Design of Linear-phase FIR Filter using POCS

- Consider designing an N-length linear-phase FIR filter with N being an even number.
- One can show that:

$$H(\omega) = A(\omega)e^{j\phi(\omega)}$$

where

$$A(\omega) = \sum_{n=0}^{N/2-1} 2h[n]\cos[(n - \frac{N-1}{2})\omega]$$

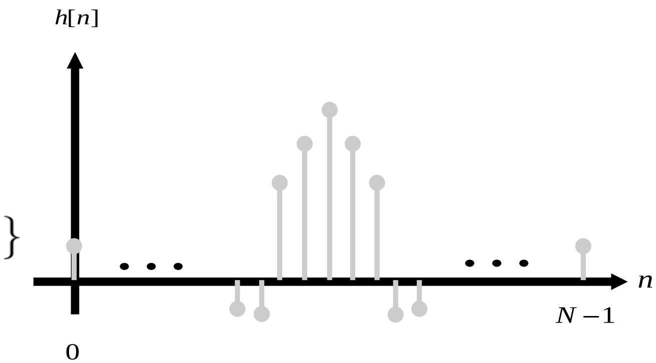
$$\phi(\omega) = -\frac{N-1}{2}\omega$$

- We next define the set of constraint sets for such a problem. Our Hilbert space is $\mathbb{R}^L$, L>>N to insure a high-resolution Fourier transform without aliasing.

Design of Linear-phase FIR Filter using POCS

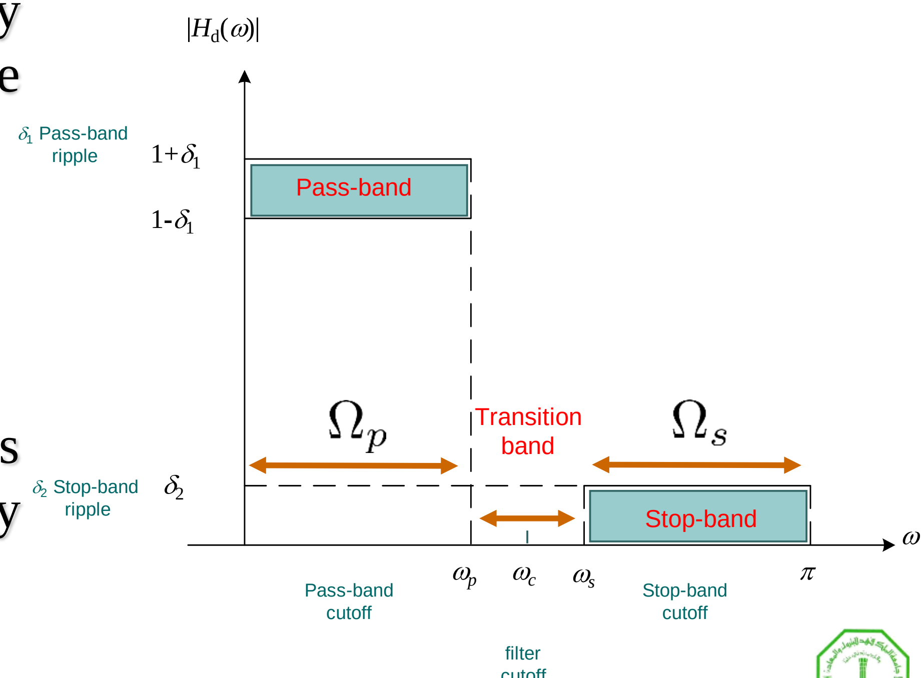
- C_1 is the set of all finite-length sequences with appropriate symmetry that imply a Fourier transform with linear phase. That is:

$$C_1 = \{\mathbf{h} \in \mathbb{R}^L : h[n] = h[N-1-n] \text{ for } n = 0, 1, \ldots, N-1\}$$
$$\text{and } h[n] = 0 \text{ for } N \leq n \leq L-1$$



- C_2 is the set of all finite-length sequences whose Fourier amplitude is appropriately constrained in the passband and whose phase is linear in that band:

$$C_2 = \{\mathbf{h} \in \mathbb{R}^L : 1 - \delta_p \leq A(\omega) \leq 1 + \delta_p$$
$$\text{and } \phi(\omega) = -\omega(N-1)/2 \text{ for } \omega \in \Omega_p\}.$$

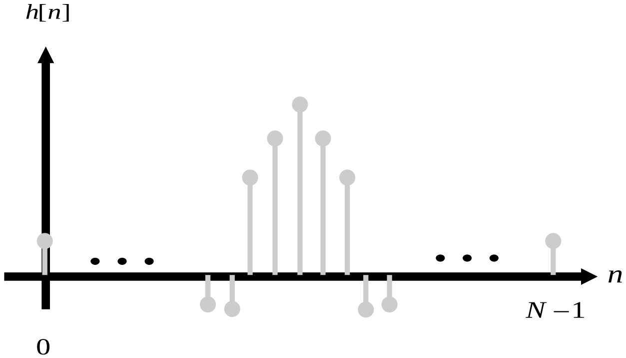


- C_3 is the set of all finite-length sequences whose Fourier amplitude is appropriately constrained in the stopband:

$$C_3 = \{\mathbf{h} \in \mathbb{R}^L : A(\omega) \leq \delta_s \text{ for } \omega \in \Omega_s\}$$

Constraint set C_1

$$C_1 = \{\mathbf{h} \in \mathbb{R}^L : h[n] = h[N-1-n] \text{ for } n = 0, 1, ..., N-1\}$$
and $h[n] = 0$ for $N \le n \le L-1$



- **Convexity:**

Let $h_1(n), h_2(n) \in C_1$

Define $h_3(n) = \mu h_1(n) + (1-\mu) h_2(n)$ for $0 \le \mu \le 1$

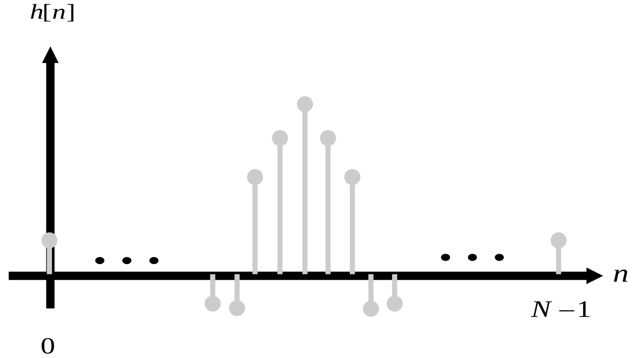
Since $h_1(n) = h_1(N-1-n)$ and $h_2(n) = h_2(N-1-n)$, we have:

$$\begin{aligned} h_3(n) &= \mu h_1(n) + (1-\mu) h_2(n) \\ &= \mu h_1(N-1-n) + (1-\mu) h_2(N-1-n) \\ &= h_3(N-1-n) \end{aligned}$$

Hence, C_1 is convex.

Constraint set C_1

$$C_1 = \{\mathbf{h} \in \mathbb{R}^L : h[n] = h[N-1-n] \text{ for } n = 0, 1, ..., N-1\}$$
and $h[n] = 0$ for $N \leq n \leq L-1$



Closeness:

Let $\mathbf{h}_k$, $k = 0, 1, \ldots$ be a sequence of vectors in C_1 with a limit point $\mathbf{h}$. Then by defintion:

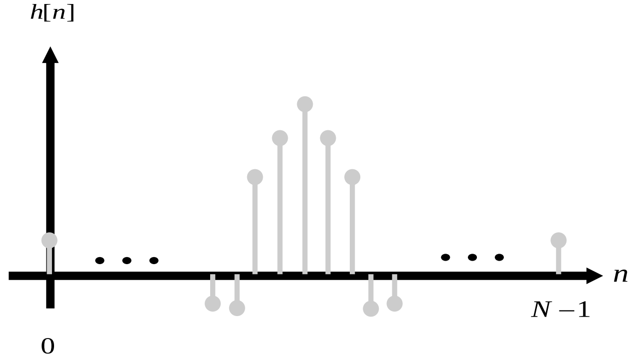
$$\sum_{n=0}^{N-1} |\mathbf{h}_k(n) - h(n)|^2 \to 0$$

However, $\mathbf{h}_k(n) = \mathbf{h}_k(N-1-n)$. Hence,

$$\sum_{n=0}^{N-1} |\mathbf{h}_k(N-1-n) - h(n)|^2 \to 0$$

which implies that $\mathbf{h}_k(N-1-n) \to h(n)$. At the same time, $\mathbf{h}_k(n) \to h(n)$. Hence, $h(n) = h(N-1-n)$. $\therefore \mathbf{h} \in C_1$ and the set C_1 is closed.

Constraint set C_1



$$C_1 = \{\mathbf{h} \in \mathbb{R}^L : h[n] = h[N-1-n] \text{ for } n = 0, 1, ..., N-1\}$$
and $h[n] = 0$ for $N \leq n \leq L-1$

- **Projection:**

Assume that all the vectors are real. Let $\mathbf{g}$ be an arbitrary vector, $\mathbf{h}$ be any vector in C_1, and $\mathbf{h}^*$ the projection of $\mathbf{g}$ onto C_1. Then,

$$\mathbf{h}^* = \arg\min_{\mathbf{h} \in C_1} \sum_{n=0}^{L-1} [g(n) - h(n)]^2$$

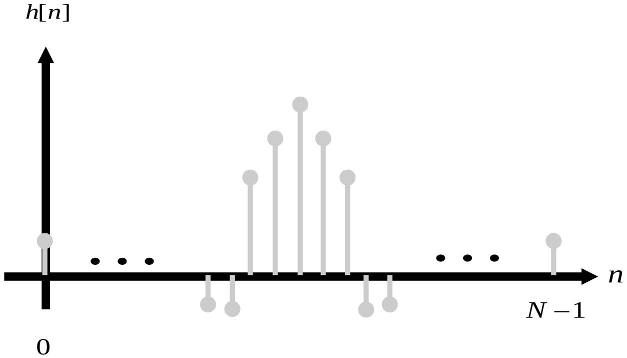
, where $h^*(n) = 0$ for $N \leq n \leq L-1$. With

$$J = \sum_{n=0}^{L-1} [g(n) - h(n)]^2$$

, the projection is easily computed.

Constraint set C_1

$$C_1 = \{\mathbf{h} \in \mathbb{R}^L : h[n] = h[N-1-n] \text{ for } n = 0, 1, ..., N-1\}$$
and $h[n] = 0$ for $N \leq n \leq L-1$



- **Projection:**

Now, by taking into account that $h^*(n) = 0$ for $n = N, \ldots, L-1$, we write (assuming N is even) the Lagrange functional as:

$$J = \sum_{n=0}^{N/2-1} \{[g(n) - h(n)]^2 + [g(n+N/2) - h(n+N/2)]^2\}$$

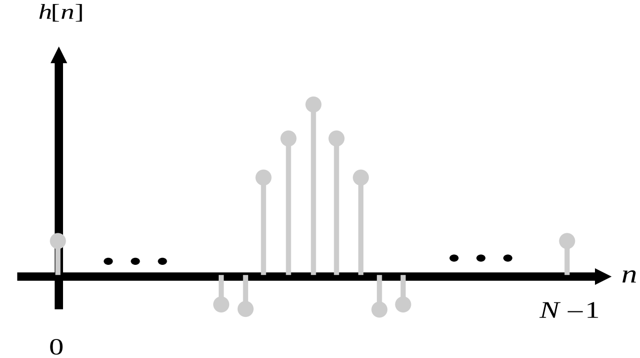
and using the fact that $[h(n+N/2) - h(N/2-1-n)]^2$. Then with $\frac{\partial J}{\partial h(l)} = 0$, $l = 0, 1, \ldots, N/2-1$, we obtain:

$$h^*(l) = \frac{g(l) + g(N-1-l)}{2}$$

. This clearly shows that $h^*(l) = h^*(N-1-l)$.

Constraint set C_1

$$C_1 = \{\mathbf{h} \in \mathbb{R}^L : h[n] = h[N-1-n] \text{ for } n = 0, 1, ..., N-1\}$$
and $h[n] = 0$ for $N \leq n \leq L-1$

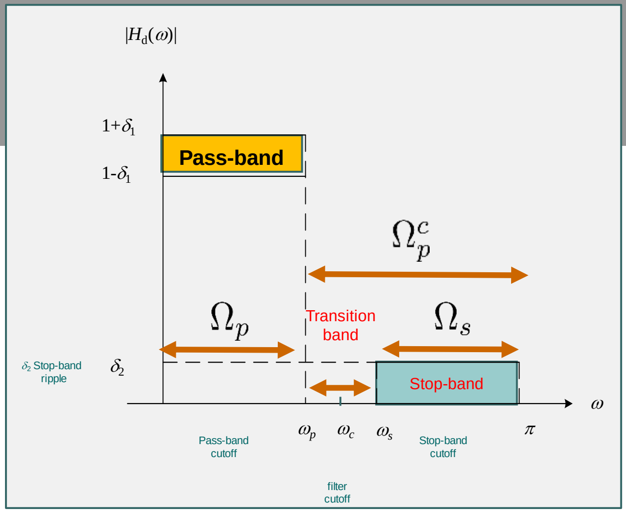


- **Projection:**

Thus, the projection $\mathbf{h}^* = P_1\mathbf{g}$ of $\mathbf{g}$ onto C_1 becomes:

$$P_{C_1}\mathbf{g} = \begin{cases} \frac{g(l)+g(N-1-l)}{2}, \text{ for } l = 0, 1, \ldots, (N-1) \\ 0, \text{ elsewhere.} \end{cases}$$

Constraint set C_2



$$C_2 = \{\mathbf{h} \in \mathbb{R}^L : 1 - \delta_p \leq A(\omega) \leq 1 + \delta_p$$
$$\text{and } \phi(\omega) = -\omega(N-1)/2 \text{ for } \omega \in \Omega_p\}.$$

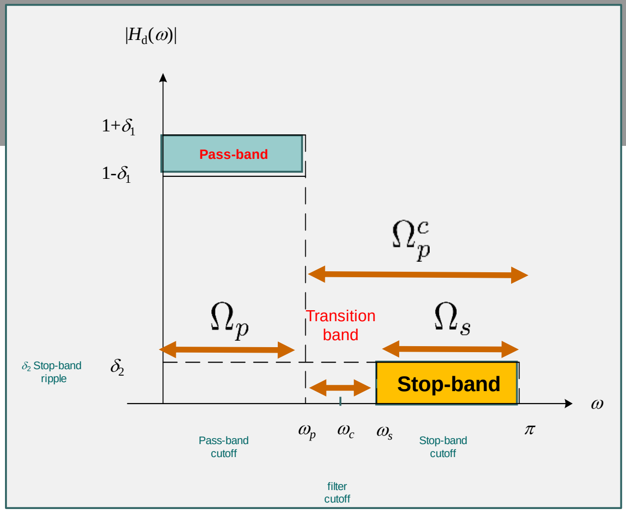
- Convexity and closeness: Exercise.
- Projection:

The projection of an arbitrary vector $\mathbf{g} \in \mathbb{R}^L$ with $\mathcal{F}\{\mathbf{g}\} = G(\omega) = |G(\omega)|e^{j\theta_{G(\omega)}}$ can be computed using the Lagrange multiplier method (exercise). The projection will be $\mathbf{h}^* = P_{C_2}\mathbf{g} \leftrightarrow H^*(\omega)$:

$$P_{C_2}\mathbf{g} \leftrightarrow H^*(\omega) = \begin{cases} (1+\delta_p)e^{j\phi(\omega)}, \text{ if cond. A} \\ (1-\delta_p)e^{j\phi(\omega)}, \text{ if cond. B} \\ |G(\omega)|\cos(\theta_{G(\omega)} - \phi(\omega))e^{j\phi(\omega)}, \text{ if cond. C} \\ G(\omega), \text{ if } \omega \in \Omega_p^c. \end{cases}$$

where, cond. A is: $|G(\omega)|\cos(\theta_{G(\omega)} - \phi(\omega)) \geq (1+\delta_p)$ and $\omega \in \Omega_p$
cond. B is: $|G(\omega)|\cos(\theta_{G(\omega)} - \phi(\omega)) \leq (1-\delta_p)$ and $\omega \in \Omega_p$
cond. C is: $(1-\delta_p) \leq |G(\omega)|\cos(\theta_{G(\omega)} - \phi(\omega)) \leq (1+\delta_p)$ and $\omega \in \Omega_p$

Constraint set C_3



$$C_3 = \{\mathbf{h} \in \mathbb{R}^L : A(\omega) \le \delta_s \text{ for } \omega \in \Omega_s\}$$

- Convexity and closeness: Exercise.
- Projection:

The projection of an arbitrary vector $\mathbf{g} \in \mathbb{R}^L$ with $\mathcal{F}\{\mathbf{g}\}$ can be computed using the Lagrange multiplier method (exercise). The projection will be $\mathbf{h}^* = P_{C_3}\mathbf{g} \leftrightarrow H^*(\omega)$:

$$P_{C_3}\mathbf{g} \leftrightarrow H^*(\omega) = \begin{cases} \frac{\delta_s G(\omega)}{|G(\omega)|}, \text{ for } |G(\omega)| > \delta_s, \ \omega \in \Omega_s \\ G(\omega), \text{ for } |G(\omega)| \le \delta_s, \ \omega \in \Omega_s \\ G(\omega), \text{ elsewhere.} \end{cases}$$

POCS algorithm for designing linear-phase FIR filters

- The FIR filter design algorithm can be given by:

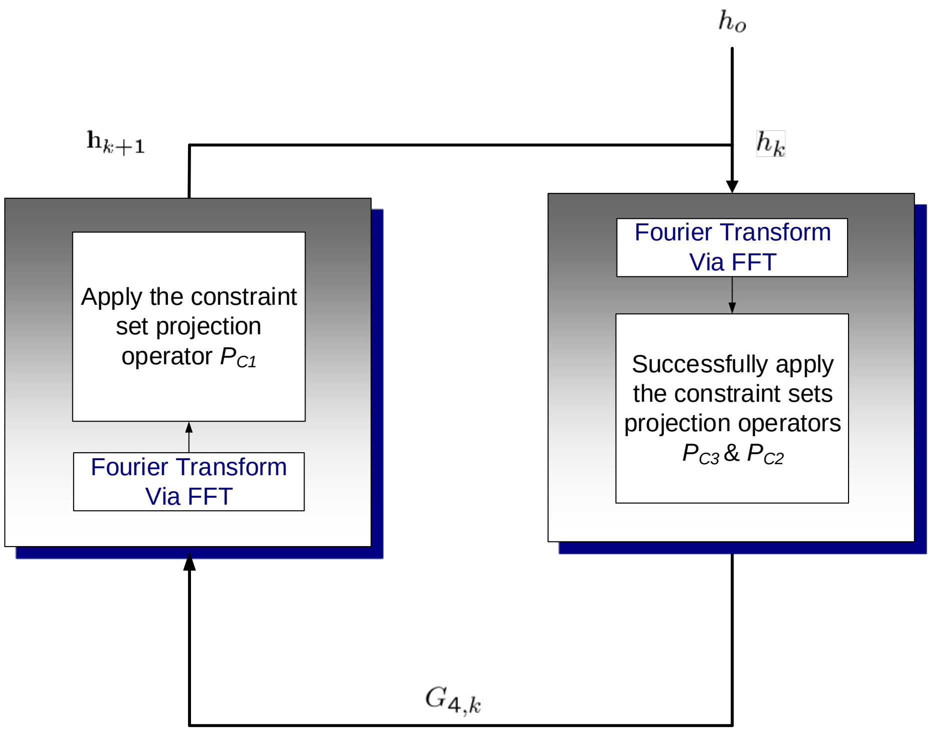
$$\mathbf{h}_{k+1} = P_{C_1} P_{C_2} P_{C_3} \mathbf{h}_k$$

where $\mathbf{h}_0$ is arbitrary.

- A good choice for $\mathbf{h}_0$ is $\mathbf{h}_0 \leftrightarrow H_0(\omega) = H_{ideal}(\omega)$, where $H_{ideal}(\omega)$ is the ideal lowpass response characteristic of the digital filter.
- The stopping criteria is:

$$\|\mathbf{h}_{k+1} - \mathbf{h}_k\| \leq \epsilon$$

where ϵ is a small positive number.



Computer Assignment 7 & 8 (due 18 April)

- Design an equiripple filter but using the POCS method with $\omega_p = 0.3\pi$ rad/s , $\omega_s = 0.35\pi$ rad/s, $\delta_p = 0.01$ and $\delta_s = 0.001$. Note that the filter order will be equal to:

$$M = \lceil \frac{-10\log(0.01 * 0.001) - 13}{14.6 * 0.05\pi/(2\pi)} \rceil = 102$$