Q1 a) Verify that $\{|+\rangle, |-\rangle\}$ form an orthonormal basis for $\mathbb{C}^2$

Solution

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Suppose $\alpha|+\rangle + \beta|-\rangle = 0$, then $\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha+\beta\\ \alpha-\beta\end{pmatrix} = 0$

So we need $\alpha - \beta = 0 = \alpha + \beta \Rightarrow \alpha = \beta = 0$.

Therefore $\{|+\rangle, |-\rangle\}$ are linearly independent.

They are normalized as
$$\langle +|+\rangle = \tfrac{1}{2} + \tfrac{1}{2} = 1$$
$$\langle -|-\rangle = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

And orthogonal $\langle +|-\rangle = \frac{1}{2} - \frac{1}{2} = 0$.

b) Express $\{|0\rangle, |1\rangle\}$ in the basis $\{|+\rangle, |-\rangle\}$

Solution

$$|0\rangle = \tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{1}{\sqrt{2}}|-\rangle \qquad |1\rangle = \tfrac{1}{\sqrt{2}}|+\rangle - \tfrac{1}{\sqrt{2}}|-\rangle$$

c) Write down the unitary transformation mapping $|+\rangle \mapsto |0\rangle$, $|-\rangle \mapsto |1\rangle$

Solution

$$U = |0\rangle\langle +| + |1\rangle\langle -| = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Reverse mapping is the same as U is Hermitian so $U = U^+$

Q2 Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$ $\quad |\phi\rangle = \frac{i}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$. Compute

a) $\langle\psi|$, $\langle\phi|$

Sol^n

$$\langle\psi| = \tfrac{1}{\sqrt{2}}\left(\langle 00| + \langle 01|\right) = \tfrac{1}{\sqrt{2}}(1 \ 1 \ 0 \ 0)$$
$$\langle\phi| = \tfrac{1}{\sqrt{2}}\left(-i\langle 01| + \langle 10|\right) = \tfrac{1}{\sqrt{2}}(0 \ -i \ 1 \ 0)$$

b) $\langle\phi|\psi\rangle$

Sol^n

$$\langle\phi|\psi\rangle = \tfrac{1}{\sqrt{2}}\left(-i\langle 01| + \langle 10|\right)\cdot\tfrac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$$
$$= \tfrac{1}{2}\Big(-i\underbrace{\langle 01|00\rangle}_{0} - i\underbrace{\langle 01|01\rangle}_{1} + \underbrace{\langle 10|00\rangle}_{0} + \underbrace{\langle 10|01\rangle}_{0}\Big)$$
$$= \frac{-i}{2}$$

c) $|\psi\rangle\langle\phi|$, $|\psi\rangle\langle\psi|$

Sol^n

$$|\psi\rangle\langle\phi| = \tfrac{1}{2}\left(-i|00\rangle\langle 01| + |00\rangle\langle 10| - i|01\rangle\langle 01| + |01\rangle\langle 10|\right)$$
$$= \frac{1}{2}\begin{pmatrix}0 & -i & 1 & 0\\ 0 & -i & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \tfrac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 01| + |01\rangle\langle 00| + |01\rangle\langle 01|\right) = \frac{1}{2}\begin{pmatrix}1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Q3 a) Let $|\psi\rangle$ be a quantum state show that $P = |\psi\rangle\langle\psi|$ is a projector.

<u>Sol^n</u> $P^\dagger = (|\psi\rangle\langle\psi|)^\dagger = (\langle\psi|)^\dagger(|\psi\rangle)^\dagger = |\psi\rangle\langle\psi| = P$

Using $(AB)^\dagger = B^\dagger A^\dagger$

$$P^2 = |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{1}\langle\psi| = |\psi\rangle\langle\psi|$$

b) Let $\{|\psi_i\rangle\}_i$ be a collection of d orthogonal quantum states on $\mathbb{C}^d$. Show that $\{P_i\}_i$ with $P_i = |\psi_i\rangle\langle\psi_i|$ forms a valid measurement.

<u>Sol^n</u> Note a set of d orthogonal quantum states on $\mathbb{C}^d$ forms an orthonormal basis for $\mathbb{C}^d$. Thus any other vector $|\omega\rangle \in \mathbb{C}^d$ can be written as $|\omega\rangle = \sum_i \omega_i|\psi_i\rangle$.

By above we know P_i are all projectors. Let $M = \sum_i P_i$ then,

$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$$M|\omega\rangle = \sum_i |\psi_i\rangle\langle\psi_i| \sum_j \omega_j|\psi_j\rangle = \sum_{ij}\omega_j|\psi_i\rangle\langle\psi_i|\psi_j\rangle = \sum_{ij}\omega_j|\psi_i\rangle\delta_{ij}$$
$$= \sum_i \omega_i|\psi_i\rangle$$

Thus $M|\omega\rangle = |\omega\rangle$ for any $|\omega\rangle \in \mathbb{C}^d$ and so $M = \mathbb{1}$.

Q4 Let U_1, U_2 be unitary operators show U_1U_2, $U_1 \otimes U_2$,  are all unitary.

<u>Sol^n</u> Let $V = U_1U_2$, $V^\dagger = U_2^\dagger U_1^\dagger$

$$VV^\dagger = U_1U_2U_2^\dagger U_1^\dagger = U_1U_1^\dagger = \mathbb{1}$$
$$V^\dagger V = U_2^\dagger U_1^\dagger U_1U_2 = U_2^\dagger U_2 = \mathbb{1}$$

Let $V = U_1 \otimes U_2$ $V^\dagger = U_1^\dagger \otimes U_2^\dagger$

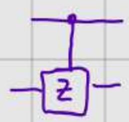
$$VV^\dagger = (U_1 \otimes U_2)(U_1^\dagger \otimes U_2^\dagger) = U_1U_1^\dagger \otimes U_2U_2^\dagger = \mathbb{1} \otimes \mathbb{1} = \mathbb{1}$$
$$V^\dagger V = (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) = U_1^\dagger U_1 \otimes U_2^\dagger U_2 = \mathbb{1} \otimes \mathbb{1} = \mathbb{1}$$

Let $V = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U_1$, $V^\dagger = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U_1^\dagger$

$$VV^\dagger = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U_1U_1^\dagger = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{1} = (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \mathbb{1}$$
$$= \mathbb{1} \otimes \mathbb{1} = \mathbb{1}.$$

Same for $V^\dagger V$...

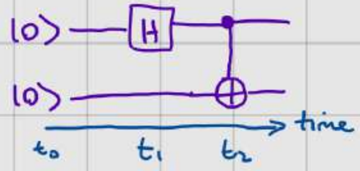
Q5 Let $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ show that

 = .

<u>Sol^n</u> Call the first gate U_1 and the second gate U_2. Then

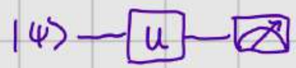
$$U_1 = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z = (|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|) + (|1\rangle\langle 1| \otimes |0\rangle\langle 0| - |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$
$$= (|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|) + (|0\rangle\langle 0| \otimes |1\rangle\langle 1| - |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$
$$= \mathbb{1} \otimes |0\rangle\langle 0| + Z \otimes |1\rangle\langle 1|$$
$$= U_2$$

Q6 Compute the final state of this circuit



Sol^n At time t_0 state is $|00\rangle$
At time t_1 state is $|+0\rangle$
At time t_2 state is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Q7 Suppose we want to measure a qubit $|\psi\rangle$ in a basis $\{|v_0\rangle, |v_1\rangle\}$. Find a unitary U such that the circuit



implements this measurement, where (meter) denotes a measurement in the computational basis.

Sol^n For arbitrary U the circuit produces results 0/1 with probabilities

$$P(0) = \langle\psi|U^\dagger |0\rangle\langle 0| U|\psi\rangle$$
$$P(1) = \langle\psi|U^\dagger |1\rangle\langle 1| U|\psi\rangle$$

We want that
$$P(0) = \langle\psi| |v_0\rangle\langle v_0| |\psi\rangle$$
$$P(1) = \langle\psi| |v_1\rangle\langle v_1| |\psi\rangle$$

Choose unitary $U = |0\rangle\langle v_0| + |1\rangle\langle v_1|$ which maps from $\{|v_0\rangle, |v_1\rangle\}$ basis to the $\{|0\rangle, |1\rangle\}$ basis. Then

$$P(0) = \langle\psi| U^\dagger |0\rangle\langle 0| U|\psi\rangle = \langle\psi| |v_0\rangle\langle v_0| |\psi\rangle$$

and similarly for $P(1)$.

Q8 Given an input qubit $|0\rangle$ design a circuit to simulate a biased coin.

Sol^n

Where $U_p = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ \sqrt{1-p} & -\sqrt{p} \end{pmatrix}$

Then $U_p|0\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$ which has a distribution $(p, 1-p)$

Q9 Consider the state $|+\rangle$. First measure it in the computational basis, compute the post measurement states then measure it in the Hadamard basis. Does the order of the measurements affect the statistics?

Sol^n Measurement $|0\rangle/|1\rangle$

Outcome	Prob	Post-measurement state
0	$\frac{1}{2}$	$|0\rangle$
1	$\frac{1}{2}$	$|1\rangle$

Case 1: Measurement 1 gave 0

Then $|+\rangle/|-\rangle$ measurement gives

Outcome	Prob	PMS
+	$\frac{1}{2}$	$\|+\rangle$
−	$\frac{1}{2}$	$\|-\rangle$

Case 2: Measurement 1 gave 1

Then $|+\rangle/|-\rangle$ measurement gives

Outcome	Prob	PMS
+	$\frac{1}{2}$	$\|+\rangle$
−	$\frac{1}{2}$	$\|-\rangle$

Overall, all measurements resulted in a uniform distribution.

If we change the order of the measurements then measuring $|+\rangle/|-\rangle$ on $|+\rangle$ will give outcome + with prob 1. So does affect the results.

Q10 Consider state $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ measure qubit 1 in the $|0\rangle/|1\rangle$ basis compute post measurement state and then measure qubit 2 in the $|+\rangle/|-\rangle$ basis. Does the order of measurements matter here?

Sol^n

Qubit 1 measurement is defined by projectors $P_0 = |0\rangle\langle 0| \otimes \mathbb{1}$ $P_1 = |1\rangle\langle 1| \otimes \mathbb{1}$.

we get

Outcome	Prob	PMS
0	$\frac{1}{2}$	$\|00\rangle$
1	$\frac{1}{2}$	$\|11\rangle$

Qubit 2 measurement is defined by projectors $Q_+ = \mathbb{1} \otimes |+\rangle\langle +|$, $Q_- = \mathbb{1} \otimes |-\rangle\langle -|$.

This measurement will produce $(\frac{1}{2},\frac{1}{2})$ distribution for both of the post-measurement states from Measurement 1.

The order of measurements does not matter in this case because the projectors from measurement 1 commute with the projectors from measurement 2. That is $P_i Q_j = Q_j P_i$ for $i \in \{0,1\}$, $j \in \{+,-\}$.

$$\begin{aligned}\mathbb{P}(M_1=i, M_2=j \mid \text{Qubit 1 measured first}) &= \langle\psi|P_i Q_j P_i|\psi\rangle \\ &= \langle\psi|Q_j P_i Q_j|\psi\rangle \\ &= \mathbb{P}(M_1=i, M_2=j \mid \text{Qubit 2 measured first})\end{aligned}$$

In general if measurements commute we can define a joint measurement $R_{ij} = P_i Q_j$.

Q11 Prove that global phase is not observable.

Sol^n Let $|\psi\rangle = e^{it}|\phi\rangle$. The measurement results of a circuit starting in a state $|\Psi\rangle$ can be described by $\langle\Psi| U^\dagger P_i U |\Psi\rangle$ for some unitary U and measurement $\{P_i\}$. For the state $|\phi\rangle$ we get

$$\mathbb{P}(i \mid |\phi\rangle) = \langle\phi| U^\dagger P_i U |\phi\rangle$$

and for the state $|\psi\rangle = e^{it}|\phi\rangle$ we get

$$\begin{aligned}\mathbb{P}(i \mid |\psi\rangle) &= \langle\psi| U^\dagger P_i U |\psi\rangle \\ &= \langle\phi| e^{-it} U^\dagger P_i U e^{it} |\phi\rangle \\ &= e^{-it} e^{it} \langle\phi| U^\dagger P_i U |\phi\rangle \\ &= \mathbb{P}(i \mid |\phi\rangle)\end{aligned}$$

And so no circuit can distinguish these two states.

Q12 Find the best measurement in the Z-X plane of the Bloch-sphere that distinguishes $|0\rangle$ from $|+\rangle$. I.e. Maximize the success probability

$$\frac{1}{2}\Big(\mathbb{P}(0 \mid |0\rangle) + \mathbb{P}(1 \mid |+\rangle)\Big).$$

Sol^n Such a measurement can be written as

$$M_0 = \frac{\mathbb{1} + \cos(t) Z + \sin(t) X}{2} \qquad \text{for } \theta \in [0, 2\pi)$$

$$M_1 = \mathbb{1} - M_0$$

$$M_0 = \begin{pmatrix} \frac{1+\cos(t)}{2} & \frac{\sin(t)}{2} \\ \frac{\sin(t)}{2} & \frac{1-\cos(t)}{2} \end{pmatrix} \qquad M_1 = \begin{pmatrix} \frac{1-\cos(t)}{2} & -\frac{\sin(t)}{2} \\ -\frac{\sin(t)}{2} & \frac{1+\cos(t)}{2} \end{pmatrix}$$

Then
$$\mathbb{P}(0 \mid |0\rangle) = \frac{1+\cos(t)}{2}$$
$$\mathbb{P}(1 \mid |+\rangle) = \frac{1-\sin(t)}{2}$$

Want to maximize $\frac{1}{2} + \frac{\cos(t) - \sin(t)}{4}$

$$f(t) = \cos(t) - \sin(t)$$
$$f'(t) = -\sin(t) - \cos(t)$$
$$f''(t) = -\cos(t) + \sin(t)$$

Assume $\cos(t) \neq 0$

$f'(t) = 0 \Rightarrow -\cos(t) = \sin(t) \Rightarrow t = \operatorname{atan}(-1)$

$t = \frac{3\pi}{4}$ or $t = \frac{7\pi}{4}$

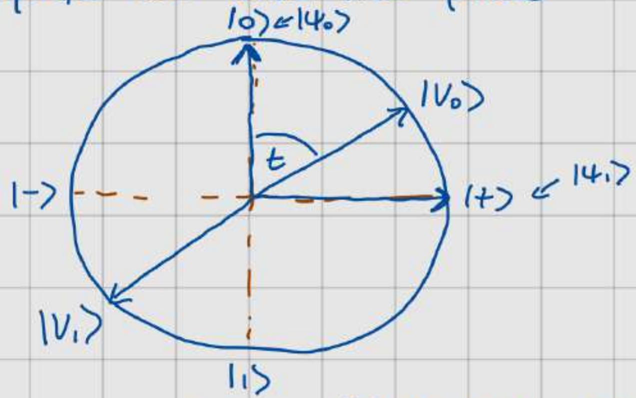
$f''\left(\frac{3\pi}{4}\right) = \sqrt{2} > 0$

$f''\left(\frac{7\pi}{4}\right) = -\sqrt{2} < 0 \quad \leftarrow$ local maximum

Max achieved for $t = \frac{7\pi}{4}$

Success probability is $\frac{1}{2} + \frac{\sqrt{2}}{4}$

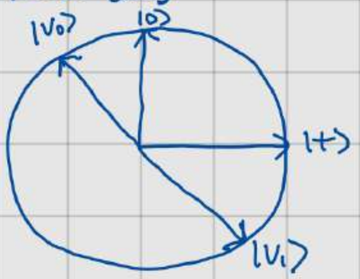
On the Bloch sphere we have the plane



Measurements M_0, M_1 are projectors $M_0 = |v_0\rangle\langle v_0|$, $M_1 = |v_1\rangle\langle v_1|$

$|v_0\rangle$ is an angle t from the Z-axis and $|v_1\rangle$ occurs at the opposite point.

The optimal measurement found was



This measurement keeps $|v_0\rangle$ far from $|+\rangle$ but close to $|0\rangle$ and $|v_1\rangle$ far from $|0\rangle$ but close to $|+\rangle$.

Q3 Construct a gate that swaps two qubits. I.e $U|\phi\rangle|\psi\rangle = |\psi\rangle|\phi\rangle$.

Sol^n Consider the computational basis. We want the following action

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |10\rangle \\ |10\rangle &\mapsto |01\rangle \\ |11\rangle &\mapsto |11\rangle \end{aligned}$$

Thus

$$U = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$

then $|\psi\rangle|\phi\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$

$$\begin{aligned} U|\psi\rangle|\phi\rangle &= \alpha\gamma|00\rangle + \alpha\delta|10\rangle + \beta\gamma|01\rangle + \beta\delta|11\rangle \\ &= (\gamma|0\rangle + \delta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\ &= |\phi\rangle|\psi\rangle. \end{aligned}$$