ACCQ206 exercises – Week 2

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1. Which states from the following family are entangled?

$$|\psi(\theta)\rangle := \cos(\theta)\,|00\rangle + \sin(\theta)\,|11\rangle \qquad 0 \leq \theta \leq \pi/4$$

2. Let $\{|\psi_i\rangle\}_i$ be and set of states and let $p(i)$ be any probability distribution. Prove that for $\rho = \sum_i p(i)\,|\psi_i\rangle\langle\psi_i|$ we have

   (a) $\mathrm{Tr}\,[\rho] = 1$

   (b) $\rho \geq 0$

   Also show that a density operator ρ is pure iff $\mathrm{Tr}\left[\rho^2\right] = 1$.

3. (a) Prove that any single qubit state ρ (density matrix) can be written as

$$\rho = \frac{I + n_x X + n_y Y + n_z Z}{2}$$

   with $n_x, n_y, n_z \in \mathbb{R}$ and $n_x^2 + n_y^2 + n_z^2 \leq 1$.

   (b) Show that a state is pure iff $n_x^2 + n_y^2 + n_z^2 = 1$.

4. Prove that for the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ we have

$$|\psi\rangle = (U \otimes U)\,|\psi\rangle$$

   for any single qubit unitary U.

5. (a) Prove that the trace (defined as $\mathrm{Tr}\,[X] = \sum_i \langle i|\,X\,|i\rangle$ with $\{|i\rangle\}$ being the computational basis) is cyclic, i.e.,

$$\mathrm{Tr}\,[XY] = \mathrm{Tr}\,[YX]\,.$$

   (b) Use this to prove that the trace is basis independent. I.e., $\mathrm{Tr}\,[X] = \sum_i \langle v_i|\,X\,|v_i\rangle$ for any orthonormal basis $\{|v_i\rangle\}_i$.

6. (Requires bonus material POVM – see notes)

   We know that we can't distinguish states perfectly if they are not orthogonal, e.g., $|0\rangle$ and $|+\rangle$. But can we make it so we never make a mistake? Define a three outcome POVM $\{M_0, M_+, M_{fail}\}$ that allows you to distinguish $|0\rangle$ from $|+\rangle$ sometimes but crucially you never make a misidentification. That is

$$\langle 0|\,M_0\,|0\rangle > 0, \qquad \langle +|\,M_+\,|+\rangle > 0$$

   and

$$\langle 0|\,M_+\,|0\rangle = 0 \qquad \langle +|\,M_0\,|+\rangle = 0.$$

7. Let $|\psi\rangle$ be a state on a bipartite system AB. Show that $|\psi\rangle$ is a product state iff ρ_A and ρ_B are pure states. (Hint: use the Schmidt decomposition – see the bonus material in the notes.)

8. (Proving the Tsirelson bound – more difficult)

Alice and Bob play the CHSH game. For convenience we label the inputs as $x, y \in \{0, 1\}$ and the outputs as $a, b \in \{+1, -1\}$. The winning condition then becomes $(-1)^x y = ab$. Let $|\psi\rangle$ be the quantum state shared by Alice and Bob. Let Alice's projective measurement on input x be $\{A_{+1|x}, A_{-1|x}\}$ and let Bob's projective measurement on input y be $\{B_{+1|y}, B_{-1|y}\}$. Finally define the observables (expectation operators) $A_x = A_{+1|x} - A_{-1|x}$ and $B_y = B_{+1|y} - B_{-1|y}$.

(a) Show that for any fixed (x, y) the expected value of the product ab is given by

$$\langle\psi| A_x \otimes B_y |\psi\rangle .$$

(b) Let $K = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$. Show that the expected winning probability is

$$\frac{1}{2} + \frac{1}{8} \langle\psi| K |\psi\rangle .$$

(c) Show that

$$K^2 = 4I + [A_0, A_1] \otimes [B_0, B_1]$$

where $[X, Y] = XY - YX$.

(d) Show that $\langle\psi| K |\psi\rangle \leq 2\sqrt{2}$. What does this say about the maximum winning probability for the CHSH game? (Hint: begin with the Cauchy-Schwarz inequality to bound $\langle\psi| K |\psi\rangle$ in terms of $\langle\psi| K^2 |\psi\rangle$).