대학원 신입생 세미나 HomeWork 1

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April 11, 2020

1. 확률론 교과서(Probability Theory and Examples, Rick Durrett)에 나와있는 정리를 하나 골라서 정리 내용과 증명을 LaTeX으로 작성하고 pdf 파일을 생성하기

Definition 1. *P is called π-system if $A,B \in P \Longrightarrow A \cap B \in P$*

Definition 2. *L is called λ-system if it satisfy following conditions*

1. $\Omega \in L$
2. A, $B \in L$, $A \subset B$ *then* B-$A \in L$
3. $A_n \nearrow A$, $A_n \in L$ *then* $A \in L$

Lemma 1 (Dynkin's π-λ Theorem)**.**

If P is a π-system and Ł is a λ-system & $P \subset$ Ł $\Longrightarrow \sigma(P) \subset$ Ł

Theorem 1 (Theorem 2.1.7)**.**

Suppose Λ_1, Λ_2, ..., Λ_n are independent and each Λ_i is a π system. Then $\sigma(\Lambda_1)$, $\sigma(\Lambda_2)$, ... ,$\sigma(\Lambda_n)$ are independent. ($\sigma(\Lambda_i)$ is the σ-algebra generated by Λ_i)

Proof. Let A_2, ... ,A_n be sets with $A_i \in \Lambda_i$. Let $F = A_2 \cap ... \cap A_n$ and Ł= {A : P(A$\cap$ F) =P(A)P(F)}

We want to show the set Ł is a λ-system.

1. $P(\Omega \cap F) = P(\Omega)P(F) \Longrightarrow \Omega \in$ Ł
2. Let A, $B \in$ Ł with $A \subset B$ then (B-A)$\cap$ F=($B \cap F$)-($A \cap F$)

$$P((B\text{-}A) \cap F) = P(B \cap F) \text{ - } P(A \cap F) \begin{aligned} &= \mathrm{P(B)P(F)-P(A)P(F)} \\ &= \mathrm{(P(B)-P(A))P(F)} = \mathrm{P(B-A)P(F)} \end{aligned} \tag{1}$$

   P((B−A)$\cap F$) = $P(B\text{-}A)P(F) \Longrightarrow B\text{-}A \in$ Ł

3. let $B_k \in$ Ł with $B_k \nearrow B$ and note that $(B_k \cap F) \nearrow (B \cap F)$
   $P(B \cap F) = \lim_{k\to+\infty} \mathrm{P}(\mathrm{B}_k \cap F) = \lim_{k\to+\infty} \mathrm{P}(\mathrm{B}_k)P(F) = P(B)P(F) \Longrightarrow B \in$ Ł

Therefore, Ł is a λ-system.

Since Λ_1, Λ_2, ..., Λ_n are independent, $\Lambda_1 \subset$ Ł. Since Λ_1 is a π-system, by Dynkin's π-λ Theorem $\sigma(\Lambda_1) \subset$ Ł

It follows that if $A_1 \in \sigma(\Lambda_1)$ *and* $\mathrm{A}_i \in \Lambda_i$ $(2 \leq i \leq n)$ then $P(\bigcap\limits_{i=1}^{n} A_i) = \mathrm{P}(A_1)\mathrm{P}(\bigcap\limits_{i=2}^{n} A_i) = \prod_{i=1}^{n} \mathrm{P}(\mathrm{A}_i)$

Therefore, if Λ_1, Λ_2, ..., Λ_n are independent then $\sigma(\Lambda_1)$, Λ_2, ..., Λ_n are independent. Applying above process to Λ_2, ..., Λ_n, $\sigma(\Lambda_1)$, then we can show that $\sigma(\Lambda_1)$, $\sigma(\Lambda_2)$, Λ_3, ..., Λ_n are independent. And after n iterations we have the desired result.

$\sigma(\Lambda_1)$, $\sigma(\Lambda_2)$, ... ,$\sigma(\Lambda_n)$ are independent.

$\square$