ACCQ206 exercises – Week 1

Peter Brown

February 18, 2022

1. (a) Verify that $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ form an orthonormal basis for $\mathbb{C}^2$.

   (b) Express the basis $\{|0\rangle, |1\rangle\}$ in terms of $\{|+\rangle, |-\rangle\}$.

   (c) Write down the unitary transformation that maps $|+\rangle \mapsto |0\rangle$ and $|-\rangle \mapsto |1\rangle$. What about the reverse mapping?

2. Let $|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle \otimes (|0\rangle + |1\rangle)$ and $|\phi\rangle = \frac{i}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle$. Compute:

   (a) $\langle\psi|, \quad \langle\phi|$

   (b) $\langle\phi|\psi\rangle$

   (c) $|\psi\rangle\langle\phi|, \quad |\psi\rangle\langle\psi|$

3. (a) Let $|\psi\rangle$ be a quantum state, show that

$$P = |\psi\rangle\langle\psi|$$

   is a projector, i.e., Hermitian and idempotent ($P^2 = P$).

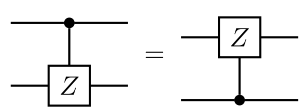
   (b) Let $\{|\psi_i\rangle\}_{i=1}^d$ be a collection of d orthogonal quantum states on $\mathbb{C}^d$, Show that $\{P_i\}_{i=1}^d$ where $P_i = |\psi_i\rangle\langle\psi_i|$ forms a valid quantum measurement on $\mathbb{C}^d$.

4. (a) Let U_1, U_2 be unitary operators. Show that

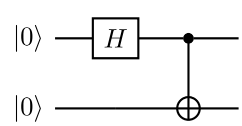
$$U_1 U_2, \qquad U_1 \otimes U_2, \qquad c(U_1),$$

   where $c(U_1)$ denotes a controlled U_1, are all unitary operators.

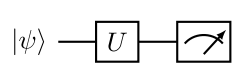
5. Let $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ show that

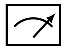


6. Compute the end state of this circuit



7. Suppose we want to measure a qubit $|\psi\rangle$ in a basis $\{|v_0\rangle, |v_1\rangle\}$. Find the unitary U such that the circuit



implements this, where  denotes measurement in the computational basis. I.e., the probability of obtaining outcome k is

$$P(k) = \langle \psi \,|\, \phi_k \rangle\langle \phi_k \,|\, \psi \rangle \,.$$

8. Given an input qubit $|0\rangle$ design a single qubit circuit to simulate a biased coin. I.e., when the final state is measured in the computational basis it should output 0 with probability p and 1 with probability $1 - p$ for $p \in [0, 1]$.

9. Consider the quantum state $|+\rangle$. First measure it in the computational basis, compute the post-measurement states and then measure them in the Hadamard basis ($\{|+\rangle, |-\rangle\}$). Does the order in which we perform the measurements affect the joint probability distribution of outcomes. That is, does the distribution change if we were to instead first measure in the Hadamard basis and then in the computational basis?

10. Consider the two-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Measure qubit 1 in the computational basis, compute the post-measurement states and then measure qubit 2 in the Hadamard basis. Does the order of the measurements affect the joint distribution in this case. Why does / doesn't this differ from the answer to question 9?

11. Prove that global phase it not observable. I.e., if $|\psi\rangle = e^{it} |\phi\rangle$ then they give the same statistics for all quantum circuits.

12. Find the best projective measurement from the $Z - X$ plane of the bloch sphere that distinguishes $|0\rangle$ from $|+\rangle$. I.e., find a measurement in the family

$$M_0 = \frac{I + \cos(\theta)Z + \sin(\theta)X}{2} \qquad M_1 = I - M_0$$

that maximizes the probability of success $\frac{1}{2}(P(0|\,|0\rangle) + P(1|\,|+\rangle))$. Try to interpret this geometrically on the Bloch sphere.

13. Construct the quantum gate U that swaps 2 qubits. I.e., for qubit states $|\psi\rangle$ and $|\phi\rangle$ we have

$$U\,|\psi\rangle\,|\phi\rangle = |\phi\rangle\,|\psi\rangle \ .$$