11 Appendix E: Derivation of RBE3 element constraint equations

11.1 Introduction

The RBE3 element is used for distributing applied loads and mass from a reference point to other points in the finite element model. The geometry and loads for a RBE3 are shown in Figure 1. Point d in the figure is the RBE3 reference (or dependent) point and is the grid where loads will be applied by the user. The RBE3 element will distribute these loads to other, independent, points i = 1,…,N, in the model, where N is the total number of independent grid points defined on the RBE3 Bulk Data entry. The RBE3 is not intended to add stiffness to the model as does a RBE2 element. As such, the RBE3 reference point should not be a grid that is attached to other elements in the model – it should be a stand alone grid only connected to other grids through the REB3 element definition. The following describes the nomenclature used in this appendix in deriving the “constraint” equations used in MYSTRAN for the RBE3 element.

Superscripts denote the location of a quantity:
“d” refers to the reference (or dependent) grid on the RBE3
“i” refers to the independent grids, the locations where the loads on point d will be distributed

$$X, Y, Z = \text{coordinate system axes}$$
$$u_x, u_y, u_z = \text{ displacements in the x, y, z directions}$$
$$\theta_x, \theta_y, \theta_z = \text{ rotations about the x, y, z axes}$$
$$F_x, F_y, F_z = \text{ forces in the x, y, z directions}$$
$$M_x, M_y, M_z = \text{ moments about the x, y, z axes}$$
$$d_x^i, d_y^i, d_z^i = \text{ position of point i relative to the RBE3 reference point, d}$$

For the sake of simplicity and clarity, the following derivation of the RBE3 equations is done for conditions where the global coordinate systems of all grid points involved in the RBE3 are the same and are rectangular. The code in the MYSTRAN program is written for general conditions where the global system of all points may be different and non-rectangular.

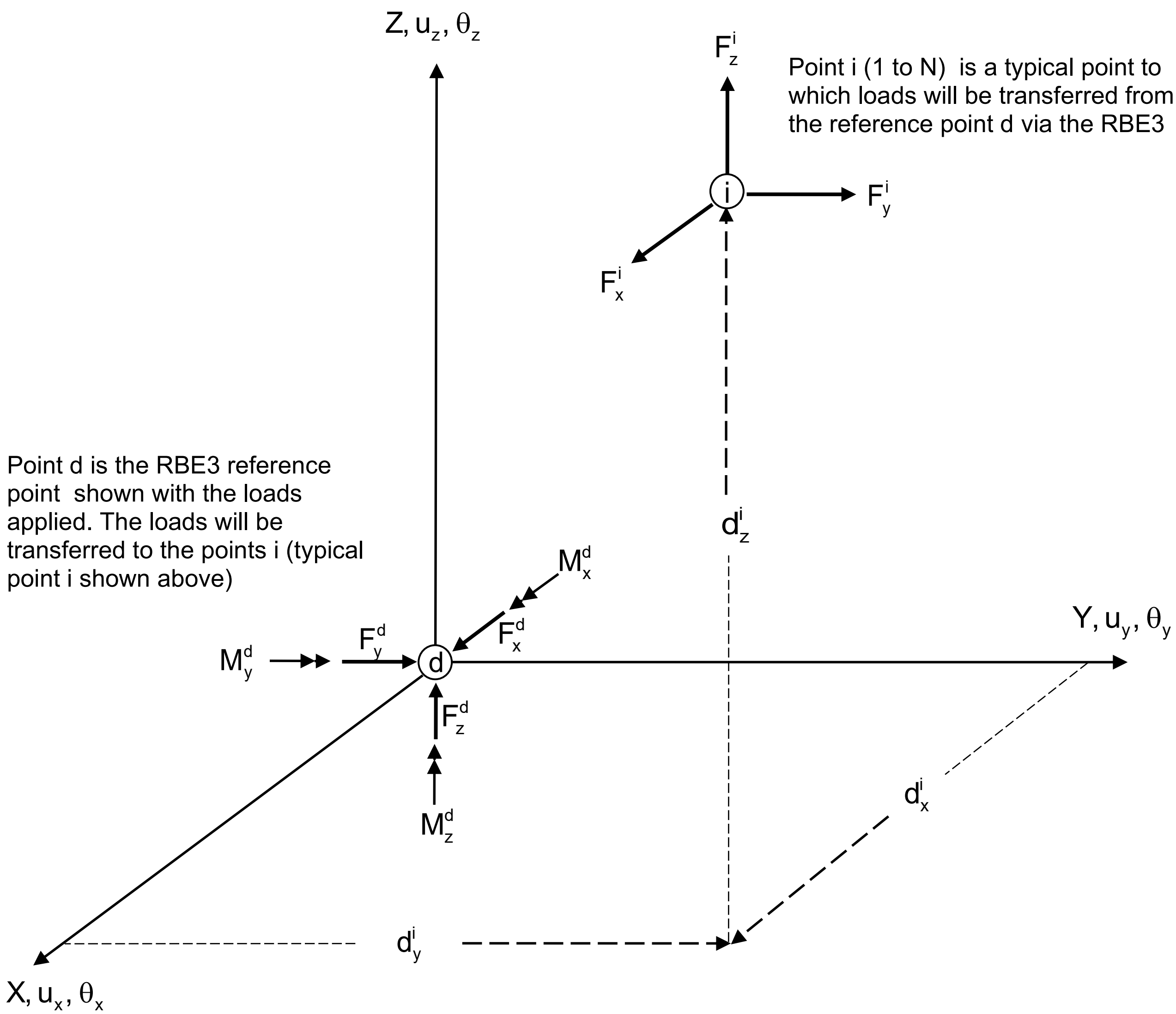

Fig 1: RBE3 geometry and loads

11.2 Equations for translational force components

In this section 3 equations will be developed that relate the forces applied at the RBE3 reference point to those where the loads will be distributed (points i = 1,…,N).

The sum of the forces on the points i = 1,…,N must equal the forces on the reference point d. Thus:

$$\sum_{i=1}^{N} F_x^i = F_x^d \quad , \quad \sum_{i=1}^{N} F_y^i = F_y^d \quad , \quad \sum_{i=1}^{N} F_z^i = F_z^d \qquad 11\text{-}1$$

The moments at reference point due to the forces at the points i are:

$$\sum_{i=1}^{N} (F_z^i d_y^i - F_y^i d_z^i) = M_x^d \quad , \quad \sum_{i=1}^{N} (F_x^i d_z^i - F_z^i d_x^i) = M_y^d \quad , \quad \sum_{i=1}^{N} (F_y^i d_x^i - F_x^i d_y^i) = M_z^d \qquad 11\text{-}2$$

Write the F_x^i, etc, as:

$$F_x^i = \frac{\omega_i}{W_T} F_x^d \quad , \quad F_y^i = \frac{\omega_i}{W_T} F_y^d \quad , \quad F_z^i = \frac{\omega_i}{W_T} F_z^d \qquad 11\text{-}3$$

where ω_i is the weighting factor (the WTi on the RBE3 Bulk Data entry) for the ith force and:

$$W_T = \sum_{i=1}^{N} \omega_i \qquad 11\text{-}4$$

Equations 3 and 4 are sufficient for equations 1. Substitute equations 3 and 4 into 2 to get the following 3 equations:

$$\frac{F_z^d}{W_T} \sum_{i=1}^{N} \omega_i d_y^i - \frac{F_y^d}{W_T} \sum_{i=1}^{N} \omega_i d_z^i = M_x^d \qquad 11\text{-}5$$

$$\frac{F_x^d}{W_T} \sum_{i=1}^{N} \omega_i d_z^i - \frac{F_z^d}{W_T} \sum_{i=1}^{N} \omega_i d_x^i = M_y^d \qquad 11\text{-}6$$

$$\frac{F_y^d}{W_T} \sum_{i=1}^{N} \omega_i d_x^i - \frac{F_x^d}{W_T} \sum_{i=1}^{N} \omega_i d_y^i = M_z^d \qquad 11\text{-}7$$

Define:

$$\bar{d}_x = \frac{1}{W_T} \sum_{i=1}^{N} \omega_i d_x^i \quad , \quad \bar{d}_y = \frac{1}{W_T} \sum_{i=1}^{N} \omega_i d_y^i \quad , \quad \bar{d}_z = \frac{1}{W_T} \sum_{i=1}^{N} \omega_i d_z^i \qquad 11\text{-}8$$

Using equation 8, equations 5-7 become:

$$F_z^d \bar{d}_y - F_y^d \bar{d}_z = M_x^d \qquad 11\text{-}9$$

$$F_x^d \bar{d}_z - F_z^d \bar{d}_z = M_y^d \tag{11-10}$$

$$F_y^d \bar{d}_x - F_x^d \bar{d}_y = M_z^d \tag{11-11}$$

The work done by the forces and moments at the reference point, d, is Ω_d:

$$\Omega_d = F_x^d u_x^d + F_y^d u_y^d + F_z^d u_z^d + M_x^d \theta_x^d + M_y^d \theta_y^d + M_z^d \theta_z^d \tag{11-12}$$

where u, θ are the displacements and rotations of the reference point in the x, y, z directions. Similarly, the work done by the forces on the points I = 1,…,N is:

$$\Omega_N = \sum_{i=1}^{N} (F_x^i u_x^i + F_y^i u_y^i + F_z^i u_z^i) \tag{11-13}$$

The u_x^i, ec, are the displacements in the x, y and z directions at point I. Substitute equation 3 into 12 and 9, 10 and 11 into 12 and equate the work done by the two systems of forces:

$$F_x^d u_x^d + F_y^d u_y^d + F_z^d u_z^d + (F_z^d \bar{d}_y - F_y^d \bar{d}_z)\theta_x^d + (F_x^d \bar{d}_z - F_z^d \bar{d}_z)\theta_y^d + (F_y^d \bar{d}_x - F_x^d \bar{d}_y)\theta_z^d =$$
$$\sum_{i=1}^{N} \left(\frac{\omega_i}{W_T} F_x^d u_x^i + \frac{\omega_i}{W_T} F_y^d u_y^i + \frac{\omega_i}{W_T} F_z^d u_z^i\right)$$

Rearrange:

$$\begin{aligned}
&(u_x^d + \bar{d}_z \theta_y^d - \bar{d}_y \theta_z^d - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_x^i) F_x^d + \\
&(u_y^d + \bar{d}_z \theta_x^d - \bar{d}_x \theta_z^d - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_y^i) F_y^d + \\
&(u_z^d + \bar{d}_y \theta_x^d - \bar{d}_x \theta_y^d - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_z^i) F_z^d = 0
\end{aligned} \tag{11-14}$$

Since the F_x^d, F_y^d and F_z^d are independent and, in general, not zero, equation 14 requires that:

$$\boxed{\begin{aligned}
&(u_x^d + \bar{d}_z \theta_y^d - \bar{d}_y \theta_z^d - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_x^i) = 0 \\
&(u_y^d - \bar{d}_z \theta_x^d + \bar{d}_x \theta_z^d - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_y^i) = 0 \\
&(u_z^d + \bar{d}_y \theta_x^d - \bar{d}_x \theta_y^d - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_z^i) = 0
\end{aligned}} \tag{11-15}$$

Equation 15 represents 3 constraint equations for the RBE3. However, there are only 3 equations and 6 unknowns. This will be resolved in the next section where we develop 3 more equations based on the moments at the reference point.

11.3 Equations for rotational moment components

In addition to the 3 equations developed in the last section there are also 3 equations that relate the moments applied at the RBE3 reference point to those where the loads will be distributed (points i = 1,…,N).

Figure 2 shows how the forces in the y-z plane relate to the RBE3 reference point moment about the x axis:

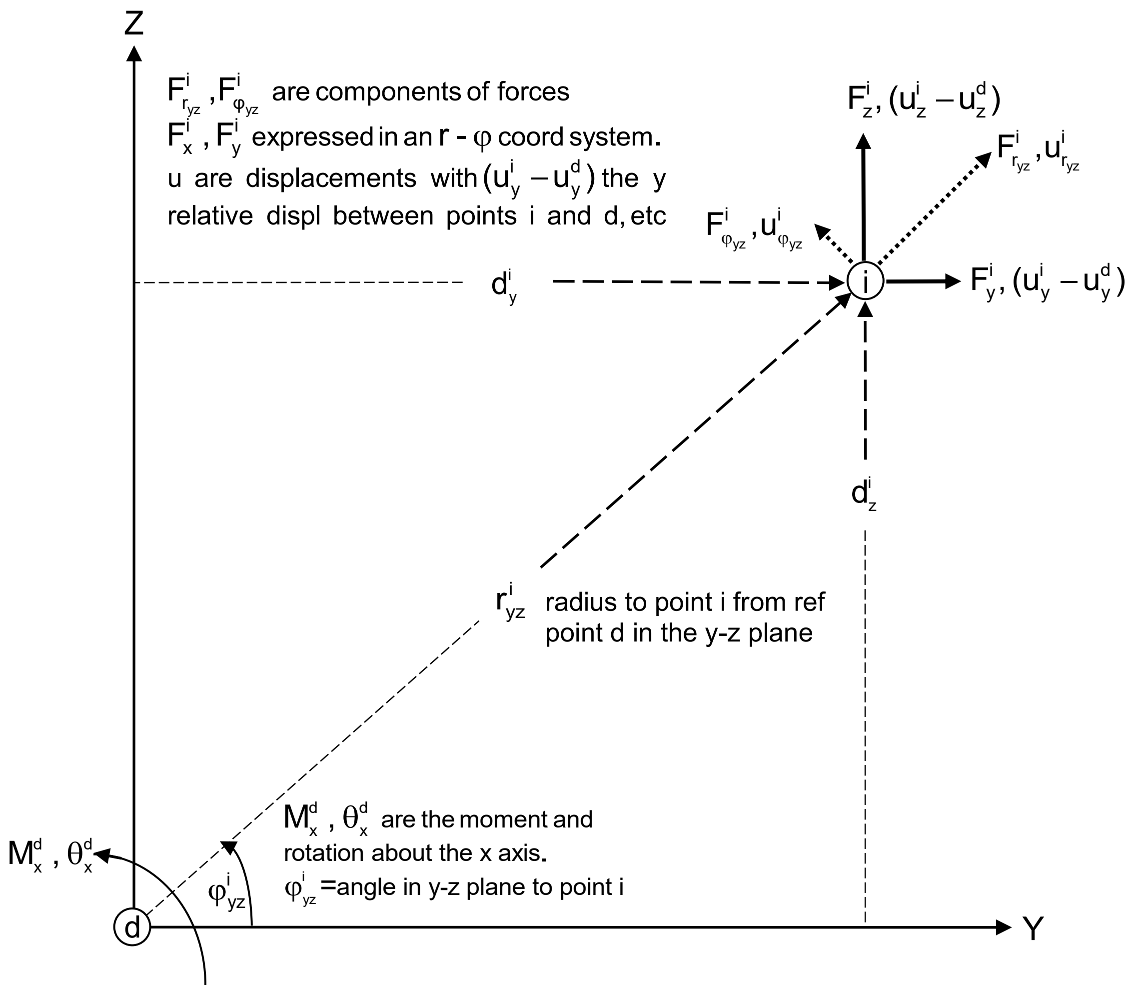

Figure 2: Relationship of moments and forces in the y-z plane

Using the r-φ components of the forces, the moments about the x axis of the forces at the i = 1,..,N points is:

$$\sum_{i=1}^{N} F^i_{\varphi_{yz}} r^i_{yz} = M^d_x \qquad \text{11-16}$$

As before, express the forces at the i points using the weighting factors, ω_i:

$$F^i_{\varphi_{yz}} = \frac{\omega^i r^i_{yz}}{\sum_{i=1}^{N} \omega^i r^{i^2}_{yz}} M^d_x \qquad \text{11-17}$$

Note that if equation 17 were substituted into 16 it would be seen that 17 is a valid representation of the tangential force components.

The work done by M^d_x must equal that due to all of the $F^i_{\varphi_{yz}}$, or:

$$\sum F^i_{\varphi_{yz}} u^i_{\varphi_{yz}} = M^d_x \theta^d_x \qquad \text{11-18}$$

where $u^i_{\varphi_{yz}}$ is the tangential component of displacement at independent grid i in the y-z plane.
Substitute equation 17 into 18:

$$\sum_{i=1}^{N} \frac{\omega^i r^i_{yz}}{\sum_{i=1}^{N} \omega^i r^{i^2}_{yz}} M^d_x u^i_{\varphi_{yz}} = M^d_x \theta^d_x$$

or:

$$\theta^d_x = \frac{\sum_{i=1}^{n} \omega^i r^i_{yz} u^i_{\varphi_{yz}}}{\sum_{i=1}^{N} \omega^i r^{i^2}_{yz}} \qquad \text{11-19}$$

From Figure 2 it can be seen that:

$$\begin{aligned} u^i_{\varphi_{yz}} &= (u^i_z - u^d_z)\cos\varphi^i_{yz} - (u^i_y - u^d_y)\sin\varphi^i_{yz} \\ &= (u^i_z - u^d_z)\frac{d^i_y}{r^i_{yz}} - (u^i_y - u^d_y)\frac{d^i_z}{r^i_{yz}} \end{aligned}$$

Therefore:

$$r^i_{yz} u^i_{\varphi_{yz}} = (u^i_z - u^d_z) d^i_y - (u^i_y - u^d_y) d^i_z \qquad \text{11-20}$$

Define:

$$\bar{e}^i_{yz} = \frac{1}{W_T} \sum_{i=1}^{N} \omega^i r^{i^2}_{\phi_{yz}} \equiv \frac{1}{W_T} \sum_{i=1}^{N} \omega^i (d^{i^2}_y + d^{i^2}_z) \qquad \text{11-21}$$

Substitute equations 20 and 21 into 19

$$
\begin{aligned}
\theta_x^d &= \frac{1}{W_T \bar{e}_{yz}^i}\left[\sum_{i=1}^{N}\omega^i(u_z^i - u_z^d)d_y^i - \sum_{i=1}^{N}\omega^i(u_y^i - u_y^d)d_z^i\right] \\
&= \frac{1}{W_T \bar{e}_{yz}^i}\left[-(\sum_{i=1}^{N}\omega^i d_y^i)u_z^d + (\sum_{i=1}^{N}\omega^i d_z^i)u_y^d + \sum_{i=1}^{N}\omega^i d_y^i u_z^i - \sum_{i=1}^{N}\omega^i d_z^i u_y^i\right] \\
&= \frac{1}{\bar{e}_{yz}^i}\left[-\bar{d}_y u_z^d + \bar{d}_z u_y^d + \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_y^i u_z^i - \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_z^i u_y^i\right]
\end{aligned}
\qquad 11\text{-}22
$$

In reference to Figures 3 and 4, define:

$$
\begin{aligned}
&\bar{e}_{zx}^i = \frac{1}{W_T}\sum_{i=1}^{N}\omega^i r_{\phi_{zx}}^{i^2} \equiv \frac{1}{W_T}\sum_{i=1}^{N}\omega^i(d_z^{i^2} + d_x^{i^2}) \\
&\text{and} \\
&\bar{e}_{xy}^i = \frac{1}{W_T}\sum_{i=1}^{N}\omega^i r_{\phi_{xy}}^{i^2} \equiv \frac{1}{W_T}\sum_{i=1}^{N}\omega^i(d_x^{i^2} + d_y^{i^2})
\end{aligned}
\qquad 11\text{-}23
$$

Then, θ_y^d and θ_z^d , by similar reasoning for θ_x^a in equation 22 are:

$$
\begin{aligned}
\theta_y^d &= \frac{1}{W_T \bar{e}_{zx}^i}\left[\sum_{i=1}^{N}\omega^i(u_x^i - u_x^d)d_z^i - \sum_{i=1}^{N}\omega^i(u_z^i - u_z^d)d_x^i\right] \\
&= \frac{1}{\bar{e}_{zx}^i}\left[-\bar{d}_z u_x^d + \bar{d}_x u_z^d + \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_z^i u_x^i - \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_x^i u_z^i\right]
\end{aligned}
\qquad 11\text{-}24
$$

and

$$
\begin{aligned}
\theta_z^d &= \frac{1}{W_T \bar{e}_{xy}^i}\left[\sum_{i=1}^{N}\omega^i(u_y^i - u_y^d)d_x^i - \sum_{i=1}^{N}\omega^i(u_x^i - u_x^a)d_y^i\right] \\
&= \frac{1}{\bar{e}_{xy}^i}\left[-\bar{d}_x u_y^d + \bar{d}_y u_x^d + \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_x^i u_y^i - \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_y^i u_x^i\right]
\end{aligned}
\qquad 11\text{-}25
$$

Thus, for the rotations:

$$
\boxed{
\begin{aligned}
&\bar{e}_{yz}\theta_x^d - \bar{d}_z u_y^d + \bar{d}_y u_z^d + \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_z^i u_y^i - \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_y^i u_z^i = 0 \\
&\bar{e}_{zx}\theta_y^d + \bar{d}_z u_x^d - \bar{d}_x u_z^d - \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_z^i u_x^i + \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_x^i u_z^i = 0 \\
&\bar{e}_{xy}\theta_z^d - \bar{d}_y u_x^d + \bar{d}_x u_y^d + \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_y^i u_x^i - \frac{1}{W_T}\sum_{i=1}^{N}\omega^i d_x^i u_y^i = 0
\end{aligned}
}
\qquad 11\text{-}26
$$

Equations 15 and 26 constitute 6 equations in the 6 unknown displacements and rotations at point a. They are summarized in matrix notation below at the end of this appendix.

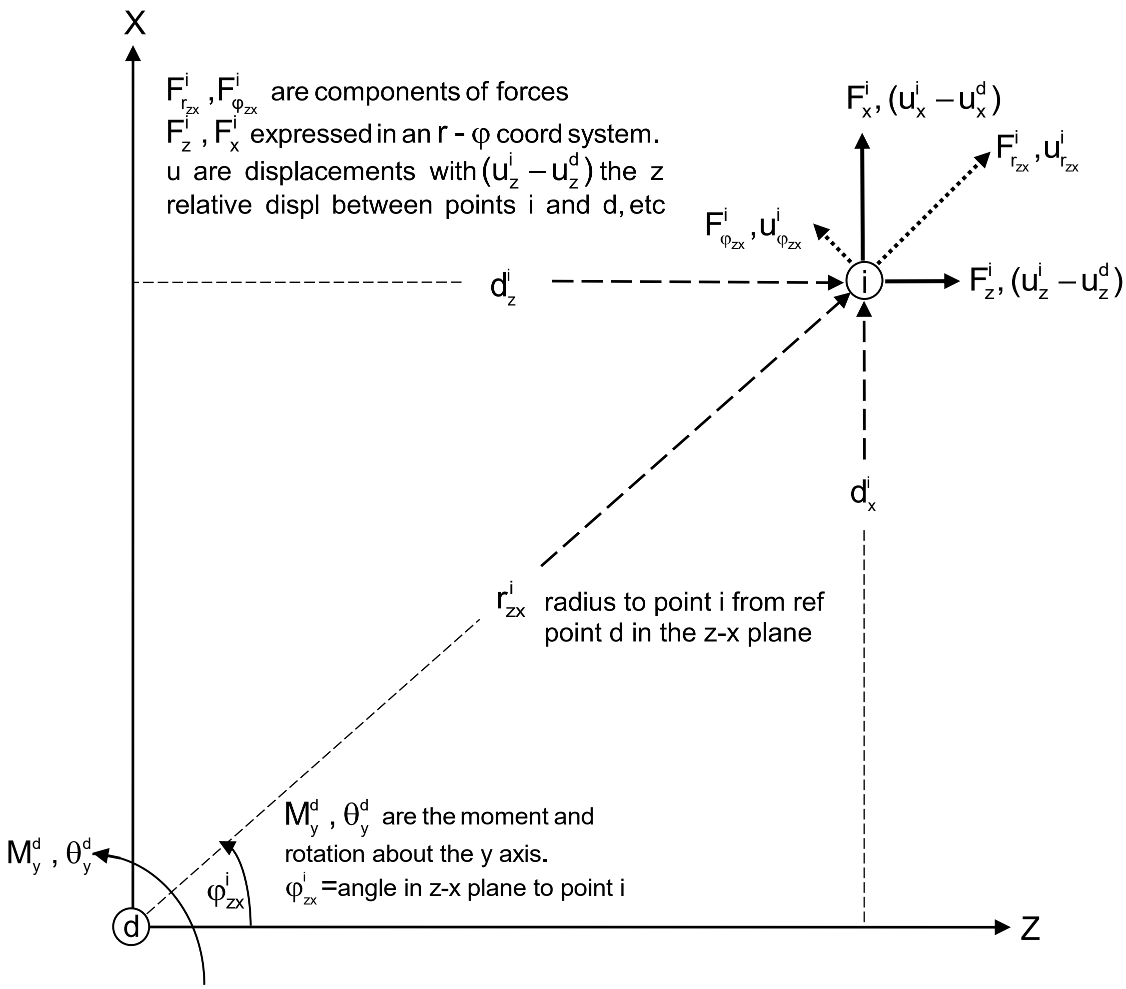

Figure 3: Relationship of moments and forces in the z-x plane

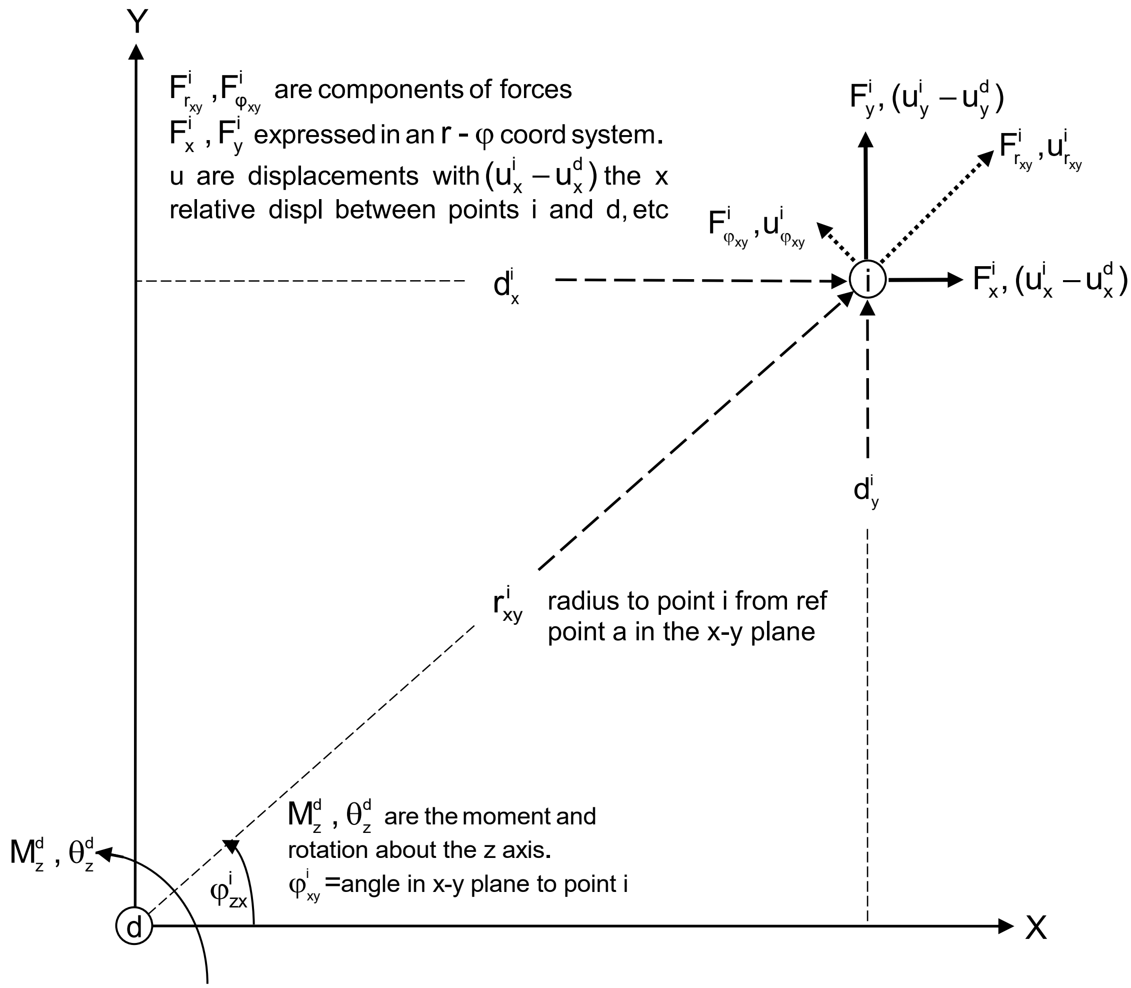

Figure 4: Relationship of moments and forces in the x-y plane

11.4 Summary of equations for the RBE3

In general, the equations for one RBE3 can be represented in matrix notation as:

$$R_{dd}U_d + R_{dN}U_N = 0 \qquad \text{11-27}$$

R_{dd} is the square, d x d, matrix of coefficients for the dependent (or reference) grid denoted as REFGRID in field 4 of the RBE3 Bulk Data entry. It can have up to d = 6 dependent components (REFC in field 5). For all 6 components, R_{dd} and U_d are:

$$R_{dd} = \begin{bmatrix} 1 & 0 & 0 & | & 0 & \bar{d}_z & -\bar{d}_y \\ 0 & 1 & 0 & | & -\bar{d}_z & 0 & \bar{d}_x \\ 0 & 0 & 1 & | & \bar{d}_y & -\bar{d}_x & 0 \\ - & - & - & | & - & - & - \\ 0 & -\bar{d}_z & \bar{d}_y & | & \bar{e}_{yz} & 0 & 0 \\ \bar{d}_z & 0 & -\bar{d}_x & | & 0 & \bar{e}_{zx} & 0 \\ -\bar{d}_y & \bar{d}_x & 0 & | & 0 & 0 & \bar{e}_{xy} \end{bmatrix} \quad , \quad U_d = \begin{Bmatrix} u_x^a \\ u_y^a \\ u_z^a \\ - \\ \theta_x^a \\ \theta_y^a \\ \theta_z^a \end{Bmatrix} \qquad \text{11-28}$$

R_{dN} is a rectangular, d x N, matrix of coefficients for the N independent grids on the RBE3

$$R_{dN} = \frac{1}{W_T}\begin{bmatrix} R_{d1} & R_{d2} & . & . & . & R_{dN} \end{bmatrix} \quad , \quad U_N = \begin{Bmatrix} U_1 \\ U_2 \\ . \\ . \\ . \\ U_N \end{Bmatrix} \qquad \text{11-29}$$

A typical sub-matrix in R_{ai} is of size d by 3 with R_{ai} and U_i. For d = 6:

$$R_{di} = \frac{1}{W_T}\begin{bmatrix} \omega^i & 0 & 0 \\ 0 & \omega^i & 0 \\ 0 & 0 & \omega^i \\ - & - & - \\ 0 & \omega^i d_z^i & -\omega^i d_y^i \\ -\omega^i d_z^i & 0 & \omega^i d_x^i \\ \omega^i d_y^i & -\omega^i d_x^i & 0 \end{bmatrix} \quad , \quad U_i = \begin{Bmatrix} u_x^i \\ u_y^i \\ u_z^i \end{Bmatrix} \qquad \text{11-30}$$

A RBE3 is processed by solving equation 27 for the dependent degrees of freedom, U_d, in terms of the independent degrees of freedom, U_N.