Q1: Which of the following states are entangled?

$$|\psi(\theta)\rangle := \cos(\theta)|00\rangle + \sin(\theta)|11\rangle \qquad 0 \leq \theta \leq \pi/4$$

Sol^n

For $\theta = 0$, $|\psi(0)\rangle = |00\rangle = |0\rangle \otimes |0\rangle$ so not entangled.

Otherwise, let $|v\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|w\rangle = \gamma|0\rangle + \delta|1\rangle$.

Then $|v\rangle \otimes |w\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$.

Thus we need

$$\begin{aligned} \alpha\gamma &= \cos(\theta) \\ \alpha\delta &= 0 \\ \beta\gamma &= 0 \\ \beta\delta &= \sin(\theta) \end{aligned}$$

if the state is not entangled. But $\alpha\delta = 0 \Rightarrow \alpha = 0$ or $\delta = 0$. If $\alpha = 0$ we need $\cos(\theta) = 0$ and if $\delta = 0$ we need $\sin(\theta) = 0$. But this is not possible for $0 < \theta \leq \pi/4$. Thus $|\psi(\theta)\rangle$ is entangled for all $\theta \in (0, \pi/4]$ ☒

Q2 Let $\{|\psi_i\rangle\}$ be a set of states and p_i be a probability distribution. Prove that for $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ we have

a) $Tr(\rho) = 1$

b) $\rho \geq 0$.

Also show ρ is pure iff $Tr(\rho^2) = 1$.

Sol^n

linearity; Compute in ONB containing $|\psi_i\rangle$.

(a) $Tr(\rho) = \sum_i p_i \, Tr[|\psi_i\rangle\langle\psi_i|] = \sum_i p_i = 1$

Probability distribution

(b) We need ρ is Hermitian and $\langle x|\rho|x\rangle \geq 0 \;\; \forall \, |x\rangle \in \mathcal{H}$.

Hermitian is clear $\rho^\dagger = \sum_i p_i (|\psi_i\rangle\langle\psi_i|)^\dagger = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho$.

And

$$\langle x|\rho|x\rangle = \sum_i p_i \langle x|\psi_i\rangle\langle\psi_i|x\rangle = \sum_i p_i |\langle x|\psi_i\rangle|^2 \geq 0.$$

For purity recall ρ is pure when $\rho = |\psi\rangle\langle\psi|$ for some state $|\psi\rangle$. Then $\rho^2 = |\psi\rangle\langle\psi|$ thus $Tr(\rho^2) = 1$ when ρ is pure. Now suppose $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ and $Tr(\rho^2) = 1$.
By the spectral theorem we can assume that $\{|\psi_i\rangle\}_i$ form an orthonormal basis. Then

$$\rho^2 = \sum_{i,j} p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|$$
$$= \sum_i p_i^2 |\psi_i\rangle\langle\psi_i|.$$

So $Tr(\rho^2) = \sum_i p_i^2$. But we know $\sum_i p_i = 1$ and so $\sum_i p_i^2 = 1$ only when $p_i = 1$ for some i and $p_j = 0 \ \forall j \neq i$. In that case $\rho = |\psi_i\rangle\langle\psi_i|$ and is pure. ☒

Q3 a) Prove that any qubit state ρ can be written as

$$\rho = \frac{\mathbb{1} + n_x X + n_y Y + n_z Z}{2}$$

with $n_x, n_y, n_z \in \mathbb{R}$ and $n_x^2 + n_y^2 + n_z^2 \leq 1$.

Sol^n

Can show that $\{\mathbb{1}, X, Y, Z\}$ form an orthogonal basis with respect to the inner-product $\langle R, S\rangle = Tr[R^\dagger S]$ for the Hilbert space of 2x2 matrices with elements in $\mathbb{C}$. Thus we can always write

$$\rho = \frac{1}{2}\left(n_0 \mathbb{1} + n_x X + n_y Y + n_z Z\right)$$

for some $n_0, n_x, n_y, n_z \in \mathbb{C}$.

Now we need $Tr[\rho] = 1$, as $Tr[X] = Tr(Y) = Tr(Z) = 0$
$\Rightarrow$ that $n_0 = 1$.

Secondly we need ρ to be Hermitian (as it is positive semidefinite), thus $\rho = \rho^\dagger$. This implies (noting X, Y, Z are all Hermitian)

$$\bar{n}_x X + \bar{n}_y Y + \bar{n}_z Z = n_x X + n_y Y + n_z Z.$$

where $\bar{\alpha}$ denote the complex conjugate of α.
As $\mathbb{1}, X, Y, Z$ are an orthogonal basis this implies that

$$\bar{n}_i = n_i \Rightarrow n_x, n_y, n_z \in \mathbb{R}$$

Thus we arrive at $\rho = \frac{1}{2}\begin{pmatrix} 1+n_z & n_x - i n_y \\ n_x + i n_y & 1 - n_z \end{pmatrix}$

Finally for $\rho \geq 0$ we require the eigenvalues of ρ to be non-negative. We find the eigenvalues of the above matrix to be

$$\left\{ \frac{1}{2}\left(1 \pm \sqrt{n_x^2 + n_y^2 + n_z^2} \right) \right\}$$

Thus we need $n_x^2 + n_y^2 + n_z^2 \leq 1$

b) Show ρ is pure iff $n_x^2 + n_y^2 + n_z^2 = 1$.

Sol^n Note that ρ is pure iff $Tr(\rho^2) = 1$.

We have $Tr(\rho^2) = \frac{1}{2}(1 + n_x^2 + n_y^2 + n_z^2)$

Thus $Tr(\rho^2) = 1 \iff n_x^2 + n_y^2 + n_z^2 = 1$ $\boxtimes$

Q4) Prove that for the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ we have for any single qubit unitary U,

$$|\psi\rangle = (U \otimes U)|\psi\rangle.$$

Proof

Write the Unitary as $U = |w\rangle\langle v| + |w^\perp\rangle\langle v^\perp|$ for two orthonormal bases $\{|v\rangle, |v^\perp\rangle\}$ and $\{|w\rangle, |w^\perp\rangle\}$. You can verify $\left(\text{letting } \begin{array}{l} |v\rangle = v_{00}|0\rangle + v_{01}|1\rangle \\ |v^\perp\rangle = v_{10}|0\rangle + v_{11}|1\rangle \end{array}\right)$

$$(\langle v| \otimes \langle v|)|\psi\rangle = 0$$
$$(\langle v^\perp| \otimes \langle v^\perp|)|\psi\rangle = 0$$
$$(\langle v| \otimes \langle v^\perp|) = \frac{1}{\sqrt{2}}(\overline{v_{00}}\,\overline{v_{11}} - \overline{v_{01}}\,\overline{v_{10}}) =: K$$
$$(\langle v^\perp| \otimes \langle v|) = -K$$

Thus $(U \otimes U)|\psi\rangle = K(|w\rangle \otimes |w^\perp\rangle - |w^\perp\rangle \otimes |w\rangle)$ for normalization we must have $K = \frac{1}{\sqrt{2}}$.

So $(U \otimes U)|\psi\rangle = \frac{1}{\sqrt{2}}(|w\rangle \otimes |w^\perp\rangle - |w^\perp\rangle \otimes |w\rangle)$

Now write $|w\rangle = w_{00}|0\rangle + w_{01}|1\rangle$ and $|w^\perp\rangle = w_{10}|0\rangle + w_{11}|1\rangle$.

Then $(U\otimes U)|\psi\rangle = \frac{1}{\sqrt{2}}(w_{00}w_{11} - w_{01}w_{10})\,(|01\rangle - |10\rangle)$.

Again by normalization (which is preserved by unitaries) we must have

$$\frac{1}{\sqrt{2}}(w_{00}w_{11} - w_{01}w_{10}) = \frac{1}{\sqrt{2}} \implies (U\otimes U)|\psi\rangle = |\psi\rangle \qquad \boxtimes$$

Q5 a) Prove the trace is cyclic, i.e. $Tr(XY) = Tr(YX)$.

Sol^n

Write $X = \sum_{ij} x_{ij}|i\rangle\langle j|$ and $Y = \sum_{ij} y_{ij}|i\rangle\langle j|$.

Then $XY = \sum_{ijk} x_{ij}y_{jk}|i\rangle\langle k|$

$YX = \sum_{ijk} y_{ij}x_{jk}|i\rangle\langle k|$

Thus $Tr[XY] = \sum_{ij} x_{ij}y_{ji} = \sum_{ij} y_{ij}x_{ji} = Tr(YX)$.

b) Use this to prove trace is Basis independent i.e.,

$Tr(X) = \sum_i \langle v_i|X|v_i\rangle$ for any orthonormal basis $\{|v_i\rangle\}_i$.

Sol^n

Let $U = \sum_i |v_i\rangle\langle i|$, then U is unitary and we have

$$\begin{aligned}
Tr[X] = Tr(UU^\dagger X) = Tr(U^\dagger X U) &= \sum_i \langle i|U^\dagger X U|i\rangle \\
&= \sum_i \langle i|\Big(\sum_j |j\rangle\langle v_j|\Big) X \Big(\sum_k |v_k\rangle\langle k|\Big)|i\rangle \\
&= \sum_i \langle i|i\rangle\langle v_i|X|v_i\rangle\langle i|i\rangle \\
&= \sum_i \langle v_i|X|v_i\rangle \qquad \boxtimes
\end{aligned}$$

6) Define a 3 outcome POVM $\{M_0, M_+, M_{fail}\}$ that allows you to distinguish $|0\rangle$ from $|+\rangle$ sometimes but never misidentifies, i.e.,

$$\langle 0|M_0|0\rangle > 0, \qquad \langle +|M_+|+\rangle > 0$$

and

$$\langle 0|M_+|0\rangle = 0 \qquad \langle +|M_0|+\rangle = 0$$

Sol^n

We need $\langle 0|M_+|0\rangle = 0$ so we take $M_+ = \alpha|1\rangle\langle 1|$ for some $\alpha > 0$. Similarly we set $M_0 = \beta|-\rangle\langle -|$.

Then take $M_{fail} = \mathbb{1} - M_0 - M_+$.

We clearly have $M_0, M_+ \geq 0$ we just need to check that $M_{fail} \geq 0$.

$$M_{fail} = \begin{pmatrix} 1-\beta/2 & \beta/2 \\ \beta/2 & 1-\alpha-\beta/2 \end{pmatrix}$$

Computing the eigenvalues we find $\frac{1}{2}\left(2-\alpha-\beta \pm \sqrt{\alpha^2+\beta^2}\right)$

thus we need α, β such that

$\alpha > 0$, $\beta > 0$ and $\frac{1}{2}\left(2-\alpha-\beta \pm \sqrt{\alpha^2+\beta^2}\right) \geq 0$

For instance we can choose $\alpha = \beta = \frac{1}{2}$.

Then the probability we succesfully distinguish (assuming $|0\rangle$ and $|+\rangle$ sent with uniform probability) is

$$\frac{1}{2}\langle 0| \frac{1}{2}|-\rangle\langle -| |0\rangle + \frac{1}{2}\langle +| \frac{1}{2}|1\rangle\langle 1| |+\rangle = \frac{1}{4}$$

Bonus Question: Whats the optimal probability?

7) Show that a bipartite state $|\psi\rangle_{AB}$ is product iff ρ_A and ρ_B are pure.

Sol^n $(\Rightarrow)$

If $|\psi\rangle = |v\rangle \otimes |w\rangle$ then $\rho_A = Tr_B\left[|v\rangle\langle v| \otimes |w\rangle\langle w|\right] = |v\rangle\langle v|$ which is pure.

Similarly $\rho_B = |w\rangle\langle w|$ is pure.

$(\Leftarrow)$ By the schmidt-decomposition $\exists$ an orthonormal basis $\{|v_i\rangle\}_i$ of A and $\{|w_i\rangle\}_i$ of B and nonnegative coefficients $\lambda_i \geq 0$ such that

$$|\psi\rangle = \sum_i \lambda_i |v_i\rangle|w_i\rangle$$

Thus $\rho_A = \sum_i \lambda_i |v_i\rangle\langle v_i|$ and $\rho_B = \sum_i \lambda_i |w_i\rangle\langle w_i|$.

If ρ_A and ρ_B are pure $\Rightarrow$ that only one λ_i is non-zero.

Thus $\rho_A = |v_i\rangle\langle v_i|$ for some i and $\rho_B = |w_i\rangle\langle w_i|$

$\Rightarrow \quad |\psi\rangle = |v_i\rangle \otimes |w_i\rangle$ is a product state.

8) (Deriving the Tsirelson bound)

Alice and Bob play the CHSH game. For convenience we let the inputs $x, y \in \{0,1\}$ and the outputs $a, b \in \{+1, -1\}$. The winning condition then becomes

$$(-1)^{xy} = ab$$

Let Alice's projective measurement on input x be $\{A_{1|x}, A_{-1|x}\}$ and Bob's projective measurement on input y be $\{B_{1|y}, B_{-1|y}\}$. Let the quantum state shared between Alice and Bob be $|\psi\rangle$

Define observables
$$A_x = A_{1|x} - A_{-1|x}$$
$$B_y = B_{1|y} - B_{-1|y}$$

a) Show that for any fixed xy the expected value of ab is given by

$$\langle\psi| A_x \otimes B_y |\psi\rangle$$

<u>Sol^n</u>

$$\begin{aligned}
\langle\psi| A_x \otimes B_y |\psi\rangle &= \langle\psi| (A_{1|x} - A_{-1|x}) \otimes (B_{1|y} - B_{-1|y}) |\psi\rangle \\
&= \langle\psi| A_{1|x} \otimes B_{1|y} |\psi\rangle - \langle\psi| A_{1|x} \otimes B_{1|y} |\psi\rangle \\
&- \langle\psi| A_{-1|x} \otimes B_{1|y} |\psi\rangle + \langle\psi| A_{-1|x} \otimes B_{-1|y} |\psi\rangle \\
&= p(a=1, b=1 | x, y) - p(a=1, b=-1 | x, y) \\
&- p(a=-1, b=1 | x, y) + p(a=-1, b=-1 | x, y) \\
&= \sum_{ab} ab\, p(ab|xy) \\
&= \mathbb{E}[ab | X=x, Y=y]
\end{aligned}$$

b) Let $K = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$. Show that Alice and Bob's winning probability is

$$\frac{1}{2} + \frac{1}{8} \langle\psi| K |\psi\rangle .$$

$\underline{\text{Sol}^n}$ $\langle\psi|A_x\otimes B_y|\psi\rangle = p(11|xy) + p(-1-1|xy) - p(1,-1|xy) - p(-1,1|xy)$

As $p(1,-1|xy) = 1 - p(11|xy) - p(-1-1|xy) - p(-1,1|xy)$ we get

$$\langle\psi|A_x\otimes B_y|\psi\rangle = 2\big(p(11|xy) + p(-1-1|xy)\big) - 1$$

Performing the same argument with $p(11|xy)$ we also arrive at

$$\langle\psi|A_x\otimes B_y|\psi\rangle = 1 - 2\big(p(1,-1|xy) + p(-1,1|xy)\big)$$

Thus $\langle\psi|K|\psi\rangle = 2\big(p(11|00) + p(-1-1|00) + p(11|01) + p(-1-1|01)$
$+ p(11|10) + p(-1-1|10) + p(1,-1|11) + p(-1,1|11)\big)$
$- 4$

$$= 2\big(4\cdot \mathbb{P}(\text{Alice and Bob win})\big) - 4$$

$$\Rightarrow \mathbb{P}(\text{Alice and Bob win}) = \frac{1}{2} + \frac{1}{8}\langle\psi|K|\psi\rangle$$

c) Show $K^2 = 4\mathbb{1} + [A_0,A_1]\otimes[B_0,B_1]$ where $[X,Y] = XY - YX$.

$\underline{\text{Sol}^n}$

$A_{1|x} + A_{-1|x} = \mathbb{1}$

First note that $A_x = A_{1|x} - A_{-1|x} = 2A_{1|x} - \mathbb{1}$

So $A_x^2 = 4A_{1|x} - 4A_{1|x} + \mathbb{1} = \mathbb{1}$.

Similarly $B_y^2 = \mathbb{1}$.

Now $K = A_0\otimes(B_0+B_1) + A_1\otimes(B_0-B_1)$

So $K^2 = A_0^2\otimes(B_0+B_1)^2 + A_0A_1\otimes(B_0+B_1)(B_0-B_1)$
$+ A_1A_0\otimes(B_0-B_1)(B_0+B_1) + A_1^2\otimes(B_0-B_1)^2$

$= \mathbb{1}\otimes(2\mathbb{1} + B_0B_1 + B_1B_0) + A_0A_1\otimes(\mathbb{1} - B_0B_1 + B_1B_0 - \mathbb{1})$
$+ A_1A_0\otimes(\mathbb{1} - B_1B_0 + B_0B_1 - \mathbb{1}) + \mathbb{1}(2\mathbb{1} - B_0B_1 - B_1B_0)$

$$= 4\mathbb{1} + A_0A_1 \otimes (B_1B_0 - B_0B_1) + A_1A_0(-B_1B_0 + B_0B_1)$$
$$= 4\mathbb{1} - A_0A_1 \otimes [B_0, B_1] + A_1A_0 \otimes [B_0, B_1]$$
$$= 4\mathbb{1} - [A_0, A_1] \otimes [B_0, B_1]$$

← Minus sign missing somewhere.

d) Show $\langle\psi| K |\psi\rangle \leq 2\sqrt{2}$. (Hint: use Cauchy-Schwarz). What is the maximum winning probability for quantum strategies?

Sol^n First use Cauchy-Schwarz $|\langle v|w\rangle| \leq \|v\| \|w\|$

So
$$\langle\psi|K|\psi\rangle \leq \| |\psi\rangle \| \, \| K|\psi\rangle \|$$
$$= \sqrt{\langle\psi|K^\dagger K|\psi\rangle} = \sqrt{\langle\psi|K^2|\psi\rangle}$$

Now we look to bound $\langle\psi|K^2|\psi\rangle$.

By previous question $\langle\psi|K^2|\psi\rangle \leq 4 + \langle\psi|[A_0, A_1]\otimes[B_0,B_1]|\psi\rangle$

Now we use operator norm $\|X\| = \sup_{|\psi\rangle} \langle\psi|X|\psi\rangle$

$$\langle\psi|K^2|\psi\rangle \leq 4 + \|[A_0,A_1] \otimes [B_0,B_1]\|$$
$$= 4 + \|[A_0,A_1]\| \, \|[B_0,B_1]\|$$

↙ $\|X\otimes Y\| = \|X\| \|Y\|$

Now $\|[A_0,A_1]\| = \|A_0A_1 - A_1A_0\| \leq \|A_0A_1\| + \|A_1A_0\|$ ↙ Triangle inequality

$\leq \|A_0\|\|A_1\| + \|A_1\|\|A_0\|$ ↙ Submultiplicativity

≤ 2 ← $\|A_x\| \leq 1$

↑ For Hermitian operators $\|X\| = \sup\{|\lambda| : \lambda \text{ is eigenvalue}\}$ And A_x has eigenvalues $\{+1, -1\}$

Thus $\langle\psi|K^2|\psi\rangle \leq 8$

$\Rightarrow \langle\psi|K|\psi\rangle \leq \sqrt{8} = 2\sqrt{2}$.

Using part (b) this shows that

$$\mathbb{P}(\text{Alice and Bob win}) \leq \frac{1}{2} + \frac{1}{8} 2\sqrt{2} = \frac{1}{2} + \frac{\sqrt{2}}{4} = \cos^2(\pi/8)$$

As the bound is for an arbitrary state and measurements we can conclude that no quantum strategy can win with a higher probability