The BAR Element

I. Stiffness Matrix

The equations for the BAR element are

$$K U = P + P_T \qquad \begin{cases} P = \text{applied loads + constraints} \\ P_T = \text{equiv. thermal loads} \end{cases} \qquad \text{I-(1)}$$

Define:

$$k_a = \frac{AE}{\ell}, \quad k_t = \frac{GJ}{\ell}, \quad \beta = \frac{12EI_{12}}{\ell^3}$$

$$\hat{k}_1 = \frac{\ell^2}{4}R_1 + \frac{EI_1}{L}, \quad \hat{k}_2 = \frac{\ell^2}{4}R_2 + \frac{EI_2}{\ell}, \quad \hat{k}_3 = \frac{\ell^2}{4}R_1 - \frac{EI_1}{\ell}, \quad \hat{k}_4 = \frac{\ell^2}{4}R_2 - \frac{EI_2}{\ell} \qquad \text{I-(2)}$$

where

$$R_1 = \frac{12EI_1}{\ell^3}(1+\gamma_1)^{-1}$$

$$R_2 = \frac{12EI_2}{\ell^3}(1+\gamma_2)^{-1} \qquad \text{I-(3)}$$

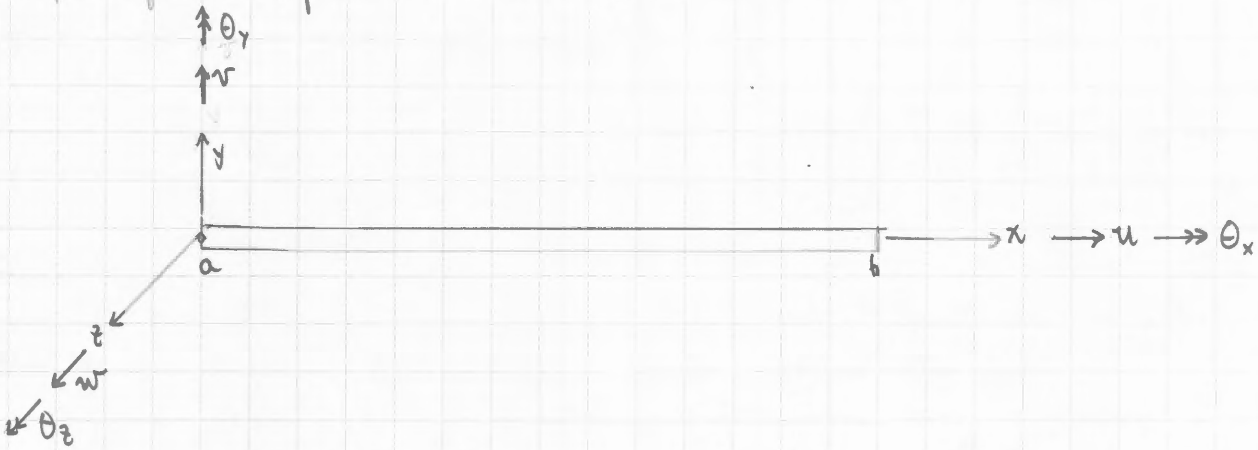
and

$$\gamma_1 = \begin{cases} \dfrac{12EI_1}{K_1 A G \ell^2} & \text{if } I_{12}=0 \text{ and } K_1 \neq 0 \\ 0 & \text{if } I_{12} \neq 0 \text{ or } K_1 = 0 \end{cases} \qquad \text{I-(4)}$$

$$\gamma_2 = \begin{cases} \dfrac{12EI_2}{K_2 A G \ell^2} & \text{if } I_{12}=0 \text{ and } K_2 \neq 0 \\ 0 & \text{if } I_{12} \neq 0 \text{ or } K_2 = 0 \end{cases}$$

and K_1, K_2 are the area factors for shear (a zero value is assigned to γ_1, γ_2 when K_1 and K_2 are zero which is interpreted to be the case of no shear flexibility (which is actually $K_1, K_2 \to \infty$).

degrees of freedom for the BAR element are:



$$
U = \begin{Bmatrix} u_a \\ v_a \\ w_a \\ \theta_{xa} \\ \theta_{Ya} \\ \theta_{za} \\ u_b \\ v_b \\ w_b \\ \theta_{xb} \\ \theta_{Yb} \\ \theta_{zb} \end{Bmatrix}, \quad P = \begin{Bmatrix} P_{xa} \\ P_{Ya} \\ P_{za} \\ M_{xa} \\ M_{Ya} \\ M_{za} \\ P_{xb} \\ P_{Yb} \\ P_{zb} \\ M_{xb} \\ M_{Yb} \\ M_{zb} \end{Bmatrix} \qquad \text{I-(5)}
$$

Eqn.(1) can be partitioned as

$$
K = \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ab}^{T} & K_{bb} \end{bmatrix}
$$

The partitions of K are shown on the following page

$$
K = \begin{array}{cccccccccccc|c}
u_a & v_a & w_a & \theta_{xa} & \theta_{ya} & \theta_{za} & u_b & v_b & w_b & \theta_{xb} & \theta_{yb} & \theta_{zb} & \\
\hline
k_a & & & & & & -k_a & & & & & & u_a \\
& R_1 & \beta & & -\frac{l}{2}\beta & \frac{l}{2}R_1 & & -R_1 & -\beta & & -\frac{l}{2}\beta & \frac{l}{2}R_1 & v_a \\
& \beta & R_2 & & -\frac{l}{2}R_2 & \frac{l}{2}\beta & & -\beta & -R_2 & & -\frac{l}{2}R_2 & \frac{l}{2}\beta & w_a \\
& & & k_t & & & & & & -k_t & & & \theta_{xa} \\
& -\frac{l}{2}\beta & -\frac{l}{2}R_2 & & \hat{k}_2 & -\frac{l^2}{3}\beta & & \frac{l}{2}\beta & \frac{l}{2}R_2 & & \hat{k}_4 & -\frac{l^2}{6}\beta & \theta_{ya} \\
& \frac{l}{2}R_1 & \frac{l}{2}\beta & & -\frac{l^2}{3}\beta & \hat{k}_1 & & -\frac{l}{2}R_1 & -\frac{l}{2}\beta & & -\frac{l^2}{6}\beta & \hat{k}_3 & \theta_{za} \\
-k_a & & & & & & k_a & & & & & & u_b \\
& -R_1 & -\beta & & \frac{l}{2}\beta & -\frac{l}{2}R_1 & & R_1 & \beta & & \frac{l}{2}\beta & -\frac{l}{2}R_1 & v_b \\
& -\beta & -R_2 & & \frac{l}{2}R_2 & -\frac{l}{2}\beta & & \beta & R_2 & & \frac{l}{2}R_2 & -\frac{l}{2}\beta & w_b \\
& & & -k_t & & & & & & k_t & & & \theta_{xb} \\
& -\frac{l}{2}\beta & -\frac{l}{2}R_2 & & \hat{k}_4 & -\frac{l^2}{6}\beta & & \frac{l}{2}\beta & \frac{l}{2}R_2 & & \hat{k}_2 & -\frac{l^2}{3}\beta & \theta_{yb} \\
& \frac{l}{2}R_1 & \frac{l}{2}\beta & & -\frac{l^2}{6}\beta & \hat{k}_3 & & -\frac{l}{2}R_1 & -\frac{l}{2}\beta & & -\frac{l^2}{3}\beta & \hat{k}_1 & \theta_{zb}
\end{array}
\equiv \left[\begin{array}{c|c} K_{aa} & K_{ab} \\ \hline K_{ab}^T & K_{bb} \end{array}\right] \tag{I-(6)}
$$

No pin flags

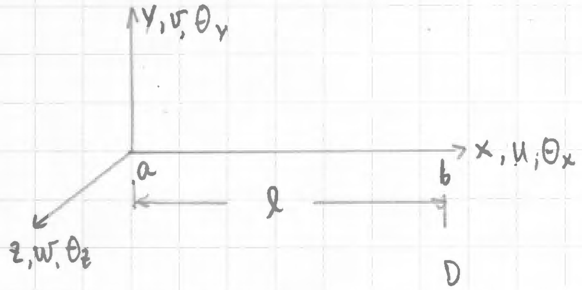
Check that partitions of K satisfy rigid body equilibrium. That is:

$$K_{ab} \stackrel{?}{=} -K_{aa} D$$

$$K_{bb} \stackrel{?}{=} -D^T K_{aa} D$$

where D is

$$u_a = D u_b \quad \text{for rigid body motion}$$



$$\begin{Bmatrix} u_a \\ v_a \\ w_a \\ \hline \theta_{xa} \\ \theta_{ya} \\ \theta_{za} \end{Bmatrix} = \overbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -l \\ 0 & 0 & 1 & 0 & l & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}^{D} \begin{Bmatrix} u_b \\ v_b \\ w_b \\ \theta_{x_b} \\ \theta_{y_b} \\ \theta_{z_b} \end{Bmatrix}$$

$$-K_{aa}D = -\left[\begin{array}{ccc|ccc} k_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_1 & \beta & 0 & -\frac{l}{2}\beta & \frac{l}{2}R_1 \\ 0 & \beta & R_2 & 0 & -\frac{l}{2}R_2 & \frac{l}{2}\beta \\ \hline 0 & 0 & 0 & k_t & 0 & 0 \\ 0 & -\frac{l}{2}\beta & -\frac{l}{2}R_2 & 0 & \hat{k}_2 & -\frac{l^2}{3}\beta \\ 0 & \frac{l}{2}R_1 & \frac{l}{2}\beta & 0 & -\frac{l^2}{3}\beta & \hat{k}_1 \end{array}\right]\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -l \\ 0 & 0 & 1 & 0 & l & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$-K_{aa}D = \left[\begin{array}{ccc|ccc} -k_a & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_1 & -\beta & 0 & -\frac{\beta l}{2} & \frac{R_1 l}{2} \\ 0 & -\beta & -R_2 & 0 & -\frac{R_2 l}{2} & \frac{\beta l}{2} \\ 0 & 0 & 0 & k_t & 0 & 0 \\ 0 & \frac{l}{2}\beta & \frac{l}{2}R_2 & 0 & -\hat{k}_2 + \frac{R_2 l^2}{2} & -\frac{\beta l^2}{6} \\ 0 & -\frac{l}{2}R_1 & -\frac{l}{2}\beta & 0 & -\frac{\beta l^2}{6} & -\hat{k}_1 + \frac{R_1 l^2}{2} \end{array}\right]$$

but:

$$-\hat{k}_2 + \frac{R_2 l^2}{2} = -\frac{R_2 l^2}{4} - \frac{EI_2}{l} + \frac{R_2 l^2}{2} = \frac{R_2 l^2}{4} - \frac{EI_2}{l} \equiv \hat{k}_4$$

and

$$-\hat{k}_1 + \frac{R_1 l^2}{2} = -\frac{R_1 l^2}{4} - \frac{EI_1}{l} + \frac{R_1 l^2}{2} = \frac{R_1 l^2}{4} - \frac{EI_1}{l} = \hat{k}_3$$

$$= \left[\begin{array}{ccc|ccc} -k_a & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_1 & -\beta & 0 & -\frac{l}{2}\beta & \frac{l}{2}R_1 \\ 0 & -\beta & -R_2 & 0 & -\frac{l}{2}R_2 & \frac{l}{2}\beta \\ 0 & 0 & 0 & -k_t & 0 & 0 \\ 0 & \frac{l}{2}\beta & \frac{l}{2}R_2 & 0 & \hat{k}_4 & -\frac{l^2}{6}\beta \\ 0 & -\frac{l}{2}R_1 & -\frac{l}{2}\beta & 0 & -\frac{l^2}{6}\beta & \hat{k}_3 \end{array}\right] \equiv K_{ab} \;!$$

Also check $K_{bb} = D^T K_{aa} D = -D^T K_{ab}$ (since $K_{ab} = -K_{aa} D$)

$$-D^T K_{ab} = -\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & l & 0 & 1 & 0 \\ 0 & -l & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} -k_a & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_1 & -\beta & 0 & -\frac{l}{2}\beta & \frac{l}{2}R_1 \\ 0 & -\beta & -R_2 & 0 & -\frac{l}{2}R_2 & \frac{l}{2}\beta \\ \hline 0 & 0 & 0 & -k_t & 0 & 0 \\ 0 & \frac{l}{2}\beta & \frac{l}{2}R_2 & 0 & \hat{k}_4 & -\frac{l^2}{6}\beta \\ 0 & -\frac{l}{2}R_1 & -\frac{l}{2}\beta & 0 & -\frac{l^2}{6}\beta & \hat{k}_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} k_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_1 & \beta & 0 & \frac{l}{2}\beta & -\frac{l}{2}R_1 \\ 0 & \beta & R_2 & 0 & \frac{l}{2}R_2 & -\frac{l}{2}\beta \\ \hline 0 & 0 & 0 & k_t & 0 & 0 \\ 0 & \frac{l}{2}\beta & \frac{l}{2}R_2 & 0 & -\hat{k}_4 + \frac{l^2}{2}R_2 & -\frac{l^2}{3}\beta \\ 0 & -\frac{l}{2}R_1 & -\frac{l}{2}\beta & 0 & -\frac{l^2}{3}\beta & -\hat{k}_3 + \frac{l^2}{2}R_1 \end{array}\right]$$

but

$$-\hat{k}_4 + \frac{R_2 l^2}{4} = -\frac{R_2 l^2}{4} + \frac{EI_2}{l} + \frac{R_2 l^2}{4}$$

$$= \frac{R_2 l^2}{4} + \frac{EI_2}{l} = \hat{k}_2$$

and

$$-\hat{k}_3 + \frac{R_1 l^2}{2} = -\frac{R_1 l^2}{4} + \frac{EI_1}{l} + \frac{R_1 l^2}{2}$$

$$= \frac{R_1 l^2}{4} + \frac{EI_1}{l} = \hat{k}_1$$

$$-D^T K_{ab} = D^T K_{aa} D = \left[\begin{array}{ccc|ccc} k_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_1 & \beta & 0 & \frac{l}{2}\beta & -\frac{l}{2}R_1 \\ 0 & \beta & R_2 & 0 & \frac{l}{2}R_2 & -\frac{l}{2}\beta \\ \hline 0 & 0 & 0 & k_t & 0 & 0 \\ 0 & \frac{l}{2}\beta & \frac{l}{2}R_2 & 0 & \hat{k}_2 & -\frac{l^2}{3}\beta \\ 0 & -\frac{l}{2}R_1 & -\frac{l}{2}\beta & 0 & -\frac{l^2}{3}\beta & \hat{k}_1 \end{array}\right] \equiv K_{bb}$$

Relationship Among Partitions of a Free-Free Stiffness Matrix

$$K U = P$$

Partition into 2 sets: a and b:

$$\begin{bmatrix} K_{aa} & K_{ab} \\ K_{ab}^T & K_{bb} \end{bmatrix} \begin{Bmatrix} U_a \\ U_b \end{Bmatrix} = \begin{Bmatrix} P_a \\ P_b \end{Bmatrix} \quad (1)$$

(P_a, P_b are all forces, incl. constr. forces on U_a, U_b and must be in equilibrium)

For rigid body motion we could take $P_a = P_b = 0$

$$U_a = D\,U_b \quad (D \text{ is a rigid body matrix})$$

and

$$P_a = P_b = 0 \quad (2)$$

Subst. (2) into the 1st of eqns (1)

$$K_{aa}(D U_b) + K_{ab} U_b = 0 \quad \text{or} \quad K_{ab} = -K_{aa} D \quad (\text{and } K_{ab}^T = -D^T K_{aa}) \quad (3)$$

Subst. (2) and (3) into the 2nd of eqns (1):

$$(-D^T K_{aa})^T (D U_b) + K_{bb} U_b = 0 \quad \text{or} \quad K_{bb} = D^T K_{aa} D \quad (4)$$

So

$$\boxed{K \equiv \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ab}^T & K_{bb} \end{bmatrix} = \begin{bmatrix} K_{aa} & -K_{aa} D \\ -D^T K_{aa} & D^T K_{aa} D \end{bmatrix}} \quad (5)$$

We can also show a relationship between P_a, P_b. Using (5) for K in (1) and a rigid body displ $U_a = D U_b$

eqn (1) is

$$\begin{bmatrix} K_{aa} & -D^T K_{aa} \\ (-D^T K_{aa})^T & D^T K_{aa} D \end{bmatrix} \begin{bmatrix} D \\ I \end{bmatrix} U_b = \begin{Bmatrix} P_a \\ P_b \end{Bmatrix}$$

Mult. by $[D^T \; I]$:

$$\underbrace{[D^T \; I] \begin{bmatrix} K_{aa} & -D^T K_{aa} \\ (-D^T K_{aa})^T & D^T K_{aa} D \end{bmatrix} \begin{bmatrix} D \\ I \end{bmatrix}}_{0} U_b = [D^T \; I] \begin{Bmatrix} P_a \\ P_b \end{Bmatrix}$$

$$\therefore [D^T \; I] \begin{Bmatrix} P_a \\ P_b \end{Bmatrix} = 0 \quad \text{or} \quad \boxed{P_b = -D^T P_a} \quad (6)$$

(for forces to be in equilibrium) when P_a, P_b incl. all forces (w/ constraint forces)

II. Bar Element Thermal Loads and Stress Recovery Matrices

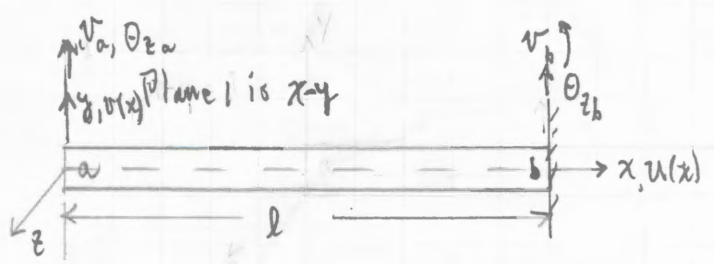
In local element coordinates the equation for the bar element in terms of degrees of freedom at the bar ends (not the grid points - possibly there are offsets that make bar ends $\neq$ gridpoints):

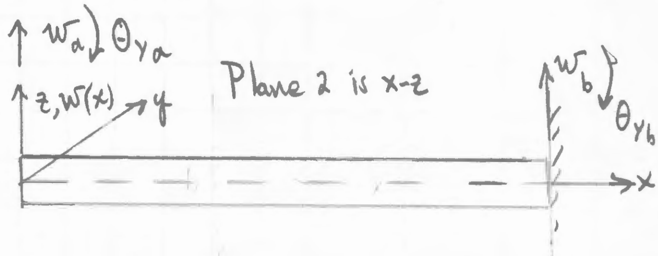
$$K_{ee} U_e = P_{A_e} + P_{T_e} \quad \begin{cases} P_{A_e} = \text{applied loads on element} \\ P_{T_e} = \text{equiv. thermal loads} \end{cases}$$

Partition

$$\begin{bmatrix} K_{aa} & K_{ab} \\ K_{ab}^T & K_{bb} \end{bmatrix} \begin{Bmatrix} U_a \\ U_b \end{Bmatrix} = \begin{Bmatrix} P_{A_a} \\ P_{A_b} \end{Bmatrix} + \begin{Bmatrix} P_{T_a} \\ P_{T_b} \end{Bmatrix} \qquad \text{II-(1)}$$

If there are no applied loads we can use (1) to determine the thermal loads. Without loss of generality we can fix one end (6 DOF) as a reference. Say we take the displs zero at end b:





Then, from 1 $(U_b = 0,\ P_{A_a} = P_{A_b} = 0)$

$$P_{T_a} = K_{aa} U_a \quad , \quad P_{T_b} = K_{ab}^T U_a \qquad \text{II-(2)}$$

where

$$U_a = \begin{Bmatrix} u_a \\ v_a \\ w_a \\ \theta_{xa} \\ \theta_{ya} \\ \theta_{za} \end{Bmatrix} = \begin{Bmatrix} u(x) \\ v(x) \\ w(x) \\ 0 \\ -dw/dx \\ dv/dx \end{Bmatrix}_{x=a} \qquad \text{II-(3)}$$

Take the temperature distribution to be

$$T(x,y,z) = T_a(y,z) + \frac{T_b(y,z) - T_a(y,z)}{l}\, x \qquad \text{II-(4)}$$

where T_a, T_b are temp distributions at ends a, b respectively

The displacements are free thermal expansion and are determined from:

(a) Thermal Displ - x direction

$$u_a = u(x)\Big|_{x=0} \quad \text{where} \quad \frac{du}{dx} = \epsilon_x = \frac{P_T}{AE} = \frac{1}{AE}\int_A E\alpha T\, dA \quad \text{(Rivello pp 158, 136)}$$

for E, α constant

$$\epsilon_x = \frac{\alpha}{A}\int_A (T - T_{ref})\, dA$$

$$= \frac{\alpha}{A}\int_A \left[T_a(y,z) + \frac{T_b(y,z) - T_a(y,z)}{l}\, x - T_{ref} \right] dA$$

$$= \frac{\alpha}{A}\left\{ \int_A T_a(y,z)\, dA + \frac{x}{l}\int_A T_b(y,z)\, dA - \frac{x}{l}\int_A T_a(y,z)\, dA - T_{ref} A \right\}$$

Define

$$\boxed{\bar{T}_a = \frac{1}{A}\int_A T_a(y,z)\, dA \;,\quad \bar{T}_b = \frac{1}{A}\int_A T_b(y,z)\, dA} \qquad \text{II-(5)}$$

Then

$$\epsilon_x = \alpha\left\{ \bar{T}_a\left(1 - \frac{x}{l}\right) + \bar{T}_b \frac{x}{l} - T_{ref} \right\} \qquad \text{II-(6)}$$

Integrate to get $u(x)$ noting that $u(x = L) = 0$

$$u(x) = \alpha\left\{ \bar{T}_a\left(x - \frac{x^2}{2l} - \frac{l}{2}\right) + \bar{T}_b\left(\frac{x^2}{2l} - \frac{l}{2}\right) - T_{ref}(x - l) \right\} \qquad \text{II-(7)}$$

$$\boxed{\therefore\; u_a = u(x=0) = -\alpha l\left(\bar{T} - T_{ref}\right),\quad \bar{T} = \frac{\bar{T}_a + \bar{T}_b}{2}} \qquad \text{II-(8)}$$

$\bar{T}_a, \bar{T}_b$ defined in eqn (5)

(1b) Displacements in Plane 1 ($v(x)$, dv/dx)

From Rivello pp 157, 136: (constant E, α and M^* = thermal moments and using $I_1 = I_{zz}$, $I_2 = I_{yy}$, $I_{12} = I_{yz}$)

$$\frac{d^2 v}{dx^2} = \frac{1}{E}\,\frac{M_{zT} I_2 - M_{yT} I_{12}}{I_1 I_2 - I_{12}^2}, \quad \begin{cases} M_{yT} = -E\alpha \int_A z(T - T_{ref})\,dA \\ M_{zT} = -E\alpha \int_A y(T - T_{ref})\,dA \end{cases} \qquad \text{II-(9)}$$

Define

$$\Delta_1 = \frac{I_2}{I_1 I_2 - I_{yz}^2}, \quad \Delta_2 = \frac{I_1}{I_1 I_2 - I_{12}^2}, \quad \Delta_{12} = \frac{I_{yz}}{I_1 I_2 - I_{yz}^2} \qquad \text{II-(10)}$$

From M_{yT}, M_{zT} in eqn (9) and the temp. distribution in eqn (4):

$$M_{yT} = -E\alpha \int_A z\left\{\left(1 - \frac{x}{\ell}\right) T_a(y,z) + \frac{x}{\ell} T_b(y,z) - T_{ref}\right\} dA$$

$$M_{yT} = -E\alpha \left\{\left(1 - \frac{x}{\ell}\right) \int_A z\, T_a(y,z)\,dA + \frac{x}{\ell} \int_A z\, T_b(y,z)\,dA \right. \qquad \text{II-(11)}$$

and

$$M_{zT} = -E\alpha \left\{\left(1 - \frac{x}{\ell}\right) \int_A y\, T_a(y,z)\,dA + \frac{x}{\ell} \int_A y\, T_b(y,z)\,dA \right. \qquad \text{II-(12)}$$

Define

$$\boxed{\begin{aligned} T'_{1a} &= \frac{1}{I_1} \int_A y\, T_a(y,z)\,dA \\ T'_{1b} &= \frac{1}{I_1} \int_A y\, T_b(y,z)\,dA \\ T'_{2a} &= \frac{1}{I_2} \int_A z\, T_a(y,z)\,dA \\ T'_{2b} &= \frac{1}{I_2} \int_A z\, T_b(y,z)\,dA \end{aligned}} \qquad \text{II-(13)}$$

Then

$$\begin{aligned} M_{yT} &= -E\alpha I_2 \left[\left(1 - \frac{x}{\ell}\right) T'_{2a} + \frac{x}{\ell} T'_{2b}\right] \\ M_{zT} &= -E\alpha I_1 \left[\left(1 - \frac{x}{\ell}\right) T'_{1a} + \frac{x}{\ell} T'_{1b}\right] \end{aligned} \qquad \text{II-(14)}$$

Substitute (10) and (14) (9) for d^2v/dx^2 eqn:

$$\frac{d^2v}{dx^2} = \frac{1}{E}\left[\Delta_1 M_{zT} - \Delta_{12} M_{yT}\right]$$

$$= -\alpha\left\{\left(1-\frac{x}{\ell}\right)\left[\Delta_1 I_1 T'_{1a} - \Delta_{12} I_2 T'_{2a}\right] + \frac{x}{\ell}\left[\Delta_1 I_1 T'_{1b} - \Delta_{12} I_2 T'_{2b}\right]\right\}$$

Integrate and use b.c. of $v = dv/dx = 0$ at $x = L$: II-(15)

$$\frac{dv}{dx} = -\alpha\left\{\left(x-\frac{x^2}{2\ell}-\frac{\ell}{2}\right)\left[\Delta_1 I_1 T'_{1a} - \Delta_{12} I_2 T'_{2a}\right] + \left(\frac{x^2}{2\ell}-\frac{\ell}{2}\right)\left[\Delta_1 I_1 T'_{1b} - \Delta_{12} I_2 T'_{2b}\right]\right\} \quad \text{II-(16)}$$

and

$$v = -\alpha\left\{\left(\frac{x^2}{2}-\frac{x^3}{6\ell}-\frac{\ell x}{2}+\frac{\ell^2}{6}\right)\left[\Delta_1 I_1 T'_{1a} - \Delta_{12} I_2 T'_{2a}\right] + \left(\frac{x^3}{6\ell}-\frac{\ell x}{2}+\frac{\ell^2}{3}\right)\left[\Delta_1 I_1 T'_{1b} - \Delta_{12} I_2 T'_{2b}\right]\right\} \quad \text{II-(17)}$$

From which we obtain:

$$\boxed{\begin{aligned} v_a = v(x)\Big|_{x=0} &= -\alpha\ell^2\left\{\frac{1}{6}\left[\Delta_1 I_1 T'_{1a} - \Delta_{12} I_2 T'_{2a}\right] + \frac{1}{3}\left[\Delta_1 I_1 T'_{1b} - \Delta_{12} I_2 T'_{2b}\right]\right\} \quad \text{II-(18)} \\ \theta_{za} = \frac{dv}{dx}\Big|_{x=0} &= +\alpha\ell\left\{\frac{1}{2}\left[\Delta_1 I_1 T'_{1a} - \Delta_{12} I_2 T'_{2a}\right] + \frac{1}{2}\left[\Delta_1 I_1 T'_{1b} - \Delta_{12} I_2 T'_{2b}\right]\right\} \quad \text{II-(19)} \end{aligned}}$$

(c) Displacements in Plane 2 ($w(x)$, dw/dx)

From Rivello pp 157, 176 (Constant E, α and M^* = thermal moments)

$$\frac{d^2w}{dx^2} = \frac{1}{E}\left[\frac{M_{yT} I_z - M_{zT} I_{yz}}{I_y I_z - I_{yz}^2}\right], \quad (M_{yT}, M_{zT} \text{ given in eqns (9), (14)}) \quad \text{II-(20)}$$

Using (10):

$$\frac{d^2w}{dx^2} = \frac{1}{E}\left[\Delta_2 M_{yT} - \Delta_{12} M_{zT}\right] \quad \text{II-(21)}$$

Substitute (10) and (~~14~~) into (9) for d^2v/dx^2 equation:

$$\frac{d^2v}{dx^2} = \frac{1}{E}\left[\Delta_1 M_{ZT} - \Delta_{12} M_{YT}\right] \qquad \text{II-(22)}$$

$$= -\alpha\left\{\left(1-\frac{x}{l}\right)\left[\Delta_2 I_2 T'_{2a} - \Delta_{12} I_1 T'_{1a}\right] + \frac{x}{l}\left[\Delta_2 I_2 T'_{2b} - \Delta_{12} I_1 T'_{1b}\right]\right\} \qquad \text{II-(23)}$$

Integrate and use b.c. of $w = dw/dx = 0$ at $x = l$

$$\frac{dw}{dx} = -\alpha\left\{\left(x - \frac{x^2}{2l} - \frac{l}{2}\right)\left[\Delta_2 I_2 T'_{2a} - \Delta_{12} I_1 T'_{1a}\right] + \left(\frac{x^2}{2l} - \frac{l}{2}\right)\left[\Delta_2 I_2 T'_{2b} - \Delta_{12} I_1 T'_{1b}\right]\right\} \qquad \text{II-(24)}$$

and

$$w = -\alpha\left\{\left(\frac{x^2}{2} - \frac{x^3}{6l} - \frac{lx}{2} + \frac{l^2}{6}\right)\left[\Delta_2 I_2 T'_{2a} - \Delta_{12} I_1 T'_{1a}\right] + \left(\frac{x^3}{6l} - \frac{lx}{2} + \frac{l^2}{3}\right)\left[\Delta_2 I_2 T'_{2b} - \Delta_{12} I_1 T'_{1b}\right]\right\} \qquad \text{II-(25)}$$

from which we obtain

$$\boxed{\begin{aligned} w_a = w(x)\Big|_{x=0} &= -\alpha l^2\left\{\frac{1}{6}\left[\Delta_2 I_2 T'_{2a} - \Delta_{12} I_1 T'_{1a}\right] + \frac{1}{3}\left[\Delta_2 I_2 T'_{2b} - \Delta_{12} I_1 T'_{1b}\right]\right\} && \text{II-(26)} \\ \theta_{Y_a} = -\frac{dw}{dx}\Big|_{x=0} &= -\alpha l\left\{\frac{1}{2}\left[\Delta_2 I_2 T'_{2a} - \Delta_{12} I_1 T'_{1a}\right] + \frac{1}{2}\left[\Delta_2 I_2 T'_{2b} - \Delta_{12} I_1 T'_{1b}\right]\right\} && \text{II-(27)} \end{aligned}}$$

Using (8), (18), (19), (26) and (27) eqn (7) can be written in matrix form as

$$\begin{Bmatrix} u_a \\ v_a \\ w_a \\ \theta_{xa} \\ \theta_{ya} \\ \theta_{za} \end{Bmatrix}_{\text{thermal}} = -\alpha l \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{l}{6}\Delta_1 I_1 & \frac{l}{3}\Delta_1 I_1 & -\frac{l}{6}\Delta_{12} I_2 & -\frac{l}{3}\Delta_{12} I_2 \\ 0 & -\frac{l}{6}\Delta_{12} I_1 & -\frac{l}{3}\Delta_{12} I_1 & \frac{l}{6}\Delta_2 I_2 & \frac{l}{3}\Delta_2 I_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\Delta_{12} I_1 & -\frac{1}{2}\Delta_{12} I_1 & \frac{1}{2}\Delta_2 I_2 & \frac{1}{2}\Delta_2 I_2 \\ 0 & -\frac{1}{2}\Delta_1 I_1 & -\frac{1}{2}\Delta_1 I_1 & \frac{1}{2}\Delta_{12} I_2 & \frac{1}{2}\Delta_{12} I_2 \end{bmatrix}}_{\text{call this matrix } \bar{A}} \begin{Bmatrix} \bar{T}-T_{ref} \\ T'_{1a} \\ T'_{1b} \\ T'_{2a} \\ T'_{2b} \end{Bmatrix} \qquad \text{II-(28)}$$

Note that, when $I_{12}=0$ eqns (10) becomes

$$\Delta_1 = \frac{1}{I_1}, \quad \Delta_2 = \frac{1}{I_2}, \quad \Delta_{12} = 0$$

In which case (28) becomes:

$$\begin{Bmatrix} u_a \\ v_a \\ w_a \\ \theta_{xa} \\ \theta_{ya} \\ \theta_{za} \end{Bmatrix}_{\substack{\text{thermal} \\ (I_{12}=0)}} = -\alpha l \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & l/6 & l/3 & 0 & 0 \\ 0 & 0 & 0 & l/6 & l/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & -1/2 & -1/2 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{T}-T_{ref} \\ T'_{1a} \\ T'_{1b} \\ T'_{2a} \\ T'_{2b} \end{Bmatrix} \qquad \text{II-(29)}$$

which agrees with eqn (41) in Section 8.2 (Bar elem) in NASTRAN Programmers manual

From (28) we can write

$$U_a = \bar{A}\, T' \qquad \text{II-(30)}$$

where matrices $\bar{A}$ and T' are

$$\bar{A} = -\alpha l \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{l}{6}\Delta_1 I_1 & \frac{l}{3}\Delta_1 I_1 & -\frac{l}{6}\Delta_{12} I_2 & -\frac{l}{3}\Delta_{12} I_2 \\ 0 & -\frac{l}{6}\Delta_{12} I_1 & -\frac{l}{3}\Delta_{12} I_1 & \frac{l}{6}\Delta_2 I_2 & \frac{l}{3}\Delta_2 I_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\Delta_{12} I_1 & -\frac{1}{2}\Delta_{12} I_1 & \frac{1}{2}\Delta_2 I_2 & \frac{1}{2}\Delta_2 I_2 \\ 0 & -\frac{1}{2}\Delta_1 I_1 & -\frac{1}{2}\Delta_1 I_1 & \frac{1}{2}\Delta_{12} I_2 & \frac{1}{2}\Delta_{12} I_2 \end{bmatrix}, \quad T' = \begin{Bmatrix} (\bar{T} - T_{ref}) + \delta_m \\ T'_{1a} \\ T'_{1b} \\ T'_{2a} \\ T'_{2b} \end{Bmatrix} \qquad \text{II-(31)}$$

Substitute (30) into (2) to get the eqn for thermal loads at end a:

$$P_{T_a} = K_{aa}\bar{A}\, T' = B_{T_a} T' \quad (\text{where } B_{T_a} = K_{aa}\bar{A})$$

Also, from (1) and (30)

$$P_{T_b} = K_{ab}^T \bar{A}\, T' = B_{T_b} T' \quad (\text{where } B_{T_b} = K_{ab}\bar{A}$$

So

$$\boxed{P_T = \begin{Bmatrix} P_{T_a} \\ P_{T_b} \end{Bmatrix} = \begin{bmatrix} K_{aa}\bar{A} \\ K_{ab}^T \bar{A} \end{bmatrix} T' = B_T T' \qquad \text{where} \qquad B_T = \begin{bmatrix} B_{Ta} \\ B_{Tb} \end{bmatrix} = \begin{bmatrix} K_{aa}\bar{A} \\ K_{ab}^T \bar{A} \end{bmatrix}}$$

This is the form of P_{Te} eqns used in MYSTRAN where K_{aa} is the partition of elem K matrix for bar end a. If there are no pinflags, this K_{aa} is given in the upper left partition of eqn I-(6). See later section on how K changes with pinflags

eqn II-32

(d) Explicit Form of B_T when there are no Pinflags

Multiply $K_{aa} A$ to get B_{T_a}. Note, all ij terms not explicitly shown below are zero.

Row 1: $B_{T_{a11}} = -\alpha l k_a$

Row 2: $B_{T_{a22}} = -\alpha l\left[R_1 \frac{l}{6}\Delta_1 I_1 - \beta\frac{l}{6}\Delta_{12}I_1 + \beta\frac{l}{4}\Delta_{12}I_1 - R_1\frac{l}{4}\Delta_1 I_1\right] = \alpha l(R_1\Delta_1 - \beta\Delta_{12})\frac{I_1 l}{12}$

$B_{T_{a23}} = -\alpha l\left[R_1\frac{l}{3}\Delta_1 I_1 - \beta\frac{l}{3}\Delta_{12}I_1 + \beta\frac{l}{4}\Delta_{12}I_1 - R_1\frac{l}{4}\Delta_1 I_1\right] = -\alpha l(R_1\Delta_1-\beta\Delta_{12})\frac{I_1 l}{12} = -B_{T_{a22}}$

$B_{T_{a24}} = -\alpha l\left[-R_1\frac{l}{6}\Delta_{12}I_2 + \beta\frac{l}{6}\Delta_2 I_2 - \beta\frac{l}{4}\Delta_2 I_2 + R_1\frac{l}{4}\Delta_{12}I_2\right] = -\alpha l(R_1\Delta_{12}-\beta\Delta_2)\frac{I_2 l}{12}$

$B_{T_{a25}} = -\alpha l\left[-R_1\frac{l}{3}\Delta_{12}I_2 + \beta\frac{l}{3}\Delta_2 I_2 - \beta\frac{l}{4}\Delta_2 I_2 + R_1\frac{l}{4}\Delta_{12}I_2\right] = \alpha l(R_1\Delta_{12}-\beta\Delta_2)\frac{I_2 l}{12} = -B_{T_{a24}}$

Row 3: $B_{T_{a32}} = -\alpha l\left[\beta\frac{l}{6}\Delta_1 I_1 - R_2\frac{l}{6}\Delta_{12}I_1 + R_2\frac{l}{4}\Delta_{12}I_1 - \beta\frac{l}{4}\Delta_1 I_1\right] = -\alpha l(R_2\Delta_{12}-\beta\Delta_1)\frac{I_1 l}{12}$

$B_{T_{a33}} = -\alpha l\left[\beta\frac{l}{3}\Delta_1 I_1 - R_2\frac{l}{3}\Delta_{12}I_1 + R_2\frac{l}{4}\Delta_{12}I_1 - \beta\frac{l}{4}\Delta_1 I_1\right] = \alpha l(R_2\Delta_{12}-\beta\Delta_1)\frac{I_1 l}{12} = -B_{T_{a32}}$

$B_{T_{a34}} = -\alpha l\left[-\beta\frac{l}{6}\Delta_{12}I_2 + R_2\frac{l}{6}\Delta_2 I_2 - R_2\frac{l}{4}\Delta_2 I_2 + \beta\frac{l}{4}\Delta_{12}I_2\right] = \alpha l(R_2\Delta_2-\beta\Delta_{12})\frac{I_2 l}{12}$

$B_{T_{a35}} = -\alpha l\left[-\beta\frac{l}{3}\Delta_{12}I_2 + R_2\frac{l}{3}\Delta_2 I_2 - R_2\frac{l}{4}\Delta_2 I_2 + \beta\frac{l}{4}\Delta_{12}I_2\right] = -\alpha l(R_2\Delta_2-\beta\Delta_{12})\frac{I_2 l}{12} = -B_{T_{a34}}$

Row 4: Null

Row 5: $B_{T_{a52}} = -\alpha l\left[-\beta\frac{l^2}{12}\Delta_1 I_1 + R_2\frac{l^2}{12}\Delta_{12}I_1 - \frac{1}{2}\hat{k}_2\Delta_{12}I_1 + \beta\frac{l^2}{6}\Delta_1 I_1\right] = \alpha l\left\{-(R_2\Delta_{12}-\beta\Delta_1)\frac{I_1 l^2}{12} + \frac{1}{2}\Delta_{12}I_1\hat{k}_2\right\}$

$B_{T_{a53}} = -\alpha l\left[-\beta\frac{l^2}{6}\Delta_1 I_1 + R_2\frac{l^2}{6}\Delta_{12}I_1 - \frac{1}{2}\hat{k}_2\Delta_{12}I_1 + \frac{\beta l^2}{6}\Delta_1 I_1\right] = \alpha l\left[-R_2\Delta_{12}\frac{I_1 l^2}{6} + \frac{1}{2}\Delta_{12}I_1\hat{k}_2\right]$

$B_{T_{a54}} = -\alpha l\left[\beta\frac{l^2}{12}\Delta_{12}I_2 - R_2\frac{l^2}{12}\Delta_2 I_2 + \frac{1}{2}\hat{k}_2\Delta_2 I_2 - \frac{\beta l^2}{6}\Delta_{12}I_2\right] = \alpha l\left\{(R_2\Delta_2+\beta\Delta_{12})\frac{I_2 l^2}{12} - \frac{1}{2}\Delta_2 I_2\hat{k}_2\right\}$

$B_{T_{a55}} = -\alpha l\left[\beta\frac{l^2}{6}\Delta_{12}I_2 - R_2\frac{l^2}{6}\Delta_2 I_2 + \frac{1}{2}\hat{k}_2\Delta_2 I_2 - \beta\frac{l^2}{6}\Delta_{12}I_2\right] = \alpha l\left[R_2\Delta_2\frac{I_2 l^2}{6} - \frac{1}{2}\Delta_2 I_2\hat{k}_2\right]$

Row 6: $B_{T_{a62}} = -\alpha l\left[R_1\frac{l^2}{12}\Delta_1 I_1 - \beta\frac{l^2}{12}\Delta_{12}I_1 + \beta\frac{l^2}{6}\Delta_{12}I_1 - \frac{1}{2}\hat{k}_1\Delta_1 I_1\right] = \alpha l\left\{-(R_1\Delta_1+\beta\Delta_{12})\frac{I_1 l^2}{12} + \frac{1}{2}\Delta_1 I_1\hat{k}_1\right\}$

$B_{T_{a63}} = -\alpha l\left[R_1\frac{l^2}{6}\Delta_1 I_1 - \beta\frac{l^2}{6}\Delta_{12}I_1 + \beta\frac{l^2}{6}\Delta_{12}I_1 - \frac{1}{2}\hat{k}_1\Delta_1 I_1\right] = \alpha l\left[-R_1\Delta_1\frac{I_1 l^2}{6} + \frac{1}{2}\Delta_1 I_1\tilde{k}_1\right]$

$B_{T_{a64}} = -\alpha l\left[-R_1\frac{l^2}{12}\Delta_{12}I_2 + \beta\frac{l^2}{12}\Delta_2 I_2 - \beta\frac{l^2}{6}\Delta_2 I_2 + \frac{1}{2}\hat{k}_1\Delta_{12}I_2\right] = \alpha l\left\{(R_1\Delta_{12}+\beta\Delta_2)\frac{I_2 l^2}{12} - \frac{1}{2}\Delta_{12}I_2\tilde{k}_1\right\}$

$B_{T_{a65}} = -\alpha l\left[-R_1\frac{l^2}{6}\Delta_{12}I_2 + \beta\frac{l^2}{6}\Delta_2 I_2 - \beta\frac{l^2}{6}\Delta_2 I_2 + \frac{1}{2}\hat{k}_1\Delta_{12}I_2\right] = \alpha l\left[R_1\Delta_{12}\frac{I_2 l^2}{6} - \frac{1}{2}\Delta_{12}I_2\hat{k}_1\right]$

Multiply $K_{ab}^{T}\bar{A} \equiv K_{ba}\bar{A}$ to get B_{T_b}. Note, all ij terms not explicitly shown are zero:

Row 1: $B_{T_{b11}} = \alpha l k_a$

Row 2: $$B_{T_{b22}} = -\alpha l\left[-R_1\frac{l}{6}\Delta_1 I_1 + \beta\frac{l}{6}\Delta_{12}I_1 - \beta\frac{l}{4}\Delta_{12}I_1 + R_1\frac{l}{4}\Delta_1 I_1\right] = -\alpha l\left(R_1\Delta_1-\beta\Delta_{12}\right)\frac{I_1 l}{12}$$

$$B_{T_{b23}} = -\alpha l\left[-R_1\frac{l}{3}\Delta_1 I_1 + \beta\frac{l}{3}\Delta_{12}I_1 - \beta\frac{l}{4}\Delta_{12}I_1 + R_1\frac{l}{4}\Delta_1 I_1\right] = \alpha l\left(R_1\Delta_1-\beta\Delta_{12}\right)\frac{I_1 l}{12} = -B_{T_{b22}}$$

$$B_{T_{b24}} = -\alpha l\left[R_1\frac{l}{6}\Delta_{12}I_2 - \beta\frac{l}{6}\Delta_2 I_2 + \beta\frac{l}{4}\Delta_2 I_2 - R_1\frac{l}{4}\Delta_{12}I_2\right] = \alpha l\left(R_1\Delta_{12}-\beta\Delta_2\right)\frac{I_2 l}{12}$$

$$B_{T_{b25}} = -\alpha l\left[R_1\frac{l}{3}\Delta_{12}I_2 - \beta\frac{l}{3}\Delta_2 I_2 + \beta\frac{l}{4}\Delta_2 I_2 - R_1\frac{l}{4}\Delta_{12}I_2\right] = -\alpha l\left(R_1\Delta_{12}-\beta\Delta_2\right)\frac{I_2 l}{12}$$

Row 3: $$B_{T_{b32}} = -\alpha l\left[-\beta\frac{l}{6}\Delta_1 I_1 + R_2\frac{l}{6}\Delta_{12}I_1 - R_2\frac{l}{4}\Delta_{12}I_1 + \beta\frac{l}{4}\Delta_1 I_1\right] = \alpha l\left(R_2\Delta_{12}-\beta\Delta_1\right)\frac{I_1 l}{12}$$

$$B_{T_{b33}} = -\alpha l\left[-\beta\frac{l}{3}\Delta_1 I_1 + R_2\frac{l}{3}\Delta_{12}I_1 - R_2\frac{l}{4}\Delta_{12}I_1 + \beta\frac{l}{4}\Delta_1 I_1\right] = -\alpha l\left(R_2\Delta_{12}-\beta\Delta_1\right)\frac{I_1 l}{12} = -B_{T_{b32}}$$

$$B_{T_{b34}} = -\alpha l\left[\beta\frac{l}{6}\Delta_{12}I_2 - R_2\frac{l}{6}\Delta_2 I_2 + R_2\frac{l}{4}\Delta_2 I_2 - \beta\frac{l}{4}\Delta_{12}I_2\right] = -\alpha l\left(R_2\Delta_2-\beta\Delta_{12}\right)\frac{I_2 l}{12}$$

$$B_{T_{b35}} = -\alpha l\left[\beta\frac{l}{3}\Delta_{12}I_2 - R_2\frac{l}{3}\Delta_2 I_2 + R_2\frac{l}{4}\Delta_2 I_2 - \beta\frac{l}{4}\Delta_{12}I_2\right] = \alpha l\left(R_2\Delta_2-\beta\Delta_{12}\right)\frac{I_2 l}{12} = -B_{T_{b34}}$$

Row 4: Null

Row 5: $$B_{T_{b52}} = -\alpha l\left[-\beta\frac{l^2}{12}\Delta_1 I_1 + R_2\frac{l^2}{12}\Delta_{12}I_1 - \frac{1}{2}\Delta_{12}I_1\hat{k}_4 + \beta\frac{l^2}{12}\Delta_1 I_1\right] = \alpha l\left[-R_2\Delta_{12}\frac{I_1 l^2}{12} + \frac{1}{2}\Delta_{12}I_1\hat{k}_4\right]$$

$$B_{T_{b53}} = -\alpha l\left[-\beta\frac{l^2}{6}\Delta_1 I_1 + R_2\frac{l^2}{6}\Delta_{12}I_1 - \frac{1}{2}\Delta_{12}I_1\hat{k}_4 + \beta\frac{l^2}{12}\Delta_1 I_1\right] = \alpha l\left\{-\left(R_2\Delta_{12}-\frac{\beta\Delta_1}{2}\right)\frac{I_1 l^2}{6} + \frac{1}{2}\Delta_{12}I_1\hat{k}_4\right\}$$

$$B_{T_{b54}} = -\alpha l\left[\beta\frac{l^2}{12}\Delta_{12}I_2 - R_2\frac{l^2}{12}\Delta_2 I_2 + \frac{1}{2}\Delta_2 I_2\hat{k}_4 - \beta\frac{l^2}{12}\Delta_{12}I_2\right] = \alpha l\left[R_2\Delta_2\frac{I_2 l^2}{12} - \frac{1}{2}\Delta_2 I_2\hat{k}_4\right]$$

$$B_{T_{b55}} = -\alpha l\left[\beta\frac{l^2}{6}\Delta_{12}I_2 - R_2\frac{l^2}{6}\Delta_2 I_2 + \frac{1}{2}\Delta_2 I_2\hat{k}_4 - \beta\frac{l^2}{12}\Delta_{12}I_2\right] = \alpha l\left\{\left(R_2\Delta_2-\frac{\beta\Delta_{12}}{2}\right)\frac{I_2 l^2}{6} - \frac{1}{2}\Delta_2 I_2\hat{k}_4\right\}$$

Row 6: $$B_{T_{b62}} = -\alpha l\left[R_1\frac{l^2}{12}\Delta_1 I_1 - \beta\frac{l^2}{12}\Delta_{12}I_1 + \beta\frac{l^2}{12}\Delta_{12}I_1 - \frac{1}{2}\Delta_1 I_1\hat{k}_3\right] = \alpha l\left[-R_1\Delta_1\frac{I_1 l^2}{12} + \frac{1}{2}\Delta_1 I_1\hat{k}_3\right]$$

$$B_{T_{b63}} = -\alpha l\left[R_1\frac{l^2}{6}\Delta_1 I_1 - \beta\frac{l^2}{6}\Delta_{12}I_1 + \beta\frac{l^2}{12}\Delta_{12}I_1 - \frac{1}{2}\Delta_1 I_1\hat{k}_3\right] = \alpha l\left\{-\left(R_1\Delta_1-\frac{\beta\Delta_{12}}{2}\right)\frac{I_1 l^2}{6} + \frac{1}{2}\Delta_1 I_1\hat{k}_3\right\}$$

$$B_{T_{b64}} = -\alpha l\left[-R_1\frac{l^2}{12}\Delta_{12}I_2 + \beta\frac{l^2}{12}\Delta_2 I_2 - \beta\frac{l^2}{12}\Delta_2 I_2 + \frac{1}{2}\Delta_{12}I_2\hat{k}_3\right] = \alpha l\left[R_1\Delta_{12}\frac{I_2 l^2}{12} - \frac{1}{2}\Delta_{12}I_2\hat{k}_3\right]$$

$$B_{T_{b65}} = -\alpha l\left[-R_1\frac{l^2}{6}\Delta_{12}I_2 + \beta\frac{l^2}{6}\Delta_2 I_2 - \beta\frac{l^2}{12}\Delta_2 I_2 + \frac{1}{2}\Delta_{12}I_2\hat{k}_3\right] = \alpha l\left\{\left(R_1\Delta_{12}-\frac{\beta\Delta_2}{2}\right)\frac{I_2 l^2}{6} - \frac{1}{2}\Delta_{12}I_2\hat{k}_3\right\}$$

Substitute the $B_{T_{a_{ij}}}, B_{T_{b_{ij}}}$ on 2 previous pages to get

$$B_T = \alpha l \left[\begin{array}{c|c|c|c|c}
-k_a & 0 & 0 & 0 & 0 \\
0 & (R_1\Delta_1 - \beta\Delta_{12})\frac{I_1 l}{12} & -(R_1\Delta_1 - \beta\Delta_{12})\frac{I_1 l}{12} & -(R_1\Delta_{12} - \beta\Delta_2)\frac{I_2 l}{12} & (R_1\Delta_{12} - \beta\Delta_2)\frac{I_2 l}{12} \\
0 & -(R_2\Delta_{12} - \beta\Delta_1)\frac{I_1 l}{12} & (R_2\Delta_{12} - \beta\Delta_1)\frac{I_1 l}{12} & (R_2\Delta_2 - \beta\Delta_{12})\frac{I_2 l}{12} & -(R_2\Delta_2 - \beta\Delta_{12})\frac{I_2 l}{12} \\
0 & 0 & 0 & 0 & 0 \\
0 & -(R_2\Delta_{12} + \beta\Delta_1)\frac{I_1 l^2}{12} + \frac{\Delta_{12} I_1}{2}\hat{k}_2 & -R_2\Delta_{12}\frac{I_1 l^2}{6} + \frac{\Delta_{12} I_1}{2}\hat{k}_2 & (R_2\Delta_2 + \beta\Delta_{12})\frac{I_2 l^2}{12} - \frac{\Delta_2 I_2}{2}\hat{k}_2 & R_2\Delta_2\frac{I_2 l^2}{6} - \frac{\Delta_2 I_2}{2}\hat{k}_2 \\
0 & -(R_1\Delta_1 + \beta\Delta_{12})\frac{I_1 l^2}{12} + \frac{\Delta_1 I_1}{2}\hat{k}_1 & -R_1\Delta_1\frac{I_1 l^2}{6} + \frac{\Delta_1 I_1}{2}\hat{k}_1 & (R_1\Delta_{12} + \beta\Delta_2)\frac{I_2 l^2}{12} - \frac{\Delta_{12} I_2}{2}\hat{k}_1 & R_1\Delta_{12}\frac{I_2 l^2}{6} - \frac{\Delta_{12} I_2}{2}\hat{k}_1 \\
\hline
k_a & 0 & 0 & 0 & 0 \\
0 & -(R_1\Delta_1 - \beta\Delta_{12})\frac{I_1 l}{12} & (R_1\Delta_1 - \beta\Delta_{12})\frac{I_1 l}{12} & (R_1\Delta_{12} - \beta\Delta_2)\frac{I_2 l}{12} & -(R_1\Delta_{12} - \beta\Delta_2)\frac{I_2 l}{12} \\
0 & (R_2\Delta_{12} - \beta\Delta_1)\frac{I_1 l}{12} & -(R_2\Delta_{12} - \beta\Delta_1)\frac{I_1 l}{12} & -(R_2\Delta_2 - \beta\Delta_{12})\frac{I_2 l}{12} & (R_2\Delta_2 - \beta\Delta_{12})\frac{I_2 l}{12} \\
0 & 0 & 0 & 0 & 0 \\
0 & -R_2\Delta_{12}\frac{I_1 l^2}{12} + \frac{\Delta_{12} I_1}{2}\hat{k}_4 & -(R_2\Delta_{12} - \frac{\beta\Delta_1}{2})\frac{I_1 l^2}{6} + \frac{\Delta_{12} I_1}{2}\hat{k}_4 & R_2\Delta_2\frac{I_2 l^2}{12} - \frac{\Delta_2 I_2}{2}\hat{k}_4 & (R_2\Delta_2 - \frac{\beta\Delta_{12}}{2})\frac{I_2 l^2}{6} - \frac{\Delta_2 I_2}{2}\hat{k}_4 \\
0 & -R_1\Delta_1\frac{I_1 l^2}{12} + \frac{\Delta_1 I_1}{2}\hat{k}_3 & -(R_1\Delta_1 - \frac{\beta\Delta_{12}}{2})\frac{I_1 l^2}{6} + \frac{\Delta_1 I_1}{2}\hat{k}_3 & R_1\Delta_{12}\frac{I_2 l^2}{12} - \frac{\Delta_{12} I_2}{2}\hat{k}_3 & (R_1\Delta_{12} - \frac{\beta\Delta_2}{2})\frac{I_2 l^2}{6} - \frac{\Delta_{12} I_2}{2}\hat{k}_3
\end{array}\right] \qquad \text{II-33}$$

Then $\quad P_T = B_T T'$

The values for $R_1, R_2, \beta, \hat{k}_i$ from section I can be substituted into the $B_{T_{ij}}$ (in rows 2,3,5,6 and 8,9,11,12):

Row 2: $B_{T_{a_{22}}} = \alpha l (R_1\Delta_1 - \beta\Delta_{12})\frac{I_1 l}{12} = \alpha \cancel{l}\left[\frac{\cancel{12}EI_1}{l^{\cancel{3}}}(1+\gamma_1)^{-1}\Delta_1 - \frac{\cancel{12}EI_{12}}{l^{\cancel{3}}}\Delta_{12}\right]\frac{I_1\cancel{l}}{\cancel{12}}$

$$= \frac{\alpha EI_1}{l}\left[(1+\gamma_1)^{-1}I_1\Delta_1 - I_{12}\Delta_{12}\right]$$

$B_{T_{a_{23}}} = -\alpha l (R_1\Delta_1 - \beta\Delta_{12})\frac{I_1 l}{12} = -\frac{\alpha EI_1}{l}\left[(1+\gamma_1)^{-1}I_1\Delta_1 - I_{12}\Delta_{12}\right]$

$B_{T_{a_{24}}} = -\alpha l (R_1\Delta_{12} - \beta\Delta_2)\frac{I_2 l}{12} = -\alpha\cancel{l}\left[\frac{\cancel{12}EI_1}{l^{\cancel{3}}}(1+\gamma_1)^{-1}\Delta_{12} - \frac{\cancel{12}EI_{12}}{l^{\cancel{3}}}\Delta_2\right]\frac{I_2\cancel{l}}{\cancel{12}}$

$$= \frac{\alpha EI_2}{l}\left[I_{12}\Delta_2 - (1+\gamma_1)^{-1}I_1\Delta_{12}\right] \quad \text{but } I_{12}\Delta_2 = I_1\Delta_{12}$$

$$= \frac{\alpha EI_1I_2\Delta_{12}}{l}\left[1-(1+\gamma_1)^{-1}\right]$$

$B_{T_{a_{25}}} = -\alpha l (R_1\Delta_{12} - \beta\Delta_2)\frac{I_2 l}{12} = -\frac{\alpha EI_1I_2\Delta_{12}}{l}\left[1-(1+\gamma_1)^{-1}\right]$

Row 3: $B_{T_{a_{32}}} = -\alpha l (R_2\Delta_{12} - \beta\Delta_1)\frac{I_1 l}{12} = -\alpha\cancel{l}\left[\frac{\cancel{12}EI_2}{l^{\cancel{3}}}(1+\gamma_2)^{-1}\Delta_{12} - \frac{\cancel{12}EI_{12}}{l^{\cancel{3}}}\Delta_1\right]\frac{I_1\cancel{l}}{\cancel{12}}$

$$= \frac{\alpha EI_1}{l}\left[I_{12}\Delta_1 - (1+\gamma_2)^{-1}I_2\Delta_{12}\right] \quad \text{but } I_{12}\Delta_1 = I_2\Delta_{12}$$

$$= \frac{\alpha EI_1I_2\Delta_{12}}{l}\left[1-(1+\gamma_2)^{-1}\right]$$

$B_{T_{a_{33}}} = \alpha l (R_2\Delta_{12} - \beta\Delta_1)\frac{I_1 l}{12} = -\frac{\alpha EI_1I_2\Delta_{12}}{l}\left[1-(1+\gamma_2)^{-1}\right]$

$B_{T_{a_{34}}} = \alpha l (R_2\Delta_2 - \beta\Delta_{12})\frac{I_2 l}{12} = \alpha\cancel{l}\left[\frac{\cancel{12}EI_2}{l^{\cancel{3}}}(1+\gamma_2)^{-1}\Delta_2 - \frac{\cancel{12}EI_{12}}{l^{\cancel{3}}}\Delta_{12}\right]\frac{I_2\cancel{l}}{\cancel{12}}$

$$= \frac{\alpha EI_2}{l}\left[(1+\gamma_2)^{-1}I_2\Delta_2 - I_{12}\Delta_{12}\right]$$

$B_{T_{a_{35}}} = -\alpha l (R_2\Delta_2 - \beta\Delta_{12})\frac{I_2 l}{12} = -\frac{\alpha EI_2}{l}\left[(1+\gamma_2)^{-1}I_2\Delta_2 - I_{12}\Delta_{12}\right]$

Row 4 - Null

<u>Row 5:</u>

$$B_{T_{a_{52}}} = \alpha l\left\{-(R_2\Delta_{12}+\beta\Delta_1)\frac{I_1 l^2}{12} + \frac{\Delta_{12} I_1}{2}\overbrace{\left(\frac{R_2 l^2}{4}+\frac{EI_2}{l}\right)}^{\hat{k}_2}\right\} = \alpha l\left\{R_2\Delta_{12}I_1 l^2\left(-\frac{1}{12}+\frac{1}{8}\right) - \frac{\cancel{12}EI_{12}}{l^3}\Delta_1\frac{I_1 l^2}{\cancel{12}} + \frac{EI_1I_2\Delta_{12}}{2l}\right\}$$

$$= \alpha \cancel{l}\left\{\frac{12EI_2}{l^3}(1+\gamma_2)^{-1}\Delta_{12}I_1 l^2\left(\frac{1}{24}\right) - \frac{EI_1I_{12}\Delta_1}{l} + \frac{EI_1I_2\Delta_{12}}{2l}\right\}$$

$$= \alpha E\left\{\frac{1}{2}(1+\gamma_2)^{-1}I_1I_2\Delta_{12} - I_1I_{12}\Delta_1 + \frac{1}{2}I_1I_2\Delta_{12}\right\}$$ but $I_2\Delta_{12} = I_{12}\Delta_1$

$$= -\frac{\alpha E I_1 I_2\Delta_{12}}{2}\left[1-(1+\gamma_2)^{-1}\right]$$

$$B_{T_{a_{53}}} = \alpha l\left[-R_2\Delta_{12}\frac{I_1 l^2}{6} + \frac{1}{2}\Delta_{12}I_1\left(\frac{R_2 l^2}{4}+\frac{EI_2}{l}\right)\right] = \alpha l\left[R_2\Delta_{12}I_1 l^2\left(-\frac{1}{6}+\frac{1}{8}\right) + \frac{1}{2}\Delta_{12}I_1\frac{EI_2}{l}\right]$$

$$= \alpha l\left[\frac{12EI_2}{l^3}(1+\gamma_2)^{-1}\Delta_{12}I_1 l^2\left(-\frac{1}{24}\right) + \frac{1}{2l}EI_1I_2\Delta_{12}\right]$$

$$= \frac{\alpha E I_1 I_2\Delta_{12}}{2}\left[1-(1+\gamma_2)^{-1}\right]$$

$$B_{T_{a_{54}}} = \alpha l\left\{(R_2\Delta_2+\beta\Delta_{12})\frac{I_2 l^2}{12} - \frac{1}{2}\Delta_2 I_2\left(\frac{R_2 l^2}{4}+\frac{EI_2}{l}\right)\right\} = \alpha l\left\{R_2\Delta_2 I_2 l^2\left(\frac{1}{12}-\frac{1}{8}\right) + \frac{\cancel{12}EI_{12}}{l^3}\Delta_{12}\frac{I_2 l^2}{\cancel{12}} - \frac{EI_2^2\Delta_2}{2l}\right\}$$

$$= \alpha \cancel{l}\left\{\frac{\cancel{12}EI_2}{\cancel{l^3}}(1+\gamma_2)^{-1}\Delta_2 I_2 \cancel{l^2}\left(-\frac{1}{\cancel{24}\,2}\right) + \frac{EI_2I_{12}\Delta_{12}}{l} - \frac{EI_2^2\Delta_2}{2l}\right\}$$

$$= \frac{\alpha EI_2}{2}\left\{-\left[(1+\gamma_2)^{-1}+1\right]I_2\Delta_2 + 2I_{12}\Delta_{12}\right\} = -\frac{\alpha EI_2}{2}\left\{\left[1+(1+\gamma_2)^{-1}\right]I_2\Delta_2 - 2I_{12}\Delta_{12}\right\}$$

$$B_{T_{a_{55}}} = \alpha l\left[R_2\Delta_2\frac{I_2 l^2}{6} - \frac{1}{2}\Delta_2 I_2\left(\frac{R_2 l^2}{4}+\frac{EI_2}{l}\right)\right] = \alpha l\left[R_2\Delta_2 I_2 l^2\left(\frac{1}{6}-\frac{1}{8}\right) - \frac{E}{2l}\Delta_2 I_2^2\right]$$

$$= \alpha \cancel{l}\left[\frac{\cancel{12}EI_2}{\cancel{l^3}}(1+\gamma_2)^{-1}\Delta_2 I_2 \cancel{l^2}\left(\frac{1}{\cancel{24}\,2}\right) - \frac{E}{2\cancel{l}}\Delta_2 I_2^2\right]$$

$$= -\frac{\alpha E I_2^2\Delta_2}{2}\left[1-(1+\gamma_2)^{-1}\right]$$

Row 8 $(B_{T_{b_{2j}}})$

$$B_{T_{b_{22}}} = -\alpha l \left(R_1\Delta_1 - \beta\Delta_{12}\right)\frac{I_1 l}{12} = -B_{T_{a_{22}}} = -\frac{\alpha E I_1}{l}\left\{(1+\gamma_1)^{-1} I_1\Delta_1 - I_{12}\Delta_{12}\right]$$

$$B_{T_{b_{23}}} = \alpha l \left(R_1\Delta_1 - \beta\Delta_{12}\right)\frac{I_1 l}{12} = -B_{T_{a_{23}}} = -\frac{\alpha E I_1}{l}\left[(1+\gamma_1)^{-1} I_1\Delta_1 - I_{12}\Delta_{12}\right]$$

$$B_{T_{b_{24}}} = \alpha l \left(R_1\Delta_{12} - \beta\Delta_2\right)\frac{I_2 l}{12} = -B_{T_{a_{24}}} = -\frac{\alpha E I_1 I_2 \Delta_{12}}{l}\left[1-(1+\gamma_1)^{-1}\right]$$

$$B_{T_{b_{25}}} = -\alpha l \left(R_1\Delta_{12} - \beta\Delta_2\right)\frac{I_2 l}{12} = -B_{T_{a_{25}}} = \frac{\alpha E I_1 I_2 \Delta_{12}}{l}\left[1-(1+\gamma_1)^{-1}\right]$$

Row 9 $B_{T_{b_{3j}}}$

$$B_{T_{b_{32}}} = \alpha l \left(R_2\Delta_{12} - \beta\Delta_1\right)\frac{I_1 l^2}{12} = -B_{T_{a_{32}}} = -\frac{\alpha E I_1 I_2 \Delta_{12}}{l}\left[1-(1+\gamma_2)^{-1}\right]$$

$$B_{T_{b_{33}}} = -\alpha l \left(R_2\Delta_{12} - \beta\Delta_1\right)\frac{I_1 l}{12} = -B_{T_{a_{33}}} = \frac{\alpha E I_1 I_2 \Delta_{12}}{l}\left[1-(1+\gamma_2)^{-1}\right]$$

$$B_{T_{b_{34}}} = -\alpha l \left(R_2\Delta_2 - \beta\Delta_{12}\right)\frac{I_2 l}{12} = -B_{T_{a_{34}}} = -\frac{\alpha E I_2}{l}\left[(1+\gamma_2)^{-1} I_2\Delta_2 - I_{12}\Delta_{12}\right]$$

$$B_{T_{b_{35}}} = \alpha l \left(R_2\Delta_2 - \beta\Delta_{12}\right)\frac{I_2 l}{12} = -B_{T_{a_{35}}} = -\frac{\alpha E I_2}{l}\left[(1+\gamma_2)^{-1} I_2\Delta_2 - I_{12}\Delta_{12}\right]$$

Row 11 $B_{T_{b_{5j}}}$

$$B_{T_{b_{52}}} = \alpha l\left\{-R_2\Delta_{12}\frac{I_1 l^2}{12} + \frac{\Delta_{12}}{2}I_1\left(\frac{R_2 l^2}{4} - \frac{EI_2}{l}\right)\right\} = \alpha l\left\{R_2\Delta_{12}I_1 l^2\left(-\frac{1}{12}+\frac{1}{8}\right) - \frac{1}{2l}E\Delta_{12}I_1I_2\right\}$$

$$= \alpha \cancel{l}\left\{\frac{\cancel{12}EI_2}{\cancel{l^3}}(1+\gamma_2)^{-1}\Delta_{12}I_1\cancel{l^2}\left(\frac{1}{\cancel{24}_{2}}\right) - \frac{1}{2\cancel{l}}E\Delta_{12}I_1I_2\right\}$$

$$= -\frac{\alpha E I_1 I_2 \Delta_{12}}{2}\left[1-(1+\gamma_2)^{-1}\right]$$

$$B_{T_{b_{53}}} = \alpha l\left\{-\left(R_2\Delta_{12} - \frac{\beta\Delta_1}{2}\right)\frac{I_1 l^2}{6} + \frac{1}{2}\Delta_{12}I_1\left(\frac{R_2 l^2}{4} - \frac{EI_2}{l}\right)\right\} = \alpha l\left\{R_2\Delta_{12}I_1 l^2\left(-\frac{1}{6}+\frac{1}{8}\right) + \frac{\beta}{12}\Delta_1 I_1 l^2 - \frac{E}{2l}\Delta_{12}I_1I_2\right\}$$

$$= \alpha\cancel{l}\left\{\frac{12EI_2}{\cancel{l^3}}(1+\gamma_2)^{-1}\Delta_{12}I_1\cancel{l^2}\left(-\frac{1}{24}\right) + \frac{\cancel{12}EI_{12}}{\cancel{12}\cancel{l^3}}\Delta_1 I_1\cancel{l^2} - \frac{E}{2\cancel{l}}\Delta_{12}I_1I_2\right\}$$

$$= \alpha E\left\{-\frac{1}{2}(1+\gamma_2)^{-1}I_1I_2\Delta_{12} + I_1\underline{I_{12}\Delta_1} - \frac{1}{2}\Delta_{12}I_1I_2\right\} \qquad \text{but } I_{12}\Delta_1 = I_2\Delta_{12}$$

$$= \alpha E I_1 I_2 \Delta_{12}\left\{-\frac{1}{2}(1+\gamma_2)^{-1} + 1 - \frac{1}{2}\right\}$$

$$= \frac{\alpha E I_1 I_2 \Delta_{12}}{2}\left[1-(1+\gamma_2)^{-1}\right]$$

$$B_{T_{b\,54}} = \alpha l\left[R_2\Delta_2\frac{I_2 l^2}{12} - \frac{\Delta_2 I_2}{2}\left(\frac{R_2 l^2}{4} - \frac{EI_2}{l}\right)\right] = \alpha l\left[R_2\Delta_2 I_2 l^2\left(\frac{1}{12}-\frac{1}{8}\right) + \frac{EI_2^2\Delta_2}{2l}\right]$$

$$= \alpha \cancel{l}\left[\frac{12EI_2}{\cancel{l^3}}(1+\gamma_2)^{-1}\Delta_2 I_2 \cancel{l^2}\left(-\frac{1}{24}\right) + \frac{EI_2^2\Delta_2}{2\cancel{l}}\right]$$

$$= \frac{\alpha E I_2^2\Delta_2}{2}\left[1 - (1+\gamma_2)^{-1}\right]$$

$$B_{T_{b\,56}} = \alpha l\left\{\left(R_2\Delta_2 - \frac{\beta\Delta_{12}}{2}\right)\frac{I_2 l^2}{6} - \frac{\Delta_2 I_2}{2}\left(\frac{R_2 l^2}{4} - \frac{EI_2}{l}\right)\right\} = \alpha l\left\{R_2\Delta_2 I_2 l^2\left(\frac{1}{6}-\frac{1}{8}\right) - \frac{12EI_{12}}{l^3}\frac{I_2 l^2}{12}\Delta_{12} + \frac{EI_2^2\Delta_2}{2l}\right\}$$

$$= \alpha\cancel{l}\left[\frac{12EI_2}{\cancel{l^3}}(1+\gamma_2)^{-1}\Delta_2 I_2 \cancel{l^2}\left(\frac{1}{\cancel{24}_2}\right) - \frac{EI_2 I_{12}\Delta_{12}}{\cancel{l}} + \frac{EI_2^2\Delta_2}{2\cancel{l}}\right]$$

$$= \frac{\alpha E I_2}{2}\left\{\left[1 + (1+\gamma_2)^{-1}\right] I_2\Delta_2 - 2 I_{12}\Delta_{12}\right\}$$

<u>Row 12</u> ($B_{T_{b6j}}$)

$$B_{T_{b\,62}} = \alpha l\left[-R_1\Delta_1\frac{I_1 l^2}{12} + \frac{\Delta_1 I_1}{2}\left(\frac{R_1 l^2}{4} - \frac{EI_1}{l}\right)\right] = \alpha l\left[R_1\Delta_1 I_1 l^2\left(-\frac{1}{12}+\frac{1}{8}\right) - \frac{EI_1^2\Delta_1}{2l}\right]$$

$$= \alpha\cancel{l}\left[\frac{12EI_1}{\cancel{l^3}}(1+\gamma_1)^{-1}\Delta_1 I_1 \cancel{l^2}\left(\frac{1}{\cancel{24}_2}\right) - \frac{EI_1^2\Delta_1}{2\cancel{l}}\right]$$

$$= -\frac{\alpha E I_1^2\Delta_1}{2}\left[1 - (1+\gamma_1)^{-1}\right]$$

$$B_{T_{b\,63}} = \alpha l\left\{-\left(R_1\Delta_1 - \frac{\beta\Delta_{12}}{2}\right)\frac{I_1 l^2}{6} + \frac{\Delta_1 I_1}{2}\left(\frac{R_1 l^2}{4} - \frac{EI_1}{l}\right)\right\} = \alpha l\left\{R_1\Delta_1 I_1 l^2\left(-\frac{1}{6}+\frac{1}{8}\right) + \frac{12EI_{12}}{l^3}\frac{\Delta_{12} I_1 l^2}{12} - \frac{\Delta_1 E I_1^2}{2l}\right\}$$

$$= \alpha l\left\{\frac{12EI_1}{\cancel{l^3}}(1+\gamma_1)^{-1}\Delta_1 I_1 \cancel{l^2}\left(-\frac{1}{\cancel{24}_2}\right) + \frac{EI_1 I_{12}\Delta_{12}}{\cancel{l}} - \frac{EI_1^2\Delta_1}{2\cancel{l}}\right\}$$

$$= -\frac{\alpha E I_1}{2}\left\{\left[1 + (1+\gamma_1)^{-1}\right] I_1\Delta_1 - 2 I_{12}\Delta_{12}\right\}$$

$$B_{T_{b\,64}} = \alpha l\left[R_1\Delta_{12}\frac{I_2 l^2}{12} - \frac{\Delta_{12} I_2}{2}\left(\frac{R_1 l^2}{4} - \frac{EI_1}{l}\right)\right] = \alpha l\left[R_1\Delta_{12} I_2 l^2\left(\frac{1}{12}-\frac{1}{8}\right) + \frac{EI_1 I_2\Delta_{12}}{2l}\right]$$

$$= \alpha\cancel{l}\left[\frac{12EI_1}{\cancel{l^3}}(1+\gamma_1)^{-1}\Delta_{12} I_2 \cancel{l^2}\left(-\frac{1}{\cancel{24}_2}\right) + \frac{EI_1 I_2\Delta_{12}}{2\cancel{l}}\right]$$

$$= \frac{\alpha E I_1 I_2\Delta_{12}}{2}\left[1 - (1+\gamma_1)^{-1}\right]$$

$$B_{T_{b\,65}} = \alpha l\left\{\left(R_1\Delta_{12} - \frac{\beta\Delta_2}{2}\right)\frac{I_2 l^2}{6} - \frac{\Delta_{12} I_2}{2}\left(\frac{R_1 l^2}{4} - \frac{EI_1}{l}\right)\right\} = \alpha l\left\{R_1\Delta_{12} I_2 l^2\left(\frac{1}{6}-\frac{1}{8}\right) - \frac{12EI_{12}}{l^3}\frac{\Delta_2 I_2 l^2}{12} + \frac{EI_1 I_2\Delta_{12}}{2l}\right\}$$

$$= \alpha\cancel{l}\left\{\frac{12EI_1}{\cancel{l^3}}(1+\gamma_1)^{-1}\Delta_{12} I_2 \cancel{l^2}\left(\frac{1}{\cancel{24}_2}\right) - \frac{EI_2 I_{12}\Delta_2}{\cancel{l}} + \frac{EI_1 I_2\Delta_{12}}{2\cancel{l}}\right\} \quad \text{but } I_{12}\Delta_2 = I_1\Delta_{12}$$

$$= -\frac{\alpha E I_1 I_2\Delta_{12}}{2}\left[1 - (1+\gamma_1)^{-1}\right]$$

Substitute the $B_{T_{a_{ij}}}$, $B_{T_{b_{ij}}}$ on previous 2 pages into eqn II-32 to get

$$
B_T = \frac{\alpha E}{\ell}
\left[\begin{array}{ccccc}
-A\ell & 0 & 0 & 0 & 0 \\
0 & I_1\left[(1+\gamma_1)^{-1} I_1\Delta_1 - I_{12}\Delta_{12}\right] & -I_1\left[(1+\gamma_1)^{-1} I_1\Delta_1 - I_{12}\Delta_{12}\right] & I_1 I_2 \Delta_{12}\left[1-(1+\gamma_1)^{-1}\right] & -I_1 I_2 \Delta_{12}\left[1-(1+\gamma_1)^{-1}\right] \\
0 & I_1 I_2 \Delta_{12}\left[1-(1+\gamma_2)^{-1}\right] & -I_1 I_2 \Delta_{12}\left[1-(1+\gamma_2)^{-1}\right] & I_2\left[(1+\gamma_2)^{-1} I_2\Delta_2 - I_{12}\Delta_{12}\right] & -I_2\left[(1+\gamma_2)^{-1} I_2\Delta_2 - I_{12}\Delta_{12}\right] \\
0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} I_1 I_2 \Delta_{12}\ell\left[1-(1+\gamma_2)^{-1}\right] & \frac{1}{2} I_1 I_2 \Delta_{12}\ell\left[1-(1+\gamma_2)^{-1}\right] & -\frac{1}{2} I_2\ell\left\{\left[1+(1+\gamma_2)^{-1}\right] I_2\Delta_2 - 2I_{12}\Delta_{12}\right\} & -\frac{1}{2} I_2^2\Delta_2\ell\left[1-(1+\gamma_2)^{-1}\right] \\
0 & \frac{1}{2} I_1\ell\left\{\left[1+(1+\gamma_1)^{-1}\right] I_1\Delta_1 - 2I_{12}\Delta_{12}\right\} & \frac{1}{2} I_1^2\Delta_1\ell\left[1-(1+\gamma_1)^{-1}\right] & \frac{1}{2} I_1 I_2 \Delta_{12}\ell\left[1-(1+\gamma_1)^{-1}\right] & -\frac{1}{2} I_1 I_2 \Delta_{12}\ell\left[1-(1+\gamma_1)^{-1}\right] \\
\hline
A\ell & 0 & 0 & 0 & 0 \\
0 & (=-B_{T_{22}}) & (=-B_{T_{23}}) & (=-B_{T_{24}}) & (=-B_{T_{25}}) \\
0 & (=-B_{T_{32}}) & (=-B_{T_{33}}) & (=-B_{T_{34}}) & (=-B_{T_{35}}) \\
0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} I_1 I_2 \Delta_{12}\ell\left[1-(1+\gamma_2)^{-1}\right] & \frac{1}{2} I_1 I_2 \Delta_{12}\ell\left[1-(1+\gamma_2)^{-1}\right] & \frac{1}{2} I_2^2\Delta_2\ell\left[1-(1+\gamma_2)^{-1}\right] & \frac{1}{2} I_2\ell\left\{\left[1+(1+\gamma_2)^{-1}\right] I_2\Delta_2 - 2I_{12}\Delta_{12}\right\} \\
0 & -\frac{1}{2} I_1^2\Delta_1\ell\left[1-(1+\gamma_1)^{-1}\right] & -\frac{1}{2} I_1\ell\left\{\left[1+(1+\gamma_1)^{-1}\right] I_1\Delta_1 - 2I_{12}\Delta_{12}\right\} & \frac{1}{2} I_1 I_2 \Delta_{12}\ell\left[1-(1+\gamma_1)^{-1}\right] & -\frac{1}{2} I_1 I_2 \Delta_{12}\ell\left[1-(1+\gamma_1)^{-1}\right]
\end{array}\right]
\qquad \text{eqn II-34}
$$

This is the final form of explicit B_T when there are no pinflags and is used only in checkout of P_{T_e}.

The form used in MYSTRAN is eqn II-32 since K_{aa}, if there are pinflags, is not what was used to get eqn II-34 above

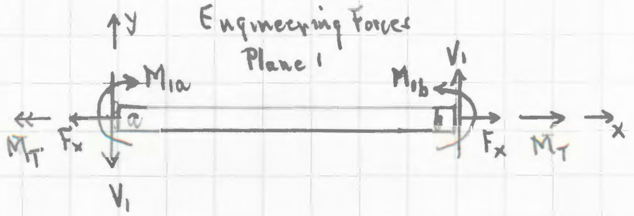
$B_T = \frac{\alpha E}{\lambda}$

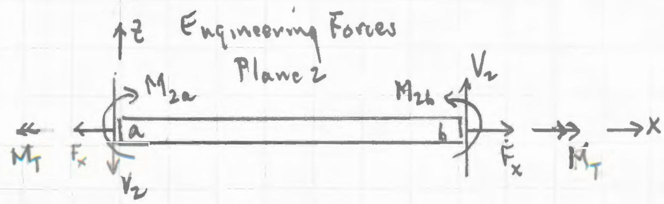
$-A\ell$				
	I_1	$-I_1$		
			I_2	$-I_2$
			$-I_2\ell$	
	$I_1\ell$			
$A\lambda$				
	$-I_1$	I_1		
			$-I_2$	I_2
				$I_2\lambda$
		$-I_1\lambda$		

when $\gamma_1 = \gamma_2 = 0$

and using $\begin{cases} I_1\Delta_1 - I_{12}\Delta_{12} = 1 \\ I_2\Delta_2 - I_{12}\Delta_{12} = 1 \end{cases}$

Stresses





The axial stress (Rivello) is (assume centroidal, but not principal, axis)

$$\sigma_x = \frac{F_x}{A} - \underbrace{\frac{M_1 I_2 - M_2 I_{12}}{I_1 I_2 - I_{12}^2}}_{k_1'} y - \underbrace{\frac{M_2 I_1 - M_1 I_{12}}{I_1 I_2 - I_{12}^2}}_{k_2'} z - E\alpha\left[T(x,y,z) - T_{ref}\right] \qquad (1)$$

where

$$\left.\begin{aligned} F_x &= \bar{F}_x + F_{x_T} \\ M_1 &= \bar{M}_1 + M_{1_T} \\ M_2 &= \bar{M}_2 + M_{2_T} \end{aligned}\right\} \qquad (2)$$

$\bar{F}_x, \bar{M}_1, \bar{M}_2$ are axial force & moments without equivalent thermal loads

$F_{x_T}, M_{1_T}, M_{2_T}$ are equivalent thermal loads

$$\left.\begin{aligned} F_{x_T} &= E\alpha \int_A (T - T_{ref})\, dA \\ M_{1_T} &= -E\alpha \int_A y(T - T_{ref})\, dA \\ M_{2_T} &= -E\alpha \int_A z(T - T_{ref})\, dA \end{aligned}\right\} \qquad (3)$$

As a temp. distribution, T, we will assume T varies linearly along the length

$$T(x,y,z) = \left(1 - \frac{x}{l}\right) T_a(y,z) + \frac{x}{l} T_b(y,z) \qquad (4)$$

where

$T_a(y,z) =$ variation in temp. at end a in y, z directions

$T_b(y,z) =$ " " " " " b " " "

Write $T_a(y,z)$ and $T_b(y,z)$ as a constant plus linear variations plus higher order terms

$$T_a(y,z) = \bar{T}_a + T_{,y_a}\, y + T_{,z_a}\, z + \delta T_a(y,z)$$
$$T_b(y,z) = \bar{T}_b + T_{,y_b}\, y + T_{,z_b}\, z + \delta T_b(y,z) \qquad (5)$$

where

$\bar{T}_a = T_a(0,0)$ (temp at end a at centroid)

$\bar{T}_b = T_b(0,0)$ (" " " b " ")

$T_{,y_a} = \left.\dfrac{dT_a(y,z)}{dy}\right|_{y=z=0}$ (slope of T in y direction at end a centroid)

$T_{,z_a} = \left.\dfrac{dT_a(y,z)}{dz}\right|_{y=z=0}$ (" " " " z " " " a ")

$T_{,y_b} = \left.\dfrac{dT_b(y,z)}{dy}\right|_{y=z=0}$ (" " " " y " " " b ") (6)

$T_{,z_b} = \left.\dfrac{dT_b(y,z)}{dz}\right|_{y=z=0}$ (" " " " z " " " " ")

$\delta T_a(y,z) =$ Variation in T (from linear distribution) at end a

$\delta T_b(y,z) =$ " " " " " " " " b

Substitute (5) into (4)

$$T(x,y,z) = \left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b\right] + \left(1-\frac{x}{l}\right)\left(T_{,y_a}\, y + T_{,z_a}\, z\right) + \frac{x}{l}\left(T_{,y_b}\, y + T_{,z_b}\, z\right)$$
$$+ \left(1-\frac{x}{l}\right)\delta T_a(y,z) + \frac{x}{l}\,\delta T_b(y,z) \qquad (7)$$

Substitute (7) into (3)

$$F_{x_T} = E\alpha \int_A \left\{ \left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b \right] - T_{ref} \right\} dA + E\alpha \int_A \left\{ \left(1-\frac{x}{l}\right)\left(T_{,y_a} y + T_{,z_a} z\right) + \frac{x}{l}\left(T_{,y_b} y + T_{,z_b} z\right) \right\} dA$$
$$+ E\alpha \int_A \left\{ \left(1-\frac{x}{l}\right)\delta T_a(y,z) + \frac{x}{l}\delta T_b(y,z) \right\} dA$$

$$M_{1_T} = -E\alpha \int_A \left\{ \left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b \right] - T_{ref} \right\} y\,dA - E\alpha \int_A \left\{ \left(1-\frac{x}{l}\right)\left(T_{,y_a} y + T_{,z_a} z\right) + \frac{x}{l}\left(T_{,y_b} y + T_{,z_b} z\right) \right\} y\,dA$$
$$- E\alpha \int_A \left\{ \left(1-\frac{x}{l}\right)\delta T_a(y,z) + \frac{x}{l}\delta T_b(y,z) \right\} y\,dA$$

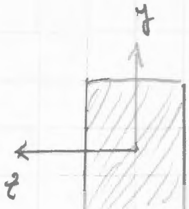
$$M_{2_T} = -E\alpha \int_A \left\{ \left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b \right] - T_{ref} \right\} z\,dA - E\alpha \int_A \left\{ \left(1-\frac{x}{l}\right)\left(T_{,y_a} y + T_{,z_a} z\right) + \frac{x}{l}\left(T_{,y_b} y + T_{,z_b} z\right) \right\} z\,dA$$
$$- E\alpha \int_A \left\{ \left(1-\frac{x}{l}\right)\delta T_a(y,z) + \frac{x}{l}\delta T_b(y,z) \right\} z\,dA$$

Since the axes are centroidal $\int_A y\,dA = \int_A z\,dA = 0$. Also $\int_A dA = A$ so:

$$\left.\begin{aligned}
F_{x_T} &= AE\alpha \left\{ \left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b \right] - T_{ref} \right\} + E\alpha \int_A \left\{ \left(1-\frac{x}{l}\right)\delta T_a(y,z) + \frac{x}{l}\delta T_b(y,z) \right\} dA \\
M_{1_T} &= -E\alpha \Bigg\{ \left(1-\frac{x}{l}\right)\left[T_{,y_a} \int_A y^2\,dA + T_{,z_a} \int_A yz\,dA \right] + \frac{x}{l}\left[T_{,y_b} \int_A y^2\,dA + T_{,z_b} \int_A yz\,dA \right] \\
&\quad + \left(1-\frac{x}{l}\right) \int_A y\,\delta T_a(y,z)\,dA + \frac{x}{l} \int_A y\,\delta T_b(y,z)\,dA \Bigg\} \\
M_{2_T} &= -E\alpha \Bigg\{ \left(1-\frac{x}{l}\right)\left[T_{,y_a} \int_A yz\,dA + T_{,z_a} \int_A z^2\,dA \right] + \frac{x}{l}\left[T_{,y_b} \int_A yz\,dA + T_{,z_b} \int_A z^2\,dA \right] \\
&\quad + \left(1-\frac{x}{l}\right) \int_A z\,\delta T_a(y,z)\,dA + \frac{x}{l} \int_A z\,\delta T_b(y,z)\,dA \Bigg\}
\end{aligned}\right\} \quad (8)$$

where

$$\int_A y^2\,dA = I_1 \qquad (I_1 \text{ is } I_{zz})$$



$$\int_A z^2\,dA = I_2 \qquad (I_2 \text{ is } I_{yy}) \qquad (9)$$

$$\int_A yz\,dA = I_{12}$$

define

$$\delta T = \left(1-\frac{x}{l}\right)\delta T_a(y,z) + \frac{x}{l}\,\delta T_b(y,z) \quad \cdots \text{higher order temp. terms (higher than linear)} \qquad (10)$$

Substitute (9), (10) into (8)

$$F_{x_T} = AE\alpha\left\{\left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b\right] - T_{ref}\right\} + E\alpha\int_A \delta T\,dA \qquad (11)$$

$$M_{1T} = -E\alpha\left\{\left(1-\frac{x}{l}\right)\left(I_1 T_{,y_a} + I_{12}T_{,z_a}\right) + \frac{x}{l}\left(I_1 T_{,y_b} + I_{12}T_{,z_b}\right)\right\} - E\alpha\int_A y\,\delta T\,dA \qquad (12)$$

$$M_{2T} = -E\alpha\left\{\left(1-\frac{x}{l}\right)\left(I_{12} T_{,y_a} + I_2 T_{,z_a}\right) + \frac{x}{l}\left(I_{12} T_{,y_b} + I_2 T_{,z_b}\right)\right\} - E\alpha\int_A z\,\delta T\,dA \qquad (13)$$

Equation (1) can be written as

$$\sigma_x = \sigma_{x_p} + \Delta\sigma_x \qquad (14)$$

where

$$\sigma_{x_p} = \frac{\bar{F}_x}{A} - \frac{\bar{M}_1 I_2 - \bar{M}_2 I_{12}}{I_1 I_2 - I_{12}^2}\,y - \frac{\bar{M}_2 I_1 - \bar{M}_1 I_{12}}{I_1 I_2 - I_{12}^2}\,z \qquad (15)$$

$$\Delta\sigma_x = \frac{F_{x_T}}{A} - \frac{M_{1T} I_2 - M_{2T} I_{12}}{I_1 I_2 - I_{12}^2}\,y - \frac{M_{2T} I_1 - M_{1T} I_{12}}{I_1 I_2 - I_{12}^2}\,z - E\alpha\left[T(x,y,z) - T_{ref}\right] \qquad (16)$$

Evaluate the parts of (16)

$$\frac{F_{x_T}}{A} = E\alpha\left\{\left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b\right] - T_{ref}\right\} + \frac{E\alpha}{A}\int \Delta T\, dA \qquad (17)$$

and

$$\frac{M_{1T}I_2 - M_{2T}I_{12}}{I_1I_2 - I_{12}^2} = -\frac{E\alpha}{I_1I_2-I_{12}^2}\left\{\left(1-\frac{x}{l}\right)\left[\left(I_1T_{,ya} + I_{12}T_{,za}\right)I_2 - \left(I_{12}T_{,ya} + I_2T_{,za}\right)I_{12}\right]\right.$$
$$+ \frac{x}{l}\left[\left(I_1T_{,yb} + I_{12}T_{,zb}\right)I_2 - \left(I_{12}T_{,yb} + I_2T_{,zb}\right)I_{12}\right]$$
$$\left. + I_2\int_A y\,\delta T\, dA - I_{12}\int_A z\,\delta T\, dA\right\}$$

$$= -\frac{E\alpha}{I_1I_2-I_{12}^2}\left\{\left(1-\frac{x}{l}\right)\left[I_1I_2T_{,ya} + \cancel{I_2I_{12}T_{,za}} - I_{12}^2T_{,ya} - \cancel{I_2I_{12}T_{,za}}\right]\right.$$
$$+ \frac{x}{l}\left[I_1I_2T_{,yb} + \cancel{I_2I_{12}T_{,zb}} - I_{12}^2T_{,yb} - \cancel{I_2I_{12}T_{,zb}}\right]$$
$$\left. + I_2\int_A y\,\delta T\, dA - I_{12}\int_A z\,\delta T\, dA\right\}$$

$$\therefore \frac{M_{1T}I_2 - M_{2T}I_{12}}{I_1I_2 - I_{12}^2} = -E\alpha\left[\left(1-\frac{x}{l}\right)T_{,ya} + \frac{x}{l}T_{,yb}\right] - \frac{E\alpha}{I_1I_2-I_{12}^2}\left\{I_2\int_A y\,\delta T\, dA - I_{12}\int_A z\,\delta T\, dA\right\} \qquad (18)$$

and

$$\frac{M_{2T}I_1 - M_{1T}I_{12}}{I_1I_2 - I_{12}^2} = -\frac{E\alpha}{I_1I_2-I_{12}^2}\left\{\left(1-\frac{x}{l}\right)\left[\left(\cancel{I_{12}T_{,ya}} + I_2T_{,za}\right)I_1 - \left(\cancel{I_1T_{,ya}} + I_{12}T_{,za}\right)I_{12}\right]\right.$$
$$+ \frac{x}{l}\left[\left(\cancel{I_{12}T_{,yb}} + I_2T_{,zb}\right)I_1 - \left(\cancel{I_1T_{,yb}} + I_{12}T_{,zb}\right)I_{12}\right]$$
$$\left. + I_1\int_A z\,\Delta T\, dA - I_{12}\int_A y\,\Delta T\, dA\right\}$$

$$= -E\alpha\left[\left(1-\frac{x}{l}\right)T_{,za} + \frac{x}{l}T_{,zb}\right] - \frac{E\alpha}{I_1I_2-I_{12}^2}\left\{I_1\int_A z\,\delta T\, dA - I_{12}\int_A y\,\delta T\, dA\right\} \qquad (19)$$

Substitute (9) and (17)-(19) into (16)

$$\Delta\sigma_x = E\alpha\left\{\cancel{\left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b\right] - T_{ref}}\right\} + \frac{E\alpha}{A}\int_A \delta T\,dA$$
$$+ E\alpha\left\{\cancel{\left[\left(1-\frac{x}{l}\right)T_{,ya} + \frac{x}{l}T_{,yb}\right]} + \frac{E\alpha}{I_1 I_2 - I_{12}^2}\left[I_2\int_A y\,\delta T\,dA - I_{12}\int_A z\,\delta T\,dA\right]\right\} y$$
$$+ E\alpha\left\{\cancel{\left[\left(1-\frac{x}{l}\right)T_{,za} + \frac{x}{l}T_{,zb}\right]} + \frac{E\alpha}{I_1 I_2 - I_{12}^2}\left[I_1\int_A z\,\delta T\,dA - I_{12}\int_A y\,\delta T\,dA\right]\right\} z$$
$$- E\alpha\left\{\cancel{\left[\left(1-\frac{x}{l}\right)\bar{T}_a + \frac{x}{l}\bar{T}_b\right]} - T_{ref} + \left(1-\frac{x}{l}\right)\left(T_{,ya}\,y + T_{,za}\,z\right) + \frac{x}{l}\left(T_{,yb}\,y + T_{,zb}\,z\right) + \underbrace{\left(1-\frac{x}{l}\right)\delta T_a(y,z) + \frac{x}{l}\delta T_b(y,z)}_{\delta T}\right\}$$

$$\Delta\sigma_x = \frac{E\alpha}{A}\int_A \delta T\,dA + \frac{E\alpha}{I_1 I_2 - I_{12}^2}\left\{\left[I_2\int_A y\,\delta T\,dA - I_{12}\int_A z\,\delta T\,dA\right]y + \left[I_1\int_A z\,\delta T\,dA - I_{12}\int_A y\,\delta T\,dA\right]z\right\} - E\alpha\,\delta T \qquad (20)$$

$\Delta\sigma_x$ is not the thermal stress. It is the portion of the total thermal stress due to temperature that is higher order than linear.

The integral terms will be ignored in (19) and we will approximate:

$$\Delta\sigma_x \approx -E\alpha\,\delta T \qquad (21)$$

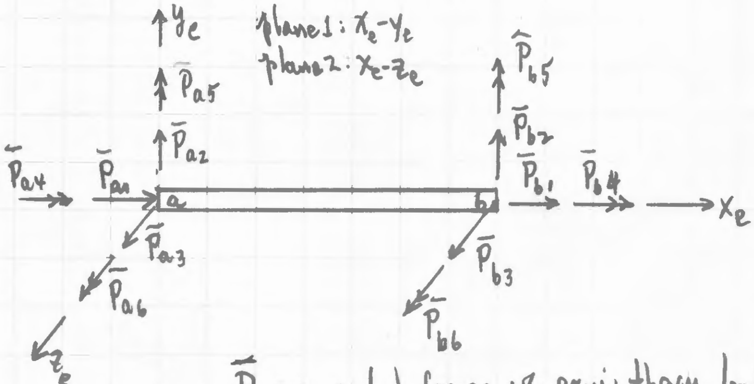
hus, in summary

$$\boxed{\begin{gathered}\sigma_x = \frac{\bar F_x}{A} - \frac{\bar M_1 I_2 - \bar M_2 I_{12}}{I_1 I_2 - I_{12}^2}\, y - \frac{\bar M_2 I_1 - \bar M_1 I_{12}}{I_1 I_2 - I_{12}^2}\, z - E\alpha\,\delta T \\ \bar F_x,\ \bar M_1,\ \bar M_2 \text{ are elem forces \underline{without} equivalent thermal loads} \\ \delta T = \left(1 - \frac{x}{\ell}\right)\delta T_a(y,z) + \frac{x}{\ell}\,\delta T_b(y,z) \\ \delta T_a(y,z),\ \delta T_b(y,z) \text{ are temp. variations \underline{above} linear}\end{gathered}} \qquad (22)$$

The output from MYSTRAN is for stresses at ends a $(x=0)$ and b $(x=\ell)$ denoted as σ_a and σ_b ;

$$\left.\begin{aligned}\sigma_a = \sigma_x(x=0) &= \frac{\bar F_x}{A} - \left(\frac{\bar M_{1a} I_2 - \bar M_{2a} I_{12}}{I_1 I_2 - I_{12}^2}\right) y - \left(\frac{\bar M_{2a} I_1 - \bar M_{1a} I_{12}}{I_1 I_2 - I_{12}^2}\right) z - E\alpha\,\delta T_a \\ \sigma_b = \sigma_x(x=\ell) &= \frac{\bar F_x}{A} - \left(\frac{\bar M_{1b} I_2 - \bar M_{2b} I_{12}}{I_1 I_2 - I_{12}^2}\right) y - \left(\frac{\bar M_{2b} I_1 - \bar M_{1b} I_{12}}{I_1 I_2 - I_{12}^2}\right) z - E\alpha\,\delta T_b\end{aligned}\right\} \qquad (23)$$

We need to relate $\bar F_x, \bar M_{1a}, \bar M_{2a}, \bar M_{1b}, \bar M_{2b}$ to element nodal forces. Comparing figure below with one on page III-1:



$\bar P$ are nodal forces w/o equiv. therm. loads

$\bar P = P - P_{Te}$

$$\left.\begin{aligned}
&7 \quad \bar F_x = -\bar P_{a1} \\
&1 \quad \bar M_{1a} = -\bar P_{a6} \\
&2 \quad \bar M_{2a} = \bar P_{a5} \\
&3 \quad \bar M_{1b} = \bar M_{1a} - V_1 \ell = -\bar P_{a6} + \bar P_{a2}\,\ell \\
&4 \quad \bar M_{2b} = \bar M_{2a} - V_2 \ell = \bar P_{a5} + \bar P_{a3}\,\ell
\end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned}
&\text{plane 1} \quad 5 \quad V_1 = \bar P_{b2} \\
&\text{plane 2} \quad 6 \quad V_2 = \bar P_{b6}
\end{aligned}\right\} \quad 24\ (a)$$

Substitute (24) into (23) and use

$$\Delta_1 = \frac{I_2}{I_1 I_2 - I_{12}^2}, \quad \Delta_2 = \frac{I_1}{I_1 I_2 - I_{12}^2}, \quad \Delta_{12} = \frac{I_{12}}{I_1 I_2 - I_{12}^2} \qquad (25)$$

to get:

$$\begin{aligned} \sigma_a &= -\frac{\bar{P}_{a1}}{A} - \left[-\Delta_1 \bar{P}_{a6} - \Delta_{12}\bar{P}_{a5}\right] y - \left[\Delta_2 \bar{P}_{a5} + \Delta_{12}\bar{P}_{a6}\right] - E\alpha\,\delta T_a(y,z) \\ \sigma_b &= -\frac{\bar{P}_{a1}}{A} - \left[\Delta_1(-\bar{P}_{a6} + \bar{P}_{a2} l) - \Delta_{12}(\bar{P}_{a5} + \bar{P}_{a3} l)\right] y \\ &\quad - \left[\Delta_2(\bar{P}_{a5} + \bar{P}_{a3} l) - \Delta_{12}(-\bar{P}_{a6} + \bar{P}_{a2} l)\right] z - E\alpha\,\delta T_b(y,z) \end{aligned} \qquad (26)$$

define

$$\begin{aligned} \sigma_{axial} &= -\frac{\bar{P}_{a1}}{A} \\ k'_{1a} &= -\Delta_1 \bar{P}_{a6} - \Delta_{12}\bar{P}_{a5} = \frac{\bar{M}_{1a} I_2 - \bar{M}_{2a} I_{12}}{I_1 I_2 - I_{12}^2} \\ k'_{2a} &= \Delta_2 \bar{P}_{a5} + \Delta_{12}\bar{P}_{a6} = \frac{\bar{M}_{2a} I_1 - \bar{M}_{1a} I_{12}}{I_1 I_2 - I_{12}^2} \\ k'_{1b} &= \Delta_1(-\bar{P}_{a6} + \bar{P}_{a2} l) - \Delta_{12}(\bar{P}_{a5} + \bar{P}_{a3} l) = \frac{\bar{M}_{1b} I_2 - \bar{M}_{2b} I_{12}}{I_1 I_2 - I_{12}^2} \\ k'_{2b} &= \Delta_2(\bar{P}_{a5} + \bar{P}_{a3} l) - \Delta_{12}(-\bar{P}_{a6} + \bar{P}_{a2} l) = \frac{\bar{M}_{2b} I_1 - \bar{M}_{1b} \Delta_{12}}{I_1 I_2 - I_{12}^2} \end{aligned} \qquad (27)$$

Then

$$\begin{aligned} \sigma_a &= \sigma_{axial} - k'_{1a} y - k'_{2a} z - E\alpha\,\delta T_a(y,z) \\ \sigma_b &= \sigma_{axial} - k'_{1b} y - k'_{2b} z - E\alpha\,\delta T_b(y,z) \end{aligned} \qquad (28)$$

In addition we can also calculate a torsional stress, τ:

$$\tau = \frac{c}{J} M_T = -\frac{c}{J}\bar{P}_{a4} \qquad (29)$$

Eqns (27) and (29) can be put into matrix form:

$$\left\{\begin{array}{c} \sigma_{axial} \\ k'_{1a} \\ k'_{2a} \\ \hline k'_{1b} \\ k'_{2b} \\ \tau \end{array}\right\} = \left[\begin{array}{cccccc} -1/A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta_{12} & -\Delta_1 \\ 0 & 0 & 0 & 0 & \Delta_2 & \Delta_{12} \\ \hline 0 & \Delta_1 l & -\Delta_{12} l & 0 & -\Delta_{12} & -\Delta_1 \\ 0 & -\Delta_{12} l & \Delta_2 l & 0 & \Delta_2 & \Delta_{12} \\ 0 & 0 & 0 & -C/J & 0 & 0 \end{array}\right] \left\{\begin{array}{c} \bar{P}_{a1} \\ \bar{P}_{a2} \\ \bar{P}_{a3} \\ \bar{P}_{a4} \\ \bar{P}_{a5} \\ \bar{P}_{a6} \end{array}\right\} \qquad (30)$$

Define

$$\sigma_1 = \left\{\begin{array}{c} \sigma_{axial} \\ k'_{1a} \\ k'_{2a} \end{array}\right\}, \quad \sigma_2 = \left\{\begin{array}{c} k'_{1b} \\ k'_{2b} \\ \tau \end{array}\right\}, \quad \bar{P}_a = \left\{\begin{array}{c} \bar{P}_{a1} \\ \vdots \\ \bar{P}_{a6} \end{array}\right\} \qquad (31)$$

$$B_1 = \left[\begin{array}{cccccc} -1/A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta_{12} & -\Delta_1 \\ 0 & 0 & 0 & 0 & \Delta_2 & \Delta_{12} \end{array}\right] \qquad (32)$$

$$B_2 = \left[\begin{array}{cccccc} 0 & \Delta_1 l & -\Delta_{12} l & 0 & -\Delta_{12} & -\Delta_1 \\ 0 & -\Delta_{12} l & \Delta_2 l & 0 & \Delta_2 & \Delta_{12} \\ 0 & 0 & 0 & -C/J & 0 & 0 \end{array}\right] \qquad (33)$$

Then

$$\left.\begin{aligned} \sigma_1 &= \begin{Bmatrix} \sigma_{axial} \\ k'_{1a} \\ k'_{2a} \end{Bmatrix} = B_1 \bar{P}_a \\ \sigma_2 &= \begin{Bmatrix} k'_{1b} \\ k'_{2b} \\ \tau \end{Bmatrix} = B_2 \bar{P}_a \end{aligned}\right\} \qquad (34)$$

The element force-displ relationship, partitioned into a, b are:

$$\begin{bmatrix} K_{aa} & K_{ab} \\ K_{ab}^T & K_{bb} \end{bmatrix} \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} = \begin{Bmatrix} P_a \\ P_b \end{Bmatrix} = \begin{Bmatrix} \bar{P}_a \\ \bar{P}_b \end{Bmatrix} + \begin{Bmatrix} P_{Ta} \\ P_{Tb} \end{Bmatrix} \qquad (35)$$

where P_{Ta}, P_{Tb} are the equivalent thermal loads from Section II. Thus

$$\bar{P}_a = K_{aa} u_a + K_{ab} u_b - P_{Ta} \qquad (36)$$

Substitute this into (34):

$$\left.\begin{aligned} \sigma_1 &= \left[B_1 K_{aa} \;\vdots\; B_1 K_{ab} \right] \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} - B_1 P_{Ta} \\ \sigma_2 &= \left[B_2 K_{aa} \;\vdots\; B_2 K_{ab} \right] \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} - B_2 P_{Ta} \end{aligned}\right\} \qquad (37)$$

but from Section II:

$$P_{Ta} = K_{aa} \bar{A} T' \qquad (38)$$

Thus, using (38) we can write (37) as

$$\sigma_1 = S_{e_1} U - S_{T_{e_1}} = \begin{Bmatrix} \sigma_{axial} \\ k'_{1a} \\ k'_{2a} \end{Bmatrix} \qquad \begin{cases} k'_{1a} = \dfrac{\bar{M}_{1a} I_2 - \bar{M}_{2a} I_{12}}{I_1 I_2 - I_{12}^2} \\ k'_{2a} = \dfrac{\bar{M}_{2a} I_1 - \bar{M}_{1a} I_{12}}{I_1 I_2 - I_{12}^2} \end{cases} \tag{39}$$

$$\sigma_2 = S_{e_2} U - S_{T_{e_2}} = \begin{Bmatrix} k'_{1b} \\ k'_{2b} \\ ? \end{Bmatrix} \qquad \begin{cases} k'_{1b} \quad \dfrac{\bar{M}_{1b} I_2 - \bar{M}_{2b} I_{12}}{I_1 I_2 - I_{12}^2} \\ k'_{2b} \quad \dfrac{\bar{M}_{2b} I_1 - \bar{M}_{1b} I_{12}}{I_1 I_2 - I_{12}^2} \end{cases} \tag{40}$$

where

$$S_{e_1} = \left[B_1 K_{aa} \,\vdots\, B_1 K_{ab} \right] \tag{41}$$

$$S_{e_2} = \left[B_2 K_{aa} \,\vdots\, B_2 K_{ab} \right] \tag{42}$$

$$I_1 = \int y^2 \, dA \qquad I_2 = \int z^2 \, dA$$

$$S_{T_{e_1}} = B_{T_1} T', \quad B_{T_1} = B_1 K_{aa} \bar{A} \tag{43}$$

$$S_{T_{e_2}} = B_{T_2} T' \quad B_{T_2} = B_2 K_{aa} \bar{A} \tag{44}$$

with $\bar{A}$ defined in Section II and $U = \begin{Bmatrix} u_a \\ u_b \end{Bmatrix}$ (45)

Using (39) we can calc. actual stresses at y, z as (eqn (28)):

$$\text{end } a: \quad \sigma_a = \sigma_{axial} - k'_{1a} y - k'_{2a} z - E \alpha \Delta T_a(y,z) \tag{46}$$

$$\text{end } b: \quad \sigma_b = \sigma_{axial} - k'_{1b} y - k'_{2b} z - E \alpha \Delta T_b(y,z) \tag{47}$$

IV Effect of Offsets on K, P_T (No pinflags, global coord sys = local coord sys)

When the global coord sys is same as local, the element global stiffness matrix and the thermal loads are the same as those developed in sections I and II.

Partition the K, P_T (prior to offsets):

$$K = \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ab}^T & K_{bb} \end{bmatrix}$$

$$P_T = \begin{Bmatrix} P_{Ta} \\ P_{Tb} \end{Bmatrix}$$

The effect of offsets are

$$K_{aa_{offset}} = E_a^T K_{aa} E_a \;,\; K_{ab_{offset}} = E_a^T K_{ab} E_b \;,\; K_{bb_{offset}} = E_b^T K_{bb} E_b$$

$$P_{Ta_{offset}} = E_a^T P_{Ta} \;,\; P_{Tb_{offset}} = E_b^T P_{Tb}$$

where

$$E_a = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & a_z & -a_y \\ 0 & 1 & 0 & -a_z & 0 & a_x \\ 0 & 0 & 1 & a_y & -a_x & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] , \quad E_b = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & b_z & -b_y \\ 0 & 1 & 0 & -b_z & 0 & b_x \\ 0 & 0 & 1 & b_y & -b_x & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

where a_x, a_y, a_z are the offsets of the bar from the grid point at end a and
b_x, b_y, b_z " " " " " " " " " " " " b

$$E_a^T P_{T_{e_a}} = \alpha l \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -a_z & a_y & 1 & 0 & 0 \\ a_z & 0 & -a_x & 0 & 1 & 0 \\ -a_y & a_x & 0 & 0 & 0 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} -k_a & 0 & 0 & 0 & 0 \\ 0 & R_1 l/12 & -R_1 l/12 & \beta l/12 & -\beta l/12 \\ 0 & \beta l/12 & -\beta l/12 & R_2 l/12 & -R_2 l/12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta l^2/12 & 0 & -\left(\frac{R_2 l^2}{24} + \frac{EI_2}{2l}\right) & \left(\frac{R_2 l^2}{24} - \frac{EI_2}{2l}\right) \\ 0 & \left(\frac{R_1 l^2}{24} + \frac{EI_1}{2l}\right) & -\left(\frac{R_1 l^2}{24} - \frac{EI_1}{2l}\right) & \beta l^2/12 & 0 \end{bmatrix}}^{\text{Matrix } B_{T_{e_a}} \text{ before offset}} \begin{Bmatrix} (\bar{T} - T_{ref}) + \delta_m \\ T'_{1a} \\ T'_{1b} \\ T'_{2a} \\ T'_{2b} \end{Bmatrix}$$

$$= \alpha l \overbrace{\begin{bmatrix} -k_a & 0 & 0 & 0 & 0 \\ 0 & R_1 l/12 & -R_1 l/12 & \beta l/12 & -\beta l/12 \\ 0 & \beta l/12 & -\beta l/12 & R_2 l/12 & -R_2 l/12 \\ 0 & (-R_1 a_z + \beta a_y) l/12 & (R_1 a_z - \beta a_y) l/12 & (R_2 a_y - \beta a_z) l/12 & (-R_2 a_y + \beta a_z) l/12 \\ -k_a a_z & -\beta(a_x + l) l/12 & \beta a_x l/12 & -R_2\left(a_x + \frac{l}{2}\right)\frac{l}{12} - \frac{EI_2}{2l} & R_2\left(a_x + \frac{l}{2}\right)\frac{l}{12} - \frac{EI_2}{2l} \\ k_a a_y & R_1\left(a_x + \frac{l}{2}\right)\frac{l}{12} + \frac{EI_1}{2l} & -R_1\left(a_x + \frac{l}{2}\right)\frac{l}{12} + \frac{EI_1}{2l} & \beta(a_x + l)\frac{l}{12} & -\beta a_x l/12 \end{bmatrix}}^{\text{Matrix } B_{T_{e_a}} \text{ after offset}} \begin{Bmatrix} (\bar{T} - T_{ref}) + \delta_m \\ T'_{1a} \\ T'_{1b} \\ T'_{2a} \\ T'_{2b} \end{Bmatrix}$$

$$
{}^{-T}_{-b}P_{T_{e_b}} = \alpha l
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & -b_z & b_y & 1 & 0 & 0 \\
b_z & 0 & -b_x & 0 & 1 & 0 \\
-b_y & b_x & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c|c|c|c|c}
k_a & 0 & 0 & 0 & 0 \\
0 & -R_1 l/12 & R_1 l/12 & -\beta l/12 & \beta l/12 \\
0 & -\beta l/12 & \beta l/12 & -R_2 l/12 & R_2 l/12 \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta l^2/12 & -\left(\frac{R_2 l^2}{24} - \frac{EI_2}{2l}\right) & \left(\frac{R_2 l^2}{24} + \frac{EI_2}{2l}\right) \\
0 & \left(\frac{R_1 l^2}{24} - \frac{EI_1}{2l}\right) & -\left(\frac{R_1 l^2}{24} + \frac{EI_1}{2l}\right) & 0 & -\beta l^2/12
\end{array}\right]
\begin{Bmatrix}
(\bar{T} - T_{ref}) + \delta_m \\
T'_{1a} \\
T'_{1b} \\
T'_{2a} \\
T'_{2b}
\end{Bmatrix}
$$

$$
= \alpha l
\left[\begin{array}{c|c|c|c|c}
k_a & 0 & 0 & 0 & 0 \\
0 & -R_1 l/12 & R_1 l/12 & -\beta l/12 & \beta l/12 \\
0 & -\beta l/12 & \beta l/12 & -R_2 l/12 & R_2 l/12 \\
\hline
0 & (R_1 b_z - \beta b_y)\frac{l}{12} & (-R_1 b_z + \beta b_y)\frac{l}{12} & (-R_2 b_y + \beta b_z)\frac{l}{12} & (R_2 b_y - \beta b_z)\frac{l}{12} \\
k_a b_z & \beta b_x l/12 & -\beta(b_x - l)\frac{l}{12} & R_2\left(b_x - \frac{l}{2}\right)\frac{l}{12} + \frac{EI_2}{2l} & -R_2\left(b_x - \frac{l}{2}\right)\frac{l}{12} + \frac{EI_2}{2l} \\
-k_a b_y & -R_1\left(b_x - \frac{l}{2}\right)\frac{l}{12} - \frac{EI_1}{2l} & R_1\left(b_x - \frac{l}{2}\right)\frac{l}{12} - \frac{EI_1}{2l} & -\beta b_x l/12 & \beta(b_x - l)\frac{l}{12}
\end{array}\right]
\begin{Bmatrix}
(\bar{T} - T_{ref}) + \delta_m \\
T'_{1a} \\
T'_{1b} \\
T'_{2a} \\
T'_{2b}
\end{Bmatrix}
$$

$$
E_a^T K_{aa} E_a =
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & -a_z & a_y & 1 & 0 & 0 \\
a_z & 0 & -a_x & 0 & 1 & 0 \\
-a_y & a_x & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{ccc|ccc}
k_a & 0 & 0 & 0 & 0 & 0 \\
0 & R_1 & \beta & 0 & -\beta l/2 & -R_1 l/2 \\
0 & \beta & R_2 & 0 & -R_2 l/2 & \beta l/2 \\
\hline
0 & 0 & 0 & k_t & 0 & 0 \\
0 & -\beta l/2 & -R_2 l/2 & 0 & \hat{k}_2 & -\beta l^2/3 \\
0 & R_1 l/2 & \beta l/2 & 0 & -\beta l^2/3 & \hat{k}_1
\end{array}\right]
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & a_z & -a_y \\
0 & 1 & 0 & -a_z & 0 & a_x \\
0 & 0 & 1 & a_y & -a_x & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
=
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & -a_z & a_y & 1 & 0 & 0 \\
a_z & 0 & -a_x & 0 & 1 & 0 \\
-a_y & a_x & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c|c|c|c|c|c}
k_a & 0 & 0 & 0 & k_a a_z & -k_a a_y \\
0 & R_1 & \beta & -R_1 a_z + \beta a_y & -\beta(a_x + l/2) & R_1(a_x + l/2) \\
0 & \beta & R_2 & R_2 a_y - \beta a_z & -R_2(a_x + l/2) & \beta(a_x + l/2) \\
\hline
0 & 0 & 0 & k_t & 0 & 0 \\
0 & -\beta l/2 & -R_2 l/2 & (-R_2 a_y + \beta a_z) l/2 & \hat{k}_2 + R_2 a_x l/2 & -\beta\left(a_x + \frac{2l}{3}\right)\frac{l}{2} \\
0 & R_1 l/2 & \beta l/2 & (-R_1 a_z + \beta a_y) l/2 & -\beta\left(a_x + \frac{2l}{3}\right)\frac{l}{2} & \hat{k}_1 + R_1 a_x l/2
\end{array}\right]
$$

$$E_a^T K_{aa} E_a = \left[\begin{array}{ccc|ccc}
k_a & 0 & 0 & 0 & k_a a_z & -k_a a_y \\
0 & R_1 & \beta & -R_1 a_z + \beta a_y & -\beta(a_x + l/2) & R_1(a_x + l/2) \\
0 & \beta & R_2 & R_2 a_y - \beta a_z & -R_2(a_x + l/2) & \beta(a_x + l/2) \\
\hline
0 & -R_1 a_z + \beta a_y & R_2 a_y - \beta a_z & \begin{array}{c} k_t + R_1 a_z^2 + R_2 a_y^2 \\ -2\beta a_z a_y \end{array} & (-R_2 a_y + \beta a_z)(a_x + \frac{l}{2}) & (-R_1 a_z + \beta a_y)(a_x + \frac{l}{2}) \\
k_a a_z & -\beta(a_x + l/2) & -R_2(a_x + l/2) & (-R_2 a_y + \beta a_z)(a_x + \frac{l}{2}) & \begin{array}{c} \hat{k}_2 + k_a a_z^2 \\ + R_2 a_x (a_x + l) \end{array} & \begin{array}{c} -k_a a_y a_z \\ -\beta[a_x(a_x + l) + \frac{l^2}{3}] \end{array} \\
-k_a a_y & R_1(a_x + l/2) & \beta(a_x + l/2) & (-R_1 a_z + \beta a_y)(a_x + \frac{l}{2}) & \begin{array}{c} -k_a a_y a_z \\ -\beta[a_x(a_x + l) + \frac{l^2}{3}] \end{array} & \begin{array}{c} \hat{k}_1 + k_a a_y^2 \\ + R_1 a_x (a_x + l) \end{array}
\end{array}\right]$$

(provided that global = basic)

$$E_b^T K_{bb} E_b = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -b_z & b_y & 1 & 0 & 0 \\ b_z & 0 & -b_x & 0 & 1 & 0 \\ -b_y & b_x & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} k_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_1 & \beta & 0 & \beta\frac{l}{2} & -R_1\frac{l}{2} \\ 0 & \beta & R_2 & 0 & R_2\frac{l}{2} & -\beta\frac{l}{2} \\ \hline 0 & 0 & 0 & k_t & 0 & 0 \\ 0 & \beta\frac{l}{2} & R_2\frac{l}{2} & 0 & \hat{k}_2 & -\beta\frac{l^2}{3} \\ 0 & -R_1\frac{l}{2} & -\beta\frac{l}{2} & 0 & -\beta\frac{l^2}{3} & \hat{k}_1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & b_z & -b_y \\ 0 & 1 & 0 & -b_z & 0 & b_x \\ 0 & 0 & 1 & b_y & -b_x & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -b_z & b_y & 1 & 0 & 0 \\ b_z & 0 & -b_x & 0 & 1 & 0 \\ -b_y & b_x & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} k_a & 0 & 0 & 0 & k_a b_z & -k_a b_y \\ 0 & R_1 & \beta & -R_1 b_z + \beta b_y & -\beta(b_x - \frac{l}{2}) & R_1(b_x - \frac{l}{2}) \\ 0 & \beta & R_2 & R_2 b_y - \beta b_z & -R_2(b_x - \frac{l}{2}) & \beta(b_x - \frac{l}{2}) \\ \hline 0 & 0 & 0 & k_t & 0 & 0 \\ 0 & \beta\frac{l}{2} & R_2\frac{l}{2} & (R_2 b_y - \beta b_z)\frac{l}{2} & \hat{k}_2 - R_2 b_z \frac{l}{2} & \beta(b_x - \frac{2l}{3})\frac{l}{2} \\ 0 & -R_1\frac{l}{2} & -\beta\frac{l}{2} & (R_1 b_z - \beta b_y)\frac{l}{2} & \beta(b_x - \frac{2l}{3})\frac{l}{2} & \hat{k}_1 - R_1 b_x \frac{l}{2} \end{array}\right]$$

$E_b^T K_{bb} E_b =$

k_a	0	0	0	$k_a b_z$	$-k_a b_y$
0	R_1	β	$-R_1 b_z + \beta b_y$	$-\beta(b_x - l/2)$	$R_1(b_x - \frac{l}{2})$
0	β	R_2	$R_2 b_y - \beta b_z$	$-R_2(b_x - \frac{l}{2})$	$\beta(b_x - \frac{l}{2})$
0	$-R_1 b_z + \beta b_y$	$R_2 b_y - \beta b_z$	$k_t + R_1 b_z^2 + R_2 b_y^2$ $- 2\beta b_y b_z$	$(-R_2 b_y + \beta b_z)(b_x - \frac{l}{2})$	$(-R_1 b_z + \beta b_y)(b_x - \frac{l}{2})$
$k_a b_z$	$-\beta(b_x - \frac{l}{2})$	$-R_2(b_x - \frac{l}{2})$	$(-R_2 b_y + \beta b_z)(b_x - \frac{l}{2})$	$\hat{k}_2 + k_a b_z^2$ $+ R_2 b_x(b_x - l)$	$-k_a b_y b_z$ X $-\beta[b_x(b_x - l) + \frac{l^2}{3}]$
$-k_a b_y$	$R_1(b_x - \frac{l}{2})$	$\beta(b_x - \frac{l}{2})$	$(-R_1 b_z + \beta b_y)(b_x - \frac{l}{2})$	$-k_a b_y b_z$ $-\beta[b_x(b_x - l) + \frac{l^2}{3}]$	$\hat{k}_1 + k_a b_y^2$ $+ R_1 b_x(b_x - l)$

$$E_a^T K_{ab} E_b = \left[\begin{array}{ccc|ccc}
-k_a & 0 & 0 & 0 & -k_a b_z & k_a b_y \\
0 & -R_1 & -\beta & R_1 b_z - \beta b_y & \beta(b_x - \frac{l}{2}) & -R_1(b_x - \frac{l}{2}) \\
0 & -\beta & -R_2 & -R_2 b_y + \beta b_z & R_2(b_x - \frac{l}{2}) & -\beta(b_x - \frac{l}{2}) \\
\hline
0 & R_1 a_z - \beta a_y & -R_2 a_y + \beta a_z & \begin{array}{c} -k_t - (R_1 b_z - \beta b_y) a_z \\ - (R_2 b_y - \beta b_z) a_y \end{array} & (R_2 a_y - \beta a_z)(b_x - \frac{l}{2}) & (R_1 a_z - \beta a_y)(b_x - \frac{l}{2}) \\
-k_a a_z & \beta(a_x + \frac{l}{2}) & R_2(a_x + \frac{l}{2}) & (R_2 b_y - \beta b_z)(a_x + \frac{l}{2}) & \begin{array}{c} \hat{k}_4 - k_a a_z b_z + R_2 a_x \frac{l}{2} \\ - R_2 b_x (a_x + \frac{l}{2}) \end{array} & \begin{array}{c} k_a a_z b_y \\ + \beta\left[b_x(a_x + \frac{l}{2}) - (a_x + \frac{l}{3})\frac{l}{2}\right] \end{array} \\
k_a a_y & -R_1(a_x + \frac{l}{2}) & -\beta(a_x + \frac{l}{2}) & (R_1 b_z - \beta b_y)(a_x + \frac{l}{2}) & \begin{array}{c} k_a a_y b_z \\ + \beta\left[b_x(a_x + \frac{l}{2}) - (a_x - \frac{l}{3})\frac{l}{2}\right] \end{array} & \begin{array}{c} \hat{k}_3 - k_a a_y b_y \\ + R_1\left[a_x \frac{l}{2} - b_x(a_x + \frac{l}{2})\right] \end{array}
\end{array}\right]$$

$$E_a^T K_{ab} E_b = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -a_z & a_y & 1 & 0 & 0 \\ a_z & 0 & -a_x & 0 & 1 & 0 \\ -a_y & a_x & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} -k_a & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_1 & -\beta & 0 & -\beta\frac{l}{2} & R_1\frac{l}{2} \\ 0 & -\beta & -R_2 & 0 & -R_2\frac{l}{2} & \beta\frac{l}{2} \\ \hline 0 & 0 & 0 & -k_t & 0 & 0 \\ 0 & \beta\frac{l}{2} & R_2\frac{l}{2} & 0 & \hat{k}_4 & -\beta\frac{l^2}{6} \\ 0 & -R_1\frac{l}{2} & -\beta\frac{l}{2} & 0 & -\beta\frac{l^2}{6} & \hat{k}_3 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & b_z & -b_y \\ 0 & 1 & 0 & -b_z & 0 & b_x \\ 0 & 0 & 1 & b_y & -b_x & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -a_z & a_y & 1 & 0 & 0 \\ a_z & 0 & -a_x & 0 & 1 & 0 \\ -a_y & a_x & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|c|c|c|c|c} -k_a & 0 & 0 & 0 & -k_a b_z & k_a b_y \\ 0 & -R_1 & -\beta & R_1 b_z - \beta b_y & \beta(b_x - \frac{l}{2}) & -R_1(b_x - \frac{l}{2}) \\ 0 & -\beta & -R_2 & -R_2 b_y + \beta b_z & R_2(b_x - \frac{l}{2}) & -\beta(b_x - \frac{l}{2}) \\ \hline 0 & 0 & 0 & -k_t & 0 & 0 \\ 0 & \beta\frac{l}{2} & R_2\frac{l}{2} & (R_2 b_y - \beta b_z)\frac{l}{2} & \hat{k}_4 - R_2 b_x \frac{l}{2} & \beta(b_x - \frac{l}{3})\frac{l}{2} \\ 0 & -R_1\frac{l}{2} & -\beta\frac{l}{2} & (R_1 b_z - \beta b_y)\frac{l}{2} & \beta(b_x - \frac{l}{3})\frac{l}{2} & \hat{k}_3 - R_1 b_x \frac{l}{2} \end{array}\right]$$