RBE3 Equations

Pt A is the ref. point where loads to be distributed are applied

i is a typical point to which loads will be transfered to via the RBE3. (There are N points i, located at distances d_{x_i}, d_{y_i}, d_{z_i} from the ref. pt A

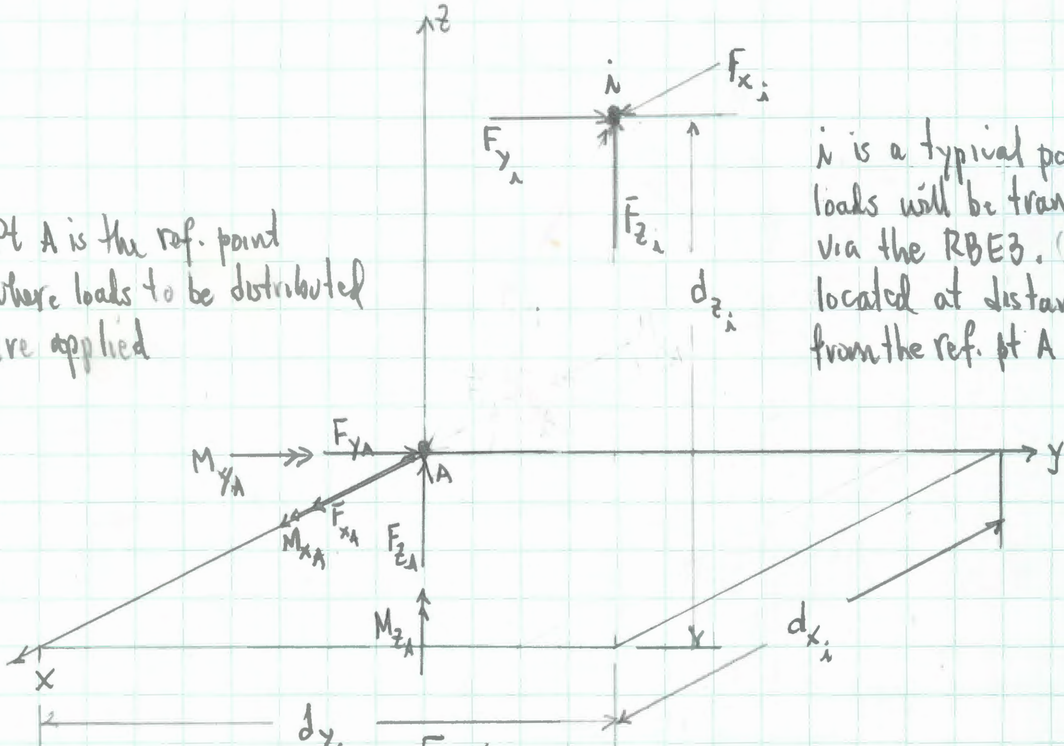


Fig 1

The forces on the i points must add up to the forces on A:

$$\sum_{i=1}^{N} F_{x_i} = F_{x_A} \qquad \sum_{i=1}^{N} F_{y_i} = F_{y_A} \qquad \sum_{i=1}^{N} F_{z_i} = F_{z_A} \qquad (1)\ a,b,c$$

Moments at A due to the forces at i:

$$\sum_{i=1}^{N} (F_{z_i} d_{y_i} - F_{y_i} d_{z_i}) = M_{x_A}, \quad \sum_{i=1}^{N} (F_{x_i} d_{z_i} - F_{z_i} d_{x_i}) = M_{y_A}, \quad \sum_{i=1}^{N} (F_{y_i} d_{x_i} - F_{x_i} d_{y_i}) = M_{z_A} \qquad (2)\ a,b,c$$

Assume that the F_{x_i}, etc can be written as

$$F_{x_i} = \frac{\omega_i}{\sum_{i=1}^{N} \omega_i} F_{x_A}, \quad F_{y_i} = \frac{\omega_i}{\sum_{i=1}^{N} \omega_i} F_{y_A}, \quad F_{z_i} = \frac{\omega_i}{\sum_{i=1}^{N} \omega_i} F_{z_A} \qquad (3)\ a,b,c$$

Define $\qquad W_T = \sum_{i=1}^{N} \omega_i \qquad$ (ω_i are weighting factors) $\qquad (4)$

Substitute (3) into (1) and we see (1) is satisfied since, for example

$$\sum_{i=1}^{N} F_{x_i} = \frac{\sum \omega_i F_{x_A}}{\sum \omega_i} = F_{x_A} \quad (\text{since } F_{x_A} \text{ constant})$$

Substitute (3)a, (4) into (2) a:

$$+\sum^{N}\left(\frac{w_i}{W_T}F_{z_A}dy_i - \frac{w_i}{W_T}F_{y_A}dz_i\right) = M_{x_A}$$

or

$$+\left[\frac{F_{zA}}{W_T}\sum^{N} w_i dy_i - \frac{F_{YA}}{W_T}\sum^{N} w_i dz_i\right] = M_{x_A} \qquad (5)$$

Substitute (3)b, 4 into (2) b:

$$+\left[\frac{F_{xA}}{W_T}\sum^{N} w_i d_{z_i} - \frac{F_{zA}}{W_T}\sum^{N} w_i dx_i\right] = M_{Y_A} \qquad (6)$$

Substitute (3)c, (4) into (2)c:

$$+\left[\frac{F_{YA}}{W_T}\sum^{N} w_i dx_i - \frac{F_{xA}}{W_T}\sum^{N} w_i d_{y_i}\right] = M_{z_A} \qquad (7)$$

<u>Define</u>

$$\bar{d}_x = \frac{1}{W_T}\sum_{i=1}^{N} w_i dx_i \quad , \bar{d}_{y_i} = \frac{1}{W_T}\sum_{i=1}^{N} w_i d_{y_i} \quad , \bar{d}_{z_i} = \frac{1}{W_T}\sum_{i=1}^{N} w_i d_{z_i} \qquad (8)$$

Then (5) - (7) become

$$\left(F_{z_A}\bar{d}_y - F_{Y_A}\bar{d}_z\right) = M_{x_A} \qquad \cancel{(8)}\ (9)$$

$$\left(F_{x_A}\bar{d}_z - F_{z_A}\bar{d}_x\right) = M_{Y_A} \qquad \cancel{(9)}\ (10)$$

$$\left(F_{Y_A}\bar{d}_x - F_{x_A}\bar{d}_y\right) = M_{z_A} \qquad \cancel{(10)}\ (11)$$

The total work done by the forces and moments on A must equal the work on the i points (to which F_{xA}, etc have been distributed) due to the $F_{x_i}, F_{Y_i}, F_{z_i}$

The work due to forces, moments on A is

$$W_A = F_{x_A} u_{x_A} + F_{y_A} u_{y_A} + F_{z_A} u_{z_A} + M_{x_A} \theta_{x_A} + M_{y_A} \theta_{y_A} + M_{z_A} \theta_{z_A} \qquad \cancel{(11)}\ (12)$$

where $u_{x_A}, \cdots \theta_{z_A}$ are the displ's and rotations at the ref. pt. A

The work done by the $F_{x_i}, F_{y_i}, F_{z_i}$ is

$$W_N = \sum_{i=1}^{N} \left(F_{x_i} u_{x_i} + F_{y_i} u_{y_i} + F_{z_i} u_{z_i} \right) \qquad \cancel{(12)}\ (13)$$

where $u_{x_i}, u_{y_i}, u_{z_i}$ are the 3 translation displ components at i

Substitute (3) into ~~(12)~~ (13) and ~~(8)~~-~~(10)~~ (9)-(11) into ~~(11)~~ (12) and equate $W_A = W_N$

$$F_{x_A} u_{x_A} + F_{y_A} u_{y_A} + F_{z_A} u_{z_A} + (F_{z_A} \bar{d}_y - F_{y_A} \bar{d}_z)\, \theta_{x_A} + (F_{x_A} \bar{d}_z - F_{z_A} \bar{d}_x)\, \theta_{y_A} + (F_{y_A} \bar{d}_x - F_{x_A} \bar{d}_y)\, \theta_{z_A} =$$
$$\sum_{i=1}^{N} \left(\frac{\omega_i}{W_T} F_{x_A} u_{x_i} + \frac{\omega_i}{W_T} F_{y_A} u_{y_i} + \frac{\omega_i}{W_T} F_{z_A} u_{z_i} \right)$$

Regroup

$$\left(u_{x_A} + \bar{d}_z \theta_{y_A} - \bar{d}_y \theta_{z_A} - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_{x_i} \right) F_{x_A} + \left(u_{y_A} - \bar{d}_z \theta_{x_A} + \bar{d}_x \theta_{z_A} - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_{y_i} \right) F_{y_A}$$
$$+ \left(u_{z_A} + \bar{d}_y \theta_{x_A} - \bar{d}_x \theta_{y_A} - \sum_{i=1}^{N} \frac{\omega_i}{W_T} u_{z_i} \right) F_{z_A} = 0 \qquad \cancel{(13)}\ (14)$$

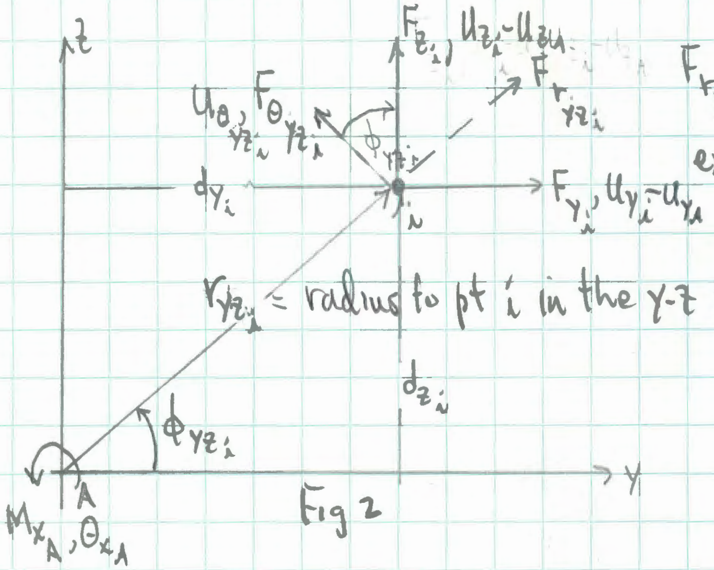
Since the $F_{x_A}, F_{y_A}, F_{z_A}$ are independent and, in general, not zero, (13) requires

$$\boxed{\begin{aligned} u_{x_A} + \bar{d}_z \theta_{y_A} - \bar{d}_y \theta_{z_A} - \sum \frac{\omega_i}{W_T} u_{x_i} &= 0 \\ u_{y_A} - \bar{d}_z \theta_{x_A} + \bar{d}_x \theta_{z_A} - \sum \frac{\omega_i}{W_T} u_{y_i} &= 0 \\ u_{z_A} + \bar{d}_y \theta_{x_A} - \bar{d}_x \theta_{y_A} - \sum \frac{\omega_i}{W_T} u_{z_i} &= 0 \end{aligned}} \qquad \cancel{(14)}\ (15)$$

There are 3 constraint eqns for the RBE3 (but we need others for θ's, since (14) represents 3 eqns but with 6 unknowns

There are also eqns relating the $F_{x_i}, F_{y_i}, F_{z_i}$ to the actual moments at A.

To see this look at the forces in 1 plane (say y-z):



$F_{r_{yz_i}}, F_{\theta_{yz_i}}$ are components of F_{y_i}, F_{z_i} expressed in an r-θ coord. system

Fig 2

Using F_{r_i}, F_{θ_i} components we can take moments about A to get

$$\sum_{i=1}^{N} F_{\theta_{yz_i}} r_{yz_i} = M_{x_A} \qquad (16)$$

If we take

$$F_{\theta_{yz_i}} = \frac{w_i r_{yz_i}}{\sum w_i r_{yz_i}^2} M_{x_A} \qquad (17)$$

and subst (17) into (16) we see that

$$\sum \left(\frac{w_i r_{yz_i}}{\sum w_i r_{yz_i}^2} \right) M_{x_A} r_i = M_{x_A}$$

or

$$\frac{\sum w_i r_{yz_i}^2}{\sum w_i r_{yz_i}^2} M_{x_A} = M_{x_A}$$

$$\therefore M_{x_A} = M_{x_A}$$

so that (17) is a valid expression for the F_{θ_i} components of force

The work done by M_{x_A} must equal that due to all of the F_{θ_i}:

$$\sum F_{\theta_{yz_i}} u_{\theta_{yz_i}} = M_{x_A}\theta_{x_A} \qquad \cancel{(17)}\ (18)$$

where $u_{\theta_{yz_i}}$ is the tangential displ at i in the y-z plane

Subst ~~(16)~~ (17) into ~~(17)~~ (18)

$$\sum \frac{\omega_i r_{yz_i}}{\sum \omega_i r^2_{yz_i}} M_{x_A} u_{\theta_{yz_i}} = \cancel{M_{x_A}}\theta_{x_A}$$

or

$$\theta_{x_A} = \frac{\sum \omega_i r_{yz_i} u_{\theta_{yz_i}}}{\sum \omega_i r^2_{yz_i}} \qquad \cancel{(18)}\ (19)$$

similarly for the x-z and x-y planes we can get

$$\theta_{y_A} = \frac{\sum \omega_i r_{zx_i} u_{\theta_{zx_i}}}{\sum \omega_i r^2_{zx_i}} \qquad (20)\ \cancel{(19)}$$

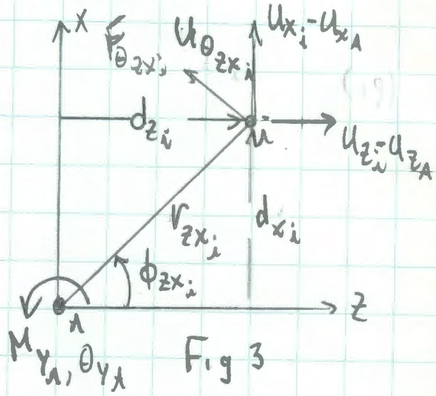


Fig 3

and

$$\theta_{z_A} = \frac{\sum \omega_i r_{xy_i} u_{\theta_{xy_i}}}{\sum \omega_i r^2_{xy_i}} \qquad (21)\ \cancel{(20)}$$

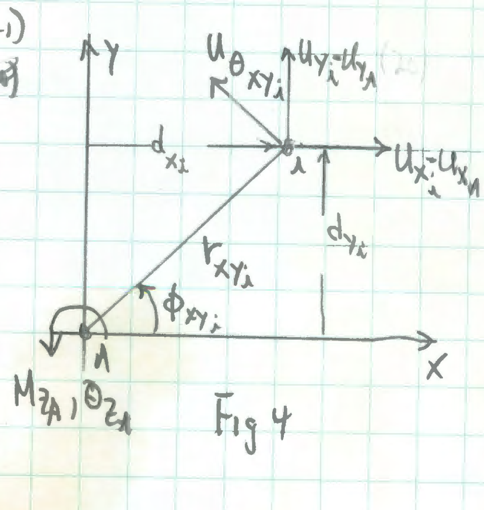


Fig 4

We can rewrite ~~(18)~~ (19) - (21) ~~(20)~~ in terms of $u_{x_i}, u_{y_i}, u_{z_i}$ (rel to $u_{x_A}, u_{y_A}, u_{z_A}$) instead of the u_θ. From Fig 2:

$$u_{\theta_{yz_i}} = (u_{z_i} - u_{z_A})\cos\phi_{yz_i} - (u_{y_i} - u_{y_A})\sin\phi_{yz_i}$$

$$= (u_{z_i} - u_{z_A})\frac{d_{y_i}}{r_{yz_i}} - (u_{y_i} - u_{y_A})\frac{d_{z_i}}{r_{yz_i}}$$

$$\therefore r_{yz_i} u_{\theta_{yz_i}} = (u_{z_i} - u_{z_A}) d_{y_i} - (u_{y_i} - u_{y_A}) d_{z_i} \qquad \cancel{(21)}\ (22)$$

From Fig 3

$$u_{\theta_{zx_i}} = (u_{x_i} - u_{x_A})\cos\phi_{zx_i} - (u_{z_i} - u_{z_A})\sin\phi_{zx_i}$$

$$= (u_{x_i} - u_{x_A})\frac{d_{z_i}}{r_{zx_i}} - (u_{z_i} - u_{z_A})\frac{d_{x_i}}{r_{zx_i}}$$

$$\therefore r_{zx_i} u_{\theta_{zx_i}} = (u_{x_i} - u_{x_A}) d_{z_i} - (u_{z_i} - u_{z_A}) d_{x_i} \qquad \cancel{(22)}\ (23)$$

From Fig 4

$$u_{\theta_{xy_i}} = (u_{y_i} - u_{y_A})\cos\phi_{xy_i} - (u_{x_i} - u_{x_A})\sin\phi_{xy_i}$$

$$= (u_{y_i} - u_{y_A})\frac{d_{x_i}}{r_{xy_i}} - (u_{x_i} - u_{x_A})\frac{d_{y_i}}{r_{xy_i}}$$

$$\therefore r_{xy_i} u_{\theta_{xy_i}} = (u_{y_i} - u_{y_A}) d_{x_i} - (u_{x_i} - u_{x_A}) d_{y_i} \qquad \cancel{(23)}\ (24)$$

where

$$\left.\begin{aligned} r_{xy_i} &= \sqrt{d_{x_i}^2 + d_{y_i}^2} \\ r_{yz_i} &= \sqrt{d_{y_i}^2 + d_{z_i}^2} \\ r_{zx_i} &= \sqrt{d_{z_i}^2 + d_{x_i}^2} \end{aligned}\right\} \qquad \cancel{(24)}\ (25)$$

Define

$$\bar{e}_{XY} = \frac{1}{W_T}\sum^{N} \omega_i r^2_{XY_i} = \frac{1}{W_T}\sum^{N} \omega_i (d^2_{X_i} + d^2_{Y_i})$$

$$\bar{e}_{YZ} = \frac{1}{W_T}\sum^{N} \omega_i r^2_{YZ_i} = \frac{1}{W_T}\sum^{N} \omega_i (d^2_{Y_i} + d^2_{Z_i})$$

$$\bar{e}_{ZX} = \frac{1}{W_T}\sum^{N} \omega_i r^2_{ZX_i} = \frac{1}{W_T}\sum^{N} \omega_i (d^2_{Z_i} + d^2_{X_i})$$

~~(23)~~ (24)

Using ~~(8), (10), (21), (25) in (18)~~ (9), (11), (22), (26) in (19)

$$\Theta_{X_A} = \frac{1}{W_T \bar{e}_{YZ}}\left[\sum^{N} \omega_i (u_{Z_i} - u_{Z_A}) d_{Y_i} - \sum^{N} \omega_i (u_{Y_i} - u_{Y_A}) d_{Z_i}\right]$$

$$= \frac{1}{W_T \bar{e}_{YZ}}\left[-\left(\sum^{N} \omega_i d_{Y_i}\right) u_{Z_A} + \left(\sum^{N} \omega_i d_{Z_i}\right) u_{Y_A} + \sum \omega_i d_{Y_i} u_{Z_i} - \sum \omega_i d_{Z_i} u_{Y_i}\right]$$

$$\therefore \Theta_{X_A} = \frac{1}{W_T \bar{e}_{YZ}}\left[-W_T \bar{d}_Y u_{Z_A} + W_T \bar{d}_Z u_{Y_A} + \sum^{N} \omega_i d_{Y_i} u_{Z_i} - \sum^{N} \omega_i d_{Z_i} u_{Y_i}\right] \qquad \text{~~(26)~~ (27)}$$

Using ~~(8), (10), (22), (25) in (19)~~ (9) (11) (23) (26) in (20):

$$\Theta_{Y_A} = \frac{1}{W_T \bar{e}_{ZX}}\left[\sum^{N} \omega_i (u_{X_i} - u_{X_A}) d_{Z_i} - \sum^{N} \omega_i (u_{Z_i} - u_{Z_A}) d_{X_i}\right]$$

$$\therefore \Theta_{Y_A} = \frac{1}{W_T \bar{e}_{ZX}}\left[-W_T \bar{d}_Z u_{X_A} + W_T \bar{d}_X u_{Z_A} + \sum^{N} \omega_i d_{Z_i} u_{X_i} - \sum^{N} \omega_i d_{X_i} u_{Z_i}\right] \qquad \text{~~(27)~~ (28)}$$

Using ~~(8), (10), (23), (25) in (20)~~ (9) (11) (24) (26) in (22)

$$\Theta_{Z_A} = \frac{1}{W_T \bar{e}_{XY}}\left[\sum^{N} \omega_i (u_{Y_i} - u_{Y_A}) d_{X_i} - \sum^{N} \omega_i (u_{X_i} - u_{X_A}) d_{Y_i}\right]$$

$$\therefore \Theta_{Z_A} = \frac{1}{W_T \bar{e}_{XY}}\left[-W_T \bar{d}_X u_{Y_A} + W_T \bar{d}_Y u_{X_A} + \sum^{N} \omega_i d_{X_i} u_{Y_i} - \sum^{N} \omega_i d_{Y_i} u_{X_i}\right] \qquad \text{~~(28)~~ (29)}$$

The constraint eqns on $U_{X_A}, U_{Y_A}, U_{Z_A}, \theta_{X_A}, \theta_{Y_A}, \theta_{Y_Z}$ are summarized below:

$$u_{X_A} + \bar{d}_z \theta_{Y_A} - \bar{d}_y \theta_{Z_A} - \frac{1}{W_T}\sum^N \omega_i u_{X_i} = 0$$

$$u_{Y_A} - \bar{d}_z \theta_{X_A} + \bar{d}_x \theta_{Z_A} - \frac{1}{W_T}\sum^N \omega_i u_{Y_i} = 0$$

$$u_{Z_A} + \bar{d}_y \theta_{X_A} - \bar{d}_x \theta_{Y_A} - \frac{1}{W_T}\sum \omega_i u_{Z_i} = 0$$

$$\bar{e}_{YZ}\theta_{X_A} - \bar{d}_z u_{Y_A} + \bar{d}_y u_{Z_A} + \frac{1}{W_T}\sum^N \omega_i d_{Z_i} u_{Y_i} - \frac{1}{W_T}\sum^N \omega_i d_{Y_i} u_{Z_i} = 0$$

$$\bar{e}_{ZX}\theta_{Y_A} + \bar{d}_z u_{X_A} - \bar{d}_x u_{Z_A} - \frac{1}{W_T}\sum^N \omega_i d_{Z_i} u_{X_i} + \frac{1}{W_T}\sum \omega_i d_{X_i} u_{Z_i} = 0$$

$$\bar{e}_{XY}\theta_{Z_A} - \bar{d}_y u_{X_A} + \bar{d}_x u_{Y_A} - \frac{1}{W_T}\sum^N \omega_i d_{Y_i} u_{X_i} - \frac{1}{W_T}\sum \omega_i d_{X_i} u_{Y_i} = 0$$

where

a) $W_T = \sum^N \omega_i$, $\bar{e}_{XY} = \frac{1}{W_T}\sum^N \omega_i (d_{X_i}^2 + d_{Y_i}^2)$, $\bar{e}_{YZ} = \frac{1}{W_T}\sum^N \omega_i (d_{Y_i}^2 + d_{Z_i}^2)$, $\bar{e}_{ZX} = \frac{1}{W_T}\sum^N \omega_i (d_{Z_i}^2 + d_{X_i}^2)$

b) $\bar{d}_x = \frac{1}{W_T}\sum^N \omega_i d_{X_i}$, $\bar{d}_y = \frac{1}{W_T}\sum^N \omega_i d_{Y_i}$, $\bar{d}_z = \frac{1}{W_T}\sum^N \omega_i d_{Z_i}$

c) $d_{X_i}, d_{Y_i}, d_{Z_i}$ are the coords of pt i relative to pt A in the global coord. sys. at A

d) $u_{X_i}, u_{Y_i}, u_{Z_i}$ are displ's at i transformed from global coord sys. at i to global coord sys at A

e) $u_{X_A}, u_{Y_A}, u_{Z_A}, \theta_{X_A}, \theta_{Y_A}, \theta_{Z_A}$ are displ's at A in the global coord. sys. at A (obviously)

f) The weights ω_i are assigned by the user (and are often 1.0)