Selecting the Condorcet Winner: Single-Stage versus Multi-Stage Voting Rules
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Source: *Public Choice*, Vol. 137, No. 1/2 (Oct., 2008), pp. 207-220
Published by: Springer
Stable URL: http://www.jstor.org/stable/40270859

Public Choice (2008) 137: 207–220
DOI 10.1007/s11127-008-9321-y

# Selecting the Condorcet Winner: single-stage versus multi-stage voting rules

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Received: 11 July 2007 / Accepted: 15 May 2008 / Published online: 29 May 2008


**Abstract** In this paper, I study elections where voters are strategic. I find that the commonly used voting rules, such as Plurality Rule, Majority Rule, Approval Voting, and Single Transferable Vote, do not always select the Condorcet Winner and suffer from multiple equilibria. Multi-stage voting rules offer a way to get around this problem. I introduce two voting rules—Multi-Stage Runoff and the Nominate-Two Rule—that select the Condorcet Winner as the unique equilibrium outcome under mild conditions. I show that a third class of voting rules—Binary Voting Trees—also select the Condorcet Winner.

**Keywords** Voting rules · Strategic voting · Approval voting · Single transferable vote

## 1 Introduction

Plurality Rule is used for elections (or group decision making) in a wide variety of circumstances. A number of alternative voting rules are in current use and many more have been proposed. The most prominent alternatives are Majority Rule, Approval Voting, and Single Transferable Vote.[1] The fact that Plurality Rule is used in so many situations would seem to suggest its superiority, but there is some evidence to suggest the contrary. Majority Rule, Approval Voting, and Single Transferable Vote have been widely suggested as superior alternatives, both by political reformers and academics.

I will evaluate voting rules based on whether they are guaranteed to select the Condorcet Winner in equilibrium (Brams and Fishburn 1978). Such voting rules are said to be Condorcet Consistent. I will begin by considering a number of commonly used voting rules-Plurality Rule, Majority Rule, Approval Voting, and Single Transferable Vote. All of these

[1] Single Transferable Vote goes by a number of different names including Hare voting, Preferential Voting, the Alternative Vote (in Australia), Rank-Choice Voting (in San Francisco) and Instant Runoff. Instant Runoff has also been used to refer to a slight variant of Single Transferable Vote, where all but two candidates are eliminated from consideration after the first round. This variant is also called the Contingent Vote.

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suffer from multiple Undominated Nash Equilibria, and are therefore not Condorcet Consistent.

I will consider multi-stage voting rules as alternatives to the commonly used voting rules. I introduce two multi-stage voting rules that select the Condorcet Winner under suitable conditions. Multi-Stage Runoff will select the Condorcet Winner so long as the Majority Social Preference is transitive (a condition weaker than single–peakedness). The Nominate-Two Rule will select the Condorcet Winner under weak conditions as well. In addition, I show that a third class of voting rules—Binary Voting Trees—are Condorcet Consistent.

## 2 Single stage voting rules

I begin by considering single-stage voting rules. I consider multi-stage voting rules in the next section. Plurality Rule, Approval Voting, and Single Transferable Vote are single-stage voting rules, while Majority Rule is a multi-stage voting rule when there are three or more candidates. For the first three voting rules, the voters visit the polling booth only once, while under Majority Rule, the voter may return for a runoff election. The main feature distinguishing single-stage and multi-stage voting rules is that for multi-stage voting rules, it is reasonable to apply subgame perfection as a refinement, because voters can reasonably be assumed to condition their current stage strategies on votes taken in the previous stages.

The strategic voting game consists of a finite set of voters $N = \{1, 2, \ldots, n\}$ who have preferences over a finite set of candidates (or policy alternatives) $X = \{1, 2, \ldots, k\}$. Assume that voters have strict preferences and denote the strict preferences of voter $i$ by $\succ_i$. Let $(x, y)$ denote a lottery between candidates $x$ and $y$, and assume that if $x \succ_i y$, then $x \succ_i (x, y) \succ_i y$. Throughout, I will consider the case where $k \geq 3$. The Majority Social Preference relation will be given by $\succ_i$, where $x \succ y$ if and only if $|\{i : x \succ_i y\}| > \frac{1}{2}n$. The Majority Social Indifference relation will be denoted by $\sim$, where $x \sim y$ if and only if $|\{i : x \succ_i y\}| > \frac{1}{2}n$. Let $x \in X$ be the Condorcet Winner if $x \succ y$ for all $y \in X - \{x\}$. Let $x \in X$ be the Condorcet Loser if $y \succ x$ for all $y \in X - \{x\}$. The relation $\succ$ is transitive if for all $x, y, z \in X$, $x \succ y$ and $y \succ z$ implies $x \succ z$.

Let $S_i$ be the (finite) set of strategies available to voter $i$. Denote $S = S_1 \times S_2 \times \ldots S_n$ and $S_{-i} = (S_1 \times \ldots \times S_{i-1} \times \ldots \times S_n$. Let $s_i \in S_i$ denote a strategy played by voter $i$, and let $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_{-i}$ denote the strategy profile of all other voters. Let $u_i(s_i; s_{-i})$ denote the utility voter $i$ receives when he plays strategy $s_i \in S_i$ and the other players play the vector of strategies $s_{-i} \in S_{-i}$.

Define a strategy $s_i \in S_i$ for player $i$ to be *Undominated* (or weakly undominated) if there does not exist a strategy $s_i' \in S_i$ such that $u_i(s_i'; s_{-i}) \geq u_i(s_i; s_{-i})$ for all $s_{-i} \in S_{-i}$ and $u_i(s_i'; s_{-i}) > (s_i; s_{-i})$ for some $s_{-i} \in S_{-i}$. The vector of strategies $s^* = s_1^*, \ldots, s_n^*$ constitutes an *Undominated Nash Equilibrium* if for each $i$ we have $u_i(s_i; s_{-i}^*) \leq u_i(s_i^*; s_{-i}^*)$ for all $s_i \in S_i$, and the strategies in $s^*$ are undominated for each player. By employing this equilibrium concept, I am effectively assuming that voters have complete information about the structure of the game and the preferences of the other voters.[2] The same equilibrium concept has been used in some of the recent work on strategic voting (Dellis 2006; Bag et al. 2007), but differs from Farquharson's (1969) iterative elimination of weakly dominated strategies.[3]

[2]See Palfrey (1989), Myerson and Weber (1993), Cox (1994, 1997), and Fey (1997) for work on voting games with incomplete information.

[3]See Dhillon and Lockwood (2004), and Buenrostro et al. (2007) for examples of iterative elimination of weakly dominated strategies applied to strategic voting games. Niou (2001) considers iterative elimination of dominated strategies by bloc, where blocs are groups of voters who have the same preference orderings.

In general, I will look for voting rules that are guaranteed to select the Condorcet Winner in Undominated Nash Equilibrium, when a Condorcet Winner exists. As Farquharson (1969) argues, almost any outcome (winning candidate) can be sustained under Nash Equilibrium. Requiring strategies to be Undominated is therefore helpful in reducing the set of equilibrium outcomes.

### 2.1 Plurality rule

Under Plurality Rule, each voter is allowed to vote for a single candidate and the candidate with the most votes wins the election (the winner is selected randomly if there is a tie). It is well known that under Plurality Rule, the only weakly dominated strategy is to vote for your last choice (Brams and Fishburn 1978). It is therefore not surprising that even if these strategies are eliminated, Plurality Rule will admit multiple Undominated Nash Equilibrium outcomes.

**Proposition 1** *Under Plurality Rule, if $n \geq 4$ and candidate $x$ is not a Condorcet Loser, there exists an Undominated Nash Equilibrium where candidate $x$ wins the election.*

### 2.2 Approval voting

Under Approval Voting, voters may vote for (approve) as many candidates as they like, and the candidate with the largest number of votes is declared the winner. As usual, assume that ties are broken randomly. Under Approval Voting, not approving of your first choice is weakly dominated since this can only have the effect of swinging the election from your first choice to another candidate. Similarly, approving your last choice is weakly dominated since this can only have the effect of swinging the election from another candidate to your last choice (Brams and Fishburn 1978). Since there are more weakly dominated strategies under Approval Voting than Plurality Rule, the set of equilibrium outcomes under Approval Voting will typically be smaller than the set of equilibrium outcomes under Plurality Rule.

**Proposition 2** *Under Approval Voting,* (i) *if the number of voters for whom $x$ is not the last choice is at least two greater than the number of voters who prefer $y$ first for all $y \neq x$, then there exists an Undominated Nash Equilibrium where candidate $x$ wins the election and* (ii) *if the number of voters for whom $x$ is not the last choice is less than the number of voters who prefer $y$ first for some $y \neq x$, then there does not exist an Undominated Nash Equilibrium where candidate $x$ wins the election.*

Unlike Plurality Rule, Approval Voting *may* have a unique Undominated Nash Equilibrium even when there are more than two candidates competing, as I show in the example below.

*Example 1* Suppose that there are three candidates running for office, $x$, $y$, and $z$. If 25 voters have preferences $x \succ_i y \succ_i z$, 30 voters have preferences $y \succ_i x \succ_i z$, 30 voters have preferences $y \succ_i z \succ_i x$, and 15 voters have preferences $z \succ_i y \succ_i x$, then under Approval Voting, there exists a unique Undominated Nash Equilibrium outcome where $y$ wins the election. Under Plurality Rule, both $x$ and $y$ can win in equilibrium.

To see that only $y$ can win under Approval Voting, notice that if voters play undominated strategies, $y$ must receive at least 60 votes, while $x$ can receive at most 55 votes and $z$ can receive at most 45 votes. To see that the second claim holds, notice that $y \succ x \succ z$. Hence, Proposition 1 implies that either $x$ or $y$ (neither of whom is a Condorcet Loser) can win in equilibrium.

### 2.3 Single transferable vote

Under Single Transferable Vote, each voter ranks all the candidates.[4] In the first round, candidates receive votes based on the number of times they were ranked first by a voter. If no candidate receives a majority of the votes, the candidate with the least number of votes is eliminated and this candidate's votes are transferred to the other candidates based on the rankings. This process is continued until one candidate has a majority of the votes. All $j$-way ties are broken by selecting each of these $j$ candidates to advance with equal probability.

Advocates of Single Transferable Vote often stress two benefits of this voting rule. They claim that it eliminates the 'wasted vote' phenomenon that occurs under Plurality Rule, and that it eliminates the need for strategic behavior. Contrary to what one might expect, this voting rule need not select the Condorcet Winner. It is difficult to fully characterize the set of weakly dominated strategies. Fortunately, it is not necessary to do this in order to show that this voting rule suffers from multiple equilibria. We can show that for every candidate $j$, each voter has a strategy that is not weakly dominated, which involves ranking candidate $j$ first. We can then use this result to construct multiple equilibria.

**Proposition 3** *Under Single Transferable Vote, if the number of candidates $k$ satisfies $n - (k+1)\lfloor (n-1)/k \rfloor + 3 < 0$ and $x$ is not a Condorcet Loser, then there exists an Undominated Nash Equilibrium where candidate $x$ wins the election.*

The consequence of the condition $n - (k+1)\lfloor (n-1)/k \rfloor + 3 < 0$ is not immediately apparent. We can show that for each $k$, this condition will hold if $n$ is large enough. For example, it will hold if $n \geq 22$ for three candidate races, $n \geq 33$ for four candidate races, and $n \geq 46$ for five candidate races.[5] Hence, except in small elections, we have a result for Single Transferable Vote that is as unappealing as the result we have for Plurality Rule.

A direct consequence of Proposition 3 is that voting sincerely is not a dominant strategy. More importantly, the result casts doubt on claims that we should not observe strategic voting under STV because this would require too much sophistication on the part of voters (Bartholdi and Orlin 1991).[6] In fact, the equilibria I construct in the proof of Proposition 3 imply that strategic voting under STV can take the same form as strategic voting under PR, with all voters giving their first-ranked choice to one of the two leading candidates.

### 2.4 Other voting rules

I showed that a large number of strategies survive weak dominance under Single Transferable Vote, which was sufficient to indicate that almost any equilibrium outcome could result. Characterizing weakly undominated strategies for other voting rules can be difficult. Bag et al. (2007) prove, by counterexample, that a class including many of the commonly used voting rules fail always to select the Condorcet Winner. The results do not tell us whether

[4] It will be necessary to distinguish between the voter's preferences (which is a ranking of the candidates) and the voter's strategy under Single Transferable Vote (which is also a ranking of the candidates). To minimize confusion, I will use variations of 'prefer' to denote the voter's preferences and variations of 'rank' to denote the voter's strategy under Single Transferable Vote.

[5] Weaker conditions on $n$ are possible, but require a messier construction in the proof of Proposition 3.

[6] Of course, I am not claiming that Bartholdi and Orlin's result in incorrect. I simply believe that the conclusion that STV resists manipulation is unwarranted. Voters need not recognize that monotonicity is violated in order to play the types of strategies suggested by Proposition 3.

the failure to select the Condorcet Winner is common or due to a few unlikely preference configurations.[7] In other words, it is possible that although these voting rules fail to always select the Condorcet Winner, they are nonetheless useful, because they select the Condorcet Winner most of the time.

My results show that Plurality Rule and Single Transferable Vote almost never have unique equilibrium outcomes, while under Approval Voting, a unique equilibrium is less rare. Dellis (2006) takes on the formidable task of characterizing weakly undominated strategies under a large class of voting rules. His class includes arbitrary scoring rules, and he allows for cumulation and truncation. The results indicate that here, too, many strategies are weakly undominated, and hence, multiple equilibrium outcomes are quite typical for single stage voting rules. The shortcomings of single-stage voting rules lead us to consider multi-stage voting rules in the next section.

## 3 Multi-stage voting rules

The single-stage voting rules considered in the previous section often failed to guarantee that the Condorcet Winner would be selected. This section considers one possible solution to this problem: multi-stage voting rules. Multi-stage voting rules differ from single-stage voting rules because the voters make multiple decisions and receive new information throughout the stages of the game. For example, before the runoff stage of Majority Rule, voters will know which two candidates received the most votes in the first round. Thus, multi-stage voting rules are those voting rules where it makes sense to apply subgame perfection in order to eliminate extraneous equilibria.

Unlike the three voting rules considered in Sect. 2, Majority Rule is a multi-stage voting rule (voters know the outcome of the first stage before they have to commit to a vote for the runoff stage). Under Majority Rule, a first round is held where each voter votes for a single candidate. The votes are tallied up and if one candidate receives a majority of the votes (more than half), then this candidate is declared the winner. Otherwise, a runoff election is held between the two candidates who received the most votes in the first round.

Each multi-stage voting rule is a $T$-stage finite action dynamic game of complete information. Let $A^t(h^t)$ denote the set of actions available to each voter given the history $h^t$. Let $a^t = (a_1^t, \dots, a_n^t) \in A^t(h^t)^n$ denote the action profile in stage $t$. Let $h^t = (a^{t-1}, \dots, a^1)$ denote the history at stage $t$. Let $H^t$ denote the set of all histories in stage $t$ and define $A^t(H^t) = \bigcup_{h^t \in H^t} A^t(h^t)$.

A stage $t$ strategy for player $i$ is a map from histories into actions, $s_i^t : H^t \to A^t(H^t)$. A strategy for player $i$ is given by $s_i = (s_i^1, \dots, s_i^T) \in S$ where $S$ is the set of all strategies. A strategy profile is denoted by $s = (s_1, \dots, s_n) \in S^n$. Define a subgame $G(h^t)$ as the restriction of the game to strategy profiles consistent with $h^t$. Define a *Subgame Perfect Undominated Nash Equilibrium* (SPUNE) as a strategy profile $s^* \in S^n$ where the restriction of $s^*$ to the history $h^t \in H^t$ is an Undominated Nash Equilibrium at each subgame $G(h^t)$.[8]

[7] Like this essay, Bag et al. (2007) argue that single stage voting rules are ineffective in selecting the Condorcet Winner, and propose multi-stage voting rules as a solution. Since multi-stage voting rules are inherently more complicated to implement, it is important to demonstrate that the shortcomings of single-stage voting rules are not isolated to a few voting rules or a few preference configurations.

[8] This definition is closely related to that of Subgame Perfect Equilibrium in Fudenberg and Tirole (1991). The difference is that we require the strategies to be consistent with Undominated Nash Equilibrium rather than Nash Equilibrium, at each subgame.

I solve for the equilibrium of the strategic voting game by applying backwards induction, and eliminating weakly dominated strategies at each stage.

### 3.1 Multi-stage runoff

Unlike the other voting rules considered so far, Multi-Stage Runoff will select the Condorcet Winner under Subgame Perfect Undominated Nash Equilibrium as long as $\succ$ is transitive.[9] Two-Stage Runoff is a slight variation of Majority Rule. Under Majority Rule, a runoff between the top two vote-getters in the first round is only held only if no candidate has received a majority of the votes. Under Two-Stage Runoff, a runoff is always held. If there are three candidates competing, the Condorcet Winner (provided one exists) will win the election. A generalization of Two-Stage Runoff is Multi-Stage Runoff, where the election is held in $k-1$ stages if there are $k$ candidates. In each stage, the candidate with the least number of votes is eliminated.

**Proposition 4** *Under Multi-Stage Runoff, if $\succ$ is transitive, then the Condorcet Winner must win the election in any Subgame Perfect Undominated Nash Equilibrium.*

There are examples where a Condorcet Winner exists, but Multi-Stage Runoff does not select the Condorcet Winner. To find such an example, we must rely on a situation where $\succ$ is not transitive. When $k=3$, the existence of a Condorcet Winner implies transitivity, so we must use an example where $k>3$.

*Example 2* Suppose there are 4 candidates, $w$, $x$, $y$, and $z$, where 30 voters prefer $x \succ_i w \succ_i y \succ_i z$, 40 voters prefer $y \succ_i w \succ_i z \succ_i x$, and 20 voters prefer $z \succ_i w \succ_i x \succ_i y$. In this case, $w \succ x$, $w \succ y$, $w \succ z$, $x \succ y$, $z \succ x$, and $y \succ z$. So $w$ is a Condorcet Winner, but $\succ$ is not transitive. Consider what would happen in the second stage if $\{x, y, z\}$ were to advance. A voter with $x \succ_i w \succ_i y \succ_i z$ has two undominated strategies: vote for $x$ and vote for $y$. Voting for $x$ is advisable if $x$ and $z$ trail $y$ with an equal number of votes. Voting for $y$ is advisable if $y$ and $z$ both trail $x$ with an equal number of votes. A similar analysis can be used to show that voters with $y \succ_i w \succ_i z \succ_i x$ have $y$ and $z$ as undominated strategies and voters with $z \succ_i w \succ_i x \succ_i y$ have $z$ and $x$ as undominated strategies.

There is an Undominated Nash Equilibrium to the $\{x, y, z\}$ subgame where all voters vote between $x$ and $y$, and $x$ wins. Similarly, there is an equilibrium where $y$ wins and an equilibrium where $z$ wins. Because of this, no candidates have weakly dominated strategies in the first or second stages. Thus there exist equilibria where any of these candidates wins the election. If $w$ does not receive the fewest votes in the first stage, then it will advance to the second stage and will win the election. If $w$ receives the fewest votes, then $\{x, y, z\}$ will advance and there exist equilibria corresponding to each of these candidates winning the election.

Bag et al. (2007) independently proved a result similar to Proposition 4 for Multi-stage Runoff (or Weakest Link Voting in their terminology). They eliminate the transitivity condition, use a slightly different equilibrium concept, and impose a restriction to

[9] While transitivity is quite restrictive when the policy space is infinite, it is a more reasonable restriction when the policy space has a small, finite number of alternatives.

"Markov" equilibria.[10] Multi-stage Runoff always selects the Condorcet Winner under a Markov equilibrium. Bag et al. show that this result holds for a more general class of voting rules, which only eliminate one candidate at each stage. They also show that Multi-stage Runoff selects an element from the top-cycle set when a Condorcet Winner does not exist.

Bag et al. (2007) argue in favor of the Markov restriction by appealing to simplicity. This restriction is most sensible it settings where there are a large number of voters, or where it is difficult for the voters to communicate. Multi-stage voting rules are less useful for large elections, however (because more than two rounds of voting would be impractical). There is little justification for removing these equilibria when studying small elections (such as committee voting). To get a more robust result, further conditions (such as transitivity) are necessary.

### 3.2 Majority rule

Unlike Multi-Stage Runoff, Majority Rule need not select the Condorcet Winner. This holds even if there are only three candidates and preferences are single-peaked. This is surprising because Majority Rule seems to resemble Multi-Stage Runoff very closely when there are three candidates competing. The difference between these two voting rules is that under Two-Stage Runoff, a runoff is always held, while under Majority Rule, no runoff is held if some candidate receives a majority in the first round.

**Proposition 5** *Under Majority Rule with $k = 3$ candidates,* (i) *if $x \succ y$, $y \succ z$, and $|\{i : \neg(x \succ_i y \succ_i z)\}| \geq (n+1)/2 + 1$, there exists a Subgame Perfect Undominated Nash Equilibrium where the Condorcet Winner does not win, and* (ii) *if $x \succ y$, $y \succ z$, and $|\{i : (x \succ_i y \succ_i z)\}| \geq (n+1)/2 + 1$, there exists a unique Subgame Perfect Undominated Nash Equilibrium outcome where the Condorcet Winner wins the election.*

When there are more than three candidates, the results are significantly more complicated. In general, increasing $k$ will make the set of possible equilibria larger.

Consider the situation where the voter's preferences are generated from a one-dimensional spatial model and the center candidate is the Condorcet Winner. The Condorcet Winner will be the unique equilibrium if the positions taken by the left and right candidates are extreme enough. The condition is restrictive, but is significantly less restrictive than the condition necessary for a unique equilibrium under Approval Voting.

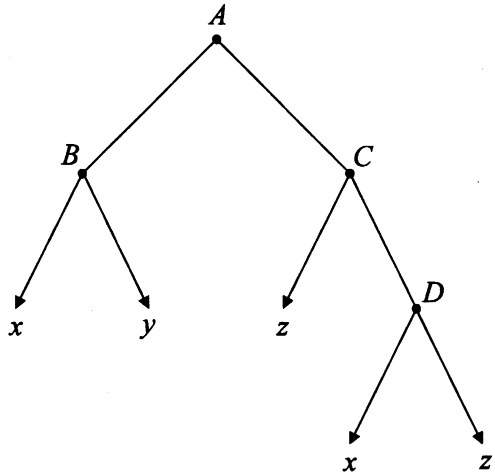
### 3.3 Nomination procedures

Nomination Procedures provide another class of multi-stage voting rules that will select the Condorcet Winner under appropriate conditions. Barberá and Coelho (2004) consider a class of voting rules which they refer to as the Rule of $k$ Names. Under the Rule of $k$ Names, one electorate proposes $k$ candidates to a second electorate and the second electorate chooses one of the nominated candidates. Barberá and Coelho solve for the strong equilibria for this class of voting rules.

I will focus on the special case where two "names" are selected to advance and both electorates use Plurality Rule to decide the winner. I will refer to this voting rule as the

[10] Markov equilibria require that the strategies of the players at stage $t$, as a function of the histories, depend only on the identities of the candidates who have not yet been eliminated from contention.

**Fig. 1** A Binary Voting Tree




Nominate-Two Rule. Thus, there is a first stage where a set of voters $N_1$ each vote for a candidate in $X$. The two candidates with the largest vote totals advance to the next stage. Then, a second set of voters, $N_2$ with $N_1 \cap N_2 = \emptyset$, select a candidate from the nominated candidates.

Think of the first group of voters as a small legislature or committee and the second group of voters as a large electorate. This suggests solving the first stage using cooperative equilibrium concepts and then solving the second stage using non-cooperative concepts. I will refer to such an equilibrium as a Subgame Perfect First-Stage Strong Equilibrium. Let $\succ^1$ and $\succ^2$ represent the Majority Social Preference for voters in groups $N_1$ and $N_2$.

**Proposition 6** *Under the Nominate-Two Rule, if $x \succ^2 y$ for all $y \in X$ and $|\{i \in N_1 : x \succ_i y\}| > n_1/3$, then there exists a unique Subgame Perfect First Stage Strong Equilibrium outcome where $x$ wins the election.*

The conditions of Proposition 6 are rather easy to meet, indicating that this voting rule is effective in selecting the Condorcet Winner so long as the preferences of the nominating committee do not differ too much from the preferences of the electorate.

### 3.4 Binary voting trees

Binary Voting Trees provide a third class of voting rules that are guaranteed to select the Condorcet Winner. Binary Voting Trees form a sequence of binary elections defined by a tree structure (an example is given in Fig. 1). If there is an even number of voters, then it is possible to have ties. Following McKelvey and Niemi (1978), I designate a deterministic tie-breaking rule. For example, in the Binary Voting Tree pictured in Fig. 1, we may impose the condition that we follow the left branch in the event of a tie.

A Binary Voting Tree is defined as a finite collection of nodes $V$ such that (i) each node points to an outcome $x \in X$ or contains two branches pointing to other nodes in $V$ (one branch is labeled left and the other branch is labeled right) and (ii) there does not exist a cycle among the nodes in $V$.[11] Each time the voters reach a node, they choose whether

[11] See McKelvey and Niemi (1978) for a definition of Binary Voting Trees specified in terms of histories.

to proceed along the left branch or the right branch. Left is chosen if left receives at least half the votes and otherwise, right is chosen. Thus, ties are broken by selecting the left branch. Refer to the nodes that point to an outcome as terminal nodes. As before, I solve for the equilibrium of this multi-stage voting rule by backwards induction, eliminating weakly dominated strategies at each stage.

The following result holds for Binary Voting Trees.

**Proposition 7** *Suppose that $x$ is the Condorcet Winner and $x$ appears is some terminal node of the Binary Voting Tree. Then there exists a unique Subgame Perfect Undominated Nash Equilibrium outcome where $x$ is selected as the winner.*

Proposition 7 shows that electing a Condorcet Winner can be achieved under weaker conditions than are required for Multi-Stage Runoff or the Nominate-Two Rule. This result is a direct consequence of Corollary 2 in McKelvey and Niemi (1978).[12] Notice also that we can design a Binary Voting Tree with only $\lceil \log_2 k \rceil$ stages of voting that selects the Condorcet Winner.[13] This compares favorably with the $k - 1$ stages of voting that would be required for Multi-stage Runoff.

## 4 Discussion

I have argued that the commonly used voting rules suffer from multiple equilibria. Multiple equilibria are pervasive under Plurality Rule and Single Transferable Vote. The result for Single Transferable Vote is somewhat surprising given the enthusiasm that political reformers have for this voting rule. Contrary to what is popularly believed, the incentives for strategic voting under Single Transferable Vote are great. Approval Voting and Majority Rule perform somewhat better: there is sometimes a unique equilibrium in which the Condorcet Winner is elected.

The solutions I considered in this paper took the form of multi-stage voting rules. Three multi-stage voting rules will guarantee that Condorcet Winners are elected: Multi-Stage Runoff, the Nominate-Two Rule, and Binary Voting Trees. Multi-stage voting rules have the obvious disadvantage of requiring multiple stages of voting, which is particularly cumbersome when the number of candidates is large. If there are $k$ candidates, Multi-Stage Runoff requires $k - 1$ stages, the Nominate-Two Rule requires two stages, and Binary Voting Trees require at least $\lceil \log_2 k \rceil$ stages.

Though single-stage voting rules are deficient in terms of selecting Condorcet winners, the lack of practical alternatives means that single-stage voting rules will continue to be used in many circumstances. However, there is a range of circumstances where multi-stage voting rules remain useful. The Nominate-Two Rule is the most workable for large elections since

[12] Miller (1977) proves that the equilibrium outcomes of amendment procedures and successive procedures must be contained in the Top-cycle set (or the minimal undominated or Condorcet set is his terminology). The Top-cycle set reduces to the Condorcet Winner when one exists. Miller (1980) shows that the equilibrium outcomes for amendment procedures must be located in the Uncovered set. Banks (1985) shows that the equilibrium outcomes of amendment procedures must be located in a smaller set, later termed the Banks set. See Miller (1995) for further results.

[13] Here, $\lceil j \rceil$ represents the least integer greater than or equal to $j$. The expression derives from the fact that a $T$-stage Binary Voting Tree can have at most $2^T$ terminal nodes, and hence can select as many as $2^T$ outcomes.

it requires only a single-stage of voting in the large electorate. However, it presumes the existence of a small committee whose preferences do not differ too much from the electorate as a whole. Multi-Stage Runoff is more practical for decision making in small committees or organizations. Binary Voting Trees are already widely for decision making in legislatures.

The Nominate-Two Rule requires that a small committee exists, but there are a number of important situations where this condition is met. When electing a prime minister in a parliamentary democracy, the parliament can serve as the small committee. For electing judges, the national and state legislatures can act as the committee. In addition, the Nominate-Two Rule offers an alternative form for direct democracy, with national and state legislative chambers serving as the nominating committee.

**Acknowledgements** I would like to thank Steven Brams, Ron Goettler, Brett Gordon, Bill Keech, Robert Norman, John Patty, Maggie Penn, Uday Rajan, Remzi Sanver, Sinan Sarpca, Holger Sieg, and an anonymous referee, as well as participants of seminars at Carnegie Mellon University and the Public Choice Society meeting (Baltimore, 2004), for many helpful suggestions.

## Appendix: Proofs of the propositions

*Proof of Proposition* 1 Brams and Fishburn (1978) show that if a candidate is not the last choice of voter $i$, then voting for that candidate is an undominated strategy. Suppose that candidate $x \in X$ is not a Condorcet Loser. Then there is a candidate $y \notin x$ such that $x \succ y$. We can construct an equilibrium where all voters vote for $x$ if $x \succ_i y$ and $y$ otherwise. Switching one's vote to $z \notin \{x, y\}$ can only increase a voter's utility if $z$ wins the election with some positive probability. This cannot be the case since $n \geq 4$ implies that $x$ must be receiving at least two votes. Similarly, switching one's vote to the least preferred candidate cannot increase one's utility since this is a weakly dominated strategy. Thus, we have constructed a Nash Equilibrium where $x$ wins with probability 1. Since this equilibrium does not involve weakly dominated strategies, we have an Undominated Nash Equilibrium. □

*Proof of Proposition* 2 (i) Brams and Fishburn (1978) show that a strategy is Undominated if and only if it involves approving the most preferred candidate and not approving the least preferred candidate. Suppose that the number of voters who don't prefer $x$ last is at least two more than the number of voters who prefer $y$ first for all $y \in X$. We can construct an equilibrium where $x$ wins the election. Suppose that all voters who don't prefer $x$ last approve only $x$, and that all other voters approve only for their first preference. Clearly, all voters are playing undominated strategies. Since $x$ has at least two more votes than all the other candidates, this is an Undominated Nash Equilibrium, since no single voter can change the outcome of the election by deviating.

(ii) Notice that if voters play undominated strategies, the maximum number of votes that $x$ can get (which is equal to number of voters that do not prefer $x$ last) is less than the minimum number of votes $y$ can get (which is equal to the number of voters that prefer $y$ first). Since $x$ will always have fewer votes than $y$, $x$ cannot win in equilibrium. □

*Proof of Proposition* 3 Let us denote the set of candidates by $\{x, y, z_1, z_2, \ldots, z_{k-2}\}$. Let $n' = n - 1$. Suppose that among a set of $n'$ voters, $\alpha$ voters rank $x$ first and $y$ second, $\alpha$ voters rank $y$ first and $x$ second, and $\beta_j$ voters rank $z_j$ first, $x$ second, and $y$ third, for all $j = 1, 2, \ldots, k-2$. Suppose further that $\alpha < \beta_j < 2\alpha$ for $j = 1, 2, \ldots, k-2$. If a voter were to rank $x$ first, then $y$ would be eliminated in the first round, and $x$ would go on to win the election. If a voter where to rank $y$ first, then $x$ would be eliminated, and $y$ would go on to win the election. If the voter were to rank any other candidate first, a lottery would determine if $x$ or $y$ is eliminated first. Hence, the final outcome would be a lottery between $x$ and $y$. In this situation, any voter who prefers $x$ to $y$ must rank $x$ first. Thus, provided that this setup is possible, for this voter there must exist some strategy ranking $x$ first that is not weakly dominated.

Next, we must check whether there exist non-negative integers $\alpha, \beta_1, \ldots, \beta_{k+2}$ such that $\alpha < \beta_j < 2\alpha$ for $j = 1, 2, \ldots, k-2$ and $2\alpha + \beta_1 + \ldots + \beta_{k-2} = n'$. Without loss of generality, let us select $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{k-2}$. We can select $\alpha \lfloor n'/k \rfloor - 1$, $\beta_j = \lfloor = n'/k \rfloor$ for $j = 1, \ldots, k-3$, and $\beta_{k-2} = n' + 2 - (k-1)\lfloor n'/k \rfloor$. Clearly, $\alpha < \beta_1$, so we must only check that $\beta_{k-3} \leq \beta_{k-2}$ and $\beta_{k-2} < 2\alpha$. In the first case, we require $\lfloor n'/k \rfloor (n'+2)/k$, which clearly holds. In the second case, we require $n' - (k+1)\lfloor n'/k \rfloor + 4 < 0$, which holds by assumption. Thus, the setup described above is possible, and for this voter there must exist some strategy ranking $x$ first that is not weakly dominated.

Next, we would like to show that $n' - (k+1)\lfloor n'/k \rfloor + 4 < 0$ and $k \geq 3$ imply that $n \geq 4$. Notice that the first inequality implies

$$n' - (k+1)\lfloor n'/k \rfloor + 4 \leq n'(1 - (k+1)/k) + 4 = -n'/k + 4 < 0.$$

From this, we have $n' > 4k$ which clearly implies that $n \geq 4$ when $k \geq 3$ . Finally, note that $n' - (k+1)\lfloor n'/k \rfloor + 4 < 0$ is equivalent to $n - (k+1)\lfloor (n-1)/k \rfloor + 3 < 0$.

Now let $x$ be a candidate that is not a Condorcet Loser and let $x \succ y$. Suppose all voters who prefer $x$ to $y$ rank $x$ first and all other voters rank $y$ first. Switching to a strategy where $z \notin \{x, y\}$ is ranked first cannot improve one's utility since candidate $z$ is still guaranteed to be eliminated before $x$ and $y$. Similarly, switching to a strategy that ranks the least preferred candidate first cannot improve one's utility. We showed above that this equilibrium involves undominated strategies, so we have found an Undominated Nash Equilibrium where $x$ wins. □

*Proof of Proposition* 4 We can show that this holds by induction. The base case of $T = 1$ stages and $k = 2$ candidates holds trivially. Now suppose that the proposition holds for $T = t$ stages. We can show that it must also hold for $T = t + 1$ stages. Let $x$ be the Condorcet Winner and let $y$ be the Condorcet Winner among $X - \{x\}$. We can show that for all voters who prefer $x \succ_i y$, voting for $x$ is a weakly dominant strategy in the stage 1 subgame. Suppose that the voter is pivotal between $x$ and some other candidate $x \notin \{x, y\}$. If this voter votes for $x$ in the first round, then $x$ will advance to the next stage. By the induction hypothesis, $x$ will win the election. If $x$ does not advance to the next stage, then by the induction hypothesis, $y$ will win the election. So for all voters with $x \succ_i y$, voting for $x$ is a weakly dominant strategy in the stage 1 subgame. Since $x \succ y$, more than half of the voters will vote for $x$, and $x$ is guaranteed to advance to the second stage. By the induction hypothesis, $x$ will win the election. □

*Proof of Proposition* 5 (i) We can begin by characterizing the outcome of the final stage. Voters have weakly dominated strategies of voting sincerely in this subgame, as they are choosing between two candidates (Brams and Fishburn 1978). Hence, the unique Undominated Nash Equilibrium outcome of the runoff subgame will be $x$ if $x$ advances to the runoff stage and $y$ otherwise.

Assume that $n$ is odd (the result when $n$ is even can be proved analogously). Let $\#x$ denote the number of voters among the $n - 1$ other voters who vote for candidate $x$ in the

**Table 1**

| Scenario | First-stage vote | | |
|---|---|---|---|
| | $x$ | $y$ | $z$ |
| (1) $\#z = (n-1)/2 > \#y > \#x + 1$ | $y$ | $y$ | $z$ |
| (2) $(n-1)/2 > \#y > \#x + 1 = \#z + 1$ | $x$ | $(x, y)$ | $y$ |

first stage. Table 1 summarizes two voting scenarios. For each scenario, the outcome of the election is listed for each possible first stage vote of player $i$. Here, $(x, y)$ denotes a lottery between candidate $x$ and candidate $y$. In scenario (1), if voter $i$ votes for candidate $z$, then $z$ will win the election without a runoff. If the voter votes for $x$ or $y$, then candidates $y$ and $z$ will advance to the runoff stage, and candidate $y$ will win. In scenario (2), if voter $i$ votes for candidate $x$, candidates $x$ and $y$ will advance to the runoff stage, and candidate $x$ will win. If voter $i$ votes for candidate $z$, candidates $y$ and $z$ will advance to the runoff stage, and candidate $y$ will win. If voter $i$ votes for candidate $y$, a lottery will determine whether $x$ or $z$ advances to the runoff stage, and the outcome will be a lottery between $x$ and $y$.

From scenario (1), we can determine that if a voter prefers $z \succ_i y$, voting for $z$ in the first stage is an Undominated Strategy. From scenario (2), we can determine that if a voter prefers $y \succ_i x$, then voting for $z$ is the first stage is a Undominated Strategy. Hence, any voter who does not have preferences $x \succ_i y \succ_i z$ has an undominated strategy of voting for $z$ in the first stage.

Now, we can show that there exists a Subgame Perfect Undominated Nash Equilibrium where $z$ wins the election. Since $n$ is odd, a candidate needs at least $(n+1)/2$ first round votes to win without a runoff. We can ensure that no voter is pivotal in the first round if at least $(n+1)/2+1$ voters vote for candidate $z$ in the first round. If all voters for which $x \succ_i y \succ_i z$ does not hold vote for candidate $z$ in the first round, this will correspond to a Subgame Perfect Undominated Nash Equilibrium if $|\{i : \neg(x \succ_i y \succ_i z)\}| \geq (n+1)/2+1$, which holds by assumption. Hence, there exists a SPUNE where candidate $z$ (who is not the Condorcet Winner) wins the election.

(ii) We can show that in any SPUNE, a voter with $x \succ_i y \succ_i z$ must vote for $x$ in the first stage. As before, backwards induction indicates that $x$ will win the election if $x$ advances to the second stage and $y$ will win otherwise.

Voting for $y$ in the first stage is weakly dominated. Suppose that among the other voters, $x$ and $y$ are each receiving $(n-1)/4$ votes and candidate $z$ is receiving $(n-1)/2$ votes in the first stage. In this case, voting for $x$ yields strictly higher utility than voting for $y$. To show that voting for $y$ is weakly dominated, we must show that there does not exist any case where voting for $y$ yields strictly higher utility than voting for $x$. We can focus on the situations where the voter is pivotal. If candidate $x$ has $(n-1)/2$ first round votes, then voting for $x$ clearly yields at least as much utility as voting for $y$. If candidate $y$ has $(n-1)/2$ first round votes, then voting for $x$ clearly yields at least as much utility voting for $y$, since if the game proceeds to the second round, only $x$ and $y$ can win. A similar argument holds if $z$ has $(n-1)/2$ first round votes. The voter may also be pivotal about which candidate advances. If he is pivotal about whether $y$ or $x$ advances, he clearly receives at least as much utility by voting for $x$, since this assures that $x$ advances to the runoff stage (and thus wins the election). A similar argument holds if he is pivotal about whether $z$ or $x$ advances.

A similar argument can be used to show that voting for $z$ is the first stage is a weakly dominated strategy. Now, since we have shown that all voters with $x \succ_i y \succ_i z$ must vote for $x$ in the first stage in any SPUNE, it must be the case that candidate $x$ receives at least

$(n+1)/2$ first-stage votes. This ensures that $y$ and $z$ do not win in the first stage and ensures that $x$ advances to the second stage in the event of a runoff. This guarantees that candidate $x$ wins the election in any SPUNE.

Next, we can show that there exists a SPUNE where all voters with $x \succ_i y \succ_i z$ vote for $x$ in the first round. All voters with $x \succ_i y \succ_i z$ are playing best responses since $x$ is their most preferred outcome. None of the other voters is pivotal, so they are playing best responses as well. Hence, there exists a unique SPUNE where $x$ wins the election. □

*Proof of Proposition* 6 Let $a_i^1 \in X$ for $i \in N_1$ denote the action of a first-stage voter. Let $a^1 = (a_1^1, a_2^1, \dots, a_{n_1}^1)$ denote the vector of such actions. Since all the second-stage voters face a plurality rule vote between two candidates, we can determine that they will vote sincerely. Using backwards induction, we can determine that if $\{x, y\}$ are selected to advance to the second stage, then the outcome will be $x$ if $x \succ^2 y$, $y$ if $y \succ^2 x$, and a lottery over $\{x, y\}$ otherwise. Let $\phi(x, y)$ denote this outcome. The choice $\{x, y\}$ will satisfy the conditions of equilibrium if and only if there does not exist a coalition of voters $L \subset N_1$ and first-stage actions $b^1$ such that $a_n^1 = b_n^1$ for all $n \notin L$ and $\phi(b^1) \succ_n^1 \phi(a^1)$ for all $n \in L$. To construct an such an equilibrium, suppose that $a_i^1 = x$ for all $i \in N_1$. If the coalition $L$ has $|L| < \frac{2}{3}n_1$, then it must still be the case that $x$ advances to the second stage, which implies $\phi(a^1) = x$. Alternatively, $|L| \geq \frac{2}{3}n_1$, then there must be an $i \in N_1$ such that $x \succ_i^1 y$ so that $\phi(b^1) \succ_i^1 \phi(a^1)$ for all $i \in L$ cannot hold. Thus, $a_1$ must be an equilibrium. We also need to show that $x \neq y$ cannot be an equilibrium. Since $x \succ^2 y$, $y$ could be selected only if $x$ does not advance to the second stage. In this case, there is a coalition $L \subset N_1$ or voters such that $x \succ_n^1 y$ for all $i \in L$ and $|L| > \frac{1}{3}n_1$ by assumption. If these voters all nominate $x$, then $x$ will win the election. Thus, $y \neq x$ cannot be an equilibrium. □

*Proof of Proposition* 7 Corollary 2 of McKelvey and Niemi (1978) shows that for a more general class of games, all equilibrium outcomes must be contained in the Top-cycle set. Subgame Perfect Undominated Nash Equilibrium imposes a further requirement, so the set of SPUNE to the Binary Voting Tree must be a subset of the Top-cycle set. When a Condorcet winner exists, the Top-cycle set includes only the Condorcet Winner. Hence, the result follows directly from Corollary 2 of McKelvey and Niemi. □

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