

Theory Manual

for the

MYSTRAN General Purpose Finite Element Structural Analysis Computer Program (Open Source Version)

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MYSTRAN

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1 Equations for the reduction of the G-set to the A-set and solution for displacements and constraint forces

1.1 Introduction

As discussed in Section 3.6, MYSTRAN builds the original stiffness and mass matrices based on the G-set, which has 6 degrees of freedom per grid specified in the Bulk Data deck. The stiffness matrix is by definition singular as, at this point, there have been no constraints imposed. There are two type of constraints MYSTRAN allows; single point constraints and multi-point constraints as discussed earlier in this manual. In order to apply boundary conditions that restrain the model from rigid body motion, single point constraints must be used. Multi-point constraints (using rigid elements or Bulk Data MPC entries) are used to express some degrees of freedom (DOF's) of the model as being rigidly restrained to some other DOF's. Thus, MYSTRAN must reduce the G-set stiffness, mass, and loads to the independent A-set DOF's

The discussion below shows the process that MYSTRAN uses to solve for the displacements and constraint forces by going through a systematic reduction of the G-set to the N-set then to the F-set and finally to the L-set which represent the independent DOF's. These equations can then be solved for the L-set DOF's. The other DOF displacements, as well as constraint forces, can then be recovered. Element forces and stresses are obtained from the displacements as discussed in Appendix C. The process in this appendix uses the displacement set notation developed in Section 3.6 which should be reviewed prior to this section. In general, the matrix notation used in this development is such that the matrix subscripts describe the matrix size. Thus, K_{GG} is a matrix which has G rows and columns, R_{CG} is a matrix that has C rows and G columns and R_{CG}^T is the transpose of R_{CG} and has G rows and C columns. If a matrix has only one column, it would exhibit only one subscript, as in Y_S which is an S x 1 matrix of single point constraint values

1.2 Reduction of the G-set to the N-set

In terms of this G-set, the equations of motion for the structure can be written as:

$$\begin{array}{l} M_{GG}\ddot{U}_G + K_{GG}U_G = P_G + R_{CG}^T q_C \\ R_{CG}U_G = Y_C \end{array} \quad (8-1)$$

In the first of equations 8.1 M_{GG} is the G-set mass matrix, K_{GG} is the G-set stiffness matrix, U_G are the G-set displacements, P_G are the applied loads on the G-set DOF's and q_C are the independent, generalized, constraint forces (due to single and multi-point constraints). The second of 8.1 expresses the constraints (both single and multi-point constraints) wherein C is the number of constraint equations, R_{CG} is a constraint coefficient matrix and Y_C is a vector of constraint values. For example, if all of the constraints were single point constraints, then all of the coefficients in any one row of R_{CG} would be zero except for one unity value. In addition, if all of these single point constraints were for DOF's that are grounded, then all of the Y_C values would be zero and these single point constraints would all have the form of $u_i = 0$.

The unknowns in 8.1 are the U_G displacements and the q_C generalized constraint forces and there are G+C equations to solve for these unknowns. As will be explained later, direct solution of the q_C constraint forces will not be made.

The q_C generalized forces of constraint do not necessarily have any physical meaning. Rather, the G-set nodal forces of constraint are of interest and are expressed in terms of the q_C as:

$$Q_G = R_{CG}^T q_C \quad (8-2)$$

In order to reduce 8.1 the G-set is partitioned into the N and M-sets, where the M DOF's are to be eliminated using the multi-point constraints (from rigid elements as well as MPC Bulk Data entries defined by the user in the input data deck). The U_N are the remainder of the DOF's in the G-set. Thus, write U_G as:

$$U_G = \begin{Bmatrix} U_N \\ U_M \end{Bmatrix} \quad (8-3)$$

The number of constraints is C which is equal to M+S (where S is the number of DOF's in the S set). Thus, partition q_C and Y_C as:

$$q_C = \begin{Bmatrix} q_S \\ q_M \end{Bmatrix} \quad (8-4)$$

$$Y_C = \begin{Bmatrix} Y_S \\ 0_M \end{Bmatrix}$$

0_M is a column vector of M zeros. That is, only the S-set can have nonzero constraint values.

With the second of 8.4 in mind, partition the second of equations 8.1 using 8.3 as:

$$\begin{bmatrix} R_{SN} & 0_{SM} \\ R_{MN} & R_{MM} \end{bmatrix} \begin{Bmatrix} U_N \\ U_M \end{Bmatrix} = \begin{Bmatrix} Y_S \\ 0_M \end{Bmatrix} \quad (8-5)$$

The 0_{SM} partition is an S x M matrix of zero's. This is required by the form of the single point constraint equations which are all of the form $u_i = Y_i$ where Y_i is a constant (zero or some enforced displacement value).

Using 8.3, partition the first of equations 8.1 as:

$$\begin{bmatrix} \bar{M}_{NN} & M_{NM} \\ M_{NM}^T & M_{MM} \end{bmatrix} \begin{Bmatrix} \ddot{U}_N \\ \ddot{U}_M \end{Bmatrix} + \begin{bmatrix} \bar{K}_{NN} & K_{NM} \\ K_{NM}^T & K_{MM} \end{bmatrix} \begin{Bmatrix} U_N \\ U_M \end{Bmatrix} = \begin{Bmatrix} \bar{P}_N \\ P_M \end{Bmatrix} + \begin{bmatrix} R_{SN}^T & R_{MN}^T \\ 0_{SM}^T & R_{MM}^T \end{bmatrix} \begin{Bmatrix} q_S \\ q_M \end{Bmatrix} \quad (8-6)$$

The bars over the N-set mass, stiffness and loads matrices are used for convenience to distinguish these terms from those that will result from the reduction of the G-set to the N-set. From the second of the constraint equations in 8.5 solve for U_M in terms of U_N :

$$U_M = G_{MN} U_N \quad (8-7)$$

where

$$G_{MN} = -(R_{MM}^{-1} R_{MN}) \quad (8-8)$$

Using 8.7, equation 8.3 can be written as:

$$U_G \equiv \begin{Bmatrix} U_N \\ U_M \end{Bmatrix} = \begin{bmatrix} I_{NN} \\ G_{MN} \end{bmatrix} U_N \quad (8-9)$$

where I_{NN} is an identity matrix of size N.

Substitute 8.9 into 8.6 and premultiply the result by the transpose of the coefficient matrix in 8.9. The result can be written as:

$$M_{NN} \ddot{U}_N + K_{NN} U_N = P_N + \begin{bmatrix} R_{SN}^T & (R_{MN}^T + G_{MN}^T R_{MM}^T) \end{bmatrix} \begin{Bmatrix} q_S \\ q_M \end{Bmatrix} \quad (8-10)$$

where:

$$\begin{aligned}
K_{NN} &= \bar{K}_{NN} + K_{NM}G_{MN} + (K_{NM}G_{MN})^T + G_{MN}^T K_{MM}G_{MN} \\
M_{NN} &= \bar{M}_{NN} + M_{NM}G_{MN} + (M_{NM}G_{MN})^T + G_{MN}^T M_{MM}G_{MN} \\
P_N &= \bar{P}_N + G_{MN}^T P_M
\end{aligned} \tag{8-11}$$

M_{NN} , K_{NN} and P_N are the reduced N-set mass stiffness and loads. Note that P_N is not the set of applied loads on the N-set if there are applied loads on the M-set as expressed by the second of equations 8.11 (\bar{P}_N are the applied loads on the N set).

In addition, the second term in the square brackets in 8.10 is zero by the definition of G_{MN} in 8.8 so that 8.10 and 8.5 can be written as:

$$\boxed{M_{NN}\ddot{U}_N + K_{NN}U_N = P_N + R_{SN}^T q_S} \tag{8-12}$$

1.3 Reduction of the N-set to the F-set

The N-set can now be partitioned into the F and S-sets where the S DOF's are to be eliminated using the single point constraints identified by the user in the input data deck. The F-set are the remainder of the DOF's in the N-set and are known as the "free" DOF's (i.e. those that have no constraints imposed on them). Thus, partition U_N into U_F and U_S :

$$U_N = \begin{Bmatrix} U_F \\ U_S \end{Bmatrix} \tag{8-13}$$

Rewrite equation 8.5 in terms of the F, S and M-sets with the restriction that the single point constraints are of the form $u_i = Y_i$ where Y_i is a constant (zero or some enforced displacement value), using:

$$\begin{aligned}
R_{SN} &= \begin{bmatrix} 0_{SF} & I_{SS} \end{bmatrix} \\
R_{MN} &= \begin{bmatrix} R_{MF} & R_{MS} \end{bmatrix}
\end{aligned} \tag{8-14}$$

where 0_{SF} is an $S \times F$ matrix of zeros and I_{SS} is an S size identity matrix. Equation 8.5 can be written as:

$$\begin{bmatrix} 0_{SF} & I_{SS} & 0_{SM} \\ R_{MF} & R_{MS} & R_{MM} \end{bmatrix} \begin{Bmatrix} U_F \\ U_S \\ U_M \end{Bmatrix} = \begin{Bmatrix} Y_S \\ 0_M \end{Bmatrix} \tag{8-15}$$

Substitute 8.13 and the first of 8.14 into 8.12 and partition the mass, stiffness and load matrices into the F and S-sets to get:

$$\begin{bmatrix} M_{FF} & M_{FS} \\ M_{FS}^T & M_{SS} \end{bmatrix} \begin{Bmatrix} \ddot{U}_F \\ \ddot{U}_S \end{Bmatrix} + \begin{bmatrix} K_{FF} & K_{FS} \\ K_{FS}^T & K_{SS} \end{bmatrix} \begin{Bmatrix} U_F \\ U_S \end{Bmatrix} = \begin{Bmatrix} \bar{P}_F \\ P_S \end{Bmatrix} + \begin{bmatrix} 0_{FS} \\ I_{SS} \end{bmatrix} q_S \tag{8-16}$$

Note that 0_{SF} is the transpose of 0_{FS} and is an $S \times F$ matrix of zero's. From the first of 8.15 it is seen that the single point constraints are of the form:

$$U_S = Y_S = \text{constants} \quad (8-17)$$

where Y_S is a column matrix of known constant displacement values (either zero or some enforced displacement). This agrees with the single point constraint form discussed above; that is, single point constraints express one DOF as being equal to a constant.

Substituting 8.17 into the first of 8.16 results in the equations for the F-set displacements:

$$\boxed{M_{FF}\ddot{U}_F + K_{FF}U_F = P_F} \quad (8-18)$$

where

$$P_F = \bar{P}_F - K_{FS}Y_S \quad (8-19)$$

At this point the F-set equations in 8.18 can be solved for since there are F unknowns and F equations with which to solve for them. However, MYSTRAN also allows for a Guyan reduction which, although not generally used in static analysis, may be relevant for eigenvalue analysis. In eigenvalue analyses by the GIV method (see EIGR Bulk Data entry), the mass matrix must be nonsingular. In a situation where the model has no mass for the rotational DOF's, the mass matrix would be singular. Guyan reduction to statically condense massless DOF's will result in a nonsingular mass matrix. Thus, if the user identifies an O set, there is a further reduction; that from the F-set to the A-set

1.4 Reduction of the F-set to the A-set

The F-set is partitioned into the A and O-sets where the O DOF's are to be eliminated using Guyan reduction identified by the user either through the use of ASET/ASET1 or OMIT/OMIT1 entries in the input data deck. The A-set are the remainder of the DOF's in the F-set and are known as the "analysis" DOF's. Thus, partition U_F into U_A and U_O :

$$U_F = \begin{Bmatrix} U_A \\ U_O \end{Bmatrix} \quad (8-20)$$

Substitute 8.20 into 8.18 and partition the stiffness and load matrices into the A and O-sets to get:

$$\begin{bmatrix} \bar{M}_{AA} & M_{AO} \\ M_{AO}^T & M_{OO} \end{bmatrix} \begin{Bmatrix} \ddot{U}_A \\ \ddot{U}_O \end{Bmatrix} + \begin{bmatrix} \bar{K}_{AA} & K_{AO} \\ K_{AO}^T & K_{OO} \end{bmatrix} \begin{Bmatrix} U_A \\ U_O \end{Bmatrix} = \begin{Bmatrix} \bar{P}_A \\ P_O \end{Bmatrix} \quad (8-21)$$

Guyan reduction is only exact, in general, for a statics problem. In a dynamic problem it is only exact if there is no mass on the O-set. In order to explain the Guyan reduction, consider equation 8.21 for a statics problem:

In a static analysis ($\ddot{U}=0$) the second of 8.21 can be used to get:

$$\begin{bmatrix} \bar{K}_{AA} & K_{AO} \\ K_{AO}^T & K_{OO} \end{bmatrix} \begin{Bmatrix} U_A \\ U_O \end{Bmatrix} = \begin{Bmatrix} \bar{P}_A \\ P_O \end{Bmatrix} \quad (8-22)$$

From the 2nd of 8.22 we can solve for U_O in terms of U_A . We can then write:

$$\begin{Bmatrix} U_A \\ U_O \end{Bmatrix} = \begin{bmatrix} I_{AA} \\ G_{OA} \end{bmatrix} U_A + \begin{Bmatrix} 0 \\ U_O^0 \end{Bmatrix}$$

where

$$G_{OA} = -K_{OO}^{-1} K_{AO}^T \quad (8-23)$$

and

$$U_O^0 = K_{OO}^{-1} P_O$$

The first part of the first equation in 8.23 suggests the possibility of using:

$$\begin{Bmatrix} U_A \\ U_O \end{Bmatrix} = \begin{bmatrix} I_{AA} \\ G_{OA} \end{bmatrix} U_A \quad (8-24)$$

Using 8.24 in 8.22 and premultiplying by the transpose of the coefficient matrix in 8.24 yields:

$$\begin{bmatrix} I_{AA} & G_{OA}^T \end{bmatrix} \begin{bmatrix} \bar{K}_{AA} & K_{AO} \\ K_{AO}^T & K_{OO} \end{bmatrix} \begin{bmatrix} I_{AA} \\ G_{OA} \end{bmatrix} \begin{Bmatrix} U_A \\ U_O \end{Bmatrix} = \begin{bmatrix} I_{AA} & G_{OA}^T \end{bmatrix} \begin{Bmatrix} \bar{P}_A \\ P_O \end{Bmatrix}$$

or

$$K_{AA} U_A = P_A$$

where

$$K_{AA} = \bar{K}_{AA} + K_{AO} G_{OA} + (K_{AO} G_{OA})^T + G_{OA}^T K_{OO} G_{OA} = \bar{K}_{AA} + K_{AO} G_{OA} \quad (\text{by virtue of definition of } G_{OA})$$

and

$$P_A = \bar{P}_A + G_{OA}^T P_O \quad (8-25)$$

Which is exactly what would have been found if 8.23 had been substituted into 8.22 for U_O .

Equation 8.24 to can be used as a way to eliminate the O-set degrees of freedom for the dynamic system of equations in 8.21. This would be an approximation unless there was no mass associated with the O-set degrees of freedom and is the classic Guyan reduction approximation made in dynamic analyses in which the O-set is eliminated by static condensation (i.e. using the G_{OA} in equation 8.23). Using 8.24 in 8.21 yields

$$\begin{bmatrix} I_{AA} & G_{OA}^T \end{bmatrix} \begin{bmatrix} \bar{M}_{AA} & M_{AO} \\ M_{AO}^T & M_{OO} \end{bmatrix} \begin{bmatrix} I_{AA} \\ G_{OA} \end{bmatrix} \begin{Bmatrix} \ddot{U}_A \\ \ddot{U}_O \end{Bmatrix} + \begin{bmatrix} I_{AA} & G_{OA}^T \end{bmatrix} \begin{bmatrix} \bar{K}_{AA} & K_{AO} \\ K_{AO}^T & K_{OO} \end{bmatrix} \begin{bmatrix} I_{AA} \\ G_{OA} \end{bmatrix} \begin{Bmatrix} U_A \\ U_O \end{Bmatrix} = \begin{bmatrix} I_{AA} & G_{OA}^T \end{bmatrix} \begin{Bmatrix} \bar{P}_A \\ P_O \end{Bmatrix} \quad (8-26)$$

where:

$$\begin{aligned}
& \mathbf{M}_{AA} \ddot{\mathbf{U}}_A + \mathbf{K}_{AA} \mathbf{U}_A = \mathbf{P}_A \\
& \text{where} \\
& \mathbf{M}_{AA} = \bar{\mathbf{M}}_{AA} + \mathbf{M}_{AO} \mathbf{G}_{OA} + (\mathbf{M}_{AO} \mathbf{G}_{OA})^T + \mathbf{G}_{OA}^T \mathbf{M}_{OO} \mathbf{G}_{OA} \\
& \mathbf{K}_{AA} = \bar{\mathbf{K}}_{AA} + \mathbf{K}_{AO} \mathbf{G}_{OA} \\
& \mathbf{P}_A = \bar{\mathbf{P}}_A + \mathbf{G}_{OA}^T \mathbf{P}_O
\end{aligned} \tag{8-27}$$

Now, equation 8.27 can be solved for the A-set DOF displacements. The process of recovering the displacements of the O, S and M-set displacements is accomplished by reversing the process we just went through in the reduction. First, the O set displacements are recovered using 8.23. The combination of the A and O-sets yields the F-set. The S-set is given by 8.17. The combination of the F and S-sets yields the N-set. The M-set is recovered from the N-set by 8.7 and the combination of the N and M-sets yield the complete model displacements in the G-set.

1.5 Reduction of the A-set to the L-set

The A-set is partitioned into the L and R-sets where the R DOF's are boundary DOF's where one substructure attaches to another in Craig-Bampton (CB) analyses. The modal properties of the substructure in CB analysis are fixed boundary modes so that, for the modal portion of CB, the R-set are constrained to zero. The development of the subsequent CB equations of motion in terms of the modal and boundary DOF's will not be presented here. See Appendix D and reference 11 for a complete discussion of CB analyses. For other analyses there is no R-set so that the L set is the same as the A set for solution of the independent degrees of freedom

$$\mathbf{U}_A = \begin{Bmatrix} \mathbf{U}_L \\ \mathbf{U}_R \end{Bmatrix}$$

1.6 Solution for constraint forces

The constraint forces are recovered as follows. Rewrite 8.2 by partitioning \mathbf{Q}_G into \mathbf{Q}_F , \mathbf{Q}_N and \mathbf{Q}_M and partitioning \mathbf{q}_C into \mathbf{q}_S and \mathbf{q}_M . Using the coefficient matrix in 8.15 for \mathbf{R}_{CG} we get, for \mathbf{Q}_G :

$$\mathbf{Q}_G = \begin{Bmatrix} \mathbf{Q}_F \\ \mathbf{Q}_S \\ \mathbf{Q}_M \end{Bmatrix} = \begin{bmatrix} 0_{FS} & \mathbf{R}_{MF}^T \\ \mathbf{I}_{SS} & \mathbf{R}_{MS}^T \\ 0_{MS} & \mathbf{R}_{MM}^T \end{bmatrix} \begin{Bmatrix} \mathbf{q}_S \\ \mathbf{q}_M \end{Bmatrix} \tag{8-28}$$

As discussed earlier, the distinction between the \mathbf{q} and \mathbf{Q} is that the former are generalized forces of constraint and the later are physical constraint forces on the DOF's of the model. It is the \mathbf{Q} constraint forces that are of interest.

Rewrite 8.28 as:

$$Q_G = \begin{Bmatrix} 0_F \\ q_S \\ 0_M \end{Bmatrix} + \begin{bmatrix} R_{MF}^T \\ R_{MS}^T \\ R_{MM}^T \end{bmatrix} q_M \quad (8-29)$$

where 0_F and 0_M are null column matrices of size F and M .

Equation 8.29 can be written as:

$$Q_G = Q_{G_{SPC}} + Q_{G_{MPC}} \quad (8-30)$$

The first term in 8.30 represents the forces of single point constraint and the second the forces of multi-point constraint. Comparing 8.29 and 8.30:

$$Q_{G_{SPC}} = \begin{Bmatrix} 0_F \\ q_S \\ 0_M \end{Bmatrix} \quad (8-31)$$

$$Q_{G_{MPC}} = \begin{bmatrix} R_{MF}^T \\ R_{MS}^T \\ R_{MM}^T \end{bmatrix} q_M$$

From the first of 8.31 it is seen that the grid point SPC constraint forces are equal to the generalized q_S forces. Using 8.17 and the second of 8.16 (keeping in mind that the derivatives of the S-set degrees of freedom are zero due to 8.17) the q_S , or Q_S is:

$$Q_{G_{SPC}} \equiv \begin{Bmatrix} 0_F \\ Q_{S_{SPC}} \\ 0_M \end{Bmatrix} = \begin{Bmatrix} 0_F \\ M_{FF}\ddot{U}_{FF} + K_{FS}^T U_F + K_{SS} Y_S - P_S \\ 0_M \end{Bmatrix} \quad (8-32)$$

Thus, there are SPC forces only on the S-set DOF's

From the second of 8.31 and using 8.14 it is seen that the MPC forces can be written as:

$$Q_{G_{MPC}} = \begin{bmatrix} R_{MN}^T \\ R_{MM}^T \end{bmatrix} q_M \quad (8-33)$$

From 8.7 and the second of 8.6, solve for q_M :

$$q_M = R_{MM}^{-T} [(M_{NM}^T + M_{MM} G_{MN}) \ddot{U}_N + (K_{NM}^T + K_{MM} G_{MN}) U_N - P_M] \quad (8-34)$$

Substituting 8.34 into 8.33 yields:

$$Q_{G_{MPC}} = \begin{bmatrix} R_{MN}^T R_{MM}^{-T} \\ I_{MM} \end{bmatrix} [(M_{NM}^T + M_{MM} G_{MN}) \ddot{U}_N + (K_{NM}^T + K_{MM} G_{MN}) U_N - P_M] \quad (8-35)$$

Using 8.8 this becomes:

$$\mathbf{Q}_{G_{MPC}} \equiv \begin{Bmatrix} \mathbf{Q}_{N_{MPC}} \\ \mathbf{Q}_{M_{MPC}} \end{Bmatrix} = \begin{bmatrix} -\mathbf{G}_{MN}^T \\ \mathbf{I}_{MM} \end{bmatrix} [(\mathbf{M}_{NM}^T + \mathbf{M}_{MM}\mathbf{G}_{MN})\ddot{\mathbf{U}}_N + (\mathbf{K}_{NM}^T + \mathbf{K}_{MM}\mathbf{G}_{MN})\mathbf{U}_N - \mathbf{P}_M] \quad (8-36)$$

This can also be written as:

$$\begin{aligned} \mathbf{Q}_{G_{MPC}} &\equiv \begin{Bmatrix} \mathbf{Q}_{N_{MPC}} \\ \mathbf{Q}_{M_{MPC}} \end{Bmatrix} \\ \text{with} \\ \mathbf{Q}_{M_{MPC}} &= \mathbf{L}_{MN}\ddot{\mathbf{U}}_N + \mathbf{H}_{mn}\mathbf{U}_n - \mathbf{P}_m \\ \mathbf{Q}_{N_{MPC}} &= -\mathbf{G}_{mn}^T \mathbf{Q}_{M_{MPC}} \\ \text{where} \\ \mathbf{H}_{mn} &= (\mathbf{K}_{NM}^T + \mathbf{K}_{MM}\mathbf{G}_{MN}) \\ \mathbf{L}_{MN} &= (\mathbf{M}_{NM}^T + \mathbf{M}_{MM}\mathbf{G}_{MN}) \end{aligned} \quad (8-37)$$

There are MPC forces on the N-set (which includes the F and S-sets) as well as on the M-set. Equations 8.32 and 8.36 (or 8.37) are used to determine the constraint forces once the \mathbf{U}_G are found.

This completes the derivation of the solution for the G-set displacements and the constraint forces. However, it is of interest to demonstrate that the constraint forces satisfy the principal of virtual work (that is, constraint forces do no virtual work).

Let W_C be the work done by the constraint forces and δW_C the virtual work done by the constraint forces. Write δW_C as:

$$\delta W_C = \delta W_{SPC} + \delta W_{MPC} = 0$$

where

$$\delta W_{SPC} = \text{virtual work of the SPC single point constraint forces} \quad (8-38)$$

and

$$\delta W_{MPC} = \text{virtual work of the MPC multi-point constraint forces}$$

The virtual work of the constraint forces is equal to the constraint forces moving through a virtual displacement, $\delta \mathbf{U}$. Thus:

$$\delta W_{SPC} = \mathbf{Q}_{SPC}^T \delta \mathbf{U}_S \quad (8-39)$$

By virtue of 8.17:

$$\delta U_S = \delta Y_S = 0_S \quad (8-40)$$

That is, the virtual displacements of the S-set are zero since Y_S contains specified values (zero or some enforced displacement). Therefore:

$$\delta W_{\text{spc}} = 0 \quad (8-41)$$

Thus δW_{MPC} must also be zero by virtue of the first of 8.38. This virtual work of the MPC forces can be written as a combination of the virtual work of the MPC forces on the N and M-sets as follows:

$$\delta W_{\text{MPC}} = Q_{N_{\text{MPC}}}^T \delta U_N + Q_{M_{\text{MPC}}}^T \delta U_M \quad (8-42)$$

Using 8.7 this can be written as:

$$\delta W_{\text{MPC}} = (Q_{N_{\text{MPC}}}^T + Q_{M_{\text{MPC}}}^T G_{MN}) \delta U_N \quad (8-43)$$

using 8-41:

$$\delta W_{\text{MPC}} = (Q_{N_{\text{MPC}}} + G_{MN}^T Q_{M_{\text{MPC}}})^T \delta U_N = 0 \quad (8-44)$$

Since the virtual displacements of the N-set are not necessarily zero this requires that:

$$Q_{N_{\text{MPC}}} = -G_{MN}^T Q_{M_{\text{MPC}}} \quad (8-45)$$

This agrees with 8.36. Thus, the constraint forces developed above are consistent with the principal of virtual work.

2 Equations for Element Stress/Strain Recovery

2.1 General discussion

For the 2D plate elements and 3D solid elements arrays called STRAIN and STRESS are calculated for each element. For 1D elements.. like the rod and beam. only the STRESS array is calculated. Both arrays STRAIN and STRESS can contain up to 9 rows and there is one of each these calculated for every subcase. The STRAIN and STRESS arrays are further subdivided as shown below:

$$\text{STRAIN} = \begin{Bmatrix} \text{STRAIN}_1 \\ \text{STRAIN}_2 \\ \text{STRAIN}_3 \end{Bmatrix}, \quad \text{STRESS} = \begin{Bmatrix} \text{STRESS}_1 \\ \text{STRESS}_2 \\ \text{STRESS}_3 \end{Bmatrix} \quad (9-1)$$

where STRAIN_i and STRESS_i each have 3 rows

for 2D and 3D elements:

$$\text{STRAIN}_i = (\text{BE}_i) * U_e$$

and

$$\text{STRESS}_i = (\text{DE}_i) * \text{STRAIN}_i - (\text{STE}_i) \quad (9-2)$$

for 1D elements stresses are calculated directly from displacements:

$$\text{STRESS}_i = (\text{SE}_i) * U_e - (\text{STE}_i)$$

U_e are the displacements of the nodes of the element in the local element coordinate system (see Figures 3-2 through 3-6 in the main body of this manual) and are obtained from the G-set displacements, the solution for which is discussed in Appendix B. These G-set displacements for the nodes of an element are transformed to the local element coordinate system to obtain U_e which has a number of rows equal to $6n$ where n is the number of nodes for the element (e.g. $n=4$ for a quadrilateral plate element). There is one U_e for each subcase in the solution. The BE_i arrays each have 3 rows and $6n$ columns and are based on the strain-displacement relationships for individual elements. The SE_i are equal to material matrices times the BE_i . The STE_i arrays contain the thermal stress effects, if there are any, and have 3 rows and as many columns as there are thermal subcases.. That is, if the input data deck has 5 subcases and two of these have thermal loads, then STE_i will have only 2 columns while U_e will have 5 columns. If a user outputs the SE_i and STE_i arrays, it is their responsibility to keep track of which subcases the columns of STE_i belong. MYSTRAN does this internally for its stress output calculations.

The following sections show what is contained in arrays STRESS_i for each of the element types. In that manner, it will be obvious how MYSTRAN uses the SE_i and STE_i arrays, generated internally in MYSTRAN, to obtain stresses. If desired, they are available to be output to a text or unformatted binary file through use of the Case Control entry ELDATA. They need not be output for the user to obtain element stresses, however, which are available in the normal text output file through use of the Case Control entry STRESS.

2.2 Rod element

The rod geometry and loading is shown in Figure 3-2 in the main body of this manual. It is a very simple element and has only two stresses that can be output: the axial stress and the torsional stress. It only uses the first 2 rows of array STRESS_1 with row 1 being the axial stress and row 2 the torsional stress. Array STRESS_1 is:

$$\text{STRESS}_1 = \begin{Bmatrix} \sigma_{\text{axial}} \\ \tau \\ 0 \end{Bmatrix} \quad (9-3)$$

As an example of what is in arrays SE1 and STE1 for a simple element, the arrays are shown below for this rod element. More complicated elements won't have a simple closed form for these matrices and will not be shown.

Array SE1 for the rod element is:

$$SE1 = \frac{1}{L} \begin{bmatrix} -E & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C*G & 0 & 0 & 0 & 0 & 0 & C*G & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (9-4)$$

E and G are Young's modulus and shear modulus from the Bulk Data material entry for the element, L is the element length and C is the torsional stress recovery coefficient from a PROD entry.

Array STE1 would have the following column for each subcase that has a thermal load:

$$STE1 = E\alpha(\bar{T} - T_{ref}) \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad (9-5)$$

α and T_{ref} are the coefficient of thermal expansion and reference temperature from the material Bulk Data entry for the element and \bar{T} is the average element temperature for the thermal subcase.

2.3 Bar element

The bar element geometry and loading is shown in Figures 3-3 and 3-4 in the main body of this manual. For the bar element, array STRESS uses all 3 rows of STRESS₁ and STRESS₂. The first row of STRESS₁ contains the actual axial stress in the bar and the third row of STRESS₂ contains the actual torsional stress. The second and third rows of STRESS₁ and the first two rows of STRESS₂ are not actual stress values. Rather, they are the four independent parameters needed to determine the bending stresses at points in the bar cross-section. Thus:

$$STRESS_1 = \begin{Bmatrix} \sigma_{axial} \\ \kappa'_{1a} \\ \kappa'_{1b} \end{Bmatrix}, \quad STRESS_2 = \begin{Bmatrix} \kappa'_{2a} \\ \kappa'_{2b} \\ \tau \end{Bmatrix} \quad (9-6)$$

where

$$\kappa'_{1a} = \frac{M_{1a}l_2 - M_{2a}l_{12}}{l_1l_2 - l_{12}^2}, \quad \kappa'_{1b} = \frac{M_{1b}l_2 - M_{2b}l_{12}}{l_1l_2 - l_{12}^2}$$

$$\kappa'_{2a} = \frac{M_{2a}l_1 - M_{1a}l_{12}}{l_1l_2 - l_{12}^2}, \quad \kappa'_{2b} = \frac{M_{2b}l_1 - M_{1b}l_{12}}{l_1l_2 - l_{12}^2}$$

and

σ_{axial} = Axial stress at the neutral axis

τ = Torsional stress

l_1, l_2, l_{12} = the moments of inertia of the bar on the PBAR entry for this bar element

$M_{1a}, M_{2a}, M_{1b}, M_{2b}$ = the moments in planes 1 and 2 at ends a and b of the bar

(9-7)

This can be put into the form of equation 9.2 as:

$$\text{STRESS}_1 = \text{SE1} * U_e - \text{STE1}$$

$$\text{STRESS}_2 = \text{SE2} * U_e - \text{STE2}$$

where

$$\text{SE1} = \begin{bmatrix} B_1 K_{aa} & B_1 K_{ab} \end{bmatrix}, \quad \text{STE1} = B_1 K_{aa} \bar{A} T'$$

$$\text{SE2} = \begin{bmatrix} B_2 K_{aa} & B_2 K_{ab} \end{bmatrix}, \quad \text{STE2} = B_2 K_{aa} \bar{A} T'$$

K_{aa} and K_{ab} are 6x6 partitions from the 1st 6 rows of the bar element stiffness matrix and B_1 , B_2 and \bar{A} are matrices of element properties as shown below:

$$B_1 = \begin{bmatrix} -1/A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta_{12} & -\Delta_1 \\ 0 & 0 & 0 & 0 & \Delta_2 & \Delta_{12} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & \Delta_1 L & -\Delta_{12} L & 0 & -\Delta_{12} & -\Delta_1 \\ 0 & -\Delta_{12} L & \Delta_2 L & 0 & \Delta_2 & \Delta_{12} \\ 0 & 0 & 0 & -C/J & 0 & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & L\Delta_1 l_1/6 & L\Delta_1 l_1/3 & -L\Delta_{12} l_2/6 & L\Delta_{12} l_2/3 \\ 0 & -L\Delta_{12} l_1/6 & -L\Delta_{12} l_1/3 & L\Delta_2 l_2/6 & L\Delta_2 l_2/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\Delta_{12} l_1/2 & -\Delta_{12} l_1/2 & \Delta_2 l_2/2 & \Delta_2 l_2/2 \\ 0 & -\Delta_1 l_1/2 & -\Delta_1 l_1/2 & \Delta_{12} l_2/2 & \Delta_{12} l_2/2 \end{bmatrix}$$

and

$$T' = \begin{Bmatrix} \bar{T} - T_{\text{ref}} \\ T'_{1a} \\ T'_{1b} \\ T'_{2a} \\ T'_{2b} \end{Bmatrix} = \begin{Bmatrix} \text{avg bulk temp above material ref temp} \\ \text{gradient through bar in plane 1 at end a} \\ \text{gradient through bar in plane 1 at end b} \\ \text{gradient through bar in plane 2 at end a} \\ \text{gradient through bar in plane 2 at end b} \end{Bmatrix}$$

with the following bar properties:

L = bar length
 A = cross-sectional area
 I_1 = area moment of inertia in plane 1
 I_2 = area moment of inertia in plane 1
 I_{12} = product of inertia

$$\Delta_1 = \frac{I_2}{I_1 I_2 - I_{12}^2}$$

$$\Delta_2 = \frac{I_1}{I_1 I_2 - I_{12}^2}$$

$$\Delta_{12} = \frac{I_{12}}{I_1 I_2 - I_{12}^2}$$

Stresses due to bending (i.e. not including axial stress at the neutral axis) at ends a and b of the bar element are obtained from:

$$\sigma_a = -(\kappa'_{1a} \bar{y}_e + \kappa'_{2a} \bar{z}_e) \quad , \quad \sigma_b = -(\kappa'_{1b} \bar{y}_e + \kappa'_{2b} \bar{z}_e) \quad (9-8)$$

where σ_a , σ_b are the bending stresses at ends a and b of the bar and \bar{y}_e , \bar{z}_e are the coordinates of a point on the bar cross section as measured in the local element coordinate system (see Figure 3-3 in the main body of this manual). It should be noted that temperature distributions through the depth of the bar that are higher order than linear are ignored

2.4 Plate elements

Triangular and quadrilateral plate element geometry, loading and stress conventions are shown in Figures 3-5 and 3-6 in the main body of this manual. They can use all three of the STRESS_i arrays.

2.4.1 Membrane stresses

STRESS₁ contains the membrane stresses (at the plate mid-plane)

$$\text{STRESS}_1 = \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\}_{z=0} \quad (9-9)$$

This can be put into the form of equation 9.2 as:

$$\text{STRESS}_1 = (\text{SE1}) * U_e - (\text{STE1})$$

where

$$\text{SE1} = E_m B_m \quad \text{and} \quad \text{STE1} = E_m \alpha (T - T_{\text{ref}}) \quad (9-10)$$

E_m is the 3x3 membrane material matrix, B_m is the element membrane strain-displacement matrix (developed internally in MYSTRAN), α is the 3x1 vector of coefficients of thermal expansion for the material, T is the element average bulk temperature and T_{ref} is the reference temperature for the element material.

2.4.2 Bending stresses

STRESS₂, times a fiber distance, contains the stresses due to bending, where:

$$\text{STRESS}_2 = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \quad (9-11)$$

This can be put into the form of equation 9.2 as:

$$\begin{aligned} \text{STRESS}_2 &= (\text{SE2}) * U_e - (\text{STE2}) \\ \text{where} \\ \text{SE2} &= E_b B_b \quad \text{and} \quad \text{STE2} = E_b \alpha T' \end{aligned} \quad (9-12)$$

E_b is the 3x3 bending material matrix, B_b is the element bending strain-displacement matrix (developed internally in MYSTRAN), α is the 3x1 vector of coefficients of thermal expansion for the material and T' is the temperature gradient through the thickness of the plate element.

2.4.3 Combined membrane and bending stresses

The total bending and in-plane shear stresses at a fiber distance z are obtained from STRESS₁ and STRESS₂ as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \text{STRESS}_1 + z(\text{STRESS}_2) \quad (9-13)$$

2.4.4 Transverse shear stresses

The average transverse shear stresses through the thickness of the plate (for TRIA3 and QUAD4 elements only) are obtained from STRESS₃:

$$\text{STRESS}_3 = \begin{Bmatrix} \tau_{zx} \\ \tau_{zy} \\ 0 \end{Bmatrix} \quad (9-14)$$

This can be put into the form of equation 9.2 as

$$\text{STRESS}_3 = \text{SE3}$$

where

$$\text{SE3} = E_s B_s$$

E_s is the 3x3 transverse shear material matrix and B_s is the element transverse shear strain-displacement matrix (developed internally in MYSTRAN).

The transverse shear stresses are not output in the normal output file even if stress output is requested in Case Control. However, the transverse shear stress resultants (integrals of shear stress through thickness) are output if there is a request in Case Control for element engineering forces

2.5 Solid elements

For the 3D solid elements HEXA, PENTA and TETRA arrays STRAIN and STRESS contain only the 6 actual strains and stresses for a 3D solid:

$$\text{STRAIN} = (\text{BE}) * \text{U}_e = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

The BE are strain-displacement matrices that are based on the shape functions chosen for the particular 3D solid element. Once the strains have been calculated the stresses are determined from:

$$\text{STRESS} = (\text{ES}) * (\text{STRAIN} - \text{ALPT}) = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

ES is the 6x6 material matrix for a solid and ALPT is the thermal distortion portion of the strains. For a homogeneous isotropic material these are:

$$\text{ES} = \begin{bmatrix} (1-\nu)E_0 & \nu E_0 & \nu E_0 & 0 & 0 & 0 \\ \nu E_0 & (1-\nu)E_0 & \nu E_0 & 0 & 0 & 0 \\ \nu E_0 & \nu E_0 & (1-\nu)E_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}, \quad E_0 = \frac{E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad G = \frac{E}{2(1+\nu)}$$

$$\text{ALPT} = \begin{Bmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{\text{ref}})$$

MYSTRAN does allow anisotropic element properties for solids and, in that case, ES and ALPT are different

3 Craig-Bampton Model Generation

3.1 Craig-Bampton Equations of Motion for Substructures

MYSTRAN has the capability to generate Craig-Bampton (CB) models via SOL 31 (or SOL GEN CB MODEL). This solution sequence calculates the fixed-base modes of a substructure and generates all of the matrices needed to couple the substructure to other CB models. This appendix describes the Craig-Bampton method and its implementation in MYSTRAN and includes an example problem to explain the input and output for SOL 31.

Craig and Bampton¹ are credited with the first unified approach to modal synthesis, or substructuring for dynamic analysis, using fixed interface flexible modes augmented by boundary constraint modes to describe each substructure. Their work was a simplification of earlier work by Hurty² who first introduced the concept for substructures with redundant boundary degrees of freedom (DOF's).

In order to explain the Craig-Bampton (CB) method, consider a structure represented by the picture below that is comprised of several (in this case 5) substructures connected at an arbitrary number of points:

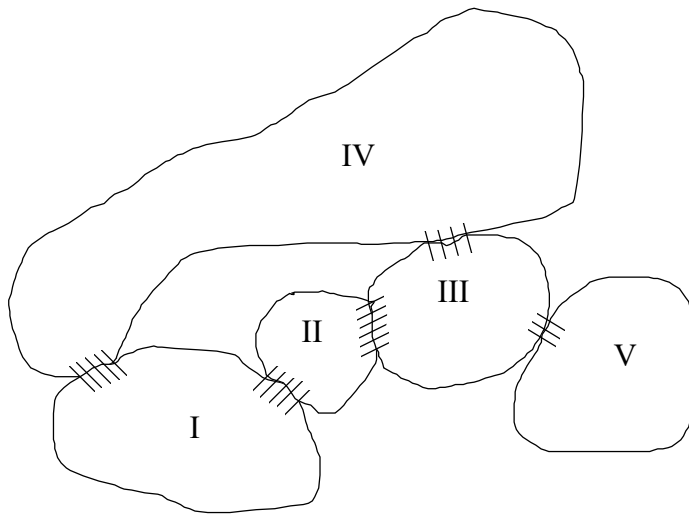


Figure 10.1 - Overall Structure Composed of Several Substructures

Each substructure is joined to one or more other substructures at some number of interface, or boundary, DOF's (indicated by the hatched areas in the above picture. The complete structure, consisting of the connected substructures, may or may not be restrained from free body motion. For any one of the substructures ($j = I, II, III, \text{ etc.}$) the G-set equations of motion (ignoring damping for the moment) are:

¹ Craig, R.R. and Bampton, M.C.C. "Coupling of Substructures for Dynamic Analysis", AIAA Journal, Vol. 6, No. 7, July 1968, pp 1313-1319

² Hurty, W.C. "Dynamic Analysis of Structural Systems Using Component Modes", AIAA Journal, Vol. 3, No. 4, April 1965, pp 678-685

$$M_{GG}^j \ddot{u}_G^j + K_{GG}^j u_G^j = P_G^j + Q_G^j$$

where

$$Q_G^j = Q_G^{mj} + Q_G^{sj} + Q_G^{rj}$$

$$u_G^j = \begin{Bmatrix} u_A^j \\ u_O^j \\ u_S^j \\ u_M^j \end{Bmatrix} = \begin{Bmatrix} \text{analysis DOF's} \\ \text{omitted DOF's} \\ \text{SPC'd DOF's} \\ \text{MPC'd DOF's} \end{Bmatrix}$$

and

10-1

P_G^j = applied loads on the G-set

Q_G^{mj} = constraint forces due to multi-point constraints (MPC's)

Q_G^{sj} = constraint forces due to single point constraints (SPC's)

Q_G^{rj} = interface forces at boundaries between substructures

In MYSTRAN nomenclature, the G-set is reduced to the A-set by the elimination of the M-set multi-point constraints, the S-set single point constraints and the O-set omitted DOF's (using OMIT's or ASET's). The A-set DOF's for this substructure must contain all DOF's that will be connected to other substructures. The resulting A-set equations of motion (dropping the j superscript notation for each substructure) are:

$$M_{AA} \ddot{u}_A + K_{AA} u_A = P_A + Q_A^r \quad 10-2$$

where the A set matrices are mathematical reductions from the G-set (see Appendix B for details)

Partition 2 into the R-set and L-set, where, the R-set represents the boundary DOF's in which this substructure connects with other substructures and the L-set are all free interior DOF's in this substructure

$$\begin{bmatrix} M_{RR} & M_{LR}^T \\ M_{LR} & M_{LL} \end{bmatrix} \begin{Bmatrix} \ddot{u}_R \\ \ddot{u}_L \end{Bmatrix} + \begin{bmatrix} K_{RR} & K_{LR}^T \\ K_{LR} & K_{LL} \end{bmatrix} \begin{Bmatrix} u_R \\ u_L \end{Bmatrix} = \begin{Bmatrix} P_R \\ P_L \end{Bmatrix} + \begin{Bmatrix} Q_R^r \\ o \end{Bmatrix} \quad 10-3$$

Notice at this point that there remain forces of constraint only at the substructure attach points as the L-set represents all free DOF's for this substructure.

At this point we can introduce the transformation from the physical displacements in equation (3) to what are known as the CB DOF's; namely the flexible mode DOF's and the boundary (R-set) DOF's. In order to show that this is not any further approximation to equation 3, consider the following argument:

- 1) the $u_A = \begin{Bmatrix} u_R \\ u_L \end{Bmatrix}$ DOF's are clearly a complete set of DOF's for the substructure in that, once they are known, the complete g-set DOF's for this substructure can be determined.

2) similarly, a new set of DOF's for the substructure,

$$\mathbf{u}_X = \begin{Bmatrix} \mathbf{u}_R \\ \xi_N \end{Bmatrix} \quad 10-4$$

are a complete set of DOF's if ξ_N are the generalized DOF's for flexible modes when $\mathbf{u}_R = 0$

3) Thus we can take \mathbf{u}_L to be a linear combination of \mathbf{u}_R and ξ_N or:

$$\mathbf{u}_L = \mathbf{D}_{LR}\mathbf{u}_R + \Phi_{LN}\xi_N \quad 10-5$$

if we insist that:

a) Φ_{LN} are shapes when $\mathbf{u}_R = 0$ and ξ_N are modal DOF's. That is, the columns of Φ_{LN} are the flexible modes, ϕ_L^i , when the boundary is fixed. The i-th column of the modal matrix Φ_{LN} is ϕ_L^i .

b) \mathbf{D}_{LR} are shapes when $\xi_N = 0$. That is, the columns of \mathbf{D}_{LR} are the L-set shapes for unit motions of the R-set when the flexible mode DOF's are zero.

The ϕ_L^i are easy to understand. They are the eigenvectors resulting from solving an eigenvalue problem from equations 3 with $\mathbf{u}_R = 0$. This eigenvalue problem would be:

$$(\mathbf{K}_{LL} - \omega^2 \mathbf{M}_{LL})\phi_L = 0 \quad 10-6$$

This requires that the determinant of the coefficient matrix on the left side of equation 6 be zero:

$$|\mathbf{K}_{LL} - \omega^2 \mathbf{M}_{LL}| = 0 \quad \text{which yields } N \text{ eigenvalues } \omega_1^2, \omega_2^2, \dots, \omega_N^2 > 0 \quad 10-7$$

The i-th eigenvector, ϕ_L^i , is then determined by solving the equation:

$$(\mathbf{K}_{LL} - \omega_i^2 \mathbf{M}_{LL})\phi_L^i = 0 \quad \text{for } i = 1, 2, \dots, N \quad 10-8$$

Solution of equation 8 requires that one element of ϕ_L^i be arbitrarily set (the ϕ_L^i are shapes and their amplitude does not matter). Once equation 8 is solved, the modal matrix is:

$$\Phi_{LN} = \begin{bmatrix} \phi_L^1 & \phi_L^2 & \dots & \phi_L^N \end{bmatrix} \quad 10-9$$

The \mathbf{D}_{LR} can also be explained easily. As stated above, the \mathbf{D}_{LR} are shapes when the flexible mode response is zero. We can see from equation 5 that a column of \mathbf{D}_{LR} represents the displacements at the L-set DOF's due to motion at one of the R-set DOF's while all other R-set DOF's are zero (as well

as all $\xi_N = 0$). We can therefore solve for D_{LR} from equation 3 by taking all applied forces and accelerations equal to zero and solving the statics problem:

$$\begin{bmatrix} K_{RR} & K_{LR}^T \\ K_{LR} & K_{LL} \end{bmatrix} \begin{Bmatrix} u_R \\ u_L^s \end{Bmatrix} = \begin{Bmatrix} Q_R^r \\ 0 \end{Bmatrix} \quad 10-10$$

where u_L^s are static displacements of the L-set. From the second row of equation 10, solve for u_L^s in terms of u_R :

$$\begin{aligned} u_L^s &= -K_{LL}^{-1} K_{LR} u_R = D_{LR} u_R \\ \text{or} \\ D_{LR} &= -K_{LL}^{-1} K_{LR} \end{aligned} \quad 10-11$$

Thus, the CB DOF's are contained in u_X (equation 4) and the transformation between u_X and u_A is:

$$\begin{Bmatrix} u_R \\ u_L \end{Bmatrix} = \begin{bmatrix} I & 0 \\ D_{LR} & \Phi_{LN} \end{bmatrix} \begin{Bmatrix} u_R \\ \xi_N \end{Bmatrix} \quad 10-12$$

where I is an R x R identity matrix. Equation 12 can be written as:

$$\begin{aligned} u_A &= \Psi_{AX} u_X \\ \text{where} \\ \Psi_{AX} &= \begin{bmatrix} I & 0 \\ D_{LR} & \Phi_{LN} \end{bmatrix}, \quad u_A = \begin{Bmatrix} u_R \\ u_L \end{Bmatrix}, \quad u_X = \begin{Bmatrix} u_R \\ \xi_N \end{Bmatrix} \end{aligned} \quad 10-13$$

Ψ_{AX} is the CB transformation matrix and is of A-set size. In MYSTRAN this is called matrix PHIXA. When expanded to G-set size, PHIXA becomes matrix PHIXG:

$\begin{aligned} u_G &= \Psi_{GX} u_X \\ \Psi_{GX} &= \text{matrix data block PHIXG} \\ \text{PHIXG} &= \text{PHIXA expanded to G-set} \end{aligned}$

10-14

Note that when all flexible modes of the substructure are used in u_X equation 13 is exact. In practice, all modes are never used since this would defeat the purpose of making the transformation (which is to find a smaller set of DOF's which are nonetheless an accurate representation of the A-set). Substituting equation 13 into equation 2 and premultiplying the result by the transpose of Ψ_{AX} yields:

$$M_{xx} \ddot{u}_X + K_{xx} u_X = P_X + Q_X^r \quad 10-15$$

where:

$$\mathbf{M}_{XX} = \Psi_{AX}^T \mathbf{M}_{AA} \Psi_{AX} = \begin{bmatrix} \mathbf{I} & \mathbf{D}_{LR}^T \\ 0 & \Phi_{LN}^T \end{bmatrix} \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{LR}^T \\ \mathbf{M}_{LR} & \mathbf{M}_{LL} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{D}_{LR} & \Phi_{LN} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_{RR} & \mathbf{m}_{NR}^T \\ \mathbf{m}_{NR} & \mathbf{m}_{NN} \end{bmatrix}$$

$$\mathbf{K}_{XX} = \Psi_{AX}^T \mathbf{K}_{AA} \Psi_{AX} = \begin{bmatrix} \mathbf{I} & \mathbf{D}_{LR}^T \\ 0 & \Phi_{LN}^T \end{bmatrix} \begin{bmatrix} \mathbf{K}_{RR} & \mathbf{K}_{LR}^T \\ \mathbf{K}_{LR} & \mathbf{K}_{LL} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{D}_{LR} & \Phi_{LN} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{RR} & 0 \\ 0 & \mathbf{k}_{NN} \end{bmatrix}$$

10-16

$$\mathbf{P}_X = \Psi_{AX}^T \mathbf{P}_A = \begin{bmatrix} \mathbf{I} & \mathbf{D}_{LR}^T \\ 0 & \Phi_{LN}^T \end{bmatrix} \begin{Bmatrix} \mathbf{P}_R \\ \mathbf{P}_L \end{Bmatrix} = \begin{Bmatrix} \mathbf{P}'_R \\ \Xi_N \end{Bmatrix}, \quad \mathbf{P}'_R = \mathbf{P}_R + \mathbf{D}_{LR}^T \mathbf{P}_L, \quad \Xi_N = \Phi_{LN}^T \mathbf{P}_L$$

$$\mathbf{Q}_X^r = \Psi_{AX}^T \mathbf{Q}_A^r = \begin{bmatrix} \mathbf{I} & \mathbf{D}_{LR}^T \\ 0 & \Phi_{LN}^T \end{bmatrix} \begin{Bmatrix} \mathbf{Q}_R^r \\ \mathbf{o} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_R^r \\ 0 \end{Bmatrix}$$

and:

$$\mathbf{m}_{RR} = \mathbf{M}_{RR} + \mathbf{M}_{LR}^T \mathbf{D}_{LR} + (\mathbf{M}_{LR}^T \mathbf{D}_{LR})^T + \mathbf{D}_{LR}^T \mathbf{M}_{LL} \mathbf{D}_{LR}$$

$$\mathbf{m}_{NR} = \Phi_{LN}^T (\mathbf{M}_{LR} + \mathbf{M}_{LL} \mathbf{D}_{LR})$$

$$\mathbf{m}_{NN} = \Phi_{LN}^T \mathbf{M}_{LL} \Phi_{LN}$$

10-17

$$\mathbf{k}_{RR} = \mathbf{K}_{RR} + \mathbf{K}_{LR}^T \mathbf{D}_{LR}$$

$$\mathbf{k}_{NN} = \Phi_{LN}^T \mathbf{K}_{LL} \Phi_{LN}$$

\mathbf{m}_{NN} , \mathbf{k}_{NN} are diagonal matrices of generalized masses and stiffnesses, respectively.

Equations 15 for the i-th substructure can be written as:

$$\begin{bmatrix} \mathbf{m}_{RR} & \mathbf{m}_{NR}^T \\ \mathbf{m}_{NR} & \mathbf{m}_{NN} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_R \\ \ddot{\xi}_N \end{Bmatrix} + \begin{bmatrix} \mathbf{k}_{RR} & 0 \\ 0 & \mathbf{k}_{NN} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_R \\ \xi_N \end{Bmatrix} = \begin{Bmatrix} \mathbf{P}'_R \\ \Xi_N \end{Bmatrix} + \begin{Bmatrix} \mathbf{Q}_R^r \\ 0 \end{Bmatrix}$$

10-18

The off-diagonal terms in the above stiffness matrix are zero due to the definition of \mathbf{D}_{LR} in equation 11. In addition, matrix \mathbf{k}_{RR} in equation 18 is null if the boundary is a determinant interface. Equations 14 and 18 are the Craig-Bampton equations of motion for the i-th substructure. The \mathbf{P}'_R are due to applied loads on the R and L-set DOF's (see equation 16) and the \mathbf{Q}_R^r are the interface forces where substructures connect. Once the equations are developed for all substructures, the individual substructures can be connected and the resulting equations solved for the combined R-set and N-set DOF's \mathbf{u}_R and ξ_N for all substructures. Once this is done, the forces of inter-connection, or

substructure interface forces, (that is, the \mathbf{Q}_R^f) can be solved from the individual substructure equations in the top row of equation 18. Equation 14 is used to obtain displacements for all G-set DOF's.

Each organization that is developing a substructure in CB format would deliver the above coefficient matrices in equations 14 and 18 to the organization that is doing the combined structure analysis. In addition, Displacement and Load Transformation Matrices (DTM's and LTM's) collectively known as Output Transformation Matrices, (OTM's), described below, are also delivered as part of the CB model.

3.2 Development of Displ Output Transformation Matrices (Displ OTM's)

Typically, a set of displacement output transformation matrices (displ OTM's, or DTM's for short), is delivered with a Craig-Bampton model to the organization that will couple all substructures and solve for the primary unknowns (\mathbf{u}_R and ξ_N and \mathbf{Q}_R^f) in order that desired displacements at some of the substructure G-set DOF's may be obtained along with the coupled solution.

Once the combined structure has been solved for the primary variables, the original \mathbf{u}_L physical DOF's could be determined from equation 5 and then element forces and stresses could be determined from the \mathbf{u}_R and \mathbf{u}_L displacements. This is called recovery of the \mathbf{u}_L DOF's and element forces and stresses using the Modal Displacement Method (MDM). However, as is often the case, equations 18 are solved using a severely truncated set of modes for each substructure. While this may not compromise the accuracy of the solutions for \mathbf{u}_R and ξ_N , it could compromise the accuracy of element forces and stresses calculated using displacements determined from equation 5 with the truncated set of modes. In order to avoid this problem, the \mathbf{u}_L DOF's can be found using the Modal Acceleration Method (MAM), described below. It should be noted that the MAM described below *ignores* damping forces so that it is only useful when the damping is small (e.g. less than 10% or so).

From the bottom row of equation 3, solve for \mathbf{u}_L in terms of the other variables in the equation:

$$\begin{aligned}\mathbf{u}_L &= -\mathbf{K}_{LL}^{-1}(\mathbf{M}_{LR}\ddot{\mathbf{u}}_R + \mathbf{M}_{LL}\ddot{\mathbf{u}}_L) - \mathbf{K}_{LL}^{-1}\mathbf{K}_{LR}\mathbf{u}_R + \mathbf{K}_{LL}^{-1}\mathbf{P}_L \\ &= -\mathbf{K}_{LL}^{-1}(\mathbf{M}_{LR}\ddot{\mathbf{u}}_R + \mathbf{M}_{LL}\ddot{\mathbf{u}}_L) + \mathbf{D}_{LR}\mathbf{u}_R + \mathbf{K}_{LL}^{-1}\mathbf{P}_L\end{aligned}\quad 10-19$$

Differentiate equation 5 twice and use the result for $\ddot{\mathbf{u}}_L$ in equation 19, to get:

$$\mathbf{u}_L = \left[-\mathbf{K}_{LL}^{-1}(\mathbf{M}_{LR} + \mathbf{M}_{LL}\mathbf{D}_{LR}) \mid -\mathbf{K}_{LL}^{-1}\mathbf{M}_{LL}\Phi_{LN} \mid \mathbf{D}_{LR} \right] \begin{Bmatrix} \ddot{\mathbf{u}}_R \\ \xi_N \\ \mathbf{u}_R \end{Bmatrix} + \mathbf{K}_{LL}^{-1}\mathbf{P}\quad 10-20$$

The term $\mathbf{K}_{LL}^{-1}\mathbf{M}_{LL}\Phi_{LN}$ in equation 20. can be written in a form more convenient for calculation. From equation 8 it can be seen that:

$$\mathbf{K}_{LL}^{-1}\mathbf{M}_{LL}\phi_L^i = \frac{1}{\omega_i^2} \phi_L^i$$

so that

$$K_{LL}^{-1} M_{LL} \begin{bmatrix} \phi_1^1 & \phi_1^2 & \cdots & \phi_L^N \end{bmatrix} = \begin{bmatrix} \phi_1^1 & \phi_1^2 & \cdots & \phi_L^N \end{bmatrix} \begin{bmatrix} \omega_1^{-2} & & & \\ & \omega_2^{-2} & & \\ & & \ddots & \\ & & & \omega_N^{-2} \end{bmatrix}$$

or

$$K_{LL}^{-1} M_{LL} \Phi_{LN} = \Phi_{LN} \Omega_{NN}^{-2} \quad 10-21$$

where

$$\Omega_{NN}^{-2} = \begin{bmatrix} \omega_1^{-2} & & & \\ & \omega_2^{-2} & & \\ & & \ddots & \\ & & & \omega_N^{-2} \end{bmatrix} \quad 10-22$$

substitute equation 21 into equation 20 to get:

$$u_L = \left[-K_{LL}^{-1} (M_{LR} + M_{LL} D_{LR}) \mid -\Phi_{LN} \Omega_{NN}^{-2} \mid D_{LR} \right] \begin{Bmatrix} \ddot{u}_R \\ \ddot{\xi}_N \\ u_R \end{Bmatrix} + K_{LL}^{-1} P_L \quad 10-23$$

The various terms in the coefficient matrices in equation 23 are known as Displacement Transformation Matrices (DTM's). Equation 23 can be written as:

$$u_L = \left[DTM1_{LR} \mid DTM2_{LN} \mid DTM3_{LR} \right] \begin{Bmatrix} \ddot{u}_R \\ \ddot{\xi}_N \\ u_R \end{Bmatrix} + DTM4_{LL} P_L \quad 10-24$$

where

$$\begin{aligned} DTM1_{LR} &= -K_{LL}^{-1} (M_{LR} + M_{LL} D_{LT}) \\ DTM2_{LN} &= -\Phi_{LN} \Omega_{NN}^{-2} \\ DTM3_{LR} &= D_{LR} \\ DTM4_{LL} &= K_{LL}^{-1} \end{aligned} \quad 10-25$$

Equations 24 and 25 represent the MAM for recovering displacements for the L-set, for the i-th substructure, once the assembled substructure equations have been solved for the u_R and q_N DOF's. Once the L-set displacements have been found, recovery of the remaining displacements in the G-set is accomplished through the transformation matrices used in their elimination from equation 1 (for details see Appendix B). At the G-set level, equation 24 is:

$$u_G = [DTM1_{GR} \mid DTM2_{GN} \mid DTM3_{GR}] \begin{Bmatrix} \ddot{u}_R \\ \ddot{q}_N \\ u_R \end{Bmatrix} + DTM4_{GL} P_L$$

or

$$u_G = \Gamma_{GZ} u_Z + DTM4_{GL} P_L$$

where

10-26

$$\Gamma_{GZ} = [DTM1_{GR} \mid DTM2_{GN} \mid DTM3_{GR}] = DTM_{GZ}$$

and

$$u_Z = \begin{Bmatrix} \ddot{u}_R \\ \ddot{q}_N \\ u_R \end{Bmatrix}, \quad \text{where } u_Z \text{ are the Craig-Bampton Degrees of freedom (CB_DOF's)}$$

where each of the G-set DTM's in equation 26 is obtained from the L-set DTM's in equation 25 through the normal recovery operations to build back up to the G-set from the L-set. The coefficient matrix in equation 26 that has DTM's 1 - 3 in it is called matrix PHIZG. The table below explains the meaning of each of the DTM's in equation 26:

Table 10.1

i-th col of:	Represents:
$DTM1_{GR}$	displ's of G-set due to a unit accel of the i-th interface DOF (all other R, N set DOF's zero)
$DTM2_{GN}$	displ's of G-set due to a unit accel of the i-th flex mode DOF (all other R, N set DOF's zero)
$DTM3_{GR}$	displ's of G-set due to a unit displ of the i-th interface DOF (all other R, N set DOF's zero)
$DTM4_{GL}$	displ's of G-set due to a unit force on the i-th L-set DOF (all other L-set forces zero)

3.3 Development of Load Output Transformation Matrices (Load OTM's)

Once the G-set displacements have been found, substructure element forces and stresses, as well as grid point forces, can be recovered and assembled into a Loads Output Transformation Matrix, or Load OTM (more commonly referred to as LTM). There are several types of quantities one may desire in an LTM. Equations are developed, below, for several types of LTM quantities typically used in CB analyses.

3.3.1 LTM Terms for Substructure Interface Forces

From the top row of equation 18, the interface forces can be determined once the substructures have been coupled and the u_R and ξ_N solved. The interface forces are:

$$Q_R^r = m_{RR} \ddot{u}_R + m_{NR}^T \ddot{\xi}_N + k_{RR} u_R - P_R'$$

or

$$Q_R^r = \begin{bmatrix} m_{RR} & m_{NR}^T & k_{RR} \end{bmatrix} \begin{Bmatrix} \ddot{u}_R \\ \ddot{\xi}_N \\ u_R \end{Bmatrix} - I_{RR} P_R' \quad 10-27$$

where I_{RR} is an $R \times R$ identity matrix. Equation 27 can be written as:

$$Q_R^r = \begin{bmatrix} \text{LTM21}_{RR} & \text{LTM22}_{RN} & \text{LTM23}_{RR} \end{bmatrix} \begin{Bmatrix} \ddot{u}_R \\ \ddot{\xi}_N \\ u_R \end{Bmatrix} - \text{LTM24}_{RR} P_R'$$

or

$$Q_R^r = J_{RZ} U_Z - I_{RR} P_R'$$

where

$$J_{RZ} = \begin{bmatrix} \text{LTM21}_{RR} & \text{LTM22}_{RN} & \text{LTM23}_{RR} \end{bmatrix} = \text{LTM2}_{RZ} \quad 10-28$$

$$\text{LTM21}_{RR} = m_{RR}$$

$$\text{LTM22}_{RN} = m_{NR}^T$$

$$\text{LTM23}_{RR} = k_{RR}$$

$$\text{LTM24}_{RR} = I_{RR}$$

3.3.2 LTM Terms for Net cg Loads

Terms can also be included in the overall LTM that will recover what are known as “net” accelerations at the center of gravity (cg) of the CB model. These are termed Net Load factors (NLF's) and represent rigid body accelerations of the cg due to the reaction (or interface) forces, Q_R^r . The development below demonstrates how these are determined.

Define:

$u_{cg} = 6 \times 1$ matrix of rigid body displacements of the cg of the substructure

$u_{R_{rb}} = r \times 1$ vector of rigid body displacements at the r DOF

$T_{R6} = r \times 6$ matrix where each column represents rigid body displacements of the r DOF due to a unit motion in one DOF at the cg 10-29

$Q_{cg} = 6 \times 1$ vector of forces at the cg that are static equivalents of Q_r^r

Then:

$$\begin{aligned} u_{R_{rb}} &= T_{R6} u_{cg} \\ \text{and} \\ Q_{cg} &= T_{R6}^T Q_r^r \end{aligned} \quad 10-30$$

Substitute equation 27 into 30 for Q_r^r :

$$Q_{cg} = T_{R6}^T (m_{RR} \ddot{u}_R + m_{NR}^T \ddot{\xi}_N + k_{RR} u_R - P_R') \quad 10-31$$

For rigid body motion:

$$Q_{cg} = m_{cg} \ddot{u}_{cg} \quad 10-32$$

where m_{cg} is the 6×6 rigid body mass matrix relative to the cg and is equal to:

$$m_{cg} = T_{R6}^T m_{RR} T_{R6} \quad 10-33$$

and m_{RR} is given in equation 17. From equations 31 through 33 we can write the cg acceleration net load factors (NLF's) as:

$$\ddot{u}_{cg} = m_{cg}^{-1} Q_{cg} = m_{cg}^{-1} T_{R6}^T \begin{bmatrix} m_{RR} & m_{NR}^T & k_{RR} \end{bmatrix} \begin{Bmatrix} \ddot{u}_R \\ \ddot{\xi}_N \\ u_R \end{Bmatrix} - m_{cg}^{-1} T_{R6}^T P_R' \quad 10-34$$

However, $T_{R6}^T k_{RR} = 0$ since the columns of T_{R6} are rigid body modes. Therefore:

$$\ddot{u}_{cg} = m_{cg}^{-1} Q_{cg} = m_{cg}^{-1} T_{R6}^T \begin{bmatrix} m_{RR} & m_{NR}^T & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u}_R \\ \ddot{\xi}_N \\ u_R \end{Bmatrix} - m_{cg}^{-1} T_{R6}^T P_R' \quad 10-35$$

which can be written as:

$$\ddot{u}_{cg} = \begin{bmatrix} \text{LTM11}_{6R} & \text{LTM12}_{6N} & 0_{6R} \end{bmatrix} \begin{Bmatrix} \ddot{u}_R \\ \ddot{\zeta}_N \\ u_R \end{Bmatrix} - [\text{LTM14}_{6R}] P'_R$$

where

$$\begin{aligned} \text{LTM11}_{6R} &= m_{cg}^{-1} T_{R6}^T m_{RR} \\ \text{LTM12}_{6N} &= m_{cg}^{-1} T_{R6}^T m_{NR}^T \\ \text{LTM14}_{6R} &= m_{cg}^{-1} T_{R6}^T \\ \text{LTM1}_{6Z} &= [\text{LTM11}_{6R} \quad \text{LTM12}_{6N} \quad 0] \end{aligned}$$

10-36

3.3.3 LTM Terms for Element Forces and Stresses

In MYSTRAN, element forces and stresses are obtained from the G-set displacement vector and the individual element stiffness matrices. Equation 26 is the G-set displacement vector:

$$u_G = [\text{DTM1}_{GR} \mid \text{DTM2}_{GN} \mid \text{DTM3}_{GR}] \begin{Bmatrix} \ddot{u}_R \\ \ddot{\zeta}_N \\ u_R \end{Bmatrix} + \text{DTM4}_{GL} P_L = \Gamma_{GZ} u_Z + \text{DTM4}_{GL} P_L$$

Thus the columns of each of the DTM's represents G-set displacements per unit value of one of the variables $\ddot{u}_R, \ddot{\zeta}_N, u_R, P_L$ as described in Table 10.1. Therefore, each of the DTM's can be used as if they were a matrix of displacements in calculating element forces and stresses to give:

$$f_e = [\text{LTM31}_{eR} \mid \text{LTM32}_{eN} \mid \text{LTM33}_{eR}] \begin{Bmatrix} \ddot{u}_R \\ \ddot{\zeta}_N \\ u_R \end{Bmatrix} + \text{LTM34}_{eL} P_L$$

where

f_e = vector of element forces and stresses (e = number of finite elements)

LTM31_{eR} = matrix of element forces and stresses due to G-set displ's DTM1_{GR}

LTM32_{eN} = matrix of element forces and stresses due to G-set displ's DTM2_{GN}

LTM33_{eR} = matrix of element forces and stresses due to G-set displ's DTM3_{GR}

LTM34_{eL} = matrix of element forces and stresses due to G-set displ's DTM4_{GL}

$\text{LTM3}_{eZ} = [\text{LTM31}_{eR} \mid \text{LTM32}_{eN} \mid \text{LTM33}_{eR}]$

10-37

3.3.4 LTM Terms for Grid Point Forces due to multi-point constraints (MPC's)

There are cases in CB analyses in which the forces due to MPC's are of interest. As an example, if a user wishes to determine a load in a bolt at an interface between components, it is common to model the bolt as an MPC where two coincident grids are constrained to have the same displacements. This section develops the equations for determining an LTM for grid point MPC forces.

Equation 1 for the i-th substructure (dropping the superscript-j notation):

$$\mathbf{M}_{GG}\ddot{\mathbf{u}}_G + \mathbf{K}_{GG}\mathbf{u}_G = \mathbf{P}_G + \mathbf{Q}_G^s + \mathbf{Q}_G^m + \mathbf{Q}_G^r \quad 10-38$$

As described in section 10.1 the Q constraint forces on the right side of equation 38 are the constraint forces on the S-set SPC DOF's, the M-set MPC DOF's and on the R-set boundary DOF's respectively. Since all of the boundary DOF's are contained in the R-set there should be no constraint forces on the S-set. That is, all S-set DOF's should be the result of removing singularities and not the result of grounding the model³. With this assumption, as well as the assumption that there are no applied loads on the M-set degrees of freedom the following equation is valid for the MPC forces on the M-set grids:

$$\mathbf{Q}_G^m = \mathbf{M}_{GG}\ddot{\mathbf{u}}_G + \mathbf{K}_{GG}\mathbf{u}_G - \mathbf{Q}_G^r \quad 10-39$$

We want to get 39 in a form like the other LTM'; that is, in terms of \mathbf{u}_Z .

From equation 26 with applied loads zero:

$$\mathbf{u}_G = \Gamma_{GZ}\mathbf{u}_Z, \quad \mathbf{u}_Z = \begin{Bmatrix} \ddot{\mathbf{u}}_R \\ \ddot{\xi}_N \\ \mathbf{u}_R \end{Bmatrix} \quad 10-40$$

The g-set DOF vector can also be written using equation 14:

$$\mathbf{u}_G = \Psi_{GX}\mathbf{u}_X, \quad \mathbf{u}_X = \begin{Bmatrix} \mathbf{u}_R \\ \xi_N \end{Bmatrix} \quad 10-41$$

Differentiating twice:

$$\ddot{\mathbf{u}}_G = \Psi_{GX}\ddot{\mathbf{u}}_X$$

This can also be written as:

$$\ddot{\mathbf{u}}_G = \begin{bmatrix} \Psi_{GX} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_X \\ \mathbf{u}_R \end{Bmatrix} \quad 10-42$$

Partition the x DOF's into R and N as in equation 13. This will require partitioning Ψ_{GX} into sub-matrices for the R and N also, so that equation 42 can be written as:

³ This should be verified by the user by inspection of the forces of single point constraint in the output from the analysis

$$\ddot{\mathbf{u}}_G = \begin{bmatrix} \Psi_{GR} & \Psi_{GN} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_R \\ \ddot{\xi}_N \\ \mathbf{u}_R \end{Bmatrix} = \Psi'_{GZ} \mathbf{u}_Z$$

where

$$\Psi'_{GZ} = \begin{bmatrix} \Psi_{GR} & \Psi_{GN} & 0 \end{bmatrix} = \begin{bmatrix} \Psi_{GX} & 0 \end{bmatrix}$$

10-43

Substitute equations 40 and 43 into 39 for \mathbf{u}_G and $\ddot{\mathbf{u}}_G$ respectively to get:

$$\mathbf{Q}_G^m = \mathbf{M}_{GG} \Psi'_{GZ} \mathbf{u}_Z + \mathbf{K}_{GG} \Gamma_{GZ} \mathbf{u}_Z - \mathbf{Q}_G^r \quad 10-44$$

We need to express the boundary constraint forces in equation 44 in terms of the \mathbf{u}_Z vector as we did for the inertia and stiffness terms. From 28:

$$\mathbf{Q}_R^r = \mathbf{J}_{RZ} \mathbf{u}_Z - \mathbf{I}_{RR} \mathbf{P}_R \quad 10-45$$

The \mathbf{Q}_R^r boundary forces on the R-set can be expanded from the R-set to the G-set \mathbf{Q}_G^r by adding zero rows to 45 for the M, S, O-sets (all of the G-set but the R degrees of freedom) to give

$$\mathbf{Q}_G^r = \mathbf{J}_{GZ} \mathbf{u}_Z - \mathbf{I}_{GR} \mathbf{P}_R \quad 10-46$$

where \mathbf{J}_{GZ} is \mathbf{J}_{RZ} expanded to G-set size by addition of zero rows for M, S, O-sets and \mathbf{I}_{GR} is expanded from \mathbf{I}_{RR} in the same fashion (recall \mathbf{I}_{RR} is an R size identity matrix). Substituting 46 into 44 we get::

$$\mathbf{Q}_G^m = (\mathbf{M}_{GG} \Psi'_{GZ} + \mathbf{K}_{GG} \Gamma_{GZ} - \mathbf{J}_{GZ}) \mathbf{u}_Z$$

or

$$\mathbf{Q}_G^m = \mathbf{LTM4}_{GZ} \mathbf{u}_Z \quad 10-47$$

where

$$\mathbf{LTM4}_{GZ} = (\mathbf{M}_{GG} \Psi'_{GZ} + \mathbf{K}_{GG} \Gamma_{GZ} - \mathbf{J}_{GZ})$$

$\mathbf{LTM4}_{GZ}$ is the LTM for MPC forces at grids that have no applied load

3.4 Development of Acceleration Output Transfer Matrices (Accel OTM)

In addition to the displacement and load output transformation matrices (DTM's and LTM's) it is common to supply acceleration output transformation matrices (accel OTM's or ATM's for short). From equation 10-12 and differentiating twice we obtain:

$$\begin{Bmatrix} \ddot{u}_R \\ \ddot{u}_L \end{Bmatrix} = [ATM] \begin{Bmatrix} \ddot{u}_R \\ \ddot{\xi}_N \end{Bmatrix}$$

where

10-48

$$ATM = \begin{bmatrix} I & 0 \\ D_{LR} & \Phi_{LN} \end{bmatrix}$$

ATM is the acceleration transfer matrix. Notice that the “degrees of freedom” for the ATM are the accelerations of the boundary and modal degrees of freedom whereas all of the other OTM's have as degrees of freedom: boundary accelerations, modal accelerations and boundary displacements. This is due to the use of the modal acceleration method for recovery of displacements and element forces.

3.5 Correspondence between matrix names and CB Equation Variables

The table below shows the correspondence between variables introduced in the above equations and matrix data block names in the DMAP program in Section 10.5. Any of these may be output in a MYSTRAN CB model generation analysis using the Executive Control entry OUTPUT4.

Table 10-2
Matrices that can be written to OUTPUT4 files

	MYSTRAN Matrix Name (OUTPUT4 matrices)	NASTRAN DMAP Name	CB equation variable in Appendix D (where applicable)	Matrix size ¹	Partition rows and/or cols
1	CG_LTM		$[LTM11_{6r} \quad LTM12_{6N} \quad 0]$	$6 \times (2R+N)$	
2	DLR	DM	D_{LR}	$L \times R$	rows and cols
3	EIGEN_VAL	LAMA	Ω_{NN}^2	$N \times N$	
4	EIGEN_VEC	PHIG	Φ_{GN} , (Φ_{LN} with rows expanded to G-set)	$G \times N$	rows
5	GEN_MASS	MI	m_{NN}	$N \times 1$ vector of diag. terms	
6	IF_LTM		$[LTM21_{RR} \quad LTM22_{RN} \quad LTM23_{RR}]$	$R \times (2R+N)$	rows
7	KAA	KAA	K_{AA}	$A \times A$	rows and cols
8	KGG	KGG	K_{GG}	$G \times G$	rows and cols
9	KLL	KLL	K_{LL}	$L \times L$	rows and cols
10	KRL	KLR(t)	K_{LR}	$L \times R$	rows and cols
11	KRR	KRR	K_{RR}	$R \times R$	rows and cols
12	KRRcb	KBB	$k_{RR} = K_{RR} + K_{LR}^T D_{LR}$	$R \times R$	rows and cols
13	KXX	KRRGN	K_{XX}	$(R+N) \times (R+N)$	
14	LTM	LTM	CG_LTM and IF_LTM merged	$(6+R) \times (2R+N)$	
15	MCG	RBMCG	m_{cg}	6×6	
16	MEFFMASS		Modal effective mass	$N \times 6$	
17	MPFACTOR		Modal participation factors	$N \times 6$ or $N \times R$	
18	MAA		M_{AA}	$A \times A$	rows and cols
19	MGG		M_{GG}	$G \times G$	rows and cols
20	MLL	MLL	M_{LL}	$L \times L$	rows and cols
21	MRL	MRL	M_{RL}	$R \times L$	rows and cols
22	MRN		$m_{RN} = m_{NR}^T$	$R \times N$	rows
23	MRR	MRR	M_{RR}	$R \times R$	rows and cols

Table 10-2 (con't)

	MYSTRAN Matrix Name (OUTPUT4 matrices)	NASTRAN DMAP Name	CB equation variable in Appendix D (where applicable)	Matrix size ⁴	Partition rows and/or cols
24	MRRcb	MBB	$m_{RR} = M_{RR} + M_{LR}^T D_{LR} + (M_{LR}^T D_{LR})^T + D_{LR}^T M_{LL} D_{LR}$	RxR	rows and cols
25	MXX	MRRGN	$M_{XX} = \begin{bmatrix} m_{RR} & m_{NR}^T \\ m_{NR} & m_{NN} \end{bmatrix}$	(R+N)x(R+N)	
26	PA		(A-set static reduced loads - only used in statics)		Rows
27	PG		(G-set static loads - only used in statics)		Rows
28	PL		(L-set static reduced loads - only used in statics)		rows
29	PHIXG	PHIXG	Ψ_{AX} , (Ψ_{AX} with rows expanded to G-set)	Gx(R+N)	rows
30	PHIZG		The G-set displacement transformation matrix is written out in the F06 file under "C B D I S P L A C E M E N T O T M"	Gx(2R+N)	rows
31	RBM0		Rigid body mass matrix relative to the basic origin	6x6	
32	TR6_0	RBR	T_{R6} : rigid body displacement matrix for R-set relative to the model basic coordinate system	Rx6	rows
33	TR6_CG	RBR CG	T_{R6} : rigid body displacement matrix for R-set relative to the model CG	Rx6	rows

Notes:

- (t) indicates matrix transposition
- Matrix m_{RR} will be singular if there are rotational DOF's but no rotational inertia in the R-set, in which case small rotational inertias may have to be added at these DOF's.
- Matrix k_{RR} is null if the boundary is a determinant set of DOF's.
- Matrix m_{RR} is the rigid body mass matrix if the boundary is a determinant set of DOF's

⁴ Matrix size given in rows x columns where R means the size of the R-set, L is the size of the L-set, A is the size of the A-set, G is the size of the G-set and N is the number of eigenvectors. See section 3.6 for definition of the complete displacement set notation

3.6 Craig-Bampton model generation example problem

The figure below shows a small example problem that is a frame made of CBAR's that is a substructure assumed to be attached to some other structure in DOF's 1,2,3 at grids 11 and 13 and in DOF's 2,3 at grid 12. The example problem F06 file (with the input echo'd) is shown on the following pages. This section will discuss the input and output in an effort to explain the Craig-Bampton model generation process.

Equation 10.26 defines the Craig-Bampton degrees of freedom (CB-DOF's) as U_z which, for this example, consists of the 18 DOF's:

- 8 boundary acceleration DOF's, \ddot{U}_R
- 2 modal acceleration DOF's, $\ddot{\xi}_N$ (see EIGRL request for 2 modes to be extracted)
- 8 boundary displacement DOF's, U_R

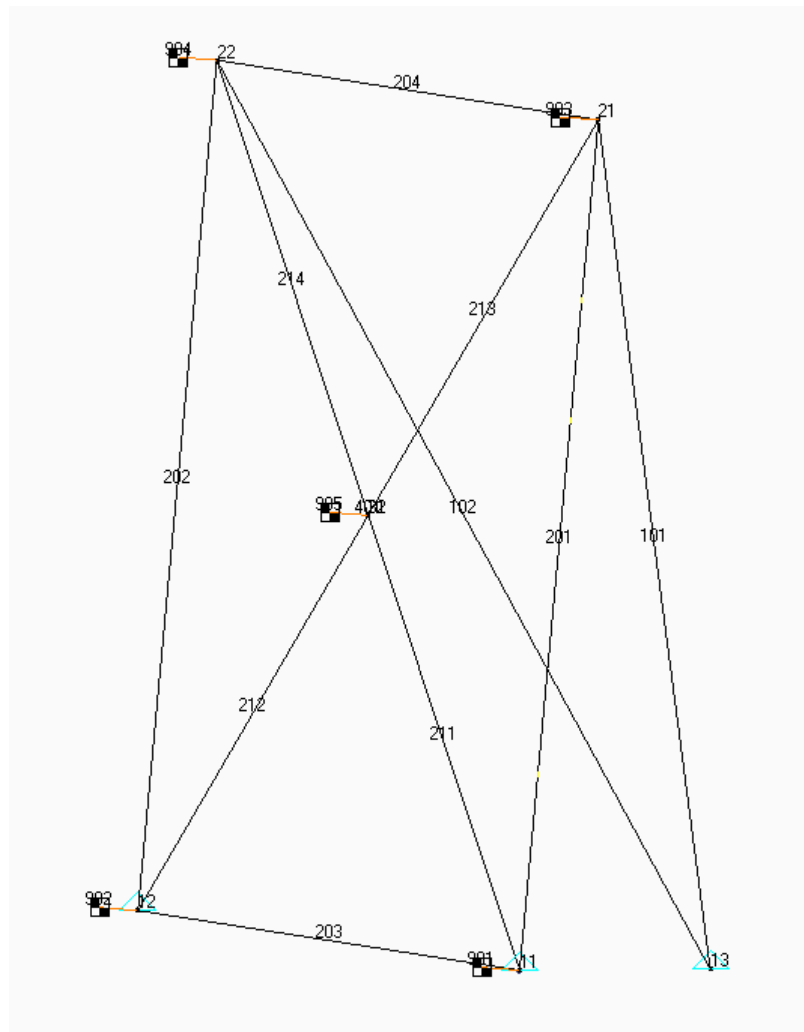


Figure 10.2 – Example CB model: CB-EXAMPLE-12b.DAT

Notes on section 10.6.1: CB-EXAMPLE-12b.F06

The echo of the input shows the following salient points for a CB model generation (much like a SOL 3 eigenvalue analysis in terms of input data):

- Executive Control:
 - SOL 31 indicates CB model generation
 - The OUTPUT4 commands show the matrices that will be written in a format the same as NASTRAN OUTPUT4 files. These matrix data blocks are ones that are listed on Table 10.2 as allowable OUTPUT4 matrices. Notice that several are written to unit 21 while others are written to unit 22. As explained in section 5.1 of the MYSTRAN Users Reference Manual, unit numbers 21 through 27 are valid for writing OUTPUT4 matrices.
- Case Control:
 - METHOD = 1 is to be used for a normal eigenvalue analysis (same as if SOL were 3)
 - Outputs (ACCE, DISP, ELFORCE, STRESS) are for Output Transformation Matrices (OTM's) for the specified sets. These will be written to the text F06 file. In addition they will be written to binary files (same name, CB-EXAMPLE-12b) with extension OP8 for the element related OTM's (ELFORCE, STRESS in this case and OP9 for the grid related OTM's (ACCE, DISP in this case)
- Bulk Data:
 - Shows the model for this example (notice it has mostly CBAR's but there is also a RBE2)
 - Degrees of freedom at the boundary where this substructure attaches to other substructures are defined with the SUPORT Bulk Data entry. This is the same procedure that is used in CB analyses by the NASTRAN DMAP (Direct Matrix Abstraction Program) method familiar to NASTRAN CB analysts.
 - Eigenvalue extraction, EIGRL requesting 2 modes to be extracted

The delineated F06 output begins on the page following the input model echo and shows the following:

- Eigenvalues extracted
- Messages on the matrices requested to be written to OUTPUT4 files
- For the first 3 of the 18 CB_DOF's in this example the following output (requested in Case Control) is shown (other 15 were left out for clarity):
 - Displacement OTM for the requested grids (see Case Control command DISP = 102)
 - Element engineering force OTM (see Case Control command ELFORCE = 201)
 - Element stress OTM (see Case Control command STRESS = 202)
- Acceleration OTM. As shown in equation 10.48 the acceleration OTM has columns for \ddot{u}_R and $\ddot{\xi}_N$ but not u_R . For this example, there are 10 columns in the acceleration OTM (8 boundary acceleration DOF's and 2 modal acceleration DOF's)

Notes on section 10.6.2: OUTPUT4 matrices written to CB-EXAMPLE-12b.OP1 and OP2

As shown in the Executive Control section of the F06 file in section 10.6.1, there were 3 matrices requested to be written to unit 21 and 4 to unit 22. These binary files, translated to text, are shown in section 10.6.2. The number of actual columns for each matrix is indicated in Table 10.2 but only the first 5 of the columns are shown here for the sake of brevity. These are several of the important CB matrices needed to couple this CB substructure to other substructures in a combined analysis. The binary OUTPUT4 files are written in the same format as the NASTRAN OUTPUT4 binary files.

Notes on section 10.6.3: Displ and elem force/stress OTM's written to CB-EXAMPLE-12b.OP1, OP2

Any output requests in Case Control for grid related outputs (e.g. DISPL, ACCEL) and element force/stress outputs (e.g. ELFORCE, STRESS) are written to the text F06 file and also written to OUTPUT4 binary files (automatically; that is, no formal OUTPUT4 request is needed). The element related OTM's are always written to a file with the same filename as the F06 file but with extension OP8. The grid related OTM's are written to a file with extension OP9.

The first page of section 10.6.3 is a text translation of the element related OTM's written to file CB-EXAMPLE-12b.OP8. The values are the same as was written to the F06 file for element forces and stresses but are also written to binary files in OUTPUT4 format to be used in analyses that couple the CB substructures. In order to explain the contents of the binary OP8 file, a text file with extension OT8 is also automatically written (provided any Case Control requests are included for element forces/stresses) describing the contents of the OP8 binary file. This OT8 text file gives an overview of the OP8 binary file and then goes on to describe each row written to the OP8 file.

The next several pages show the same type of information on the grid related OTM's written to binary file with extension OP9 (with text description in OT9). Again, this is the grid related outputs requested in Case Control and also written to the F06 text file.

*

3.6.1 CB-EXAMPLE-12-b.F06

(delineated – some output not included here for the sake of clarity)


```

$
$ Frame support bars
$
CBAR      101      1      13      21      0.0      0.5      1.0      +C1
+C1       56      456
CBAR      102      1      13      22      0.0      0.5      1.0      +C2
+C2       56      456
$
$ Edge bars
$
CBAR      201      2      11      21      0.0      0.0      1.0
CBAR      202      2      12      22      0.0      0.0      1.0
CBAR      203      2      11      12      0.0      0.0      1.0
CBAR      204      2      21      22      0.0      0.0      1.0
$
$ Diag bars
$
CBAR      211      3      11      31      0.0      0.0      1.0
CBAR      212      3      12      31      0.0      0.0      1.0
CBAR      213      3      21      31      0.0      0.0      1.0
CBAR      214      3      22      31      0.0      0.0      1.0
$
PBAR      1        1      0.36    0.09    0.09    0.18
PBAR      2        1      0.10    10.0    10.0    20.0
PBAR      3        1      6.0     6.0     6.0     12.0
$
MAT1      1        10.+6      0.3     0.1
*INFORMATION: MAT1 ENTRY      1 HAD FIELD FOR G BLANK. MYSTRAN CALCULATED G = 3.846154E+06
$
CONM2     901      11      150.0    0.0     0.0     -5.0
CONM2     902      12      150.0    0.0     0.0     -5.0
CONM2     903      21      150.0    0.0     0.0     -5.0
CONM2     904      22      150.0    0.0     0.0     -5.0
CONM2     905      32      150.0    0.0     0.0     -5.0
$
SPC1      1        456      13
$
$ BOUNDARY DOF'S
$
SUPT      11      123      12      23      13      123
$
PARAM     WTMASS   .002591
$
ENDDATA

```

E I G E N V A L U E A N A L Y S I S S U M M A R Y (LANCZOS Mode 2 DPB Shift eigen = -1.00E+00)

NUMBER OF EIGENVALUES EXTRACTED 2

LARGEST OFF-DIAGONAL GENERALIZED MASS TERM -2.7E-13 (Vecs renormed to 1.0 for gen masses)

MODE PAIR 2

. 1

NUMBER OF OFF DIAGONAL GENERALIZED MASS
TERMS FAILING CRITERION OF 1.0E-04. 0

MODE NUMBER	EXTRACTION ORDER	EIGENVALUE	R E A L RADIANS	E I G E N V A L U E S CYCLES	GENERALIZED MASS	GENERALIZED STIFFNESS
1	1	3.895211E+03	6.241163E+01	9.933119E+00	1.000000E+00	3.895211E+03
2	2	7.011163E+03	8.373269E+01	1.332647E+01	1.000000E+00	7.011163E+03

>> LINK 4 END

>> LINK 6 BEGIN

*INFORMATION: THE FOLLOWING 7 MATRICES WILL BE WRITTEN TO 2 OUTPUT4 FILES IN THE ORDER LISTED BELOW:

OUTPUT4 file on unit 21 has been created as: CB-EXAMPLE-12-b.OP1 and will contain the matrices:

(1) CG_LTM	:	6 rows and	18 cols	This is MYSTRAN matrix CG_LTM
(2) IF_LTM	:	8 rows and	18 cols	This is MYSTRAN matrix IF_LTM
(3) MR	:	8 rows and	8 cols	This is MYSTRAN matrix MRRcb

OUTPUT4 file on unit 22 has been created as: CB-EXAMPLE-12-b.OP2 and will contain the matrices:

(1) KRRGN	:	10 rows and	10 cols	This is MYSTRAN matrix KXX
(2) RBMCG	:	6 rows and	6 cols	This is MYSTRAN matrix MCG
(3) MRRGN	:	10 rows and	10 cols	This is MYSTRAN matrix MXX
(4) RBRCG	:	8 rows and	6 cols	This is MYSTRAN matrix TR6

>> LINK 6 END

>> LINK 5 BEGIN

>> LINK 5 END

>> LINK 9 BEGIN

C B D I S P L A C E M E N T O T M (in global coordinate system at each grid)								
GRID	COORD	T1	T2	T3	R1	R2	R3	
	SYS							
22	0	-1.412939E-05	1.622140E-05	8.242222E-05	5.883709E-07	-1.667433E-06	5.125151E-07	
32	0	1.051041E-05	-9.465944E-06	-3.182887E-06	-1.086181E-07	-9.450720E-07	2.106009E-07	

C B E L E M E N T E N G I N E E R I N G F O R C E O T M F O R E L E M E N T T Y P E B A R								
Element	Bend-Moment	End A	Bend-Moment	End B	- Shear -	Axial	Torque	
ID	Plane 1	Plane 2	Plane 1	Plane 2	Plane 1	Plane 2	Force	
211	2.091876E-01	7.894539E-01	1.515607E+00	-1.439344E+00	-1.847556E-02	3.151997E-02	6.266800E-01	9.672846E-03
212	-1.133151E-01	-1.008960E-02	-1.725401E+00	-6.166148E-02	2.279833E-02	7.293366E-04	-2.953611E-01	-4.720428E-03

C B E L E M E N T S T R E S S O T M I N L O C A L E L E M E N T C O O R D I N A T E S Y S T E M F O R E L E M E N T T Y P E B A R								
Element	SA1	SA2	SA3	SA4	Axial	SA-Max	SA-Min	M.S.-T
ID	SB1	SB2	SB3	SB4	Stress	SB-Max	SB-Min	M.S.-C
201	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	-2.748670E+00	-2.748670E+00	-2.748670E+00	
	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00		-2.748670E+00	-2.748670E+00	

C B D I S P L A C E M E N T O T M (in global coordinate system at each grid)								
GRID	COORD	T1	T2	T3	R1	R2	R3	
	SYS							
22	0	-7.600290E-05	8.243595E-05	3.128787E-04	1.925291E-06	2.220055E-06	1.292053E-07	
32	0	-5.990878E-05	6.308617E-05	3.224179E-04	3.643362E-06	4.904270E-07	3.218612E-08	

C B E L E M E N T E N G I N E E R I N G F O R C E O T M F O R E L E M E N T T Y P E B A R								
Element	Bend-Moment	End A	Bend-Moment	End B	- Shear -	Axial	Torque	
ID	Plane 1	Plane 2	Plane 1	Plane 2	Plane 1	Plane 2	Force	
211	3.640634E+00	-2.875040E+00	-7.752079E+00	4.486528E+00	1.611173E-01	-1.041083E-01	1.906435E+00	-5.333935E-03
212	3.789705E+00	2.992877E+00	-6.061077E+00	-4.713484E+00	1.393111E-01	1.089844E-01	1.808077E+00	5.333935E-03

C B E L E M E N T S T R E S S O T M I N L O C A L E L E M E N T C O O R D I N A T E S Y S T E M F O R E L E M E N T T Y P E B A R								
Element	SA1	SA2	SA3	SA4	Axial	SA-Max	SA-Min	M.S.-T
ID	SB1	SB2	SB3	SB4	Stress	SB-Max	SB-Min	M.S.-C
201	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	7.582667E+00	7.582667E+00	7.582667E+00	
	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00		7.582667E+00	7.582667E+00	

C B D I S P L A C E M E N T O T M
(in global coordinate system at each grid)

GRID	COORD SYS	T1	T2	T3	R1	R2	R3
22	0	3.800145E-05	-4.121798E-05	-1.564393E-04	-9.626456E-07	-1.110028E-06	-6.460267E-08
32	0	2.995439E-05	-3.154308E-05	-1.612090E-04	-1.821681E-06	-2.452135E-07	-1.609306E-08

C B	E L E M E N T	E N G I N E E R I N G	F O R C E	O T M
	F O R	E L E M E N T	T Y P E	B A R
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5	5	5	5	5
6	6	6	6	6
7	7	7	7	7
8	8	8	8	8
9	9	9	9	9
10	10	10	10	10
11	11	11	11	11
12	12	12	12	12
13	13	13	13	13
14	14	14	14	14
15	15	15	15	15
16	16	16	16	16
17	17	17	17	17
18	18	18	18	18
19	19	19	19	19
20	20	20	20	20
21	21	21	21	21
22	22	22	22	22
23	23	23	23	23
24	24	24	24	24
25	25	25	25	25
26	26	26	26	26
27	27	27	27	27
28	28	28	28	28
29	29	29	29	29
30	30	30	30	30
31	31	31	31	31
32	32	32	32	32
33	33	33	33	33
34	34	34	34	34
35	35	35	35	35
36	36	36	36	36
37	37	37	37	37
38	38	38	38	38
39	39	39	39	39
40	40	40	40	40
41	41	41	41	41
42	42	42	42	42
43	43	43	43	43
44	44	44	44	44
45	45	45	45	45
46	46	46	46	46
47	47	47	47	47
48	48	48	48	48
49	49	49	49	49
50	50	50	50	50
51	51	51	51	51
52	52	52	52	52
53	53	53	53	53
54	54	54	54	54
55	55	55	55	55
56	56	56	56	56
57	57	57	57	57
58	58	58	58	58
59	59	59	59	59
60	60	60	60	60
61	61	61	61	61
62	62	62	62	62
63	63	63	63	63
64	64	64	64	64
65	65	65	65	65
66	66	66	66	66
67	67	67	67	67
68	68	68	68	68
69	69	69	69	69
70	70	70	70	70
71	71	71	71	71
72	72	72	72	72
73	73	73	73	73
74	74	74	74	74
75	75	75	75	75
76	76	76	76	76
77	77	77	77	77
78	78	78	78	78
79	79	79	79	79
80	80	80	80	80
81	81	81	81	81
82	82	82	82	82
83	83	83	83	83
84	84	84	84	84
85	85	85	85	85
86	86			

Element	Bend-Moment End A		Bend-Moment End B		- Shear -		Axial	Torque
ID	Plane 1	Plane 2	Plane 1	Plane 2	Plane 1	Plane 2	Force	
211	-1.820317E+00	1.437520E+00	3.876039E+00	-2.243264E+00	-8.055864E-02	5.205414E-02	-9.532175E-01	2.666968E-03
212	-1.894852E+00	-1.496438E+00	3.030538E+00	2.356742E+00	-6.965554E-02	-5.449220E-02	-9.040385E-01	-2.666968E-03

C B	E L E M E N T	S T R E S S	O T M	I N	L O C A L	E L E M E N T	C O O R D I N A T E	S Y S T E M
		F O R		E L E M E N T	T Y P E	B A R		

Element	SA1	SA2	SA3	SA4	Axial	SA-Max	SA-Min	M.S.-T
ID	SB1	SB2	SB3	SB4	Stress	SB-Max	SB-Min	M.S.-C
201	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	-3.791334E+00	-3.791334E+00	-3.791334E+00	
	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00		-3.791334E+00	-3.791334E+00	

-
-
-
-
-
-
-
-

(output for the 4th – 18th CB DOF deleted)

OUTPUT FOR CRAIG-BAMPTON ACCEL OTM COL 1 OF 10

C B A C C E L E R A T I O N O T M							
(in global coordinate system at each grid)							
GRID	COORD	T1	T2	T3	R1	R2	R3
	SYS						
32	0	2.199853E-02	-2.028331E-02	-1.681579E-02	-3.363157E-04	8.006145E-03	5.254334E-04

OUTPUT FOR CRAIG-BAMPTON ACCEL OTM COL 2 OF 10

C B A C C E L E R A T I O N O T M							
(in global coordinate system at each grid)							
GRID	COORD	T1	T2	T3	R1	R2	R3
	SYS						
32	0	0.000000E+00	0.000000E+00	-1.000000E+00	-2.000000E-02	0.000000E+00	0.000000E+00

OUTPUT FOR CRAIG-BAMPTON ACCEL OTM COL 3 OF 10

C B A C C E L E R A T I O N O T M							
(in global coordinate system at each grid)							
GRID	COORD	T1	T2	T3	R1	R2	R3
	SYS						
32	0	0.000000E+00	0.000000E+00	5.000000E-01	1.000000E-02	0.000000E+00	0.000000E+00

.
.
.
.
.
.
.
.

(output for the 4th – 10th Accel OTM columns deleted)


```

MODAL PARTICIPATION FACTORS
      (dimensionless, in coordinate sys 0)
MODE   T1      T2      T3      R1      R2      R3
NUM
1  1.227574E-01 -1.758352E+00  8.791759E-01  1.259087E+00  6.535370E-02 -5.341716E-01
2  6.061630E-01  1.829524E-01 -9.147622E-02 -4.910542E-01 -1.366914E-01 -4.626569E-01

```

```

EFFECTIVE MODAL MASSES OR WEIGHTS
      (in coordinate system 0)
Units are same as units for mass input in the Bulk Data Deck
MODE   T1      T2      T3      R1      R2      R3
NUM
1  6.532677E+01  4.179096E+01  4.694259E+02  3.836785E+05  3.287406E+04  3.611917E+02
2  7.948285E+00  9.016521E-01  1.363070E+01  1.674257E+00  6.082279E+05  4.781873E+05
-----
Sum all modes:  7.327506E+01  4.269261E+01  4.830566E+02  3.836801E+05  6.411019E+05  4.785485E+05
Total model mass:  9.325238E+02  9.325238E+02  9.325238E+02  4.105260E+06  4.094237E+06  8.139951E+06
Modes % of total mass*:      7.86      4.58      51.80      9.35      15.66      5.88

```

*If all modes are calculated the % of total mass should be 100% of the free mass (i.e. not counting mass at constrained DOF's).
Percentages are only printed for components that have finite model mass.

>> LINK 9 END

>> MYSTRAN END : 10/30/2006 at 18: 3:31.562

3.6.2 OUTPUT4 matrices written to CB-EXAMPLE-12-b.OP1 and OP2

(OUTPUT4 matrices requested in Exec Control)

OUTPUT4 matrices requested in Exec Control to be written to file CB-EXAMPLE-12-b.OP1 (on unit 21)

(note: only 1st 5 columns written here for the sake of clarity)

	CG_LTM	NCOLS =	18	NROWS =	6	FORM =	2	PREC =	2
	1	2	3	4	5				
1	-6.65821789802521E-05	1.29562159612018E-17	-6.47810798060089E-18	-1.29549999999999E-03	6.47766872193621E-05			
2	-2.99785601343913E-05	-1.96135553418977E-04	1.04193052213477E-04	1.39356777670951E-03	-6.70858061739371E-05			
3	-4.35697030582909E-05	-2.59100000000000E-03	1.30775055100798E-03	1.29550000000001E-03	6.19839872966866E-04			
4	-3.33844454038618E-04	-2.00000000000000E-02	9.80743672854175E-03	1.00000000000000E-02	-5.07064059129018E-03			
5	8.13687816036514E-03	1.47885176327023E-16	-7.39425881635114E-17	-7.78457159844592E-17	-5.93156091981744E-03			
6	5.63393757592496E-04	8.55130582230051E-17	-4.27565291115026E-17	9.99999999999996E-03	2.81696878796245E-04			

	IF_LTM	NCOLS =	18	NROWS =	8	FORM =	2	PREC =	2
	1	2	3	4	5				
1	6.02957424769077E-01	7.32039059471622E-02	-3.66019529735811E-02	3.35492666170908E-02	-7.19015457719424E-02			
2	7.32039059471623E-02	4.25469107253153E+00	-2.12163357113457E+00	-2.21879607113459E+00	-1.10665832128050E-01			
3	-3.66019529735811E-02	-2.12163357113457E+00	1.07224071582968E+00	1.10939803556729E+00	5.53329160640251E-02			
4	3.35492666170908E-02	-2.21879607113459E+00	1.10939803556729E+00	3.26418464157067E+00	1.75366508593570E-02			
5	-7.19015457719424E-02	-1.10665832128050E-01	5.53329160640251E-02	1.75366508593570E-02	4.96481812094837E-01			
6	-6.65046890695409E-01	-7.32039059471504E-02	3.66019529735752E-02	-1.24163383728600E+00	1.32307347677584E-01			
7	-1.34708893096271E-01	-2.21879607113459E+00	1.10939803556729E+00	2.54146535101691E-01	3.05700710026811E-02			
8	6.78737140960850E-02	-1.83869738075211E-01	9.19348690376054E-02	8.11498842422746E-02	2.62006997196796E-02			

	MR	NCOLS =	8	NROWS =	8	FORM =	1	PREC =	2
	1	2	3	4	5				
1	6.02957424769077E-01	7.32039059471622E-02	-3.66019529735811E-02	3.35492666170908E-02	-7.19015457719424E-02			
2	7.32039059471623E-02	4.25469107253153E+00	-2.12163357113457E+00	-2.21879607113459E+00	-1.10665832128050E-01			
3	-3.66019529735811E-02	-2.12163357113457E+00	1.07224071582968E+00	1.10939803556729E+00	5.53329160640251E-02			
4	3.35492666170908E-02	-2.21879607113459E+00	1.10939803556729E+00	3.26418464157067E+00	1.75366508593570E-02			
5	-7.19015457719424E-02	-1.10665832128050E-01	5.53329160640251E-02	1.75366508593570E-02	4.96481812094837E-01			
6	-6.65046890695409E-01	-7.32039059471504E-02	3.66019529735752E-02	-1.24163383728600E+00	1.32307347677584E-01			
7	-1.34708893096271E-01	-2.21879607113459E+00	1.10939803556729E+00	2.54146535101691E-01	3.05700710026811E-02			
8	6.78737140960850E-02	-1.83869738075211E-01	9.19348690376054E-02	8.11498842422746E-02	2.62006997196796E-02			

OUTPUT4 matrices requested in Exec Control to be written to file CB-EXAMPLE-12-b.OP2 (on unit 22)

(note: only 1st 5 columns written here the sake of clarity)

	KRRGN	NCOLS =	10	NROWS =	10	FORM =	1	PREC =	2	
	1	2	3	4	5					
1	1.19504240447136E+03	-3.63797880709171E-12	1.81898940354586E-12	1.54614099301398E-11	5.97521202235677E+02				
2	-5.45696821063757E-12	0.00000000000000E+00	0.00000000000000E+00	1.81898940354586E-12	0.00000000000000E+00				
3	2.72848410531878E-12	0.00000000000000E+00	0.00000000000000E+00	-9.09494701772928E-13	0.00000000000000E+00				
4	2.08011385893769E-11	0.00000000000000E+00	0.00000000000000E+00	-1.16415321826935E-10	9.43778388773353E-12				
5	5.97521202235677E+02	-1.13686837721616E-13	5.68434188608080E-14	-1.59161572810262E-12	2.98760601117838E+02				
6	-1.19504240447137E+03	0.00000000000000E+00	0.00000000000000E+00	-1.79397829924710E-10	-5.97521202235685E+02				
7	-2.98427949019242E-13	0.00000000000000E+00	0.00000000000000E+00	-4.31782609666698E-10	-2.76401124210679E-12				
8	-5.97521202235677E+02	-1.81898940354586E-12	9.09494701772928E-13	1.36424205265939E-12	-2.98760601117839E+02				
9	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00				
10	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00				

	RBMCG	NCOLS =	6	NROWS =	6	FORM =	2	PREC =	2	
	1	2	3	4	5					
1	2.41616914133782E+00	-3.35287353436797E-14	-6.52256026967279E-15	-1.34114941374719E-13	-3.97903932025656E-13				
2	-3.30846461338297E-14	2.41616914133786E+00	2.33146835171283E-14	7.74491581978509E-13	2.89102075612391E-13				
3	-6.52256026967279E-15	2.27734497926235E-14	2.41616914133783E+00	-9.59232693276135E-14	-7.10542735760100E-14				
4	-1.35891298214119E-13	7.81374964731185E-13	-1.24344978758018E-13	4.56169135583651E+03	-3.86535248253495E-12				
5	-3.92130772297605E-13	2.88435941797616E-13	-6.75015598972095E-14	-4.09272615797818E-12	4.53313153018053E+03				
6	1.99662508748588E-12	4.26325641456060E-14	-3.62376795237651E-13	-1.36424205265939E-11	2.85598256559946E+01				

	MRRGN	NCOLS =	10	NROWS =	10	FORM =	1	PREC =	2	
	1	2	3	4	5					
1	6.02957424769077E-01	7.32039059471622E-02	-3.66019529735811E-02	3.35492666170908E-02	-7.19015457719424E-02				
2	7.32039059471623E-02	4.25469107253153E+00	-2.12163357113457E+00	-2.21879607113459E+00	-1.10665832128050E-01				
3	-3.66019529735811E-02	-2.12163357113457E+00	1.07224071582968E+00	1.10939803556729E+00	5.53329160640251E-02				
4	3.35492666170908E-02	-2.21879607113459E+00	1.10939803556729E+00	3.26418464157067E+00	1.75366508593570E-02				
5	-7.19015457719424E-02	-1.10665832128050E-01	5.53329160640251E-02	1.75366508593570E-02	4.96481812094837E-01				
6	-6.65046890695409E-01	-7.32039059471504E-02	3.66019529735752E-02	-1.24163383728600E+00	1.32307347677584E-01				
7	-1.34708893096271E-01	-2.21879607113459E+00	1.10939803556729E+00	2.54146535101691E-01	3.05700710026811E-02				
8	6.78737140960850E-02	-1.83869738075211E-01	9.19348690376054E-02	8.11498842422746E-02	2.62006997196796E-02				
9	1.22757372107055E-01	-1.75835189695839E+00	8.79175948479194E-01	1.25908689725916E+00	6.53537005701318E-02				
10	6.06162990294928E-01	1.82952442095713E-01	-9.14762210478567E-02	-4.91054200271590E-01	-1.36691428775775E-01				

	RBRCG	NCOLS =	6	NROWS =	8	FORM =	2	PREC =	2	
	1	2	3	4	5					
1	1.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	5.37849392786371E+01				
2	0.00000000000000E+00	1.00000000000000E+00	0.00000000000000E+00	-5.37849392786371E+01	0.00000000000000E+00				
3	0.00000000000000E+00	0.00000000000000E+00	1.00000000000000E+00	-5.00000000000000E+01	0.00000000000000E+00				
4	0.00000000000000E+00	1.00000000000000E+00	0.00000000000000E+00	-3.78493927863709E+00	0.00000000000000E+00				
5	0.00000000000000E+00	0.00000000000000E+00	1.00000000000000E+00	-5.00000000000000E+01	-5.00000000000000E+01				
6	1.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	3.78493927863709E+00				
7	0.00000000000000E+00	1.00000000000000E+00	0.00000000000000E+00	-3.78493927863709E+00	0.00000000000000E+00				
8	0.00000000000000E+00	0.00000000000000E+00	1.00000000000000E+00	-5.00000000000000E+01	5.00000000000000E+01				

3.6.3 Displ and Element force/stress OTM's written to CB-EXAMPLE-12-b.OP8 and OP9

(OTM's requested in Case Control)

CB-EXAMPLE-12-b.OP8 binary file of element force/stress OTM's requested in Case Control

(note: only 1st 5 columns written here the sake of clarity)

	OTM_ELFE	NCOLS =	18	NROWS =	16	FORM =	2	PREC =	2
	1	2	3	4	5				
1	2.09187572390564E-01	3.64063384390388E+00	-1.82031692195194E+00	-1.84227921264778E+00	-9.14925412689932E-01			
2	7.89453912890167E-01	-2.87503976462738E+00	1.43751988231369E+00	1.92080844772306E+00	-1.26234542491864E-01			
3	1.51560714339846E+00	-7.75207867487571E+00	3.87603933743785E+00	3.62690741509324E+00	1.45527637571713E+00			
4	-1.43934432738336E+00	4.48652751792572E+00	-2.24326375896286E+00	-2.73874759882899E+00	2.35906653084923E-01			
5	-1.84755627546901E-02	1.61117285562758E-01	-8.05586427813792E-02	-7.73459790410093E-02	-3.35197151472623E-02			
6	3.15199669918811E-02	-1.04108282913086E-01	5.20541414565432E-02	6.58960735567147E-02	-5.12144990278700E-03			
7	6.26679968599842E-01	1.90643492900070E+00	-9.53217464500349E-01	-1.19040949990613E-01	-1.14791218537626E-01			
8	9.67284596743351E-03	-5.33393540270422E-03	2.66696770135211E-03	-5.34876839175438E-02	8.35971431688627E-04			
9	-1.13315069892136E-01	3.78970456518829E+00	-1.89485228259414E+00	-1.26147862482940E+00	-9.55864075040792E-01			
10	-1.00896004659258E-02	2.99287680850590E+00	-1.49643840425295E+00	-4.03697533588189E+00	-1.41398274167766E-02			
11	-1.72540058669802E+00	-6.06107677196644E+00	3.03053838598322E+00	2.53928832803047E+00	1.96715396237338E+00			
12	-6.16614847670031E-02	-4.71348398353008E+00	2.35674199176504E+00	6.82365970711492E+00	3.39064169416761E-02			
13	2.27983320157212E-02	1.39311085669760E-01	-6.96555428348799E-02	-5.37509617215390E-02	-4.13377175157231E-02			
14	7.29336582157196E-04	1.08984399486375E-01	-5.44921997431877E-02	-1.53592573737906E-01	-6.79476503928156E-04			
15	-2.95361107284698E-01	1.80807707871691E+00	-9.04038539358453E-01	-1.95832712226347E+00	3.00896480121837E-03			
16	-4.72042770150405E-03	5.33393540270377E-03	-2.66696770135189E-03	-1.12160973347287E-01	-3.69369770142806E-03			

	OTM_STRE	NCOLS =	18	NROWS =	18	FORM =	2	PREC =	2
	1	2	3	4	5				
1	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
2	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
3	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
4	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
5	-2.74867035744303E+00	7.58266712433821E+00	-3.79133356216910E+00	-1.07520478850513E+00	4.30045958649968E-01			
6	-2.74867035744303E+00	7.58266712433821E+00	-3.79133356216910E+00	-1.07520478850513E+00	4.30045958649968E-01			
7	-2.74867035744303E+00	7.58266712433821E+00	-3.79133356216910E+00	-1.07520478850513E+00	4.30045958649968E-01			
8	-1.00000000000000E+00	-1.00000000000000E+00	-1.00000000000000E+00	-1.00000000000000E+00	-1.00000000000000E+00			
9	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
10	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
11	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
12	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
13	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00			
14	-2.74867035744303E+00	7.58266712433821E+00	-3.79133356216910E+00	-1.07520478850513E+00	4.30045958649968E-01			
15	-2.74867035744303E+00	7.58266712433821E+00	-3.79133356216910E+00	-1.07520478850513E+00	4.30045958649968E-01			
16	-2.74867035744303E+00	7.58266712433821E+00	-3.79133356216910E+00	-1.07520478850513E+00	4.30045958649968E-01			
17	-1.00000000000000E+00	-1.00000000000000E+00	-1.00000000000000E+00	-1.00000000000000E+00	-1.00000000000000E+00			
18	1.00000000000000E+10	1.00000000000000E+10	1.00000000000000E+10	1.00000000000000E+10	1.00000000000000E+10			

CB-EXAMPLE-12-b.OT8 text file descriptor of rows in above binary file for element related OTM's

This text file describes the rows of the elem related OTM matrices written to unformatted file: CB-EXAMPLE-12-b.OP8

The description for each of the matrices has the headers:

ROW : row number in the individual OTM described
DESCRIPTION: what OTM is this
TYPE : element type
EID : element ID

Then, for the element nodal force OTM:

GRID : grid number of the element that the OTM is for
COMP : displacement component number (1,2,3 translations and 4,5,6 rotations)

and for element engineering force and element stress OTMs:

ITEM : element force or stress item (axial force, torque, etc)

The number of rows for each OTM depends on the output requests, by the user, in Case Control

The number of cols for each OTM depends on the number of support DOFs (NDOFR) and the number of eigenvectors (NVEC) where:

NDOFR = 8
NVEC = 2

This text file has descriptions for the following element related OTMs from CB-EXAMPLE-12-b.OP8

Element engr force OTM (matrix OTM_ELFE) with $2 \times \text{NDOFR} + \text{NVEC} = 18$ cols
Element stress OTM (matrix OTM_STRE) with $2 \times \text{NDOFR} + \text{NVEC} = 18$ cols

Explanation of rows of 16 row by 18 col matrix OTM_ELFE				
ROW	DESCRIPTION	TYPE	EID	ITEM

1	Element engineering force	BAR	211	M1a: Mom Plane1 EndA
2	Element engineering force	BAR	211	M1b: Mom Plane2 EndA
3	Element engineering force	BAR	211	M2a: Mom Plane1 EndB
4	Element engineering force	BAR	211	M2b: Mom Plane2 EndB
5	Element engineering force	BAR	211	V1 : Shear Plane1
6	Element engineering force	BAR	211	V2 : Shear Plane2
7	Element engineering force	BAR	211	FX : Axial force
8	Element engineering force	BAR	211	T : Torque
9	Element engineering force	BAR	212	M1a: Mom Plane1 EndA
10	Element engineering force	BAR	212	M1b: Mom Plane2 EndA
11	Element engineering force	BAR	212	M2a: Mom Plane1 EndB
12	Element engineering force	BAR	212	M2b: Mom Plane2 EndB
13	Element engineering force	BAR	212	V1 : Shear Plane1
14	Element engineering force	BAR	212	V2 : Shear Plane2
15	Element engineering force	BAR	212	FX : Axial force
16	Element engineering force	BAR	212	T : Torque

Explanation of rows of		18 row by	18 col matrix OTM_STRE	
ROW	DESCRIPTION	TYPE	EID	ITEM
1	Element stress	BAR	201	SA1: Stress Pt1 EndA
2	Element stress	BAR	201	SA2: Stress Pt2 EndA
3	Element stress	BAR	201	SA3: Stress Pt3 EndA
4	Element stress	BAR	201	SA4: Stress Pt4 EndA
5	Element stress	BAR	201	Axial Stress
6	Element stress	BAR	201	SA-Max
7	Element stress	BAR	201	SA-Min
8	Element stress	BAR	201	MS-Tension
9	Element stress	BAR	201	Torsional Stress
10	Element stress	BAR	201	SB1: Stress Pt1 EndB
11	Element stress	BAR	201	SB2: Stress Pt2 EndB
12	Element stress	BAR	201	SB3: Stress Pt3 EndB
13	Element stress	BAR	201	SB4: Stress Pt4 EndB
14	Element stress	BAR	201	Axial stress
15	Element stress	BAR	201	SB-Max
16	Element stress	BAR	201	SB-Min
17	Element stress	BAR	201	MS-Compression
18	Element stress	BAR	201	MS-Torsion

CB-EXAMPLE-12-b.OP9 binary file of displacement OTM's requested in Case Control

(note: only 1st 5 columns written here the sake of clarity)

	OTM_ACCE	NCOLS =	10	NROWS =	6	FORM =	2	PREC =	2
	1	2	3	4	5				
1	2.19985250269592E-02	0.00000000000000E+00	0.00000000000000E+00	-5.00000000000004E-01	1.09992625134795E-02			
2	-2.02833087802606E-02	0.00000000000000E+00	0.00000000000000E+00	5.00000000000004E-01	-1.01416543901302E-02			
3	-1.68157865913898E-02	-1.00000000000000E+00	5.00000000000000E-01	5.00000000000005E-01	2.41592106704306E-01			
4	-3.36315731827796E-04	-2.00000000000000E-02	1.00000000000000E-02	1.00000000000001E-02	-5.16815786591390E-03			
5	8.00614495648658E-03	0.00000000000000E+00	0.00000000000000E+00	0.00000000000000E+00	-5.99692752175671E-03			
6	5.25433423070610E-04	0.00000000000000E+00	0.00000000000000E+00	9.99999999999992E-03	2.62716711535305E-04			

	OTM_DISP	NCOLS =	18	NROWS =	12	FORM =	2	PREC =	2
	1	2	3	4	5				
1	-1.41293911043985E-05	-7.60029025912968E-05	3.80014512956484E-05	1.29492635368416E-04	3.14571590643487E-06			
2	1.62214021120513E-05	8.24359519633505E-05	-4.12179759816752E-05	-1.30161832591346E-04	-3.52963231517632E-06			
3	8.24222187730972E-05	3.12878663301563E-04	-1.56439331650781E-04	-2.40634384994669E-04	-1.68993616070736E-05			
4	5.88370868696758E-07	1.92529119983460E-06	-9.62645599917302E-07	-2.07019101770705E-06	1.88916538580397E-07			
5	-1.66743323917105E-06	2.22005501168008E-06	-1.11002750584004E-06	-1.14971054599053E-06	-8.88454144573320E-08			
6	5.12515138397389E-07	1.29205343624621E-07	-6.46026718123106E-08	-1.07589130445167E-06	-9.61720937623318E-08			
7	1.05104109813473E-05	-5.99087762260462E-05	2.99543881130231E-05	6.53233961326989E-05	-1.57813540011406E-06			
8	-9.46594436701425E-06	6.30861677743807E-05	-3.15430838871904E-05	-6.55217977160166E-05	1.38681670255135E-06			
9	-3.18288681491121E-06	3.22417925611894E-04	-1.61208962805947E-04	-1.96081126486432E-04	-3.61627931263323E-05			
10	-1.08618067423320E-07	3.64336233382231E-06	-1.82168116691115E-06	-2.63986785628832E-06	-3.24126419085498E-08			
11	-9.45071958677177E-07	4.90427017653186E-07	-2.45213508826593E-07	-2.21449664764883E-07	1.36502293189118E-07			
12	2.10600905814006E-07	3.21861205426993E-08	-1.60930602713497E-08	-6.09852683088454E-07	-3.82285587596693E-08			

CB-EXAMPLE-12-b.OT9 text file descriptor of rows in above binary file for grid related OTM's

This text file describes the rows of the grid related OTM matrices written to unformatted file: CB-EXAMPLE-12-b.OP9

The description for each of the matrices has the headers:

ROW : row number in the individual OTM described
DESCRIPTION: what OTM is this
GRID : grid number for this row of the OTM
COMP : displacement component number (1,2,3 translations and 4,5,6 rotations)

The number of rows for each OTM depends on the output requests, by the user, in Case Control

The number of cols for each OTM depends on the number of support DOFs (NDOFR) and the number of eigenvectors (NVEC) where:

NDOFR = 8
NVEC = 2

This text file has descriptions for the following grid related OTMs from CB-EXAMPLE-12b.OP9

Acceleration OTM (matrix OTM_ACCE) with NDOFR + NVEC = 10 cols
Displacement OTM (matrix OTM_DISP) with 2*NDOFR + NVEC = 18 cols

Explanation of rows of 6 row by 10 col matrix OTM_ACCE

ROW	DESCRIPTION	GRID	COMP
1	Acceleration	32	1
2	Acceleration	32	2
3	Acceleration	32	3
4	Acceleration	32	4
5	Acceleration	32	5
6	Acceleration	32	6

Explanation of rows of 12 row by 18 col matrix OTM_DISP

ROW	DESCRIPTION	GRID	COMP
1	Displacement	22	1
2	Displacement	22	2
3	Displacement	22	3
4	Displacement	22	4
5	Displacement	22	5
6	Displacement	22	6
7	Displacement	32	1
8	Displacement	32	2
9	Displacement	32	3
10	Displacement	32	4
11	Displacement	32	5

4 Derivation of RBE3 Element Constraint Equations

4.1 Introduction

The RBE3 element is used for distributing applied loads and mass from a reference point to other points in the finite element model. The geometry and loads for a RBE3 are shown in Figure 1. Point d in the figure is the RBE3 reference (or dependent) point and is the grid where loads will be applied by the user. The RBE3 element will distribute these loads to other, independent, points $i = 1, \dots, N$, in the model, where N is the total number of independent grid points defined on the RBE3 Bulk Data entry. The RBE3 is not intended to add stiffness to the model as does a RBE2 element. As such, the RBE3 reference point should not be a grid that is attached to other elements in the model – it should be a stand alone grid only connected to other grids through the RBE3 element definition. The following describes the nomenclature used in this appendix in deriving the “constraint” equations used in MYSTRAN for the RBE3 element.

Superscripts denote the location of a quantity:

“d” refers to the reference (or dependent) grid on the RBE3

“i” refers to the independent grids, the locations where the loads on point d will be distributed

X, Y, Z = coordinate system axes

u_x, u_y, u_z = displacements in the x, y, z directions

$\theta_x, \theta_y, \theta_z$ = rotations about the x, y, z axes

F_x, F_y, F_z = forces in the x, y, z directions

M_x, M_y, M_z = moments about the x, y, z axes

d_x^i, d_y^i, d_z^i = position of point i relative to the RBE3 reference point, d

For the sake of simplicity and clarity, the following derivation of the RBE3 equations is done for conditions where the global coordinate systems of all grid points involved in the RBE3 are the same and are rectangular. The code in the MYSTRAN program is written for general conditions where the global system of all points may be different and non-rectangular.

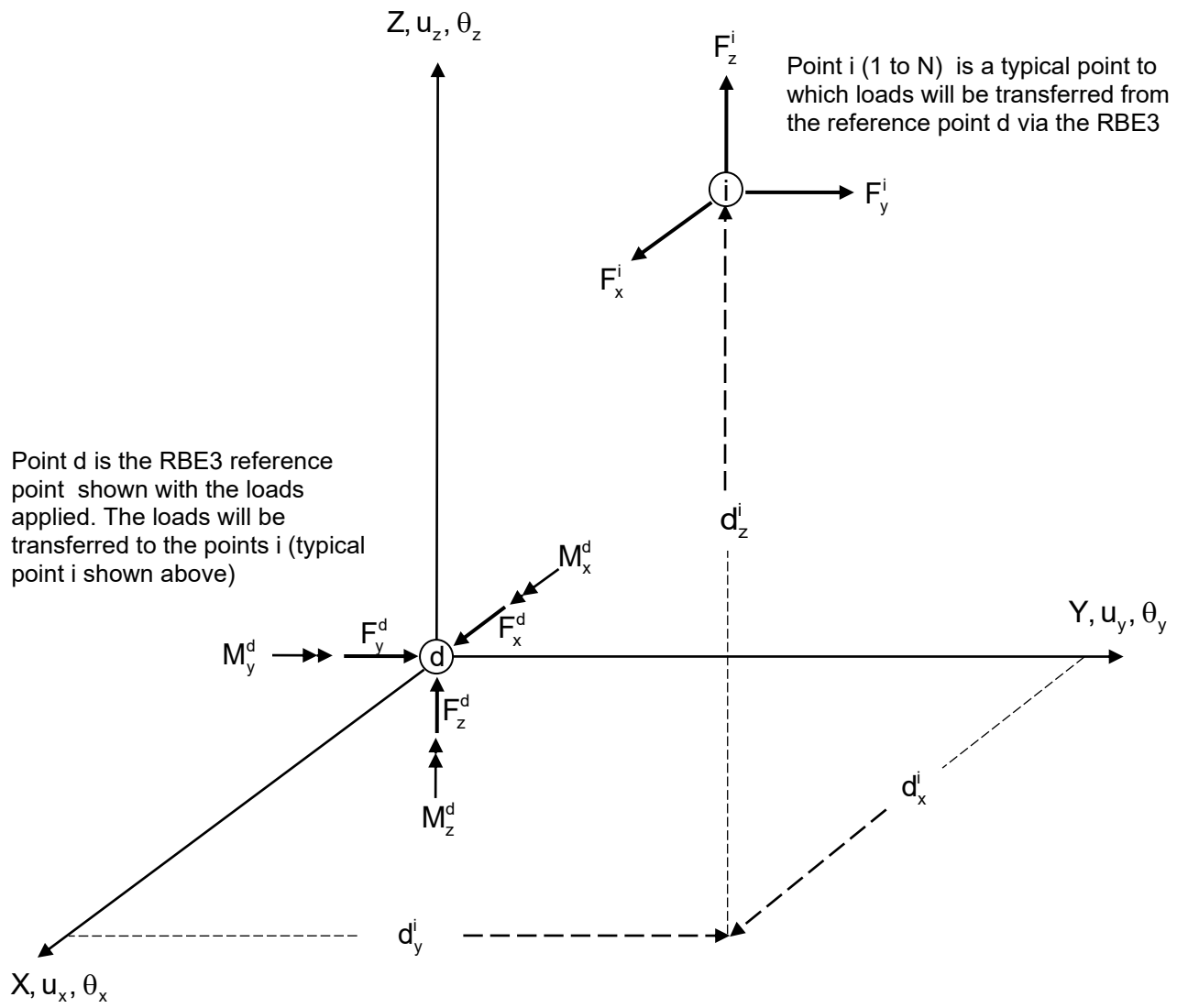


Fig 1: RBE3 geometry and loads

4.2 Equations for translational force components

In this section 3 equations will be developed that relate the forces applied at the RBE3 reference point to those where the loads will be distributed (points $i = 1, \dots, N$).

The sum of the forces on the points $i = 1, \dots, N$ must equal the forces on the reference point d. Thus:

$$\sum_{i=1}^N F_x^i = F_x^d, \quad \sum_{i=1}^N F_y^i = F_y^d, \quad \sum_{i=1}^N F_z^i = F_z^d \quad 11-1$$

The moments at reference point due to the forces at the points i are:

$$\sum_{i=1}^N (F_z^i d_y^i - F_y^i d_z^i) = M_x^d, \quad \sum_{i=1}^N (F_x^i d_z^i - F_z^i d_x^i) = M_y^d, \quad \sum_{i=1}^N (F_y^i d_x^i - F_x^i d_y^i) = M_z^d \quad 11-2$$

Write the F_x^i , etc, as:

$$F_x^i = \frac{\omega_i}{W_T} F_x^d, \quad F_y^i = \frac{\omega_i}{W_T} F_y^d, \quad F_z^i = \frac{\omega_i}{W_T} F_z^d \quad 11-3$$

where ω_i is the weighting factor (the WTi on the RBE3 Bulk Data entry) for the i th force and:

$$W_T = \sum_{i=1}^N \omega_i \quad 11-4$$

Equations 3 and 4 are sufficient for equations 1. Substitute equations 3 and 4 into 2 to get the following 3 equations:

$$\frac{F_z^d}{W_T} \sum_{i=1}^N \omega_i d_y^i - \frac{F_y^d}{W_T} \sum_{i=1}^N \omega_i d_z^i = M_x^d \quad 11-5$$

$$\frac{F_x^d}{W_T} \sum_{i=1}^N \omega_i d_z^i - \frac{F_z^d}{W_T} \sum_{i=1}^N \omega_i d_x^i = M_y^d \quad 11-6$$

$$\frac{F_y^d}{W_T} \sum_{i=1}^N \omega_i d_x^i - \frac{F_x^d}{W_T} \sum_{i=1}^N \omega_i d_y^i = M_z^d \quad 11-7$$

Define:

$$\bar{d}_x = \frac{1}{W_T} \sum_{i=1}^N \omega_i d_x^i, \quad \bar{d}_y = \frac{1}{W_T} \sum_{i=1}^N \omega_i d_y^i, \quad \bar{d}_z = \frac{1}{W_T} \sum_{i=1}^N \omega_i d_z^i \quad 11-8$$

Using equation 8, equations 5-7 become:

$$F_z^d \bar{d}_y - F_y^d \bar{d}_z = M_x^d \quad 11-9$$

$$F_x^d \bar{d}_z - F_z^d \bar{d}_x = M_y^d \quad 11-10$$

$$F_y^d \bar{d}_x - F_x^d \bar{d}_y = M_z^d \quad 11-11$$

The work done by the forces and moments at the reference point, d, is Ω_d :

$$\Omega_d = F_x^d u_x^d + F_y^d u_y^d + F_z^d u_z^d + M_x^d \theta_x^d + M_y^d \theta_y^d + M_z^d \theta_z^d \quad 11-12$$

where u, θ are the displacements and rotations of the reference point in the x, y, z directions. Similarly, the work done by the forces on the points $I = 1, \dots, N$ is:

$$\Omega_N = \sum_{i=1}^N (F_x^i u_x^i + F_y^i u_y^i + F_z^i u_z^i) \quad 11-13$$

The u_x^i , etc, are the displacements in the x, y and z directions at point I. Substitute equation 3 into 12 and 9, 10 and 11 into 12 and equate the work done by the two systems of forces:

$$\begin{aligned} & F_x^d u_x^d + F_y^d u_y^d + F_z^d u_z^d + (F_z^d \bar{d}_y - F_y^d \bar{d}_z) \theta_x^d + (F_x^d \bar{d}_z - F_z^d \bar{d}_x) \theta_y^d + (F_y^d \bar{d}_x - F_x^d \bar{d}_y) \theta_z^d = \\ & \sum_{i=1}^N \left(\frac{\omega_i}{W_T} F_x^d u_x^i + \frac{\omega_i}{W_T} F_y^d u_y^i + \frac{\omega_i}{W_T} F_z^d u_z^i \right) \end{aligned}$$

Rearrange:

$$\begin{aligned} & (u_x^d + \bar{d}_z \theta_y^d - \bar{d}_y \theta_z^d - \sum_{i=1}^N \frac{\omega_i}{W_T} u_x^i) F_x^d + \\ & (u_y^d + \bar{d}_x \theta_z^d - \bar{d}_z \theta_x^d - \sum_{i=1}^N \frac{\omega_i}{W_T} u_y^i) F_y^d + \\ & (u_z^d + \bar{d}_y \theta_x^d - \bar{d}_x \theta_y^d - \sum_{i=1}^N \frac{\omega_i}{W_T} u_z^i) F_z^d = 0 \end{aligned} \quad 11-14$$

Since the F_x^d , F_y^d and F_z^d are independent and, in general, not zero, equation 14 requires that:

$$\begin{aligned} & (u_x^d + \bar{d}_z \theta_y^d - \bar{d}_y \theta_z^d - \sum_{i=1}^N \frac{\omega_i}{W_T} u_x^i) = 0 \\ & (u_y^d + \bar{d}_x \theta_z^d - \bar{d}_z \theta_x^d - \sum_{i=1}^N \frac{\omega_i}{W_T} u_y^i) = 0 \\ & (u_z^d + \bar{d}_y \theta_x^d - \bar{d}_x \theta_y^d - \sum_{i=1}^N \frac{\omega_i}{W_T} u_z^i) = 0 \end{aligned} \quad 11-15$$

Equation 15 represents 3 constraint equations for the RBE3. However, there are only 3 equations and 6 unknowns. This will be resolved in the next section where we develop 3 more equations based on the moments at the reference point.

4.3 Equations for rotational moment components

In addition to the 3 equations developed in the last section there are also 3 equations that relate the moments applied at the RBE3 reference point to those where the loads will be distributed (points $i = 1, \dots, N$).

Figure 2 shows how the forces in the y-z plane relate to the RBE3 reference point moment about the x axis:

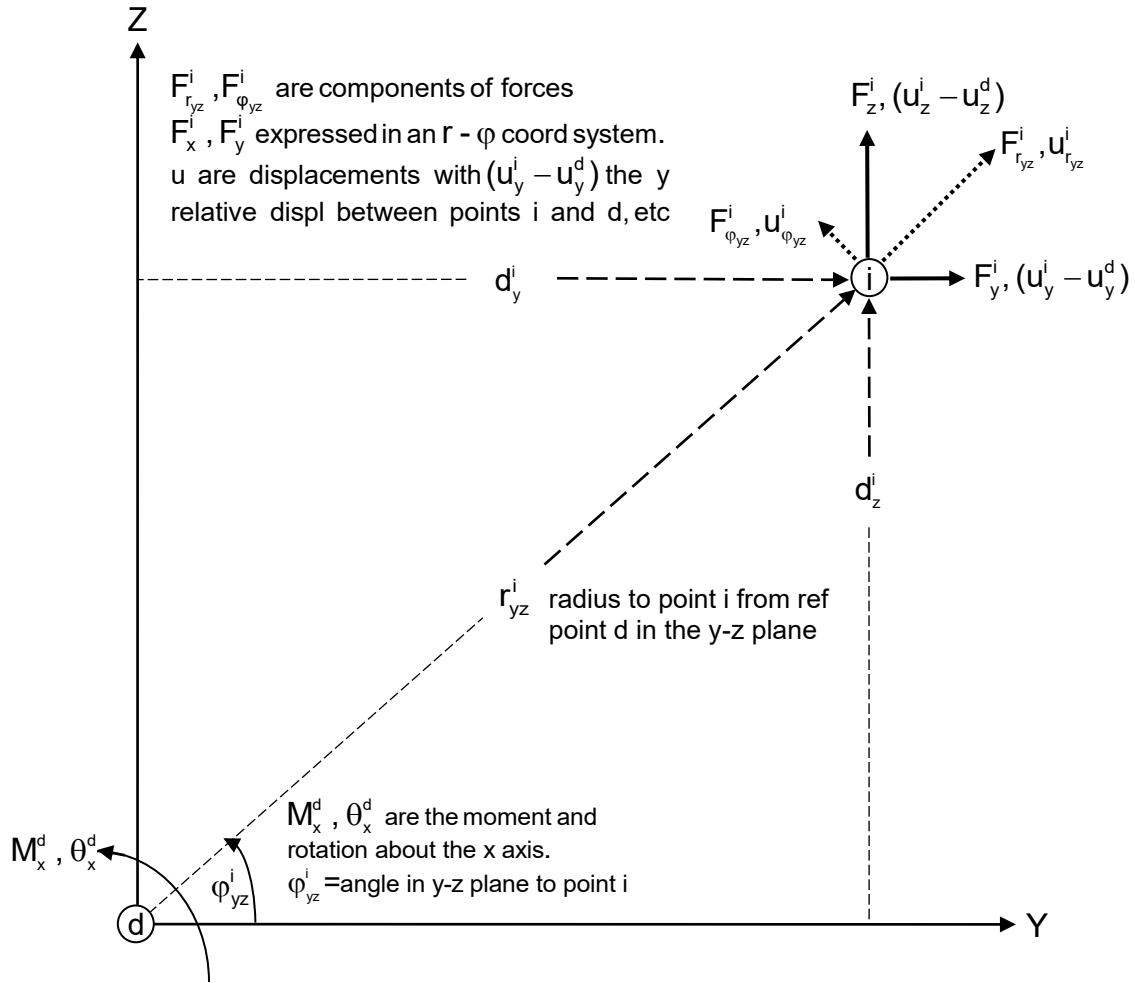


Figure 2: Relationship of moments and forces in the y-z plane

Using the r - ϕ components of the forces, the moments about the x axis of the forces at the $i = 1, \dots, N$ points is:

$$\sum_{i=1}^N F_{\phi_{yz}}^i r_{yz}^i = M_x^d \quad 11-16$$

As before, express the forces at the i points using the weighting factors, ω_i :

$$F_{\phi_{yz}}^i = \frac{\omega_i r_{yz}^i}{\sum_{i=1}^N \omega_i r_{yz}^{i^2}} M_x^d \quad 11-17$$

Note that if equation 17 were substituted into 16 it would be seen that 17 is a valid representation of the tangential force components.

The work done by M_x^d must equal that due to all of the $F_{\phi_{yz}}^i$, or:

$$\sum F_{\phi_{yz}}^i u_{\phi_{yz}}^i = M_x^d \theta_x^d \quad 11-18$$

where $u_{\phi_{yz}}^i$ is the tangential component of displacement at independent grid i in the y - z plane.

Substitute equation 17 into 18:

$$\sum_{i=1}^N \frac{\omega_i r_{yz}^i}{\sum_{i=1}^N \omega_i r_{yz}^{i^2}} M_x^d u_{\phi_{yz}}^i = M_x^d \theta_x^d$$

or:

$$\theta_x^d = \frac{\sum_{i=1}^N \omega_i r_{yz}^i u_{\phi_{yz}}^i}{\sum_{i=1}^N \omega_i r_{yz}^{i^2}} \quad 11-19$$

From Figure 2 it can be seen that:

$$\begin{aligned} u_{\phi_{yz}}^i &= (u_z^i - u_z^d) \cos \phi_{yz}^i - (u_y^i - u_y^d) \sin \phi_{yz}^i \\ &= (u_z^i - u_z^d) \frac{d_y^i}{r_{yz}^i} - (u_y^i - u_y^d) \frac{d_z^i}{r_{yz}^i} \end{aligned}$$

Therefore:

$$r_{yz}^i u_{\phi_{yz}}^i = (u_z^i - u_z^d) d_y^i - (u_y^i - u_y^d) d_z^i \quad 11-20$$

Define:

$$\bar{e}_{yz}^i = \frac{1}{W_T} \sum_{i=1}^N \omega_i r_{yz}^{i^2} \equiv \frac{1}{W_T} \sum_{i=1}^N \omega_i (d_y^{i^2} + d_z^{i^2}) \quad 11-21$$

Substitute equations 20 and 21 into 19

$$\begin{aligned}
\theta_x^d &= \frac{1}{W_T \bar{e}_{yz}^i} \left[\sum_{i=1}^N \omega^i (u_z^i - u_z^d) d_y^i - \sum_{i=1}^N \omega^i (u_y^i - u_y^d) d_z^i \right] \\
&= \frac{1}{W_T \bar{e}_{yz}^i} \left[-(\sum_{i=1}^N \omega^i d_y^i) u_z^d + (\sum_{i=1}^N \omega^i d_z^i) u_y^d + \sum_{i=1}^N \omega^i d_y^i u_z^i - \sum_{i=1}^N \omega^i d_z^i u_y^i \right] \\
&= \frac{1}{\bar{e}_{yz}^i} \left[-\bar{d}_y u_z^d + \bar{d}_z u_y^d + \frac{1}{W_T} \sum_{i=1}^N \omega^i d_y^i u_z^i - \frac{1}{W_T} \sum_{i=1}^N \omega^i d_z^i u_y^i \right]
\end{aligned} \tag{11-22}$$

In reference to Figures 3 and 4, define:

$$\begin{aligned}
\bar{e}_{zx}^i &= \frac{1}{W_T} \sum_{i=1}^N \omega^i r_{\phi_{zx}}^2 \equiv \frac{1}{W_T} \sum_{i=1}^N \omega^i (d_z^2 + d_x^2) \\
&\text{and} \\
\bar{e}_{xy}^i &= \frac{1}{W_T} \sum_{i=1}^N \omega^i r_{\phi_{xy}}^2 \equiv \frac{1}{W_T} \sum_{i=1}^N \omega^i (d_x^2 + d_y^2)
\end{aligned} \tag{11-23}$$

Then, θ_y^d and θ_z^d , by similar reasoning for θ_x^a in equation 22 are:

$$\begin{aligned}
\theta_y^d &= \frac{1}{W_T \bar{e}_{zx}^i} \left[\sum_{i=1}^N \omega^i (u_x^i - u_x^d) d_z^i - \sum_{i=1}^N \omega^i (u_z^i - u_z^d) d_x^i \right] \\
&= \frac{1}{\bar{e}_{zx}^i} \left[-\bar{d}_z u_x^d + \bar{d}_x u_z^d + \frac{1}{W_T} \sum_{i=1}^N \omega^i d_z^i u_x^i - \frac{1}{W_T} \sum_{i=1}^N \omega^i d_x^i u_z^i \right]
\end{aligned} \tag{11-24}$$

and

$$\begin{aligned}
\theta_z^d &= \frac{1}{W_T \bar{e}_{xy}^i} \left[\sum_{i=1}^N \omega^i (u_y^i - u_y^d) d_x^i - \sum_{i=1}^N \omega^i (u_x^i - u_x^d) d_y^i \right] \\
&= \frac{1}{\bar{e}_{xy}^i} \left[-\bar{d}_x u_y^d + \bar{d}_y u_x^d + \frac{1}{W_T} \sum_{i=1}^N \omega^i d_x^i u_y^i - \frac{1}{W_T} \sum_{i=1}^N \omega^i d_y^i u_x^i \right]
\end{aligned} \tag{11-25}$$

Thus, for the rotations:

$$\begin{aligned}
&\bar{e}_{yz} \theta_x^d - \bar{d}_z u_y^d + \bar{d}_y u_z^d + \frac{1}{W_T} \sum_{i=1}^N \omega^i d_z^i u_y^i - \frac{1}{W_T} \sum_{i=1}^N \omega^i d_y^i u_z^i = 0 \\
&\bar{e}_{zx} \theta_y^d + \bar{d}_z u_x^d - \bar{d}_x u_z^d - \frac{1}{W_T} \sum_{i=1}^N \omega^i d_z^i u_x^i + \frac{1}{W_T} \sum_{i=1}^N \omega^i d_x^i u_z^i = 0 \\
&\bar{e}_{xy} \theta_z^d - \bar{d}_y u_x^d + \bar{d}_x u_y^d + \frac{1}{W_T} \sum_{i=1}^N \omega^i d_y^i u_x^i - \frac{1}{W_T} \sum_{i=1}^N \omega^i d_x^i u_y^i = 0
\end{aligned} \tag{11-26}$$

Equations 15 and 26 constitute 6 equations in the 6 unknown displacements and rotations at point a. They are summarized in matrix notation below at the end of this appendix.

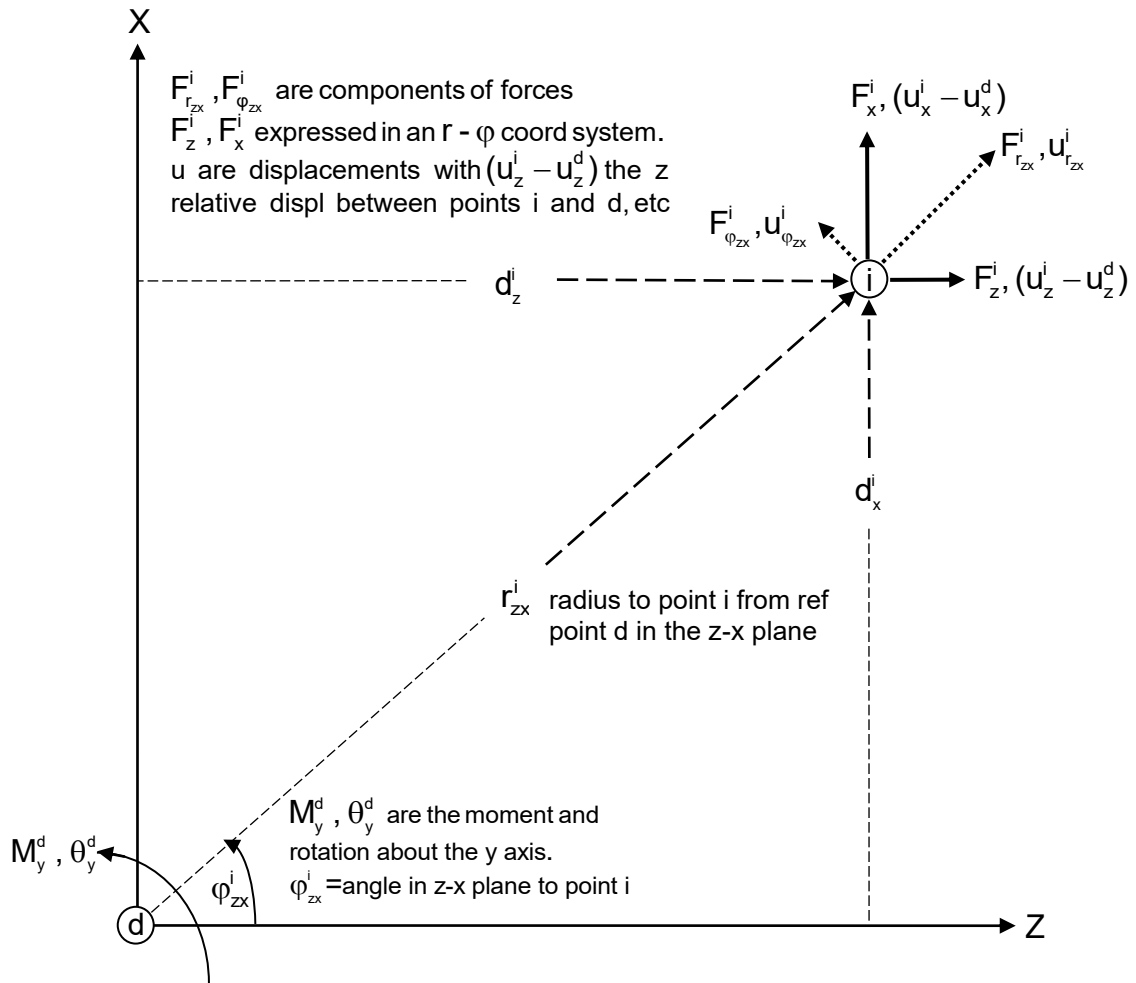


Figure 3: Relationship of moments and forces in the z - x plane

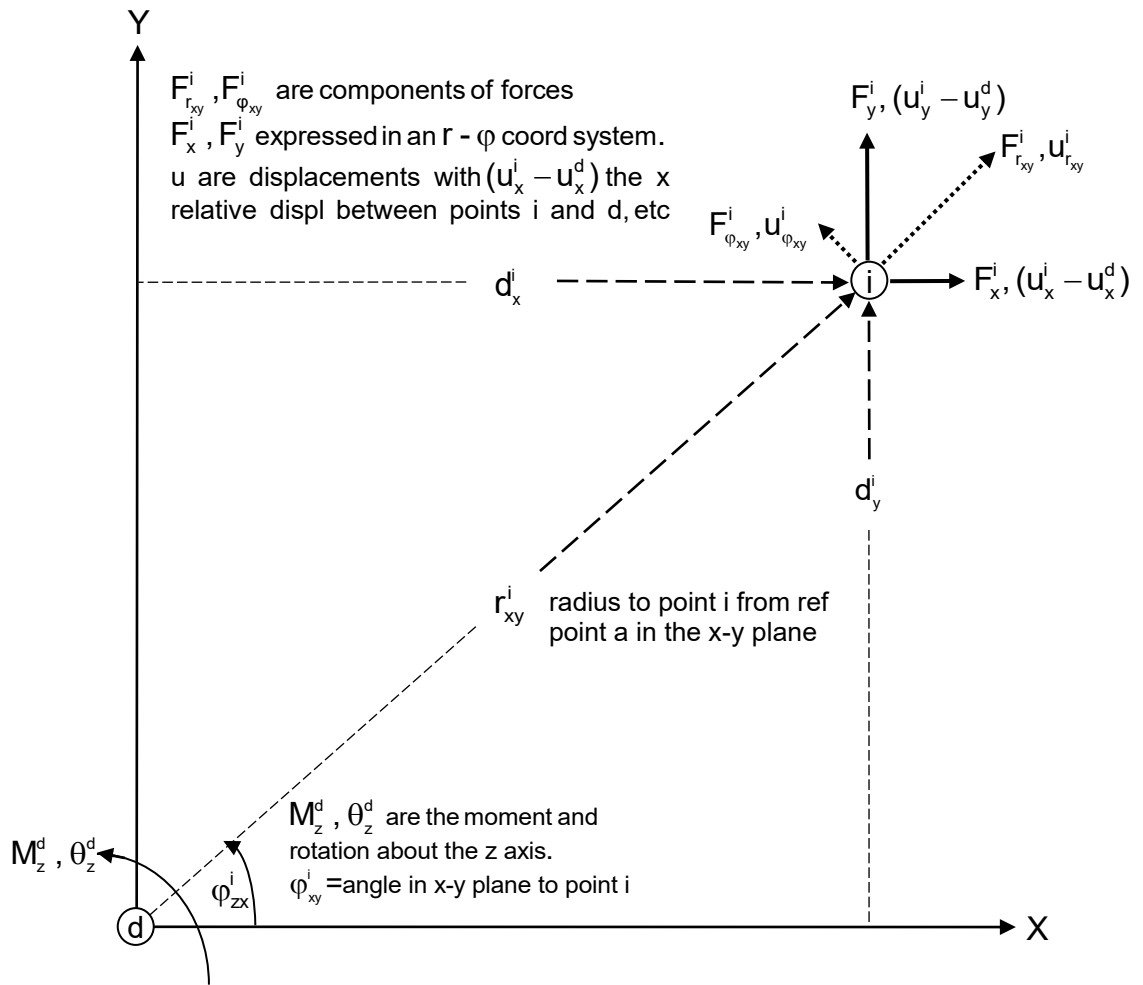


Figure 4: Relationship of moments and forces in the x-y plane

4.4 Summary of equations for the RBE3

In general, the equations for one RBE3 can be represented in matrix notation as:

$$R_{dd}U_d + R_{dN}U_N = 0 \quad 11-27$$

R_{dd} is the square, $d \times d$, matrix of coefficients for the dependent (or reference) grid denoted as REFGRID in field 4 of the RBE3 Bulk Data entry. It can have up to $d = 6$ dependent components (REFC in field 5). For all 6 components, R_{dd} and U_d are:

$$R_{dd} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \bar{d}_z & -\bar{d}_y \\ 0 & 1 & 0 & -\bar{d}_z & 0 & \bar{d}_x \\ 0 & 0 & 1 & \bar{d}_y & -\bar{d}_x & 0 \\ - & - & - & - & - & - \\ 0 & -\bar{d}_z & \bar{d}_y & \bar{e}_{yz} & 0 & 0 \\ \bar{d}_z & 0 & -\bar{d}_x & 0 & \bar{e}_{zx} & 0 \\ -\bar{d}_y & \bar{d}_x & 0 & 0 & 0 & \bar{e}_{xy} \end{array} \right], \quad U_d = \left\{ \begin{array}{c} u_x^a \\ u_y^a \\ u_z^a \\ - \\ \theta_x^a \\ \theta_y^a \\ \theta_z^a \end{array} \right\} \quad 11-28$$

R_{dN} is a rectangular, $d \times N$, matrix of coefficients for the N independent grids on the RBE3

$$R_{dN} = \frac{1}{W_T} [R_{d1} \quad R_{d2} \quad . \quad . \quad . \quad R_{dN}] \quad , \quad U_N = \left\{ \begin{array}{c} U_1 \\ U_2 \\ . \\ . \\ . \\ U_N \end{array} \right\} \quad 11-29$$

A typical sub-matrix in R_{ai} is of size d by 3 with R_{ai} and U_i . For $d = 6$:

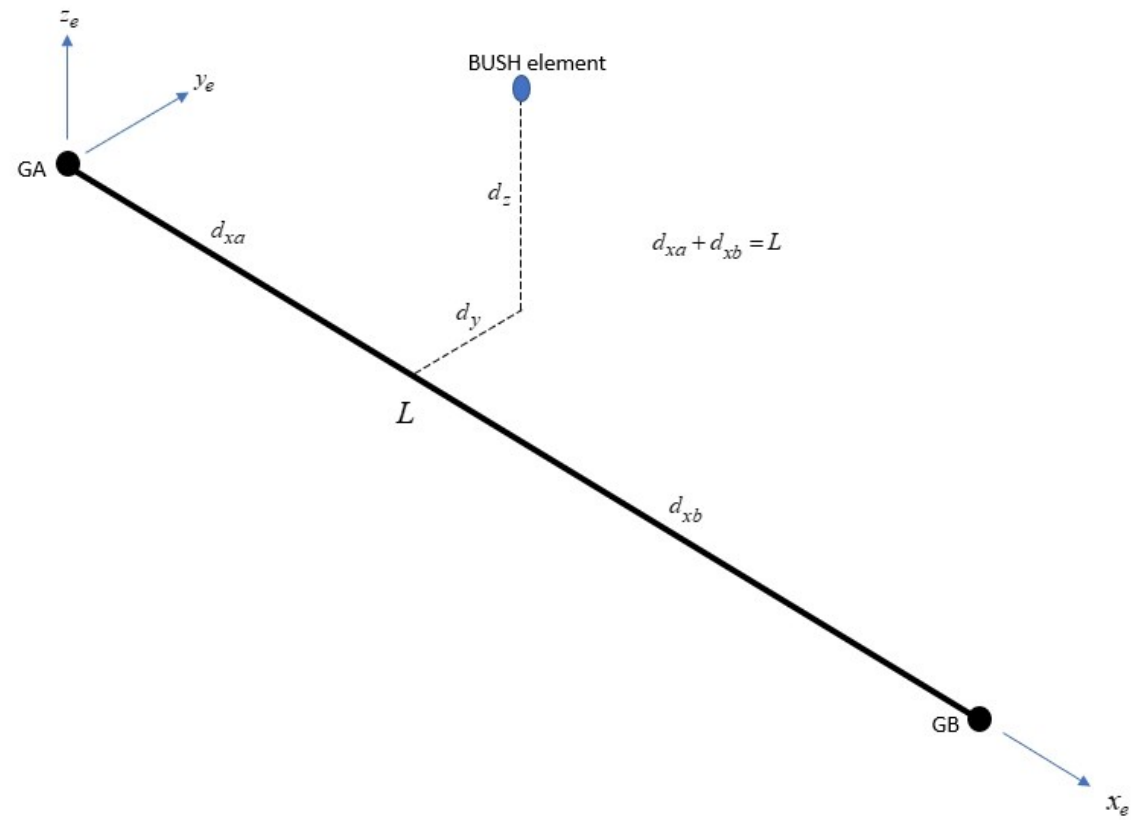
$$R_{di} = \frac{1}{W_T} \left[\begin{array}{ccc} \omega^i & 0 & 0 \\ 0 & \omega^i & 0 \\ 0 & 0 & \omega^i \\ - & - & - \\ 0 & \omega^i d_z^i & -\omega^i d_y^i \\ -\omega^i d_z^i & 0 & \omega^i d_x^i \\ \omega^i d_y^i & -\omega^i d_x^i & 0 \end{array} \right], \quad U_i = \left\{ \begin{array}{c} u_x^i \\ u_y^i \\ u_z^i \end{array} \right\} \quad 11-30$$

A RBE3 is processed by solving equation 27 for the dependent degrees of freedom, U_d , in terms of the independent degrees of freedom, U_N .

5 Equations for the BUSH Element

BUSH Element Geometry

(in local element coordinates)



The stiffness equations for the BUSH element can be expressed as:

$$Ku = F$$

where K is a 12x12 matrix and u and F are the 12 degree of freedom (6 at each of the 2 grids) displacements and node forces. For the sake of clarity, rather than showing the whole 12x12 stiffness matrix, express the above equation in grid partitioned form as:

$$\begin{bmatrix} K_{aa} & K_{ab} \\ K_{ab}^T & K_{bb} \end{bmatrix} \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} = \begin{Bmatrix} F_a \\ F_b \end{Bmatrix}$$

If we denote κ_i (i=1,...6) as the 6 stiffness values from the PBUSH Bulk Data entry then the above partitions are:

$$K_{aa} = \begin{bmatrix} \kappa_1 & 0 & 0 & 0 & d_z \kappa_1 & -d_y \kappa_1 \\ 0 & \kappa_2 & 0 & -d_z \kappa_2 & 0 & d_{xa} \kappa_2 \\ 0 & 0 & \kappa_3 & d_y \kappa_3 & -d_{xa} \kappa_3 & 0 \\ 0 & -d_z \kappa_2 & d_y \kappa_3 & \kappa_4 + d_y^2 \kappa_3 + d_z^2 \kappa_2 & -d_{xa} d_y \kappa_3 & -d_{xa} d_z \kappa_2 \\ d_z \kappa_1 & 0 & -d_{xa} \kappa_3 & -d_{xa} d_y \kappa_3 & \kappa_5 + d_{xa}^2 \kappa_3 + d_z^2 \kappa_1 & -d_y d_z \kappa_1 \\ -d_y \kappa_1 & d_{xa} \kappa_2 & 0 & -d_{xa} d_z \kappa_2 & -d_y d_z \kappa_1 & \kappa_6 + d_{xa}^2 \kappa_2 + d_y^2 \kappa_1 \end{bmatrix}$$

$$K_{ab} = \begin{bmatrix} -\kappa_1 & 0 & 0 & 0 & -d_z \kappa_1 & d_y \kappa_1 \\ 0 & -\kappa_2 & 0 & d_z \kappa_2 & 0 & d_{xb} \kappa_2 \\ 0 & 0 & -\kappa_3 & -d_y \kappa_3 & -d_{xb} \kappa_3 & 0 \\ 0 & d_z \kappa_2 & -d_y \kappa_3 & -(\kappa_4 + d_y^2 \kappa_3 + d_z^2 \kappa_2) & -d_{xb} d_y \kappa_3 & -d_{xb} d_z \kappa_2 \\ -d_z \kappa_1 & 0 & d_{xa} \kappa_3 & d_{xa} d_y \kappa_3 & -\kappa_5 + d_{xa} d_{xb} \kappa_3 - d_z^2 \kappa_1 & d_y d_z \kappa_1 \\ d_y \kappa_1 & -d_{xa} \kappa_2 & 0 & d_{xa} d_z \kappa_2 & d_y d_z \kappa_1 & -\kappa_6 + d_{xa} d_{xb} \kappa_2 - d_y^2 \kappa_1 \end{bmatrix}$$

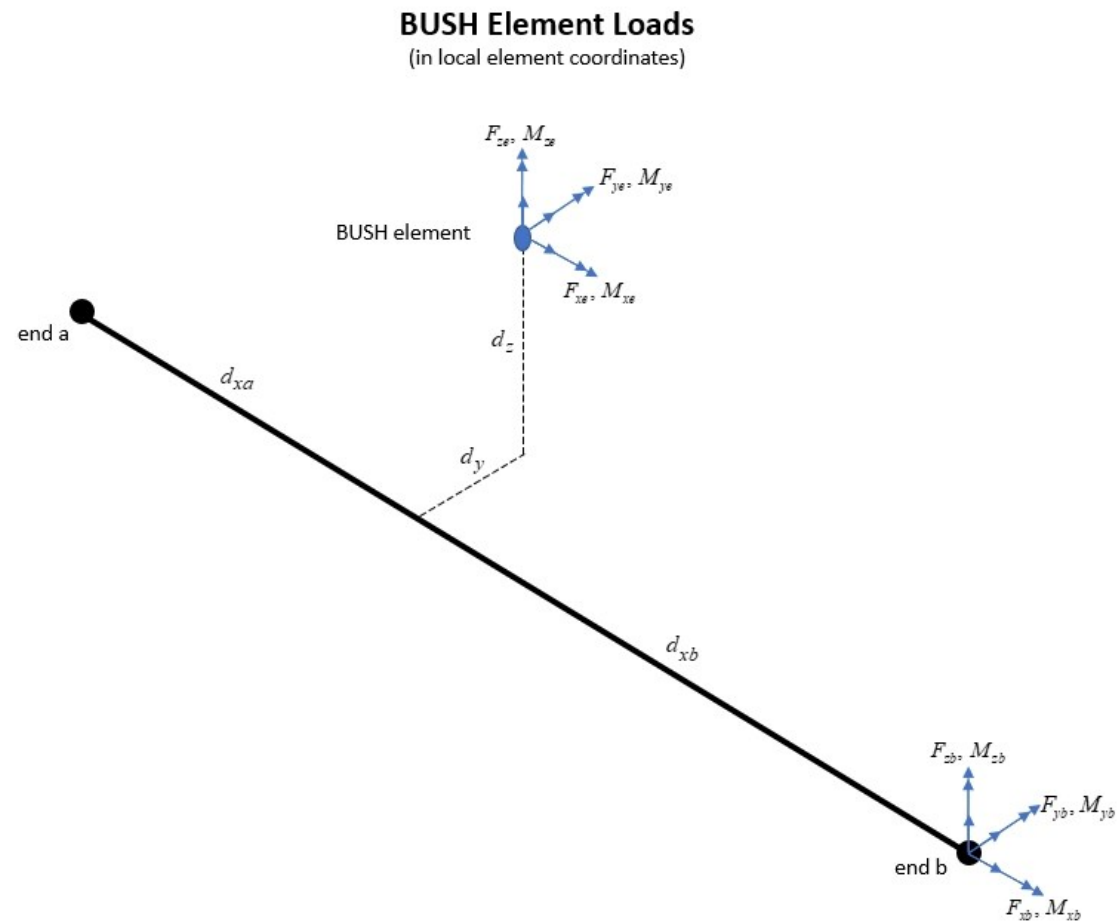
$$K_{bb} = \begin{bmatrix} \kappa_1 & 0 & 0 & 0 & d_z \kappa_1 & -d_y \kappa_1 \\ 0 & \kappa_2 & 0 & -d_z \kappa_2 & 0 & -d_{xb} \kappa_2 \\ 0 & 0 & \kappa_3 & d_y \kappa_3 & d_{xb} \kappa_3 & 0 \\ 0 & -d_z \kappa_2 & d_y \kappa_3 & \kappa_4 + d_y^2 \kappa_3 + d_z^2 \kappa_2 & d_{xa} d_y \kappa_3 & d_{xb} d_z \kappa_2 \\ d_z \kappa_1 & 0 & -d_{xb} \kappa_3 & d_{xb} d_y \kappa_3 & \kappa_5 + d_{xb}^2 \kappa_3 + d_z^2 \kappa_1 & -d_y d_z \kappa_1 \\ -d_y \kappa_1 & d_{xb} \kappa_2 & 0 & d_{xb} d_z \kappa_2 & -d_y d_z \kappa_1 & \kappa_6 + d_{xb}^2 \kappa_2 + d_y^2 \kappa_1 \end{bmatrix}$$

An image of the full 12x12 matrix with the above partitions is shown below:

$$K = \left[\begin{array}{cccccc|cccccc} K_1 & 0 & 0 & 0 & d_z K_1 & -d_y K_1 & -K_1 & 0 & 0 & 0 & -d_z K_1 & d_y K_1 \\ 0 & K_2 & 0 & -d_z K_2 & 0 & d_{yb} K_2 & 0 & -K_2 & 0 & d_z K_2 & 0 & d_{yb} K_2 \\ 0 & 0 & K_3 & d_y K_3 & -d_{yb} K_3 & 0 & 0 & 0 & -K_3 & -d_z K_3 & -d_{yb} K_3 & 0 \\ 0 & -d_z K_2 & d_y K_3 & K_4 + d_y^2 K_3 + d_z^2 K_2 & -d_{yb} d_z K_3 & -d_{yb} d_z K_2 & 0 & d_z K_2 & -d_y K_3 & -(K_4 + d_y^2 K_3 + d_z^2 K_2) & -d_{yb} d_y K_3 & -d_{yb} d_z K_2 \\ d_z K_1 & 0 & -d_{yb} K_3 & -d_{yb} d_z K_3 & K_5 + d_{yb}^2 K_3 + d_z^2 K_1 & -d_y d_z K_1 & -d_z K_1 & 0 & d_{yb} K_3 & d_{yb} d_z K_3 & -K_5 + d_{yb} d_{yb} K_1 - d_z^2 K_1 & d_y d_z K_1 \\ -d_y K_1 & d_{yb} K_2 & 0 & -d_{yb} d_z K_2 & -d_y d_z K_1 & K_6 + d_{yb}^2 K_2 + d_y^2 K_1 & d_y K_1 & -d_{yb} K_2 & 0 & d_{yb} d_z K_2 & d_y d_z K_1 & -K_6 + d_{yb} d_{yb} K_2 - d_y^2 K_1 \\ \hline -K_1 & 0 & 0 & 0 & -d_z K_1 & d_y K_1 & K_1 & 0 & 0 & 0 & d_z K_1 & -d_y K_1 \\ 0 & -K_2 & 0 & d_z K_2 & 0 & -d_{yb} K_2 & 0 & K_2 & 0 & -d_z K_2 & 0 & -d_{yb} K_2 \\ 0 & 0 & -K_3 & -d_y K_3 & d_{yb} K_3 & 0 & 0 & 0 & K_3 & d_y K_3 & d_{yb} K_3 & 0 \\ 0 & d_z K_2 & -d_y K_3 & -(K_4 + d_y^2 K_3 + d_z^2 K_2) & d_{yb} d_z K_3 & d_{yb} d_z K_2 & 0 & -d_z K_2 & d_y K_3 & K_4 + d_y^2 K_3 + d_z^2 K_2 & d_{yb} d_y K_3 & d_{yb} d_z K_2 \\ -d_z K_1 & 0 & -d_{yb} K_3 & -d_{yb} d_z K_3 & -K_5 + d_{yb}^2 K_3 + d_z^2 K_1 & d_y d_z K_1 & d_z K_1 & 0 & d_{yb} K_3 & d_{yb} d_z K_3 & K_5 + d_{yb}^2 K_3 + d_z^2 K_1 & -d_y d_z K_1 \\ d_y K_1 & d_{yb} K_2 & 0 & -d_{yb} d_z K_2 & d_y d_z K_1 & -K_6 + d_{yb} d_{yb} K_2 - d_y^2 K_1 & -d_y K_1 & -d_{yb} K_2 & 0 & d_{yb} d_z K_2 & -d_y d_z K_1 & K_6 + d_{yb}^2 K_2 + d_y^2 K_1 \end{array} \right]$$

Note that the partitions K_{aa} and K_{bb} are symmetric

The element engineering forces can be derived using the figure below:



The engineering forces in the BUSH element are:

$$\begin{aligned}
 F_{xe} &= F_{xb} \\
 F_{ye} &= F_{yb} \\
 F_{ze} &= F_{zb} \\
 M_{xe} &= F_{yb}d_z - F_{zb}d_y + M_{xb} \\
 M_{ye} &= -F_{xb}d_z - F_{zb}d_{xb} + M_{yb} \\
 M_{ze} &= F_{xb}d_y + F_{yb}d_{xb} + M_{zb}
 \end{aligned}$$

This can be put into a form which includes all nodal forces as:

$$\begin{Bmatrix} F_{xe} \\ F_{ye} \\ F_{ze} \\ M_{xe} \\ M_{ye} \\ M_{ze} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_z & -d_y & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_z & 0 & -d_{xb} & 0 & . & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_y & d_{xb} & 0 & 0 & 0 & . \end{bmatrix} \begin{Bmatrix} F_{xa} \\ F_{ya} \\ F_{za} \\ M_{xa} \\ M_{ya} \\ M_{za} \\ F_{xb} \\ F_{yb} \\ F_{zb} \\ M_{xb} \\ M_{yb} \\ M_{zb} \end{Bmatrix}$$

The 6x12 transformation matrix in the above equation is used in the MYSTRAN code to transform the element nodal forces to element engineering forces

The engineering forces in the BUSH element are:

$$\begin{aligned}
 F_{xe} &= F_{xa} \\
 F_{ye} &= F_{ya} \\
 F_{ze} &= F_{za} \\
 M_{xe} &= F_{ya}d_z - F_{za}d_y + M_{xa} \\
 M_{ye} &= -F_{xa}d_z + F_{za}d_{xa} + M_{ya} \\
 M_{ze} &= F_{xa}d_y - F_{yb}d_{xa} + M_{za}
 \end{aligned}$$

This can be put into a form which includes all nodal forces as:

$$\begin{Bmatrix} F_{xe} \\ F_{ye} \\ F_{ze} \\ M_{xe} \\ M_{ye} \\ M_{ze} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d_z & -d_y & 1 & 0 & 0 \\ -d_z & 0 & d_{xa} & 0 & 1 & 0 \\ d_y & -d_{xa} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_{xa} \\ F_{ya} \\ F_{za} \\ M_{xa} \\ M_{ya} \\ M_{za} \end{Bmatrix}$$

The 6x transformation matrix in the above equation is used in the MYSTRAN code to transform the element nodal forces to element engineering forces