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On the Existence of Steiner Triple Systems in 3-uniform Hypergraphs

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ABSTRACT

At the beginning of the last century, discrete mathematics and combinatorics established themselves as foundational areas of mathematics with numerous applications, especially in computer science and operations research. One particularly active field is extremal combinatorics and specifically extremal graph theory. There, one is typically interested in determining the *minimum density* (measured by the number of edges for example) to always force some structure inside the host graph.

In this thesis, we are interested in generalizing one of the most fundamental classical results in extremal graph theory: Dirac's theorem. Informally, if our graph has an even number of vertices, Dirac's theorem states that if every vertex has at least half of the vertices as neighbors, then the graph must contain a perfect matching. This result is tight and was one of the starting points of what are now called *Dirac-type results*.

As more and more problems have been resolved in the graph setting, interest naturally shifted to uniform hypergraphs. In light of Dirac's theorem, a lot of work has been devoted into determining the *minimum degree* or *minimum codegree threshold* of hypergraphs matchings, provided that the number of vertices satisfies the right divisibility conditions.

However, recently a new generalization was proposed: If the fundamental property of a perfect matching is that every vertex is covered by exactly one edge, why not consider the structure where every pair of vertices is covered by exactly one edge in the 3-uniform case? These objects are called *Steiner triple systems* and are a type of *combinatorial design*.

In this thesis, we will give a detailed overview on the first paper making progress in determining the minimum codegree threshold of Steiner triple systems. Furthermore, we will improve on the results by finding a better upper bound on the minimum codegree threshold for, essentially, the LP-relaxation of Steiner triple systems. In particular, our results imply that if n is sufficiently large and satisfies some necessary divisibility conditions, then a 3-uniform, n-vertex hypergraph H contains a Steiner triple system if every pair of vertices forms an edge in H with at least $0.858n$ other vertices.

CONTENTS

§ 1. INTRODUCTION

Extremal graph theory is one of the most important and active research areas of combinatorics. Classically, questions can be usually categorized into three types:

Turán-type problems ask how *dense*[1] a graph can be without containing a certain substructure. In other words, we want to know at what density the substructure always emerges. Meanwhile, *Dirac-type problems* study the conditions under which the existence of a spanning structure in the graph are guaranteed. Typically, as density purely given by the number of edges is generally not sufficient for these problems, minimum degree conditions are considered. Lastly, there are *decomposition problems*, where one is interested in conditions such that the edges of the graph can be partitioned into copies of a fixed substructure or family of substructures.

As a lot of progress for the graph case has been made, it is only natural to consider more generally *hypergraphs* with a special focus on *uniform* hypergraphs. However, it turns out that the problems become much harder in the hypergraph setting. For instance, while the *Turán-density* is completely determined for all graphs with the infamous Erdős-Stone theorem ([16]), it is still unknown what the exact Turán-density of the *tetrahedron*, the 3-uniform hypergraph on four vertices with four edges, is.

Following this line of research, we present some new results about Dirac-type problems in the 3-uniform hypergraph setting.

1.1 DIRAC'S THEOREM AND ITS GENERALIZATIONS

Arguably the most natural spanning structure is a Hamiltonian cycle. As determining whether a graph contains such a cycle is a $\mathcal{NP}$-complete problem (see [17, Thm. 3.4]), research is instead devoted to finding sufficient conditions for their existence. One of the first and seminal Dirac-type results, *Dirac's theorem*, gives such a sufficient condition.

Theorem 1.1 (Dirac 1952, [9])**.** Every graph with $n \geqslant 3$ vertices and minimum degree at least $n/2$ has a Hamiltonian cycle.

Note that the minimum degree condition is best possible: $K_{\lfloor n/2 \rfloor - 1, \lceil n/2 \rceil + 1}$ has minimum degree $\lfloor n/2 \rfloor - 1$ and is bipartite. Hence, every cycle in the graph contains equally many vertices of both partition classes. However, as the partition classes differ in size, there can't be a Hamiltonian cycle.

By taking every second edge in the Hamiltonian cycle, Dirac's theorem implies a sufficient minimum degree condition for the existence of perfect matchings under the necessary parity conditions.

Corollary 1.2. Every graph with $n \geqslant 2$ vertices, $n \equiv 0 \pmod 2$, and minimum degree at least $n/2$ has a perfect matching.

[1] In terms of the number of edges.

Again, the minimum degree condition is best possible by the aforementioned example.

In light of higher uniformities, the question how Dirac's theorem for Hamiltonian cycles should be generalized is not clear at first. Hence, there have been numerous results extending the theorem to so-called *loose* or *tight cycles*. Meanwhile, the extension for the perfect matching result seems more obvious. Indeed, a lot of results (see for instance [34, 43, 44, 19, 48, 29, 30, 23]) studied sufficient conditions for the existence of perfect hypergraph matchings, where every vertex is now covered by exactly one hyperedge.

Inspired by Linal's presentation on high-dimensional combinatorics (see [36]), Lee proposes a different generalization. Namely, going from 2-uniformity to 3-uniformity, if the primary property of a perfect matching is that every vertex[2] is covered by an edge, why not consider the structure where instead every pair of vertices[3] gets covered by a hyperedge?

1.2 LEE'S RESULTS AND CONJECTURE

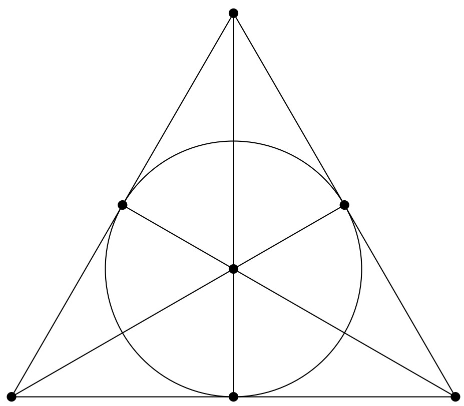

Figure 1.1: The Fano plane – the most prominent example of a Steiner triple system

These structures are known as *Steiner triple systems* and are of independent interest in the theory of *combinatorial designs*. For such a design to exist, the corresponding parameters must satisfy certain divisibility conditions.[4] For instance, for there to be a perfect matching on n vertices, n must be an even number. Similarly, it turns out that Steiner triple systems can only exist for $n \equiv 1, 3 \pmod 6$ and then have $n(n-1)/6$ edges.[5]

Generally speaking, it was clear that these divisibility conditions are necessary for a design to exist. It was a long-standing problem, however, whether these conditions were sufficient for the existence of designs. Only in recent years, using tools such as the *absorption method* or the *iterative absorption method* did Keevash (see [27]) and Glock, Kühn, Lo, Osthus (see [18]) independently gave an affirmative answer to that question if the parameters are sufficiently large.

With the *existence conjecture* now proven, interest shifted to proving the existence of designs – in our case a Steiner triple system – in more restrictive settings. Indeed, it turns out that both proofs of the existence conjecture already yield the existence of spanning

[2] Which one may consider a 0-dimensional object.

[3] Which one may consider 1-dimensional.

[4] See Fact 2.4.

[5] See Corollary 2.5 and Lemma 2.6.

Steiner triple systems in n-vertex, 3-uniform hypergraphs that have sufficiently high minimum codegree for sufficiently large n satisfying $n \equiv 1, 3 \pmod 6$. However, the minimum codegree condition given by both proofs are close to the maximum codegree possible, rendering them far from optimal.

In [35], Lee establishes a significant improvement on the *minimum codegree threshold* for Steiner triple systems in 3-uniform hypergraphs. In particular, he proves the following:

Theorem 1.3 (Lee 2023, [35, Thm. 1.2])**.** For any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that the following holds for every $n \geqslant n_0$ satisfying $n \equiv 1, 3 \pmod 6$: Let H be a 3-uniform hypergraph on n vertices. If the minimum codegree of H is at least

$$\left(\frac{3+\sqrt{57}}{12} + \varepsilon\right) n = (0.879\ldots + \varepsilon)\, n,$$

then H contains a Steiner triple system.

In fact, Lee proves in [35] a *transversal* or *rainbow* version of this result:

Theorem 1.4 (Lee 2023, [35, Thm. 1.6, Thm. 1.8])**.** For any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that the following holds for every $n \geqslant n_0$ satisfying $n \equiv 1, 3 \pmod 6$: Let $\mathcal{H} = \{H_1, \ldots, H_{n(n-1)/6}\}$ be a family of 3-uniform hypergraph on n vertices sharing the same vertex set V. If the minimum codegree of H_i is at least

$$\left(\frac{3+\sqrt{57}}{12} + \varepsilon\right) n = (0.879\ldots + \varepsilon)\, n$$

for all $i \in [n(n-1)/6]$, then there exists a *transversal* Steiner triple system S. Specifically, the vertex set of S is V and there exists a bijective function $\varphi \colon E(S) \longrightarrow [n(n-1)/6]$ such that $e \in E(H_{\varphi(e)})$ for all $e \in E(S)$.

Remark 1.5. Unlike with perfect hypergraph matchings (see [19]), no such condition can be established purely using the minimum degree. Indeed, the 3-uniform hypergraph H where all possible hypergraphs are present except for the ones containing a fixed pair p has minimum degree

$$\binom{v(H)-1}{2} - (v(H)-2) = (1+o(1))\binom{v(H)}{2},$$

but does not contain a Steiner triple system as there is no edge covering p.

Though Theorem 1.4 is a big improvement, Lee conjectures that the minimum codegree threshold for Steiner triple systems is actually $3n/4 + C$ for some constant C:

Conjecture 1.6 (Lee 2023, [35, Conj. 7.1])**.** There is a constant $n_0 \in \mathbb{N}$ and $C \in \mathbb{R}$ such that the following holds for all $n \geqslant n_0$ satisfying $n \equiv 1, 3 \pmod 6$: Let $\mathcal{H} = \{H_1, \ldots, H_{n(n-1)/6}\}$ be a family of 3-uniform hypergraph on n vertices sharing the same vertex set V. If the minimum codegree of H_i is at least $3n/4 + C$ for all $i \in [n(n-1)/6]$, then there exists a transversal Steiner triple system S.

This conjectured minimum codegree would be optimal as evident by the following construction:

Lemma 1.7 (Lee 2023, [35, Prop. 1.7])**.** For every $n \geqslant 3$ with $n \equiv 1,3 \pmod 6$, there is an n-vertex 3-uniform hypergraph H with

$$\frac{3n}{4} - \frac{7}{2} \leqslant \delta_2(H) \leqslant \Delta_2(H) \leqslant \frac{3n}{4} + \frac{3}{2}$$

that does not contain a Steiner triple system.

Apart from this construction, there are other reasons to believe that $(3/4 + o(1))n$ is at least asymptotically the correct minimum codegree threshold, which we will expand on in later chapters.

1.3 MAIN RESULTS

The goal of this thesis is twofold. First, after some preliminaries in Chapter 2, we will give a detailed overview of Lee's results and provide sketches to the main proofs in Chapter 3. Afterwards, in Chapter 4, we will quantitatively improve on Theorem 1.4 by improving estimates on the so-called *fractional threshold* for Steiner triple systems, a parameter on which Lee's proof heavily depends. In particular, we show:

Theorem 1.8. Let x^* be the unique root of the polynomial $p(x) = 8x^3 - 22x^2 + 10x - 1$ in $[0, 1/6]$. Then, Theorem 1.4 is true even if the minimum codegree of each H_i is at least

$$(1 - x^* + \varepsilon)\, n = (0.8578\ldots + \varepsilon)\, n.$$

Lastly, we propose some further directions, both in regards to solving Conjecture 1.6 but also possible generalizations and variations.

§ 2. PRELIMINARIES

2.1 TERMINOLOGY AND NOTATION

We expect that the reader has a decent background in graph theory. For all notation not defined here, we refer to [8] and note that all (hyper-)graphs considered are simple[1] and finite, unless explicitly stated otherwise. While some of the following notation is also adapted from [35], some of the conventions were changed.

- For integers $a, b \in \mathbb{Z}$, $a \leqslant b$, we let $[a, b] = \{n \in \mathbb{N}\colon a \leqslant n \leqslant b\}$.[2] In particular, we set $[n] = [1, n]$ for all $n \in \mathbb{N}$.
- For a set X and $k \in \mathbb{N}_0$, we let $X^{(k)} = \{Y \subseteq X\colon |Y| = k\}$.
- For a hypergraph H, we set $v(H) = |V(H)|$ and $e(H) = |E(H)|$.
- Let $q \in \mathbb{N}$. We call a hypergraph H q-*uniform* if $E(H) \subseteq V(H)^{(q)}$. For the remainder of this summary, let H always denote a q-uniform hypergraph.
- Given H as above, we will think of N_H as the function

$$\begin{aligned} \bigcup_{r=0}^{q} V(H)^{(r)} &\longrightarrow \bigcup_{r=0}^{q} \mathcal{P}\left(V(H)^{(r)}\right) \\ u &\longmapsto \{e \setminus u\colon u \subseteq e \in E(H)\} \end{aligned}$$

  and refer to $N_H(u)$ as the *neighborhood of* u. In other words, $N_H(u)$ is the set of $(q - |u|)$-subsets of $V(H)$ that form an edge with u.
- $\deg_H\colon \bigcup_{r=0}^{q} V(H)^{(r)} \longrightarrow \mathbb{N}_0$ is then given by $\deg_H(u) = |N_H(u)|$ for all $u \subseteq V(H)$ with $|u| \leqslant q$. Additionally, for $u \in V(H)$, we will refer to $\deg_H(u)$ as the *degree of* u, and for a pair $p \in V(H)^{(2)}$, we will refer to $\deg_H(p)$ as the *codegree of* p.
- As a shorthand, we also define

$$\delta_i(H) = \min_{u \in V^{(i)}} \deg_H(u), \qquad \Delta_i(H) = \max_{u \in V^{(i)}} \deg_H(u)$$

  for all $i \in [0, q]$ and refer to them as the *minimum* i*-degree* and *maximum* i*-degree of* H respectively. We will also refer to them as the *minimum* and *maximum degree* for the case $i = 1$ and, in the case $i = 2$, as the *minimum* and *maximum codegree* respectively.
- Additionally, as we are occasionally interested in the minimum i-degree among all i-sets covered by an edge, we define the *essential minimum* i*-degree* to be

$$\delta_i^{\mathrm{ess}}(H) = \min_{u \in V^{(i)}\colon \deg_H(u) > 0} \deg_H(u)$$

[1] Though all of the concepts below can be extended to multi-hypergraphs by using multi-sets.
[2] Whether $[a, b]$ denotes an integer interval or real interval will always be clear from context.

for all $i \in [0,q]$. Similar to before, we will refer to $\delta_i^{\mathrm{ess}}(H)$ as the *essential minimum degree* for $i = 1$ and *essential minimum codegree* for $i = 2$.

- Sometimes, for $i \in [0,q]$, it will also be useful to think of the i-degree restricted to a subset of vertices. Hence, we define for an i-set $p \in V(H)^{(i)}$ and subset $U \subseteq V(H)$

$$\deg_H(p; U) = \left| U^{(q-i)} \cap N_H(p) \right| .$$

- To understand the structure induced by the neighborhood of $u \subseteq V(H)$, $1 \leqslant |u| \leqslant q$, we define the *link of* u *(with respect to* H*)* to be the $(q - |u|)$-uniform graph $L(u)$ given by

$$V(L(u)) = V(H) \setminus u, \qquad E(L(u)) = \{e \setminus u \colon u \subseteq e \in E(H)\}.$$

- To "go down a uniformity", we define ∂H as the $(q-1)$-uniform hypergraph with

$$V(\partial H) = V(H), \qquad E(\partial H) = \left\{ e \in V^{(q-1)} \colon \deg_H(e) \geqslant 1 \right\}$$

  and refer to ∂H as the *shadow of* H.

- As another short hand, we set for $r \geqslant q$

$$\mathcal{K}_r(H) = \left\{ K \subseteq H \colon K \simeq K_r^{(q)} \right\} .$$

- Lastly, we say that H is *linear* if $\Delta_2(H) \leqslant 1$. In other words, hyperedges intersect at most in one vertex.

If the ambient hypergraph H is clear from context, we will also drop the subscript and write deg for example instead of $\deg_H$.

Arguably less standard is the notion of *weighted subhypergraphs* which are used in the proof of Theorem 1.4. Roughly speaking, by identifying subhypergraphs of a given hypergraph H via mappings of the type $E(H) \longrightarrow \{0,1\}$, weighted subhypergraphs are the generalization where fractional values are allowed for the mappings.

Definition 2.1 (Weighted subhypergraphs, [35, Sec. 2.2])**.** Let H be a q-uniform hypergraph. We call a function $\psi \colon E(H) \longrightarrow [0,1]$ a *weighted subhypergraph of* H. For $u \subseteq V(H)$, $|u| \leqslant q$, we define

$$\deg_H^{\psi}(u) = \sum_{u \subseteq e} \psi(e)$$

and refer to it as the *weighted* $|u|$*-degree of* u or simply the *weight of* u. Furthermore, for $i \in [0,q]$, we define $\delta_i^{\psi}(H)$ and $\Delta_i^{\psi}(H)$ analogously to $\delta_i(H)$ and $\Delta_i(H)$ by simply letting $\deg_H^{\psi}$ play the role of deg. Lastly, we set

$$\|\psi\|_1 = \sum_{e \in E(H)} \psi(e), \qquad \|\psi\|_\infty = \max_{e \in E(H)} \psi(e).$$

Remark 2.2. As before, if H is clear from context, we may drop the subscript for $\deg_H^{\psi}$. Similar conventions used for hypergraphs will also be employed for weighted subhypergraphs. Also, in a slight abuse of notation, we will use these conventions as long as the domain of ψ is a subset of $\mathbb{R}$.

2.2 COMBINATORIAL DESIGNS

In this section, we will define and show some basic properties of *combinatorial designs*. Around these objects, two distinct communities in combinatorics have formed. One considers them special types of set families, the other hypergraphs with particular properties. While these viewpoints are equivalent, different questions are of interest. For us, it is more natural to take a hypergraph point of view.

Definition 2.3 ((n, q, r, λ)-design)**.** Let $n \geqslant q > r \geqslant 1$, $\lambda \geqslant 1$ and H be an n-vertex, q-uniform hypergraph. We call H an (n, q, r, λ)-*design* if every r-subset of vertices is contained in exactly λ hyperedges. In other words, $\delta_r(H) = \Delta_r(H) = \lambda$. In the case of $\lambda = 1$, we will also refer to the given design as a *Steiner system* or more concretely an (n, q, r)-*Steiner system*.[3]

Note that perfect matchings in an n-vertex graph correspond to $(n, 2, 1, 1)$-designs. More generally, perfect matchings in an n-vertex, q-uniform hypergraph correspond to $(n, q, 1, 1)$-designs. Lee's question then deals with $(n, 3, 2)$-Steiner systems, which are also called *Steiner triple systems*. As with perfect matchings, the existence of a (n, q, r, λ)-design necessitates certain *divisibility conditions*.

Fact 2.4 (Divisibility conditions)**.** If there exists an (n, q, r, λ)-design, then we must have for all $0 \leqslant i \leqslant r - 1$

$$\binom{q-i}{r-i} \,\Big|\, \lambda\binom{n-i}{r-i}.$$

Proof. Let $H = (V, E)$ be a (n, q, r, λ)-design and fix some $I \in V^{(i)}$. We double count

$$\mathcal{S} = \left\{(R, e) \in V^{(r)} \times E \colon I \subseteq R \subseteq e\right\}.$$

On the one hand, note that there are $\binom{n-i}{r-i}$ ways to extend I to a set in $V^{(r)}$. As each of those r-sets is covered by exactly λ hyperedges, we get

$$|\mathcal{S}| = \lambda\binom{n-i}{r-i}.$$

On the other hand, for fixed $I \subseteq e \in E$, there are $\binom{q-i}{r-i}$ many r-sets R that satisfy $I \subseteq R \subseteq e$. Hence,

$$|\mathcal{S}| = \deg(I)\binom{q-i}{r-i}.$$

This concludes the proof. □

From the case $i = 0$, we are able to deduce the number of edges in a design:

Corollary 2.5. For $n \in \mathbb{N}$, $q > r \geqslant 1$ and $\lambda \geqslant 1$, an (n, q, r, λ)-design must have exactly $\lambda\binom{n}{r}/\binom{q}{r}$ edges. In particular, a Steiner triple system on n vertices has precisely $n(n-1)/6$ hyperedges.

Even before the existence conjecture was posed, Kirkman showed that the conjecture is true for Steiner triple systems:

[3] Classically, one would require $r \geqslant 2$ for a Steiner system.

Lemma 2.6 (Kirkman 1847, [31])**.** A Steiner triple system on $n \in \mathbb{N}$ vertices exists if and only if $n \equiv 1, 3 \pmod 6$.

We conclude with the following useful fact about Steiner triple systems:

Fact 2.7. Let H be a q-uniform hypergraph on n vertices where $n \geqslant q \geqslant 3$. Then, H is an $(n, q, q-1)$-Steiner system if and only if for every vertex $v \in V(H)$ the link $L(v)$ is an $(n-1, q-1, q-2)$-Steiner system. In particular, for $q = 3$, H is a Steiner triple system if and only if for every vertex v the link $L(v)$ is a perfect matching.

2.3 THE RESOLUTION OF THE EXISTENCE CONJECTURE

As we are interested in the existence of Steiner triple systems, which are particular combinatorial designs, it is helpful to look back at tools developed for the existence conjecture. Recall that by Fact 2.4, the parameters n, q, r, λ must necessarily satisfy certain divisibility conditions for an (n, q, r, λ)-design to exist. In the other direction, it was a longstanding problem to determine whether those divisibility conditions are also (almost always) *sufficient*:

Conjecture 2.8 (Existence conjecture 1800s)**.** Given $q > r \geqslant 1$ and $\lambda \geqslant 1$, there exist $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ where (n, q, r, λ) satisfy the conditions in Fact 2.4 there exists an (n, q, r, λ)-design.

Note that the existence conjecture is true for $q = 3$, $r = 2$ and $\lambda = 1$ by Lemma 2.6. The first general improvement for Steiner systems (i.e. $\lambda = 1$ and $r \geqslant 2$) was done by Rödl, showing that the approximate version of the existence conjecture (also known as the Erdős-Hanani conjecture [15]) is true:

Theorem 2.9 (Rödl 1985, [41])**.** Let $q > r \geqslant 1$ and $1 > \varepsilon > 0$.

Packing version: For all sufficiently large n, there exists a q-uniform hypergraph S with $e(S) \geqslant (1-\varepsilon)\binom{n}{r}/\binom{q}{r}$ edges such that $\deg(f) \leqslant 1$ for all $f \in V(S)^{(r)}$.

Covering version: For all sufficiently large n, there exists a q-uniform hypergraph S with $e(S) \leqslant (1+\varepsilon)\binom{n}{r}/\binom{q}{r}$ edges such that $\deg(f) \geqslant 1$ for all $f \in V(S)^{(r)}$.

One of the key ingredients to Rödl's proof was the following auxiliary hypergraph that reduced the existence conjecture to a question about the existence of a matching:

Definition 2.10 (H_{aux})**.** Let $q > r \geqslant 1$ and let H be a q-uniform hypergraph. The auxiliary hypergraph H_{aux} is defined by

$$V(H_{\mathrm{aux}}) = \left\{ f \in V(H)^{(r)} : \deg_H(f) > 0 \right\}, \quad E(H_{\mathrm{aux}}) = \{\{f \in V(H_{\mathrm{aux}}) : f \subseteq e\} : e \in E(H)\}.$$

Note that the definition of H_{aux} does depend on r, which we will nevertheless notationally surpress.[4]

It turns out that a Steiner system in H is equivalent to a perfect matching in H_{aux}:

Fact 2.11. Let $q > r \geqslant 1$ and let H be a q-uniform hypergraph with $\delta_r(H) > 0$. Then H contains an (n, q, r)-Steiner system if and only if H_{aux} contains a perfect matching.

[4] In fact, we will mostly deal with $q = 3$ and $r = 2$.

Proof. We show the following correspondence:

$$\begin{aligned} & S = \left(V(H), \left\{ \bigcup_{f \in e'} f \colon e' \in E(S') \right\} \right) \text{ is an } (n, q, r)\text{-Steiner system.} \\ \iff & S' = S_{\mathrm{aux}} \text{ is a perfect matching.} \end{aligned}$$

Note that, by construction, $S \subseteq H$ if $S' \subseteq H_{\mathrm{aux}}$ and vice versa.

"$\implies$": Let $S \subseteq H$ be an (n, q, r)-Steiner system, meaning that $\deg_S(f) = 1$ for any $f \in V(H)^{(r)}$. Let $e_f \in E(S)$ be the unique edge with $f \subseteq e_f$. Then the unique edge of S' covering f (as a vertex) in S' is

$$\left\{ f' \in V(H)^{(r)} \colon f' \subseteq e_f \right\} \in E(S').$$

Hence, S' is a perfect matching.

"$\impliedby$": Let $S' \subseteq H_{\mathrm{aux}}$ be a perfect matching in H_{aux}, meaning that every $f \in V(H_{\mathrm{aux}}) = V(H)^{(r)}$ is covered by exactly one edge e'_f. By construction, there must be some $e_f \in E(H)$ such that

$$e'_f = \left\{ f' \in V(H)^{(r)} \colon f' \subseteq e_f \right\} \implies e_f = \bigcup_{f' \in e'_f} f'.$$

Therefore, e'_f uniquely determines e_f, meaning that e_f is also unique for every f. In particular, $\deg_S(f) = 1$ for every $f \in V(H)^{(r)}$, making S an (n, q, r)-Steiner system.

This concludes the proof. □

Remark 2.12. "Packing" or "covering" versions of Fact 2.11 also hold and can be analogously proven.

Though the transformation from H to H_{aux} reduces the problem (for the packing version) to a matching problem, it is not a priori clear how finding a matching in H_{aux} should be easier. Indeed, one can verify that, simply by construction, the edges of H and H' are in correspondence to each other, meaning that H_{aux} is generally very sparse.

However, what we lose in density we gain in stronger structural properties: Consider the case $H = K_n^{(q)}$. Then, H_{aux} is $\binom{n-r}{q-r}$-regular and every pair $f_1 f_2$ has at most codegree

$$\binom{n - |f_1 \cup f_2|}{q - |f_1 \cup f_2|} \leqslant \binom{n - (r+1)}{q - (r+1)} \in o\left(\binom{n-r}{q-r} \right).$$

In other words, H_{aux} is regular and has a small codegree relative to its degree. More explicitly, each vertex sees the same amount of edges, and two vertices have few edges covering both of them. In light of these properties, one hopes to generate the matching by simply choosing the edges at random. However, it turns out that applying the probabilistic method naively, i.e. every edge are independently chosen with some fixed probability, is not sophisticated enough to obtain Theorem 2.9. Instead, Rödl proceeds as follows: Instead of trying to get the desired matching in one step, we probabilistically get a small matching and remove the covered vertices at each step.

As we don't remove too many vertices, the properties of H_{aux} (regularity and small codegree) can be approximately guaranteed for the remaining hypergraph, making iterating possible.

Both the method and this general problem setting have substantially impacted the field. The method, now widely known as *Rödl's nibble method*, has become one of the standard tools in extremal graph theory. Because the method gained wider attention through [40] by Pippenger and Spencer, we will also refer to the setting and similar results as being of *Pippenger-Spencer-type*. Meanwhile, finding matchings in hypergraphs with these properties has also sparked a large line of research. For one, Rödl's result was generalized as follows:

Theorem 2.13 (Pippenger 1980s)**.** For every $q \geqslant 2$ and $\varepsilon > 0$ there exists $\gamma > 0$ such that the following holds: If H is an q-uniform $(1 \pm \gamma)D$-regular hypergraph on n vertices with codegrees at most γD, then there is a matching in H covering all but at most εn vertices.

It is common that hypergraphs with these properties are called *pseudorandom*. Indeed, one can check that the *random, q-uniform hypergraph* $G^{(q)}(n, p)$[5] has those properties with high probability.

Pippenger's result has been generalized in several ways. One direction (see for example [40]) is to show that such hypergraphs can be nearly optimally decomposed into matchings, i.e. their chromatic number is $(1 + o(1))D$. Another direction establishes additional properties to the matching generated (see [2, 13]). Namely, under slightly stronger assumptions, recent work has dealt with finding almost-perfect matchings that are in some sense also *pseudorandom*, e.g. that among a large set of edges, the expected number of edges are also contained in the matching. Having the latter property will be crucial for the proof of Theorem 1.4, as it allows us to make certain guarantees for the almost-perfect matching (or, going back to H, a partial design).

Though Rödl was able to prove the approximate version of the existence conjecture, the question remained open for nearly another 30 years. In fact, no (n, q, r)-Steiner systems were known for $r > 5$ until Keevash in [27] and Glock, Kühn, Osthus, and Lo in [18] independently proved the conjecture:

Theorem 2.14 (Keevash 2014 / Glock, Kühn, Lo, Osthus 2016, [27, 18])**.** Conjecture 2.8 is true.

What both proofs have in common is that they rely on the so-called *absorption method*, which has also become a standard tool since its first explicit use in [42]. This method is used whenever the sought structure involves the whole hypergraph, e.g. finding spanning structures or say a perfect decomposition into matchings. Usually, especially in higher uniformities, the best way to solve these problems is by involving probabilistic or greedy arguments that may get us, say, a matching that covers all except for a small fraction of vertices. To overcome this, we try to find an *absorber*, a special structure that essentially is itself well-structured and stays that way even if it has to "absorb" some leftover. For instance, for finding a perfect matching, the absorber could be a set of vertices V' that can not only be covered by a perfect submatching, but such that the same is true for $V' \cup V''$ as long as V'' is sufficiently small and of suitable size.

The rough outline of the absorption method thus goes as follows:

[5]Meaning the hypergraph with vertex set $[n]$ where each edge $e \in [n]^{(q)}$ has probability p of being present, independent from the presence of other edges.

- Find an absorber A and set it aside.
- Without touching the absorber, find a partial structure S in the rest of the hypergraph, leaving some small leftover L.
- Use the absorption property of A to absorb L and construct with S the desired structure.

Apart from this strategy, Keevash's and Glock, Kühn, Osthus, and Lo's proof differ quite significantly. While the former additionally uses algebraic methods, the latter is purely combinatorial in nature. However, this was only achievable because instead of simply absorption, *iterative absorption* was used.

Iterative absorption, first used by Knox, Kühn, and Osthus in [32], can be thought of as applying the absorption method with more of a "Nibble mentality": Sometimes, finding a suitable absorber for a "one-step" absorption and arbitrary leftover is too difficult. In these situations, in addition to finding a partial structure to cover almost everything, we wish to restrict the location of the possible leftover. As a result, iterating this process, our final leftover is drastically smaller and lives in a set for which finding and placing an absorber at the beginning becomes a managable task. However, this requires that we know beforehand the set where our final leftover configuration will live. Hence, we fix the vertex set sequence, $V = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_l$, representing the location where our leftover lives after each iteration, at the beginning. This sequence is called a *vortex* and has the property that those U_i's will rapidly decrease in their size and the hypergraph restricted onto each U_i has roughly the same properties as the initial hypergraph. After setting aside the absorber, we then need to show a *cover down lemma*. This lemma should certify that, if only (almost all of) U_i remains to be covered, we can cover $U_i \setminus U_{i+1}$ while leaving U_{i+1} mostly uncovered. Using this lemma, we are then able to iteratively find a partial structure that covers everything except (almost all of) U_l. But U_l is small and known at the start of the procedure. Hence, it can be dealt with by the absorber.

2.4 THE NASH-WILLIAMS CONJECTURE

Related to both the existence conjecture and Conjecture 1.6 is the so-called *Nash-Williams conjecture*. The connection is not obvious at first glance as it deals with clique-decompositions of graphs. However, there is a natural transition from one problem to the other: For a Steiner system S with parameters n and $q > r \geqslant 1$, we can think of the r-subsets of $V(S)$ as elements of $K_n^{(r)}$ and each edge $e \in E(S)$ as a q-clique of $K_n^{(r)}$. The following fact thus follows:

Fact 2.15. Let $n \in \mathbb{N}$ and $q \geqslant r \geqslant 2$. There exists an (n, q, r)-Steiner system if and only if there exists a $K_q^{(r)}$-decomposition of $K_n^{(r)}$. Specifically, the latter means that we can partition the edges of $K_n^{(r)}$ using $K_q^{(r)}$-copies.

Hence, the existence of a Steiner triple system on n vertices is equivalent to the existence of a K_3-decomposition of K_n. Even before the existence conjecture was posed, Kirkman already showed that the divisibility conditions are sufficient, i.e. in "decomposition terms" that for every $n \equiv 1, 3 \pmod 6$ the complete graph K_n can be partitioned into triangles, see Lemma 2.6. Hence, one was interested in the structural properties G needs to satisfy to be K_3-decomposable or in general F-decomposable for some fixed graph F.

To keep things simple, we will focus on the K_3-setting. Two necessary properties G must have immediately come to mind: For one, as each K_3 has three edges, we must have $3 \mid e(G)$. Furthermore, as K_3 is 2-regular, G must be *even*, meaning that every vertex has even degree. We will refer to such candidate graphs as being K_3*-divisible*:

Definition 2.16 (K_3-divisible)**.** A graph G is called K_3*-divisible* if $3 \mid e(G)$ and G is even.

Clearly, those conditions can't be sufficient, as C_{3k} is for all $k \geqslant 2$ K_3-divisible but also K_3-free. In fact, determining whether a graph is K_3-decomposable is $\mathcal{NP}$-hard, see [10]. Hence, in the spirit of Dirac's theorem, the question became at what density (quantified by the minimum degree) K_3-divisibility and being K_3-decomposable coincide. This finally leads us to the famous Nash-Williams conjecture:

Conjecture 2.17 (Nash-Williams 1970, [37])**.** For all sufficiently large $n \in \mathbb{N}$ the following holds: If G is K_3-divisible and has minimum degree at least $3n/4$, then G is K_3-decomposable.

Remark 2.18. One indicator for the hardness of the conjecture is the fact that the extremal construction is not unique. Indeed, there are many extremal constructions showing that this minimum degree condition would be optimal. The standard one (as constructed for example in [4]) goes as follows: Given any $k \in \mathbb{N}$, consider vertex-disjoint K_{6k+3}-copies $F_1, \dots, F_4$. Furthermore, let F_5 be the complete bipartite graph with parts $V(F_1) \cup V(F_2)$ and $V(F_3) \cup V(F_4)$. Finally, let $G = F_1 \cup \cdots \cup F_5$. Clearly, $\Delta(G) = \delta(G) = 3v(G)/4 - 1 = 18k + 8$ as every vertex is adjacent to all vertices in two other K_{6k+3}-copies and all vertices of its own K_{6k+3}-copy except itself. Furthermore, we have

$$e(G) = 4\binom{6k+3}{2} + (12k+6)^2 \equiv 0 \pmod 3.$$

Hence, G is also K_3-divisible. However, every triangle of G contains at least one edge in $E(F_1) \cup \cdots \cup E(F_4)$. Since

$$2\,|E(F_1) \cup \cdots \cup E(F_4)| = 8\binom{6k+3}{2} < (12k+6)^2 = e(F_5),$$

G cannot be K_3-decomposable.

A lot of progress has been made towards solving the Nash-Williams conjecture, especially using the iterative absorption method. Asymptotically, the problem has been reduced to *fractional* K_3*-decompositions*, i.e. where we instead seek a function $w \colon \mathcal{K}_3(G) \longrightarrow [0,1]$ such that $\sum_{T \in \mathcal{K}_3(G) \colon e \subseteq V(T)} w(T) = 1$ for every edge $e \in E(G)$.

Theorem 2.19 (Barber, Glock, Kühn, Lo, Montgomery, Osthus 2020, [4])**.** Let δ^* be the infimum of $\delta \in [0,1]$ satisfying the following: There exists $n_0 \in \mathbb{N}$ such that every graph G on $n \geqslant n_0$ vertices with minimum degree δn has a fractional K_3-decomposition.

Then the following holds: For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all K_3-divisible graphs G on $n \geqslant n_0$ vertices with minimum degree $(\max\{3/4, \delta^*\} + \varepsilon)n$ has a K_3-decomposition.

The proof utilizes the iterative absorption method and Lee follows their approach closely for Theorem 1.4. This makes sense given that, interpreting the hyperedges of our host graph H as triangles, we are in some sense looking for a K_3-decomposition of ∂H. This motivates the following definition:

Definition 2.20 (K_3-decomposition (Hypergraph version))**.** Let G be a graph and H be a 3-uniform hypergraph. We call H a K_3*-decomposition of* G if H is a linear hypergraph and $\partial H = G$.

Remark 2.21. Note that H is required to be linear as every pair / edge of G should be covered exactly once. Furthermore, if G is complete, then a K_3-decomposition H of G is a Steiner triple system. Thus, K_3-decompositions naturally generalize Steiner triple systems.

Of course, despite their similarities, Lee's setting deviates from the setting of the Nash-Williams conjecture in significant ways: While we gain more control with the minimum codegree than simply the minimum degree, not every triangle in the shadow corresponds to a hyperedge of our hypergraph. These and other obstacles and how to overcome them will be discussed in the next chapter.

§ 3. OVERVIEW OF LEE'S RESULTS

In this chapter, we will discuss in more detail the results from [35]. First, we will prove Lemma 1.7 which shows that the minimum codegree condition in Conjecture 1.6 would be optimal. Next, we will give an outline of the proof of Theorem 1.4. This will be of expository nature with a focus on the ideas involved rather than the quantitative details. In fact, to show the essence of the ideas involved, we will restrict ourselves to the non-transversal version, i.e. Theorem 1.3.

3.1 LOWER BOUND CONSTRUCTION

Together with Conjecture 1.6, Lee gave a corresponding lower bound construction, showing that the conjectured minimum codegree condition would be tight up to an additive constant. Intriguingly enough, it turns out that in the construction the codegree is actually roughly equal to $3n/4$ for every pair.

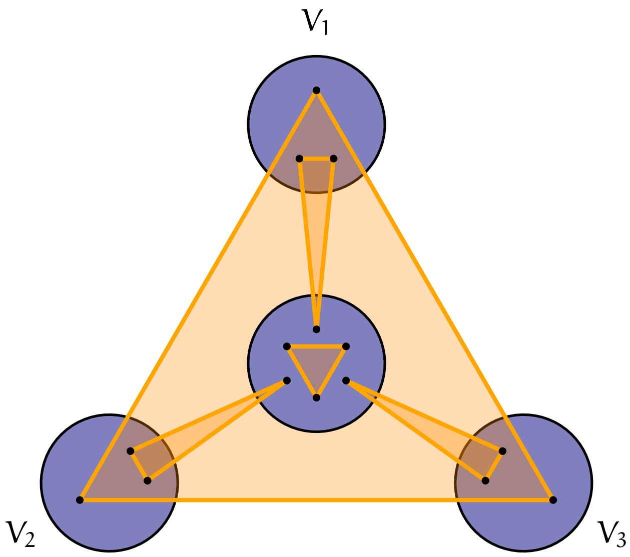

Figure 3.1: Lee's construction with the missing edges drawn in

Proof of Lemma 1.7. Let $n \in \mathbb{N}$ with $n \equiv 1, 3 \pmod 6$ be given. We construct a hypergraph H with vertex set $V = [n]$: Partition $V = V_0 \cup V_1 \cup V_2 \cup V_3$ such that

- $n/4 - 3/2 \leqslant |V_i| \leqslant n/4 + 3/2$ for all $i \in [0, 3]$, and
- $|V_0|$ is even and $|V_1|, |V_2|, |V_3|$ are odd.

Such a partition does exist: Start with a partition $V_0, \dots, V_3$ such that the parity conditions are met. Suppose that $\min_{i\in[0,3]} |V_i| < n/4 - 3/2$ and let $i = \arg\min_{i\in[0,3]} |V_i|$. By the pigeonhole principle, there must be some $j \in [0,3]$, $i \neq j$, such that

$$\left|V_j\right| \geqslant \frac{\frac{3n}{4} + \frac{3}{2}}{3} = \frac{n}{4} + \frac{1}{2}.$$

Take two elements from V_j and insert them into V_i. The new partition W_0, W_1, W_2, W_3 satisfies the parity conditions, $|W_i| \geqslant |W_i| + 2$ and $\left|W_j\right| \geqslant n/4 - 3/2$. By iterating this argument as long as $\min_{i\in[0,3]} |W_i| < n/4 - 3/2$, we finally arrive at a partition $\tilde{V}_0, \dots, \tilde{V}_3$ satisfying the parity conditions such that $\min_{i\in[0,3]} |\tilde{V}_i| \geqslant n/4 - 3/2$. By applying a similar argument, one can additionally guarantee the upper bound.

Finally, let $E = E_0 \uplus E_1 \uplus E_2 \uplus E_3$ and $H = (V, E)$, where

- $E_0 = \left\{e \in V^{(3)} \colon |e \cap V_0| = 2\right\}$,
- $E_1 = \bigcup_{1\leqslant i<j\leqslant 3} \left\{e \in V^{(3)} \colon |e \cap V_0| = |e \cap V_i| = \left|e \cap V_j\right| = 1\right\}$,
- $E_2 = \bigcup_{i\in[3]} V_i^{(3)}$, and
- $E_3 = \bigcup_{i,j\in[3]\colon i\neq j} \left\{e \in V^{(3)} \colon |e \cap V_i| = 1 \text{ and } \left|e \cap V_j\right| = 2\right\}$.

In other words, $e \in V^{(3)}$ is not in E if and only if

- $e \subseteq V_0$,
- $|e \cap V_0| = 1$ and $|e \cap V_i| = 2$ for some $i \in [3]$, or
- $|e \cap V_1| = |e \cap V_2| = |e \cap V_3| = 1$.

We first show the bounds for $\delta_2(H)$ and $\Delta_2(H)$: Let $u, v \in V$ be distinct vertices.

Case 1: $u, v \in V_0$. By construction, $e \in V^{(3)} \setminus E$ covers $\{u, v\}$ if and only if $e \in V_0^{(3)}$. Hence,

$$\deg(u, v) = (n-2) - |V_0 \setminus \{u, v\}| = n - |V_0| \in \left[\frac{3n}{4} - \frac{3}{2}, \frac{3n}{4} + \frac{3}{2}\right].$$

Case 2: $u \in V_0, v \in V_i$ for some $i \in [3]$. By construction, $e \in V^{(3)} \setminus E$ covers $\{u, v\}$ if and only if $|e \cap V_i| = 2$. Thus,

$$\deg(u, v) = (n-2) - |V_i \setminus \{v\}| = n - 1 - |V_i| \in \left[\frac{3n}{4} - \frac{5}{2}, \frac{3n}{4} + \frac{1}{2}\right].$$

Case 3: $u, v \in V_i$ for some $i \in [3]$. By construction, $e \in V^{(3)} \setminus E$ covers $\{u, v\}$ if and only if $|e \cap V_0| = 1$. Therefore,

$$\deg(u, v) = (n-2) - |V_0| \in \left[\frac{3n}{4} - \frac{7}{2}, \frac{3n}{4} - \frac{1}{2}\right].$$

Case 4: $u \in V_i, v \in V_j$ and $1 \leqslant i < j \leqslant 3$. By construction, $e \in V^{(3)} \setminus E$ covers $\{u, v\}$ if and only if $|e \cap V_k| = 1$ for $k \in [3] \setminus \{i, j\}$. In particular, we get

$$\deg(u, v) = (n-2) - |V_k| \in \left[\frac{3n}{4} - \frac{7}{2}, \frac{3n}{4} - \frac{1}{2}\right].$$

Lastly, we show that H doesn't contain a Steiner triple system: Assume that there exists a Steiner triple system $S \subseteq H$. Set $E_i' = E(S) \cap E_i$ for all $0 \leqslant i \leqslant 3$.

1. E_1' covers exactly $|V_0|(n - 2|V_0| + 1)$ pairs in $\{e \in V^{(2)} : |e \cap V_0| = 1\}$. Indeed, the pairs in $V_0^{(2)}$ can only be covered by edges in E_0'. So, as each edge in E_0' covers precisely one pair in $V_0^{(2)}$, we have
$$|E_0'| = \binom{|V_0|}{2}.$$
Furthermore, every edge in E_0' also covers two pairs in $\{e \in V^{(2)} : |e \cap V_0| = 1\}$. Hence, out of the $|V_0|(|V| - |V_0|)$ total pairs, $|V_0|(|V_0| - 1)$ are covered by E_0', leaving the remainder to be covered by E_1'.
2. E_3' covers
$$(|V_1||V_2| + |V_1||V_3| + |V_2||V_3|) - \frac{|V_0|(n - 2|V_0| + 1)}{2}$$
pairs in $\bigcup_{1 \leqslant i < j \leqslant 3} \{uv : u \in V_i, v \in V_j\}$: Each edge in E_1 covers two pairs in $\{e \in V^{(2)} : |e \cap V_0| = 1\}$ and one pair in $\bigcup_{1 \leqslant i < j \leqslant 3} \{uv : u \in V_i, v \in V_j\}$. Thus, $|E_1'| = |V_0|(n - 2|V_0| + 1)/2$ and
$$(|V_1||V_2| + |V_1||V_3| + |V_2||V_3|) - |E_1'|$$
pairs of that type must be covered by E_3'.
3. S cannot exist: Per edge in E_3', two pairs in $\bigcup_{1 \leqslant i < j \leqslant 3} \{uv : u \in V_i, v \in V_j\}$ get covered. In particular, this means that
$$(|V_1||V_2| + |V_1||V_3| + |V_2||V_3|) - \frac{|V_0|(n - 2|V_0| + 1)}{2}$$
is even. However, as n is odd, $|V_0|$ even and the other partition classes odd, we get
$$(|V_1||V_2| + |V_1||V_3| + |V_2||V_3|) - \frac{|V_0|(n - 2|V_0| + 1)}{2} \equiv 1 + 1 + 1 - 0 \equiv 1 \pmod 2. \lightning$$

Thus, H doesn't contain a Steiner triple system, which shows the claim. □

Remark 3.1. Interestingly enough, the argument ultimately boils down to a *parity barrier*: We get a contradiction due to some quantity being supposedly both odd and even.

This suggests that the minimum codegree threshold of finding a partial Steiner triple system[1] or a fractional Steiner triple system θ^*_{STS} (see Definition 3.4) are not just trivially at most θ_{STS} (see Definition 3.2) from the fact that Steiner triple systems are in particular partial and fractional Steiner triple systems, but actually smaller than that. In fact, it is possible to show that for $n \geqslant 21$ the construction above always contains a fractional Steiner triple system, see Proposition A.1.

[1]As in a linear subhypergraph of our host hypergraph covering all but at most $o(n^2)$ pairs of vertices.

3.2 OUTLINE OF LEE'S MAIN THEOREM

As all presented results are asymptotic, we will formally define the *minimum codegree threshold* for Steiner triple systems as follows:

Definition 3.2 (θ_{STS})**.** Let θ_{STS} be the infimum over all $\delta \in [0,1]$ for which there exists $n_0 \in \mathbb{N}$ such that for every 3-uniform hypergraph H on $n \geqslant n_0$ vertices with $\delta_2(H) \geqslant \delta n$ and $n \equiv 1, 3 \pmod 6$ contains a Steiner triple system. We will refer to θ_{STS} as the *minimum codegree threshold (for Steiner triple systems)*.

Following [4], specifically the proof of Theorem 2.19, Lee basically reduces the problem to its fractional relaxation:

Definition 3.3 (Perfect fractional Steiner triple systems)**.** Let H be a 3-uniform hypergraph. A weighted subhypergraph $\psi\colon E(H) \longrightarrow [0,1]$ is a *perfect fractional Steiner triple system in* H if every pair $p \in E(\partial H)$ satisfies $\deg^{\psi}(p) = 1$.

If the host graph H is obvious from the context, we will simply refer to w as a perfect fractional Steiner triple system or just fractional Steiner triple system.[2] Analogous to Steiner triple systems, we may also consider the minimum codegree threshold for fractional Steiner triple systems, which leads to the definition of $\theta^{f}_{\mathrm{STS}}$ and $\theta^{*}_{\mathrm{STS}}$.

Definition 3.4 ($\theta^{f}_{\mathrm{STS}}$, $\theta^{*}_{\mathrm{STS}}$, [35, Def. 1.5])**.** We define the function $\theta^{f}_{\mathrm{STS}}\colon [0,1) \longrightarrow [0,1]$ as follows: Let $\theta^{f}_{\mathrm{STS}}(\varepsilon)$ be the infimum over all $\delta \in [0,1]$ for which there exists $n_0 \in \mathbb{N}$ such that for every 3-uniform hypergraph H on $n \geqslant n_0$ vertices with $\delta(\partial H) \geqslant (1-\varepsilon)(n-1)$ and $\delta_2^{\mathrm{ess}}(H) \geqslant \delta(n-2)$ contains a perfect fractional Steiner triple system. Furthermore, let $\theta^{*}_{\mathrm{STS}} = \lim_{\varepsilon \downarrow 0} \theta^{f}_{\mathrm{STS}}(\varepsilon)$. We refer to $\theta^{*}_{\mathrm{STS}}$ as the *fractional threshold*.

Remark 3.5. While $\theta^{f}_{\mathrm{STS}}$ is obviously monotonically decreasing and thus $\theta^{*}_{\mathrm{STS}} \geqslant \theta^{f}_{\mathrm{STS}}(0)$, it is not clear whether equality holds. We also note that, in the definition of $\theta^{f}_{\mathrm{STS}}$, ∂H not necessarily being complete is necessary for Lee's proof to go through. Furthermore, while dealing with nearly identical structures, computing $\theta^{*}_{\mathrm{STS}}$ should, in theory, be a much simpler feat than computing θ_{STS} directly. Indeed, on the one hand, finding a fractional Steiner triple system inside a 3-uniform hypergraphs reduces to a linear programming problem that is solvable in polynomial time. Meanwhile, by referring to a result by Dor and Tarsi in [10], finding a Steiner triple system inside a 3-uniform hypergraph is $\mathcal{NP}$-complete.

The proof of Theorem 1.3 then boils down into two steps: Reducing the problem of determining θ_{STS} to $\theta^{*}_{\mathrm{STS}}$ and then estimating $\theta^{*}_{\mathrm{STS}}$.

Theorem 3.6 (Lee 2023, [35, Simplification of Thm. 1.6])**.** $\theta_{\mathrm{STS}} = \max\left\{\theta^{*}_{\mathrm{STS}}, 3/4\right\}$.

Theorem 3.7 (Lee 2023, [35, Thm. 1.8])**.** $\theta^{f}_{\mathrm{STS}}(\varepsilon) \leqslant (3+\sqrt{57})/12 < 0.88$ for all $\varepsilon \in [0,1)$.

As we will improve on the latter result in Chapter 4, we will only focus on giving a proof sketch of Theorem 3.6. As with all proofs involving the iterative absorption method, this can be again broken down into

- constructing a vortex, (Lemma 3.9)
- constructing and embedding an absorber, (Lemma 3.12 & Lemma 3.13)
- proving a corresponding cover down lemma. (Lemma 3.19)

[2] Here, we deviate slightly from the naming convention established in [35].

3.2.1 *Constructing the vortex*

One of the main structures needed to apply the iterative absorption method is the so-called vortex. As discussed, the vortex is a sequence $V = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_l$ of vertex sets that rapidly decrease in their size such that the hypergraph induced by U_i has roughly the same properties as our host graph. This allows us with the cover down lemma to iteratively cover almost all of the vertices except some leftover in U_l which the absorber then takes care of.

Essentially, the hypergraphs we deal with in Theorem 1.4 have (apart from the divisibility conditions) one main property, a minimum codegree of at least

$$(\max\{\theta^*_{\mathrm{STS}}, 3/4\} + o(1))\, v(H).$$

Hence, we define the vortex as follows.

Definition 3.8 ((α, ε, m)-vortex, [35, Simplification of Def. 5.2])**.** Let $\alpha, \varepsilon \in \mathbb{R}_{>0}$ and $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Furthermore, let H be a 3-uniform hypergraph on n vertices. An (α, ε, m)-*vortex in* H is a sequence $U_0 \supseteq U_1 \cdots \supseteq U_l$ such that

- $U_0 = V(H)$,
- $|U_i| = \lfloor \varepsilon |U_{i-1}| \rfloor$ for all $i \in [l]$,
- $|U_l| = m$,
- $\deg_H(p; U_i) \geqslant \alpha |U_i|$ for all $i \in [l]$ and $p \in U^{(2)}_{i-1}$.

In other words, the size of the vertex sets shrinks by a factor of ε each time, and every pair in U_{i-1} should have the right codegree with respect to U_i. Since $U_{i-1} \supseteq U_i$, this in particular implies that the minimum codegree of $H[U_i]$ is, as desired, at least an α-fraction of $|U_i|$.

In a fairly standard manner, by choosing U_i randomly from U_{i-1} and applying Theorem B.3, Lee proves the following lemma:

Lemma 3.9 (Lee 2023, [35, Simplification of Lem. 5.3])**.** Let $m, n \in \mathbb{N}$ be integers such that

$$0 < 1/n \ll 1/m' \ll \alpha, \varepsilon < 1.$$

Moreover, let H be a 3-uniform hypergraph on n vertices with $\delta_2(H) \geqslant (\alpha + \varepsilon)n$. Then H has an (α, ε, m)-vortex for some $\varepsilon m' < m \leqslant m'$.

3.2.2 *Constructing (exclusive) absorbers*

In this subsection, we will discuss the construction of the absorber. To briefly summarize, after iteratively applying the cover down lemma / Lemma 3.19, all the pairs that are not contained in the last subset of the vortex U_l, where $|U_l|$ is of fixed size (in comparison to the order n of our hypergraph). Because of the initial parity conditions on n, $n \equiv 1, 3 \pmod 6$, the uncovered pairs induce a K_3-divisible graph.

Indeed, if $n \equiv 1, 3 \pmod 6$, then we have that the total number of pairs is divisible by 3 since

$$\binom{n}{2} = \frac{n-1}{2} \cdot n \equiv \begin{Bmatrix} 0 \cdot n, & n \equiv 1 \pmod 6 \\ \frac{n-1}{2} \cdot 0, & n \equiv 3 \pmod 6 \end{Bmatrix} \equiv 0 \pmod 3.$$

Furthermore, as the uncovered pairs initially induce a K_n, which is even since n is odd, and each added hyperedge covers exactly 3 pairs, the number of uncovered pairs in the end is also a multiple of 3 and together they induce an even graph.

To cover those remaining pairs encoded in this graph, we consider all possibilities G, $V(G) = U_l$, of this resulting graph and embed an *exclusive absorber* beforehand for each of these possibilities. As n is chosen sufficiently large after the size[3] of $|U_l|$ is already fixed, meaning that there are only $\mathcal{O}(\exp(|U_l|^2))$ many possibilities for G, it is possible to embed all those exclusive absorbers "vertex-disjointly" in our hypergraph, thus giving us our desired absorber. Of course, generally, two exclusive absorbers A, A' can't be exactly vertex-disjoint, especially if, say, they serve as absorbers of possible leftovers G and G' respectively, where $G \neq G'$ but $V(G) \cap V(G') \neq \emptyset$. Instead what we precisely mean is that $(V(A) \setminus V(G)) \cap (V(A') \setminus V(G')) = \emptyset$.

To guarantee that the vertices of A and A', which are not part of the leftover, are in the "outermost layer" of our vortex, $U_0 \setminus U_1$, we also require $V(A) \setminus V(G) \subseteq U_0 \setminus U_1$ and $V(A') \setminus V(G') \subseteq U_0 \setminus U_1$. Having this kind of control concerning how the exclusive absorbers are embedded will be necessary to formulate the cover down lemma.

For the absorber, and thus our exclusive absorbers A for G, to truly be effective and also embeddable in our host hypergraph, we need to satisfy the following conditions:

- A should be *sufficiently sparse* to guarantee embeddability. This will be formalized with the notion of *G-rooted edge-degeneracy*.
- A should be K_3-*disjoint* from G, meaning $E(A) \cap \mathcal{K}_3(G) = \emptyset$.[4] This property guarantees us that we do not need to rely on the hyperedges induced by U_l, simplifying our embedding scheme.
- T should be an induced subgraph of the shadow of A, i.e. $\partial A[V(G)] \simeq G$. In short, this avoids making potential mistakes by covering a pair in U_l twice.
- A should contain K_3-decompositions for ∂A and $\partial A \setminus G$ respectively. In other words, A should contain a hypergraph which only covers pairs in $\partial A \setminus G$ exactly once in the case that G is not the "graph of leftover pairs", and another one which covers all pairs in ∂A exactly once if G is the "leftover".

In short, ignoring sparsity, we define an absorber as follows:

Definition 3.10 (K_3-absorber, [35, Def. 4.4]). Let G be a graph. We say a 3-uniform hypergraph A is a K_3-*absorber for* G if A and G are K_3-disjoint, G is induced in ∂A and both ∂A and $\partial A \setminus G$ have a K_3-decomposition in A.

To formalize sparsity, we will use the following notion:

Definition 3.11 (G-rooted edge-degeneracy, [35, Def. 4.5]). Let H be a 3-uniform hypergraph and let G be an induced subgraph of ∂H. The G-*rooted edge-degeneracy* of H is the smallest $d \in \mathbb{N}_0$ such that there is an ordering $v_1, v_2, \dots$ of $V(H) \setminus V(G)$ such that for all $i = 1, 2, \dots$ the following holds:

$$\left|\left\{\{u, w\} \subseteq \left(V(G) \cup \{v_j : 1 \leqslant j < i\}\right) : uv_iw \in E(H)\right\}\right| \leqslant d.$$

Essentially, this naturally generalizes the notion of edge-degeneracy for graphs. Informally, H has G-rooted edge-degeneracy d if, following that order, each new vertex induces at most d hyperedges with the previous vertices, including the vertices in G.

[3]More precisely, an upper bound on that size.

[4]Here, we identify hyperedges with their spanned triangle.

To give a taste in what way this notion of degeneracy can be used for embedability, we show in larger generality the following:

Lemma 3.12 (Embeddability of hypergraphs with G-rooted edge-degeneracy d)**.** Let H be a 3-uniform hypergraph and G be a graph with $V(G) \subseteq V(H)$. Furthermore, let A be a 3-uniform hypergraph with G-rooted edge-degeneracy at most d. If

$$\delta_2(H) \geqslant \left(1 - \frac{1}{d}\right) \cdot \nu(H) + \frac{\nu(A)}{d}$$

for all $p \in V(H)^{(2)}$, then A is embeddable into H such that the embedding $\tau\colon V(A) \longrightarrow V(H)$ satisfies $\tau(\nu) = \nu$ for all $\nu \in V(G)$.

Proof. W.l.o.g. we may assume that $V(H) \cap V(A) = V(G)$. Now, to construct the embedding τ, let $\nu_1, \nu_2, \ldots$ be the ordering of $V(A) \setminus V(G)$ witnessing that A has G-rooted edge-degeneracy at most d. Since $\tau(\nu) = \nu$ for all $\nu \in V(G)$, we focus on embedding the vertices in $V(A) \setminus V(G)$.

For that, we proceed inductively, extending a partial embedding of $A[V(G) \cup \{\nu_1, \ldots, \nu_{i-1}\}]$ to an embedding of $A[V(G) \cup \{\nu_1, \ldots, \nu_i\}]$ for all $1 \leqslant i \leqslant |V(A) \setminus V(G)|$: Let $\tau\colon V(G) \cup \{\nu_1, \ldots, \nu_{i-1}\} \longrightarrow V(H)$ be the partial embedding. Furthermore, let $p_1, \ldots, p_{d'} \in (V(G) \cup \{\nu_1, \ldots, \nu_{i-1}\})^{(2)}$, $0 \leqslant d' \leqslant d$, be the pairs of vertices that induce edges in H with ν_i. W.l.o.g. we may assume that $d' = d$. Consider the common neighbors of the corresponding pairs, i.e.

$$N = N_H(\tau(p_1)) \cap \cdots \cap N_H(\tau(p_d)).$$

To show that we can extend our embedding, it suffices to show that $|N| \geqslant \nu(A)$ as $N \setminus (\tau(V(G) \cup \{\nu_1, \ldots, \nu_{i-1}\}))$ would consequently be non-empty. Finally, observe that

$$\begin{aligned}
|N_H(\tau(p_1)) \cap N_H(\tau(p_2))| &= |N_H(\tau(p_1))| + |N_H(\tau(p_2))| - |N_H(\tau(p_1)) \cup N_H(\tau(p_2))| \\
&\geqslant |N_H(\tau(p_1))| + |N_H(\tau(p_2))| - |\nu(H)| \\
&\geqslant \left(1 - \frac{2}{d}\right) \cdot \nu(H) + \frac{2\nu(A)}{d} \\
&\vdots \\
|N| &= |N_H(\tau(p_1)) \cap \cdots \cap N_H(\tau(p_{d-1}))| + |N_H(\tau(p_d))| \\
&\quad - |(N_H(\tau(p_1)) \cap \cdots \cap N_H(\tau(p_{d-1}))) \cup N_H(\tau(p_d))| \\
&\geqslant \left(1 - \frac{d-1}{d}\right) \cdot |V(H)| + \frac{(d-1)\,|V(A)|}{d} \\
&\quad + \left(1 - \frac{1}{d}\right) \cdot |V(H)| + \frac{|V(A)|}{d} - |V(H)| \\
&\geqslant |V(A)|,
\end{aligned}$$

concluding the proof. □

Following the absorber construction in [4], Lee achieves the following:

Lemma 3.13 (Lee 2023, [35, Lem. 4.6])**.** For every K_3-divisible graph G, there exists a K_3-absorber A for G whose G-rooted edge-degeneracy is at most 4.

By Lemma 3.12, we get that the K_3-absorber can be embedded in a hypergraph as given in Theorem 1.3.

The basic idea behind Lemma 3.13 is to "*transform* a given leftover [. . .] into a new leftover" ([4]). Unsurprisingly, the structures enabling us to do so are called *transformers*. As absorbers basically transform the given leftover to an empty leftover, the definition for a transformer naturally goes as follows:

Definition 3.14 ((S, S')-absorber, [35, Def. 4.7])**.** Let S and S' be two vertex-disjoint K_3-divisible graphs. A 3-uniform hypergraph T is a *transformer of* (S, S') or (S, S')*-transformer* if

- T and $S \cup S'$ are K_3-disjoint,
- ∂T contains $S \cup S'$ as an induced subgraph, and
- both $\partial T \setminus S$ and $\partial T \setminus S'$ have a K_3-decomposition in T.

What is very convenient about transformers is that you can "stack" them.

Fact 3.15. Let S, S', and S'' be three vertex-disjoint K_3-divisible graphs such that there exists an (S, S')-transformer and an (S', S'')-transformer. Then there exists an (S, S'')-transformer.

Indeed, let T be the (S, S')-transformer and T' the (S', S'')-transformer. W.l.o.g. we may assume that

- T and S'' are vertex-disjoint,
- T' and S are vertex-disjoint, and
- $V(T) \cap V(T') = V(S')$.

Then, taking the union $T \cup T'$ yields an (S, S'')-transformer.

Hence, we wish to transform our given leftover step by step into a leftover for which finding a becomes obvious. That "final" leftover in Lemma 3.13 will be the disjoint union of $m = e(G)/3$ K_3's, denoted by mK_3, which obviously can be absorbed by a hypergraph matching of size $e(G)/3$. The proof of Lemma 3.13 breaks down into the following steps:

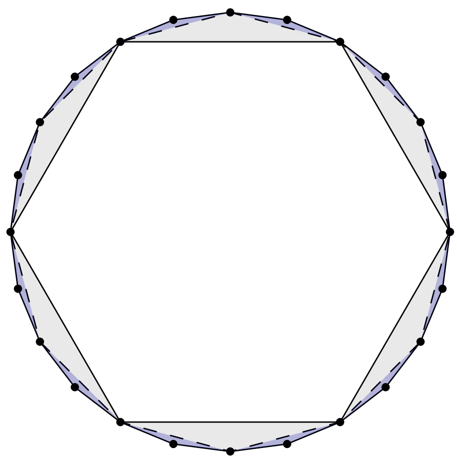

Figure 3.2: $T = T' \cup T''$ in Lemma 3.16 for $G = C_6$ with $\partial T' = \partial T \setminus G^{**}$ and $\partial T'' = \partial T \setminus G$

Lemma 3.16. Let F^* denote the 1-subdivision of a graph F. For every K_3-divisible graph G there exists a hypergraph T with G-rooted edge-degeneracy 1 that, ignoring $V(G) \cap V(G^*) \neq \emptyset$, *acts*[5] like a (G, G^{**})-transformer.

Proof. Convert G to a 3-uniform hypergraph T' by adding to each edge of G its own private vertex. Clearly, $\partial T' = G \cup G^*$. Now, apply the same procedure to G^* and obtain the 3-uniform hypergraph T''. Clearly, $\partial T'' = G^* \cup G^{**}$.

Now, let $T = T' \cup T''$. One can check that T acts like a (G, G^{**})-transformer, where T' and T'' are the K_3-decompositions for $\partial T \setminus G^{**}$ and $\partial T \setminus G$ respectively. Furthermore, the ordering where we first place the vertices in $T' \setminus V(G)$ and then the vertices $T'' \setminus V(G')$ witnesses that T has G-rooted edge-degeneracy at most 1. $\square$

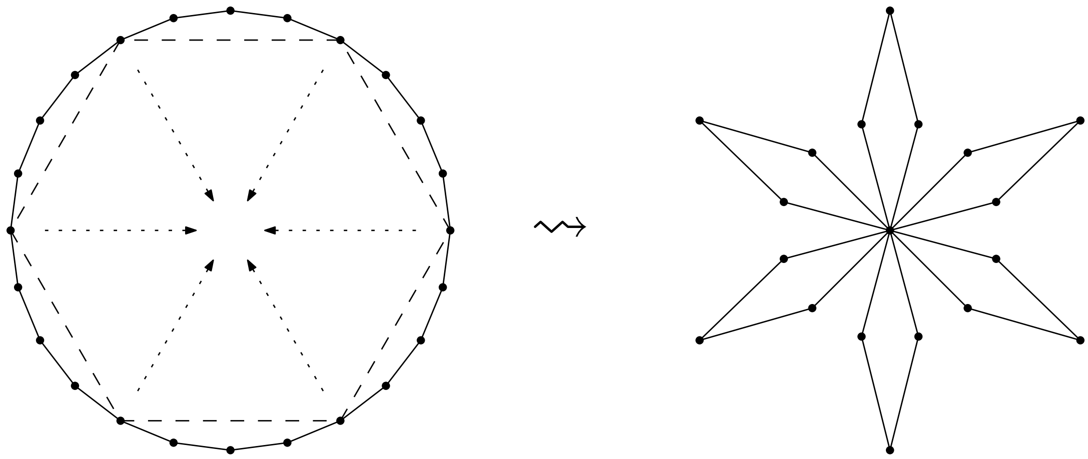

Figure 3.3: $C_6^{**} \rightsquigarrow S_6$

Though not obvious at first, this turns out to be a huge simplification. Namely, by identifying all the vertices of $V(G)$ in G^{**}, the resulting graph is the union of $e(G)$ C_4's. We will denote this graph as $S_{e(G)}$. Note that if there exists a $(G^{**}, S_{e(G)})$-transformer of small G^{**}-rooted edge-degeneracy, then we are done: We could then transform G into G^{**} and G^{**} into $S_{e(G)}$. Additionally, since an (S, S')-transformer is also an (S', S)-transformer, we could transform $S_{e(G)}$ into $(mK_3)^{**}$ and $(mK_3)^{**}$ into mK_3 where $m = e(G)/3$. The latter can then be absorbed by a hypergraph matching.

To construct a $(G^{**}, S_{e(G)})$-transformer, Lee proves in greater generality that if there exists an *edge-bijective graph homomorphism* between vertex-disjoint, K_3-divisible graphs S, S', then an (S, S')-transformer with $(S \cup S')$-rooted edge-degeneracy at most 4 exists. Conceptually, the existence of such a homomorphism means that we can construct S' starting from S by iteratively identifying non-adjacent vertices that don't share neighbors.

Lemma 3.17 ([35, Lem. 4.8])**.** Let (S, S') be a pair of vertex-disjoint K_3-divisible graphs for which there exists an edge-bijective homomorphism $\varphi\colon V(S) \longrightarrow V(S')$. Then there exists an (S, S')-transformer T which has the $(S \cup S')$-rooted edge-degeneracy at most 4.

The crucial property used to prove this lemma is the following fact:

Fact 3.18. Every even graph can be decomposed into edge-disjoint cycles.

[5]In the sense that it has all the properties of a transformer.

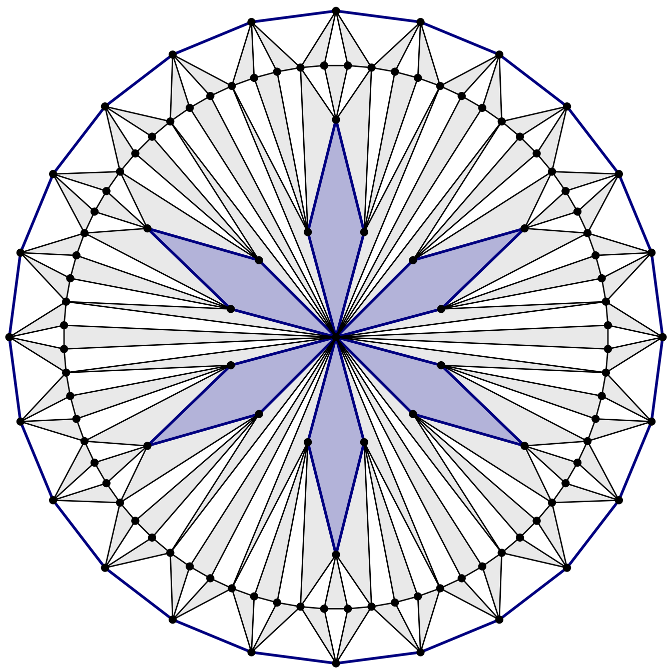

Figure 3.4: A (C_6^{**}, S_6)-transformer with $C_6^{**} \cup S_6$-rooted edge-degeneracy at most 4

Hence, it suffices to show the lemma for cycles for which an explicit construction is given, see Figure 3.4 for example.

As every transformer (or transformer acting structure) constructed has rooted edge-degeneracy at most 4, it is not hard to show that the resulting K_3-absorber for the leftover G has G-rooted edge-degeneracy at most 4, concluding the proof of Lemma 3.13.

3.2.3 *Cover down lemma*

To finish up Theorem 1.3, we need to show a cover down lemma. Let H denote the given hypergraph as in Theorem 1.3. Given the vortex $V(H) = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_l$, the cover down lemma should not only allow us to cover all pairs incident to some vertex in $U_0 \setminus U_1$ using a partial Steiner triple system, but we wish to do so in a way that preserves the structure well enough to be able to iterate, i.e., the first cover down step was suitable enough that we can apply the cover down lemma again to cover all remaining pairs incident to a vertex in $U_1 \setminus U_2$ and so on.

For that, we recall under what conditions the cover down lemma is first applied: After the construction of the vortex, the next step of iterative absorption is embedding the absorber A, which is really just a suitable union of exclusive absorbers. To "set aside" A, let $H' = H \setminus E(A)$. As we embed the vertices of the absorber that are not in U_l into $U_0 \setminus U_1$, certain pairs are already gonna be covered, regardless of which K_3-decomposition per exclusive absorber we take to complete the Steiner triple system. These pairs are precisely the ones in $\partial A \setminus U_l^{(2)}$. Hence, to avoid overlap, we delete all the edges containing such pairs from H'. The resulting hypergraph H'' therefore now has minimum codegree zero with the graph $G = \partial H''$ induced by the uncovered pairs not forming a complete graph. To compensate, we use the essential minimum codegree which is still at least $\left(\max\left\{\theta^*_{\mathrm{STS}}, 3/4\right\} + o(1)\right)|U_0|$. Additionally, every pair $p \in E(\partial H'')$

satisfies

$$\deg_{H''}(p, U_1) = (\max\{\theta^*_{STS}, 3/4\} + o(1)) |U_1|$$

by the properties of our vortex. Furthermore, as $n \gg |U_l|$, we only "lost" few pairs, which in turn means that the graph G still has a high minimum degree of $(1-o(1))n$ and is complete on U_1. Due to the structure of the vortex, every vertex also has $(1-o(1))$ of all vertices in U_1 as neighbors. Note that this is exactly the type of setting under which θ^f_{STS} guarantees a perfect fractional Steiner triple system.

Hence, as

$$\lim_{\varepsilon\downarrow 0} (\max\{\theta^*_{STS}, 3/4\} + \varepsilon) = \max\{\theta^*_{STS}, 3/4\} = \lim_{\varepsilon\downarrow 0} \left(\max\left\{\theta^f_{STS}(3\varepsilon), 3/4\right\} + 10\varepsilon\right),$$

it suffices by continuity to prove the following:

Lemma 3.19 (Cover down lemma, [35, Simplification of Lem. 5.4])**.** Let $0 < 1/n \ll \varepsilon < 1$ and let H be a 3-uniform hypergraph with vertex set V where $|V| = n$ with

$$\delta_2^{ess}(H) \geqslant \left(\max\left\{\theta^f_{STS}(3\varepsilon), 3/4\right\} + 10\varepsilon\right) n.$$

Furthermore, let $U' \subseteq U \subseteq V$ be subsets such that $|U'| = \lfloor \varepsilon\, |U| \rfloor$ and $|U| = \lfloor \varepsilon\, |V| \rfloor$. Additionally, the shadow $G = \partial H$ is K_3-divisible and satisfies the following:

- $\delta(G) \geqslant (1-\varepsilon)n$;
- $\deg_G(v; U) \geqslant (1-2\varepsilon)\, |U|$ for all $v \in V$;
- $G[U]$ is complete.

Lastly, we assume that for every $p \in E(G)$

$$\deg_H(p; U) \geqslant \left(\max\left\{\theta^f_{STS}(3\varepsilon), 3/4\right\} + 9\varepsilon\right) |U|.$$

Then there exists a 3-uniform, linear hypergraph $T \subseteq H$ on V that satisfies the following:

- $\delta((G \setminus \partial T)[U]) \geqslant (1-\varepsilon)\, |U|$,
- $\deg_{(G\setminus\partial T)[U]}(u; U') \geqslant (1-2\varepsilon)\, |U'|$ for all $u \in U$,
- $(G \setminus G[U]) \subseteq \partial T \subseteq (G \setminus G[U'])$.

In particular, $G \setminus \partial T$ is complete on U'.

Remark 3.20. Metaphorically, we may think of the vertices in V as being in *different layers*:

- We say that a vertex is *in the outer layer* (short: *outer vertex*) if it is in $V \setminus U$.
- We say that a vertex is *in the middle layer* (short: *middle vertex*) if it is in $U \setminus U'$.
- We say that a vertex is *in the inner layer* (short: *inner vertex*) if it is in U'.

Using our new terminology, the goal is that all pairs involving outer vertices are covered without covering any pairs between inner vertices.

As common with absorption, we need a *reservoir*. The goal of the reservoir, which is just a family of carefully chosen sets of vertices, is similar to that of the (global) absorber: Cover all the leftover pairs. Since we are the most flexible with the middle vertices, we will place the reservoir in the middle layer. For concreteness, let us enumerate the vertices $V = \{v_1, \ldots, v_n\}$ such that $\{v_1, \ldots, v_{n'}\} = V \setminus U$ for some $\mathbb{N} \ni n' \leqslant n$. To handle the leftover pairs incident to v_i for each outer vertex v_i, we want to choose some $A_i \subseteq U \setminus U'$ for our reservoir.

Before going into more detail, let us informally first review the basic steps:

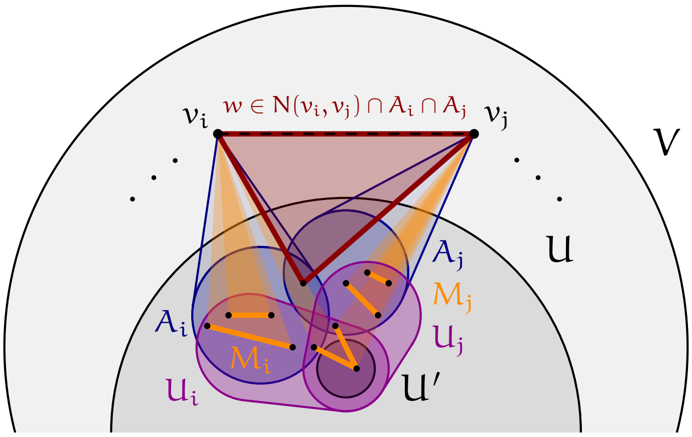

Figure 3.5: Sketch of how the reservoir is used to deal with the leftover

- After finding and setting aside the reservoir, we establish using the definition of θ^f_{STS} that there is a highly pseudorandom fractional Steiner triple system ψ in H, meaning that no edge in H gets a high value by that fractional Steiner triple system.
- By representing the weights of ψ by multi-edges, we obtain a 3-uniform, multi-hypergraph $\hat{H}$, whose *simplification*[6] is a subhypergraph of H. By the properties of ψ, $\hat{H}$ roughly covers each pair of vertices by the same number of edges and per triple $V(H)^{(3)}$, there are only few multi-edges representing that triple.
- This means that $\hat{H}_{aux}$ is a pseudorandom hypergraph, where we can apply Pippenger-Spencer-type results to get a large matching M in $\hat{H}_{aux}$.
- In fact, we can guarantee that this matching will be pseudorandom in the sense that among a large set of vertices, the *expected* number of vertices are contained in M. We apply this to $\{p \in V(\hat{H}_{aux}) : v \in p\}$ for all outer vertices $v \in V \setminus U$, which in particular gives us that almost all except for $o(n)$ many pairs incident to v are covered.
- Now, we use the reservoir: For every uncovered pair $v_i v_j$ between two outer vertices, we greedily select a vertex in $N(v_i, v_j) \cap A_i \cap A_j$.
- After that, only a few pairs between outer vertices and non-outer vertices remain to be covered. For that, consider for each outer vertex v_i the set of vertices $u \in U$ such that uv_i is still not covered. Denote this set by U_i. Inspired by Fact 2.7, we consider the auxiliary graph F_i with vertex set U_i and

$$E(F_i) = \left\{ uw \in U_i^{(2)} \setminus U'^{(2)} : uv_i w \in E(H) \right\}.$$

  Note that uw as above is always uncovered since $G[U]$ is complete and all pairs covered until that point are incident to an outer vertex.

  Since the pairs we covered until now form a K_3-divisible graph and G is K_3-divisible by assumption, it will be the case that $|U_i|$ is even. Hence, it remains to show that F_i has for every outer vertex v_i a perfect matching M_i. This can be achieved by using the resevoir, which in particular will guarantee for every outer vertex v_i and for every $u \in U_i$ that a good fraction of vertices $w \in A_i$ form an edge $uv_i w \in E(H)$.

[6] I.e. where all multi-edges are removed.

This corresponds to a high minimum degree in F_i such that Dirac's theorem for matchings applies. However, we require that $E(M_i) \cap E(M_j) = \emptyset$ for all outer vertices $v_i \neq v_j$ to avoid overlaps. To overcome this issue, we use that the A_i's don't overlap too much so that disjointness can be guaranteed by a slightly more complicated generalisation of Dirac's theorem.

We will now proceed with a more detailed rundown of the argument. First, we show that a reservoir exists:

Proposition 3.21 ([35, Simplification of Clm. 1 in proof of Lem. 5.4])**.** Let H, n, ε, and $U' \subseteq U \subseteq V$ be given as in Lemma 3.19. Let $\mu = \varepsilon^{10}$. There exists subsets $A_1, \dots, A_{n'} \in U \setminus U'$ such that for all $1 \leqslant i < j \leqslant n'$ the following holds:

(a) $A_i \subseteq N_G(v_i)$,

(b) $|A_i| \geqslant \frac{\mu|U|}{2}$,

(c) $|A_i \cap A_j| \leqslant 2\mu^2 |U|$,

(d) $|N_H(u, v_i) \cap A_i| \geqslant \frac{2|A_i|}{3}$ for all $u \in U$,

(e) $|N_H(v_i, v_j) \cap A_i \cap A_j| \geqslant \frac{\mu^2|U|}{8}$ if $v_i v_j \in E(G)$, and

(f) each $u \in U$ is contained in A_i for at most $2\mu n$ numbers $i \in [n']$.

Remark 3.22. Condition (a) is necessary to ensure that we don't cover a pair twice. Condition (e) will be used to cover leftover pairs between two outer vertices. Conditions (b), (c), (d) and (f) are used to get that family of edge-disjoint perfect matchings at the end, see Lemma 3.28. Concerning the proof of this proposition, it is a straightforward application of probabilistic methods. Lastly, in the original statement, there is one additional condition (g) that becomes relevant in the transversal version.

Recall that we are primarily interest in covering pairs in $E(G)$ incident to the outer vertices. The graph formed by these pairs is precisely $G \setminus G[U]$. To "set aside" the reservoir as well, we want to only consider pairs in

$$G' = G \setminus (R \cup G[U]),$$

where $R = \{av_i : i \in [n'], a \in A_i\}$. In H, this means that we first consider the subhypergraph $H' \subseteq H$ where we keep precisely the edges corresponding to triangles in G'.

We wish to now construct the linear hypergraph covering for each outer vertex v almost all of its incident pairs: H' still satisfies the conditions of θ^f_{STS}, meaning that it contains a fractional Steiner triple system. However, as indicated, we can guarantee more. Namely, we can exploit that the set of fractional Steiner triple systems are closed under convex combinations. Using this observation, Lee proves the following:

Lemma 3.23 (Lee 2023, [35, Lem. 4.1])**.** For every real number $\varepsilon > 0$, there is $n_0 = n_0(\varepsilon)$ such that the following holds for all $n \geqslant n_0$: Let H be an n-vertex, 3-uniform hypergraph with

$$\delta_2^{ess}(H) \geqslant \left(\theta^f_{STS}(\varepsilon) + \varepsilon\right) n$$

and $\delta(\partial H) \geqslant (1 - \varepsilon)n$. Then there is a fractional Steiner triple system ψ of H such that $\|\psi\|_\infty \leqslant \log^2(n)/n$.

In other words, what this lemma implies is that there is a fractional Steiner triple system where the contribution of each edge is not too large.

Remark 3.24 (Comparison to Theorem 2.19)**.** Recall that Lee's proof of Theorem 1.3 follows closely the proof of Theorem 2.19 concerning Conjecture 2.17. However, this step of the cover down lemma is where Lee deviates the most from that proof: Due to a theorem by Haxell and Rödl in [20], it directly follows that if a graph is dense enough (quantified using the minimum degree) to have a fractional K_3-decomposition (in the graph-sense), then that same density suffices to guarantee a partial K_3-decomposition, i.e. a K_3-packing that misses $o(n^2)$ edges. However, no such analogue theorem is known for the setting of Theorem 1.3. As such, Lee's approach to prove the lemma above is novel and may be of independent interest in the future.

Applying Lemma 3.23 to H', we obtain this fractional Steiner triple system ψ. In H'_{aux} ψ corresponds to a *perfect fractional matching*, i.e. where the weight each vertex gets is exactly one. Additionally, as the maximum weight assigned to an edge by ψ is at most $\log^2(n)/n$, the weighted codegree of this fractional perfect matching is at most $\log^2(n)/n$.

As a next step, we wish to obtain a pseudorandom multi-hypergraph F that is, ignoring the multiplicities of edges, a subhypergraph of H'_{aux}. If we get to this situation, we can apply Pippenger-Spencer-type results to obtain the large matching. Going back to H', this then corresponds to a linear hypergraph that covers almost all pairs in G'.

To reach our goal, we need to somehow get away from the fractional weights. In some sense, we wish to represent the weight of an edge by the multiplicity it gets in F. Hence, scaling up ψ by some appropriate factor in $\mathrm{poly}(n)$, we apply the following lemma:

Lemma 3.25 ([35, Lem. 5.5])**.** Let $k \in \mathbb{N}$ and $D, d, \tau, \delta \in \mathbb{R}_{\geqslant 0}$. Let H be a k-uniform hypergraph that contains a (d, τ, δ)*-pseudorandom weighted subgraph* $\psi \colon E(H) \longrightarrow \mathbb{R}_{\geqslant 0}$, i.e. $\deg^{\psi}(v) = (1 \pm \tau)d$ for all $v \in V(H)$ and $\Delta_2^{\psi}(H) \leqslant \delta d$. Furthermore, assume that $\Delta(H) \leqslant \tau D$. Then there exists a multi-hypergraph F such that the simplification of F F_{simp} is a subhypergraph of H and F is an $(D, 2\tau, \delta)$-pseudorandom multi-hypergraph, i.e. where $\deg_F(v) = (1 \pm \tau)D$ for all $v \in V(H)$ and $\Delta_2(F) \leqslant \delta D$.

Remark 3.26. We note that the definition of degree and codegree are naturally extended to multi-hypergraphs by accounting for the multiplicities of the edges. Furthermore, the proof of this lemma basically goes by replacing each edge $e \in E(H)$ by multiple edges with multiplicity $\lfloor D\psi(e)/d \rfloor$.

Given F, we want to now construct a matching that leaves *few* incident pairs uncovered per outer vertex. More formally, we want for every outer vertex v_i that all but at most $o(n)$ many pairs in $E_{v_i} = \{v_i w \colon w \in N_{G'}(v_i)\} \subseteq V(H'_{\text{aux}})$ get covered. This can be achieved using the following lemma:

Lemma 3.27 ([26, Simplification of Thm. 7.1])**.** Let $k > 3$ be an integer, $D, \tau, \delta, \mu, \varepsilon, \gamma, K \in \mathbb{R}_{>0}$ and $0 < \varepsilon < (k-2)/(k-1)$. Then there exists $n_0 = n_0(k, K, \gamma, \mu, \varepsilon)$ such that the following holds for every $n \geqslant n_0$: Let H be a k-uniform, n-vertex, (D, τ, δ)-pseudorandom multi-hypergraph. Assume $D \geqslant \exp(\log^{\mu}(n))$, $\delta \leqslant D^{-\gamma}$, and $\tau \leqslant K\delta^{1-\varepsilon}$. Let $\mathcal{F} \subseteq \mathcal{P}(V(H))$ be a collection of vertex subsets of $V(H)$ such that $|F| \geqslant \delta^{-1/2}\log(n)$ for all $F \in \mathcal{F}$ and $|\mathcal{F}| \leqslant \exp(\log^{4/3} n)$. Then H has a $(\mathcal{F}, \delta^{1/(k-1)})$*-pseudorandom* matching M: $|F \setminus V(M)| \leqslant \delta^{1/(k-1)}\,|\mathcal{F}|$ for every $F \in \mathcal{F}$.

Note that, while the uniformity k needs to be greater than 3 here, since Lee's proof (of Theorem 1.4) also involves colors and thus works with 4-uniform multi-hypergraphs, we can still apply Lemma 3.27. Hence, by taking $E_{v_1}, \dots, E_{v_{n'}} \in \mathcal{F}$, appropriate choices for the constants, and some (higher uniformity version of) F, we get a matching M in H'_{aux} such that $E_{v_i} \setminus V(M) \leqslant (\varepsilon n)^{3/4} \in o(n)$.

Transferred back to H', M corresponds to a linear hypergraph T_1 such that the degree of every outer vertex in $G'' = G' \setminus \partial T_1$ is at most $(\varepsilon n)^{3/4} \ll \varepsilon^{20} |U| / 8$. At this point, we first deal with the uncovered pairs between two outer vertices by using the reservoir: For every uncovered pair of vertices $v_i v_j \in E(G''[V \setminus U])$, we greedily choose $w \in N_H(v_i, v_j) \cap A_i \cap A_j$. This is possible due to property (e) in Proposition 3.21 and $(\varepsilon n)^{3/4} \ll \varepsilon^{20} |U| / 8$.

Let T_2 be the corresponding linear hypergraph induced by those edges. The uncovered pairs which we still wish to cover are precisely the ones in $R' = (R \cup G'') \setminus \partial T_2$. For outer vertices v_i, let $A'_i = N_{R'}(v_i)$. Due to the properties of T_1 and the construction of T_2, we have that

$$|A_i| - (\varepsilon n)^{3/4} \leqslant |A'_i| \leqslant |A_i| + (\varepsilon n)^{3/4}. \tag{$\star$}$$

Furthermore, note that $|A'_i|$ is even: Indeed, we have by assumption that G is K_3-divisible. In particular, $\deg_G(v_i)$ is even. As both ∂T_1 and ∂T_2 are K_3-divisible as well and edge-disjoint, we thus get that

$$\deg_{G \setminus \partial(T_1 \cup T_2)}(v_i) = \deg_{R'}(v_i) = |A'_i|$$

is even. Hence, define for every $i \in [n']$ the auxiliary graph

$$F_i = \left\{ uw \in A_i'^{(2)} \setminus U'^{(2)} \colon u v_i w \in E(H) \right\} = (G \setminus G[U'])[A'_i].$$

It suffices to show that there are edge-disjoint perfect matchings $M_i \subseteq F_i$. Indeed, by adding v_i to the edges of M_i, we get that

$$T_3 = \left(V(H), \left\{ u v_i w \colon uw \in M_i, i \in [n'] \right\}\right) \subseteq H$$

is a linear hypergraph with $R' \subseteq \partial T_3$. By construction, it is clear that T_1, T_2, and T_3 are edge-disjoint, linear hypergraphs such that $G \setminus G[U] \subseteq \partial T$ where $T = T_1 \cup T_2 \cup T_3 \subseteq H$. It is also clear by construction that $\partial(T_1 \cup T_2) \subseteq G \setminus G[U]$. Additionally, due to the definition of F_i for all $i \in [n']$, we have that $\partial T_3 \subseteq G \setminus G[U']$. Hence, in total we have that

$$G \setminus G[U] \subseteq \partial T \subseteq G \setminus G[U'].$$

Furthermore, due to (f) of Proposition 3.21, we have due to $|U| = \lfloor \varepsilon n \rfloor$

$$\Delta(\partial(T_2 \cup T_3)[U]) \leqslant \Delta(G'') + 2\varepsilon^{10} n \leqslant |U|^{\frac{3}{4}} + 3\varepsilon^9 |U| \leqslant \frac{\varepsilon}{2} |U'|.$$

Since $G[U]$ is complete, we also get

$$\begin{aligned}
\delta((G \setminus \partial T)[U]) &= |U| - 1 - \Delta(\partial(T_2 \cup T_3)[U]) \\
&\geqslant (1 - \varepsilon) |U|, \\
\deg_{(G \setminus \partial T)[U]}(u; U') &\geqslant |U'| - 1 - \Delta(\partial(T_2 \cup T_3)[U]) \\
&\geqslant (1 - 2\varepsilon) |U'|
\end{aligned}$$

for all $u \in U$, showing that we would indeed be done.

To show that those edge-disjoint, perfect matchings $M_i \subseteq F_i$ exist, we apply the following lemma, which can be seen as a generalization of Dirac's theorem for perfect matchings:

Lemma 3.28 ([35, Simplification of Lem. 5.7])**.** Let $\mu > 0$ and let $N \in \mathbb{N}$. Then there exists $n_0 = n_0(\mu)$ such that the following holds for all $n \geqslant n_0$: Let $G = (V, E)$ be an n-vertex graph and assume that $A_1, \ldots, A_N \subseteq V$ satisfy the following:

- $|A_i|$ is even and $\delta(G[A_i]) \geqslant \left(\frac{1}{2} + 4\mu^{\frac{1}{6}}\right)|A_i|$ for all $i \in [N]$;
- $|A_i| \geqslant \mu^{\frac{4}{3}} n$ for all $i \in [N]$;
- $\left|A_i \cap A_j\right| \leqslant \mu^2 n$ for all $1 \leqslant i < j \leqslant N$;
- every $v \in V$ is contained in at most μn of the sets A_i.

Then for every $i \in [N]$, the graph $G[A_i]$ contains a perfect matching M_i such that $E(M_i) \cap E(M_j) = \emptyset$ for every $1 \leqslant i < j \leqslant N$.

Remark 3.29. For a proof of this lemma, we refer to [4, Lem. 3.10].

It can be shown that we can apply the lemma with $G \setminus G[U']$, n', $A'_1, \ldots, A'_{n'} \subseteq V(G)$, and $2\varepsilon^9$ as the parameters G, N, $A_1, \ldots, A_N$, and μ respectively: Together with $(\star)$ and all A'_i having even size, the first condition follows from (d) and the second condition from (b) in Proposition 3.21; the third condition follows from (c) in Proposition 3.21; the fourth condition follows from (f) in Proposition 3.21 and the fact that $\Delta(G'') \leqslant (\varepsilon n)^{3/4}$.

§ 4. IMPROVING THE FRACTIONAL THRESHOLD

In previous chapters, we focused our attention on Steiner triple systems and the minimum codegree threshold θ_{STS} which guarantees their existence in our host graph H. In particular, we have shown that determining the threshold θ_{STS} can basically be reduced to the fractional threshold θ^*_{STS}. In this chapter, we will quantitatively improve Lee's result by analyzing more closely this parameter θ^*_{STS}.

Recall that the significance of θ^*_{STS} lies in the fact that – if provably below 3/4 – Theorem 3.6 would immediately give us the asymptotically optimal minimum codegree threshold for Steiner triple systems $\theta_{\mathrm{STS}} = 3/4$. As such, its importance is evident.

The main result of this chapter is the following.

Theorem 4.1. Let x^* be the unique root of the polynomial $p(x) = 8x^3 - 22x^2 + 10x - 1$ in $[0, 1/6]$. Then, $\theta^{f}_{\mathrm{STS}}(\varepsilon) \leqslant 1 - x^* < 0.8579$ for any $\varepsilon \in [0, 1)$. In particular, $\theta^*_{\mathrm{STS}} \leqslant 1 - x^* < 0.8579$.

To achieve this result, we closely follow Delcourt and Postle's approach in [6]. In that paper, they established the best known minimum degree threshold for a fractional K_3-decomposition of a graph. Together with the transversal version of Theorem 3.6 (see [35, Thm. 1.6]), we immediately obtain Theorem 1.8.

4.1 COMPARISON OF PREVIOUS APPROACHES

To establish the upper bound for θ^*_{STS} given in Theorem 3.7, Lee closely follows Dross' approach in [11]. From 2015 to 2020, this approach would lead to the best upper bound for the minimum degree threshold for a graph to have a fractional K_3-decomposition. Recall that the estimation of this threshold is of equal importance to the Nash-Williams conjecture as determining θ^*_{STS} is to Conjecture 1.6, see Theorem 2.19. Dross' result was independently improved by Dukes and Horsley in [12] and Delcourt and Postle in [6], the latter holding the best upper bound (of roughly $0.827327 < 5/6$) to this date.

Dross' approach (for fractional K_3-decompositions of a graph G) can be summed up as follows:

- Start with a uniform weighting on the triangles.
- The goal is to shift the weights in such a way that the total weight is after each operation preserved[1], the weighting stays non-negative, and the demand for each pair is met.
- For that, consider (scaled versions of) gadgets $\psi\colon \mathcal{K}_3(G) \to \mathbb{R}$ that are supported on a K_4 and shift exactly the weight from one matching edge to another matching edge in that K_4.

[1] Note that this invariance implies the value of the initial uniform weighting.

- Introduce an auxiliary network where a flow of maximum value implies the existence of a sequence of (scaled) gadgets that modify our uniform weight to a fractional Steiner triple system.
- Use the max-flow min-cut theorem[2] to show that such an optimal exists if $\delta(G) \geqslant 9n/10$.

Conceptually, Dukes and Horsley do a more careful analysis of Dross' approach. Meanwhile, Delcourt and Postle's approach is already different on a conceptual level which may be the reason for its strength.

For one, Delcourt and Postle exploit the (conjectured) minimum degree threshold to its absolute maximum. Namely, given a minimum degree greater than $3n/4$, it is easy to see that every edge is contained in at least one K_5. So, instead of using K_4 as the support for modification, they use K_5's which in turn leads to less drastic changes in the weights.

Additionally, Delcourt and Postle's approach ultimately leads to a non-linear optimization program, in contrast to the linear programming behind the max-flow min-cut theorem and thus Dross' approach.

Though we are able to improve on Theorem 1.4 using Delcourt and Postle's approach, it is important to highlight that their approach can a priori not be used to (asymptotically) resolve Conjecture 1.6. Indeed, it turns out that for every pair of positive codegree to be contained in a clique of order five requires the essential minimum codegree to be greater than $5n/6$. In particular, our application of Delcourt and Postle's method unavoidably achieves weaker bounds in comparison to their original result in [6]. Surprisingly, it was the opposite for Lee's application of Dross's approach: Whereas Dross derived an upper bound of $9/10$, Theorem 3.7 yields an upper bound of roughly $0.88 < 9/10$.

4.2 EDGE-GADGETS

One fundamental ingredient are the so-called *edge-gadgets* which were first introduced in [5]. We naturally modify the definition for our setting.

Definition 4.2 (Edge-gadget)**.** Let H be a 3-uniform hypergraph. For $K \in \mathcal{K}_5(H)$ and $p \in E(\partial K)$, let $E_j(K, e) = \{e \in E(K) \colon |e \cap p| = j\}$ for all $j \in \{0, 1, 2\}$. The *edge-gadget of* p *in* K is the function

$$\psi_{K,p} \colon E(H) \longrightarrow \mathbb{R}$$
$$e \longmapsto \begin{cases} +\frac{1}{3}, & e \in E_0(K,p) \\ -\frac{1}{6}, & e \in E_1(K,p) \\ +\frac{1}{3}, & e \in E_2(K,p) \\ 0, & \text{otherwise.} \end{cases}$$

What makes these gadgets so useful is that they allow us to alter the weight of the pair p without changing the weight of other pairs. Indeed, it turns out that $\psi_{K,p}$ acts like an indicator of p, meaning that it assigns weight one to p and zero to all the other pairs. However, this comes at the cost of introducing negative weights for the edges.

[2] See [24, Thm. 5.1].

Proposition 4.3. Let $K \in \mathcal{K}_5(H)$ and $p \in E(\partial K)$. Then we have for all $q \in E(\partial H)$

$$\deg^{\psi_{K,p}}(q) = \mathbb{1}_{p=q}.$$

Proof. Clearly, $\deg^{\psi_{K,p}}(q) = 0$ for all $q \in E(\partial H) \setminus E(\partial K)$. So, consider $q \in E(\partial K)$.

If $|p \cap q| = 0$, then there is exactly one edge in $E_0(K, p)$ and exactly two edges in $E_1(K, p)$ containing q, hence

$$\deg^{\psi_{K,p}}(q) = \frac{1}{3} - \frac{2}{6} = 0.$$

If $|p \cap q| = 1$, then there are exactly two edges in $E_1(K, p)$ and exactly one edge in $E_2, (K, p)$ containing q, hence

$$\deg^{\psi_{K,p}}(q) = \frac{-2}{6} + \frac{1}{3} = 0.$$

Lastly, if $|p \cap q| = 2$, i.e. $p = q$, then there are three edges in $E_2(K, p)$ containing p, so

$$\deg^{\psi_{K,p}}(q) = \frac{3}{3} = 1. \qquad \square$$

From Proposition 4.3, we can immediately construct weightings on the edges that satisfy the pair condition if we have no restrictions on our weights.

Corollary 4.4. Let H be a 3-uniform hypergraph such that every pair $p \in E(\partial H)$ is contained in at least one 5-clique. Then there exists $w \colon E(H) \longrightarrow \mathbb{R}$ such that

$$\deg^{w}(p) = 1$$

for all pairs $p \in V(\partial H)$.

Proof. For every pair p, let K_p be a 5-clique containing p. Then, by Proposition 4.3,

$$w = \sum_{p \in V(\partial H)} \psi_{K_p, p}$$

is as desired. $\square$

4.3 THE WEIGHTING

Before we begin to define our weighting, let us first give a couple more definitions.

Definition 4.5 ($\mathcal{K}_r(H, F)$, $\mathcal{K}_r(H, S)$)**.** For a subhypergraph $F \subseteq H$, $S \subseteq V(H)$ and $r \in \{3, 4, 5\}$, let

$$\mathcal{K}_r(H, F) = \{K \in \mathcal{K}_r(H) \colon F \subseteq K\},$$
$$\mathcal{K}_r(H, S) = \mathcal{K}_r(H, H[S]).$$

Definition 4.6 (CN)**.** Given $P \subseteq V(H)^{(2)}$, let the *common co-neighborhood* equal

$$\mathrm{CN}(P) = \bigcap_{p \in P} N(p).$$

Furthermore, for $S \subseteq V(H)$ we define

$$\mathrm{CN}(S) = \mathrm{CN}\left(S^{(2)}\right).$$

For the discussion below, assume for now that $\delta_2^{\text{ess}}(H) > 5n/6$ with $n = \nu(H) \geqslant 5$. It is then evident that

$$\mathcal{K}_5(H, p) \neq \emptyset$$

for every pair $p \in E(\partial H)$. Indeed, by definition, there must be an edge $e \in E(H)$ witnessing $p \in E(\partial H)$; e can be extended to some tetrahedron K since

$$|CN(e)| \geqslant \delta_2^{\text{ess}}(H) - 3 \cdot (1 - \delta_2^{\text{ess}}(H)) > 0.$$

Similarly, K can be extended to a $K_5^{(3)}$ since

$$|CN(K)| \geqslant \delta_2^{\text{ess}}(H) - 5 \cdot (1 - \delta_2^{\text{ess}}(H)) > 6 \cdot \frac{5}{6}\nu(H) - 5\nu(H) = 0.$$

As seen in Corollary 4.4, the only constraint not immediately satisfied in our usage of edge-gadgets is non-negativity. Indeed, if every edge has a non-negative weight, then the condition on the pairs already implies that the weight of each edge is at most one, making it a fractional Steiner triple system. For this, we want a more general approach than the one in the proof of Corollary 4.4. Namely, to be more flexible, instead of relying on a single edge-gadget per pair, it seems advantageous to distribute the demand of the pair over *multiple* edge-gadgets of p. Hence, one natural approach would be to make the ansatz

$$w = \sum_{p \in E(\partial H)} \sum_{K \in \mathcal{K}_5(H,p)} \lambda_{K,p} \cdot \psi_{K,p} \qquad (\lambda_{K,p} \in \mathbb{R})$$

for our fractional Steiner triple system. One obvious constraint on the scalars is

$$\sum_{K \in \mathcal{K}_5(H,p)} \lambda_{K,p} = 1$$

for every pair $p \in E(\partial H)$. Indeed, using Proposition 4.3, we see that

$$\begin{aligned}
\deg^w(q) &= \sum_{e \in E(H):\ q \subseteq e} \sum_{p \in E(\partial H)} \sum_{K \in \mathcal{K}_5(H,p)} \lambda_{K,p} \cdot \psi_{K,p}(e) \\
&= \sum_{p \in E(\partial H)} \sum_{K \in \mathcal{K}_5(H,p)} \lambda_{K,p} \cdot \left(\sum_{e \in E(H):\ q \subseteq e} \psi_{K,p}(e) \right) \\
&= \sum_{p \in E(\partial H)} \sum_{K \in \mathcal{K}_5(H,p)} \lambda_{K,p} \cdot \mathbb{1}_{p=q} \\
&= \sum_{K \in \mathcal{K}_5(H,q)} \lambda_{K,q}.
\end{aligned}$$

From this calculation, it is also clear that this condition is sufficient for every pair to get weight one.

For the scalars, we introduce a non-uniform distribution that utilizes the structure of H. Namely, we imagine how at first every pair with positive codegree holds its demand of 1. Then, each pair *distributes* that demand uniformly among the $K_3^{(3)}$'s containing that pair. Those triples in turn distribute the demand uniformly among all $K_4^{(3)}$'s containing them, until every $K_5^{(3)}$ got from every pair a certain fraction of the pair's demand. This fraction will then serve as $\lambda_{K,p}$.

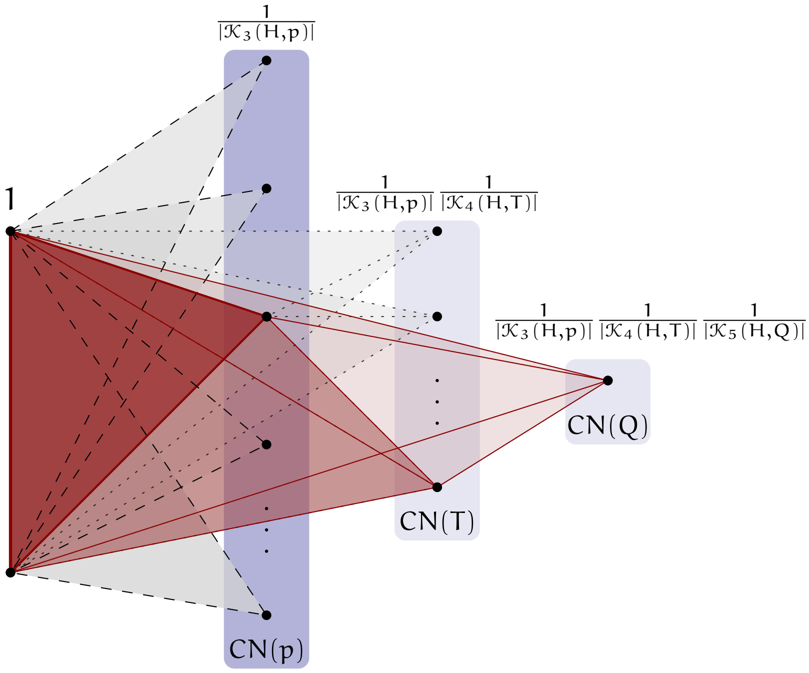

Figure 4.1: Sketch for how the demand of p is distributed among the $K_5^{(3)}$'s

Note that a $K_4^{(3)}$ containing a pair p will get a fraction of p's demand through two distinct triples, where the fraction received from each triple may be different. Hence, to make this distribution[3] of the demand formal, it seems natural to introduce *ordered* variants of the previous definitions.

Definition 4.7. Let H be a 3-uniform hypergraph. For $r \in \{2,3,4,5\}$, an *ordered* r*-clique* of G is an r-tuple $K = (v_1, \dots, v_r) \in V(H)^r$ such that $H[\{v_1, \dots, v_r\}] \in \mathcal{K}_r(H)$.

The vertex set of the ordered clique is $V(K) = \{v_1, \dots, v_r\}$ and the "$\subseteq$"-relation will be extended in a straightforward way:

- If K, K' are ordered cliques, then $K' \subseteq K$ if K' is a subsequence of K.
- If F is a subhypergraph and K an ordered clique, then $F \subseteq K$ holds if $F \subseteq H[V(K)]$.

The set of ordered r-cliques in H is denoted by $\mathcal{OK}_r(H)$. Furthermore, for an ordered r-clique K and $s \geqslant r$, let $\mathcal{OK}_s(H, K)$ denote the set of ordered s-cliques containing K in the sense of the "$\subseteq$"-relation and let $\mathcal{OK}_s(H, S)$ denote the set of ordered s-cliques whose vertex set contains $S \subseteq V(H)$.

The weight for an ordered clique is now defined as follows:

Definition 4.8 ($W(K)$)**.** Let H be a 3-uniform hypergraph and let $r \in \{2,3,4\}$. For every $K = (v_1, \dots, v_r) \in \mathcal{OK}_r(H)$, we define the weight of K to be

$$W(K) = \prod_{i=2}^{r} \frac{1}{|\mathcal{K}_{i+1}(H, \{v_1, \dots, v_i\})|}.$$

For the sake of clarity, we may also write $W(v_1, \dots, v_r)$ for $W(K)$.

[3] Or delegation, as Delcourt and Postle call it.

For the sake of completeness, we will also define edge-gadgets for ordered cliques. Instead of having the pair explicitly given, the pair for whom the edge-gadget covers the demand is implicitly given by the ordered clique:

Definition 4.9 (ψ_K)**.** Let H be a 3-uniform hypergraph. For $K = (v_1, \ldots, v_5) \in \mathcal{OK}_5(H)$ and $T \in \mathcal{K}_3(H)$, we define

$$\psi_K(T) = \psi_{H[V(K)], v_1 v_2}(T).$$

Similarly, we define $\psi_K(O) = \psi_K(H[V(O)])$ for $O \in \mathcal{OK}_3(H)$.

We can finally define our weighting:

Definition 4.10 (w_H)**.** Let H be a 3-uniform hypergraph. Define

$$\begin{aligned} w_H \colon E(H) &\longrightarrow \mathbb{R} \\ e &\longmapsto \frac{1}{2} \sum_{K=(v_1,\ldots,v_5) \in \mathcal{OK}_5(H,e)} W(v_1, \ldots, v_4) \cdot \psi_K(e). \end{aligned}$$

Note that the $1/2$ is there since a pair can be ordered in two different ways.

Proposition 4.11. Let H be a 3-uniform hypergraph on $n \geqslant 5$ vertices with $\delta_2^{\mathrm{ess}}(H) > 5n/6$. Then, the weight of each pair $p \in E(\partial H)$ equals

$$\deg^{w_H}(p) = 1.$$

Proof. By the discussion above, it suffices to show that

$$\frac{1}{2} \sum_{\substack{K=(v_1,\ldots,v_5) \in \mathcal{OK}_5(H,p): \\ p = v_1 v_2}} W(v_1, \ldots, v_4) = 1.$$

Indeed, as $W(v_1, \ldots, v_4)$ doesn't depend on v_5, $W(v_1, v_2, v_3)$ doesn't depend on v_4 etc., we compute that

$$\begin{aligned} \frac{1}{2} \sum_{\substack{K=(v_1,\ldots,v_5) \in \mathcal{OK}_5(H,p): \\ p = v_1 v_2}} W(v_1, \ldots, v_4) &= \frac{1}{2} \sum_{\substack{K=(v_1,\ldots,v_4) \in \mathcal{OK}_4(H,p): \\ p = v_1 v_2}} W(v_1, \ldots, v_3) \\ &= \frac{1}{2} \sum_{\substack{K=(v_1,\ldots,v_3) \in \mathcal{OK}_3(H,p): \\ p = v_1 v_2}} W(v_1, v_2) \\ &= \frac{1}{2} \sum_{\substack{K=(v_1,v_2) \in \mathcal{OK}_2^{(3)}(H,p): \\ p = v_1 v_2}} 1 \\ &= 1. \end{aligned}$$

□

4.4 REFORMULATION

The goal of the rest of this chapter is to show that w_H is non-negative for sufficiently large (essential) minimum codegree. Instead of showing this directly, however, we consider an ordered variant of the weighting.

Definition 4.12 ($w_H(O)$)**.** Let H be a 3-uniform hypergraph on $n \geqslant 5$ vertices with $\delta_2^{\mathrm{ess}}(H) > 5n/6$. For an ordered edge $O \in \mathcal{OK}_3(H)$, let

$$w_H(O) = \frac{1}{2} \sum_{K=(v_1,\dots,v_5)\in\mathcal{OK}_5(H,O)} W(v_1,\dots,v_4)\cdot\psi_K(O).$$

By our extension of the "$\subseteq$"-relation to ordered cliques, it is evident that

$$w_H(e) = \sum_{O\in\mathcal{OK}_3(H,e)} w_H(O).$$

Hence, we may prove the following, stronger result.

Theorem 4.13. Let x^* be defined as in Theorem 4.1 and let H be a 3-uniform hypergraph satisfying $\delta_2^{\mathrm{ess}}(H) \geqslant (1-x^*)\cdot v(H)$ and $v(H) \geqslant 5$. Then, $w_H(O) \geqslant 0$ for all $O \in \mathcal{OK}_3(H)$.

It will be useful to represent $w_H(O)$ in a more explicit manner.

Lemma 4.14. Let H be a 3-uniform hypergraph on $n \geqslant 5$ vertices with $\delta_2^{\mathrm{ess}}(H) > 5n/6$. If $O = (x_1,x_2,x_3) \in \mathcal{OK}_3(H)$, then $w_H(O)$ can be written as

$$\frac{1}{6}\Bigg(W(x_1,x_2) - \sum_{y\in CN(x_1,x_2,x_3)}\Bigg(W(x_1,y,x_2) - W(x_1,x_2,y) +$$
$$\sum_{z\in CN(x_1,x_2,x_3,y)} (W(x_1,y,x_2,z) - W(x_1,x_2,y,z) + W(x_1,y,z,x_2) - W(z,y,x_1,x_2))\Bigg)\Bigg).$$

In this representation of $w_H(O)$, we can see what Delcourt and Postle refer to as *cancellation*: Apart from the positive leading term $W(x_1,x_2)$, similar weights are paired up to (hopefully) cancel each other in the inner terms of the expression. For example, $W(x_1,y,x_2,z)$ and $W(x_1,x_2,y,z)$ differ in their first product, which are $1/|\mathcal{K}_3(H,\{x_1,y\})|$ and $1/|\mathcal{K}_3(H,\{x_1,x_2\})|$ respectively. Hence, as long as the (essential) minimum codegree is high enough, these terms can't differ too much.

It is also worth noting that, while a substantial simplification, considering the weights of ordered triples instead of triples doesn't give us worse bounds, at least when proceeding the same as below: One could consider $w_H(e)$ instead of $w_H(O)$ by summing up the weight of each ordering of e and then using the above formula for each of those weights of orderings; ignoring what we will later define as $w_{H,1}$, it could still be reduced to an optimization problem where we would try to minimize $w_H(e)$; namely, it would still be possible to simplify the problem to only depend on what we will later call *common co-neighborhood densities*; it would even be possible to use the same symmetrization arguments to reduce the number of densities considered and theoretically get tighter bounds on those. However, it turns out that choosing those densities for each ordered weight as given in Theorem 4.24 and Lemma 4.29 would still satisfy the bounds and witness for (essential) minimum codegree smaller than $(1-x^*)\cdot v(H)$ that $w_H(e)$ may become negative.

Proof of Lemma 4.14. By explicitly using Definition 4.2, we see that $w_H(O)$ equals

$$\frac{1}{2}\sum_{\substack{y\in CN(x_1,x_2,x_3),\\ z\in CN(x_1,x_2,x_3,y)}} \Bigg(W(x_1,x_2,x_3,y)\cdot\left(+\frac{1}{3}\right) \quad + W(x_1,x_2,y,x_3)\cdot\left(+\frac{1}{3}\right)$$

$$
\begin{aligned}
&+ W(x_1, x_2, y, z) \cdot \left(+\frac{1}{3}\right) \quad + W(x_1, y, x_2, x_3) \cdot \left(-\frac{1}{6}\right) \\
&+ W(x_1, y, x_2, z) \cdot \left(-\frac{1}{6}\right) \quad + W(x_1, y, z, x_2) \cdot \left(-\frac{1}{6}\right) \\
&+ W(y, x_1, x_2, x_3) \cdot \left(-\frac{1}{6}\right) + W(y, x_1, x_2, z) \cdot \left(-\frac{1}{6}\right) \\
&+ W(y, x_1, z, x_2) \cdot \left(-\frac{1}{6}\right) \quad + W(y, z, x_1, x_2) \cdot \left(+\frac{1}{3}\right) \Bigg).
\end{aligned}
$$

Recall that the order of the starting pair doesn't change the value of $W(\cdot)$. In particular, we have by that symmetry

$$
\begin{aligned}
W(x_1, y, x_2, x_3) &= W(y, x_1, x_2, x_3), \\
W(x_1, y, x_2, z) &= W(y, x_1, x_2, z), \\
W(x_1, y, z, x_2) &= W(y, x_1, z, x_2).
\end{aligned}
$$

Hence, the above expression for $w_H(O)$ can be rewritten as

$$
\begin{aligned}
\frac{1}{6} \sum_{y \in CN(x_1,x_2,x_3)} \sum_{z \in CN(x_1,x_2,x_3,y)} \Bigg(& W(x_1, x_2, x_3, y) \quad + W(x_1, x_2, y, x_3) \\
& + W(x_1, x_2, y, z) \quad - W(x_1, y, x_2, x_3) \\
& - W(x_1, y, x_2, z) \quad - W(x_1, y, z, x_2) \\
& + W(y, z, x_1, x_2) \Bigg).
\end{aligned}
$$

Note that three of the inner terms do not depend on z, so summing over all $z \in CN(x_1, x_2, x_3, y)$ cancels out the last product term in e.g. $W(x_1, x_2, y, x_3)$.

Thus, we get

$$
\begin{aligned}
&\frac{1}{6} \sum_{y \in CN(x_1,x_2,x_3)} \Bigg(W(x_1, x_2, x_3) + W(x_1, x_2, y) - W(x_1, y, x_2) + \\
&\sum_{z \in CN(x_1,x_2,x_3,y)} \left(W(x_1, x_2, y, z) - W(x_1, y, x_2, z) - W(x_1, y, z, x_2) + W(y, z, x_1, x_2) \right) \Bigg).
\end{aligned}
$$

Furthermore, note that $W(x_1, x_2, x_3)$ doesn't depend on y, so similar manipulations and additional rearranging give us the desired expression for $w_H(O)$. □

For the optimization step, it will be more convenient to work with the following function:

Definition 4.15. Let H be a 3-uniform hypergraph on $n \geqslant 5$ vertices with $\delta_2^{\text{ess}}(H) > 5n/6$. For $O = (x_1, x_2, x_3) \in \mathcal{OK}_3(H)$, let

$$
\begin{aligned}
w_{H,1}(O) &= 1 - 6 |\mathcal{K}_3(H, \{x_1, x_2\})| \, w_H(O) \\
&= |CN(x_1, x_2)| \sum_{y \in CN(x_1,x_2,x_3)} \Big(W(x_1, y, x_2) - W(x_1, x_2, y)
\end{aligned}
$$

$$+ \sum_{z \in CN(x_1,x_2,x_3,y)} (W(x_1,y,x_2,z) - W(x_1,x_2,y,z)$$
$$+ W(x_1,y,z,x_2) - W(z,y,x_1,x_2))\Bigg).$$

Clearly, it suffices to show the following:

Theorem 4.16. Let x^* be defined as in Theorem 4.1 and let H be a 3-uniform hypergraph satisfying $\delta_2^{\text{ess}}(H) \geqslant (1-x^*) \cdot v(H)$ and $v(H) \geqslant 5$. Then, $w_{H,1}(O) \leqslant 1$ for all $O \in \mathcal{OK}_3(H)$.

4.5 OPTIMIZATION

Ideally, our goal can be phrased as follows[4]: Determine

$$\sup \left\{ d \in \left[0, \frac{1}{6}\right) : \lim_{n \longrightarrow \infty} \sup_{\substack{H \text{ 3-uniform}: \\ v(H) \geqslant n, \\ \delta_2^{\text{ess}}(H) \geqslant (1-d)v(H)}} \max_{O \in \mathcal{K}_3(H)} w_{H,1}(O) \leqslant 1 \right\}.$$

This turns out to still be too difficult, so we will one by one relax the problem. First, to abstract away from H, we will rewrite $w_{H,1}$ using *scaled* versions of CN and W.

Definition 4.17. Let H be a 3-uniform hypergraph with $V(H) \neq \emptyset$. For $S \subseteq V(H)$, we define the *common co-neighborhood density* to be

$$\widehat{CN}(S) = \frac{|CN(S)|}{v(H)} \in [0,1].$$

Similarly, for an ordered r-clique $K = (v_1, \dots, v_r) \in \mathcal{OK}_r(H)$, let

$$\hat{W}(K) = v(H)^{r-1} W(K) = \prod_{i=2}^{r} \frac{v(H)}{|\mathcal{OK}_{i+1}(H, \{v_1, \dots, v_i\})|} = \prod_{i=2}^{r} \frac{1}{\left|\widehat{CN}(v_1, \dots, v_i)\right|}.$$

The following is then immediate:

Proposition 4.18. Let H be a 3-uniform hypergraph with $v(H) \geqslant 5$ and $\delta_2^{\text{ess}}(H) \geqslant 5v(H)/6$. We have

$$w_{H,1}(O) = \left|\widehat{CN}(x_1,x_2)\right| \cdot \frac{1}{v(H)} \sum_{y \in CN(x_1,x_2,x_3)} \Bigg(\hat{W}(x_1,y,x_2) - \hat{W}(x_1,x_2,y)$$
$$+ \frac{1}{v(H)} \sum_{z \in CN(x_1,x_2,x_3,y)} (\hat{W}(x_1,y,x_2,z) - \hat{W}(x_1,x_2,y,z)$$
$$+ \hat{W}(x_1,y,z,x_2) - \hat{W}(z,y,x_1,x_2))\Bigg).$$

[4]Technically, we additionally have a minimum degree condition on the shadow of H.

In this representation of $w_{H,1}$, all the information of H we really need are the common co-neighborhood densities of a small number of vertex subsets. So, instead of considering H itself, we will work with the common co-neighborhood densities directly using the bounds implied by the properties of H.

To establish these bounds, we review some properties of $\widehat{\mathrm{CN}}$: Obviously, $\widehat{\mathrm{CN}}$ is monotonically decreasing with respect to $\subseteq$. Furthermore, we have that $\widehat{\mathrm{CN}}$ is *supermodular*.

Proposition 4.19. Let H be a 3-uniform hypergraph with $V(H) \neq \emptyset$. If $A, B \subseteq V(H)^{(2)}$, then
$$\widehat{\mathrm{CN}}(A \cup B) \geqslant \widehat{\mathrm{CN}}(A) + \widehat{\mathrm{CN}}(B) - \widehat{\mathrm{CN}}(A \cap B).$$

Proof. Set $X = \bigcap_{a \in A} N(a)$ and $Y = \bigcap_{b \in B} N(b)$. Obviously,
$$|X \cap Y| = |X| + |Y| - |X \cup Y|.$$

Note however, that $|X| = v(H) \cdot \widehat{\mathrm{CN}}(A)$ and $|Y| = v(H) \cdot \widehat{\mathrm{CN}}(B)$. Moreover,
$$|X \cap Y| = \left| \bigcap_{p \in A \cup B} N(p) \right| = v(H) \cdot \widehat{\mathrm{CN}}(A \cup B).$$

Similarly, we have
$$|X \cup Y| = \left| \left(\bigcap_{a \in A} N(a) \right) \cup \left(\bigcap_{b \in B} N(b) \right) \right| \leqslant \left| \bigcap_{p \in A \cap B} N(p) \right| = v(H) \cdot \widehat{\mathrm{CN}}(A \cap B).$$

Hence, dividing by $v(H)$, we are done. $\square$

Hence, for vertices $v_1, \dots, v_4 \in V(H)$, where H satisfies the usual assumptions, we have
$$\begin{aligned}
\widehat{\mathrm{CN}}(v_1, v_2, v_3) &\geqslant \widehat{\mathrm{CN}}(\{v_1 v_2, v_1 v_3\}) + \widehat{\mathrm{CN}}(v_2, v_3) - \widehat{\mathrm{CN}}(\emptyset) \\
&\geqslant \left(\widehat{\mathrm{CN}}(v_1, v_2) + \widehat{\mathrm{CN}}(v_1, v_3) - \widehat{\mathrm{CN}}(\emptyset) \right) + \widehat{\mathrm{CN}}(v_2, v_3) - 1 \\
&\geqslant \widehat{\mathrm{CN}}(v_1, v_2) + \widehat{\mathrm{CN}}(v_1, v_3) + \widehat{\mathrm{CN}}(v_2, v_3) - 2 \\
&\geqslant \widehat{\mathrm{CN}}(v_1, v_2) + \widehat{\mathrm{CN}}(v_1, v_3) - 1 - d, \\
\widehat{\mathrm{CN}}(v_1, \dots, v_4) &\geqslant \widehat{\mathrm{CN}}\left(\{v_1, v_2, v_3\}^{(2)} \cup \{v_1, v_2, v_4\}^{(2)} \right) + \widehat{\mathrm{CN}}(v_3, v_4) - \widehat{\mathrm{CN}}(\emptyset) \\
&\geqslant \left(\widehat{\mathrm{CN}}(v_1, v_2, v_3) + \widehat{\mathrm{CN}}(v_1, v_2, v_4) - \widehat{\mathrm{CN}}(v_1, v_2) \right) + \widehat{\mathrm{CN}}(v_3, v_4) - 1 \\
&\geqslant \widehat{\mathrm{CN}}(v_1, v_2, v_3) + \widehat{\mathrm{CN}}(v_1, v_2, v_4) - \widehat{\mathrm{CN}}(v_1, v_2) - d.
\end{aligned}$$

So, our goal is now to solve the following program (P1) for $d \in [0, 1/6)$:

Maximize $w_{H,1}(O)$ *such that for all* $y_i \in \mathrm{CN}(x_1, x_2, x_3)$ *and* $z_{i,j} \in \mathrm{CN}(x_1, x_2, x_3, y_i)$

$$\widehat{\mathrm{CN}}(x_1, x_2) \in [1 - d, 1],$$
$$\widehat{\mathrm{CN}}(x_1, y_i) \in [1 - d, 1],$$

$$
\begin{aligned}
\widehat{CN}(y_i, z_{i,j}) &\in [1-d, 1],\\
\widehat{CN}(x_1, x_2, y_i) &\in \left[\widehat{CN}(x_1, x_2) + \widehat{CN}(x_1, y_i) - 1 - d, \widehat{CN}(x_1, x_2)\right],\\
\widehat{CN}(x_1, y_i, z_{i,j}) &\in \left[\widehat{CN}(x_1, y_i) + \widehat{CN}(y_i, z_{i,j}) - 1 - d, \widehat{CN}(x_1, y_i)\right],\\
\widehat{CN}(x_1, x_2, y_i, z_{i,j}) &\in \left[\widehat{CN}(x_1, x_2, y_i) + \widehat{CN}(x_1, y_i, z_{i,j}) - \widehat{CN}(x_1, y_i) - d, \widehat{CN}(x_1, x_2, y_i)\right].
\end{aligned}
$$

We note that $w_{H,1}(O)$ is under our (essential) minimum codegree assumption a well-defined, continuous function on the domain of (P1). Indeed, it's clear that all the common co-neighborhood densities are strictly positive and at most 1.

To emphasize that we now think in variables, fix an enumeration on $CN(x_1, x_2, x_3)$ and $CN(x_1, x_2, x_3, y)$ for every $y \in CN(x_1, x_2, x_3)$. Let $R_0 = |CN(x_1, x_2, x_3)|$ and $R_i = |CN(x_1, x_2, x_3, y_i)|$ for $i \in [R_0]$. We may now relabel the common co-neighborhood densities as follows:

$$
\begin{aligned}
\widehat{CN}(x_1, x_2) &\longrightarrow e_0,\\
\widehat{CN}(x_1, y_i) &\longrightarrow e_i,\\
\widehat{CN}(y_i, z_{i,j}) &\longrightarrow f_{i,j},\\
\widehat{CN}(x_1, x_2, y_i) &\longrightarrow q_{i,0} \text{ for } i \in [R_0],\\
\widehat{CN}(x_1, y_i, z_{i,j}) &\longrightarrow q_{i,j} \text{ for } i \in [R_0], j \in [R_i],\\
\widehat{CN}(x_1, x_2, y_i, z_{i,j}) &\longrightarrow p_{i,j} \text{ for } i \in [R_0], j \in [R_i].
\end{aligned}
$$

Changing the variable names accordingly in $w_{H,1}$ gives

$$
\begin{aligned}
\hat{W}_1 = \frac{e_0}{v(H)} \sum_{i=1}^{R_0} \Bigg(&\frac{1}{q_{i,0}} \left(\frac{1}{e_i} - \frac{1}{e_0} \right) \\
&+ \frac{1}{v(H)} \sum_{j=1}^{R_i} \left(\frac{1}{p_{i,j}} \left(\frac{1}{q_{i,j}} \left(\frac{1}{e_i} - \frac{1}{f_{i,j}} \right) + \frac{1}{q_{i,0}} \left(\frac{1}{e_i} - \frac{1}{e_0} \right) \right) \right) \Bigg).
\end{aligned}
$$

Thus, (P1) now reads:

Maximize $\hat{W}_1$ *such that for all* $y_i \in [R_0]$ *and* $z_{i,j} \in [R_i]$

$$
\begin{aligned}
e_0 &\in [1-d, 1],\\
e_i &\in [1-d, 1],\\
f_{i,j} &\in [1-d, 1],\\
q_{i,0} &\in [e_0 + e_i - 1 - d, e_0],\\
q_{i,j} &\in [e_i + f_{i,j} - 1 - d, e_i],\\
p_{i,j} &\in [q_{i,0} + q_{i,j} - e_i - d, q_{i,0}],\\
R_0 &\in \left[\frac{1}{2} \cdot v(H), e_0 \cdot v(H)\right],\\
R_i &\in [0, q_{i,0} \cdot v(H)].
\end{aligned}
$$

Note that the bounds from R_0 and R_i are obtained by considering the properties of $\widehat{CN}$ and the fact that $d \in [0, 1/6)$.

4.5.1 *Reduction to 8 variables*

To reduce the number of variables involved in (P1), we first employ some symmetrization arguments.

Lemma 4.20. The maximum value of (P1) is achieved by a point where for all $i \in [R_0]$ and $j, j' \in [R_i]$, we have

$$f_{i,j} = f_{i,j'}, q_{i,j} = q_{i,j'}, p_{i,j} = p_{i,j'}.$$

Proof. Since the domain of (P1) is closed and bounded and $\hat{W}_1$ is well-defined and continuous on the domain of (P1), we find that (P1) has a global maximum. Let P_0 be a point that achieves this maximum. For each i, let $j_i \in [R_i]$ be the index for which the inner term

$$\frac{1}{p_{i,j}}\left(\frac{1}{q_{i,j}}\left(\frac{1}{e_i}-\frac{1}{f_{i,j}}\right)+\frac{1}{q_{i,0}}\left(\frac{1}{e_i}-\frac{1}{e_0}\right)\right)$$

is maximized over all $j \in [R_i]$. Then the point P_0' obtained from P_0 by setting

$$\begin{aligned} f_{i,j} &= f_{i,j_i}, \\ q_{i,j} &= q_{i,j_i}, \\ p_{i,j} &= p_{i,j_i} \end{aligned}$$

for all $i \in [R]$ and $j \in [R_i]$ is also a point that achieves this maximum. Moreover, since the constraints for $f_{i,j}, q_{i,j}, p_{i,j}$ are identical for each $j \in [R_i]$, it follows that P_0' also satisfies the constraints of (P1) as desired. □

Letting $r_i = R_i/\nu(H)$, $f_{i,j} = f_{i,j_i}$, $p_i = p_{i,j_i}$, $q_i = q_{i,j_i}$, we form a new program (P2) with a new objective function that has the same optimum value as (P1):

$$\hat{W}_2 = \frac{e_0}{\nu(H)}\sum_{i=1}^{R_0}\left(\frac{1}{q_{i,0}}\left(\frac{1}{e_i}-\frac{1}{e_0}\right)+r_i\left(\frac{1}{p_i}\left(\frac{1}{q_i}\left(\frac{1}{e_i}-\frac{1}{f_i}\right)+\frac{1}{q_{i,0}}\left(\frac{1}{e_i}-\frac{1}{e_0}\right)\right)\right)\right).$$

Thus, (P2) reads:

Maximize $\hat{W}_2$ *such that for all* $y_i \in [R_0]$

$$\begin{aligned} e_0 &\in [1-d, 1], \\ e_i &\in [1-d, 1], \\ f_i &\in [1-d, 1] \\ q_{i,0} &\in [e_0 + e_i - 1 - d, e_0], \\ q_i &\in [e_i + f_i - 1 - d, e_i], \\ p_i &\in [q_{i,0} + q_i - e_i - d, q_{i,0}], \\ R_0 &\in \left[\frac{1}{2}\cdot\nu(H), e_0\cdot\nu(H)\right], \\ r_i &\in [0, q_{i,0}]. \end{aligned}$$

Corollary 4.21. $\mathrm{OPT(P1)} = \mathrm{OPT(P2)}$.

Lemma 4.22. The maximum value of (P2) is achieved by a point where for all $i, i' \in [R_0]$

$$e_i = e_{i'}, f_i = f_{i'}, q_{i,0} = q_{i',0}, p_i = p_{i'}, r_i = r_{i'}.$$

Proof. Again, let P_0 be a point that achieves this maximum. Let $i' \in [R_0]$ be the index for which the inner term

$$\frac{1}{q_{i,0}}\left(\frac{1}{e_i}-\frac{1}{e_0}\right)+r_i\left(\frac{1}{p_i}\left(\frac{1}{q_i}\left(\frac{1}{e_i}-\frac{1}{f_i}\right)+\frac{1}{q_{i,0}}\left(\frac{1}{e_i}-\frac{1}{e_0}\right)\right)\right)$$

is maximized over all $i \in [R_0]$. Then the point P_0' obtained from P_0 by setting

$$\begin{aligned} e_i &= e_{i'}, \\ f_i &= f_{i'}, \\ q_{i,0} &= q_{i',0}, \\ q_i &= q_{i'}, \\ r_i &= r_{i'} \end{aligned}$$

for all $i \in [R]$ and $j \in [R_i]$ is also a point that achieves this maximum. Moreover, since the constraints for e_i, f_i, $q_{i,0}$, p_i, r_i are identical for each $j \in [R_i]$, it follows that P_0' also satisfies the constraints of (P2) as desired. □

Letting $r_0 = R_0/\nu(H)$, $e = e_{i'}$, $f = f_{i'}$, $q_0 = q_{i',0}$, $q = q_{i'}$, $p = p_{i'}$, $r = r_{i'}$, we form a new program (P3) with a new objective function, yet the same optimum value as (P2):

$$\hat{W}_3 = e_0 \cdot r_0\left(\frac{1}{q_0}\left(\frac{1}{e}-\frac{1}{e_0}\right)+r\left(\frac{1}{p}\left(\frac{1}{q}\left(\frac{1}{e}-\frac{1}{f}\right)+\frac{1}{q_0}\left(\frac{1}{e}-\frac{1}{e_0}\right)\right)\right)\right).$$

Thus, (P3) reads:

Maximize $\hat{W}_3(e_0, e, f, q_0, q, p, r_0, r)$ *subject to*

$$\begin{aligned} e_0 &\in [1-d, 1], \\ e &\in [1-d, 1], \\ f &\in [1-d, 1] \\ q_0 &\in [e_0 + e - 1 - d, e_0], \\ q &\in [e + f - 1 - d, e], \\ p &\in [q_0 + q - e - d, q_0], \\ r_0 &\in \left[\frac{1}{2}, e_0\right], \\ r &\in [0, q_0]. \end{aligned}$$

Corollary 4.23. $\mathrm{OPT}(P1) = \mathrm{OPT}(P3)$.

At this point, the program (P3) would be small enough to be numerically solved by a commercial solver. Such an implementation agreeing with our result can be found in Section C.1. However, the advantage of our proof is that we get an exact solution for Theorem 4.1.

4.5.2 *Reduction to two variables*

Theorem 4.24. $\mathsf{OPT}(\mathrm{P3}) = \mathsf{OPT}(\mathrm{P4})$ where (P4) is defined as follows:

$$\begin{aligned} &\textit{Maximize} \quad \widehat{W}_4(e_0, f) = e_0^2 \left(\frac{\frac{1}{1-d} - \frac{1}{e_0}}{e_0 - 2d} + \frac{e_0 - 2d}{e_0 + f - 1 - 4d} \left(\frac{\frac{1}{1-d} - \frac{1}{f}}{f - 2d} + \frac{\frac{1}{1-d} - \frac{1}{e_0}}{e_0 - 2d} \right) \right) \\ &\textit{subject to} \qquad e_0 \in [1-d, 1], \\ &\qquad\qquad\qquad\quad f \in [1-d, 1]. \end{aligned}$$

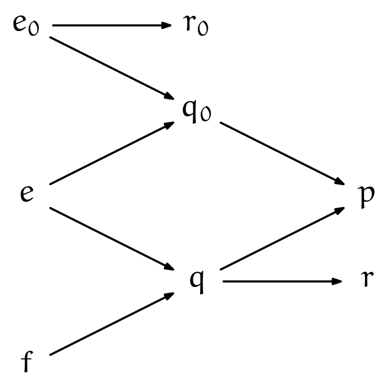

Figure 4.2: Dependency diagram of the variables' constraints in (P3)

Proof. For a function f, let the *ramp function of* f be $f^+ = \max(f, 0)$. Instead of $\widehat{W}_3$, we will consider

$$\widehat{W}_4 = e_0 \cdot r_0 \left(\frac{1}{q_0} \left(\frac{1}{e} - \frac{1}{e_0} \right)^+ + r \left(\frac{1}{p} \left(\frac{1}{q} \left(\frac{1}{e} - \frac{1}{f} \right)^+ + \frac{1}{q_0} \left(\frac{1}{e} - \frac{1}{e_0} \right)^+ \right) \right) \right)$$

as our objective function. Note that $\widehat{W}_4$ is also well-defined and continuous on the domain of (P3). In particular, if $e \leqslant e_0$, $e \leqslant f$, then $\widehat{W}_4$ and $\widehat{W}_3$ correspond for that point.

Consider first r_0. With the ramp functions, it is clear that the term with which r_0 is multiplied with is positive, meaning that $\widehat{W}_4$ is monotonically increasing if all the remaining variables are fixed. So, to attain the maximum, we must have $r_0 = e_0$. By the same logic, $r = q_0$. To keep track of the substitutions, we let $\sigma = \{r_0 \longrightarrow e_0, r \longrightarrow q_0\}$ be the set of all subsitutions we have made.[5] Plugging this in, we get for $\sigma(\widehat{W}_4)$

$$e_0^2 \left(\frac{1}{q_0} \left(\frac{1}{e} - \frac{1}{e_0} \right)^+ + q_0 \left(\frac{1}{p} \left(\frac{1}{q} \left(\frac{1}{e} - \frac{1}{f} \right)^+ + \frac{1}{q_0} \left(\frac{1}{e} - \frac{1}{e_0} \right)^+ \right) \right) \right).$$

Again, by the same logic, we must have that p is as small as possible, so $p = q_0 + q - e - d$. Updating σ accordingly, we get for $\sigma(\widehat{W}_4)$

$$e_0^2 \left(\frac{1}{q_0} \left(\frac{1}{e} - \frac{1}{e_0} \right)^+ + q_0 \left(\frac{1}{q_0 + q - e - d} \left(\frac{1}{q} \left(\frac{1}{e} - \frac{1}{f} \right)^+ + \frac{1}{q_0} \left(\frac{1}{e} - \frac{1}{e_0} \right)^+ \right) \right) \right).$$

[5] σ will be implicitly updated after each subsequent subsitution.

Next, consider q. As $q_0 + q - e - d > 0$, we see that, for all other parameters fixed, the functions

$$q \longmapsto \frac{1}{q_0 + q - e - d}, \quad q \longmapsto \left(\frac{1}{q}\left(\frac{1}{e} - \frac{1}{f}\right)^+ + \frac{1}{q_0}\left(\frac{1}{e} - \frac{1}{e_0}\right)^+\right)$$

are both positive and monotonically decreasing. Hence, as both the sum and product of positive, monotonically decreasing functions is positive and montononically decreasing, we see that $\sigma(\widehat{W}_4)$ as a whole is monotonically decreasing in q with all other parameters fixed. Thus, we set $q = e + f - 1 - d$ which yields for $\sigma(\widehat{W}_4)$

$$e_0^2\left(\frac{1}{q_0}\left(\frac{1}{e} - \frac{1}{e_0}\right)^+ + \frac{q_0}{q_0 + f - 1 - 2d}\left(\frac{\left(\frac{1}{e} - \frac{1}{f}\right)^+}{e + f - 1 - d} + \frac{1}{q_0}\left(\frac{1}{e} - \frac{1}{e_0}\right)^+\right)\right).$$

Consider now q_0. Since $f - 1 - 2d \leqslant -2d < 0$ on the domain of f and

$$q_0 \geqslant e_0 + e - 1 - d \geqslant 1 - 3d > \frac{1}{2}$$

on the domain of e_0 and e respectively, we have that

$$q_0 \longmapsto \frac{q_0}{q_0 + f - 1 - 2d}$$

is a positive, monotonically decreasing function for all choices of e_0, e and f in their respective domain. As the same thing is obviously true for $q_0 \longmapsto 1/q_0$, we attain the maximum if q_0 is chosen as small as possible, i.e. $q_0 = e_0 + e - 1 - d$. This yields for $\sigma(\widehat{W}_4)$

$$e_0^2 \cdot \left(\frac{\left(\frac{1}{e} - \frac{1}{e_0}\right)^+}{e_0 + e - 1 - d} + \frac{(e_0 + e - 1 - d)\left(\frac{\left(\frac{1}{e} - \frac{1}{f}\right)^+}{e+f-1-d} + \frac{\left(\frac{1}{e} - \frac{1}{e_0}\right)^+}{e_0+e-1-d}\right)}{e_0 + e + f - 2 - 3d}\right).$$

Finally, consider e. Clearly, by similar types of arguments as before, the functions

$$e \longmapsto \frac{\left(\frac{1}{e} - \frac{1}{f}\right)^+}{e + f - 1 - d}, \quad e \longmapsto \frac{\left(\frac{1}{e} - \frac{1}{e_0}\right)^+}{e_0 + e - 1 - d}$$

are non-neagtive and monotonically decreasing in e for all e_0, f as given by the domain. Furthermore, since $f - 1 - 2d \leqslant -2d < 0$ and $e_0 + e - 1 - d \geqslant 1 - 3d > 0$, the function

$$e \longmapsto \frac{e_0 + e - 1 - d}{e_0 + e + f - 2 - 3d} = \frac{e_0 + e - 1 - d}{(e_0 + e - 1 - d) + f - 1 - 2d}$$

is positive and monotonically decreasing in e for all e_0, f as given by the domain.

Hence, as the product and sum of non-negative, monotonically decreasing functions is also non-negative and monotonically decreasing, we get an optimal solution for $e = 1 - d$. As the ramp functions are, because of the substitution $e \longrightarrow 1 - d$, obsolete, we get

$$\sigma(\widehat{W}_4) = e_0^2\left(\frac{\frac{1}{1-d} - \frac{1}{e_0}}{e_0 - 2d} + \frac{e_0 - 2d}{e_0 + f - 1 - 4d}\left(\frac{\frac{1}{1-d} - \frac{1}{f}}{f - 2d} + \frac{\frac{1}{1-d} - \frac{1}{e_0}}{e_0 - 2d}\right)\right).$$

This concludes the proof. □

Corollary 4.25. $\mathsf{OPT}(\mathrm{P1}) = \mathsf{OPT}(\mathrm{P4})$.

4.5.3 *Reduction to one variable*

Theorem 4.26. $\mathrm{OPT}(P4) = \mathrm{OPT}(P5)$ where (P5) is defined as follows:

$$\begin{aligned} &\textit{Maximize} && \widehat{W}_5(f) = \frac{\frac{1}{1-d}-1}{1-2d} + \frac{1-2d}{f-4d}\left(\frac{\frac{1}{1-d}-\frac{1}{f}}{f-2d} + \frac{\frac{1}{1-d}-1}{1-2d}\right) \\ &\textit{subject to} && f \in [1-d, 1]. \end{aligned}$$

Proof. To prove the claim, we need to show that the choice $e_0 = 1$ is optimal. For this, we need to do a somewhat careful analysis. We will show, one by one, that the functions

$$\varphi\colon [1-d, 1] \longrightarrow \mathbb{R}_{\geqslant 0}, e_0 \longmapsto e_0^2 \cdot \frac{\frac{1}{1-d}-\frac{1}{e_0}}{e_0-2d} = e_0 \cdot \frac{e_0-(1-d)}{(1-d)(e_0-2d)},$$
$$\zeta\colon [1-d, 1] \longrightarrow \mathbb{R}_{\geqslant 0}, e_0 \longmapsto e_0 \cdot \frac{e_0-2d}{e_0+f-1-4d} \cdot \left(\frac{\frac{1}{1-d}-\frac{1}{f}}{f-2d} + \frac{\frac{1}{1-d}-\frac{1}{e_0}}{e_0-2d}\right)$$

are monotonically increasing on the domain of (P4). Then, since the sum and product of non-negative, monotonically increasing functions is itself monotonically increasing, we get that $e_0 = 1$ is indeed the optimal choice.

For φ, it suffices to show that $\ln(\varphi)\colon [1-d, 1] \longrightarrow [-\infty, \infty)$ is (for fixed $d \in [0, 1/6)$) monotonically increasing in e_0. This gives

$$\ln(e_0) + \ln(e_0 - 1 + d) - \ln(1-d) - \ln(e_0 - 2d).$$

Taking the derivative with respect to e_0, we get

$$\begin{aligned} \frac{1}{e_0} + \frac{1}{e_0-1+d} - \frac{1}{e_0-2d} &\geqslant 1 + \frac{1}{1-1+d} - \frac{1}{(1-d)-2d} && (e_0 \in [1-d, 1]) \\ &\geqslant 1 + \frac{1}{d} - \frac{1}{1-3d} \\ &> 5. && (d \in [0, 1/6)) \end{aligned}$$

As the derivative is positive, $\ln(\varphi)$ and therefore φ is strictly monotonically increasing.

For ζ, let $1-d \leqslant x < y \leqslant 1$. We want to show that

$$\frac{x(x-2d)}{x+f-1-4d} \cdot \left(\frac{\frac{1}{1-d}-\frac{1}{f}}{f-2d} + \frac{\frac{1}{1-d}-\frac{1}{x}}{x-2d}\right) \leqslant \frac{y(y-2d)}{y+f-1-4d} \cdot \left(\frac{\frac{1}{1-d}-\frac{1}{f}}{f-2d} + \frac{\frac{1}{1-d}-\frac{1}{y}}{y-2d}\right).$$

Shuffling terms around, this is equivalent to

$$\begin{aligned} &\left(\frac{x(x-2d)}{x+f-1-4d} - \frac{y(y-2d)}{y+f-1-4d}\right) \cdot \frac{\frac{1}{1-d}-\frac{1}{f}}{f-2d} \\ =& \left(\frac{x(x-2d)}{x+f-1-4d} - \frac{y(y-2d)}{y+f-1-4d}\right) \cdot \frac{f-(1-d)}{(1-d)f(f-2d)} \\ \leqslant& \frac{y(y-2d)}{y+f-1-4d} \cdot \frac{\frac{1}{1-d}-\frac{1}{y}}{y-2d} - \frac{x(x-2d)}{x+f-1-4d} \cdot \frac{\frac{1}{1-d}-\frac{1}{x}}{x-2d} \\ =& \frac{1}{1-d}\left(\frac{y-(1-d)}{y+f-1-4d} - \frac{x-(1-d)}{x+f-1-4d}\right). \end{aligned}$$

We may multiply by $(1-d)(x+f-1-4d)(y+f-1-4d)>0$ to get

$$\begin{aligned}
&(x(x-2d)(y+f-1-4d)-y(y-2d)(x+f-1-4d))\cdot\frac{f-(1-d)}{f(f-2d)}\\
=\ &(x-y)\left(8d^2+2d(1-f)-((1-f)+4d)(x+y)+xy\right)\cdot\frac{f-(1-d)}{f(f-2d)}\\
\leqslant\ &(y-(1-d))(x+f-1-4d)-(x-(1-d))(y+f-1-4d)\\
=\ &(y-x)(f-1-4d)-(1-d)(x-y)\\
=\ &(y-x)(f-5d).
\end{aligned}$$

Hence, dividing by $(y-x)>0$ and multiplying by $f(f-2d)>0$, it suffices to show

$$\left(((1-f)+4d)(x+y)-\left(8d^2+2d(1-f)+xy\right)\right)(f-(1-d))\leqslant f(f-2d)(f-5d). \quad (*)$$

Note that

$$\begin{aligned}
&((1-f)+4d)(x+y)-\left(8d^2+2d(1-f)+xy\right)\\
=\ &\underbrace{((1-f)+4d-x)}_{<0}y+((1-f)+4d)x-8d^2-2d(1-f)\\
\leqslant\ &2((1-f)+4d)x-x^2-8d^2-2d(1-f),
\end{aligned}$$

where we used the assumption $y>x$ for the last inequality. Differentiating[6] with respect to $x\in[1-d,1]$, we obtain

$$2\left((1-f)+d\right)-2x=2\left(1+d-(x+f)\right)\leqslant 2\left(1+d-2(1-d)\right)=2\left(3d-1\right)<0.$$

Thus, since $f-(1-d)\geqslant 0$, the left hand side of $(*)$ is bounded by

$$\begin{aligned}
&\left(2\left((1-f)+4d\right)(1-d)-8d^2-2d(1-f)-(1-d)^2\right)(f-(1-d))\\
=\ &\left(2(2d-1)f+1+6d-17d^2\right)(f-(1-d))
\end{aligned}$$

Hence, we may show that $\eta\colon[1-d,1]\longrightarrow\mathbb{R}$ defined by

$$f\longmapsto f(f-2d)(f-5d)-\left(2(2d-1)f+1+6d-17d^2\right)(f-(1-d))$$

is non-negative for all $d\in[0,1/6)$. Note that η expands out to

$$\eta(f)=f^3+(2-11d)f^2+\left(23d^2-3\right)f+\left(17d^3-23d^2+5d+1\right)$$

It follows that the derivative of η is

$$\begin{aligned}
\eta'(f)&=3f^2+\underbrace{(4-22d)}_{>0}f+\left(23d^2-3\right)\\
&\geqslant 3\left(1-d\right)^2+\left(4-22d\right)\left(1-d\right)+\left(23d^2-3\right)\\
&=4\left(1-2d\right)\left(1-6d\right)\\
&>0.
\end{aligned}$$

As $d<1/6$, $\eta'(f)$ is positive, meaning that η strictly monotonically increasing. Hence, η is indeed non-negative as

$$\eta(1-d)=(1-d)(1-3d)(1-6d)>0.$$

Thus, we have shown that ζ is monotonically increasing in e_0, showing that $e_0=1$ yields the optimal value for (P4). Plugging that in gives the objective function of (P5). □

Corollary 4.27. $\mathrm{OPT}(P1)=\mathrm{OPT}(P5)$.

[6] For fixed $d\in[0,1/6)$ and $f\in[1-d,1]$.

4.5.4 *The final optimization*

Despite reducing our optimization program to only one variable, solving (P5) for general $d \in [0, 1/6)$ remains somewhat tedious, especially by hand. However, we will be able to determine the optimal d for which $\mathrm{OPT}(P5) \leqslant 1$.

The following proposition will prove to be crucial:

Proposition 4.28. The polynomial $p(x) = 8x^3 - 22x^2 + 10x - 1$ has a unique root $x^* \approx 0.1421657737$ within the interval $[0, 1/6]$. In particular, $p(x) < 0$ for $0 \leqslant x < x^*$ and $p(x) > 0$ for $x^* < x \leqslant 1/6$.

Proof. To prove the uniqueness, note that for $x \in [0, 1/6]$

$$p'(x) = 24x^2 - 44x + 10 \geqslant -\frac{44}{6} + 10 > 0.$$

Hence, p is strictly, monotonically increasing on the interval $[0, 1/6]$. Since, $p(0) = -1$ and $p(1/6) = 5/54 > 0$, this concludes the proof. □

Lemma 4.29. For $d \in [0, 1/6)$, we have

$$\hat{W}_5(1) \leqslant 1 \iff d \leqslant x^*.$$

In particular, $\hat{W}_5(1) = 1$ if and only if $d = x^*$.

Proof. Consider the assignment $f = 1$. We then have

$$\begin{aligned}
\hat{W}_5(1) &= \frac{\frac{1}{1-d} - 1}{1-2d} + \frac{1-2d}{1-4d}\left(\frac{\frac{1}{1-d} - 1}{1-2d} + \frac{\frac{1}{1-d} - 1}{1-2d}\right) \\
&= \frac{\frac{1}{1-d} - 1}{1-2d} + \frac{2}{1-4d}\left(\frac{1}{1-d} - 1\right) \\
&= \left(\frac{1}{1-2d} + \frac{2}{1-4d}\right)\frac{d}{1-d} \\
&= \frac{(1-4d) + 2(1-2d)}{(1-2d)(1-4d)}\frac{d}{1-d} \\
&= \frac{8d^2 - 3d}{8d^3 - 14d^2 + 7d - 1}.
\end{aligned}$$

Note that $8d^3 - 14d^2 + 7d - 1 = (d-1)(2d-1)(4d-1) < 0$ since $d \in [0, 1/6]$. However, by Proposition 4.28, we know that

$$\begin{aligned}
d > x^* \iff 8d^3 > 22d^2 - 10d + 1 \quad &\iff 8d^3 - 14d^2 + 7d - 1 > 8d^2 - 3d \\
\iff \quad 1 < \frac{8d^2 - 3d}{8d^3 - 14d^2 + 7d - 1} \quad &\iff \quad \hat{W}_5(1) > 1,
\end{aligned}$$

where we used $8d^3 - 14d^2 + 7d - 1 < 0$ for the third equivalence. In particular, these equivalences also hold if every "$<$" and "$>$" is replaced by "$=$". □

Corollary 4.30. We have $\mathrm{OPT}(P5) > 1$ for all $d > x^*$.

Remark 4.31. It is interesting to note that $f = 1$ doesn't yield the maximum for $d > x^*$. This is not too hard to show and can be seen in Figure 4.3. However, for $d \leqslant x^*$ this assignment is indeed optimal.

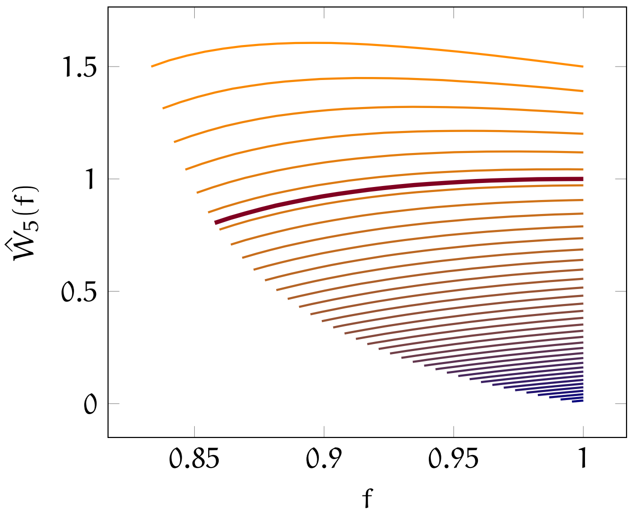

Figure 4.3: $\hat{W}_5(f)$ for $d = 0$ to $d = 1/6$ and $d = x^*$

Theorem 4.32. For $d \leqslant x^*$, $\mathrm{OPT}(P5)$ is obtained by $f = 1$. In particular,

$$\mathrm{OPT}(P5) \leqslant 1$$

where equality holds if only if $d = x^*$.

Proof. We first expand $\hat{W}_5$:

$$\begin{aligned}\hat{W}_5 &= \frac{\frac{1}{1-d} - 1}{1-2d} + \frac{1-2d}{f-4d}\left(\frac{\frac{1}{1-d} - \frac{1}{f}}{f-2d} + \frac{\frac{1}{1-d} - 1}{1-2d}\right) \\ &= \frac{d}{(1-d)(1-2d)} + \frac{1-2d}{1-d}\frac{1}{f-2d}\frac{1}{f-4d} - (1-2d)\frac{1}{f}\frac{1}{f-2d}\frac{1}{f-4d} + \frac{d}{1-d}\frac{1}{f-4d}.\end{aligned}$$

To finish the proof, we only need to show that $\hat{W}_5$ is monotonically increasing in f for fixed $d \in [0, x^*]$. The claim then follows from Lemma 4.29. To do so, we will (up to a positive constant) calculate the derivative of $\hat{W}_5$ with respect to f and show that it is non-negative. For that, we will show that actually $\hat{W}_5'$ is monotonically decreasing in f for fixed $d \in [0, x^*]$. However, it will turn out that $\hat{W}_5'(1) \geqslant 0$, completing the proof:

Multiplying $\hat{W}_5$ by $(1-d)/(1-2d) > 0$ and ignoring constant terms, consider

$$\varphi\colon [1-d, 1] \longrightarrow \mathbb{R}, f \longmapsto \frac{d}{1-2d}\frac{1}{f-4d} + \frac{1}{f-2d}\frac{1}{f-4d} - (1-d)\frac{1}{f}\frac{1}{f-2d}\frac{1}{f-4d}.$$

The derivative of φ is

$$\begin{aligned}\varphi'(f) = &\frac{-d}{1-2d}\frac{1}{(f-4d)^2} - \frac{1}{(f-2d)^2}\frac{1}{f-4d} - \frac{1}{f-2d}\frac{1}{(f-4d)^2} \\ &+ \frac{1-d}{f^2}\frac{1}{f-2d}\frac{1}{f-4d} + \frac{1-d}{f}\frac{1}{(f-2d)^2}\frac{1}{f-4d} + \frac{1-d}{f}\frac{1}{f-2d}\frac{1}{(f-4d)^2}.\end{aligned}$$

We want to show that φ' is monotonically decreasing in f for fixed $d \in [0, x^*]$. Consider the functions

$$\psi_1\colon [1-d, 1] \longrightarrow \mathbb{R}_{\geqslant 0}, f \longmapsto \left(\frac{f-(1-d)}{f}\right)\frac{1}{f-2d}\frac{1}{(f-4d)^2},$$

$$\psi_2\colon [1-d,1] \longrightarrow \mathbb{R}_{\geqslant 0}, f \longmapsto \left(\frac{f-(1-d)}{f}\right)\frac{1}{(f-2d)^2}\frac{1}{f-4d},$$
$$\psi_3\colon [1-d,1] \longrightarrow \mathbb{R}_{\geqslant 0}, f \longmapsto \frac{1}{f-4d}\left(\frac{d}{1-2d}\frac{1}{f-4d}-\frac{1-d}{f^2}\frac{1}{f-2d}\right).$$

Note that $\varphi' = -(\psi_1+\psi_2+\psi_3)$. Hence, it suffices to show that each ψ_i is monotonically increasing in f.

For ψ_1, we will equivalently show that

$$(\ln(\psi_1))(f) = \ln(f-(1-d)) - \ln(f) - \ln(f-2d) - 2\ln(f-4d)$$

is monotonically increasing. For that, we take the derivative:

$$(\ln(\psi_1))'(f) = \frac{1}{f-(1-d)} - \frac{1}{f} - \frac{1}{f-2d} - \frac{2}{f-4d}.$$

To show that the derivative is non-negative, it suffices to show

$$\begin{aligned}
& \frac{1}{f-(1-d)} \geqslant \frac{1}{f} + \frac{1}{f-2d} + \frac{2}{f-4d} \\
\iff\quad & 1 \geqslant \frac{f-(1-d)}{f} + \frac{f-(1-d)}{f-2d} + \frac{2(f-(1-d))}{f-4d} \\
& = \frac{f-(1-d)}{f} + \frac{(f-2d)-(1-3d)}{f-2d} + 2\cdot\frac{(f-4d)-(1-5d)}{f-4d}.
\end{aligned}$$

Clearly, the last expression is monotonically increasing in f, meaning that w.l.o.g. we may consider $f=1$. This gives

$$\begin{aligned}
& 1 \geqslant d + \frac{d}{1-2d} + \frac{2d}{1-4d} \\
\iff\ & 8d^2 - 6d + 1 \geqslant 8d^3 - 14d^2 + 4d \\
\iff\ & 0 \geqslant 8d^3 - 22d^2 + 10d - 1.
\end{aligned}$$

As the last inequality holds by Proposition 4.28, the derivative of $\ln(\psi_1)$ is indeed non-negative and we are done.

We proceed similarly for ψ_2:

$$\begin{aligned}
(\ln(\psi_2))(f) &= \ln(f-(1-d)) - \ln(f) - 2\ln(f-2d) - \ln(f-4d), \\
(\ln(\psi_2))'(f) &= \frac{1}{f-(1-d)} - \frac{1}{f} - \frac{2}{f-2d} - \frac{1}{f-4d}.
\end{aligned}$$

Comparing $(\ln(\psi_2))'$ with $(\ln(\psi_1))'$, we see that $(\ln(\psi_2))' \geqslant (\ln(\psi_1))' \geqslant 0$, giving us immediately that ψ_2 is monotonically increasing.

For ψ_3, we equivalently show that

$$-\psi_3(f) = \frac{1}{f-4d}\left(\frac{1-d}{f^2}\frac{1}{f-2d} - \frac{d}{1-2d}\frac{1}{f-4d}\right)$$

is monotonically decreasing. Actually, we will show that

$$\zeta(f) = \frac{1-d}{f^2}\frac{1}{f-2d} - \frac{d}{1-2d}\frac{1}{f-4d}$$

is monotonically decreasing in f for fixed $d \in [0, x^*]$. Indeed, that would imply that

$$\begin{aligned}
\frac{1-d}{f^2}\frac{1}{f-2d} - \frac{d}{1-2d}\frac{1}{f-4d} &\geqslant \frac{1-d}{1-2d} - \frac{d}{1-2d}\frac{1}{1-4d} \\
&= \frac{1}{1-2d}\left(1-d-\frac{d}{1-4d}\right) \\
&> \frac{1}{1-2d}(1-d-3d) \qquad (1-4d > 1/3) \\
&> 0,
\end{aligned}$$

meaning that ζ is also positive. As $f \longmapsto 1/(f-4d)$ is also a positive, monotonically decreasing function, it would follow that $-\psi_3$, as the product of positive, monotonically decreasing functions, is monotonically decreasing.

Hence, consider $1-d \leqslant f < g \leqslant 1$. We need to show that

$$\begin{aligned}
\frac{1-d}{f^2(f-2d)} - \frac{d}{1-2d}\frac{1}{f-4d} &\geqslant \frac{1-d}{g^2(g-2d)} - \frac{d}{1-2d}\frac{1}{g-4d} \\
\iff (1-d)\left(\frac{1}{f^2(f-2d)} - \frac{1}{g^2(g-2d)}\right) &\geqslant \frac{d}{1-2d}\left(\frac{1}{f-4d} - \frac{1}{g-4d}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{f^2(f-2d)} - \frac{1}{g^2(g-2d)} &= \frac{g^2(g-2d) - f^2(f-2d)}{f^2(f-2d)g^2(g-2d)} \\
&= \frac{(g-f)\left(f^2+fg+g^2-2d(g+f)\right)}{f^2(f-2d)g^2(g-2d)}, \\
\frac{1}{f-4d} - \frac{1}{g-4d} &= \frac{(g-4d)-(f-4d)}{(f-4d)(g-4d)} \\
&= \frac{g-f}{(f-4d)(g-4d)}.
\end{aligned}$$

Thus, it suffices to show that

$$\begin{aligned}
(1-d)\frac{(g-f)\left(f^2+fg+g^2-2d(g+f)\right)}{f^2(f-2d)g^2(g-2d)} &\geqslant \frac{d}{1-2d}\frac{g-f}{(f-4d)(g-4d)} \\
\iff \qquad \frac{f^2+fg+g^2-2d(g+f)}{f^2g^2} &\geqslant \frac{d}{(1-d)(1-2d)}\frac{f-2d}{f-4d}\frac{g-2d}{g-4d}.
\end{aligned}$$

We will lower bound and upper bound the left hand side and right hand side respectively:

$$\begin{aligned}
\frac{d}{(1-d)(1-2d)}\frac{f-2d}{f-4d}\frac{g-2d}{g-4d} &\leqslant \frac{d}{(1-d)(1-2d)}\frac{(1-3d)^2}{(1-5d)^2} \\
&= \frac{d}{(1-d)(1-2d)}\left(1+\frac{2d}{1-5d}\right)^2, \\
\frac{f^2+fg+g^2-2d(g+f)}{f^2g^2} &\geqslant \frac{f^2+fg+g^2-2d(g+f)}{f^2} \\
&= 1+\frac{g}{f}+\left(\frac{g}{f}\right)^2 - 2d\left(\frac{g}{f^2}+\frac{1}{f}\right)
\end{aligned}$$

$$\geqslant 3 - 2d\left(\frac{1}{(1-d)^2} + \frac{1}{1-d}\right),$$

where we used $f^2 + fg + g^2 - 2d(g+f) > 0$ for the second inequality.

Note that the lower bound is monotonically decreasing in d and the upper bound is monotonically increasing in d. Hence,

$$\begin{aligned}
&\frac{f^2 + fg + g^2 - 2d(g+f)}{f^2 g^2} - \frac{d}{(1-d)(1-2d)} \frac{f-2d}{f-4d} \frac{g-2d}{g-4d} \\
&\geqslant 3 - 2d\left(\frac{1}{(1-d)^2} + \frac{1}{1-d}\right) - \frac{d}{(1-d)(1-2d)}\left(1 + \frac{2d}{1-5d}\right)^2 \\
&\geqslant 3 - \frac{8}{25}\left(\left(\frac{25}{21}\right)^2 + \frac{25}{21}\right) - \frac{4}{21}\frac{25}{17}\left(1 + \frac{8}{5}\right)^2 && (d < 4/25) \\
&\geqslant \frac{2039}{7497} \\
&> 0.
\end{aligned}$$

Thus, ζ is monotonically decreasing, meaning that ψ_3 is monotonically increasing.

Altogether, we therefore have that φ', which is up to a positive constant $\widehat{W}_5'$, is indeed monotonically decreasing in f for fixed $d \in [0, x^*]$. It remains to show that $\varphi'(1) \geqslant 0$:

$$\begin{aligned}
-\psi_1(1) &= -\frac{d}{1-2d}\frac{1}{(1-4d)^2}, \\
-\psi_2(1) &= -\frac{d}{(1-2d)^2}\frac{1}{1-4d}, \\
-\psi_3(1) &= \frac{1}{1-4d}\left(\frac{1-d}{1-2d} - \frac{d}{1-2d}\frac{1}{1-4d}\right), \\
\varphi'(1) &= \frac{1}{(1-2d)(1-4d)}\left((1-d) - d\left(\frac{1}{1-2d} + \frac{2}{1-4d}\right)\right) \\
&= \frac{1}{(1-2d)^2(1-4d)^2}\left((1-d)(1-2d)(1-4d) - d\left((1-4d) + 2(1-2d)\right)\right) \\
&= \frac{-8d^3 + 22d^2 - 10d + 1}{(1-2d)^2(1-4d)^2} \\
&\geqslant 0. && \text{(Prop. 4.28)}
\end{aligned}$$

This completes the proof. □

Remark 4.33. Though this approach is inherently ineffective to resolve an asymptotic version of Conjecture 1.6, Delcourt and Postle suspect that the same is true for their approach in resolving Conjecture 2.17: Even with a more detailed analysis where additional global constraints on the densities are added, Delcourt and Postle suggest that ultimately a non-uniform distribution of each pair's demand among the cliques will prove necessary for further progress.

§ 5. CONCLUDING REMARKS

In this thesis, we gave a detailed overview and provided proof sketches to Lee's results in [35] and related the problem to existing problems in combinatorics. Furthermore, we quantitatively improved on Lee's main result, Theorem 1.4, by providing a better upper bound on θ^*_{STS} in Theorem 4.1, resulting in Theorem 1.8. Still, the resolution of Conjecture 1.6 seems very much out of reach.

5.1 MOVING TOWARDS CONJECTURE 1.6

5.1.1 *The conjectured value of* θ^*_{STS}

The obvious way on making progress is of course showing that $\theta^*_{\mathrm{STS}} \leqslant 3/4$. Together with the (transversal version of) Theorem 3.6, this would at least resolve the asymptotic version of Conjecture 1.6.

In fact, during the study of the problem, Lee originally conjectured that $\theta_{\mathrm{STS}} = 2/3$ and now believes that $\theta^*_{\mathrm{STS}} = 2/3$.

Conjecture 5.1. $\theta^*_{\mathrm{STS}} = 2/3$.

The original conjecture was made due to two reasons: On the one hand, the minimum degree threshold in the Nash-Williams conjecture implies that every edge forms a triangle with at least 2/3 of the vertices, which on the hypergraph level corresponds to a minimum codegree threshold of 2/3. On the other hand, Lee came up with the following natural construction:

Lemma 5.2. Consider the 3-uniform hypergraph $H = (V, E)$ on $n \geqslant 3$ vertices which is constructed as follows: Take an equitable[1] partition $V_1 \cup V_2 \cup V_3$ of V. Then $\delta_2(H) \geqslant 2n/3 - 8/3$ and H doesn't contain a fractional Steiner triple system.

Proof. Since the partition is equitable, we have for all $i \in [3]$

$$\frac{n}{3} - \frac{2}{3} \leqslant |V_i| \leqslant \frac{n}{3} + \frac{2}{3}.$$

Furthermore, by construction, it follows that the codegree is smallest if the vertices are taken from two different parts. Hence, we get

$$\delta_2(H) = (|V| - 2) - \max_{i \in [3]} |V_i| \geqslant \frac{2n}{3} - \frac{8}{3}.$$

[1] Meaning the size of any two partition classes differ by at most by one.

It remains to show that H doesn't contain a fractional Steiner triple system: Note that if φ is a fractional Steiner triple system, we must have that

$$3\|\varphi\|_1 = \sum_{e \in E(H)} \sum_{p \in E(\partial H)\colon p \subseteq e} \varphi(e) = \sum_{p \in E(\partial H)} \deg^{\varphi}(p) = \sum_{p \in E(\partial H)} 1 = \binom{n}{2},$$

where we have used that every edge covers exactly 3 pairs and that ∂H is complete.

However, any $\varphi\colon E(H) \longrightarrow \mathbb{R}_{\geqslant 0}$ satisfying $\deg^{\varphi}(p) \leqslant 1$ for all $p \in E(\partial H)$ must satisfy $\|\varphi\|_1 \leqslant n(n-3)/6$. Indeed, for $i \in [3]$, let $n_i = |V_i|$. Observe that any edge in H covers at least one pair in $V_1^{(2)} \cup V_2^{(2)} \cup V_3^{(2)}$. Hence, applying Jensen's inequality, we get that $\|\varphi\|_1$ is at most

$$\begin{aligned}\sum_{p \in V_1^{(2)} \cup V_2^{(2)} \cup V_3^{(2)}} \deg^{\varphi}(p) &\leqslant \binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} = 3\sum_{i=1}^{3} \frac{1}{3} \cdot \frac{n_i(n_i - 1)}{2} \\ &\leqslant n \cdot \frac{\frac{n_1+n_2+n_3}{3} - 1}{2} = \frac{n(n-3)}{6}.\end{aligned}$$

Therefore, H does not contain a fractional Steiner triple system. □

Corollary 5.3. $\theta^*_{\mathrm{STS}} \geqslant \theta^f_{\mathrm{STS}}(0) \geqslant 2/3$.

Remark 5.4. Note that, in contrast to the construction given in Lemma 1.7, our argument for why H doesn't contain a Steiner triple system relies on a *space barrier*: In some sense, there is not enough space for the edges to accommodate the demand of each pair. Furthermore, it is important to emphasize that the construction in Lemma 5.2 works for *any* n. Meanwhile, to construct the hypergraph as in Lemma 1.7, one fundamentally requires n being odd.

Since the construction given in Lemma 5.2 relies on a space barrier, which is more "robust" than a parity barrier, and works for arbitrary $n \geqslant 3$, Conjecture 5.1 seems at least reasonable. Similar conjectures involving the minimum codegree threshold of containing a partial Steiner triple system covering all but at most $o(n^2)$ pairs are also plausible.

One can also verify by hand that the construction given in Lemma 1.7 contains fractional Steiner triple systems, see Chapter A, while the construction given in Lemma 5.2 provably doesn't.[2] This is in stark contrast to the Nash-Williams conjecture where it is known that the minimum degree threshold for a fractional K_3-decomposition is at least $3/4$, see the concluding remarks of [49].

Interestingly enough, the construction given in Lemma 5.2 (with some added parity requirements) is provably optimal if we want to find *odd Steiner systems*, see Lemma 5.17.

5.1.2 *Coregular hypergraphs*

Another fruitful step towards resolving Conjecture 1.6 may be to consider hypergraphs with the following property:

Definition 5.5 ((δ, ρ)-coregular)**.** Let $\delta \in [0, 1]$ and $\rho > 0$. We call an n-vertex 3-uniform hypergraph H (δ, ρ)-*coregular* if $\deg(u, v) \in (\delta \pm \rho)n$ holds for all distinct $u, v \in V(H)$.

[2] A program to compute fractional Steiner triple systems for small instances of that construction can also be found in Section C.2.

In other words, H is coregular precisely when H_{aux} is pseudorandom in the sense of Pippenger-Spencer. Restricting to these hypergraphs should make the problem easier as applying Pippenger-Spencer-type results on H_{aux} yields partial Steiner systems that miss only $o(n^2)$ many edges. Hence, it seems reasonable that one could bypass the usage of θ^*_{STS} in the cover down step, resolving the asymptotic version of Conjecture 1.6 in this special case.

We also note that coregularity, while as far as we know not considered before, is a fairly natural thing to consider. Indeed, one can proceed similarly to the proof of Lemma 5.7 to show that $G^{(3)}(n, p)$ is for constant $p \in [0, 1]$ $(pn, o(1))$-coregular with high probability.

Additionally, the construction in Lemma 1.7 is $(3/4, o(1))$-coregular. Similarly, by deleting the edges completely contained in one of the V_i's, one obtains from the construction in Lemma 5.2 a $(2/3, o(1))$-coregular hypergraph that, by the same, reasoning, contains no Steiner triple system. Hence, Conjecture 5.1 restricted to coregular hypergraphs can also be considered. In particular, even if Conjecture 1.6 or Conjecture 5.1 would turn out to be wrong, it would be interesting to consider whether the constructions at hand are at least the extremal constructions for the coregular case.

Consequently, we may define the *codegree threshold for Steiner triple systems* as follows:

Definition 5.6. Let θ^{co}_{STS} be the infimum over all $\delta \in [0, 1]$ for which the following olds: There exist $\rho > 0$ and $n_0 \in \mathbb{N}$ such that for all 3-uniform, (δ, ρ)-coregular hypergraphs H on $n \geqslant n_0$ vertices with $n \equiv 1, 3 \pmod 6$ contain a Steiner triple system. We will refer to θ^{co}_{STS} as the *codegree threshold (for Steiner triple systems)*.

By definition and Theorem 1.8, we have that $\theta^{co}_{STS} \leqslant \theta_{STS} < 1$.

However, apriori, it is not obvious why θ^{co}_{STS} should *act* like a threshold despite the coregularity condition. So, let us briefly talk about why calling θ^{co}_{STS} a threshold is justified: For $\theta \in \mathbb{R}$ to be a threshold for some *property* $\mathcal{P}$, we would expect that at θ the behaviour of whether $\mathcal{P}$ holds drastically changes, e.g. that for every (sensible) value smaller than θ $\mathcal{P}$ does not hold and for every (sensible) value greater than θ $\mathcal{P}$ holds. For this, we first show that a large coregular hypergraph always contains a coregular hypergraph of smaller density.

Lemma 5.7. For every $0 < \delta' < \delta \leqslant 1$ and $\delta > \rho > 0$ there exists $n_0 \in \mathbb{N}$ such that the following holds: If H is a 3-uniform, (δ, ρ)-coregular hypergraph on $n \geqslant n_0$ vertices, then there exists a spanning 3-uniform, $(\delta', 2\rho)$-coregular subhypergraph $H' \subseteq H$.

Proof. Let H be as described in the lemma. Let $p = \delta'/\delta \in (0, 1)$ and let H' be the random hypergraph obtained by independently adding an edge from H with probability p. Fix distinct $u, v \in V(H)$. We have that

$$\mathbb{E}\left[\deg_{H'}(u, v)\right] = p \cdot \deg_H(u, v) \in (\delta' \pm \rho p)n \subseteq (\delta' \pm \rho)n.$$

Applying Theorem B.2, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\deg_{H'}(u, v) \notin (\delta' \pm 2\rho)n\right) \\
\leqslant\ &\mathbb{P}\left(\left|\deg_{H'}(u, v) - \mathbb{E}\left[\deg_{H'}(u, v)\right]\right| \geqslant \rho n\right) \\
\leqslant\ &\mathbb{P}\left(\left|\deg_{H'}(u, v) - \mathbb{E}\left[\deg_{H'}(u, v)\right]\right| \geqslant \rho \mathbb{E}\left[\deg_{H'}(u, v)\right]\right) \\
\leqslant\ &2 \exp\left(-\frac{\rho^2}{3}\mathbb{E}\left[\deg_{H'}(u, v)\right]\right)
\end{aligned}$$

$$\leqslant 2\exp\left(-\frac{(\delta-\rho)\rho^2 p}{3}\cdot n\right).$$

Hence, via a simple union bound we get that

$$\mathbb{P}\left(\exists uv \in V(H)^{(2)}\colon\ \deg_{H'}(u,v) \notin (\delta' \pm 2\rho)n\right) \leqslant 2\binom{n}{2}\exp\left(-\frac{(\delta-\rho)\rho^2 p}{3}\cdot n\right)$$
$$\longrightarrow 0, n \longrightarrow \infty.$$

Therefore, if n is sufficiently large, every pair has codegree $(\delta' \pm 2\rho)n$ in H' with positive probability. In particular, there must exist a subhypergraph $H' \subseteq H$ with this property. □

Remark 5.8. We remark that the proof goes through as long as the deviation of the codegree is $\omega(\sqrt{n})$. Hence, if a deviation in the $o(n)$-regime is not strong enough, one could consider stricter conditions on the range of codegrees.

Corollary 5.9. θ^{co}_{STS} is a *threshold* in the following sense: For all $\theta^{co}_{STS} < \alpha \leqslant 1$ we have that there exist $\rho > 0$ and $n_0 \in \mathbb{N}$ such that every 3-uniform, (α, ρ)-coregular hypergraphs H on $n \geqslant n_0$ vertices with $n \equiv 1, 3 \pmod 6$ contains a Steiner triple system.

Proof. Call $\alpha \in [0, 1]$ *good* if it satisfies the properties from the statement. It suffices to show that if $0 < \alpha < \beta \leqslant 1$ and α is good, then β is good as well. For the sake of contradiction, assume that there are some $0 < \alpha < \beta \leqslant 1$ for which the converse is true. In particular, for every $\rho > 0$ there exist infinitely many 3-uniform, (β, ρ)-coregular hypergraphs H on $n \in \mathbb{N}$ vertices with $n \equiv 1, 3 \pmod 6$ and H not containing a Steiner triple system.

Let $n_0 \in \mathbb{N}$ and $\rho \in (0, \alpha)$ be the witnesses that α is good. Since β is not good, there exist infinitely many 3-uniform, $(\beta, \rho/2)$-coregular hypergraphs H on $n \geqslant n_0$ vertices with $n \equiv 1, 3 \pmod 6$ and H not containing a Steiner triple system. However, applying Lemma 5.7 for H with n sufficiently lareg, we get that H contains a spanning 3-uniform, (α, ρ)-coregular hypergraph $H' \subseteq H$ which, by our choice of n_0 and ρ, contains a Steiner triple system. ↯ □

Corollary 5.10. $3/4 \leqslant \theta^{co}_{STS} \leqslant \theta_{STS}$.

Lastly, we note that, even without bypassing θ^{*}_{STS} (or the coregular analogue thereof), it should be possible to adapt Lee's approach for a better bound on θ^{co}_{STS}. Indeed, as the probabilistic inequalities used are usually concentration inequalities with bounds in both directions such as Theorem B.2, this should be very doable. Furthermore, using the approach in Chapter 4, it should be possible to get a bound on the coregular analogue of θ^{*}_{STS} that is arbitrarily close too $5/6$ since the deviation of the codegrees can be made arbitrarily close.

5.1.3 *A new absorption paradigm?*

Recently, Delcourt and Postle gave a third proof of the existence conjecture, see [7]. The proof involves what they introduce as *refined absorption*, a novel method that should have the best qualities of the previously used methods ([27, 18]): It is a one-step absorption but is purely combinatorial.

With the recency of the paper, it is hard to make any predictions as to how widely the method will be adapted. Nevertheless, it would definitely be worth a try to consider this method for Conjecture 1.6.

5.2 VARIATIONS AND STRENGTHENINGS

5.2.1 *Odd Steiner systems*

Similarly to how people consider Roth's theorem (from additive combinatorics, see [45]) in the finite field setting, it might be useful as a "toy model" to consider what we will call *odd Steiner systems*. Essentially, the concept of odd Steiner systems arises from an $\mathbb{F}_2$-relaxation: What if instead of every r-set being covered by exactly one edge, we require it to be covered by an *odd number* of edges?

In the same way that (t-)intersecting families are generalized to L-intersecting families, where $t \in \mathbb{N}$ and $L \subseteq \mathbb{N}_0$, in extremal set theory, we thus may generalize combinatorial designs in the following way:

Definition 5.11 ((n, q, r, L)-design)**.** Let $1 \leqslant r < q \leqslant n$ and $L \subseteq \mathbb{N}_0$. We say that an n-vertex, q-uniform hypergraph H is an (n, q, r, L)-*design* if $\deg_H(f) \in L$ for all $f \in V^{(r)}$.

Remark 5.12. Note that (n, q, r, λ)-designs are the same as $(n, q, r, \{\lambda\})$-designs. In such cases, we will follow the convention and omit the set brackets.

Definition 5.13 (Odd Steiner systems)**.** Let $n \in \mathbb{N}$ be given. We say that an n-vertex, q-uniform hypergraph H is an *odd* (n, q, r)-*Steiner system* if it is a $(n, q, r, 2\mathbb{N} - 1)$-design. If $q = 3$ and $r = 2$, then we will also call H an *Steiner triple system*.

Unlike with the usual Steiner triple systems, the question concerning existence is fairly straightforward. Indeed, we have the following sufficient parity condition.

Lemma 5.14. Let $n \geqslant q \geqslant 2$. There exists an odd $(n, q, q - 1)$-Steiner system of order n if and only if $n \equiv q \pmod 2$.

Proof. We first show that the parity condition is necessary: Let $S = (V, E)$ be an arbitrary odd $(n, q, q - 1)$-Steiner system. By definition, we have $\deg(p) \equiv 1 \pmod 2$ for all pairs $p \in V^{(q-1)}$. Now, let $f \in V^{(q-2)}$ be arbitrary.

We wish to double count the sum of $q - 1$-degrees for all pairs containing f. On the one hand, this obviously equals

$$\sum_{u \in V \setminus f} \deg(\{u\} \cup f).$$

On the other hand, as every hyperedge $\{u, w\} \mathbin{\dot\cup} f \in E$ incident to f must contribute to the codegrees of $\{u\} \mathbin{\dot\cup} f$ and $\{w\} \mathbin{\dot\cup} f$, it follows that this sum also equals

$$\sum_{e \in E\colon f \subseteq e} 2.$$

Equating those two expressions in Modulo 2, we get

$$0 \equiv \sum_{u \in V \setminus f} \deg(\{u\} \cup f) \equiv |V \setminus f| \equiv n - (q - 2) \equiv n - k \pmod 2$$

In particular, $n \equiv q \pmod 2$.

On the other hand, $K_n^{(q)}$ is an odd Steiner triple system for every $n \equiv q \pmod 2$, hence the parity condition is sufficient. □

As in the case of Steiner triple systems, we are interested in the minimum codegree threshold for odd Steiner triple systems, or more generally the minimum $q-1$-degree threshold for odd $(n, q, q-1)$-Steiner systems. By generalizing the construction of Lemma 5.2 with some added parity requirements, one obtains that the threshold is at least $(q-1)/q$, this time with a parity argument:

Proposition 5.15. Let $n \geqslant q \geqslant 2$, $n \equiv q \pmod 2$, and consider the hypergraph H with vertex set $V = V_1 \cup \ldots \cup V_q$, where all parts V_i are odd and pairwise differ at most by two in their size, and edge set $E = V^{(q)} \setminus \{e \in V^{(q)} \colon |e \cap V_1| = \cdots = |e \cap V_q| = 1\}$. Then $\delta_{q-1}(H) \geqslant (q-1)n/q - \mathcal{O}(1)$ and H contains no odd $(n, q, q-1)$-Steiner system.

Proof. $\delta_{q-1}(H) \geqslant (q-1)n/q$ follows directly from the fact that all V_i's have roughly equal size. Hence, let us focus on the latter claim: Assume that $S \subseteq H$ is an odd $(n, q, q-1)$.

Consider the sum

$$M = \sum_{(v_1,\ldots,v_{q-1}) \in V_1 \times \cdots \times V_{q-1}} \deg_S(v_1, \ldots, v_{q-1}).$$

Since each of those $q-1$-degrees is odd, we get

$$M \equiv |V_1||V_2| \cdots |V_{q-1}| \equiv 1 \pmod 2.$$

On the other hand, as we remove the "partite edges" from H, every edge $e \in E(S)$ covering some $(v_1, \ldots, v_{q-1}) \in V_1 \times \cdots \times V_{q-1}$ must satisfy $|e \cap V_j| = 1 + \mathbb{1}_{i=j}$ for all $j \in [q-1]$ for some $i \in [q-1]$. Let $v_i' \in (e \cap V_i) \setminus \{v_i\}$. It follows that e must contribute to M in precisely two tuples, namely $(v_1, \ldots, v_i, \ldots, v_{q-1}), (v_1, \ldots, v_i', \ldots, v_{q-1})$. Since this is the case for any edge e that contributes to M, we must have that M is even, a contradiction. □

Remark 5.16. Apart from $q = 3$, which corresponds to the construction in Lemma 5.2 with added parity requirements, the case $q = 2$ is also familiar: The construction yields two disjoint, odd cliques of nearly the same size, the standard extremal construction for Dirac's theorem for matchings apart from $K_{\lfloor n/2 \rfloor - 1, \lceil n/2 \rceil + 1}$!

Hence, we observe that for $q = 2$, the minimum degree threshold for the usual variant – a perfect matching – and the odd variant – an odd matching – coincide. The same is asymptotically true if we consider the minimum codegree threshold for perfect matchings / odd perfect matchings in 3-uniform hypergraphs, see Chapter D. As we will see this is not the case for $q = 3$ and there is a in fact a *gap*.

The following lemma was proven together with Schacht and Lee:

Lemma 5.17. Let H be a 3-uniform hypergraph on $n \geqslant 3$ vertices with $n \equiv 1 \pmod 2$ and $\delta_2(H) > 2n/3$. Then H contains an odd Steiner triple system.

Proof. With respect to a subhypergraph $F \subseteq H$, we call a pair $p \in V(H)^{(2)}$ even if $\deg_F(p) \equiv 0 \pmod 2$.

Let $F \subseteq H$ be a subhypergraph of H minimizing the number of even pairs. Assume that the number of even pairs is greater than zero, otherwise we are done. We first observe that F is P_2-free: Assume that uvw form a P_2 in the graph induced by the even pairs. Due to the codegree condition, we have

$$|N_H(u,v) \cap N_H(u,w) \cap N_H(v,w)| \geqslant 3\delta_2(H) - 2n > 0,$$

meaning that there exists some $x \in V(H)$ such that $xuv, xuw, xvw \in E(H)$. Now consider the hypergraph $F' = (V(H), E(F) \triangle \{xuv, xuw, xvw\}) \subseteq E(H)$. It can be checked that uv and vw are now odd, $\deg_{F'}(u, w) \equiv \deg_F(u, w) + 1 \pmod 2$ and $\deg_{F'}(p) \equiv \deg_F(p) \pmod 2$ for all other pairs p. Hence, we decreased the number of even pairs by at least one. ↯

This means that the even pairs induce a matching. Let uv be one of the even pairs. We know by an elementary double counting argument that

$$\deg_F(u) = \frac{\sum_{w \in V(H) \setminus \{u\}} \deg_F(u, w)}{2}.$$

In particular, $\sum_{w \in V(H) \setminus \{u\}} \deg_F(u, w)$ must be even. However, as u is incident to precisely one even pair, we see that

$$\begin{aligned} \sum_{w \in V(H) \setminus \{u\}} \deg_F(u, w) &= \deg_F(u, v) + \sum_{w \in V(H) \setminus \{u, v\}} \deg_F(u, w) \\ &\equiv 0 + |V(H) \setminus \{u, v\}| \\ &\equiv n - 2 \\ &\equiv 1 \pmod 2. \text{↯} \end{aligned}$$

Hence, there are no even pairs in F and F is the desired odd Steiner triple system. □

A similar proof due to Lee works if we go one uniformity higher, but from then on, no progress has been made. We end this subsection formally stating the conjecture:

Conjecture 5.18. For every $q \geqslant 2$ there exists $C \geqslant 0$ such that for all sufficiently large n with $n \equiv q \pmod 2$ the following holds: Let H be a q-uniform hypergraph on n-vertices with $\delta_{q-1}(H) \geqslant (q-1)n/q + C$. Then H contains an odd $(n, q, q-1)$-Steiner system.

5.2.2 *A robust version of Conjecture 1.6*

After the resolution of Conjecture 1.6 and 5.1, it would be natural to ask the following:

Question 5.19. Are the lower bound constructions given in Lemma 1.7 and Lemma 5.2 for θ_{STS} and θ^*_{STS} *unique*?

Maybe it would even be possible to establish some type of stability result for both of them, though it seems unlikely for the former given the fact that the construction in Lemma 1.7 relies on a parity barrier.

Another interesting direction would be to consider robust versions of Conjecture 1.6. For that, we need to take a small detour: While the conjecture considers under what density (measured in the minimum codegree) a 3-uniform hypergraph always contains a Steiner triple system, it is just as natural to ask at what density (measured in the fraction of edges present) a “typical” 3-uniform hypergraph will contain a Steiner triple system. More formally, we could consider the *threshold probability* for a Steiner triple system in $G^{(3)}(n, p)$. It was conjectured by Simkin in 2017 (see [47]) and independently by Keevash in his 2018 ICM talk that the threshold probability should be $\Theta(\log(n)/n)$. After the resolution of the Kahn-Kalai conjecture (see [38]), a series of results ([46, 25]) towards the resolution of that conjecture were published until Jain and Pham in [21] and Keevash in [28] independently confirmed the conjecture.

Hence, after the resolution of Conjecture 1.6, it would be interesting to combine it with the result by Jain and Pham, and Keevash. Concretely, one may show one of the following conjectures:

Conjecture 5.20. There exist $C, C' > 0$ such that for all $n \in \mathbb{N}$ with $n \equiv 1, 3 \pmod 6$ and $p \geqslant C'(\log(n)/n)$ the following holds: Let H be a 3-uniform hypergraph on n vertices such that $\delta_2(H) \geqslant 3n/4 + C$. Then the *random sparsification* H_p, where we keep each edge in $E(H)$ with probability p independent of the outcome for the other edges, contains a Steiner triple system with high probability.

Conjecture 5.21 (Transversal version of Conjecture 5.20)**.** There exist $C, C' > 0$ such that for all $n \in \mathbb{N}$ with $n \equiv 1, 3 \pmod 6$ and $p \geqslant C'(\log(n)/n)$ the following holds: Let $\mathcal{H} = \{H^1, \dots, H^{n(n-1)/6}\}$ be a family of 3-uniform hypergraph with vertex set $[n]$ such that $\delta_2(H^i) \geqslant 3n/4 + C$ for all $i \in [n(n-1)/6]$. Then there exists a transversal Steiner triple system for $H_p^1, \dots, H_p^{n(n-1)/6}$.

Such results have already been established for other spanning structures where both the threshold probability and minimum (co-)degree threshold were known, see for example [33, 3, 1].

In fact, a general framework for establishing such robust thresholds was published by a subset of the authors involved in determining the threshold probability of Steiner triple systems, see [39]. Thus, if Conjecture 1.6 has been resolved, Conjecture 5.20 and Conjecture 5.21 should be within reach.

5.2.3 *A radical generalization of Conjecture 1.6 to higher uniformities*

Another obvious, though daunting problem would of course be to consider higher uniformities. Indeed, with the existence conjecture solved barely a decade ago, estimating thresholds in the way we did for Steiner triple systems seems out of reach. However, given our usage of Fact 2.7 in the proof of Theorem 1.4 and the (supposed) extremal constructions for the perfect matching case and the Steiner triple system case, we may propose the following conjecture:

Conjecture 5.22. For any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that the following holds for every $n \geqslant n_0$ such that $(n, q, q-1, 1)$ satisfy the divisibility conditions posed in Fact 2.4: Let H be a q-uniform hypergraph on n vertices. If the minimum $q-1$-degree of H is at least

$$\left(\frac{2q-3}{2q-2} + \varepsilon\right) n,$$

then H contains a $(n, q, q-1, 1)$-Steiner system.

Note how this conjecture generalizes both Dirac's theorem for matchings and Conjecture 1.6. Apart from this numerical agreement, there are also some other reasons in favor of this conjecture.

For perfect matchings, we know two extremal constructions, one of which is the disjoint union of odd cliques that are of (roughly) equal size. In other words, the vertices get partitioned into two odd partition classes in a balanced way and forbidding exactly the edges that go across the two partition classes. This seems very reminiscient to the part of the Construction in Lemma 1.7 where we have three odd partition classes, also (roughly) of equal size, and forbid the edges going across. However, in addition to the odd partition classes, we have one more even partition class in comparison to the perfect matching case.

Hence, it seems plausible that the extremal construction for $(n, q, q-1)$-Steiner systems should involve $2q-2$ partition classes of roughly equal size, q of which are of odd size and the remaining $q-2$ classes of even size. Additionally, the edges going across all odd classes should be forbidden.

5.2.4 *Getting the minimum degree involved*

Lastly, while a minimum degree condition alone can't induce the existence of Steiner triple systems, one may consider the situation where we additionally have some mild minimum codegree conditions:

Question 5.23. Does there exist a constant $\alpha \in [0, 1]$, such that for all $\varepsilon > 0$ the following is true for sufficiently large n satisfying $n \equiv 1, 3 \pmod 6$: If H is a 3-uniform hypergraph on n vertices such that $\delta_1(H) \geqslant (\alpha + \varepsilon)\binom{n-1}{2}$ and $\delta_2(H) \geqslant \varepsilon(n-2)$, then H contains a Steiner triple system? If so, what is the infimum over all such $\alpha \in [0, 1]$?

Remark 5.24. We note that if such an α exist, it must be at least $7/9$. Indeed, it is easy to see that the construction given in Lemma 5.2 has a minimum degree of at least

$$\binom{n-1}{2} - \left(\frac{n}{3} + \frac{4}{3}\right)^2 = \frac{7}{9}\binom{n-1}{2} + o(n^2).$$

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A EXPLICIT FRACTIONAL STEINER TRIPLE SYSTEMS

In this chapter, we give some explicit fractional Steiner triple systems in the construction described in Lemma 1.7.

Proposition A.1. Let H be the hypergraph constructed in Lemma 1.7 where we only require that $n \geqslant 21$ to be odd. Then H contains a fractional Steiner triple system.

Proof. Let $n_i = |V_i|$ for all $i \in \{0, 1, 2, 3\}$. Since the sizes are chosen to be as balanced as possible, while satisfying the parity conditions, we either have

- $n_1 = n_2 = n_3$ and $|n_0 - n_1| \leqslant 1$, or
- $\max_{i \in [3]} n_i - \min_{j \in [3]} n_j = 2$ and $n_0 = \min_{j \in [3]} n_j + 1$.

We will define an explicit weighting in either case.

If the first case holds, let $l = n_0$, $m = n_1$ and let $\psi \colon E(H) \longrightarrow [0, 1]$ be defined by

$$\psi(e) = \begin{cases} \frac{1}{3m}, & e \in E_0 \\ \frac{3m-l+1}{6m^2}, & e \in E_1 \\ \frac{(3l-6)m-l^2+l}{6m(m-1)(m-2)}, & e \in E_2 \\ \frac{6m^2-(3m-l+1)l}{12m^2(m-1)}, & e \in E_3. \end{cases}$$

One can verify that since $l \geqslant 5$ and $|l - m| \leqslant 1$

$$\begin{aligned} 3m - l + 1 &\geqslant 2m, \\ (3l-6)m - l^2 + 1 &\geqslant (3l-6)(l-1) - l^2 + 1 \\ &\geqslant (2l-9)l + 7 \\ &> 0, \\ 6m^2 - (3m - l + 1)l &\geqslant 6(l-1)^2 - (3l+4)l \\ &\geqslant 3l^2 - 16l + 6 \\ &> 0, \end{aligned}$$

so ψ is non-negative. It suffices to show that ψ assigns each pair a weight of one: Let $p \in V(H)^{(2)}$ be a pair. If $p \subseteq V_0$, then p is exactly covered by the edges in E_0. Hence, we get that

$$\deg^{\psi}(p) = \frac{|V_1 \cup V_2 \cup V_3|}{3m} = 1.$$

If $|p \cap V_0| = 1$, meaning that $|p \cap V_i| = 1$ for some $i \in [3]$, then since triples e with $|e \cap V_0| = 1$ are not in $E(H)$, the edges covering p are in E_0 or E_1. There are exactly $l - 1$

edges of the first type and $2m$ of the second type, giving us

$$\deg^{\psi}(p) = \frac{l-1}{3m} + 2m \cdot \frac{3m-l+1}{6m^2} = \frac{l-1+(3m-l+1)}{3m} = 1.$$

If $|p \cap V_i| = \left|p \cap V_j\right| = 1$ for some $1 \leqslant i < j \leqslant 3$, then the edges covering p are in E_1 or E_3 as triples going across V_1, V_2, and V_3 are not in $E(H)$. There are precisely l of the first type and $2(m-1)$ of the second type. Hence,

$$\begin{aligned}\deg^{\psi}(p) &= l \cdot \frac{3m-l+1}{6m^2} + 2(m-1) \cdot \frac{6m^2-(3m-l+1)l}{12m^2(m-1)} \\ &= \frac{(3m-l+1)l + (6m^2-(3m-l+1)l)}{6m^2} \\ &= 1.\end{aligned}$$

Lastly, if $p \subseteq V_i$ for some $i \in [3]$, then the edges covering p are in E_3 or E_2 since triples e with $|e \cap V_0| = 1$ are not in $E(H)$. There are exactly $2m$ edges of the first type and $m-2$ edges of the second type. Hence,

$$\begin{aligned}\deg^{\psi}(p)(p) &= (m-2) \cdot \frac{(3l-6)m-l^2+1}{6m(m-1)(m-2)} + 2m \cdot \frac{6m^2-(3m-l+1)l}{12m^2(m-1)} \\ &= \frac{(3l-6)m-l^2+1}{6m(m-1)} + \frac{6m^2-(3m-l+1)l}{6m(m-1)} \\ &= 1.\end{aligned}$$

If the second case holds, let $m = n_0$ and w.l.o.g. let $n_1 = m-1$ and $n_3 = m+1$. We make a case distinction on $n_2 \in \{n_1, n_3\}$.

If $n_2 = m-1$, let $\psi\colon E(H) \longrightarrow [0,1]$ be defined by

$$\psi(e) = \begin{cases} \frac{m-2}{3(m-1)^2}, & e \in E_0 \text{ and } e \cap V_3 = \emptyset \\ \frac{1}{3(m-1)}, & e \in E_0 \text{ and } e \cap V_3 \neq \emptyset \\ \frac{m-2}{3(m-1)^2}, & e \in E_1 \text{ and } e \cap V_3 = \emptyset \\ \frac{1}{3(m-1)}, & e \in E_1 \text{ and } e \cap V_3 \neq \emptyset \\ \frac{2m^3-13m^2+24m-15}{6(m-1)^2(m-2)(m-3)}, & e \in E_2 \text{ and } e \cap V_3 = \emptyset \\ \frac{m}{3(m-1)^2}, & e \in E_2 \text{ and } e \cap V_3 \neq \emptyset \\ \frac{2m^2-4m+3}{6(m-1)^2(m-2)}, & e \in E_3 \text{ and } e \cap V_3 = \emptyset \\ \frac{2m-3}{6(m-1)^2}, & e \in E_3 \text{ and } e \cap V_3 \neq \emptyset. \end{cases}$$

Once again, one can verify that for $n \geqslant 21$ ψ is non-negative. So, it suffices to show that ψ assigns each pair a weight of one: Let $p \in V(H)^{(2)}$ be a pair. If $p \subseteq V_0$, then p is exactly covered by the edges in E_0. Hence, we get

$$\begin{aligned}\deg^{\psi}(p) &= 2(m-1) \cdot \frac{m-2}{3(m-1)^2} + \frac{m+1}{3(m-1)} \\ &= \frac{2(m-2)+(m+1)}{3(m-1)}\end{aligned}$$

$$= 1.$$

If $|p \cap V_0| = 1$, meaning that $|p \cap V_i| = 1$ for some $i \in [3]$, then since triples e with $|e \cap V_0| = 1$ are not in $E(H)$, the edges covering p are in E_0 or E_1. There are exactly $m-1$ edges of the first type and $n_j + n_k$ of the second type where $\{j, k\} = [3] \setminus \{i\}$. This gives us

$$\begin{aligned}
\deg^{\psi}(p) &= (m-1) \cdot \frac{m-2}{3(m-1)^2} + (m-1) \cdot \frac{m-2}{3(m-1)^2} + (m+1) \cdot \frac{1}{3(m-1)} \\
&= \frac{m-2}{3(m-1)} + \frac{m-2}{3(m-1)} + \frac{m+1}{3(m-1)} \\
&= 1, \\
\deg^{\psi}(p) &= (m-1) \cdot \frac{1}{3(m-1)} + 2(m-1) \cdot \frac{1}{3(m-1)} \\
&= 1
\end{aligned}$$

for the case $p \cap V_3 = \emptyset$ and $p \cap V_3 \neq \emptyset$ respectively.

If $|p \cap V_i| = \left|p \cap V_j\right| = 1$ for some $1 \leqslant i < j \leqslant 3$, then the edges covering p are in E_1 or E_3 as triples going across V_1, V_2, and V_3 are not in $E(H)$. There are precisely m of the first type and $(n_i - 1) + (n_j - 1)$ of the second type. Hence,

$$\begin{aligned}
\deg^{\psi}(p) &= m \cdot \frac{m-2}{3(m-1)^2} + 2(m-1) \cdot \frac{2m^2-4m+3}{6(m-1)^2(m-2)} \\
&= \frac{m^2-2m}{3(m-1)^2} + \frac{2m^2-4m+3}{2(m-1)(m-2)} \\
&= 1, \\
\deg^{\psi}(p) &= m \cdot \frac{1}{3(m-1)} + 2(m-1) \cdot \frac{2m-3}{6(m-1)^2} \\
&= \frac{m}{3(m-1)} + \frac{2m-3}{3(m-1)} \\
&= 1
\end{aligned}$$

for the case $p \cap V_3 = \emptyset$ and $p \cap V_3 \neq \emptyset$ respectively.

Lastly, if $p \subseteq V_i$ for some $i \in [3]$, then the edges covering p are in E_3 or E_2 since triples e with $|e \cap V_0| = 1$ are not in $E(H)$. There are exactly $n_j + n_k$ of the first type where $\{j, k\} = [3] \setminus \{i\}$ and $n_i - 2$ edges of the second type. Hence, $\psi(p)$ equals

$$(m-1) \cdot \frac{2m^2-4m+3}{6(m-1)^2(m-2)} + (m+1) \cdot \frac{2m-3}{6(m-1)^2} + \frac{2m^3-13m^2+24m-15}{6(m-1)^2(m-2)} = 1$$

for $p \not\subseteq V_3$ and for $p \subseteq V_3$ we get

$$2(m-1) \cdot \frac{2m-3}{6(m-1)^2} + (m-1) \cdot \frac{m}{3(m-1)^2} = \frac{2m-3}{3(m-1)} + \frac{m}{3(m-1)} = 1.$$

If $n_2 = m+1$, let $\psi\colon E(H) \longrightarrow [0,1]$ be defined by

$$\psi(e) = \begin{cases} \frac{2}{m(m+1)}, & e \in E_0 \text{ and } e \cap V_1 = \emptyset \\ \frac{m-4}{m(m-1)}, & e \in E_0 \text{ and } e \cap V_1 \neq \emptyset \\ \frac{m^2-3m+4}{m(m+1)^2}, & e \in E_1 \text{ and } e \cap V_1 = \emptyset \\ \frac{2}{m(m+1)}, & e \in E_1 \text{ and } e \cap V_1 \neq \emptyset \\ \frac{m^2-2m+3}{2m(m-1)(m+1)}, & e \in E_2 \text{ and } e \cap V_1 = \emptyset \\ 0, & e \in E_2 \text{ and } e \cap V_1 \neq \emptyset \\ \frac{5m-3}{2m(m+1)^2}, & e \in E_3 \text{ and } e \cap V_1 = \emptyset \\ \frac{1}{2(m+1)}, & e \in E_3 \text{ and } e \cap V_1 \neq \emptyset. \end{cases}$$

Once again, one can verify that for $n \geqslant 21$ ψ is non-negative. So, it suffices to show that ψ assigns each pair a weight of one: Let $p \in V(H)^{(2)}$ be a pair. If $p \subseteq V_0$, then p is exactly covered by the edges in E_0. Hence, we get

$$\begin{aligned} \deg^{\psi}(p) &= 2(m+1) \cdot \frac{2}{m(m+1)} + (m-1) \cdot \frac{m-4}{m(m-1)} \\ &= \frac{4}{m} + \frac{m-4}{m} \end{aligned}$$

If $|p \cap V_0| = 1$, meaning that $|p \cap V_i| = 1$ for some $i \in [3]$, then since triples e with $|e \cap V_0| = 1$ are not in $E(H)$, the edges covering p are in E_0 or E_1. There are exactly $m-1$ edges of the first type and $n_j + n_k$ of the second type where $\{j,k\} = [3] \setminus \{i\}$. This gives us

$$\begin{aligned} \deg^{\psi}(p) &= (m-1) \cdot \frac{2}{m(m+1)} + (m-1) \cdot \frac{2}{m(m+1)} + (m+1) \cdot \frac{m^2-3m+4}{m(m+1)^2} \\ &= \frac{4(m-1)}{m(m+1)} + \frac{m^2-3m+4}{m(m+1)} \\ &= 1, \\ \deg^{\psi}(p) &= (m-1) \cdot \frac{m-4}{m(m-1)} + 2(m+1) \cdot \frac{2}{m(m+1)} \\ &= \frac{m-4}{m} + \frac{4}{m} \\ &= 1 \end{aligned}$$

for the case $p \cap V_1 = \emptyset$ and $p \cap V_1 \neq \emptyset$ respectively.

If $|p \cap V_i| = \left|p \cap V_j\right| = 1$ for some $1 \leqslant i < j \leqslant 3$, then the edges covering p are in E_1 or E_3 as triples going across V_1, V_2, and V_3 are not in $E(H)$. There are precisely m of the first type and $(n_i - 1) + (n_j - 1)$ of the second type. Hence, we get

$$\begin{aligned} \deg^{\psi}(p) &= m \cdot \frac{m^2-3m+4}{m(m+1)^2} + 2m \cdot \frac{5m-3}{2m(m+1)^2} \\ &= \frac{m^2-3m+4}{(m+1)^2} + \frac{5m-3}{(m+1)^2} \\ &= 1, \end{aligned}$$

$$\begin{aligned}\deg^{\psi}(p) &= m \cdot \frac{2}{m(m+1)} + 2(m-1) \cdot \frac{1}{2(m+1)} \\ &= \frac{2}{m+1} + \frac{m-1}{m+1}\end{aligned}$$

for the case $p \cap V_1 = \emptyset$ and $p \cap V_1 \neq \emptyset$ respectively.

Lastly, if $p \subseteq V_i$ for some $i \in [3]$, then the edges covering p are in E_3 or E_2 since triples e with $|e \cap V_0| = 1$ are not in $E(H)$. There are exactly $n_j + n_k$ of the first type where $\{j, k\} = [3] \setminus \{i\}$ and $n_i - 2$ edges of the second type. Hence, we get

$$\begin{aligned}\deg^{\psi}(p) &= (m+1) \cdot \frac{5m-3}{2m(m+1)^2} + (m-1)\frac{1}{2(m+1)} + (m-1) \cdot \frac{m^2-2m+3}{2m(m-1)(m+1)} \\ &= \frac{5m-3}{2m(m+1)} + \frac{m^2-m}{2m(m+1)} + \frac{m^2-2m+3}{2m(m+1)} \\ &= 1, \\ \deg^{\psi}(p) &= 2(m+1) \cdot \frac{1}{2(m+1)} \\ &= 1\end{aligned}$$

for $p \not\subseteq V_1$ and $p \subseteq V_1$ respectively.

Hence, in all three cases, ψ is a fractional Steiner triple system in H. $\square$

B PROBABILISTIC INEQUALITIES

In this chapter, we briefly state some inequalities from probability theory.

Theorem B.1 (Jensen's inequality, [14, Chap. 4.1])**.** Let X be a real random variable with $\mathbb{E}[X] < \infty$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Then, $\mathbb{E}[\varphi(X)] \leqslant \varphi(\mathbb{E}[X])$.

Theorem B.2 (Two-sided Chernoff bound for binomial distributions, [22, Cor. 2.3])**.** Let $n \in \mathbb{N}, p \in [0,1]$. If $X \sim \textsc{Bin}(n,p)$ and $\varepsilon \in (0, 3/2]$, then

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geqslant \varepsilon\mathbb{E}[X]\right) \leqslant 2\exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right).$$

Theorem B.3 (Two-sided Chernoff bound for hypergeometric distributions, [22, Thm. 2.10])**.** Let $N \in \mathbb{N}, m, n \in [N]$. If $X \sim \textsc{Hyp}(N, m, n)$ and $\varepsilon \in (0, 3/2]$, then

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geqslant \varepsilon\mathbb{E}[X]\right) \leqslant 2\exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right),$$

where $\mathbb{E}[X] = mn/N$.

C GUROBI IMPLEMENTATIONS

To verify results and allow for experimentation, we wrote code throughout the writing process of this thesis. All the scripts were written and run using Python 3.8.10 and the Gurobi optimizer 10.0.3.

C.1 SOLVING (P3)

Below is an implementation of (P3) as defined above Corollary 4.23:

```
# Solving optimization program arising from Delcourt & Postle's approach.

import gurobipy as gp
from gurobipy import GRB

# Create a new model
m = gp.Model("DelcourtPostle")

# Problem is non-convex, so we need to set "NonConvex" accordingly.
m.setParam(GRB.Param.NonConvex, 2)

# (1 - d) is the minimum codegree density.
# Setting d as follows, the computed objective stays below 1.
# This agrees with our theorem.
d = 0.1421

# Create variables with given lower and upper bounds.
e0 = m.addVar(lb=1.0-d, ub=1.0, name="e_0")
e = m.addVar(lb=1.0-d, ub=1.0, name="e")
f = m.addVar(lb=1.0-d, ub=1.0, name="f")
q0 = m.addVar(name="q_0")
q = m.addVar(name="q")
p = m.addVar(name="p")
r0 = m.addVar(lb=0.5, name="r_0")
r = m.addVar(lb=0, name="r")

# Auxiliary variables representing inverses,
# Gurobi doesn't directly allow divisions.
e0_inv = m.addVar(name="e_0^-1")
e_inv = m.addVar(name="e^-1")
f_inv = m.addVar(name="f^-1")
q0_inv = m.addVar(name="q_0^-1")
q_inv = m.addVar(name="q^-1")
p_inv = m.addVar(name="p^-1")

# Auxiliary variables for nested terms,
# Gurobi doesn't directly allow the product of three or more terms.
z0 = m.addVar(name="z_1 * (z_2 + z_3)")
```

```
z1 = m.addVar(name="e_0 * r_0")
z2 = m.addVar(name="1 / q_0 * ((1 / e) - (1 / e_0)")
z3 = m.addVar(name="z_4 * z_5")
z4 = m.addVar(name="r / p")
z5 = m.addVar(name="((1 / e) - (1 / f)) / q + ((1 / e) - (1 / e_0)) / q_0")

# Additional constraints on the common co-neighborhood densities.
m.addConstr(q0 >= e0 + e - 1 - d)
m.addConstr(q0 <= e0)
m.addConstr(q >= e + f - 1 - d)
m.addConstr(q <= e)
m.addConstr(p >= q0 + q - e - d)
m.addConstr(p <= q)
m.addConstr(r0 <= e0)
m.addConstr(r <= q0)

# Auxiliary constraints for inversion.
m.addConstr(e0_inv * e0 == 1)
m.addConstr(e_inv * e == 1)
m.addConstr(f_inv * f == 1)
m.addConstr(q0_inv * q0 == 1)
m.addConstr(q_inv * q == 1)
m.addConstr(p_inv * p == 1)

# Auxiliary constraints for nested terms inside objective function.
# z0 will play the role of the objective function.
m.addConstr(z0 == z1 * (z2 + z3))
m.addConstr(z1 == e0 * r0)
m.addConstr(z2 == q0_inv * (e_inv - e0_inv))
m.addConstr(z3 == z4 * z5)
m.addConstr(z4 == r * p_inv)
m.addConstr(z5 == q_inv * (e_inv - f_inv) + q0_inv * (e_inv - e0_inv))

m.setObjective(z0, GRB.MAXIMIZE)

m.optimize()

# Prints out for every variable its value.
for v in m.getVars():
    print(f"{v.VarName} {v.X:g}")

# Prints out the best objective obtained.
print(f"Obj: {m.ObjVal:g}")
```

C.2 COMPUTING FRACTIONAL STEINER TRIPLE SYSTEMS

To find fractional Steiner triple systems for the construction described in Lemma 1.7, we may formulate the equivalent linear programming problem. Following this approach, the following Python script has been written:

```
import gurobipy as gp
from gurobipy import GRB
from itertools import chain, combinations, product
import math
import numpy as np

# See if Lee's 3/4-construction contains a fractional STS.
# Short answer: It does (for small, reasonable instances).
```

```python

# Size of V_i in the ith entry.
# Here we have |V_0| = 6, |V_i| = 7 otherwise.
sizes = [6, 7, 7, 7]

# |V|
total = sum(sizes)

# Define the vertex classes with corresponding sizes.
v0 = range(sizes[0])
v1 = range(sum(sizes[:1]), sum(sizes[:2]))
v2 = range(sum(sizes[:2]), sum(sizes[:3]))
v3 = range(sum(sizes[:3]), sum(sizes[:4]))

edges = []

# Add edges of type E_0.
for (i, j), k in product(combinations(v0, 2), chain(v1, v2, v3)):
    edges.append((i, j, k))

# Add edges of type E_1.
for i, j, k in product(v0, v1, v2):
    edges.append((i, j, k))
for i, j, k in product(v0, v1, v3):
    edges.append((i, j, k))
for i, j, k in product(v0, v2, v3):
    edges.append((i, j, k))

# Add edges of type E_2.
for i, j, k in combinations(v1, 3):
    edges.append((i, j, k))
for i, j, k in combinations(v2, 3):
    edges.append((i, j, k))
for i, j, k in combinations(v3, 3):
    edges.append((i, j, k))

# Add edges of type E_3.
for (i, j), k in product(combinations(v1, 2), v2):
    edges.append((i, j, k))
for i, (j, k) in product(v1, combinations(v2, 2)):
    edges.append((i, j, k))

for (i, j), k in product(combinations(v2, 2), v3):
    edges.append((i, j, k))
for i, (j, k) in product(v2, combinations(v3, 2)):
    edges.append((i, j, k))

for (i, j), k in product(combinations(v1, 2), v3):
    edges.append((i, j, k))
for i, (j, k) in product(v1, combinations(v3, 2)):
    edges.append((i, j, k))

# List of all pairs and total number of pairs.
pairs = combinations(range(total), 2)
total_pairs = math.comb(total, 2)

# Create incidence matrix A between pairs and edges.
# b corresponds to the desired weight of each pair, i.e. each entry is one.
A = np.zeros((total_pairs, len(edges)))
```

```
b = np.ones((total_pairs, 1))
for j, pair in enumerate(pairs):
    for k, hyper_edge in enumerate(edges):
        if set(hyper_edge).issuperset(pair):
            A[j][k] = 1

# Create new model.
# y takes non-negative real values since we deal with fractional STS.
# x takes on value of smallest weight.
m = gp.Model("LP")
y = m.addMVar(len(edges), vtype=GRB.CONTINUOUS, lb=0.0, name="weights")
x = m.addVar(vtype=GRB.CONTINUOUS, lb=0.0, name="min weight")

# Weight assigned to edges should induce that every pair has weight one.
m.addConstr(A @ y == b)
m.addConstr(x == gp.min_(y.tolist()))

# We set the objective to maximize the smallest weight.
# This forces structure into the fractional STS, which may be helpful.
# Can be turned off if we are only interested in a solution.
m.setObjective(x, GRB.MAXIMIZE)

m.optimize()

all_vars =  m.getVars()
values = m.getAttr("X", all_vars)
names = m.getAttr("VarName", all_vars)

# Print for every edge with non-zero weight its weight.
it_edges = 0
for name, val in zip(names, values):
    if name.startswith("weights"):
        if val > 0:
            print('%s %g' % (edges[it_edges], val))
        it_edges += 1
```

Apart from the smallest, arguably degenerate cases, our program is able to find a fractional Steiner triple system in small instances. From the computed fractional Steiner triples, we were able to come up with the constructions in Proposition A.1. The cases not covered by Proposition A.1 are listed in Table C.1. Overall, these findings support Conjecture 5.1.

n	$(\lvert V_0\rvert, \lvert V_1\rvert, \lvert V_2\rvert, \lvert V_3\rvert)$	Does there exist a fractional Steiner triple system?
3	$(0, 1, 1, 1)$	No.
5	$(2, 1, 1, 1)$	No.
7	$(2, 1, 1, 3)$	No.
9	$(2, 1, 3, 3)$	Yes.
11	$(2, 3, 3, 3)$	No.
13	$(4, 3, 3, 3)$	Yes.
15	$(4, 3, 3, 5)$	Yes.
17	$(4, 3, 5, 5)$	Yes.
19	$(4, 5, 5, 5)$	Yes.
21	$(6, 5, 5, 5)$	Yes.

Table C.1: Existence of fractional Steiner triple systems for all odd n with $n \leqslant 21$

D THRESHOLDS FOR ODD PERFECT MATCHINGS

In this chapter, we show that – surprisingly – there is no gap between the (asymptotic) minimum codegree threshold for perfect matchings / odd perfect matchings in 3-uniform hypergraphs. This is in contrast to the case of (odd) Steiner triple systems, see Lemma 1.7, Lemma 5.17, and Proposition 5.15.

In other words, if we consider $0 \leqslant r \leqslant 3$, there is only a gap between the (asymptotic) minimum codegree threshold for $(n, 3, r)$-Steiner systems and odd $(n, 3, r)$-Steiner systems in the case of $r = 2$.[1]

As tight Hamilton cycles always form odd perfect matchings, more specifically $(n, 3, 1, 3)$-designs, we have the following observation.

Observation D.1. For all $n \geqslant 3$ there exists an odd, 3-uniform perfect matching.

Hence, with the divisibility conditions set, we can formally define θ_{PM} and $\theta_{\mathrm{PM}}^{\mathbb{F}_2}$.

Definition D.2 (θ_{PM}, $\theta_{\mathrm{PM}}^{\mathbb{F}_2}$)**.**

- Let θ_{PM} be the infimum over all $\delta \in [0, 1]$ for which there exists $n_0 \in \mathbb{N}$ such that for every 3-uniform hypergraph H on $n \geqslant n_0$ vertices with $\delta_2(H) \geqslant \delta n$ and $n \equiv 0 \pmod 3$ contains a perfect matching. We will refer to θ_{PM} as the *minimum codegree threshold (for perfect matchings)*.
- Let $\theta_{\mathrm{PM}}^{\mathbb{F}_2}$ be the infimum over all $\delta \in [0, 1]$ for which there exists $n_0 \in \mathbb{N}$ such that for every 3-uniform hypergraph H on $n \geqslant n_0$ vertices with $\delta_2(H) \geqslant \delta n$ and contains an odd perfect matching. We will refer to $\theta_{\mathrm{PM}}^{\mathbb{F}_2}$ as the *minimum codegree threshold (for odd perfect matchings)*.

Proposition D.3. $\theta_{\mathrm{PM}} = 1/2 = \theta_{\mathrm{PM}}^{\mathbb{F}_2}$.

The following results will facilitate the proof of Proposition D.3.

Theorem D.4 (Rödl, Ruciński, Szemerédi 2009, [44, Def. 1.1, Thm. 1.1])**.** For all integers $k \geqslant 2$ and $n \geqslant k$, denote by $t(k, n)$ the smallest integer t such that every k-uniform hypergraph on n vertices with $\delta_{k-1} \geqslant t$ contains a matching of size $\lfloor n/k \rfloor$. Then, for all $k \geqslant 3$ and sufficiently large $n \in k\mathbb{N}$,

$$t(k, n) = \begin{cases} \frac{n}{2} + 3 - k, & \frac{k}{2} \in 2\mathbb{N} \text{ and } \frac{n}{2} \in 2\mathbb{N} - 1, \\ \frac{n}{2} + \frac{5}{2} - k, & k \in 2\mathbb{N} - 1 \text{ and } \frac{n-1}{2} \in 2\mathbb{N} - 1, \\ \frac{n}{2} + \frac{3}{2} - k, & k \in 2\mathbb{N} - 1 \text{ and } \frac{n-1}{2} \in 2\mathbb{N}, \\ \frac{n}{2} + 2 - k, & \text{otherwise.} \end{cases}$$

[1] The case $r = 0$ and $r = 3$ are trivial.

Theorem D.5 (Rödl, Ruciński, Szemerédi 2006, [42, Thm. 1.1])**.** For every $\gamma > 0$ there exists n_0 such that every 3-uniform hypergraph H with $n \geqslant n_0$ vertices and $\delta_1(H) \geqslant (1/2+\gamma)n$ contains a tight Hamiltonian cycle.

Proof of Proposition D.3. As perfect matchings are always odd perfect matchings, we have $\theta_{PM} \geqslant \theta_{PM}^{\mathbb{F}_2}$.

So, for the lower bound, consider $n \geqslant 3$. Let $V := [n] = A \mathbin{\dot\cup} B$ such that

- $||A| - |B||$ is minimal, and
- $|A| \equiv 1 \pmod 2$.

In particular, we always have $\min\{|A|, |B|\} \geqslant n/2 - 2$ and $\max\{|A|, |B|\} \leqslant n/2 + 1$.

Furthermore, let

$$E = \left\{ e \in V^{(3)} \colon |e \cap A| \equiv 0 \pmod 2 \right\}.$$

Finally, consider $H = (V, E)$. Assume that H contains an odd perfect matching $M \subseteq H$. We double count the number of A-edge incidences

$$I = \{(v, e) \in A \times E \colon v \in e\}.$$

On the one hand, every vertex v is covered by an odd number of edges, hence

$$|I| = \sum_{v \in A} \deg_M(v) \equiv \sum_{v \in A} 1 \equiv |A| \equiv 1 \pmod 2.$$

On the other hand, every edge e that covers some vertex A must actually cover exactly two. Thus, we get

$$|I| = \sum_{e \in E} |e \cap A| \equiv \sum_{e \in E} 0 \equiv 0 \pmod 2. \lightning$$

Therefore, H doesn't contain an odd perfect matching. Furthermore, we have

- $\deg(u, v) = |B| \in [n/2 - 2, n/2 + 1]$ for all $u, v \in A$,
- $\deg(u, v) = |A| - 1 \in [n/2 - 3, n/2]$ for all $(u, v) \in A \times B$, and
- $\deg(u, v) = |A| \in [n/2 - 2, n/2 + 1]$ for all $u, v \in B$.

Thus, H witnesses $\theta_{PM}^{\mathbb{F}_2} \geqslant 1/2$.

As not every odd perfect matching is also a perfect matching, we will prove the upper bound for both cases separately. For the upper bound of θ_{PM}, we refer to Theorem D.4 for $k = 3$. For the upper bound of $\theta_{PM}^{\mathbb{F}_2}$, we refer to Theorem D.5. $\square$