9 Appendix C: Equations for element stress/strain recovery

9.1 General discussion

For the 2D plate elements and 3D solid elements arrays called STRAIN and STRESS are calculated for each element. For 1D elements.. like the rod and beam. only the STRESS array is calculated. Both arrays STRAIN and STRESS can contain up to 9 rows and there is one of each these calculated for every subcase. The STRAIN and STRESS arrays are further subdivided as shown below:

$$\text{STRAIN} = \begin{Bmatrix} \text{STRAIN}_1 \\ \text{STRAIN}_2 \\ \text{STRAIN}_3 \end{Bmatrix} \quad , \quad \text{STRESS} = \begin{Bmatrix} \text{STRESS}_1 \\ \text{STRESS}_2 \\ \text{STRESS}_3 \end{Bmatrix} \tag{9-1}$$

where STRAIN_i and STRESS_i each have 3 rows

for 2D and 3D elements:

$$\text{STRAIN}_I = (\text{BEi}) * U_e$$

and

$$\text{STRESS}_i = (\text{DE}_i) * \text{STRAIN}_I - (\text{STEi}) \tag{9-2}$$

for 1D elements stresses are calculated directly from displacements:

$$\text{STRESS}_i = (\text{SEi}) * U_e - (\text{STEi})$$

U_e are the displacements of the nodes of the element in the local element coordinate system (see Figures 3-2 through 3-6 in the main body of this manual) and are obtained from the G-set displacements, the solution for which is discussed in Appendix B. These G-set displacements for the nodes of an element are transformed to the local element coordinate system to obtain U_e which has a number of rows equal to 6n where n is the number of nodes for the element (e.g. n=4 for a quadrilateral plate element). There is one U_e for each subcase in the solution. The BE_i arrays each have 3 rows and 6n columns and are based on the strain-displacement relationships for individual elements. The SE_i are equal to material matrices times the BE_i. The STEi arrays contain the thermal stress effects, if there are any, and have 3 rows and as many columns as there are <u>thermal</u> subcases.. That is, if the input data deck has 5 subcases and two of these have thermal loads, then STE_i will have only 2 columns while U_e will have 5 columns. If a user outputs the SE_i and STE_i arrays, it is their responsibility to keep track of which subcases the columns of STE_i belong. MYSTRAN does this internally for its stress output calculations.

The following sections show what is contained in arrays $STRESS_i$ for each of the element types. In that manner, it will be obvious how MYSTRAN uses the SEi and STEi arrays, generated internally in MYSTRAN, to obtain stresses. If desired, they are available to be output to a text or unformatted binary file through use of the Case Control entry ELDATA. They need not be output for the user to obtain element stresses, however, which are available in the normal text output file through use of the Case Control entry STRESS.

9.2 Rod element

The rod geometry and loading is shown in Figure 3-2 in the main body of this manual. It is a very simple element and has only two stresses that can be output: the axial stress and the torsional stress. It only uses the first 2 rows of array $STRESS_1$ with row 1 being the axial stress and row 2 the torsional stress. Array $STRESS_1$ is:

$$STRESS_1 = \begin{Bmatrix} \sigma_{axial} \\ \tau \\ 0 \end{Bmatrix} \tag{9-3}$$

As an example of what is in arrays SE1 and STE1 for a simple element, the arrays are shown below for this rod element. More complicated elements won't have a simple closed form for these matrices and will not be shown.

Array SE1 for the rod element is:

$$SE1 = \frac{1}{L}\begin{bmatrix} -E & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C*G & 0 & 0 & 0 & 0 & 0 & C*G & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9-4}$$

E and G are Young's modulus and shear modulus from the Bulk Data material entry for the element, L is the element length and C is the torsional stress recovery coefficient from a PROD entry.

Array STE1 would have the following column for each subcase that has a thermal load:

$$STE1 = E\alpha(\bar{T} - T_{ref})\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{9-5}$$

α and T_{ref} are the coefficient of thermal expansion and reference temperature from the material Bulk Data entry for the element and $\bar{T}$ is the average element temperature for the thermal subcase.

9.3 Bar element

The bar element geometry and loading is shown in Figures 3-3 and 3-4 in the main body of this manual. For the bar element, array STRESS uses all 3 rows of $STRESS_1$ and $STRESS_2$. The first row of $STRESS_1$ contains the actual axial stress in the bar and the third row of $STRESS_2$ contains the actual torsional stress. The second and third rows of $STRESS_1$ and the first two rows of $STRESS_2$ are not actual stress values. Rather, they are the four independent parameters needed to determine the bending stresses at points in the bar cross-section. Thus:

$$STRESS_1 = \begin{Bmatrix} \sigma_{axial} \\ \kappa'_{1a} \\ \kappa'_{1b} \end{Bmatrix} \quad , \quad STRESS_2 = \begin{Bmatrix} \kappa'_{2a} \\ \kappa'_{2b} \\ \tau \end{Bmatrix}$$

where

$$\kappa'_{1a} = \frac{M_{1a}I_2 - M_{2a}I_{12}}{I_1I_2 - I_{12}^2} \quad , \quad \kappa'_{1b} = \frac{M_{1b}I_2 - M_{2b}I_{12}}{I_1I_2 - I_{12}^2}$$
$$\kappa'_{2a} = \frac{M_{2a}I_1 - M_{1a}I_{12}}{I_1I_2 - I_{12}^2} \quad , \quad \kappa'_{2b} = \frac{M_{2b}I_1 - M_{1b}I_{12}}{I_1I_2 - I_{12}^2} \tag{9-6}$$

and

$\sigma_{axial} =$ Axial stress at the neutral axis

$\tau =$ Torsional stress

$I_1, I_2, I_{12} =$ the moments of inertia of the bar on the PBAR entry for this bar element (9-7)

$M_{1a}, M_{2a}, M_{1b}, M_{2b} =$ the moments in planes 1 and 2 at ends a and b of the bar

This can be put into the form of equation 9.2 as:

$$
\begin{aligned}
&STRESS_1 = SE1 * U_e - STE1 \\
&STRESS_2 = SE2 * U_e - STE2 \\
&\text{where} \\
&SE1 = \begin{bmatrix} B_1K_{aa} & B_1K_{ab} \end{bmatrix} \quad , \quad STE1 = B_1K_{aa}\bar{A}T' \\
&SE2 = \begin{bmatrix} B_2K_{aa} & B_2K_{ab} \end{bmatrix} \quad , \quad STE1 = B_2K_{aa}\bar{A}T'
\end{aligned}
$$

K_{aa} and K_{ab} are 6x6 partitions from the 1st 6 rows of the bar element stiffness matrix and B_1, B_2 and $\bar{A}$
are matrices of element properties as shown below:

$$
B_1 = \begin{bmatrix} -1/A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta_{12} & -\Delta_1 \\ 0 & 0 & 0 & 0 & \Delta_2 & \Delta_{12} \end{bmatrix}
$$

$$
B_2 = \begin{bmatrix} 0 & \Delta_1 L & -\Delta_{12}L & 0 & -\Delta_{12} & -\Delta_1 \\ 0 & -\Delta_{12}L & \Delta_2 L & 0 & \Delta_2 & \Delta_{12} \\ 0 & 0 & 0 & -C/J & 0 & 0 \end{bmatrix}
$$

$$
\bar{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & L\Delta_1 I_1/6 & L\Delta_1 I_1/3 & -L\Delta_{12} I_2/6 & L\Delta_{12} I_2/3 \\ 0 & -L\Delta_{12} I_1/6 & -L\Delta_{12} I_1/3 & L\Delta_2 I_2/6 & L\Delta_2 I_2/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\Delta_{12} I_1/2 & -\Delta_{12} I_1/2 & \Delta_2 I_2/2 & \Delta_2 I_2/2 \\ 0 & -\Delta_1 I_1/2 & -\Delta_1 I_1/2 & \Delta_{12} I_2/2 & \Delta_{12} I_2/2 \end{bmatrix} -
$$

and

$$
T' = \begin{Bmatrix} \bar{T} - T_{ref} \\ T'_{1a} \\ T'_{1b} \\ T'_{2a} \\ T'_{2b} \end{Bmatrix} = \begin{Bmatrix} \text{avg bulk temp above material ref temp} \\ \text{gradient through bar in plane 1 at end a} \\ \text{gradient through bar in plane 1 at end b} \\ \text{gradient through bar in plane 2 at end a} \\ \text{gradient through bar in plane 2 at end b} \end{Bmatrix}
$$

with the following bar properties:

$L =$ bar length
$A =$ cross-sectional area
$I_1 =$ area moment of inertia in plane 1
$I_2 =$ area moment of inertia in plane 1
$I_{12} =$ product of inertia

$$\Delta_1 = \frac{I_2}{I_1 I_2 - I_{12}^2}$$

$$\Delta_2 = \frac{I_1}{I_1 I_2 - I_{12}^2}$$

$$\Delta_{12} = \frac{I_{12}}{I_1 I_2 - I_{12}^2}$$

Stresses due to bending (i.e. not including axial stress at the neutral axis) at ends a and b of the bar element are obtained from:

$$\sigma_a = -(\kappa'_{1a}\bar{y}_e + \kappa'_{2a}\bar{z}_e) \quad , \quad \sigma_b = -(\kappa'_{1b}\bar{y}_e + \kappa'_{2b}\bar{z}_e) \tag{9-8}$$

where σ_a , σ_b are the <u>bending</u> stresses at ends a and b of the bar and $\bar{y}_e$, $\bar{z}_e$ are the coordinates of a point on the bar cross section as measured in the local element coordinate system (see Figure 3-3 in the main body of this manual). It should be noted that temperature distributions through the depth of the bar that are higher order than linear are ignored

9.4 Plate elements

Triangular and quadrilateral plate element geometry, loading and stress conventions are shown in Figures 3-5 and 3-6 in the main body of this manual. They can use all three of the $STRESS_i$ arrays.

9.4.1 Membrane stresses

$STRESS_1$ contains the membrane stresses (at the plate mid-plane)

$$STRESS_1 = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{z=0} \tag{9-9}$$

This can be put into the form of equation 9.2 as:

$$\begin{aligned} &STRESS_1 = (SE1) * U_e - (STE1) \\ &\text{where} \\ &SE1 = E_m B_m \quad \text{and} \qquad STE1 = E_m \alpha (T - T_{ref}) \end{aligned} \tag{9-10}$$

E_m is the 3x3 membrane material matrix, B_m is the element membrane strain-displacement matrix (developed internally in MYSTRAN), α is the 3x1 vector of coefficients of thermal expansion for the material, T is the element average bulk temperature and T_{ref} is the reference temperature for the element material.

9.4.2 Bending stresses

$STRESS_2$, times a fiber distance, contains the stresses due to bending, where:

$$STRESS_2 = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{9-11}$$

This can be put into the form of equation 9.2 as:

$$\begin{aligned} &STRESS_2 = (SE2) * U_e - (STE2) \\ &\text{where} \\ &SE2 = E_b B_b \quad \text{and} \quad STE2 = E_b \alpha T' \end{aligned} \tag{9-12}$$

E_b is the 3x3 bending material matrix, B_b is the element bending strain-displacement matrix (developed internally in MYSTRAN), α is the 3x1 vector of coefficients of thermal expansion for the material and T' is the temperature gradient through the thickness of the plate element.

9.4.3 Combined membrane and bending stresses

The total bending and in-plane shear stresses at a fiber distance z are obtained from $STRESS_1$ and $STRESS_2$ as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = STRESS_1 + z(STRESS_2) \tag{9-13}$$

9.4.4 Transverse shear stresses

The average transverse shear stresses through the thickness of the plate (for TRIA3 and QUAD4 elements only) are obtained from $STRESS_3$:

$$STRESS_3 = \begin{Bmatrix} \tau_{zx} \\ \tau_{zy} \\ 0 \end{Bmatrix} \tag{9-14}$$

This can be put into the form of equation 9.2 as

$$STRESS_3 = SE3$$

where

$$SE3 = E_s B_s$$

E_s is the 3x3 transverse shear material matrix and B_s is the element transverse shear strain-displacement matrix (developed internally in MYSTRAN).

The transverse shear stresses are not output in the normal output file even if stress output is requested in Case Control. However, the transverse shear stress resultants (integrals of shear stress through thickness) are output if there is a request in Case Control for element engineering forces

9.5 Solid elements

For the 3D solid elements HEXA, PENTA and TETRA arrays STRAIN and STRESS contain only the 6 actual strains and stresses for a 3D solid:

$$\text{STRAIN} = (\text{BE}) * \text{U}_e = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

The BE are strain-displacement matrices that are based on the shape functions chosen for the particular 3D solid element. Once the strains have been calculated the stresses are determined from:

$$\text{STRESS} = (\text{ES}) * (\text{STRAIN} - \text{ALPT}) = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

ES is the 6x6 material matrix for a solid and ALPT is the thermal distortion portion of the strains. For a homogeneous isotropic material these are:

$$\text{ES} = \begin{bmatrix} (1-\upsilon)E_0 & \upsilon E_0 & \upsilon E_0 & 0 & 0 & 0 \\ \upsilon E_0 & (1-\upsilon)E_0 & \upsilon E_0 & 0 & 0 & 0 \\ \upsilon E_0 & \upsilon E_0 & (1-\upsilon)E_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} , \quad E_0 = \frac{E}{(1+\upsilon)(1-2\upsilon)} \quad \text{and} \quad G = \frac{E}{2(1+\upsilon)}$$

$$\text{ALPT} = \begin{Bmatrix} \alpha \\ \alpha \\ \alpha \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

MYSTRAN does allow anisotropic element properties for solids and, in that case, ES and ALPT are different