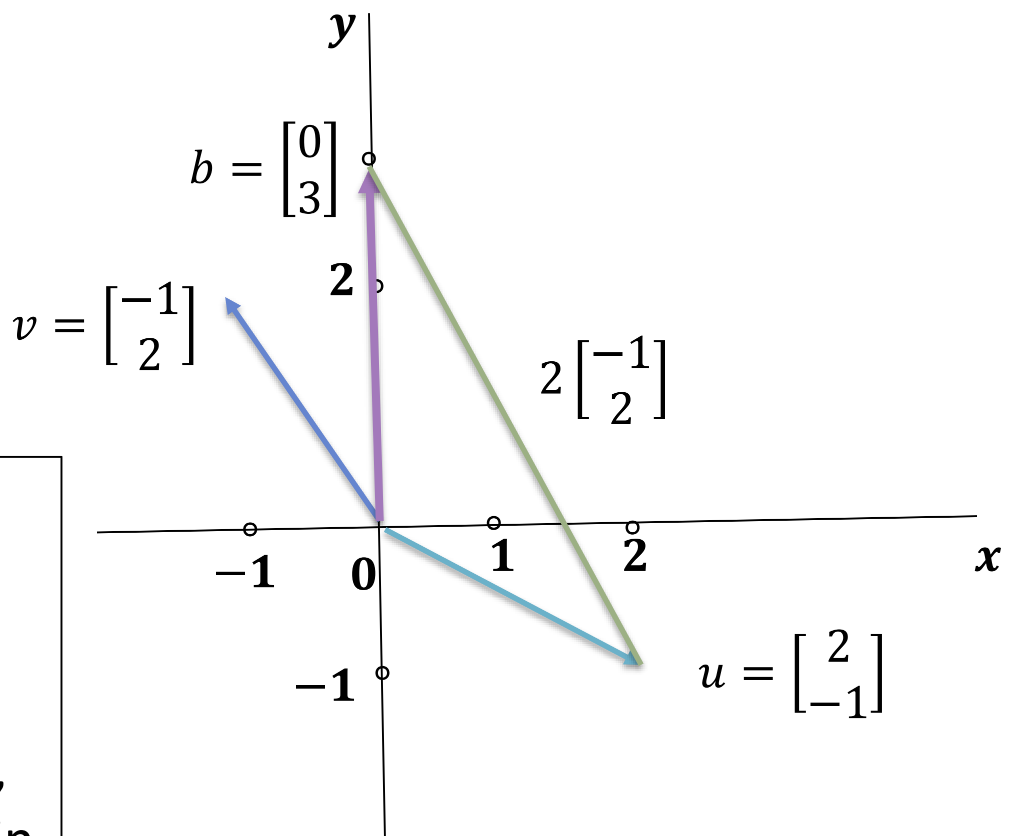
Column Vectors and Linear Combinations (Column Picture)

Consider the example 2.1

$$2x \;-\; y = 0$$
$$-x \;+ 2y = 3$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Geometrically, we want to find numbers x and y so that $x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \mathrm{y}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ equals $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.
As we see from $x = 1$ and $y = 2$, agreeing with the column picture in Figure.



Solving System of Linear equations: (Gaussian Elimination Method)

Example 2.3 Solve the following system of linear equations:

$$x_1 + 2x_2 + x_3 = 4 -----(1)$$

$$2x_1 + 2x_2 + 3x_3 = 7 -----(2)$$

$$x_1 + x_2 + 4x_3 = 6 -----(3)$$

Solution:

Step 1: Choose to pivot x_1 in (1), and eliminate x_1 in (2) & (3).

$$x_1 + 2x_2 + x_3 = 4 -----(1')$$

$$(1) \times -2 + (2) \Longrightarrow \quad -2x_2 + x_3 = -1 ----(2')$$

$$(1) \times -1 + (3) \Longrightarrow \quad -x_2 + 3x_3 = 2 -----(3')$$

Solving System of Linear equations: (Gaussian Elimination Method)

Example 2.3 Continue

$$x_1 + 2x_2 + x_3 = 4 -----(1')$$
$$-2x_2 + x_3 = -1 ----(2')$$
$$-x_2 + 3x_3 = 2 -----(3')$$

Step 2: Choose to pivot x_2 in (2'), and eliminate x_2 in (3').

$$x_1 + 2x_2 + x_3 = 4 -----(1'')$$
$$-2x_2 + x_3 = -1 -----(2'')$$
$$(2') \times -\frac{1}{2} + (3') \Longrightarrow \quad \frac{5}{2} x_3 = \frac{5}{2} -----(3'')$$

Triangular System

Solving System of Linear equations: (Gaussian Elimination Method)

Example 2.3 Continue

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 4 \text{ ------ } (1'') \\
-2x_2 + x_3 &= -1 \text{ ------ } (2'') \\
\frac{5}{2} x_3 &= \frac{5}{2} \text{ ------ } (3'')
\end{aligned}
$$

Step 3: Back-substitution

$(3'') \implies \quad x_3 = 1$

$(2'') \implies \quad x_2 = 1$

$(1'') \implies \quad x_1 = 1 .$

Consistent and Inconsistent Systems

Definition: A system of linear equations is called consistent if there exists at least one solution. It is called inconsistent if there is no solution.

Representation in Matrix Form

Solve the following system of linear equations:

$$x_1 + 2x_2 + x_3 = 4 -----(1)$$

$$2x_1 + 2x_2 + 3x_3 = 7 -----(2)$$

$$x_1 + x_2 + 4x_3 = 6 -----(3)$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}.$$

$$A\,x = b.$$

The matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ is called the **coefficient matrix**. The vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the **vector of unknowns**.

The matrix $\begin{bmatrix} 1 & 2 & 1 & \vdots & 4 \\ 2 & 2 & 3 & \vdots & 7 \\ 1 & 1 & 4 & \vdots & 6 \end{bmatrix}$ is called the **augmented matrix** of the given system of linear equations.

Example 2.4 (Singular system)

$$x_1 + x_2 + x_3 = 4 -----(1)$$
$$2x_1 + 2x_2 + 5x_3 = 7 -----(2)$$
$$4x_1 + 4x_2 + 8x_3 = 6 -----(3)$$

$$x_1 + x_2 + x_3 = 4 -----(1')$$
$$(1) \times -2 + (2) \Longrightarrow \quad +3x_3 = -1 ----(2')$$
$$(1) \times -4 + (3) \Longrightarrow \quad +4x_3 = -10 -----(3')$$

This system is unsolvable since $3x_3 = -1$ & $4x_3 = -10$ is not possible.
Hence, this system of equations is **inconsistent** or **singular.**

Multiplication of a Matrix and a Vector (Ax)

Usual way

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1\times 1+2\times 2+1\times 3 \\ 2\times 1+2\times 2+3\times 3 \\ 1\times 1+1\times 2+4\times 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 15 \end{bmatrix}$$

Other way: Considering the entries of x as the coefficients of a linear combination of the column vectors of the matrix A:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 15 \end{bmatrix}.$$

This shows the entries of x as the coefficients of a linear combination of the column vectors of the matrix A:

Linear Independence

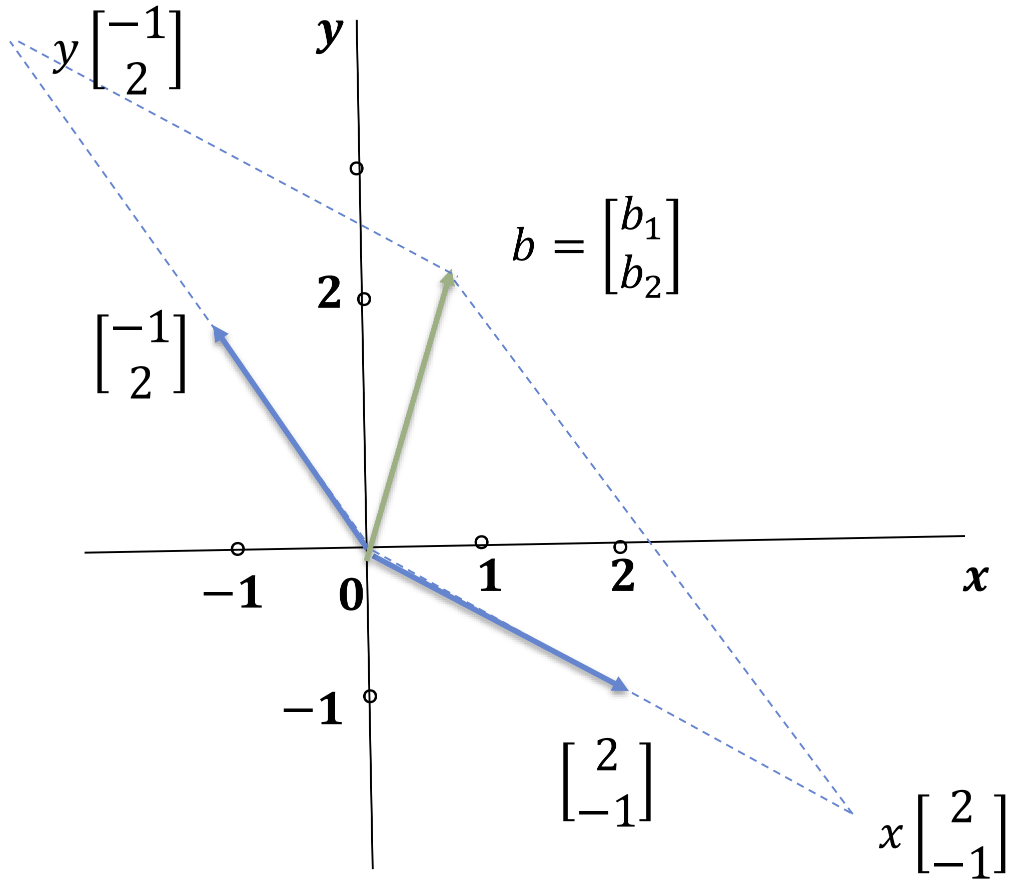
Given a matrix A, can we solve $Ax = b$ for every possible vector b?

Does $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

solvable for every b $= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The linear combinations of the column vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ fill the 2 dimensional xy-plane.



Elementary Row Operations

Let A be a real $m \times n$ matrix. The following three operations on the rows of A are called the elementary row operations:

1. Interchange the i^{th} row and j^{th} row of A $\left(R_i \leftrightarrow R_j\right)$.
2. Multiple the i^{th} row of A by a non-zero scalar k $(R_i \leftarrow kR_i)$.
3. Replace the i^{th} row by k times the j^{th} row plus the i^{th} row

   $(R_i \leftarrow R_i + kR_j)$.

Solving System of Linear equations using Elementary Row Operations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 3x_3 &= 7 \\ x_1 + x_2 + 4x_3 &= 6 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}.$$

$$A\,x = b.$$

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 4 \\ 2 & 2 & 3 & 7 \\ 1 & 1 & 4 & 6 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + (-1)R_1]{R_2 \leftarrow R_2 + (-2)R_1} \left[\begin{array}{cccc} 1 & 2 & 1 & 4 \\ 0 & -2 & 1 & -1 \\ 0 & -1 & 3 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + (-1/2)R_2} \left[\begin{array}{cccc} 1 & 2 & 1 & 4 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 5/2 & 5/2 \end{array}\right]$$

$$x_1 + 2x_2 + x_3 = 4 \qquad \Rightarrow x_1 = 1$$

$$-2\,x_2 + x_3 = -1 \qquad \Rightarrow x_2 = 1$$

$$5/2\, x_3 = 5/2 \qquad \Rightarrow x_3 = 1$$

Solving System of Linear equations using Elementary Row Operations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 3x_3 &= 7 \\ x_1 + x_2 + 4x_3 &= 6 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}.$$

$$A\,x = b.$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 2 & 3 & 7 \\ 1 & 1 & 4 & 6 \end{array}\right] \xRightarrow[R_3 \leftarrow R_3 + (-1)R_1]{R_2 \leftarrow R_2 + (-2)R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -2 & 1 & -1 \\ 0 & -1 & 3 & 2 \end{array}\right] \xRightarrow{R_3 \leftarrow R_3 + (-1/2)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 5/2 & 5/2 \end{array}\right]$$

$$\Downarrow R_2 \leftarrow (-1/2)R_2, \quad R_3 \leftarrow (2/5)R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xLeftarrow[R_1 \leftarrow R_1 + (-2)R_3]{R_2 \leftarrow R_2 + (1/2)R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xLeftarrow{R_1 \leftarrow R_1 + (-2)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Hence, the solution is $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$.

Row Echelon Form of a Matrix

Definition: A matrix U is said to be in the **echelon** (or **row echelon)** form if

1. The pivots (leading non-zero entries) are the first nonzero entries in their rows.
2. Below each pivot is a column of zeros
3. Each pivot lies to the right of the pivot in the row above.
4. All zero rows are at the bottom of the matrix.

Example: Let $U = \begin{bmatrix} 1 & 2 & 0 & 5 & 4 \\ 0 & 3 & 0 & 6 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$

U is in **row echelon** form, but B is **not** in **row echelon** form.
A column containing a pivot entry is also called a **pivot column**.

Algorithm - Gaussian elimination

Algorithm: This algorithm provides a method for using elementary row operations to take a matrix to its echelon form. We begin with the matrix in its original form.

1. Starting from the left, find the first non-zero column. This is the first pivot column, and the position at the top of this column will be the position of the first pivot entry. Switch rows if necessary to place a non-zero number in the first pivot position.

2. Use elementary row operations to make the entries below the first pivot entry (in the first pivot column) equal to zero.

3. Ignoring the row containing the first pivot entry, repeat steps 1 and 2 with the remaining rows.

Repeat the process until there are no more non-zero rows left.

Example 2.5 (Gaussian Method)

Find the row echelon form of the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 2 & 1 & -2 & 3 \\ 4 & -1 & -10 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 2 & 1 & -2 & 3 \\ 4 & -1 & -10 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{2} & 1 & -2 & 2 \\ 0 & 1 & 2 & 2 \\ 4 & -1 & -10 & 4 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + (-2)R_1} \begin{bmatrix} 2 & 1 & -2 & 3 \\ 0 & \boxed{1} & 2 & 2 \\ 0 & -3 & -6 & -2 \end{bmatrix}$$

$$\Downarrow R_3 \leftarrow R_3 + 3R_2$$

$$\begin{bmatrix} \boxed{2} & 1 & -2 & 3 \\ 0 & \boxed{1} & 2 & 2 \\ 0 & 0 & 0 & \boxed{4} \end{bmatrix}$$

This matrix is in echelon form and this is the echelon form of the given matrix A.
The numbers highlighted in red are the pivot entries. Columns corresponding to these pivots are called pivot columns.

Solving System of Linear equations

Example 2.6 - No solution:

$$\begin{aligned} y + 2z &= 2 \\ 2x + y - 2z &= 3 \\ 4x - y - 10z &= 4 \end{aligned} \Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & -2 \\ 4 & -1 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

$$A\,x = b.$$

$$\left[\begin{array}{ccccc} 0 & 1 & 2 & \vdots & 2 \\ 2 & 1 & -2 & \vdots & 3 \\ 4 & -1 & -10 & \vdots & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc} \boxed{2} & 1 & -2 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 2 \\ 4 & -1 & -10 & \vdots & 4 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + (-2)R_1} \left[\begin{array}{ccccc} 2 & 1 & -2 & \vdots & 3 \\ 0 & \boxed{1} & 2 & \vdots & 2 \\ 0 & -3 & -6 & \vdots & -2 \end{array}\right]$$

$$\Downarrow R_3 \leftarrow R_3(3)R_2$$

$$\left[\begin{array}{ccccc} \boxed{2} & 1 & -2 & \vdots & 3 \\ 0 & \boxed{1} & 2 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 4 \end{array}\right]$$

This matrix is in echelon form. The last row corresponds to the equation $0x + 0y + 0z = 4.$
This equation has no solution.
Therefore, there is no solution to the given system of equations. In other words, the system is inconsistent.

Solving System of Linear equations

Example 2.7 - Infinitely many solutions:

$$\begin{aligned} x + y + z &= 3 \\ 2x + 2y + 5z &= 9 \\ 4x + 4y + 10z &= 18 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 18 \end{bmatrix}.$$

$$A\,x = b.$$

$$\begin{bmatrix} \boxed{1} & 1 & 1 & \vdots & 3 \\ 2 & 2 & 5 & \vdots & 9 \\ 4 & 4 & 10 & \vdots & 18 \end{bmatrix} \xrightarrow[R_3 \leftarrow (-4)R_1]{R_2 \leftarrow (-2)R_1} \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 0 & \boxed{3} & \vdots & 3 \\ 0 & 0 & 6 & \vdots & 6 \end{bmatrix} \xrightarrow{R_3 \leftarrow (-2)R_2} \begin{bmatrix} \boxed{1} & 1 & 1 & \vdots & 3 \\ 0 & 0 & \boxed{3} & \vdots & 3 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

This matrix is now in echelon form. Observe that the first and third columns are pivot columns, and the second column is not. We call the corresponding variables x and z pivot variables, and the variable y is a free variable. The equations corresponding to this echelon form are

$$\begin{aligned} x + y + z &= 3 \\ 3z &= 3 \end{aligned} \Rightarrow \begin{aligned} x &= 2 - t \\ y &= t,\ t \in \mathbb{R} \\ z &= 1 \end{aligned}$$

The system has an infinite set of solutions which are given by these equations, where $t \in \mathbb{R}$ is called a parameter.

Row Reduced Echelon Form of a Matrix

Definition: A matrix R is said to be in the **row reduced echelon form** (or **reduced echelon form)** if

1. R is already in the row echelon form;
2. leading non-zero entry (pivot) in every non-zero row is 1;
3. each leading entry is the only non-zero entry in its column.

The row reduced echelon form (RREF) of a matrix is also called the **Gauss-Jordan** form.

Example: Let $R = \begin{bmatrix} 0 & \mathbf{1} & 5 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, B = \begin{bmatrix} \mathbf{1} & 5 & 3 & -1 \\ 0 & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$

R is in row reduced echelon form(RREF), but B is **not** in row reduced echelon form.

Uniqueness of the Reduced Echelon Form

Definition: Two matrices A and B of same size are said to be row equivalent if one can be obtained from the other by a finite number of elementary row operations.

Definition: The linear systems $Ax = b$ and $Cx = d$ are said to be row equivalent if their respective augmented matrices, $[A \vdots\ b]$ and $[C \vdots\ d]$, are row equivalent.

Theorem 2.1: Every matrix A is row equivalent to a unique matrix in **reduced echelon form.**

Example 2.8 (Gauss-Jordan Method)

Find the row reduced echelon form of the matrix $A = \begin{bmatrix} 0 & 0 & 2 & 6 & 0 \\ 1 & 2 & 1 & 3 & 0 \\ 2 & 4 & 3 & 9 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 0 & 2 & 6 & 0 \\ 1 & 2 & 1 & 3 & 0 \\ 2 & 4 & 3 & 9 & 1 \end{bmatrix} \xRightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \mathbf{1} & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 2 & 4 & 3 & 9 & 1 \end{bmatrix} \xRightarrow{R_3 \leftarrow (-2)R_1 + R_3} \begin{bmatrix} \mathbf{1} & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}$$

$$\Downarrow R_2 \leftarrow ({}^{1}/_{2})R_2$$

$$\begin{bmatrix} \mathbf{1} & 2 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xLeftarrow{R_1 \leftarrow (-1)R_2 + R_1} \begin{bmatrix} \mathbf{1} & 2 & 1 & 3 & 0 \\ 0 & 0 & \mathbf{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xLeftarrow{R_3 \leftarrow (-1)R_2 + R_3} \begin{bmatrix} \mathbf{1} & 2 & 1 & 3 & 0 \\ 0 & 0 & \mathbf{1} & 3 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}$$

Gauss-Jordan Elimination

Example 2.9: Solve the system of equations using Gauss-Jordan elimination.

$$x_1 + 2x_2 - 2x_3 + 2x_4 = 3$$
$$x_1 + 2x_2 - x_3 + 3x_4 = 5$$
$$x_1 + 2x_2 - 3x_3 + x_4 = 1.$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 & 2 \\ 1 & 2 & -1 & 3 \\ 1 & 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} \boxed{1} & 2 & -2 & 2 & 3 \\ 1 & 2 & -1 & 3 & 5 \\ 1 & 2 & -3 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} \boxed{1} & 2 & -2 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & -1 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} \boxed{1} & 2 & -2 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 and x_3 are pivot variables and x_2 and x_4 are free variables. We assign parameters $x_2 = s$ and $x_4 = t$ to the free variables. From the second and third equations, we have $x_3 = 2 - x_4 = 2 - t$, and $x_1 = 3 - 2s + 2(2 - t) - 2t = 7 - 2s - 4t$.

Therefore, the general solution is given by

$$x_1 = 7 - 2s - 4t$$
$$x_2 = s$$
$$x_3 = 2 - t$$
$$x_4 = t.$$

Gauss-Jordan Elimination

We reduce this to reduced echelon form by performing the following additional step:

$$\begin{bmatrix} \boxed{1} & 2 & -2 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_1 \leftarrow R_1 + 2\,R_2} \begin{bmatrix} \boxed{1} & 2 & 0 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first equation states that $x_1 = 3 - 2x_2 - 2x_4$, and the second equation states that $x_3 = 2 - x_4$.

Using the free variables x_2 and x_4 as parameters, we obtain the following general solution:

$x_1 = 7 - 2s - 4t$
$x_2 = s$
$x_3 = 2 - t$
$x_4 = t,$
where $s, t \in \mathbb{R}$ are parameters.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \end{bmatrix} t,$$

where $s, t \in \mathbb{R}$

Rank of a Matrix

Definition (Rank of a Matrix): The number of non-zero rows in the **row echelon** form of a matrix A is called the rank of the matrix and is denoted by rank (A).

$\mathrm{rank}(A) = number\ of$ non-zero rows in the row echelon form of a matrix A.

$= number\ of$ pivot entries in the row echelon form of a matrix A.

(The rank of a system of linear equations is the rank of its coefficient matrix.)

Example: Determine the row-rank of $\mathrm{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xRightarrow[R_3 \to (-1)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \xRightarrow{\mathrm{R}_3 \to (-1)\mathrm{R}_2 + R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the $\mathrm{rank}(A) = 2$.

Rank and Solutions of Consistent system of Linear Equations

Definition (Consistent and inconsistent system): A system of linear equations is called consistent if there exists at least one solution. It is called inconsistent if there is no solution.

Theorem 2.2: Consider a system of m equations in n variables, and assume that the coefficient matrix has rank r. Assume further that the system is consistent.

1. If $r = n$, then the system has a unique solution.
2. If $r < n$, then the system has infinitely many solutions, with $n - r$ parameters.

Homogeneous System of Equations

A system of equations is called homogeneous if each of the constant terms is equal to 0. A homogeneous system therefore has the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0,$$

where a_{ij} are coefficients and x_j are variables.

Remarks:

- A homogeneous system is always consistent. Indeed, it always has the solution $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. This solution is called the **trivial** solution.

- If the system has a solution in which not all of the $x_1, x_2, \ldots, x_n$ are equal to zero, then we call this solution **non-trivial**.

Solving Homogeneous System $Ax = 0$ Example 2.10

Find the general solution to the following homogeneous system of equations:

$$2x_1 + x_2 + x_3 + 4x_4 = 0$$
$$x_1 + 2x_2 - x_3 + 5x_4 = 0$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 4 & 0 \\ 1 & 2 & -1 & 5 & 0 \end{array}\right] \xRightarrow{R_1 \to (1/2)R_1} \left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 2 & 0 \\ 1 & 2 & -1 & 5 & 0 \end{array}\right]$$

$$\Downarrow R_2 \to (-1)R_1 + R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 2 & 0 \\ 0 & 1 & -1 & 2 & 0 \end{array}\right] \xLeftarrow{R_2 \to (2/3)R_2} \left[\begin{array}{cccc|c} 1 & 1/2 & 1/2 & 2 & 0 \\ 0 & 3/2 & -3/2 & 3 & 0 \end{array}\right]$$

$$\Downarrow R_1 \to (-1/2)R_2 + R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \end{array}\right]$$

This is in row reduced echelon form (RREF). Denote it by R.

Note: Solution set of $Ax = \mathbf{0}$ is equivalent to solution set of $Rx = \mathbf{0}$

Example 2.10 Continued

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{aligned} x_1 + \qquad x_3 + \ x_4 &= 0 \\ x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

$$\iff \begin{aligned} x_1 &= -\, x_3 - \ x_4 \\ x_2 &= \quad x_3 - 2\, x_4 \end{aligned}$$

$$\iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\, x_3 - \ x_4 \\ x_3 - 2\, x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, this system has infinitely many solutions, with two parameters s and t.

The general solution set $= \left\{ s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} : t, s \in \mathbb{R} \right\}.$

x_1, x_2 are called the Pivot or Basic variables, and x_2, x_4 are called the free variables.

Example 2.10 Continued

The general solution to the given system is $= \left\{ s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} : t, s \in \mathbb{R} \right\}$.

Solutions $x = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $x = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ are called **basic solutions** of the given homogeneous system. These basic solutions were obtained from the general solutions by choosing values $s = 1, \mathrm{t} = 0$ and $s = 0, t = 1$ respectively.

Solving Non-homogeneous System $Ax = b$
Example 2.11

Find the general solution to the following non-homogeneous system of equations:

$$\begin{aligned} 2x_1 + \ \ x_2 + x_3 + 4x_4 &= 2 \\ x_1 + 2x_2 - x_3 + 5x_4 &= 4 \end{aligned}$$

$$\left[\begin{array}{cccc:c} 2 & 1 & 1 & 4 & 2 \\ 1 & 2 & -1 & 5 & 4 \end{array}\right] \xRightarrow{\text{Elementary row operations}} \left[\begin{array}{cccc:c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 2 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Leftrightarrow \begin{aligned} x_1 + \qquad x_3 + x_4 &= 0 \\ x_2 - x_3 + 2x_4 &= 2 \end{aligned} \Leftrightarrow \begin{aligned} x_1 &= \quad - x_3 - x_4 \\ x_2 &= 2 + x_3 - 2x_4 \end{aligned}$$

$$\Leftrightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

Example 2.11 Continued

Hence, this non-homogeneous system has infinitely many solutions, with two parameters s and t.

The general solution set $= \left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} : t, s \in \mathbb{R} \right\}$.

$x = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a **particular solution** of the given non-homogeneous system $Ax = b$, and

$s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, where $t, s \in \mathbb{R}$ is the **general solution** of the corresponding homogeneous system $Ax = \mathbf{0}$.

The general solution of $Ax = b$ is equal to a particular solution of $Ax = b$ plus the general solution of $Ax = \mathbf{0}$.

Existence of Solution of Ax = b
Example 2.12

Consider a linear system $Ax = b$ which after the application of the Gauss-Jordan method reduces to a augmented matrix $[R\ d]$ with

$$[R \quad d] = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 & 0 & 2 & \vdots & 8 \\ 0 & 1 & 1 & 3 & 0 & 0 & 5 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & \vdots & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \vdots & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Observations:

1. The number of non-zero rows in R (also in $[R\ d]$) is 4.
2. The first non-zero entry in the non-zero rows appear in columns 1, 2, 5 and 6.
3. Thus, the respective variables x_1, x_2, x_5, and x_6 are the basic variables.
4. The remaining variables x_3, x_4, and x_7 are free variables.

5. We assign arbitrary constants (parameters) k_1, k_2, and k_3 to the free variables x_3, x_4, and x_7, respectively.

Example 2.12 Cont..

Hence, we have the set of solutions as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 8 - 2x_3 + x_4 - 2x_7 \\ 1 - x_3 - 3x_4 - 5x_7 \\ x_3 \\ x_4 \\ 2 + x_7 \\ 4 - x_7 \\ x_7 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 0 \\ 0 \\ 2 \\ 4 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ -5 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

Let $u_0 = \begin{bmatrix} 8 \\ 1 \\ 0 \\ 0 \\ 2 \\ 4 \\ 0 \end{bmatrix}$, $u_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $u_3 = \begin{bmatrix} -2 \\ -5 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Then it can be easily verified that

$\mathrm{R}u_0 = d$, and $Ru_i = 0$, for $1 \le i \le 3$.

Theorem 2.3

Theorem: Consider a linear system $Ax = b$, where A is a $m \times n$ matrix, and x, b are column vectors of orders n, and m, respectively. Suppose $rank\ (A) = r$ and $rank([A\ b]) = r'$. Then exactly one of the following statement holds:

1. if $r' = r < n$, the set of solutions of the linear system is an infinite set and has the form $\{u_0 + k_1u_1 + k_2u_2 + \cdots + k_{n-r}u_{n-r}\ | k_i \in \mathbb{R}, 1 \leq i \leq n-r\}$ where $u_0, u_1, u_2, \cdots, u_{n-r}$ are column vectors of order n satisfying $Au_0 = b$, and $Au_i = 0\ \ for\ \ 1 \leq i \leq n - r.$

2. if $r' = r = n$, the solution set of the linear system has a unique vector x_0 of order n satisfying $Ax_0 = b.$

3. if $r < r'$, the linear system has no solution.

Remark: The linear system Ax = b is consistent if and only if $rank\ (A) = rank([A\ \ b])$.