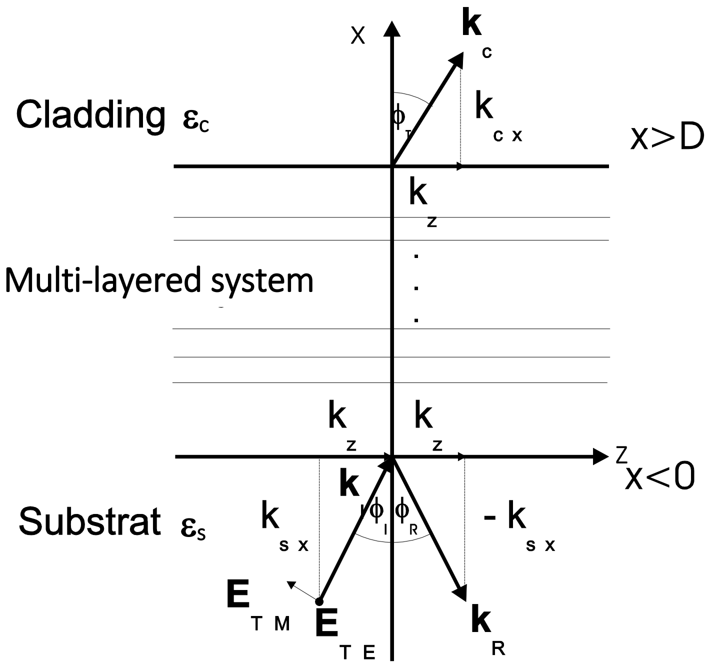
8.3 Reflection – transmission problem for layer systems

8.3.1 General layer systems

8.3.1.1 Reflection- and transmission coefficients → generalized Fresnel formulas

In the previous chapter, we have learned how to link the electromagnetic field on one side of an arbitrary multilayered system with the field on the other side. We have seen that after splitting into the TE/TM-polarizations, the continuous (transverse) field components are sufficient to describe the whole field. What we will do now is to link those field components with the fields which are accessible in an experimental configuration, i.e. the incident, reflected, and transmitted fields. In particular, we want to solve the reflection-transmission problem, which means that we have to compute the reflected and transmitted fields for a given angle of incidence, frequency, layer system and polarization.



We introduce the wave vectors of the incident ($\mathbf{k_I}$), reflected ($\mathbf{k_R}$) and transmitted ($\mathbf{k_T}$) fields:

$$\mathbf{k_I} = \begin{pmatrix} k_{\mathbf{sx}} \\ 0 \\ k_{\mathbf{z}} \end{pmatrix}, \qquad \mathbf{k_R} = \begin{pmatrix} -k_{\mathbf{sx}} \\ 0 \\ k_{\mathbf{z}} \end{pmatrix}, \qquad \mathbf{k_T} = \begin{pmatrix} k_{\mathbf{cx}} \\ 0 \\ k_{\mathbf{z}} \end{pmatrix}$$

with $k_{\mathbf{sx}} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_{\mathbf{s}} - k_{\mathbf{z}}^2} = \sqrt{k_{\mathbf{s}}^2(\omega) - k_{\mathbf{z}}^2}$, $\qquad k_{\mathbf{cx}} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_{\mathbf{c}} - k_{\mathbf{z}}^2} = \sqrt{k_{\mathbf{c}}^2(\omega) - k_{\mathbf{z}}^2}$,

where $\varepsilon_{\mathbf{s}}(\omega)$ and $\varepsilon_{\mathbf{c}}(\omega)$ are the dielectric functions of the substrate and cladding and k_z is the tangential component of the wave vector, which is continuous throughout the layer system.

As we have seen before, the k_z component of the wave vector is conserved and $\pm k_{\mathbf{x}}$ determines the propagation direction of the wave (forward or backward). The total length of the wave vector in each layer is given by the dispersion relation for dispersive, isotropic, homogeneous media. As a consequence, the k_x component must change its value in each layer.

Remark on law of reflection and transmission (Snell's law)

It is possible to derive Snell's law just from the fact that $k_{\mathbf{z}}$ is a conserved quantity:

1. $k_{\mathbf{s}} \sin \varphi_{\mathbf{I}} = k_{\mathbf{s}} \sin \varphi_{\mathbf{R}} \curvearrowright \varphi_{\mathbf{I}} = \varphi_{\mathbf{R}}$ (reflection)
2. $k_{\mathbf{s}} \sin \varphi_{\mathbf{I}} = k_{\mathbf{c}} \sin \varphi_{\mathbf{T}} \curvearrowright n_{\mathbf{s}} \sin \varphi_{\mathbf{I}} = n_{\mathbf{c}} \sin \varphi_{\mathbf{T}}$ (Snell's law)

It should be noted that these formulae have been derived for the two semi-infinite half-spaces, i.e. the substrate region and the cladding region, independently of the specific layer system in between. These formulae are derived just from the continuity of the transverse wavevector component $k_{\mathbf{z}}$ at every interface.

Let us now connect the propagating waves (incident, reflected, transmitted) to the local fields at the interfaces in order to solve the reflection transmission problem:

A) Field in substrate

The fields in the substrate $F_{\mathrm{s}}(x,z)$ and $G_{\mathrm{s}}(x,z)$ is due to contributions from the complex amplitudes of the incident $F_{\mathbf{I}}$ and reflected field $F_{\mathbf{R}}$:

$$F_{\mathbf{s}}(x,z) = \exp\left(\mathbf{i} k_{\mathbf{z}} z\right)\left[F_{\mathbf{I}} \exp\left(\mathbf{i} k_{\mathbf{sx}} x\right) + F_{\mathbf{R}} \exp\left(-\mathbf{i} k_{\mathbf{sx}} x\right)\right]$$

$$G_{\mathbf{s}}(x,z) = \mathbf{i} \alpha_{\mathbf{s}} k_{\mathbf{sx}} \exp\left(\mathbf{i} k_{\mathbf{z}} z\right)\left[F_{\mathbf{I}} \exp\left(\mathbf{i} k_{\mathbf{sx}} x\right) - F_{\mathbf{R}} \exp\left(-\mathbf{i} k_{\mathbf{sx}} x\right)\right]$$

B) Field in layer system

The fields inside the layer system can be expressed as

$$F_{\mathbf{f}}(x,z) = \exp\left(\mathbf{i} k_{\mathbf{z}} z\right) F(x)$$

$$G_{\mathbf{f}}(x,z) = \exp\left(\mathbf{i} k_{\mathbf{z}} z\right) G(x)$$

where the amplitudes $F(x)$ and $G(x)$ are given as above by the matrix method as

$$\begin{pmatrix} F \\ G \end{pmatrix}_x = \hat{\mathbf{M}}(x) \begin{pmatrix} F \\ G \end{pmatrix}_0$$

C) Field in cladding

The fields in the cladding $F_{\mathrm{c}}(x,z)$ and $G_{\mathrm{c}}(x,z)$ are due to contributions only from the complex amplitude of the transmitted field $F_{\mathbf{T}}$:

$$F_{\mathbf{c}}(x,z) = \exp\left(\mathbf{i} k_{\mathbf{z}} z\right) F_{\mathbf{T}} \exp\left[\mathbf{i} k_{\mathbf{cx}} (x - D)\right]$$

$$G_{\mathbf{c}}(x,z) = \mathbf{i} \alpha_{\mathbf{c}} k_{\mathbf{cx}} \exp\left(\mathbf{i} k_{\mathbf{z}} z\right) F_{\mathbf{T}} \exp\left[\mathbf{i} k_{\mathbf{cx}} (x - D)\right]$$

Note that in the cladding we consider a forward (transmitted) wave only. Hence, we exclude reflection at any inhomogeneities after the boundary between the layer system and the cladding. For convenience, we define the phase factor in the x-direction from this interface with the last layer at $x=D$.

Reflection-transmission problem

The aim is to compute $F_{\mathbf{R}}$ and $F_{\mathbf{T}}$ for given $F_{\mathbf{I}}$, $k_{\mathbf{z}}$ ($\sim \sin\varphi_{\mathbf{I}}$), ε_i, d_i.

We know that F and G are continuous at each interface, in particular at $x=0$ and $x=D$. We have:

$$\underbrace{\begin{pmatrix} F \\ G \end{pmatrix}_D}_{\text{Field in cladding at } x=D} = \hat{\mathbf{M}}(D) \underbrace{\begin{pmatrix} F \\ G \end{pmatrix}_0}_{\text{field in substrate at } x=0}.$$

On the other hand, we have expressions for the fields at $x=0$ and $x=D$ from our decomposition into the incident, reflected and transmitted fields above. Hence, we can write:

$$\begin{pmatrix} F_{\mathbf{T}} \\ \mathbf{i}\alpha_{\mathbf{c}} k_{\mathbf{cx}} F_{\mathbf{T}} \end{pmatrix} = \begin{Bmatrix} M_{11}(D) & M_{12}(D) \\ M_{21}(D) & M_{22}(D) \end{Bmatrix} \begin{pmatrix} F_{\mathbf{I}} + F_{\mathbf{R}} \\ \mathbf{i}\alpha_{\mathbf{s}} k_{\mathbf{sx}} \left(F_{\mathbf{I}} - F_{\mathbf{R}}\right) \end{pmatrix}.$$

We consider the incident F_{I} as known, the reflected and transmitted fields F_{R} and F_{T} as unknown and solve for F_{R} and F_{T}:

$$\boxed{F_{\mathbf{R}} = \frac{\left(\alpha_{\mathbf{s}} k_{\mathbf{sx}} M_{22} - \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{11}\right) - \mathbf{i}\left(M_{21} + \alpha_{\mathbf{s}} k_{\mathbf{sx}} \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{12}\right)}{\underbrace{\left(\alpha_{\mathbf{s}} k_{\mathbf{sx}} M_{22} + \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{11}\right) + \mathbf{i}\left(M_{21} - \alpha_{\mathbf{s}} k_{\mathbf{sx}} \alpha_{\mathbf{c}} k_{\mathbf{cx}} M_{12}\right)}_{N}} F_{\mathbf{I}}}$$

$$F_T = \frac{2\alpha_s k_{sx}(M_{11}M_{22} - M_{12}M_{21})}{(\alpha_s k_{sx} M_{22} + \alpha_c k_{cx} M_{11}) + i(M_{21} - \alpha_s k_{sx}\alpha_c k_{cx} M_{12})} F_I$$

$$\boxed{F_T = \frac{2\alpha_s k_{sx}}{N} F_I}$$

These are the general formulae for reflected and transmitted amplitudes. Please note that the matrix elements depend on the polarization direction → $M_{ij}^{\mathbf{TE}} \neq M_{ij}^{\mathbf{TM}}$.

Let us now transform back to the physical electric and magnetic fields, and write the solution for the reflection-transmission problem for TE and TM polarization:

A) TE-polarization

$$F = E = E_{\mathbf{y}}, \qquad \alpha_{\mathbf{TE}} = 1$$

i) reflected field

$$E_{\mathbf{R}}^{\mathbf{TE}} = R_{\mathbf{TE}} E_{\mathbf{I}}^{\mathbf{TE}}$$

with the reflection coefficient

$$\boxed{R_{\mathrm{TE}}=\frac{\left(k_{\mathrm{sx}}M_{22}^{\mathrm{TE}}-k_{\mathrm{cx}}M_{11}^{\mathrm{TE}}\right)-\mathrm{i}\left(M_{21}^{\mathrm{TE}}+k_{\mathrm{sx}}k_{\mathrm{cx}}M_{12}^{\mathrm{TE}}\right)}{\underbrace{\left(k_{\mathrm{sx}}M_{22}^{\mathrm{TE}}+k_{\mathrm{cx}}M_{11}^{\mathrm{TE}}\right)+\mathrm{i}\left(M_{21}^{\mathrm{TE}}-k_{\mathrm{sx}}k_{\mathrm{cx}}M_{12}^{\mathrm{TE}}\right)}_{N_{\mathrm{TE}}}}}$$

ii) transmitted field

$$E_{\mathrm{T}}^{\mathrm{TE}}=T_{\mathrm{TE}}E_{\mathrm{I}}^{\mathrm{TE}}$$

with the transmission coefficient

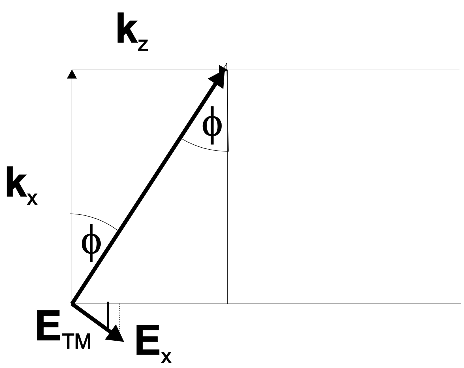
$$\boxed{T_{\mathrm{TE}}=\frac{2k_{\mathrm{sx}}}{\left(k_{\mathrm{sx}}M_{22}^{\mathrm{TE}}+k_{\mathrm{cx}}M_{11}^{\mathrm{TE}}\right)+\mathrm{i}\left(M_{21}^{\mathrm{TE}}-k_{\mathrm{sx}}k_{\mathrm{cx}}M_{12}^{\mathrm{TE}}\right)}=\frac{2k_{\mathrm{sx}}}{N_{\mathrm{TE}}},}$$

→We get complex coefficients for reflection and transmission, which determine the amplitude and phase of the reflected and transmitted light.

B) TM-polarization

$$F=H=H_{\mathrm{y}},\qquad \alpha_{\mathrm{TM}}=\frac{1}{\varepsilon}.$$

In the case of TM polarization we have the problem that an analogous calculation to TE would lead to $H_{\mathrm{R,T}}/H_{\mathrm{I}}$, i.e., relations between the magnetic field. However, we wish to instead find $E_{\mathrm{R,T}}^{\mathrm{TM}}/E_{\mathrm{I}}^{\mathrm{TM}}$. Therefore, we must convert the H-field to the E^{TM}-field:



As can be seen in the figure, we can express the amplitude of the E_{TM} field in terms of the E_x component:

$$\frac{E_{\mathrm{x}}}{E^{\mathrm{TM}}}=-\sin\varphi=-\frac{k_{\mathrm{z}}}{k}$$

$$\curvearrowright E^{\mathrm{TM}}=-\frac{k}{k_{\mathrm{z}}}E_{\mathrm{x}},$$

With Maxwell's equations we can link E_x to H_y:

$$\mathbf{E} = -\frac{1}{\omega\varepsilon_0\varepsilon}(\mathbf{k}\times\mathbf{H}) \rightarrow E_x = \frac{1}{\omega\varepsilon_0\varepsilon}k_z H_y \rightarrow E^{\mathrm{TM}} = -\frac{k}{\omega\varepsilon_0\varepsilon}H_y = -\frac{1}{c\varepsilon_0\sqrt{\varepsilon}}H_y$$

Result: $$\frac{E^{\mathrm{TM}}_{\mathrm{R,T}}}{E^{\mathrm{TM}}_{\mathrm{I}}} = \sqrt{\frac{\varepsilon_s}{\varepsilon_{s,c}}}\frac{H_{\mathrm{R,T}}}{H_{\mathrm{I}}}, \rightarrow \sqrt{\varepsilon_s/\varepsilon_c}$$ relevant only for transmission

Hence we find the following for TM polarization:

$$E^{\mathrm{TM}}_{\mathrm{R}} = R_{\mathrm{TM}}E^{\mathrm{TM}}_{\mathrm{I}}$$

with the reflection coefficient

$$\boxed{R_{\mathrm{TM}} = \frac{\left(\varepsilon_c k_{sx}M^{\mathrm{TM}}_{22} - \varepsilon_s k_{cx}M^{\mathrm{TM}}_{11}\right) - \mathrm{i}\left(\varepsilon_s\varepsilon_c M^{\mathrm{TM}}_{21} + k_{sx}k_{cx}M^{\mathrm{TM}}_{12}\right)}{\underbrace{\left(\varepsilon_c k_{sx}M^{\mathrm{TM}}_{22} + \varepsilon_s k_{cx}M^{\mathrm{TM}}_{11}\right) + \mathrm{i}\left(\varepsilon_s\varepsilon_c M^{\mathrm{TM}}_{21} - k_{sx}k_{cx}M^{\mathrm{TM}}_{12}\right)}_{N_{\mathrm{TM}}}}}$$

$$E^{\mathrm{TM}}_{\mathrm{T}} = T_{\mathrm{TM}}E^{\mathrm{TM}}_{\mathrm{I}}$$

with the transmission coefficient

$$\boxed{T_{\mathrm{TM}} = \frac{2\sqrt{\varepsilon_s\varepsilon_c}k_{sx}}{\left(\varepsilon_c k_{sx}M^{\mathrm{TM}}_{22} + \varepsilon_s k_{cx}M^{\mathrm{TM}}_{11}\right) + \mathrm{i}\left(\varepsilon_s\varepsilon_c M^{\mathrm{TM}}_{21} - k_{sx}k_{cx}M^{\mathrm{TM}}_{12}\right)} = \frac{2\sqrt{\varepsilon_s\varepsilon_c}k_{sx}}{N_{\mathrm{TM}}}}$$

In summary, we have found different complex coefficients for reflection and transmission for TE and TM polarization. The resulting generalized Fresnel formulas for multilayered systems are

$$\boxed{R_{\mathrm{TE}} = \frac{\left(k_{sx}M^{\mathrm{TE}}_{22} - k_{cx}M^{\mathrm{TE}}_{11}\right) - \mathrm{i}\left(M^{\mathrm{TE}}_{21} + k_{sx}k_{cx}M^{\mathrm{TE}}_{12}\right)}{\left(k_{sx}M^{\mathrm{TE}}_{22} + k_{cx}M^{\mathrm{TE}}_{11}\right) + \mathrm{i}\left(M^{\mathrm{TE}}_{21} - k_{sx}k_{cx}M^{\mathrm{TE}}_{12}\right)}}$$

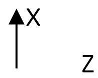
$$\boxed{T_{\mathrm{TE}} = \frac{2k_{sx}}{N_{\mathrm{TE}}}}$$

$$\boxed{R_{\mathrm{TM}} = \frac{\left(\varepsilon_c k_{sx}M^{\mathrm{TM}}_{22} - \varepsilon_s k_{cx}M^{\mathrm{TM}}_{11}\right) - \mathrm{i}\left(\varepsilon_s\varepsilon_c M^{\mathrm{TM}}_{21} + k_{sx}k_{cx}M^{\mathrm{TM}}_{12}\right)}{\left(\varepsilon_c k_{sx}M^{\mathrm{TM}}_{22} + \varepsilon_s k_{cx}M^{\mathrm{TM}}_{11}\right) + \mathrm{i}\left(\varepsilon_s\varepsilon_c M^{\mathrm{TM}}_{21} - k_{sx}k_{cx}M^{\mathrm{TM}}_{12}\right)}}$$

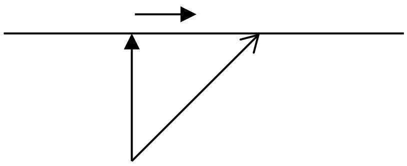
$$\boxed{T_{\mathrm{TM}} = \frac{2\sqrt{\varepsilon_s\varepsilon_c}k_{sx}}{N_{\mathrm{TM}}}.}$$

8.3.1.2 Reflectivity and transmissivity

In the previous chapter we have computed the coefficients of reflection and transmission, which relate the electric fields in TE and TM polarization of the incident, reflected and transmitted waves. However, in many situations it is more important to consider the relation between energy fluxes, the so-called reflectivity and transmissivity. In order to obtain information on these quantities we must compute the energy flux perpendicular to the interface:



– flux through a surface with $x = \text{const}$



For a monochromatic plane wave, it follows that

$$\langle \mathbf{S} \rangle \mathbf{e_x} = \frac{1}{2} \Re\left(\mathbf{E} \times \mathbf{H}^*\right) \mathbf{e_x}$$

With $$\mathbf{H}^* = \frac{1}{\omega\mu_0}\left(\mathbf{k}^* \times \mathbf{E}^*\right)$$

after evaluating the successive cross products and using the transversality of the electric field, we find that

$$\boxed{\langle \mathbf{S} \rangle \mathbf{e_x} = \frac{1}{2\omega\mu_0} \Re\left(\mathbf{k}^* \mathbf{e_x}\right) |\mathbf{E}|^2 = \frac{1}{2\omega\mu_0} \Re\left(k_x^*\right) |\mathbf{E}|^2 .}$$

Note once more that only the real part of the wave vector contributes to energy flow.

Since in an absorption-free medium the energy flux is conserved, in an absorption-free layer system the energy flux is also conserved.

In the substrate

$$k_{\mathbf{sx}} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_{\mathbf{s}} - k_{\mathbf{z}}^2} = \sqrt{k_{\mathbf{s}}^2(\omega) - k_{\mathbf{z}}^2}$$

must be real-valued, since our incident wave propagates from there. The total energy flux from the substrate to the layer system, after subtracting by the reflected flux, is given by

$$\langle \mathbf{S} \rangle_{\mathbf{s}} \mathbf{e_x} = \frac{1}{2\omega\mu_0}\left[k_{\mathbf{sx}} |\mathbf{E_I}|^2 - k_{\mathbf{sx}} |\mathbf{E_R}|^2\right]$$

In contrast, in the cladding

$$k_{\mathbf{cx}} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_{\mathbf{c}} - k_{\mathbf{z}}^2} = \sqrt{k_{\mathbf{c}}^2(\omega) - k_{\mathbf{z}}^2}$$

can in general be complex-valued. The energy flux from the layer system into the cladding is

$$\langle \mathbf{S} \rangle_{\mathbf{c}} \mathbf{e_x} = \frac{1}{2\omega\mu_0} \Re\left(k_{\mathbf{cx}}\right) |\mathbf{E_T}|^2 .$$

Because we have energy conservation:

$$\langle \mathbf{S} \rangle_{\mathbf{s}} \mathbf{e_x} = \langle \mathbf{S} \rangle_{\mathbf{c}} \mathbf{e_x} \curvearrowright$$

$$\boxed{|\mathbf{E_I}|^2 = |\mathbf{E_R}|^2 + \frac{\Re\left(k_{\mathbf{cx}}\right)}{k_{\mathbf{sx}}} |\mathbf{E_T}|^2}$$

Now we will compute the global reflectivity ρ and transmissivity τ of a layer system. Of course, we will decompose into TE and TM polarizations and use the reflectivities

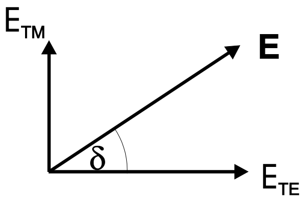
$\rho_{\mathrm{TE,TM}}$ and transmissivities $\tau_{\mathrm{TE,TM}}$. Since the TE & TM polarizations are orthogonal, we can consider their energy fluxes independently:

$$\mathbf{E}_{\mathbf{R}} = \mathbf{E}_{\mathbf{R}}^{\mathbf{TE}} + \mathbf{E}_{\mathbf{R}}^{\mathbf{TM}}, \qquad \mathbf{E}_{\mathbf{T}} = \mathbf{E}_{\mathbf{T}}^{\mathbf{TE}} + \mathbf{E}_{\mathbf{T}}^{\mathbf{TM}}$$

$$\left|\mathbf{E}_{\mathbf{I}}\right|^2 = \left|\mathbf{E}_{\mathbf{R}}^{\mathbf{TE}}\right|^2 + \left|\mathbf{E}_{\mathbf{R}}^{\mathbf{TM}}\right|^2 + \frac{\Re\left(k_{\mathbf{cx}}\right)}{k_{\mathbf{sx}}}\left(\left|\mathbf{E}_{\mathbf{T}}^{\mathbf{TE}}\right|^2 + \left|\mathbf{E}_{\mathbf{T}}^{\mathbf{TM}}\right|^2\right)$$

$$= \left\{\left|R_{\mathbf{TE}}\right|^2 + \frac{\Re\left(k_{\mathbf{cx}}\right)}{k_{\mathbf{sx}}}\left|T_{\mathbf{TE}}\right|^2\right\}\left|\mathbf{E}_{\mathbf{I}}^{\mathbf{TE}}\right|^2 + \left\{\left|R_{\mathbf{TM}}\right|^2 + \frac{\Re\left(k_{\mathbf{cx}}\right)}{k_{\mathbf{sx}}}\left|T_{\mathbf{TM}}\right|^2\right\}\left|\mathbf{E}_{\mathbf{I}}^{\mathbf{TM}}\right|^2.$$

Here, we simply substituted the reflected and transmitted field amplitudes by the incident amplitudes multiplied by Fresnel coefficients. Now, we decompose the incident field as follows:



$$E_{\mathbf{I}}^{\mathbf{TE}} = \left|\mathbf{E}_{\mathbf{I}}\right|\cos\delta, \qquad E_{\mathbf{I}}^{\mathbf{TM}} = \left|\mathbf{E}_{\mathbf{I}}\right|\sin\delta.$$

Then, we can divide by the (arbitrary) amplitude $\left|\mathbf{E}_{\mathbf{I}}\right|^2$ and write

$$1 = \left\{\left|R_{\mathbf{TE}}\right|^2 + \frac{\Re\left(k_{\mathbf{cx}}\right)}{k_{\mathbf{sx}}}\left|T_{\mathbf{TE}}\right|^2\right\}\cos^2\delta + \left\{\left|R_{\mathbf{TM}}\right|^2 + \frac{\Re\left(k_{\mathbf{cx}}\right)}{k_{\mathbf{sx}}}\left|T_{\mathbf{TM}}\right|^2\right\}\sin^2\delta$$

$$1 = \underbrace{\left(\left|R_{\mathbf{TE}}\right|^2\cos^2\delta + \left|R_{\mathbf{TM}}\right|^2\sin^2\delta\right)}_{\rho} + \underbrace{\frac{\Re\left(k_{\mathbf{cx}}\right)}{k_{\mathbf{sx}}}\left(\left|T_{\mathbf{TE}}\right|^2\cos^2\delta + \left|T_{\mathbf{TM}}\right|^2\sin^2\delta\right)}_{\tau}$$

The red (reflected) and blue (transmitted) terms can be identified as $\boxed{1 = \rho + \tau}$, which ensures energy conservation.

The global reflectivity and transmissivity are therefore given as

$$\boxed{\begin{aligned}\rho &= \rho_{\mathbf{TE}}\cos^2\delta + \rho_{\mathbf{TM}}\sin^2\delta \\ \tau &= \tau_{\mathbf{TE}}\cos^2\delta + \tau_{\mathbf{TM}}\sin^2\delta\end{aligned}}$$

with the reflectivities

$$\boxed{\rho_{\mathbf{TE,TM}} = \left|R_{\mathbf{TE,TM}}\right|^2, \qquad \tau_{\mathbf{TE,TM}} = \frac{\Re\left(k_{\mathbf{cx}}\right)}{k_{\mathbf{sx}}}\left|T_{\mathbf{TE,TM}}\right|^2.}$$

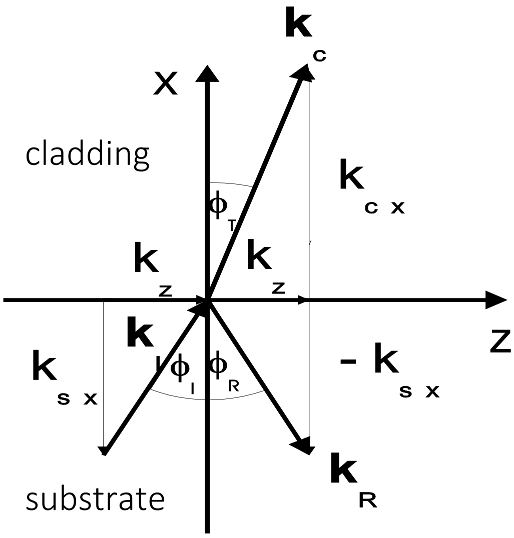
for the two polarization states TE and TM.

8.3.2 Single interface

8.3.2.1 (Classical) Fresnel formulas

Let us now consider the important example of the most simple layer system, namely the single interface. The relevant wave vectors are (as usual):

$$\mathbf{k}_{\mathrm{I}} = \begin{pmatrix} k_{\mathbf{sx}} \\ 0 \\ k_{\mathbf{z}} \end{pmatrix}, \qquad \mathbf{k}_{\mathrm{R}} = \begin{pmatrix} -k_{\mathbf{sx}} \\ 0 \\ k_{\mathbf{z}} \end{pmatrix}, \qquad \mathbf{k}_{\mathrm{T}} = \begin{pmatrix} k_{\mathbf{cx}} \\ 0 \\ k_{\mathbf{z}} \end{pmatrix}.$$



The continuous component of the wave vector, expressed in terms of the angle of incidence, is

$$k_{\mathbf{z}} = \frac{\omega}{c}\sqrt{\varepsilon_{\mathbf{s}}}\sin\varphi_{\mathrm{I}} = \frac{\omega}{c}n_{\mathbf{s}}\sin\varphi_{\mathrm{I}}.$$

Then, the discontinuous component is given by

$$\rightarrow k_{i\mathbf{x}} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_i - k_{\mathbf{z}}^2} = \sqrt{\frac{\omega^2}{c^2}\varepsilon_i - \frac{\omega^2}{c^2}\varepsilon_{\mathbf{s}}\sin^2\varphi_{\mathrm{I}}} = \frac{\omega}{c}\sqrt{n_i^2 - n_{\mathbf{s}}^2\sin^2\varphi_{\mathrm{I}}}$$

$$\boxed{\curvearrowright k_{\mathbf{sx}} = \frac{\omega}{c}n_{\mathbf{s}}\cos\varphi_{\mathrm{I}}, \qquad k_{\mathbf{cx}} = \frac{\omega}{c}\sqrt{n_{\mathbf{c}}^2 - n_{\mathbf{s}}^2\sin^2\varphi_{\mathrm{I}}} = \frac{\omega}{c}n_c\cos\varphi_{\mathrm{T}},}$$

As above, we can assume that $k_{\mathbf{sx}}$ is always real, because otherwise we have no propagating incident wave. $k_{\mathbf{cx}}$ is real for $n_{\mathbf{c}} > n_{\mathbf{s}}\sin\varphi_{\mathrm{I}}$, but imaginary for $n_{\mathbf{c}} < n_{\mathbf{s}}\sin\varphi_{\mathrm{I}}$ (total internal reflection).

The matrix for a single interface is the unit matrix

$$\hat{\mathbf{M}} = \hat{\mathbf{m}}(d=0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and it is easy to compute coefficients for reflection and transmission, and reflectivity and transmissivity. Using the formulas from above we find:

A) TE-polarization

$$R_{\mathrm{TE}}=\frac{\left(k_{\mathrm{sx}}M_{22}-k_{\mathrm{cx}}M_{11}\right)-\mathrm{i}\left(M_{21}+k_{\mathrm{sx}}k_{\mathrm{cx}}M_{12}\right)}{\underbrace{\left(k_{\mathrm{sx}}M_{22}+k_{\mathrm{cx}}M_{11}\right)+\mathrm{i}\left(M_{21}-k_{\mathrm{sx}}k_{\mathrm{cx}}M_{12}\right)}_{N_{\mathrm{TE}}}},\qquad T_{\mathrm{TE}}=\frac{2k_{\mathrm{sx}}}{N_{\mathrm{TE}}}$$

with: $\hat{\mathbf{M}}=\hat{\mathbf{m}}(d=0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$$\boxed{\begin{aligned}R_{\mathrm{TE}}&=\frac{\left(k_{\mathrm{sx}}-k_{\mathrm{cx}}\right)}{\left(k_{\mathrm{sx}}+k_{\mathrm{cx}}\right)}=\frac{n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}-\sqrt{n_{\mathrm{c}}^2-n_{\mathrm{s}}^2\sin^2\varphi_{\mathrm{I}}}}{n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}+\sqrt{n_{\mathrm{c}}^2-n_{\mathrm{s}}^2\sin^2\varphi_{\mathrm{I}}}}=\frac{n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}-n_{\mathrm{c}}\cos\varphi_{\mathrm{T}}}{n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}+n_{\mathrm{c}}\cos\varphi_{\mathrm{T}}}\\ T_{\mathrm{TE}}&=\frac{2k_{\mathrm{sx}}}{\left(k_{\mathrm{sx}}+k_{\mathrm{cx}}\right)}=\frac{2n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}}{n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}+\sqrt{n_{\mathrm{c}}^2-n_{\mathrm{s}}^2\sin^2\varphi_{\mathrm{I}}}}=\frac{2n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}}{n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}+n_{\mathrm{c}}\cos\varphi_{\mathrm{T}}}\end{aligned}}$$

$$\rho_{\mathrm{TE}}=\left|R_{\mathrm{TE}}\right|^2=\frac{\left|k_{\mathrm{sx}}-k_{\mathrm{cx}}\right|^2}{\left|k_{\mathrm{sx}}+k_{\mathrm{cx}}\right|^2}$$

$$\tau_{\mathrm{TE}}=\frac{\Re\left(k_{\mathrm{cx}}\right)}{k_{\mathrm{sx}}}\left|T_{\mathrm{TE}}\right|^2=\frac{4k_{\mathrm{sx}}\Re\left(k_{\mathrm{cx}}\right)}{\left|k_{\mathrm{sx}}+k_{\mathrm{cx}}\right|^2}.$$

$$\curvearrowright\ \rho_{\mathrm{TE}}+\tau_{\mathrm{TE}}=1$$

B) TM-Polarisation

$$R_{\mathrm{TM}}=\frac{\left(\varepsilon_{\mathrm{c}}k_{\mathrm{sx}}M_{22}-\varepsilon_{\mathrm{s}}k_{\mathrm{cx}}M_{11}\right)-\mathrm{i}\left(\varepsilon_{\mathrm{s}}\varepsilon_{\mathrm{c}}M_{21}+k_{\mathrm{sx}}k_{\mathrm{cx}}M_{12}\right)}{\underbrace{\left(\varepsilon_{\mathrm{c}}k_{\mathrm{sx}}M_{22}+\varepsilon_{\mathrm{s}}k_{\mathrm{cx}}M_{11}\right)+\mathrm{i}\left(\varepsilon_{\mathrm{s}}\varepsilon_{\mathrm{c}}M_{21}-k_{\mathrm{sx}}k_{\mathrm{cx}}M_{12}\right)}_{N_{\mathrm{TM}}}}\qquad T_{\mathrm{TM}}=\frac{2\sqrt{\varepsilon_{\mathrm{s}}\varepsilon_{\mathrm{c}}}k_{\mathrm{sx}}}{N_{\mathrm{TM}}}.$$

with: $\hat{\mathbf{M}}=\hat{\mathbf{m}}(d=0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$$\boxed{\begin{aligned}R_{\mathrm{TM}}&=\frac{\left(k_{\mathrm{sx}}\varepsilon_{\mathrm{c}}-k_{\mathrm{cx}}\varepsilon_{\mathrm{s}}\right)}{\left(k_{\mathrm{sx}}\varepsilon_{\mathrm{c}}+k_{\mathrm{cx}}\varepsilon_{\mathrm{s}}\right)}=\frac{n_{\mathrm{s}}n_{\mathrm{c}}^2\cos\varphi_{\mathrm{I}}-n_{\mathrm{s}}^2\sqrt{n_{\mathrm{c}}^2-n_{\mathrm{s}}^2\sin^2\varphi_{\mathrm{I}}}}{n_{\mathrm{s}}n_{\mathrm{c}}^2\cos\varphi_{\mathrm{I}}+n_{\mathrm{s}}^2\sqrt{n_{\mathrm{c}}^2-n_{\mathrm{s}}^2\sin^2\varphi_{\mathrm{I}}}}=\frac{n_{\mathrm{c}}\cos\varphi_{\mathrm{I}}-n_{\mathrm{s}}\cos\varphi_{\mathrm{T}}}{n_{\mathrm{c}}\cos\varphi_{\mathrm{I}}+n_{\mathrm{s}}\cos\varphi_{\mathrm{T}}}\\ T_{\mathrm{TM}}&=\frac{2k_{\mathrm{sx}}\sqrt{\varepsilon_{\mathrm{c}}\varepsilon_{\mathrm{s}}}}{\left(k_{\mathrm{sx}}\varepsilon_{\mathrm{c}}+k_{\mathrm{cx}}\varepsilon_{\mathrm{s}}\right)}=\frac{2n_{\mathrm{s}}^2n_{\mathrm{c}}\cos\varphi_{\mathrm{I}}}{n_{\mathrm{s}}n_{\mathrm{c}}^2\cos\varphi_{\mathrm{I}}+n_{\mathrm{s}}^2\sqrt{n_{\mathrm{c}}^2-n_{\mathrm{s}}^2\sin^2\varphi_{\mathrm{I}}}}=\frac{2n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}}{n_{\mathrm{c}}\cos\varphi_{\mathrm{I}}+n_{\mathrm{s}}\cos\varphi_{\mathrm{T}}},\end{aligned}}$$

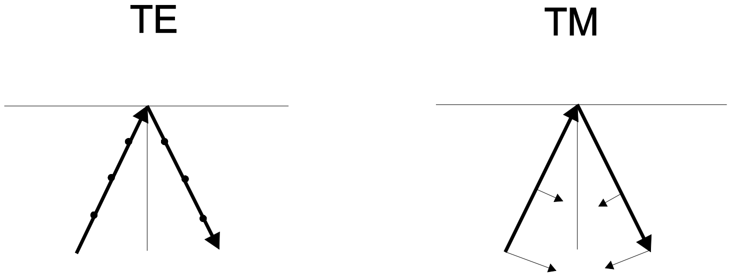
$$\rho_{\mathrm{TM}}=\left|R_{\mathrm{TM}}\right|^2=\frac{\left|k_{\mathrm{sx}}\varepsilon_{\mathrm{c}}-k_{\mathrm{cx}}\varepsilon_{\mathrm{s}}\right|^2}{\left|k_{\mathrm{sx}}\varepsilon_{\mathrm{c}}+k_{\mathrm{cx}}\varepsilon_{\mathrm{s}}\right|^2},$$

$$\tau_{\mathrm{TM}}=\frac{\Re\left(k_{\mathrm{cx}}\right)}{k_{\mathrm{sx}}}\left|T_{\mathrm{TM}}\right|^2=\frac{4k_{\mathrm{sx}}\Re\left(k_{\mathrm{cx}}\right)\varepsilon_{\mathrm{s}}\varepsilon_{\mathrm{c}}}{\left|k_{\mathrm{sx}}\varepsilon_{\mathrm{c}}+k_{\mathrm{cx}}\varepsilon_{\mathrm{s}}\right|^2}$$

$$\curvearrowright \rho_{\mathrm{TM}} + \tau_{\mathrm{TM}} = 1$$

Remark

It may seem that we have a problem for $\varphi_{\mathrm{I}} = 0$. For $\varphi_{\mathrm{I}} = 0$, the TE and TM polarizations should be equivalent, since the fields are always polarized parallel to the interface. However, formally we have $R_{\mathrm{TE}} = -R_{\mathrm{TM}}$, $T_{\mathrm{TE}} = T_{\mathrm{TM}}$. The "strange" behavior of the coefficient of reflection can be explained by the following figures:



The positive direction of the TM field is defined by the propagation direction through the right-hand rule for cross products. Even though for $\varphi_{\mathrm{I}} = 0$ the TE and TM polarizations are equivalent, the definitions of axes for their respective reflected components differ. Thus there is a relative sign between the reflection coefficients.

8.3.2.2 Total internal reflection (TIR) for $\varepsilon_{\mathrm{s}}>\varepsilon_{\mathrm{c}}$

Let us now consider the special case when all incident light is reflected from the interface. This means that the reflectivity is unity.

$$\rho_{\mathrm{TE}} = \frac{\left|k_{\mathrm{sx}} - k_{\mathrm{cx}}\right|^2}{\left|k_{\mathrm{sx}} + k_{\mathrm{cx}}\right|^2} \qquad \rho_{\mathrm{TM}} = \frac{\left|k_{\mathrm{sx}}\varepsilon_{\mathrm{c}} - k_{\mathrm{cx}}\varepsilon_{\mathrm{s}}\right|^2}{\left|k_{\mathrm{sx}}\varepsilon_{\mathrm{c}} + k_{\mathrm{cx}}\varepsilon_{\mathrm{s}}\right|^2}$$

With the normal component of the wavevector $k_{\mathrm{cx}} = \frac{\omega}{c}\sqrt{n_{\mathrm{c}}^2 - n_{\mathrm{s}}^2 \sin^2\varphi_{\mathrm{I}}} = 0$ we can compute the smallest angle of incidence with $\rho_{\mathrm{TE,TM}} = 1$:

$$k_{\mathrm{cx}} = 0 \curvearrowright n_{\mathrm{c}} = n_{\mathrm{s}} \sin\varphi_{\mathrm{Itot}}$$

$$\boxed{\sin\varphi_{\mathrm{Itot}} = \frac{n_{\mathrm{c}}}{n_{\mathrm{s}}}.}$$

For angles of incidence larger than this critical angle, $\varphi_{\mathrm{I}} > \varphi_{\mathrm{Itot}}$ we have

$$k_{\mathrm{cx}} = \mathbf{i}\frac{\omega}{c}\sqrt{n_{\mathrm{s}}^2 \sin^2\varphi_{\mathrm{I}} - n_{\mathrm{c}}^2} = \mathbf{i}\mu_{\mathrm{c}} = \mathbf{i}\sqrt{k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{\mathrm{c}}} \curvearrowright \text{imaginary}$$

$$\rightarrow \Re\left(k_{\mathrm{cx}}\right) = 0 \rightarrow \text{TIR}$$

The intuition here is that above the critical angle, the continuous tangential component k_z exceeds the cladding wavenumber as determined by the corresponding dispersion relation. As such, only evanescent waves can be excited in the cladding and thus the incident wave is entirely reflected.

Obviously, we find the same critical angle of TIR for TE and TM polarization since here we have only considered a condition on the wavevector, not the polarizations. The energy fluxes are given as (here TE, same result for TM):

$$\rho_{\mathrm{TE}} = \frac{\left|k_{\mathrm{sx}} - \mathbf{i}\mu_{\mathrm{c}}\right|^2}{\left|k_{\mathrm{sx}} + \mathbf{i}\mu_{\mathrm{c}}\right|^2} = 1 \qquad \tau_{\mathrm{TE}} = \frac{4k_{\mathrm{sx}}\Re\left(k_{\mathrm{cx}}\right)}{\left|k_{\mathrm{sx}} + k_{\mathrm{cx}}\right|^2} = 0.$$

Remark

For metals in visible range (below the plasma frequency) we have always TIR, because: $\Re\left(\varepsilon_{\mathrm{c}}\right) < 0 \rightarrow k_{\mathrm{cx}} = \frac{\omega}{c}\sqrt{\varepsilon_{\mathrm{c}} - n_{\mathrm{s}}^2 \sin^2\varphi_{\mathrm{I}}}$ is always imaginary.

In the case of TIR the modulus of the coefficient of reflection is one, but the coefficient itself is complex → nontrivial phase shift for reflected light:

A) TE-polarization

As we found above, the normal component of the wave number in the cladding is purely imaginary when total internal reflection occurs. The expression for the reflection coefficient thus becomes:

$$R_{\mathrm{TE}} = 1\cdot\exp\left(\mathbf{i}\Theta_{\mathrm{TE}}\right) = \frac{k_{\mathrm{sx}} - \mathbf{i}\mu_{\mathrm{c}}}{k_{\mathrm{sx}} + \mathbf{i}\mu_{\mathrm{c}}} = \frac{Z}{Z^*} = \frac{\exp\left(\mathbf{i}\alpha\right)}{\exp\left(-\mathbf{i}\alpha\right)} = \exp\left(2\mathbf{i}\alpha\right)$$

$$\curvearrowright \tan\alpha = \tan\frac{\Theta_{\mathrm{TE}}}{2} = -\frac{\mu_{\mathrm{c}}}{k_{\mathrm{sx}}} = -\frac{\sqrt{n_{\mathrm{s}}^2\sin^2\varphi_{\mathrm{I}} - n_{\mathrm{c}}^2}}{n_{\mathrm{s}}\cos\varphi_{\mathrm{I}}} = -\frac{\sqrt{\sin^2\varphi_{\mathrm{I}} - \sin^2\varphi_{\mathrm{Itot}}}}{\cos\varphi_{\mathrm{I}}}.$$

B) TM-polarization

$$R_{\mathrm{TM}} = 1\cdot\exp\left(\mathbf{i}\Theta_{\mathrm{TM}}\right) = \frac{k_{\mathrm{sx}}\varepsilon_{\mathrm{c}} - \mathbf{i}\mu_{\mathrm{c}}\varepsilon_{\mathrm{s}}}{k_{\mathrm{sx}}\varepsilon_{\mathrm{c}} + \mathbf{i}\mu_{\mathrm{c}}\varepsilon_{\mathrm{s}}} = \frac{Z}{Z^*} = \frac{\exp\left(\mathbf{i}\alpha\right)}{\exp\left(-\mathbf{i}\alpha\right)} = \exp\left(2\mathbf{i}\alpha\right)$$

$$\curvearrowright \tan\alpha = \tan\frac{\Theta_{\mathrm{TM}}}{2} = -\frac{\mu_{\mathrm{c}}\varepsilon_{\mathrm{s}}}{k_{\mathrm{sx}}\varepsilon_{\mathrm{c}}} = \frac{\varepsilon_{\mathrm{s}}}{\varepsilon_{\mathrm{c}}}\tan\frac{\Theta_{\mathrm{TE}}}{2},$$

In conclusion, we have seen that the phase shifts of the reflected light at TIR is different for TE and TM polarization, and because $\varepsilon_s > \varepsilon_c$, the phase shift is larger than for the TM polarization

$$\left|\Theta_{\mathrm{TM}}\right| > \left|\Theta_{\mathrm{TE}}\right|,$$

As a consequence, incident linearly polarized light in general becomes elliptically polarized after TIR $\curvearrowright$ Fresnel prism

Remark

The field in the cladding is evanescent $\sim \exp\left(\mathbf{i}k_{xc}x\right) = \exp\left(-\mu_c x\right)$.

$\curvearrowright$ The averaged energy flux in the cladding normal to the interface vanishes.

$$\langle \mathbf{S} \rangle_x = \frac{1}{2\omega\mu_0}\Re\left(\mathbf{k}|\mathbf{E}|^2\right)_x = \frac{1}{2\omega\mu_0}\underbrace{\Re\left(k_x\right)}_{=0}|\mathbf{E}|^2 = 0.$$

8.3.2.3 The Brewster angle

There exists another special angle with interesting reflection properties. For TM polarization incident at the Brewster angle φ_B we find $R_{TM} = 0$:

$$\rho_{TM} = \frac{\left|k_{sx}\varepsilon_c - k_{cx}\varepsilon_s\right|^2}{\left|k_{sx}\varepsilon_c + k_{cx}\varepsilon_s\right|^2} = 0,$$

$$\rightarrow k_{sx}\varepsilon_c = k_{cx}\varepsilon_s$$

$$\varepsilon_c^2\left(\varepsilon_s - \sin^2\varphi_B\varepsilon_s\right) = \varepsilon_s^2\left(\varepsilon_c - \sin^2\varphi_B\varepsilon_s\right)$$

$$\sin^2\varphi_B = \frac{\varepsilon_s\varepsilon_c\left(\varepsilon_s - \varepsilon_c\right)}{\varepsilon_s\left(\varepsilon_s^2 - \varepsilon_c^2\right)} = \frac{\varepsilon_c}{\left(\varepsilon_s + \varepsilon_c\right)}$$

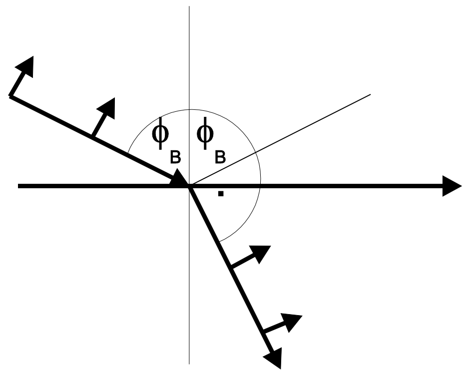
$$\cos^2\varphi_B = 1 - \sin^2\varphi_B = 1 - \frac{\varepsilon_c}{\left(\varepsilon_s + \varepsilon_c\right)} = \frac{\varepsilon_s}{\left(\varepsilon_s + \varepsilon_c\right)}$$

With the last two lines we can write the final result for the Brewster angle:

$$\boxed{\tan\varphi_B = \sqrt{\frac{\varepsilon_c}{\varepsilon_s}}.}$$

The Brewster angle exists only for TM polarization, but for any $n_s \lesssim n_c$.

There is a simple physical explanation for why there is no reflection from the interface at the Brewster angle.



$$\tan\varphi_{\mathrm{B}} = \frac{\sin\varphi_{\mathrm{B}}}{\cos\varphi_{\mathrm{B}}} = \frac{n_{\mathrm{c}}}{n_{\mathrm{s}}}$$

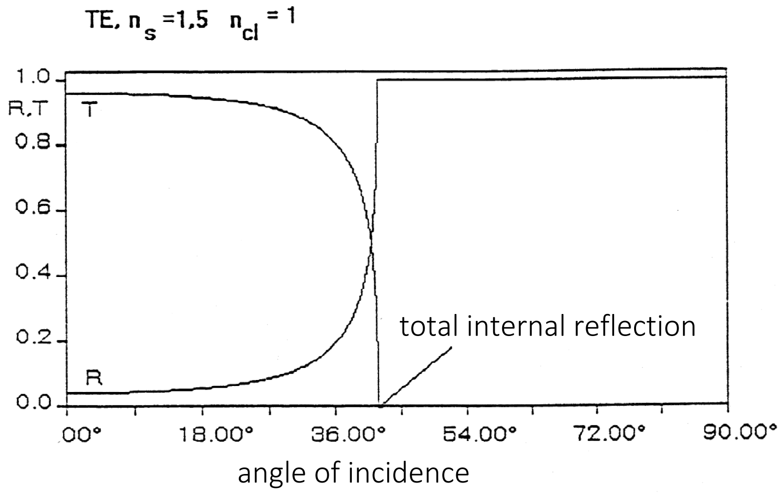
$$\curvearrowright n_{\mathrm{s}}\sin\varphi_{\mathrm{B}} = n_{\mathrm{c}}\cos\varphi_{\mathrm{B}} = n_{\mathrm{c}}\sin\left(\frac{\pi}{2} - \varphi_{\mathrm{B}}\right)$$

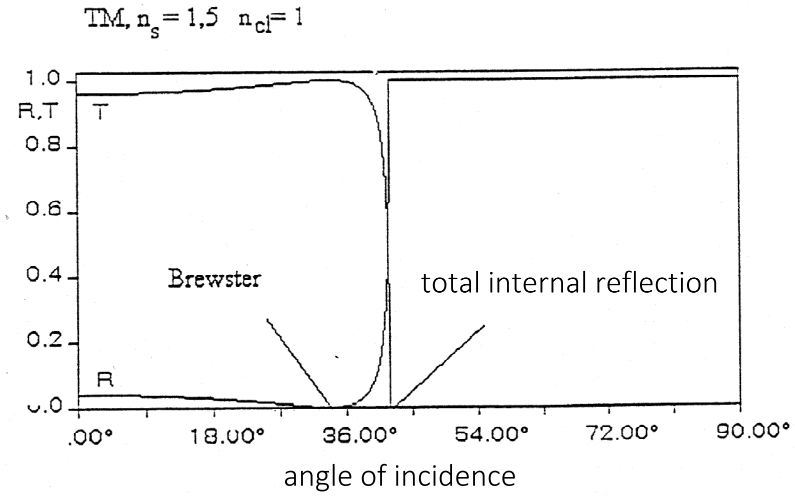
But from Snell's law, the angle of the transmitted light is always

$$n_{\mathrm{s}}\sin\varphi_{\mathrm{B}} = n_{\mathrm{c}}\sin\varphi_{\mathrm{T}} \curvearrowright \varphi_{\mathrm{T}} = \frac{\pi}{2} - \varphi_{\mathrm{B}},$$

Hence, at the Brewster angle, any reflected and transmitted waves would propagate in perpendicular directions. If we interpret any reflected light as an emission from oscillating dipoles in the cladding which are induced by the transmitted light, no reflected wave can occur for TM polarization since there can be no radiation in the direction of dipole oscillation.

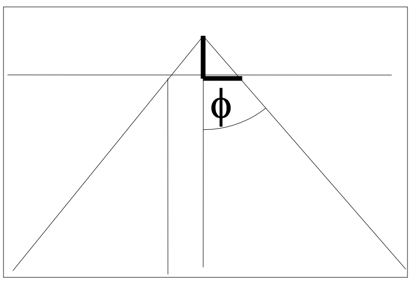
In summary, we have the following results for reflectivity and transmissivity at a single interface with $\varepsilon_s > \varepsilon_c$.





8.3.2.4 The Goos-Hänchen-Shift

The Goos-Hänchen shift is a direct consequence of the nontrivial phase shift of the reflected light during TIR. It occurs when beams undergo total internal reflection at an interface. The reflected beam appears to be shifted along the interface. As a result, it appears as if the beam penetrates the cladding and reflection occurs at a plane parallel to the interface at a certain depth, the so-called penetration depth. For the sake of simplicity, here we will treat TE-polarization only.



Let us start with an incident plane wave in TE polarization:

$$E_{\mathtt{I}}(x,z)=E_{\mathtt{I}}\exp\left[\mathtt{i}\left(k_{\mathtt{sx}}x+k_{\mathtt{z}}z\right)\right]\rightarrow E_{\mathtt{I}}(x,z)=E_{\mathtt{I}}\exp\left[\mathtt{i}\left(\alpha z+\gamma_{\mathtt{s}}x\right)\right]$$