Formal Grammar

Principles of Programming Languages
Lecture 13

CAS CS 320

Outline

Discuss briefly the **interpretation pipeline**, and how it will look in the context of this course

Introduce **formal grammars** as a mathematical framework for thinking about syntax and parsing

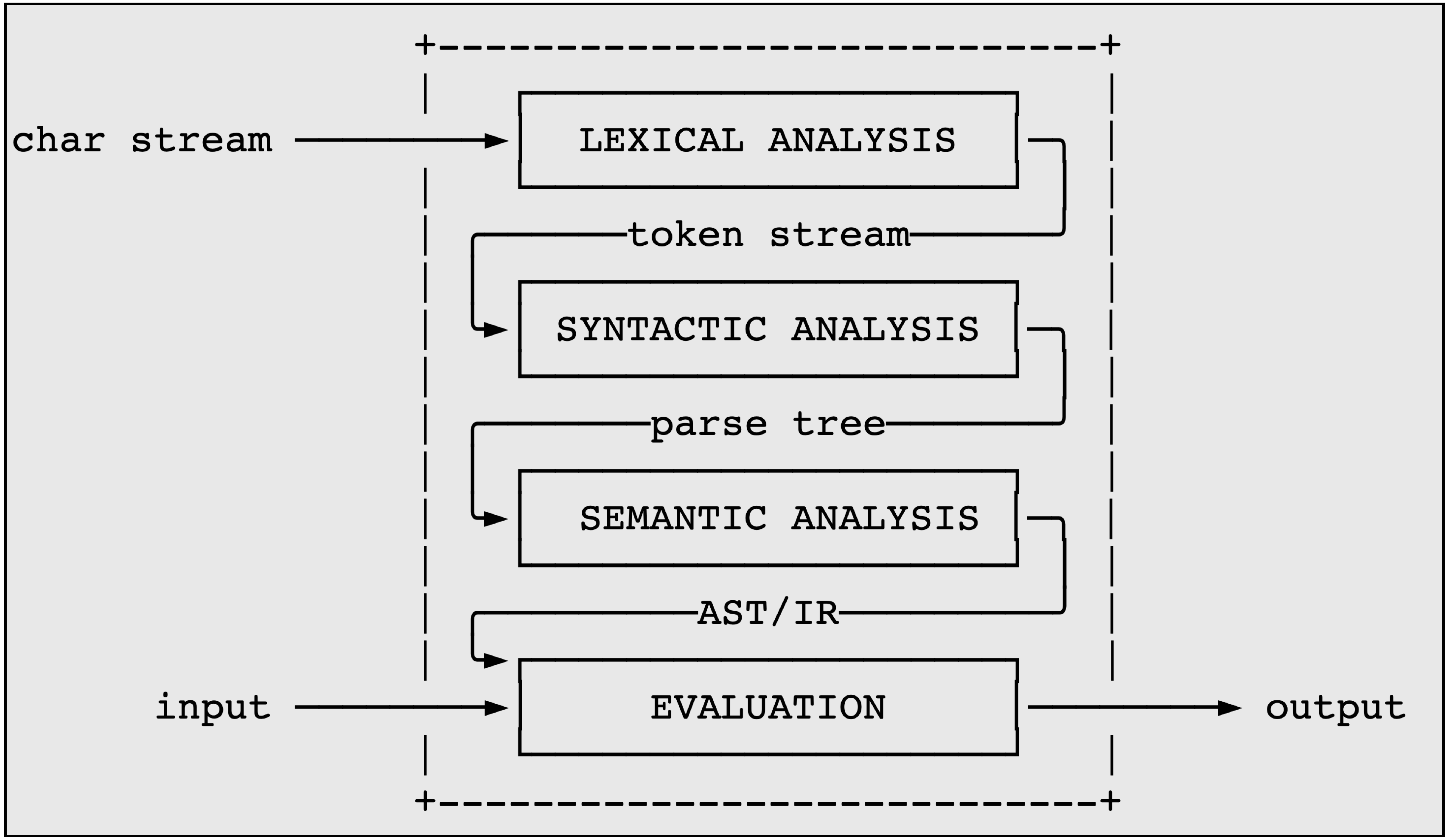
Look at what causes **ambiguity** in grammar

Learning Objectives

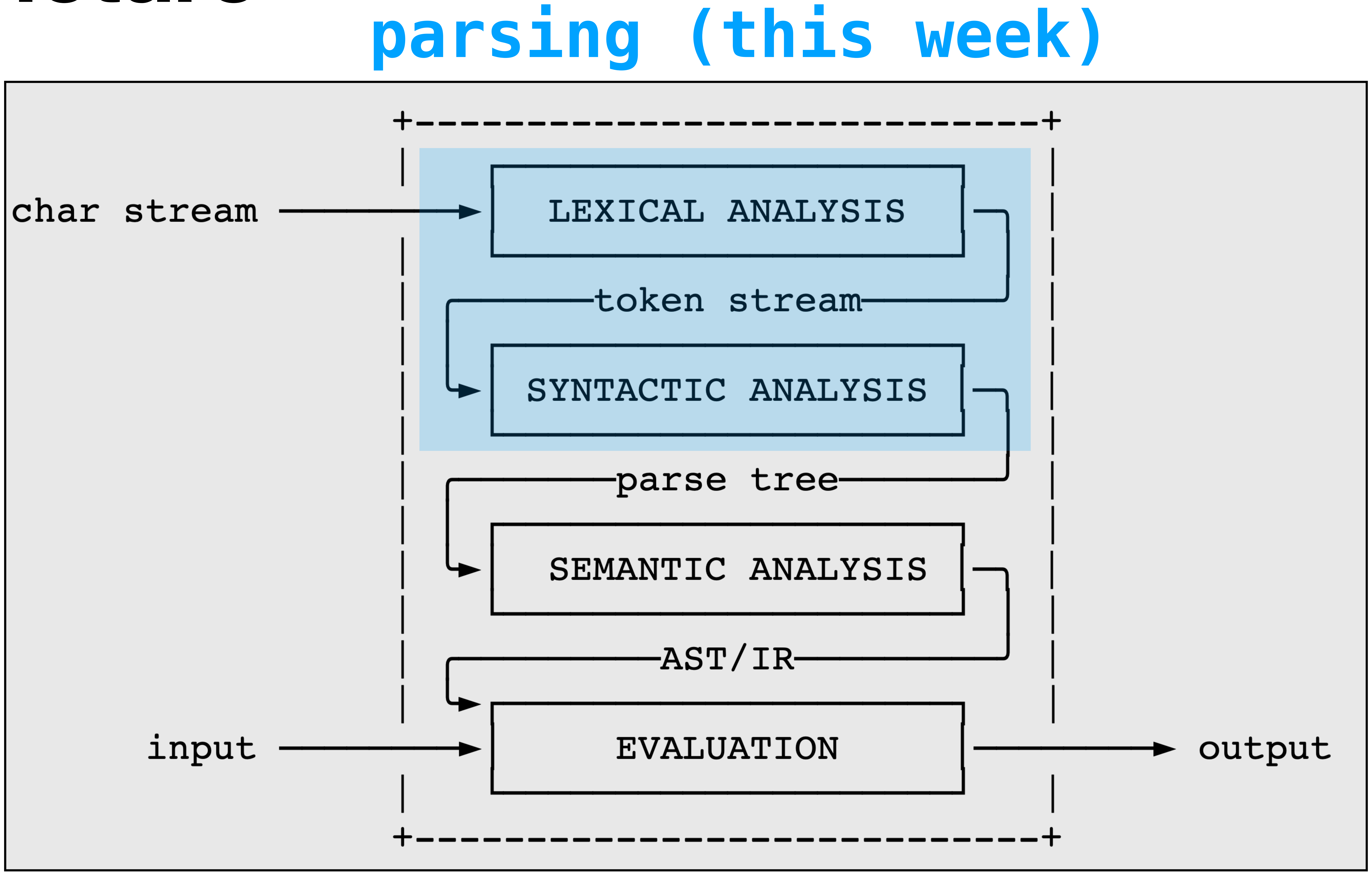
- Define sentential forms, production rules, BNF grammars, etc.
- Determine if the sentence S is recognized by the grammar $\mathcal{G}$
- Build a parse tree for S in the grammar $\mathcal{G}$
- Write a (leftmost) derivation for S in $\mathcal{G}$
- Define fixity, precedence, associativity, ambiguity, etc.
- Determine if the grammar $\mathcal{G}$ is ambiguous
- Find a sentence in an ambiguous grammar with multiple parse trees/leftmost derivations
- Based on this grammar, determine if this operator left associative or right associative
- Based on this grammar, determine if which operator has higher precedence

The Interpretation Pipeline

The Picture


char stream
LEXICAL ANALYSIS
token stream
SYNTACTIC ANALYSIS
parse tree
SEMANTIC ANALYSIS
AST/IR
input
EVALUATION
output

The Picture



parsing (this week)
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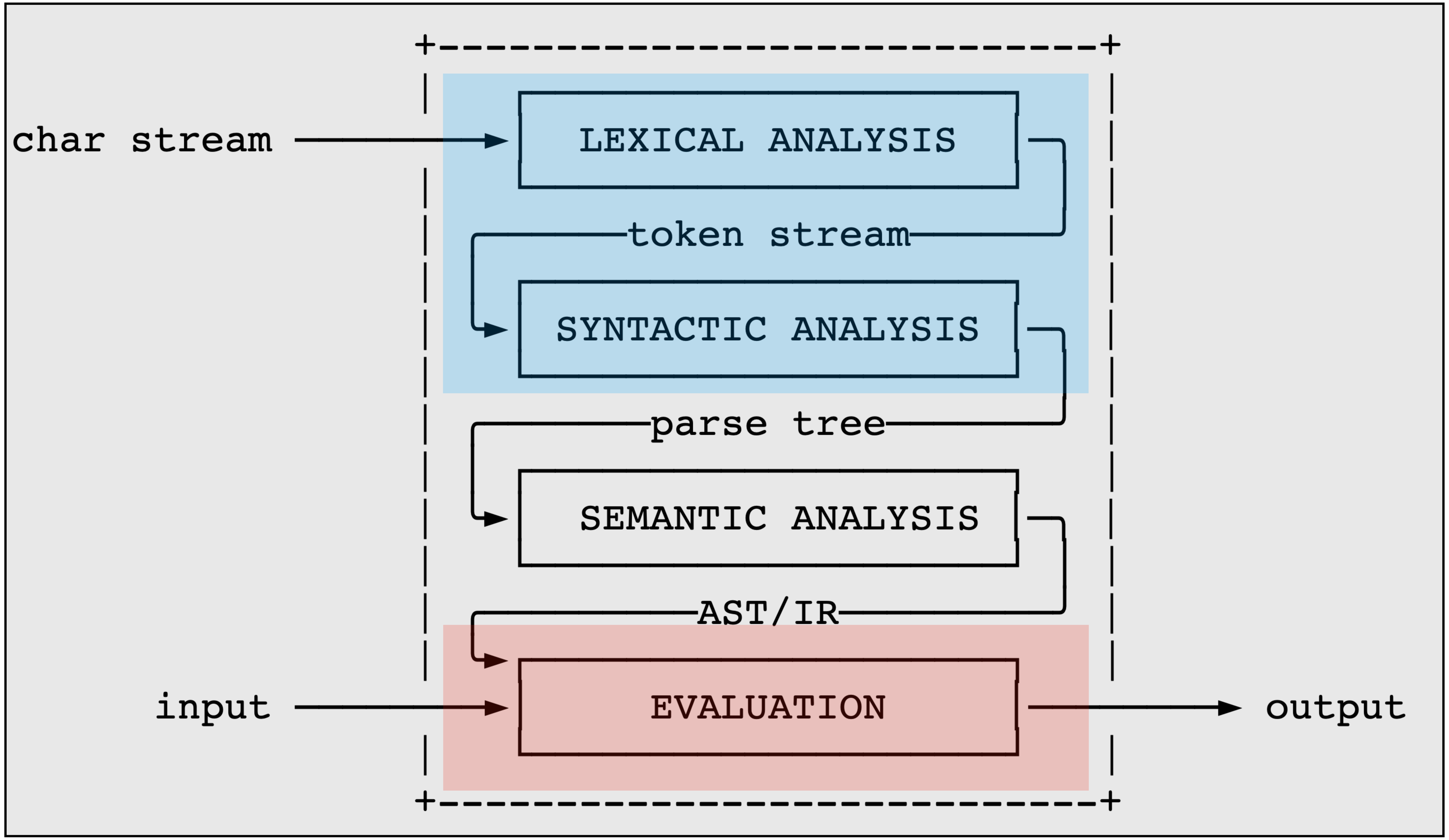
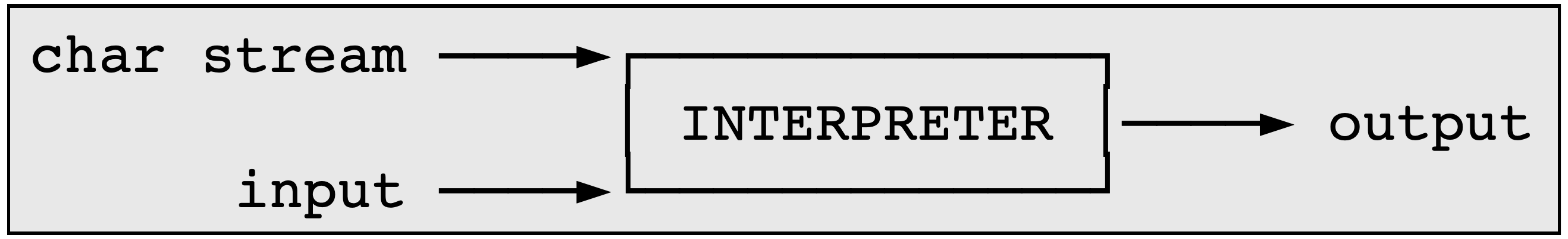
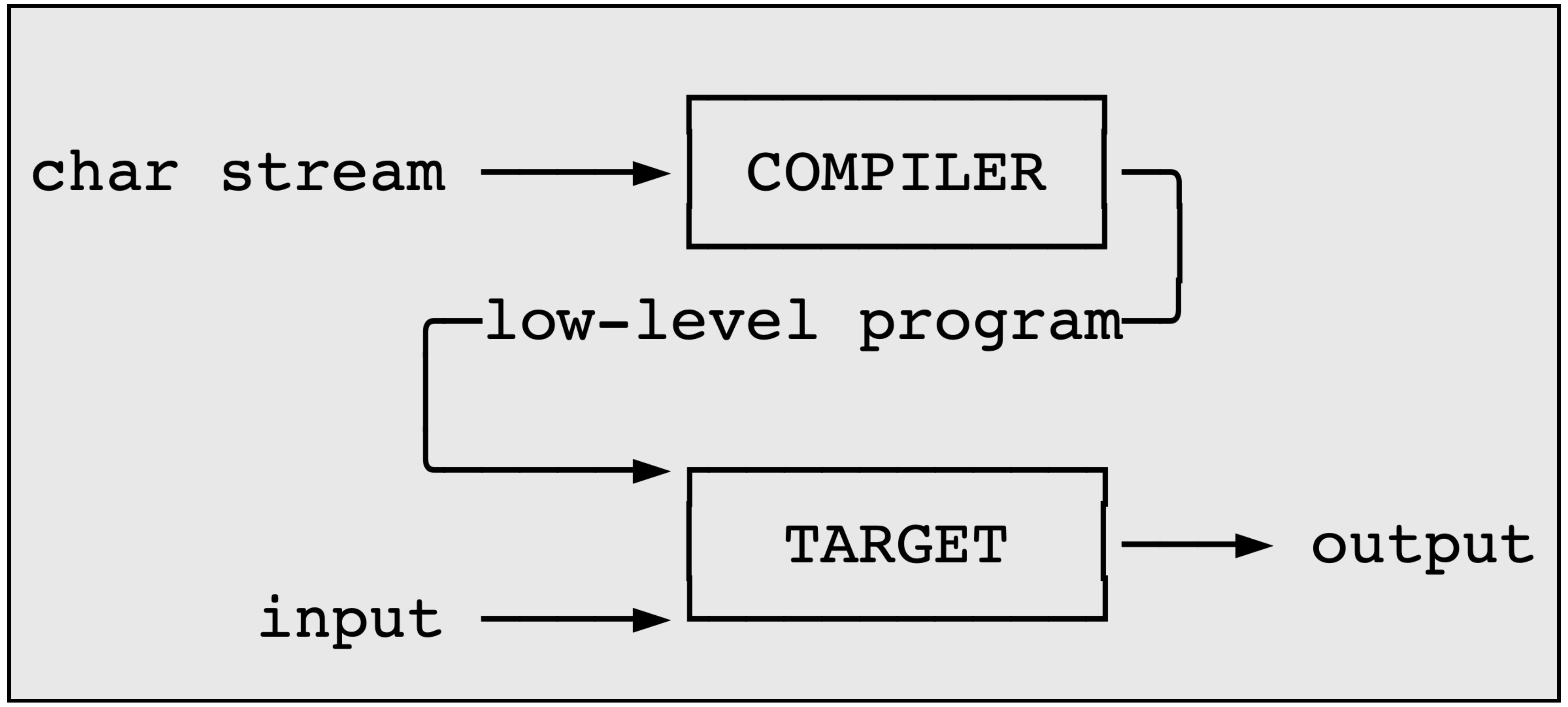


semantics (next week)

A Note on Compilation



interpretation



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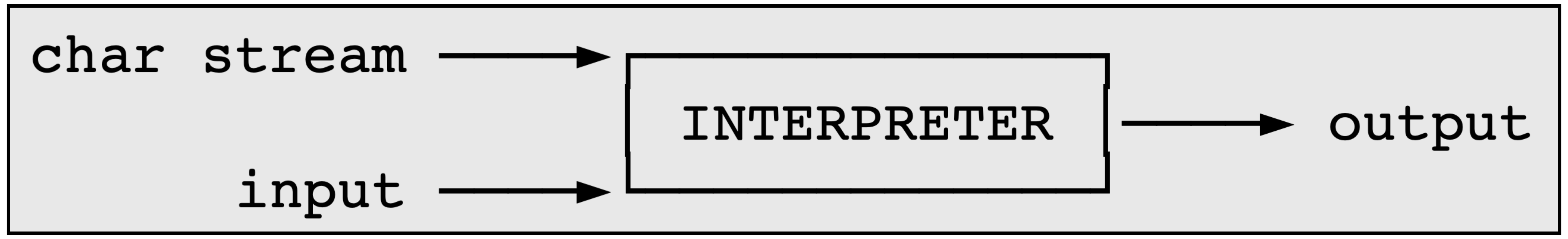

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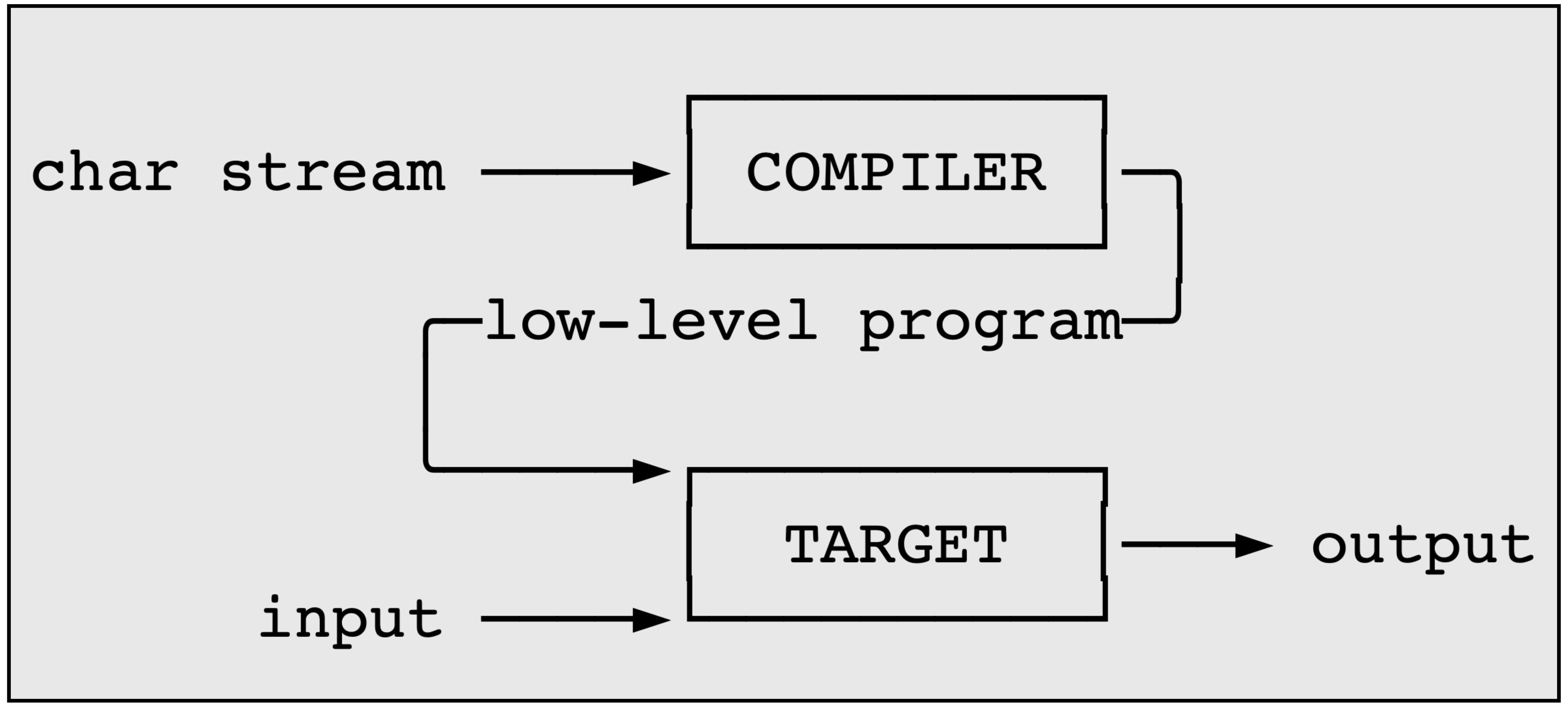

compilation

We will be building programs that directly read an evaluate programs (**interpreters**)

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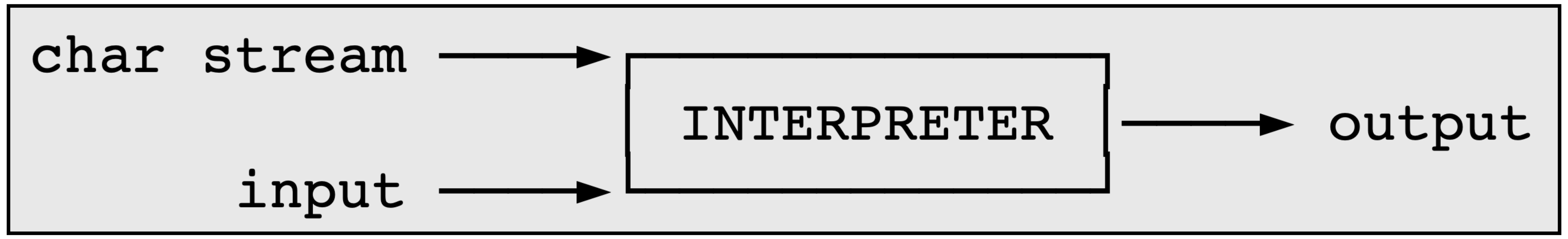

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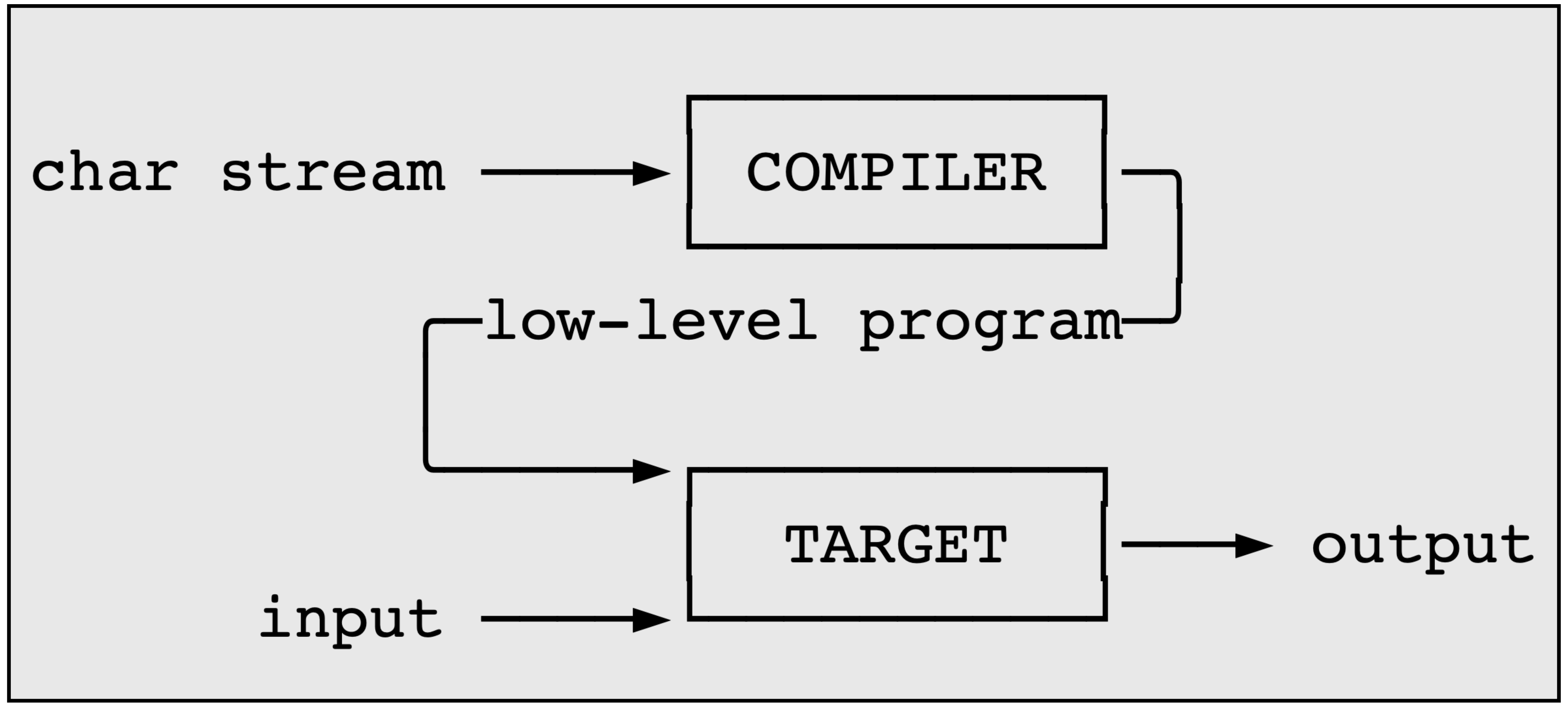


compilation

We will be building programs that directly read an evaluate programs (**interpreters**)

In a different course you may write a program which *translates* programs into another language which can then be evaluated elsewhere (**compilers**, we'll cover this briefly)

The Mini-Projects

There will be **three** mini-projects, each 2 weeks long.

For each project, you will build an interpreter.

<u>You'll be given:</u>

- the syntax
- the type rules (not in project 1)
- the semantics

Today

We need a formal language for describing the syntax of programming languages

This is part of the study of **formal language theory**

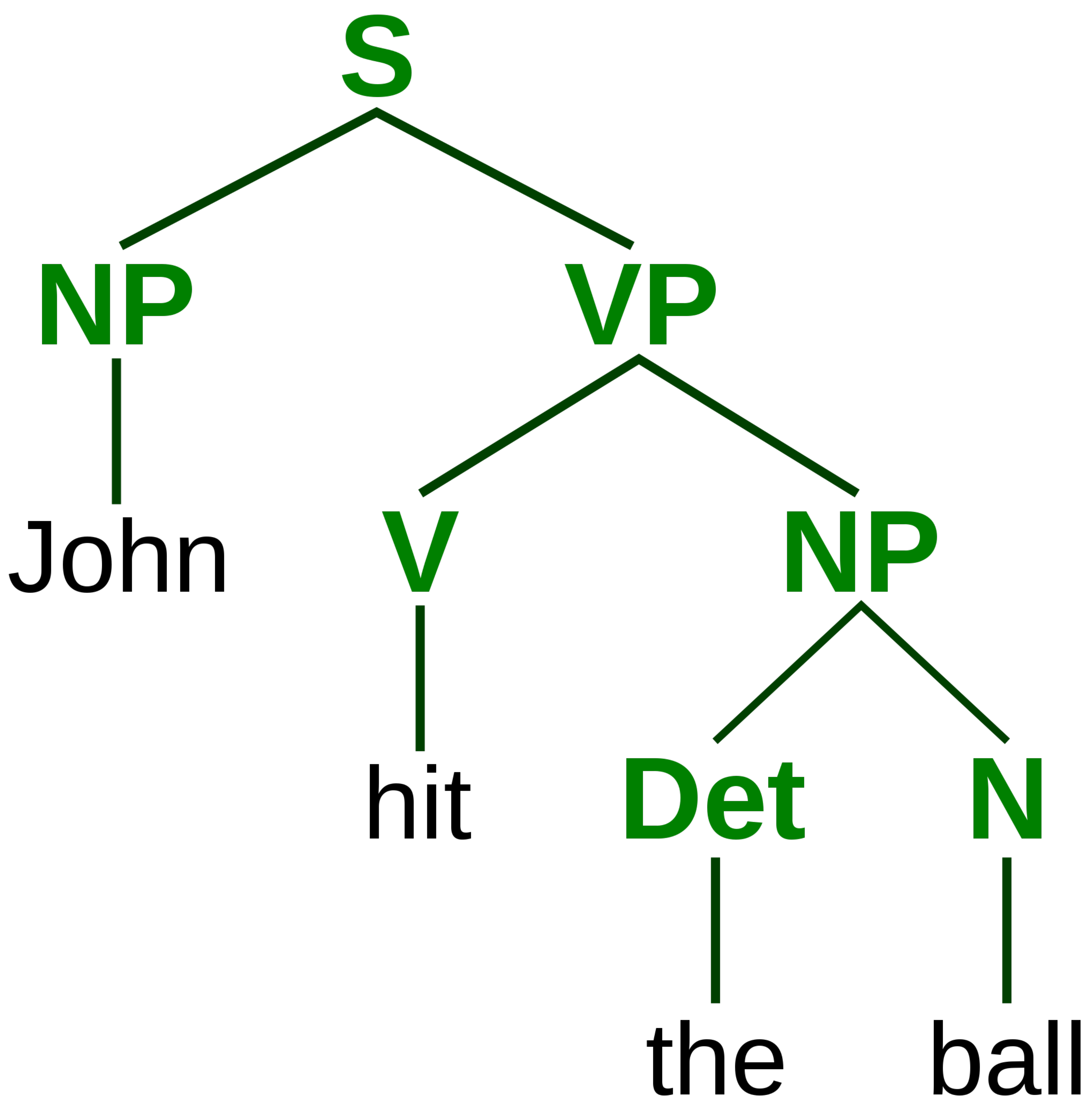
Nearly every PL out there (including OCaml) is described using **Backus–Naur Form (BNF) Grammars.**

Formal Grammar

What is Grammar?

`John hit the ball.`

⇓



`https://commons.wikimedia.org/wiki/File:ParseTree.svg`

What is Grammar?

<u>Grammar</u> refers to the rules which govern what statements are well-formed.

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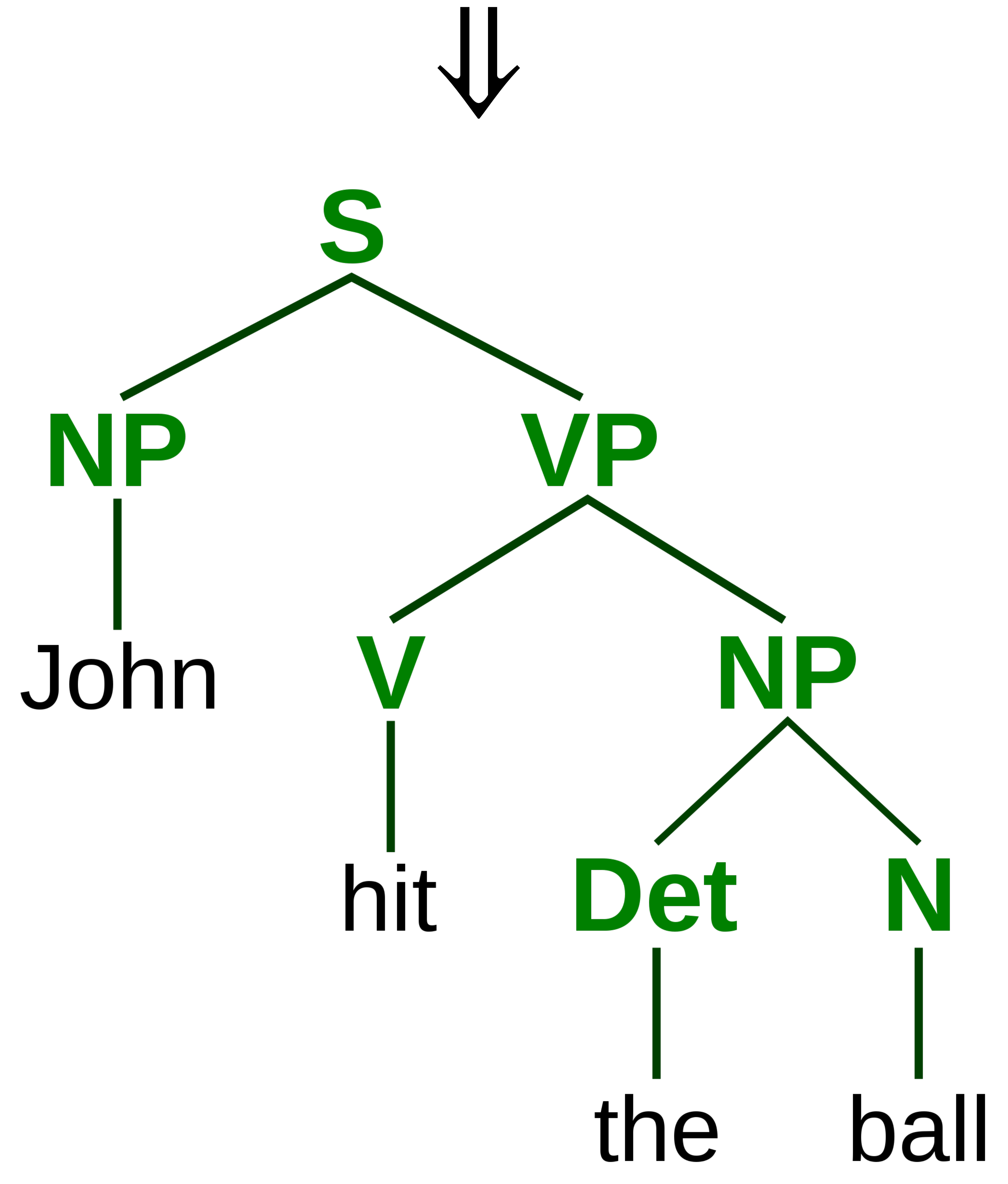

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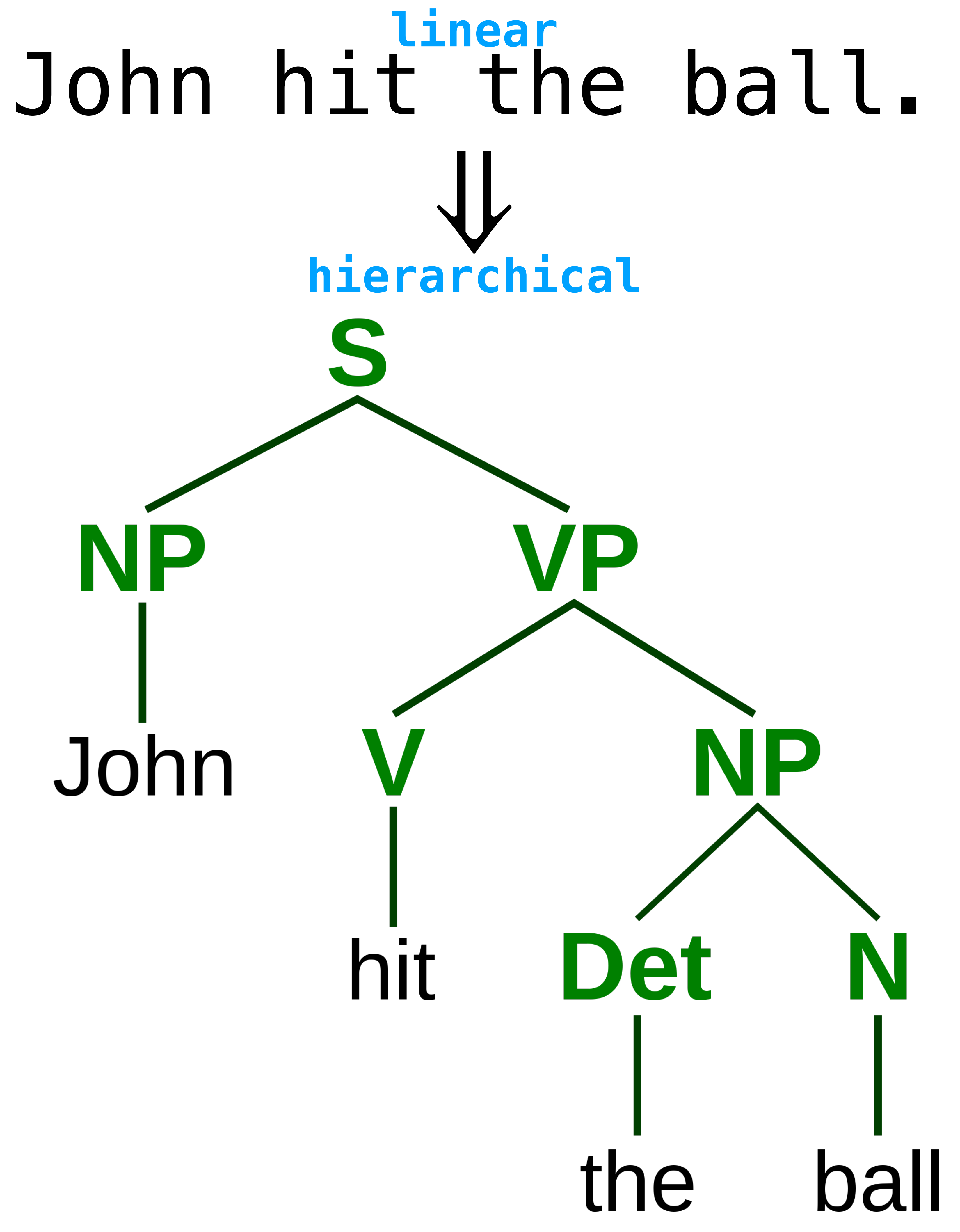
Grammar gives **linear** statements (in natural language or code) their **hierarchical** structure.



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Grammar vs. Semantics

I taught my car in the refrigerator. ✓

vs.

My the car taught I refrigerator. ✗

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Grammar is not (typically) interested in meaning, just structure.

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(As we will see, it is useful to separate these two concerns)

Grammars for Programming Languages

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# let f x = x + 1;;
val f : int -> int = <fun>
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val f : 'a -> int = <fun>
# let x = List.hd [];;
Exception: Failure "hd".
```

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(In OCaml, well-formed programs are the ones we can type-check.)

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# let x = List.hd [];;
Exception: Failure "hd".
# let x = ;;
Line 1, characters 8-10:
1 | let x = ;;
            ^^
Error: Syntax error
```

Fundamental Concern

How do we formally represent well-formed sentences?

An Example

```
the cow jumped over the moon
```

An Example

`the cow jumped over the moon`

How do we know this a well-formed sentence?

An Example

```
<article> cow jumped over the moon
```

An Example

```
<article> <noun> jumped over the moon
```

An Example

```
<noun-phrase> jumped over the moon
```

An Example

```
<noun-phrase> jumped over <article> moon
```

An Example

```
<noun-phrase> jumped over <article> <noun>
```

An Example

```
<noun-phrase> jumped over <noun-phrase>
```

An Example

`<noun-phrase> jumped over <noun-phrase>`

a thing jumped over a thing

An Example

`<noun-phrase> jumped <prep> <noun-phrase>`

An Example

```
<noun-phrase> jumped <prep-phrase>
```

An Example

```
<noun-phrase> <verb> <prep-phrase>
```

An Example

```
<noun-phrase> <verb-phrase>
```

An Example

`<noun-phrase> <verb-phrase>`

a thing did a thing

An Example

`<sentence>`

An Example

<sentence>

We know it's a sentence because it has the right kind of hierarchical structure

A Derivation

```
<sentence>
<noun-phrase> <verb-phrase>
<noun-phrase> <verb> <prep-phrase>
<noun-phrase> jumped <prep-phrase>
<noun-phrase> jumped <prep> <noun-phrase>
<noun-phrase> jumped over <noun-phrase>
<noun-phrase> jumped over <article> <noun>
<noun-phrase> jumped over <article> moon
<noun-phrase> jumped over the moon
<article> <noun> jumped over the moon
<article> cow jumped over the moon
the cow jumped over the moon
```

A Derivation

```
<sentence>
<noun-phrase> <verb-phrase>
<noun-phrase> <verb> <prep-phrase>
<noun-phrase> jumped <prep-phrase>
<noun-phrase> jumped <prep> <noun-phrase>
<noun-phrase> jumped over <noun-phrase>
<noun-phrase> jumped over <article> <noun>
<noun-phrase> jumped over <article> moon
<noun-phrase> jumped over the moon
<article> <noun> jumped over the moon
<article> cow jumped over the moon
```

```
the cow jumped over the moon
```

A Derivation

```
<sentence>
<noun-phrase> <verb-phrase>
<noun-phrase> <verb> <prep-phrase>
<noun-phrase> jumped <prep-phrase>
<noun-phrase> jumped <prep> <noun-phrase>
<noun-phrase> jumped over <noun-phrase>
<noun-phrase> jumped over <article> <noun>
<noun-phrase> jumped over <article> moon
<noun-phrase> jumped over the moon
<article> <noun> jumped over the moon
<article> cow jumped over the moon
```

```
the cow jumped over the moon
```

A Derivation

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<noun-phrase> jumped <prep> <noun-phrase>
<noun-phrase> jumped over <noun-phrase>
<noun-phrase> jumped over <article> <noun>
<noun-phrase> jumped over <article> moon
<noun-phrase> jumped over the moon
<article> <noun> jumped over the moon
```

```
<article>
        the cow jumped over the moon
```

A Derivation

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<sentence>
<noun-phrase> <verb-phrase>
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<article> <noun> jumped over the moon
```

```
<article>

    the cow jumped over the moon
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<noun-phrase> jumped over <article> <noun>
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<noun-phrase> jumped over the moon
```

```
<article> <noun>
    the cow jumped over the moon
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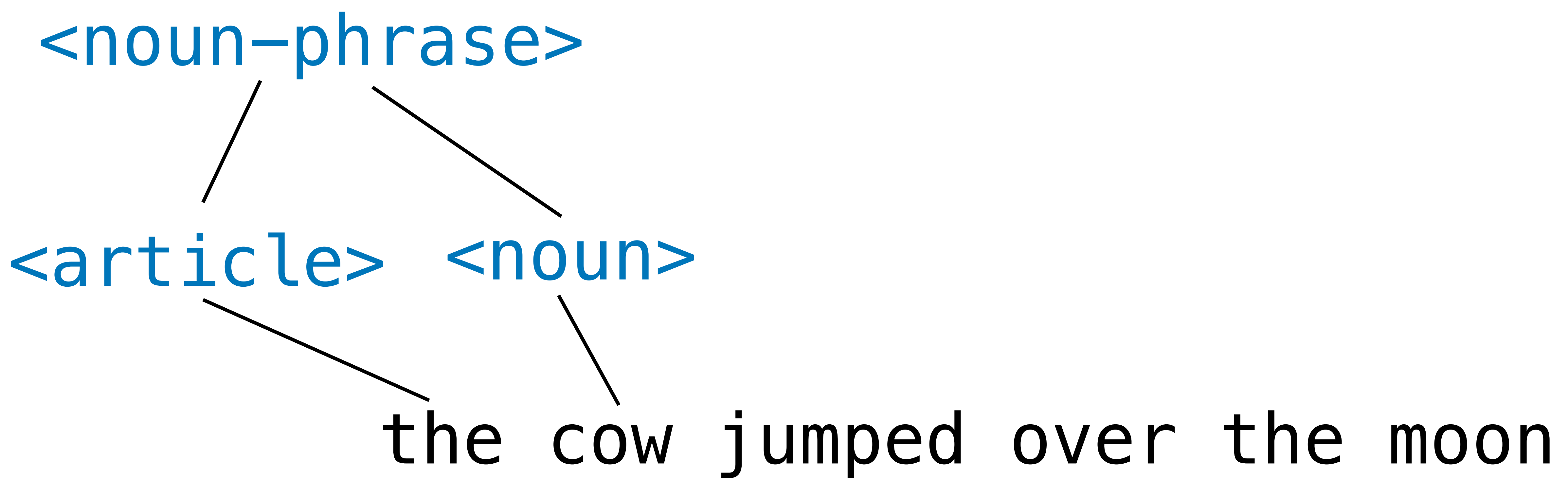

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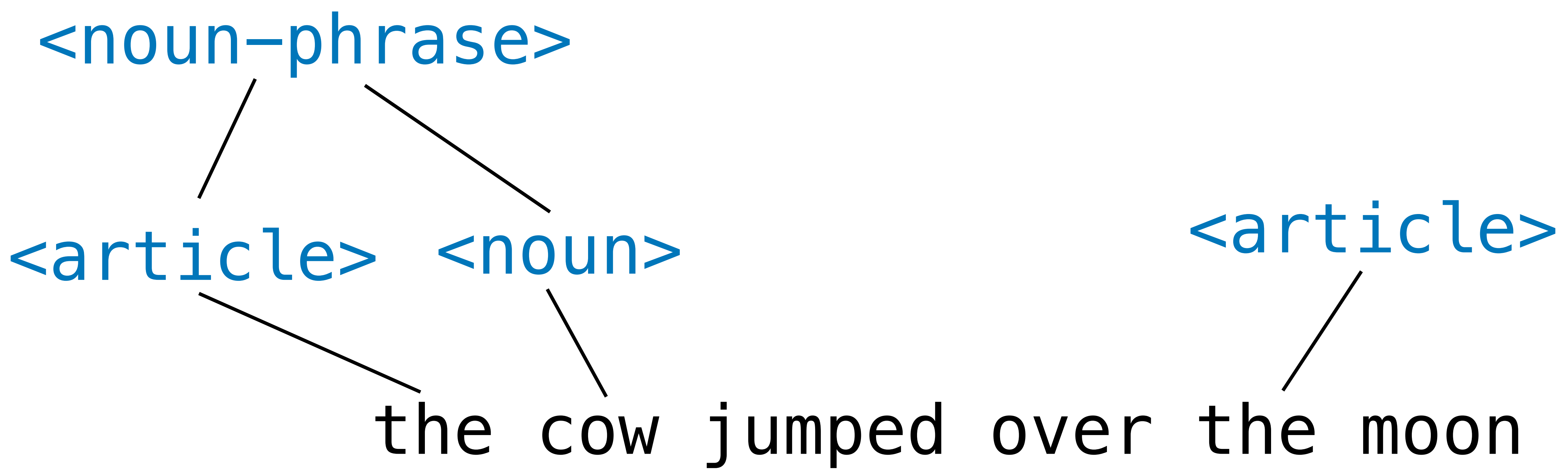

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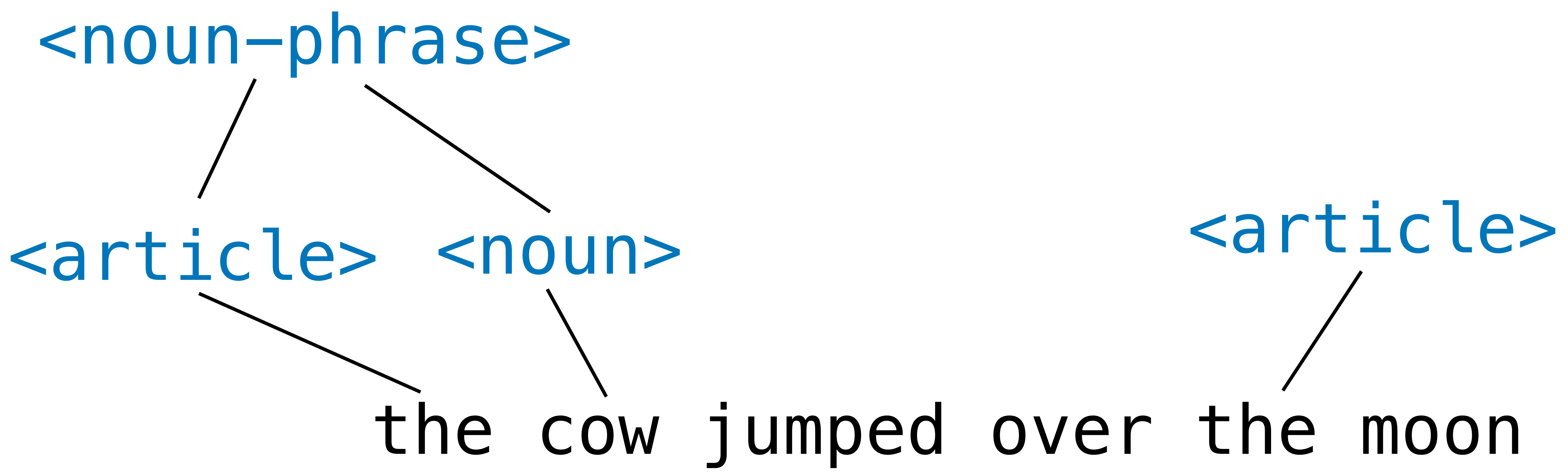

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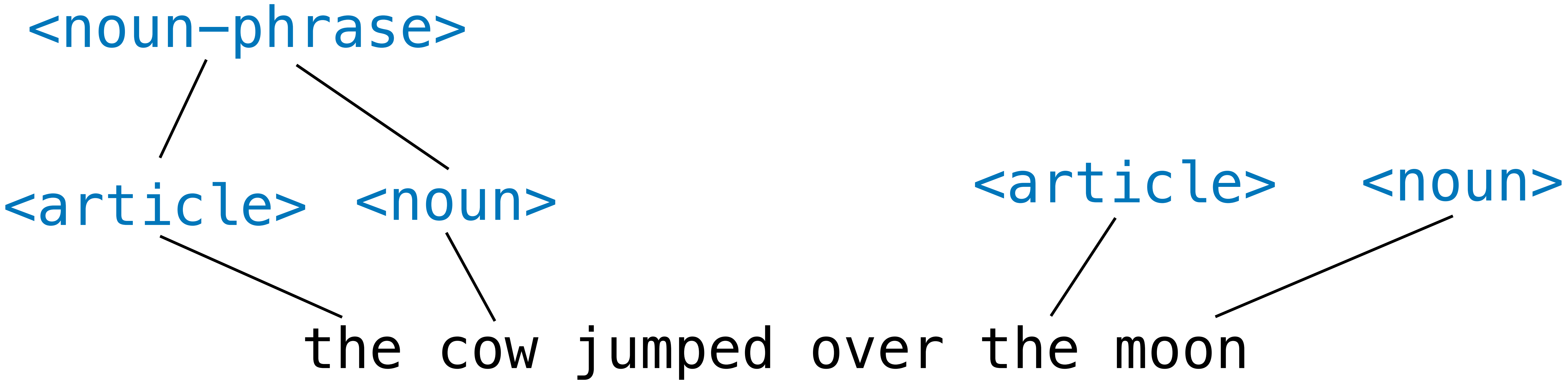

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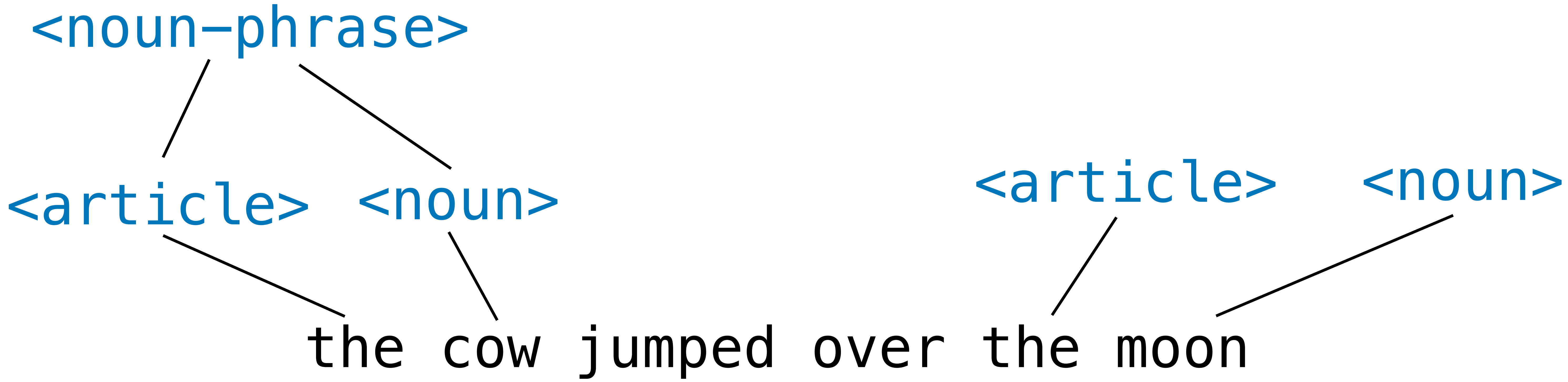

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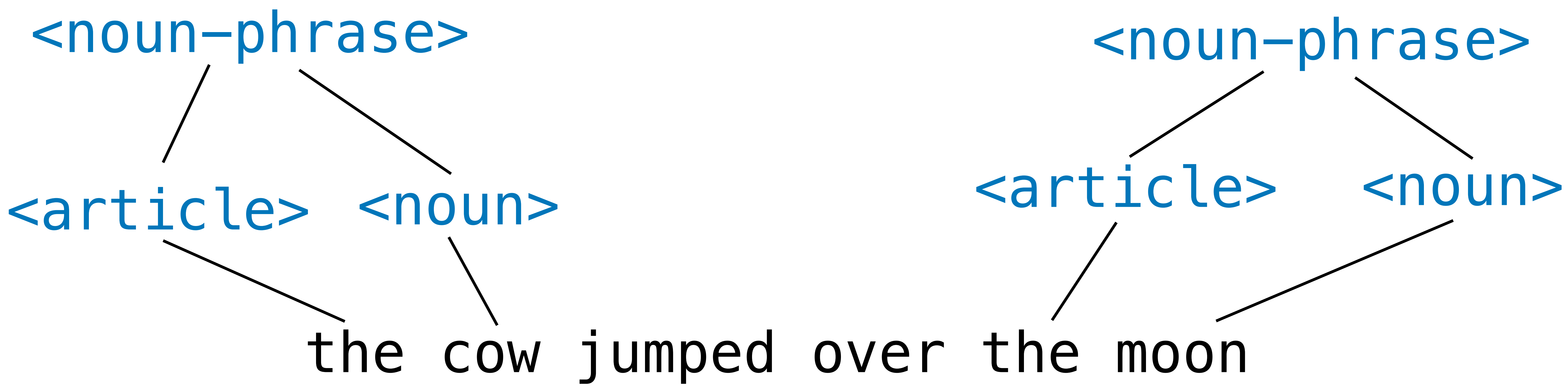

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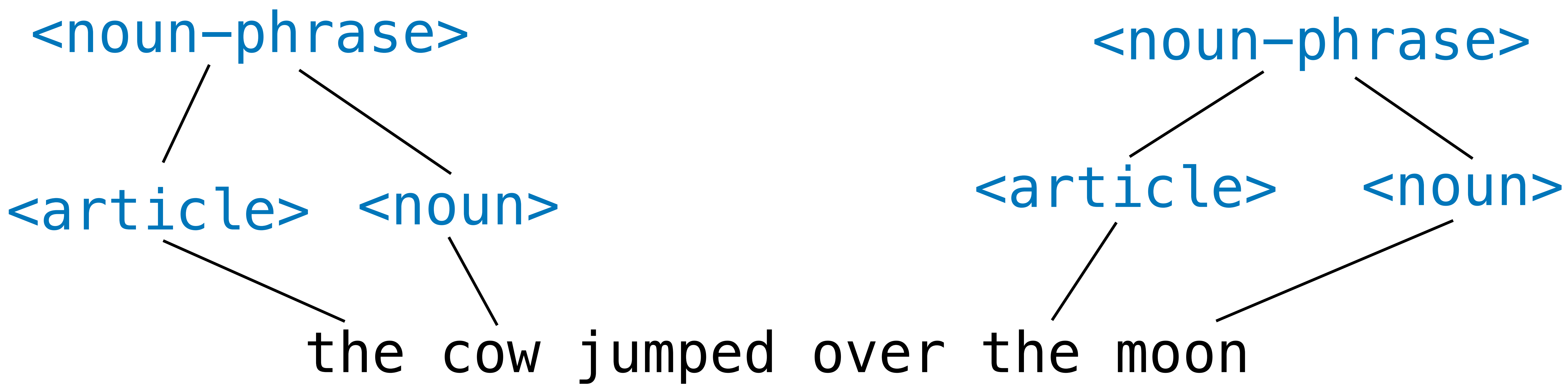

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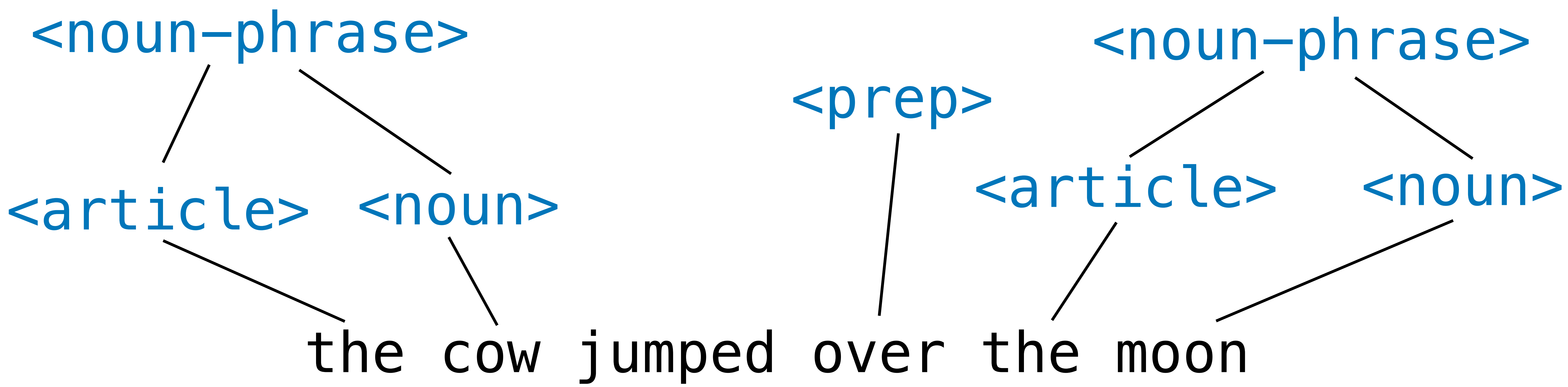

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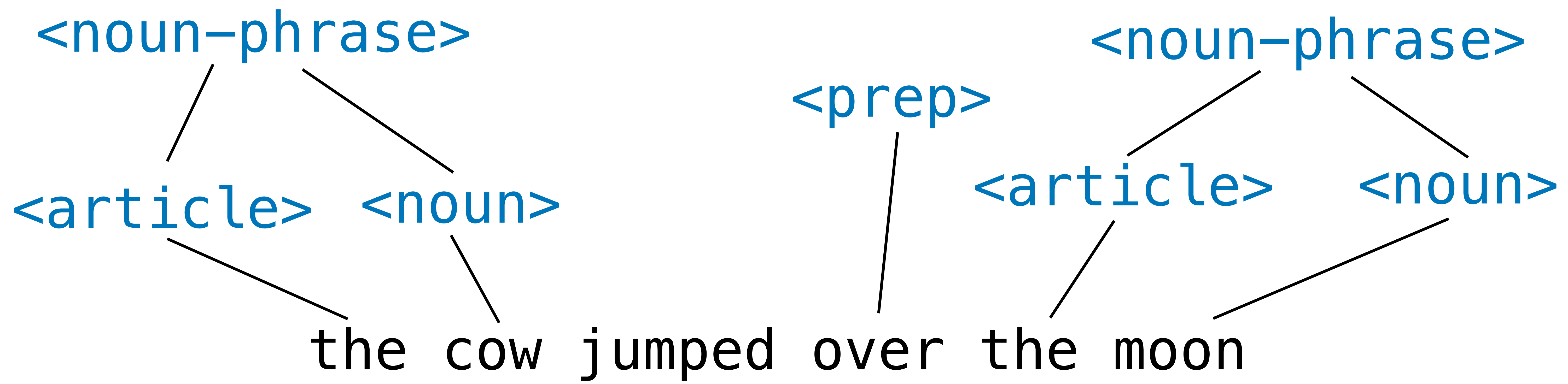

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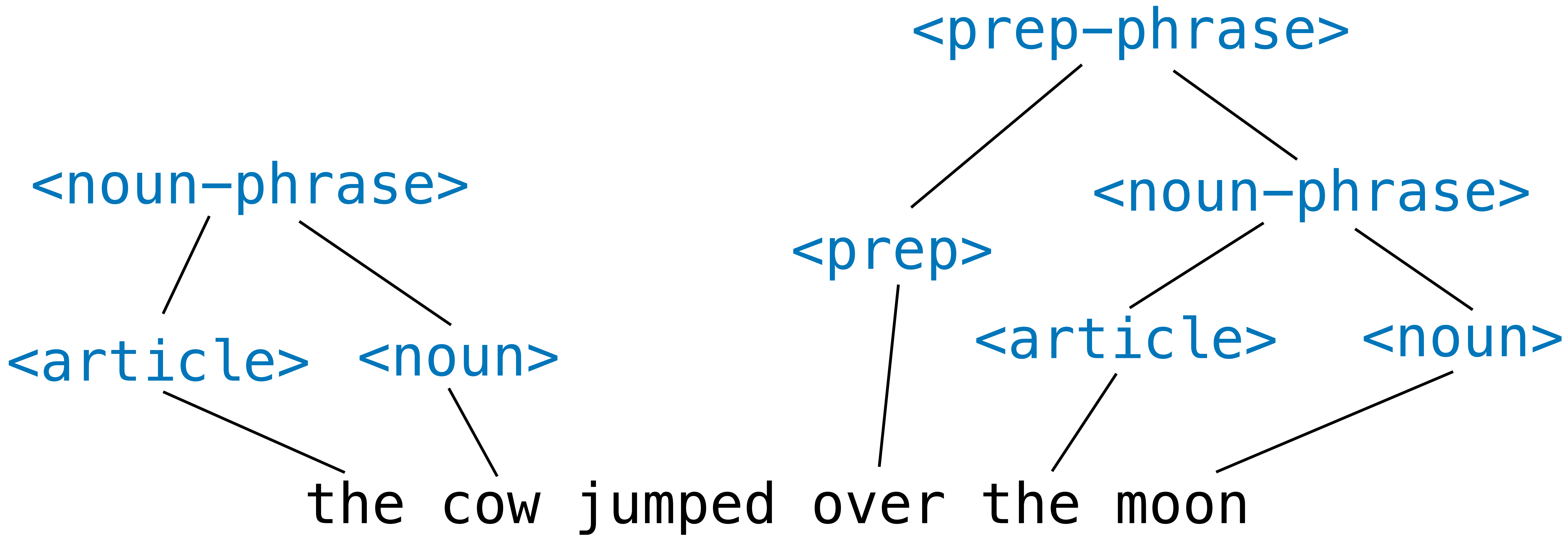

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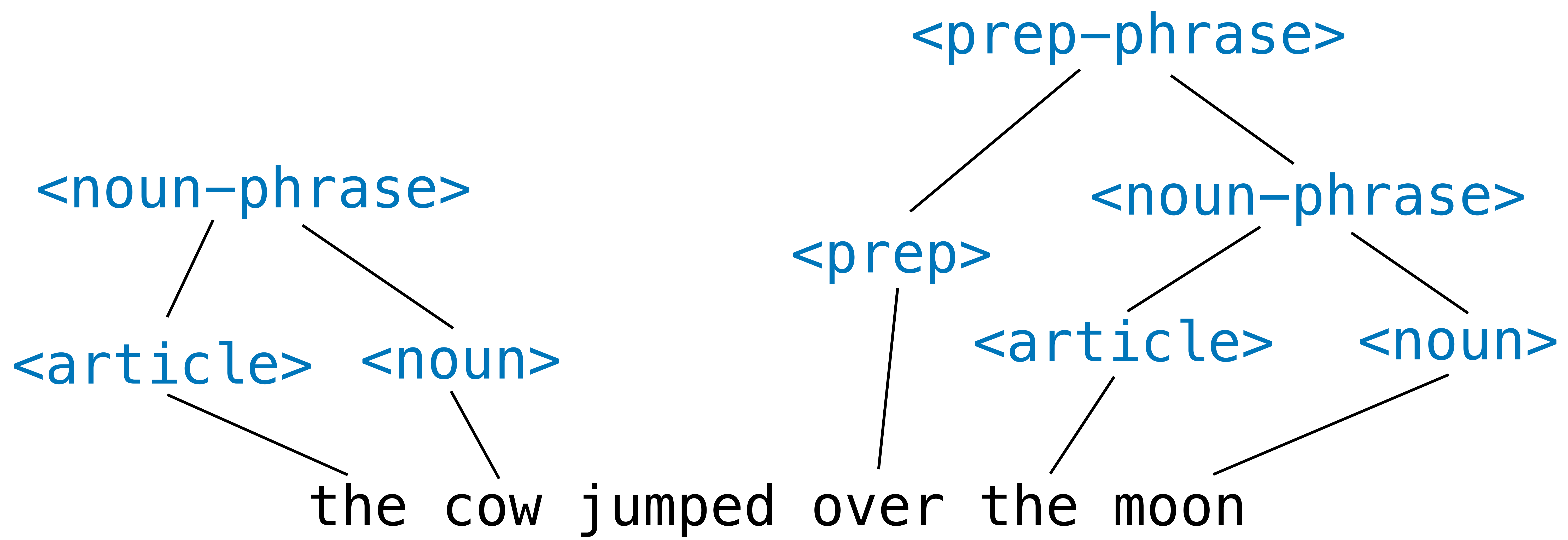

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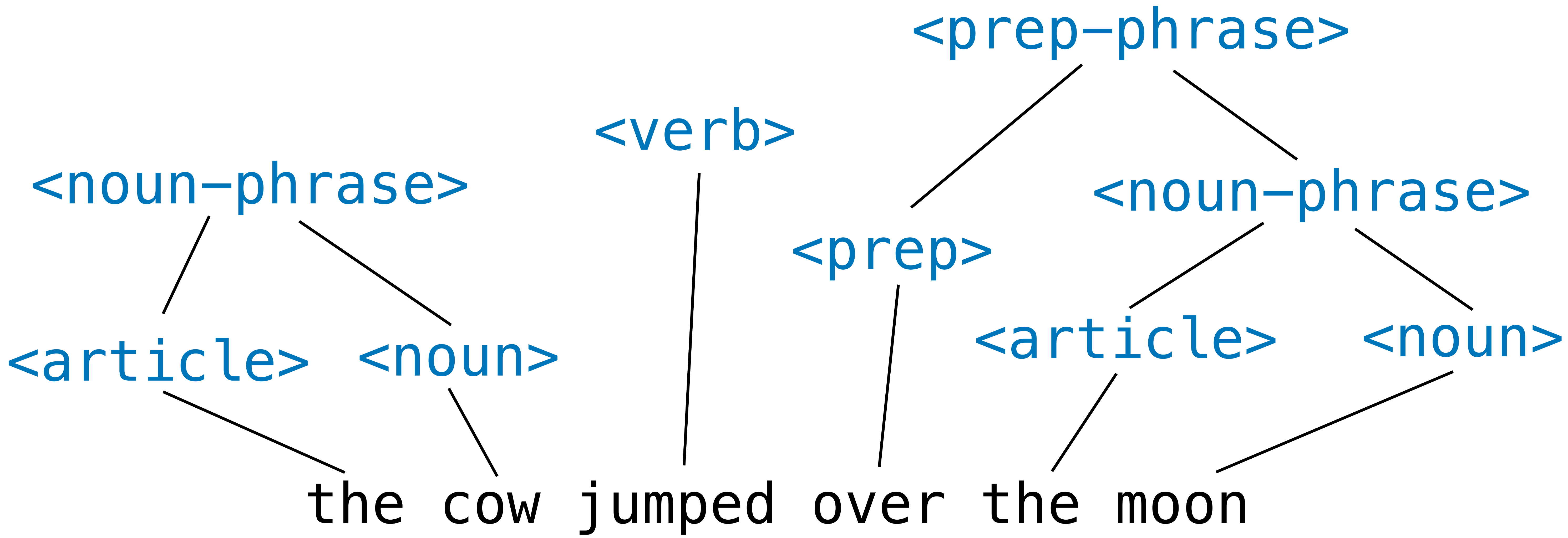

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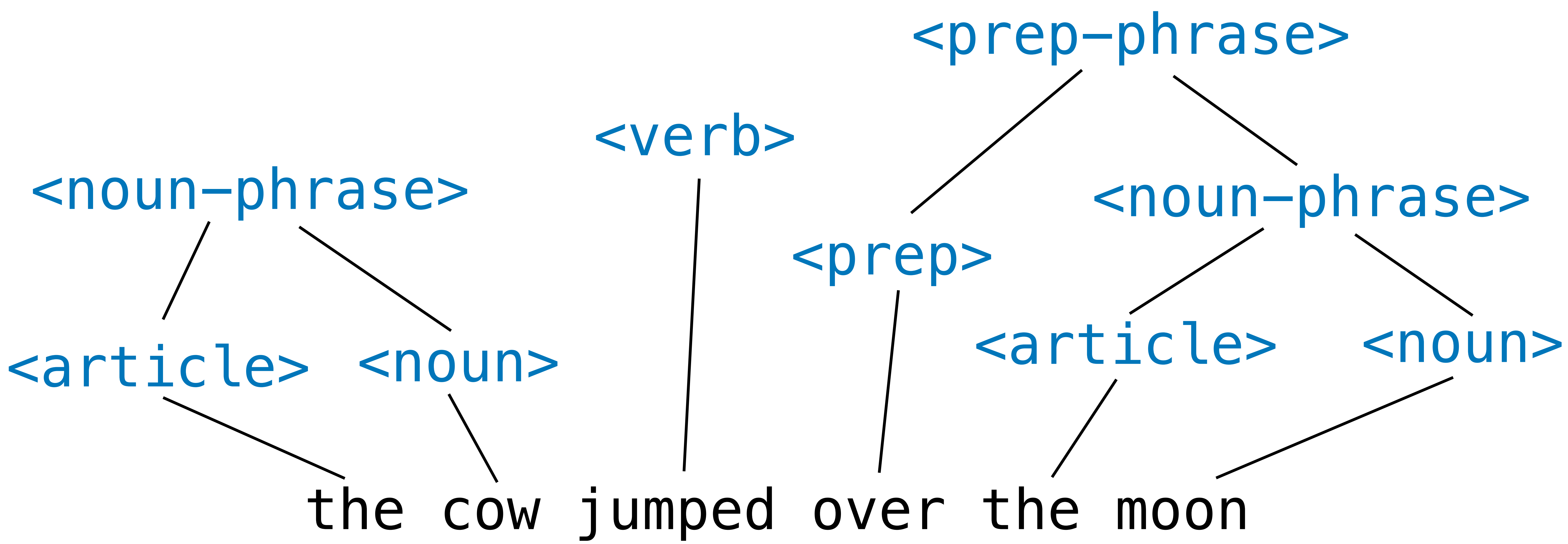

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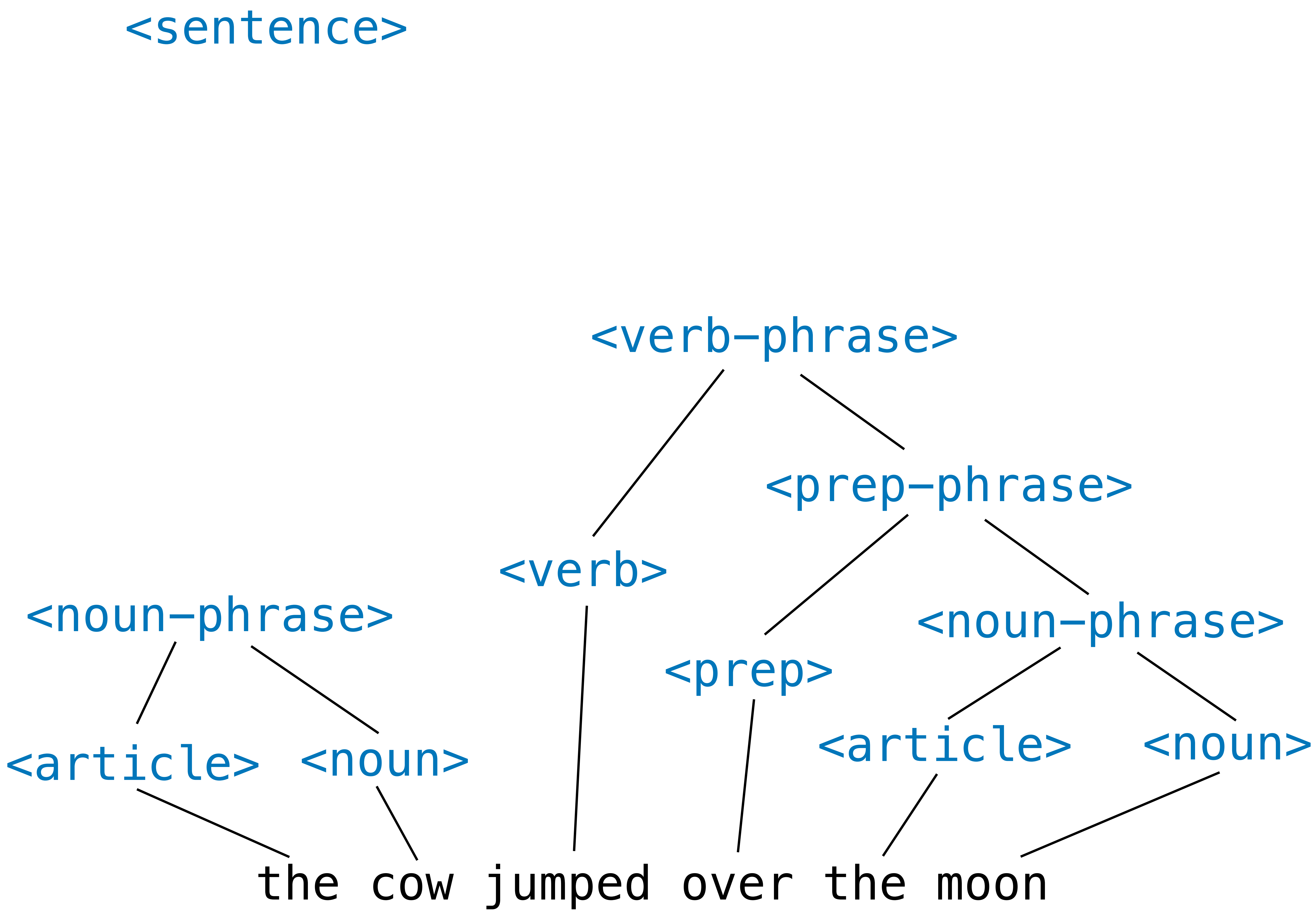

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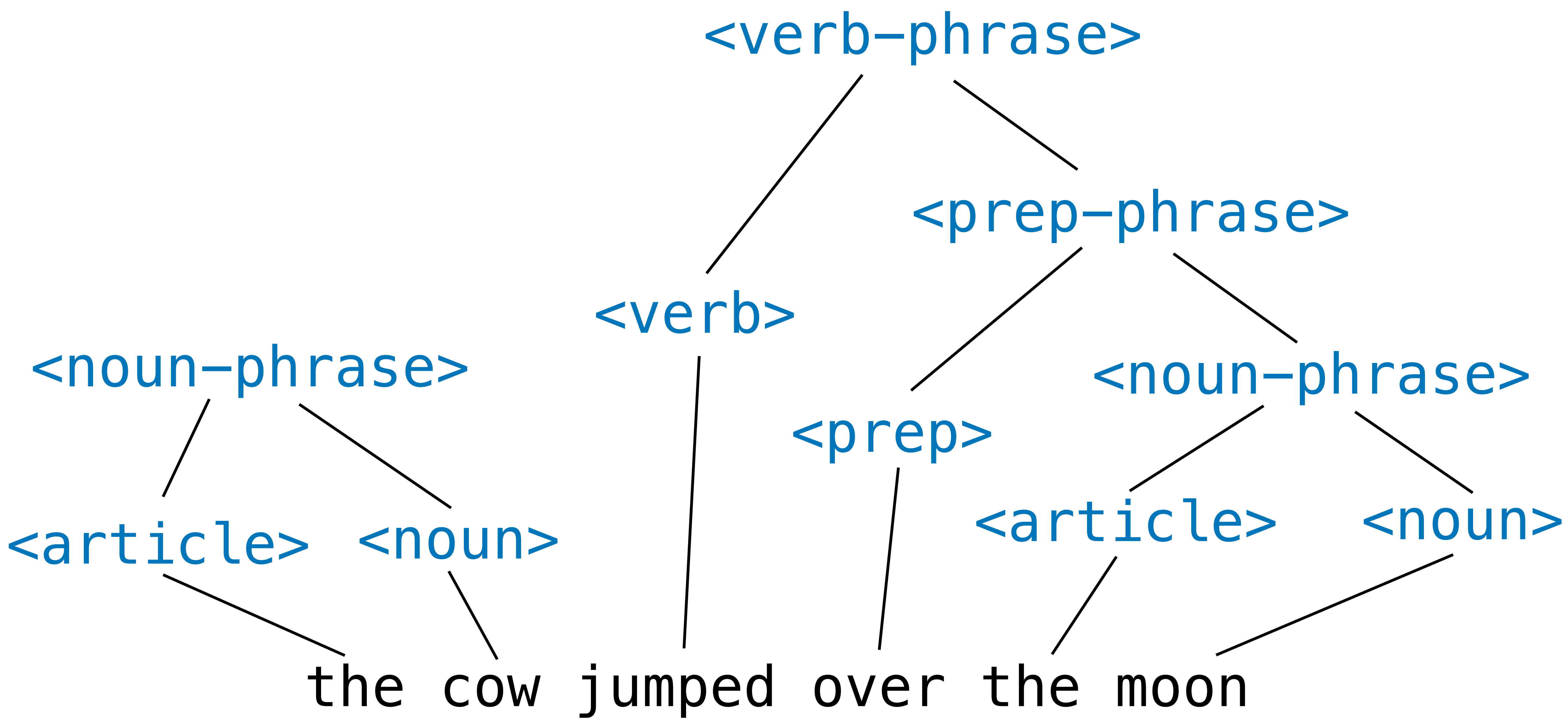

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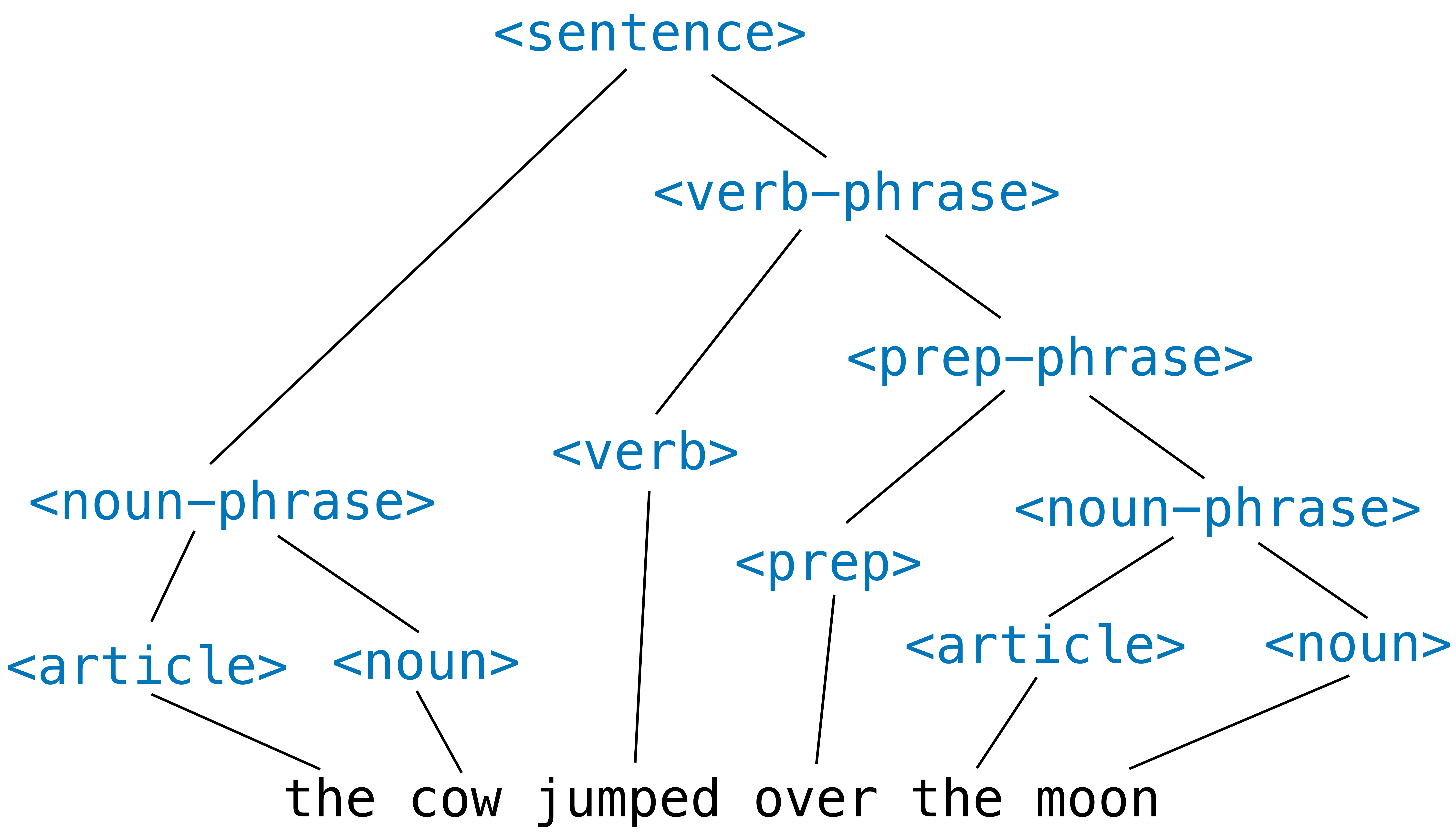
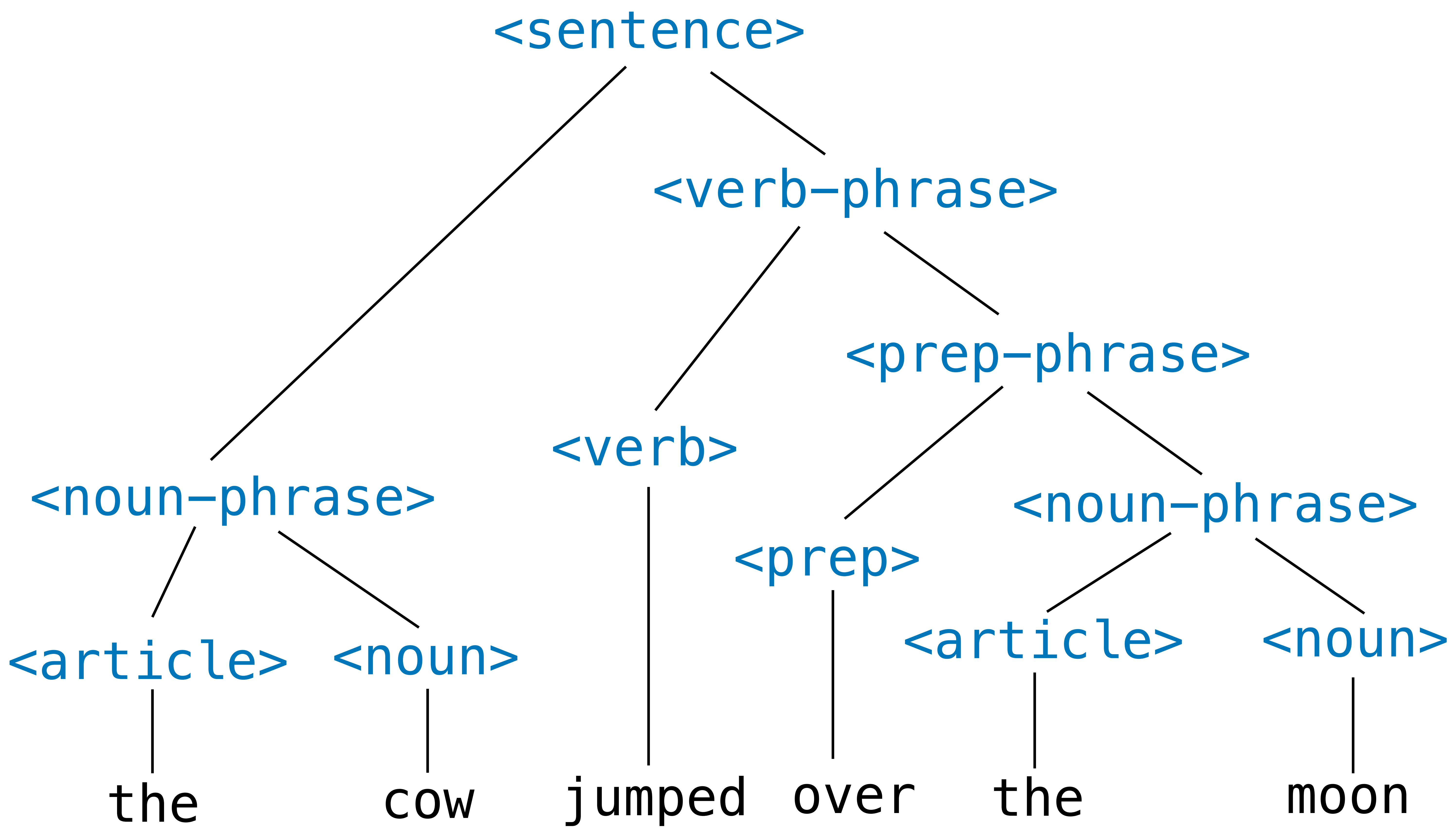


A Parse Tree



A derivation encodes hierarchical structure

Definitions (Symbols and Sentences)

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Terminal symbols are symbols cannot be further expanded, like *moon* in the previous example

A **sentential form** is a sequence of terminal or nonterminal symbols.

A **sentence** is a sequence of *only terminal* symbols

Example

```
<noun-phrase> jumped over <noun-phrase>
```

Production Rules

```
<non-term> ::= sent-form1 | sent-form2 | ...
```

Production Rules

```
<non-term> ::= sent-form1 | sent-form2 | ...
```

A **(BNF) production rule** describes what we can replace a non-terminal symbol with in a derivation.

Production Rules

```
<non-term> ::= sent-form1 | sent-form2 | ...
```

A **(BNF) production rule** describes what we can replace a non-terminal symbol with in a derivation.

The "|" means: we can replace it with one or the other sentential forms on either side of the "|".

Example

```
<sentence>    ::= <noun-phrase> <verb-phrase>

<verb-phrase> ::= <verb> <prep-phrase>

<noun>        ::= cow | moon
```

BNF Grammar

```
<sentence>    ::= <noun-phrase> <verb-phrase>
<verb-phrase> ::= <verb> <prep-phrase>
                | <verb>
<prep-phrase> ::= <prep> <noun-phrase>
<noun-phrase> ::= <article> <noun>
<article>     ::= the
<noun>        ::= cow
                | moon
<verb>        ::= jumped
<prep>        ::= over
```

BNF Grammar

A **BNF grammar** is defined by a collection of production rules and a **starting (nonterminal) symbol**

```
<sentence>    ::= <noun-phrase> <verb-phrase>
<verb-phrase> ::= <verb> <prep-phrase>
                | <verb>
<prep-phrase> ::= <prep> <noun-phrase>
<noun-phrase> ::= <article> <noun>
<article>     ::= the
<noun>        ::= cow
                | moon
<verb>        ::= jumped
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```

BNF Grammar

A **BNF grammar** is defined by a collection of production rules and a **starting (nonterminal) symbol**

Note. We don't specify the symbols of a grammar, they are implicit in the rules

```
<sentence>    ::= <noun-phrase> <verb-phrase>
<verb-phrase> ::= <verb> <prep-phrase>
                | <verb>
<prep-phrase> ::= <prep> <noun-phrase>
<noun-phrase> ::= <article> <noun>
<article>     ::= the
<noun>        ::= cow
                | moon
<verb>        ::= jumped
<prep>        ::= over
```

BNF Grammar

A **BNF grammar** is defined by a collection of production rules and a **starting (nonterminal) symbol**

Note. We don't specify the symbols of a grammar, they are implicit in the rules

Note. We don't specify the start symbol, it's the left nonterminal symbol in the **first rule**

```
<sentence>    ::= <noun-phrase> <verb-phrase>
<verb-phrase> ::= <verb> <prep-phrase>
                | <verb>
<prep-phrase> ::= <prep> <noun-phrase>
<noun-phrase> ::= <article> <noun>
<article>     ::= the
<noun>        ::= cow
                | moon
<verb>        ::= jumped
<prep>        ::= over
```

Example

```
<expr>  ::= <op1> <expr>
          | <op2> <expr> <expr>
          | <var>
<op1>   ::= not
<op2>   ::= and | or
<var>   ::= x | y | z
```

Example

```
<expr>  ::= <op1> <expr>
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          | <var>
<op1>   ::= not
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```

tokens (terminal symbols)

Example

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<expr>    ::= <op1> <expr>
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<op1>     ::= not
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```

abstractions (non-terminal symbols)

tokens (terminal symbols)

Example

production rules

```
<expr>   ::= <op1> <expr>
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           | <var>
<op1>    ::= not
<op2>    ::= and | or
<var>    ::= x | y | z
```

abstractions (non-terminal symbols)

tokens (terminal symbols)

Practice Problem

```
<sentence>     ::= <noun-phrase> <verb-phrase>
<verb-phrase>  ::= <verb> <prep-phrase> | <verb>
<prep-phrase>  ::= <prep> <noun-phrase>
<noun-phrase>  ::= <article> <noun>
<article>      ::= the
<noun>         ::= cow | moon
<verb>         ::= jumped
<prep>         ::= over
```

What are the nonterminal and terminal symbols of this grammar?

Answer

```
<sentence>    ::= <noun-phrase> <verb-phrase>
<verb-phrase> ::= <verb> <prep-phrase> | <verb>
<prep-phrase> ::= <prep> <noun-phrase>
<noun-phrase> ::= <article> <noun>
<article>     ::= the
<noun>        ::= cow | moon
<verb>        ::= jumped
<prep>        ::= over
```

Derivations and Parse Trees

Derivations and Parse Trees

Definition. A **derivation** is a sequence of sentential forms (beginning at the start symbol) in which each form is the result of **replacing a non-terminal symbol in the previous form** according to a production rule

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```
<expr>
<op2> <expr> <expr>
and <expr> <expr>
and <op1> <expr> <expr>
and not <expr> <expr>
and not <var> <expr>
and not x <expr>
and not x <var>
and not x y
```

Derivations and Parse Trees

```
<expr>  ::= <op1> <expr>
          | <op2> <expr> <expr>
          | <var>
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```

<expr>

<expr>

Derivations and Parse Trees

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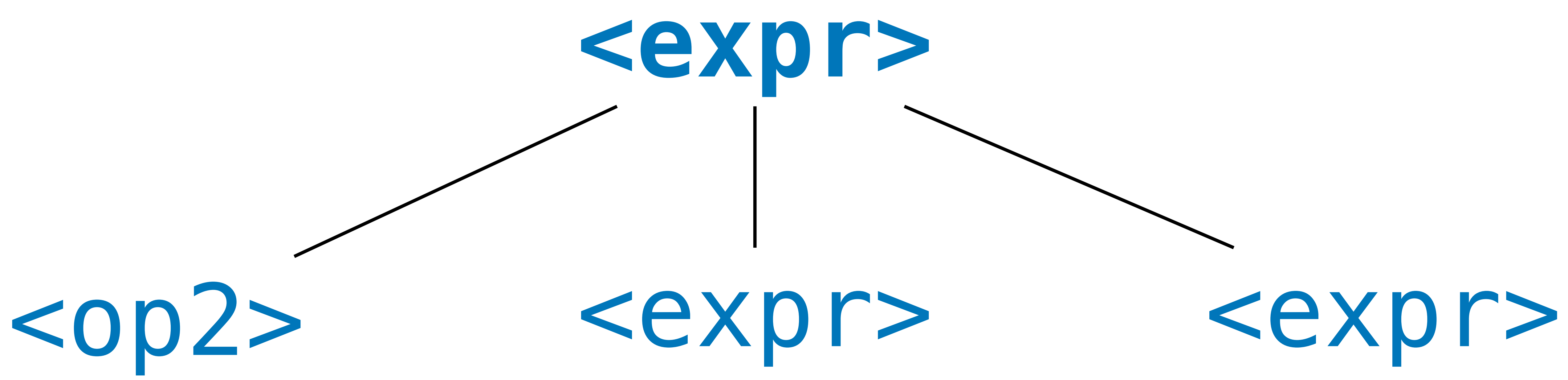
```

```
<expr>
<op2> <expr> <expr>
```



Derivations and Parse Trees

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```

```
<expr>
<op2> <expr> <expr>
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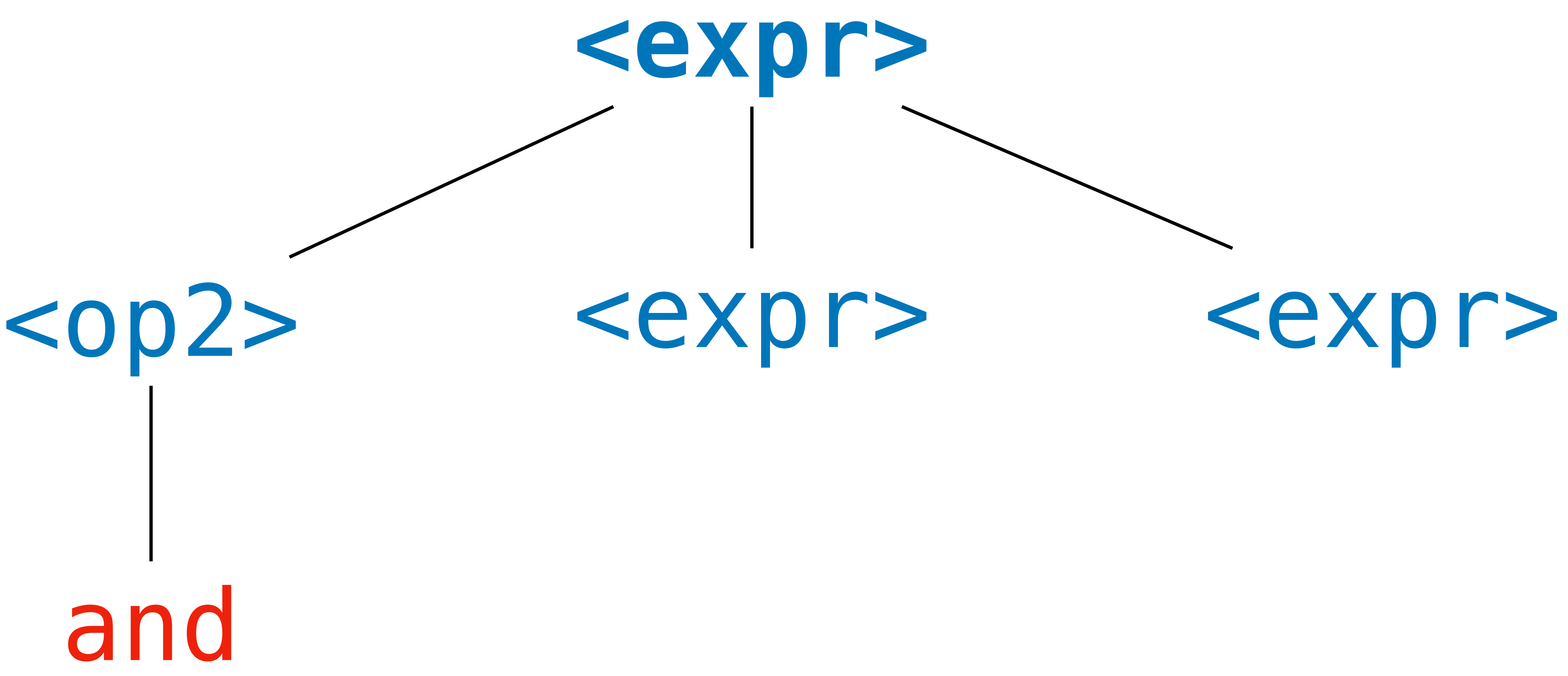
```



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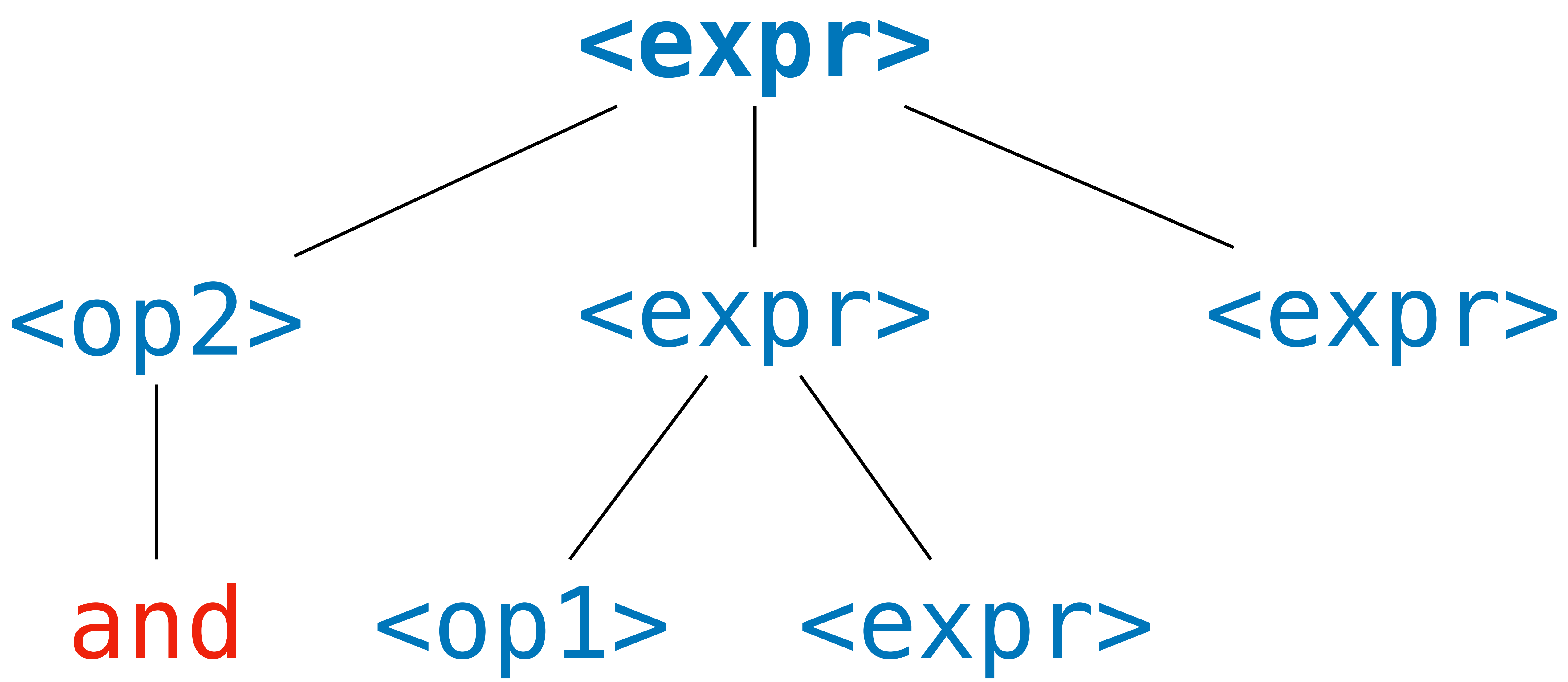

Derivations and Parse Trees

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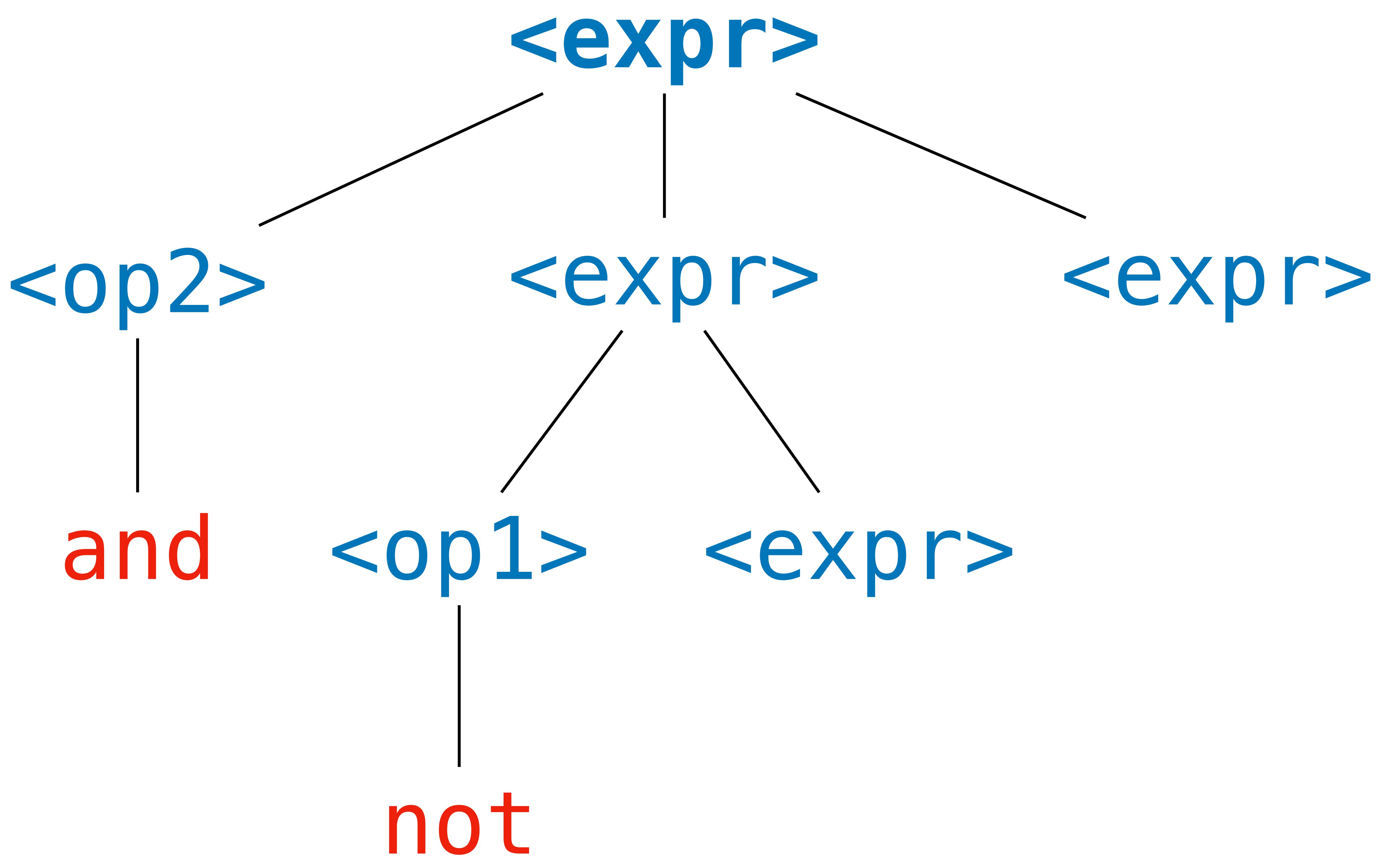
```

```
<expr>
<op2> <expr> <expr>
and <expr> <expr>
and <op1> <expr> <expr>
and not <expr> <expr>
```



Derivations and Parse Trees

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<expr>  ::= <op1> <expr>
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```

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and <op1> <expr> <expr>
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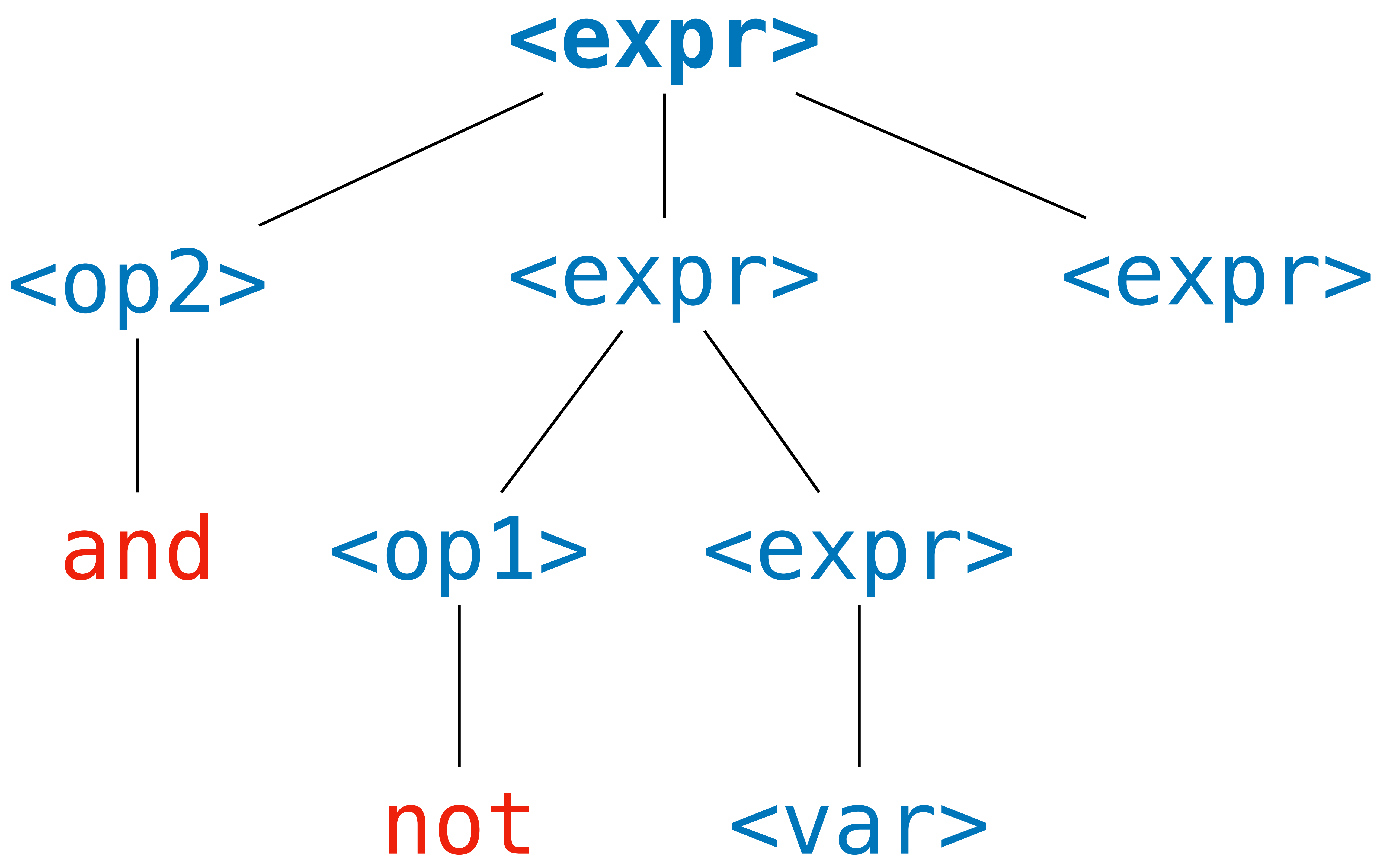
```



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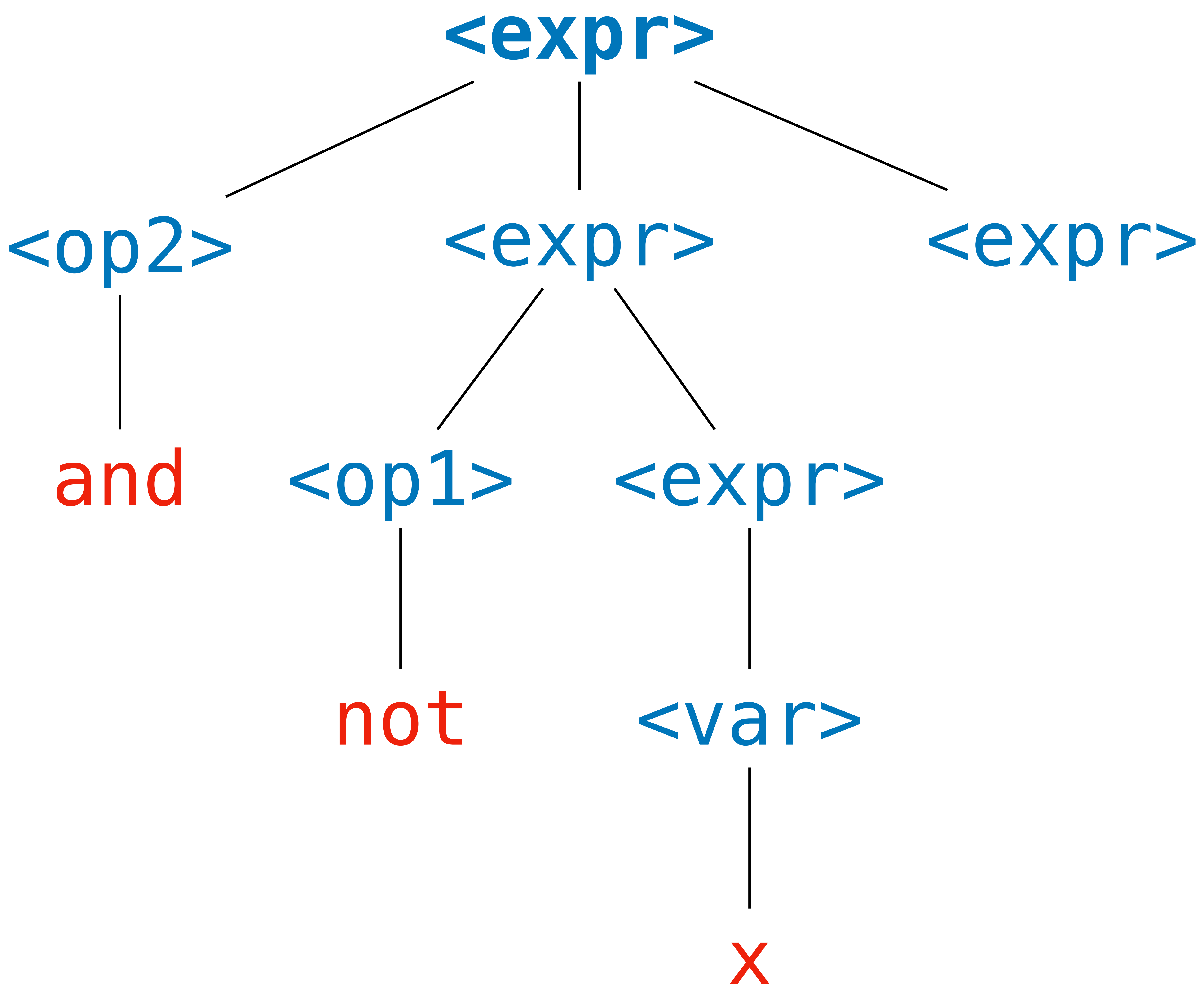
```



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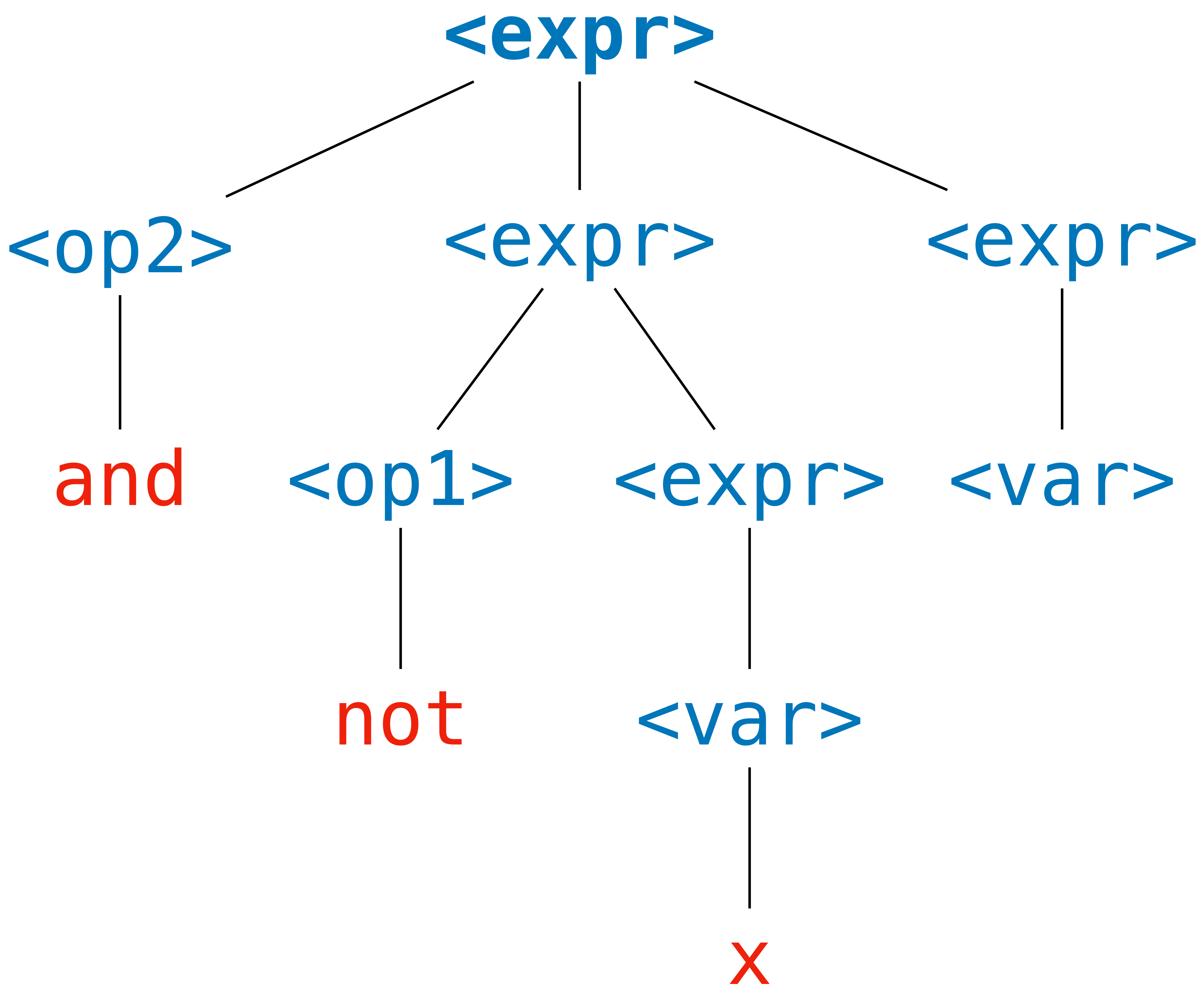
```



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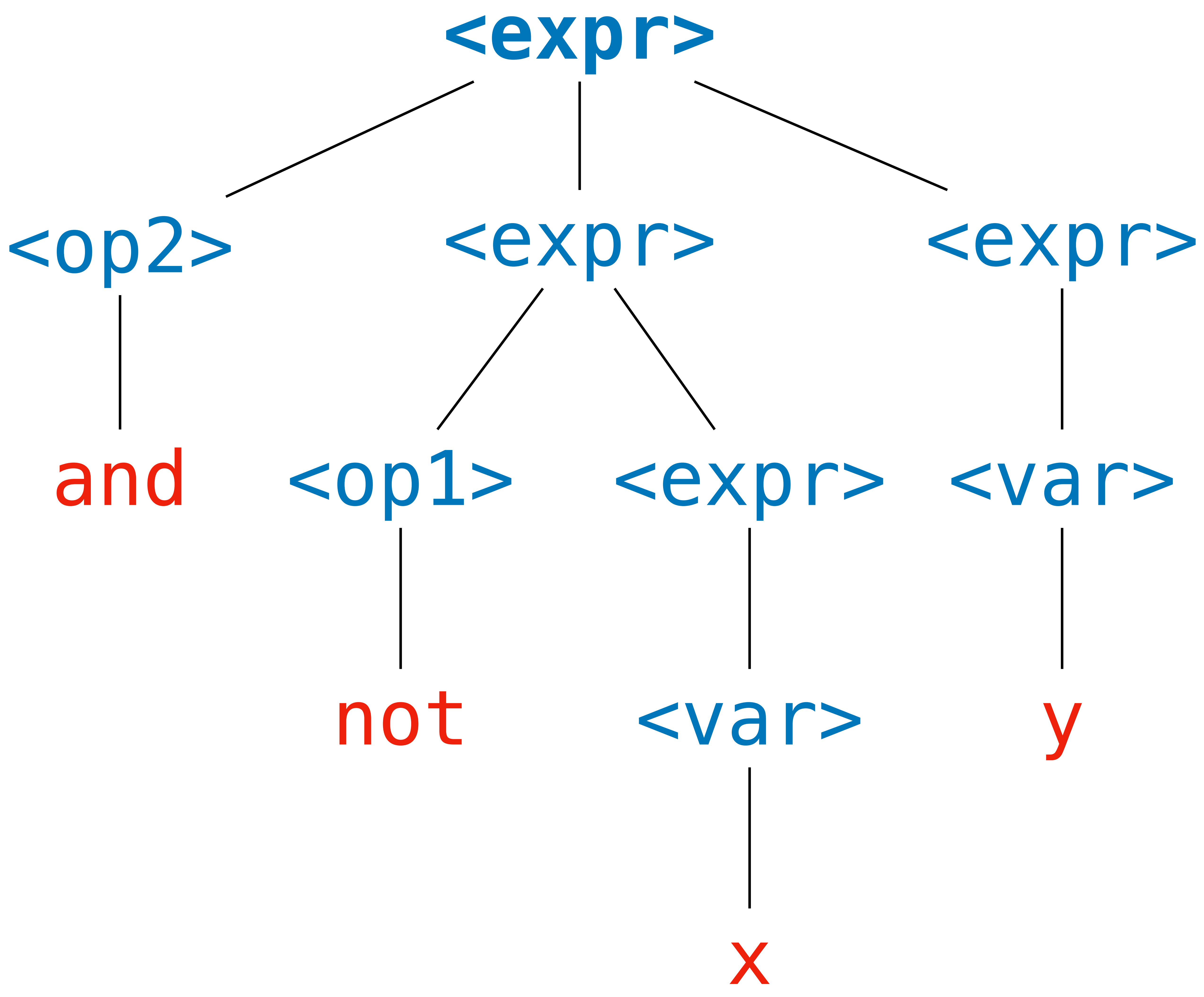

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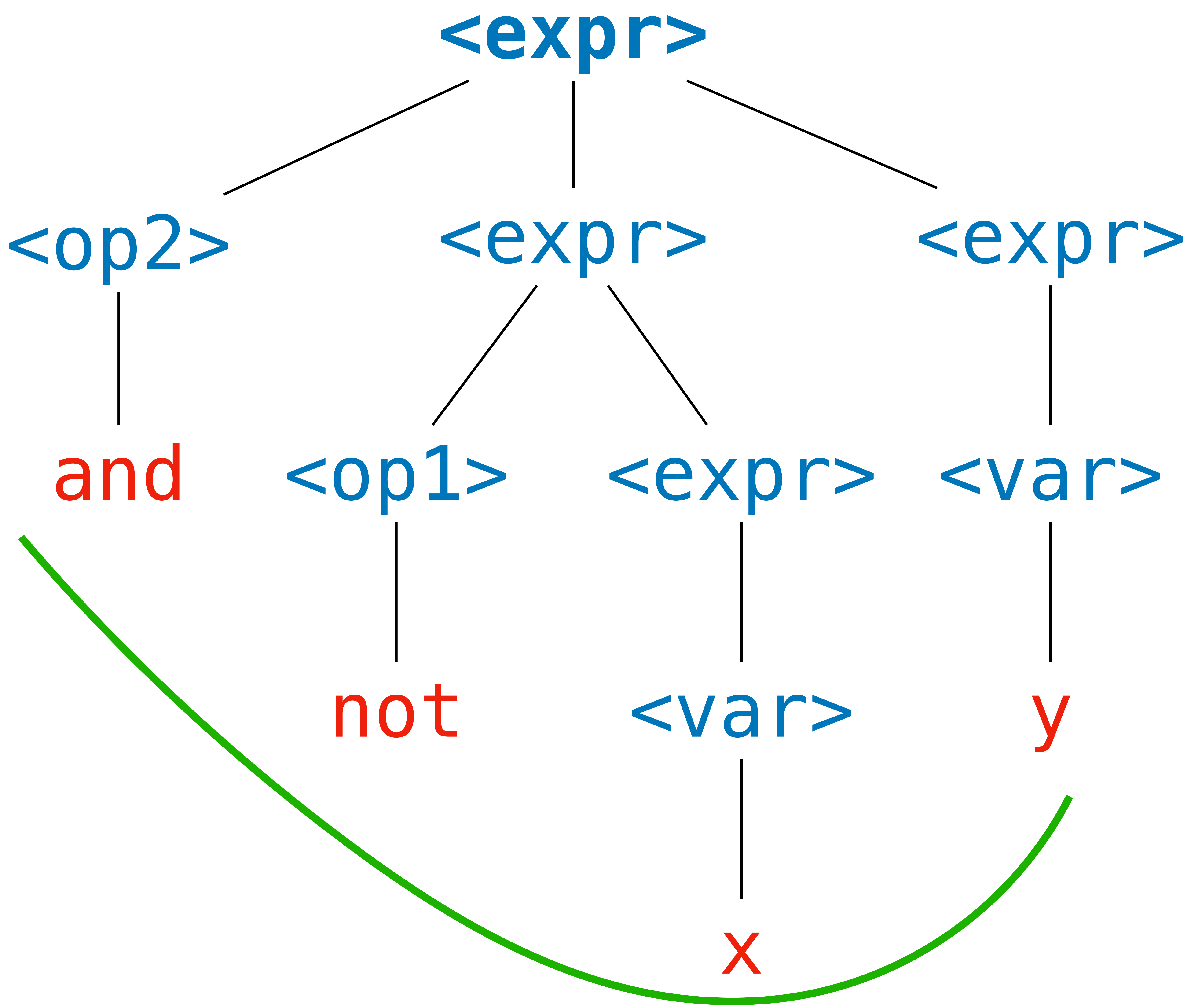

Why do we care?

`char stream` $\xrightarrow{\texttt{lexer}}$ `token stream` $\xrightarrow{\texttt{parser}}$ `parse tree` $\longrightarrow$ . . .

Why do we care?

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`char stream` —lexer→ `token stream` —parser→ `parse tree` → . . .

`token stream`: a.k.a. list of terminal symbols

`parse tree`: a.k.a. derivation in a grammar

We will parse **token streams** into **parse trees**.

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We will parse **token streams** into **parse trees**.

It is much easier to evaluate something hierarchical than something which is linear.

Practice Problem

```
<expr>  ::= <op1> <expr>
          | <expr> <op2> <expr>
          | <var>
<op1>   ::= not
<op2>   ::= and | or
<var>   ::= x | y | z
```

Give a derivation of ***not x and y or z*** *in the above grammar, both as a sequence of sentential forms and as a parse tree.*

(In Python, if ***x*** *and* ***y*** *and* ***z*** *are* ***True****, what does this expression evaluate to?)*

Answer

```
<expr>  ::= <op1> <expr>
          | <expr> <op2> <expr>
          | <var>
<op1>   ::= not
<op2>   ::= and | or
<var>   ::= x | y | z
```

not x and y or z

The Big Picture

When we specify a PL (e.g., in the projects) you will be given a *BNF grammar*

You will need to know how to translate this into a parser

So you will need *practice reading BNF specifications*

Ambiguity

Ambiguity in Natural Language

The duck is ready for dinner.

John saw the man on the mountain with a telescope.

He said on Tuesday there would be an exam.

(examples taken from around the internet)

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Natural language has ambiguities that can confuse the meaning of a sentence.

We have informal tactics for avoiding these pitfalls.

(examples taken from around the internet)

Ambiguity in Natural Language

*The duck is ready **to eat** dinner.*

*John saw the man on the mountain **using** a telescope.*

*He said the exam would **be held** on Tuesday.*

Natural language has ambiguities that can confuse the meaning of a sentence.

We have informal tactics for avoiding these pitfalls.

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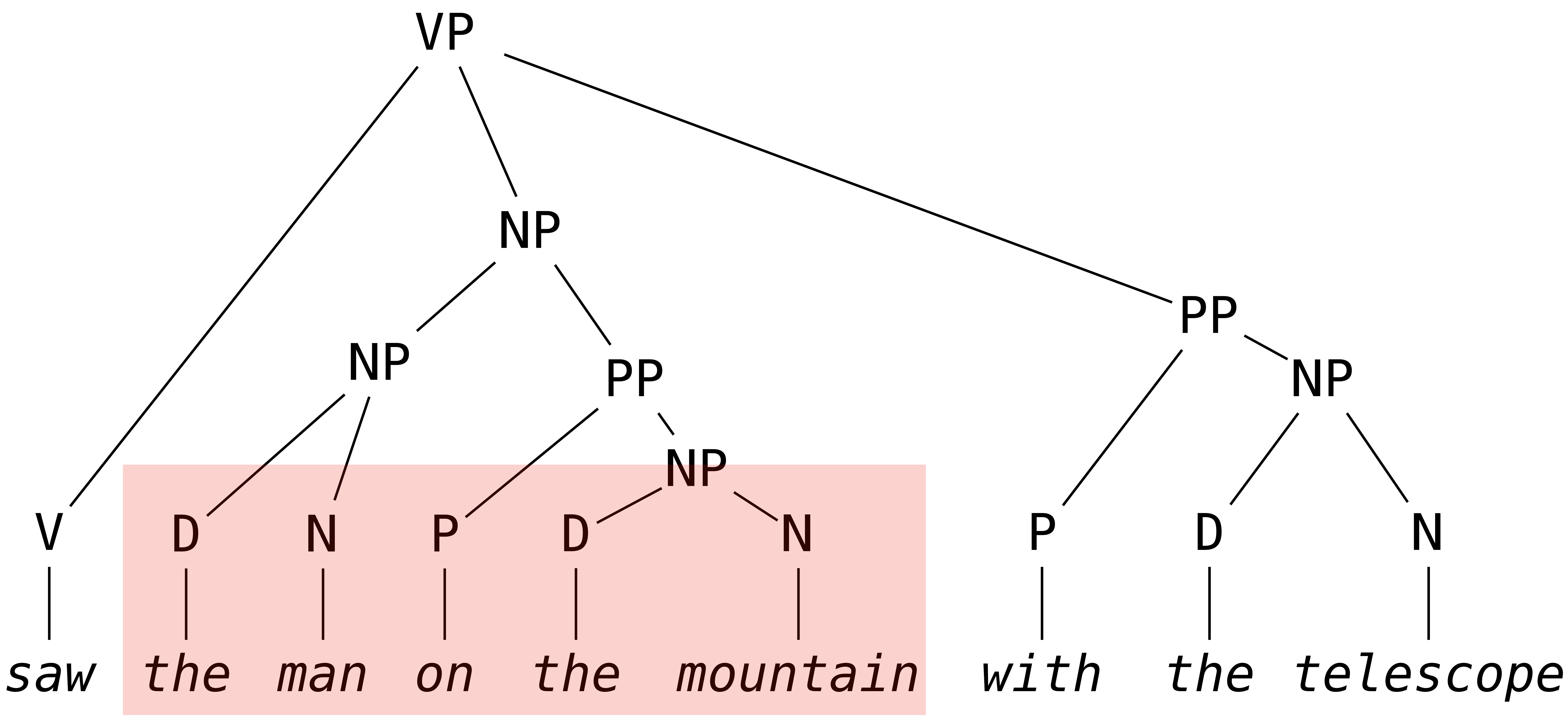
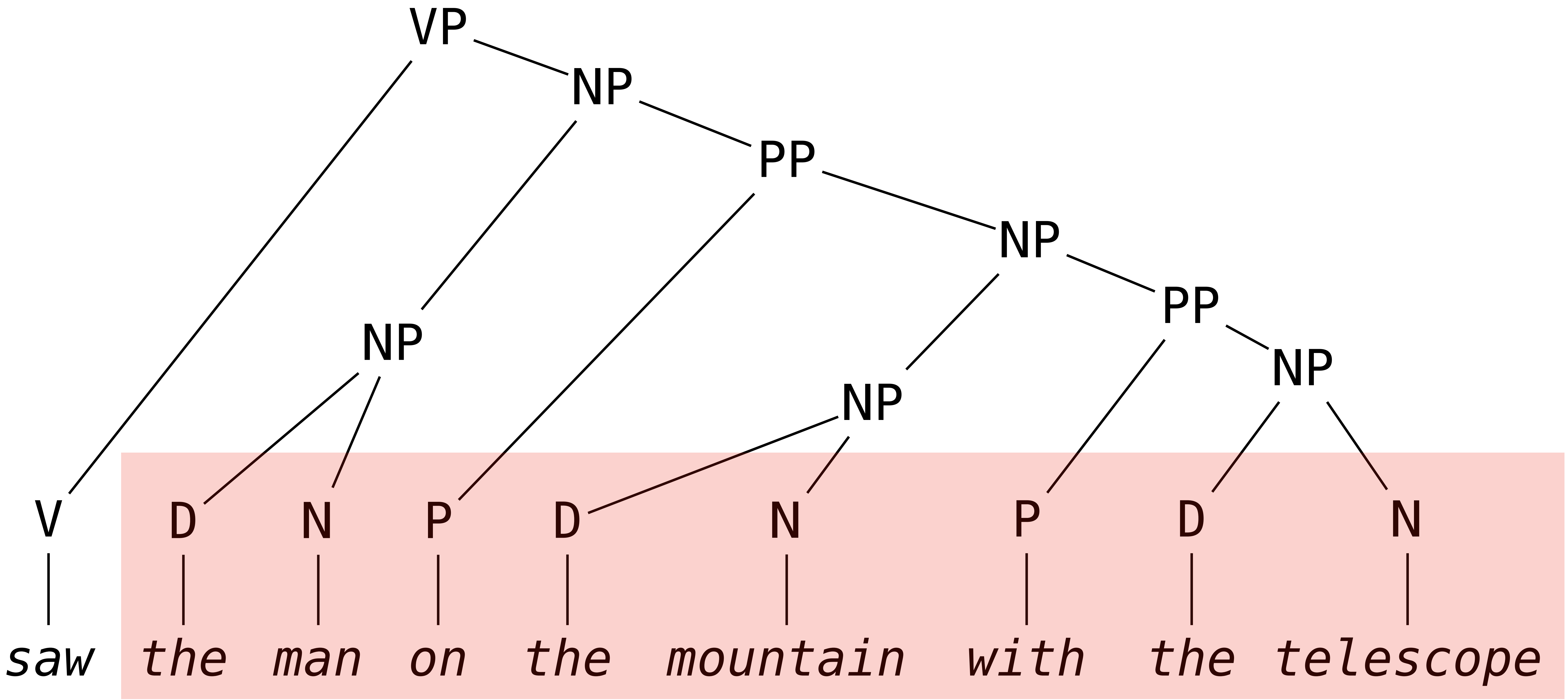
Aside: Ambiguity and Linearity



Ambiguity is caused by writing down hierarchical structures in a linear fashion.

Aside: Ambiguity and Linearity



There is no ambiguity in the grammatical parse tree of this statement.

The hierarchical structure changes the meaning of the sentence

Ambiguity in Formal Grammar

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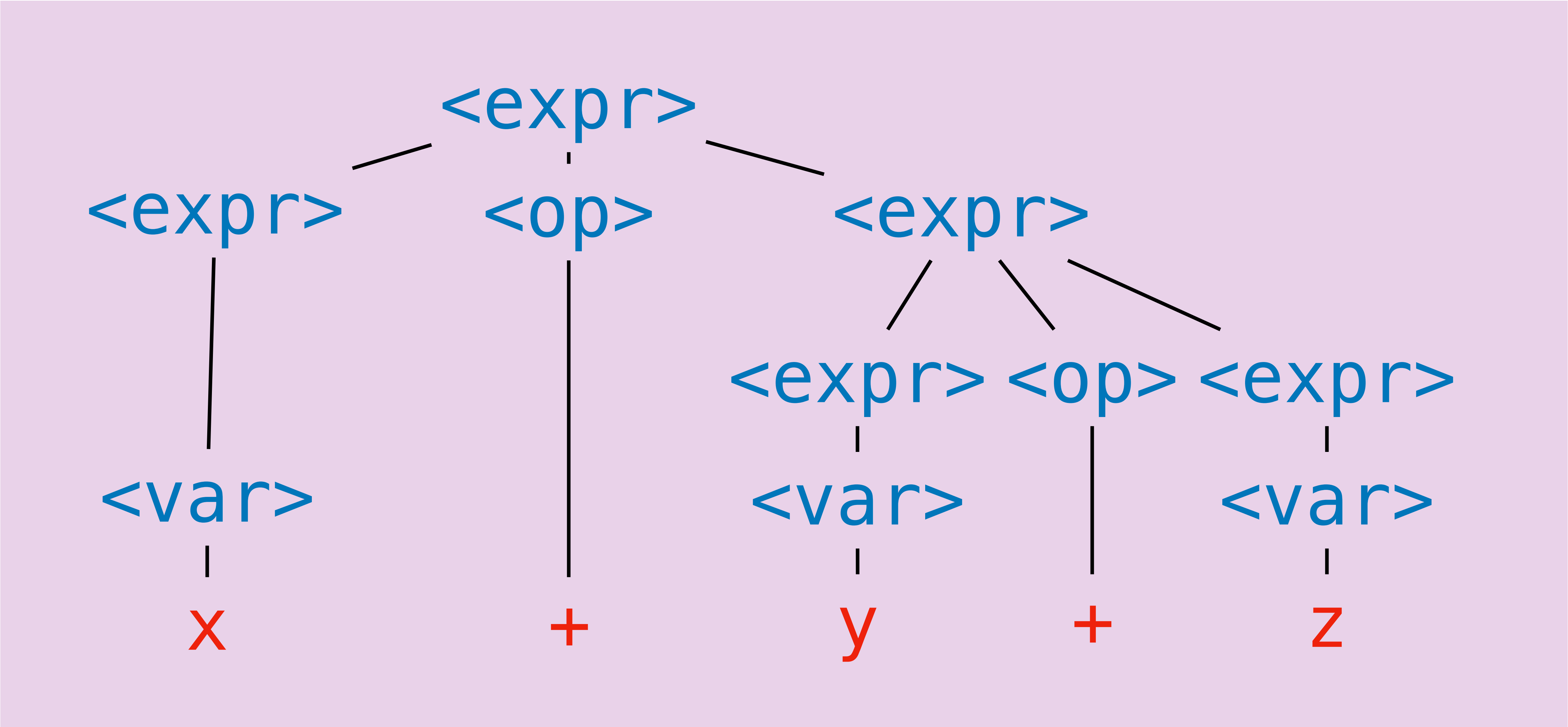

Ambiguity in Formal Grammar

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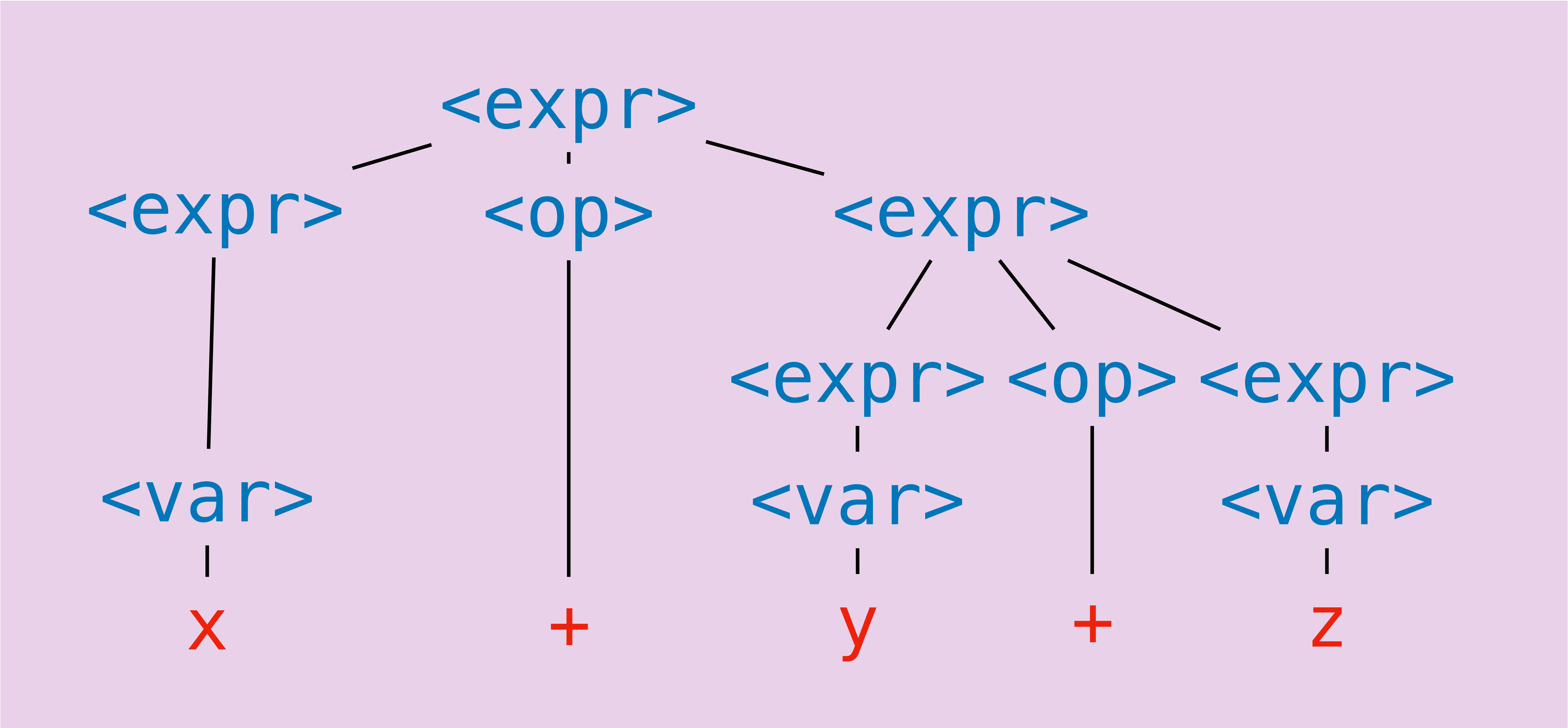

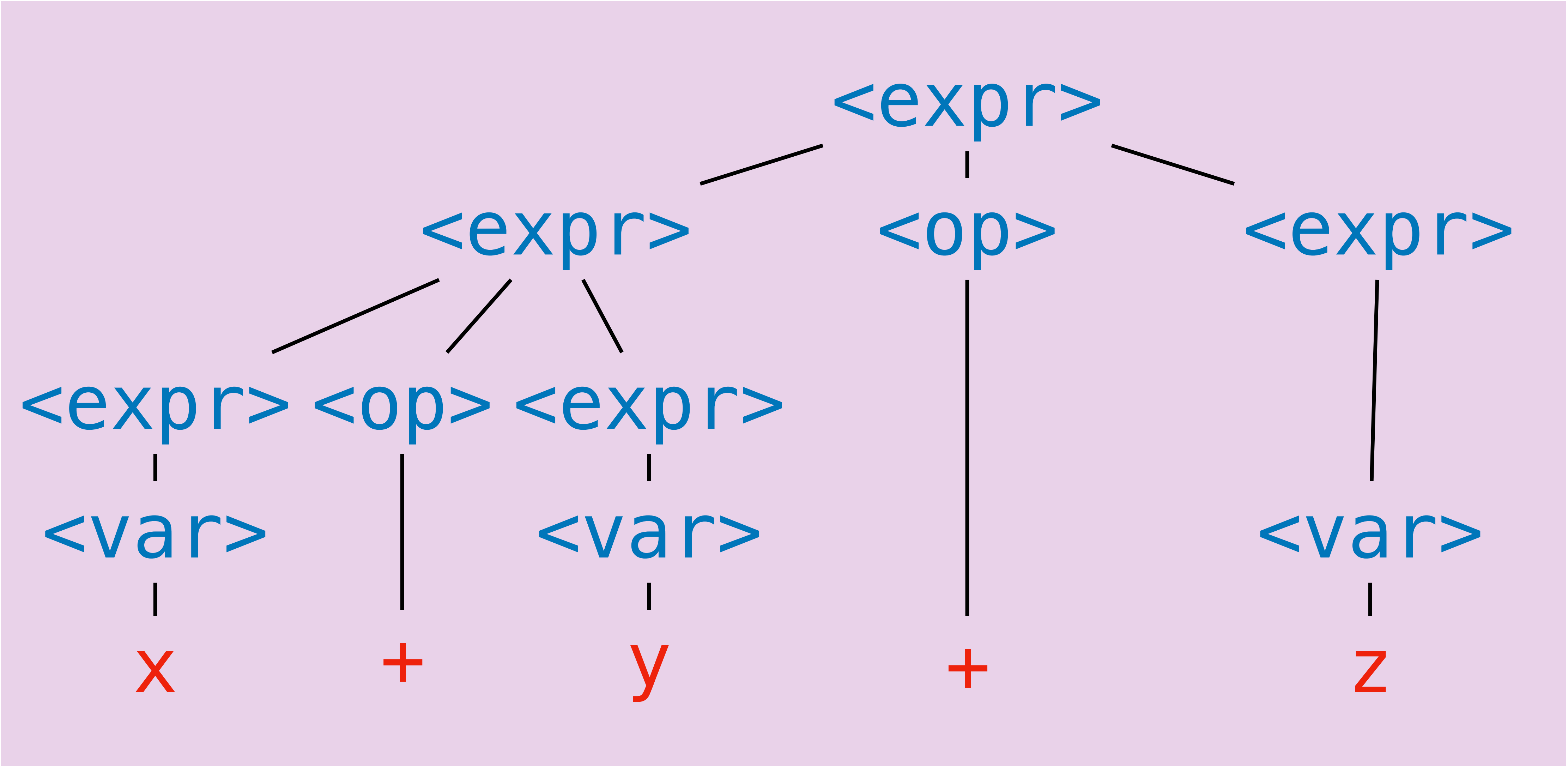

Again, why do we care?

```
false && destory_everything () || false
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Note that **1 + 1 + 1** is not ambiguous with respect to its *meaning* (it's value is **3** according to the standard definition of +)

Again, why do we care?

```
false && destory_everything () || false
```

Note that `1 + 1 + 1` is not ambiguous with respect to its *meaning* (it's value is `3` according to the standard definition of `+`)

But we make a promise to the user of a language that we won't make any unspoken assumptions about what they meant when they wrote down their program.

Practice Problem

```
<expr> ::= <bool>
         | <var>
         | if <expr> then <expr>
         | if <expr> then <expr> else <expr>
<bool> ::= tru | fls
<var>  ::= x | y | z
```

Show that the above grammar is ambiguous.

Answer

```
<expr> ::= <bool>
         | <var>
         | if <expr> then <expr>
         | if <expr> then <expr> else <expr>
<bool>::= tru | fls
<var> ::= x | y | z
```

Fundamental Concern

What can we do about ambiguity?

Aside: Ambiguity and Computability

```
let is_ambiguous(g : grammar) : bool = ???
```

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It is impossible to write a program which determines if a grammar is ambiguous.

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`It is impossible to write a program which determines if a grammar is ambiguous.`

`Not just hard, but` **`literally impossible`**`.`

Aside: Ambiguity and Computability

```
let is_ambiguous(g : grammar) : bool = ???
```

It is impossible to write a program which determines if a grammar is ambiguous.

Not just hard, but **literally impossible**.

That's not to say we can't determine that *particular* grammars are ambiguous.

Fixity

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The fixity of an operator refers to where an operator is placed relative to its arguments:

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```
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The <u>fixity</u> of an operator refers to where an operator is placed relative to its arguments:

prefix	`f x , (- x)`
postfix	`a! (get from ref)`
infix	`a * b, a + b, a mod b`
mixfix	`if b then x else y`

Polish Notation

$$- \; / \; + \; 2 \; * \; 1 \; - \; 2 \; 3$$

is equivalent to

$$- \; (2 \; + \; (1 \; * \; (- \; 2) \; / \; 3)$$

To avoid ambiguity, we can make all operators prefix (or postfix) operators. *We don't even need parentheses.*

(This how early calculators worked.)

Example

```
<expr> ::= <bool>
         | <var>
         | ifthen <expr> <expr>
         | ifthenelse <expr> <expr> <expr>
<bool>::= tru | fls
<var> ::= x | y | z
```

No more ambiguity. But programs written like this are notoriously difficult to read...

Lots of Parentheses

```
<expr>  ::= (<op1> <expr>)
          | (<expr> <op2> <expr>)
          | <var>
<op1>   ::= not
<op2>   ::= and | or
<var>   ::= x | y | z
```

Lots of Parentheses

```
<expr>   ::= (<op1> <expr>)
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If we want infix operators, we *could* add parentheses around all operators.

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```

```
(((not x) and (not (not y))) or z)

((((x or y) or z) or x) or y)

(not ((not x) and (not y)) or
     (x and z))

(x and y)
```

If we want infix operators, we *could* add parentheses around all operators.

But we run into a similar issue: *Too many parentheses are difficult to read.*

Can we get away without (or with fewer) parentheses?

Aside: The Cult of Parentheses

```
(defun fib (n)
  "Return the nth Fibonacci number."
  (if (< n 2)
      n
      (+ (fib (- n 1))
         (fib (- n 2)))))
```

Two Ingredients (or Flavors of Ambiguity)

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Associativity:

How should arguments be grouped in an expression like **1 + 2 + 3 + 4**?

Two Ingredients (or Flavors of Ambiguity)

<u>Associativity:</u>

How should arguments be grouped in an expression like `1 + 2 + 3 + 4`?

<u>Precedence:</u>

How should arguments be grouped in an expression like `1 + 2 * 3 + 4`?

Associativity

The associativity of an infix operator refers to how its arguments are grouped in the absence of parentheses:

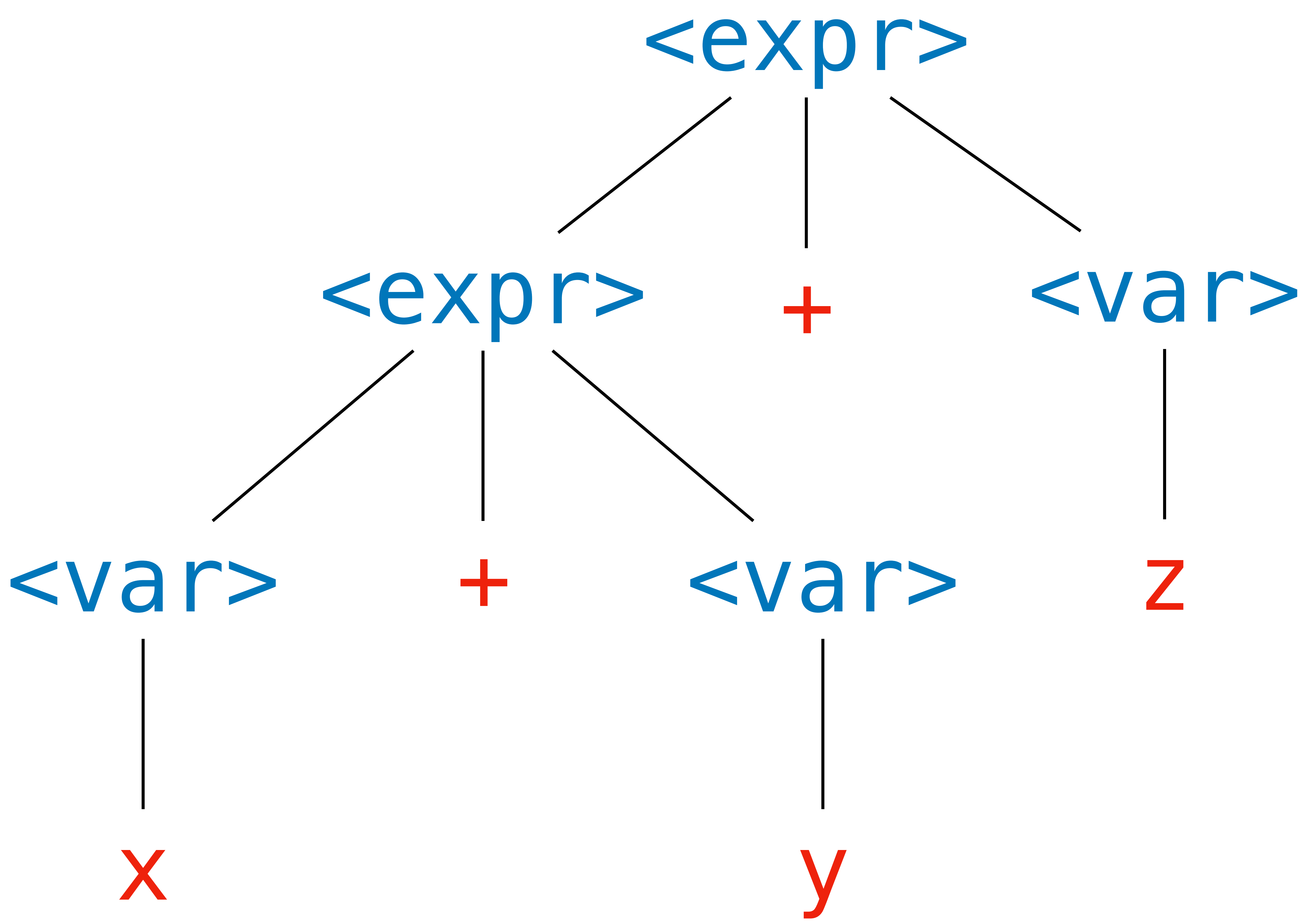
```
left associative     1 + 2 + 3 ⇒ (1 + 2) + 3

right associative    a -> b -> c ⇒  a -> (b -> c)
```

Associativity

```
<expr> ::= <expr> + <expr>
         | <var>
<var>  ::= x | y | z
```

x + y + z ⇛



"add the sum of x and y to z"

The Culprit

```
<expr> ::= <expr> + <expr>
         | <var>
<var>  ::= x | y | z
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Any time we have a rule like this, we should be suspicious...

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```

Any time we have a rule like this, we should be suspicious...

```
<expr> + <expr> ⇒ <expr> + <expr> + <expr>
```

The Culprit

```
<expr> ::= <expr> + <expr>
         | <var>
<var>  ::= x | y | z
```

`Any time we have a rule like this, we should be suspicious...`

```
<expr> + <expr> ⇒ <expr> + <expr> + <expr>
```

`Which <expr> did we replace?`

Aside: Dealing with Associativity within the Grammar

```
<expr> ::= <expr> + <var>
         | <var>
<var>  ::= x | y | z
```

```
<expr>
<expr> + <var>
<expr> + z
<expr> + <var> + z
<expr> + y + z
<var> + y + z
x + y + z
```

By enforcing that the second argument is a <var>, we will get the left-associative parse tree.

Question. *What about the right associative?*

And Right Associativity

```
<type> ::= <base> -> <type>
         | <base>
<base> ::= ()
```

```
<type>
<base> -> <type>
() -> <type>
() -> <base> -> <type>
() -> () -> <type>
() -> () -> <base>
() -> () -> ()
```

For right associativity, we break symmetry by "factoring out" the *left* argument.

Example Parse Tree

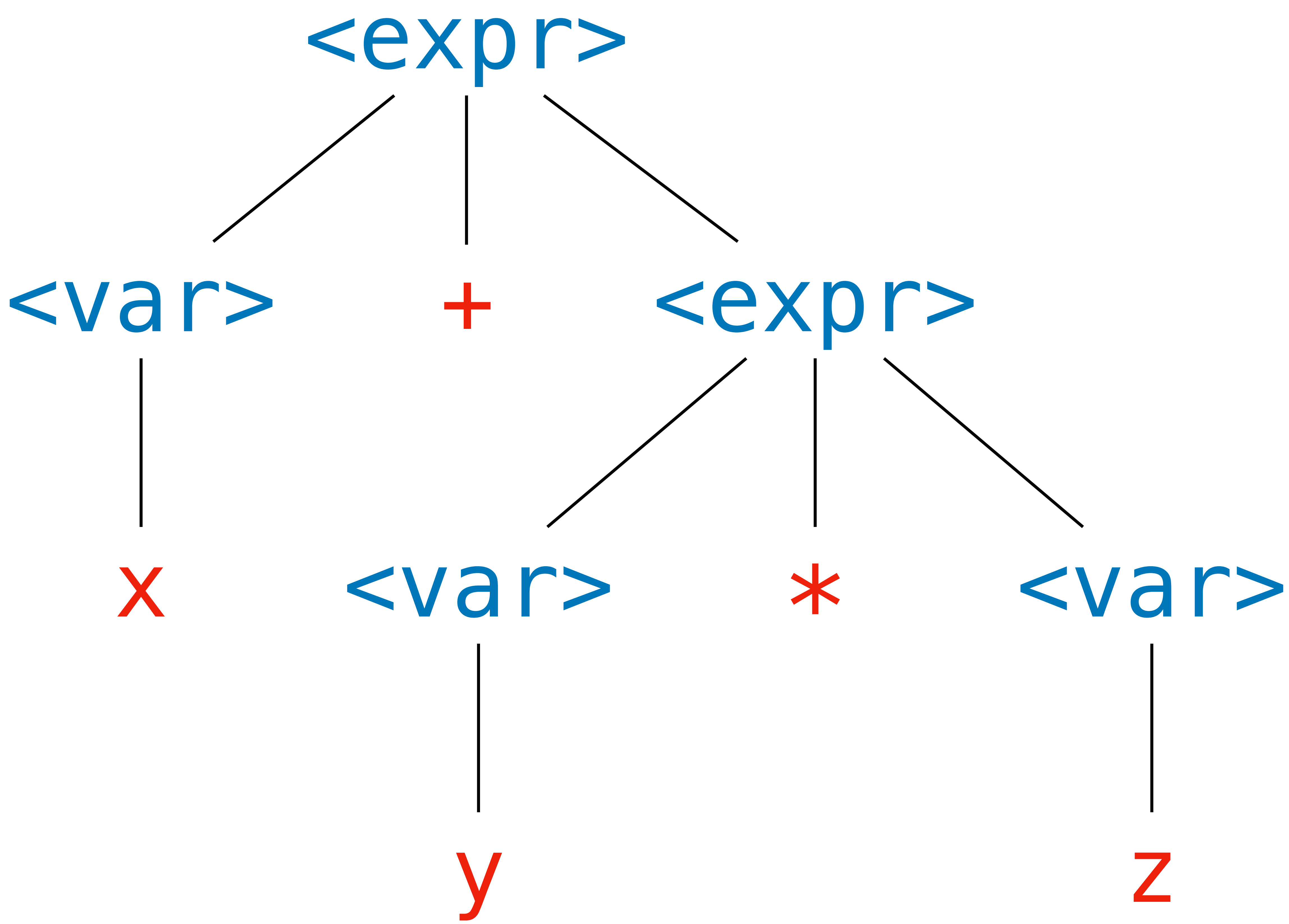
```
<type> ::= <base> -> <type>
         | <base>
<base> ::= ()
```

```
<type>
<base> -> <type>
() -> <type>
() -> <base> -> <type>
() -> () -> <type>
() -> () -> <base>
() -> () -> ()
```

Multiple Operators

```
<expr> ::= <expr> <op> <var>
         | <var>
<op>   ::= + | *
<var>  ::= x | y | z
```

x + y * z ⇒



"add x to the product of y and z"

Question. *What if we have multiple operators? Which one should "bind tighter"?*

Precedence

$$2 + 3 \times 6 = 2 + (3 \times 6) = 20$$

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Example. **PEMDAS (paren, exp, mul, div, add, sub)**

Precedence

$$2 + 3 \times 6 = 2 + (3 \times 6) = 20$$

The <u>precedence</u> of an operator refers to order in which an operator should be considered, relative to other operators.

Example. **PEMDAS (paren, exp, mul, div, add, sub)**

Higher precedence means it "binds tighter".

Aside: Dealing with Precedence within the Grammar

```
<expr> ::= <expr> + <term>
         | <term>
<term> ::= <term> * <var>
         | <var>
<var>  ::= x | y | z
```

We factor out the * part of the <expr> rule.

Note that we handle *lower* precedence terms first, since terms *deeper* in the parse tree are evaluated first.

Understanding Check

```
<expr> ::= <expr> + <term>
         | <term>
<term> ::= <term> * <var>
         | <var>
<var>  ::= x | y | z
```

Write down the parse tree for ***x + y * z****.*

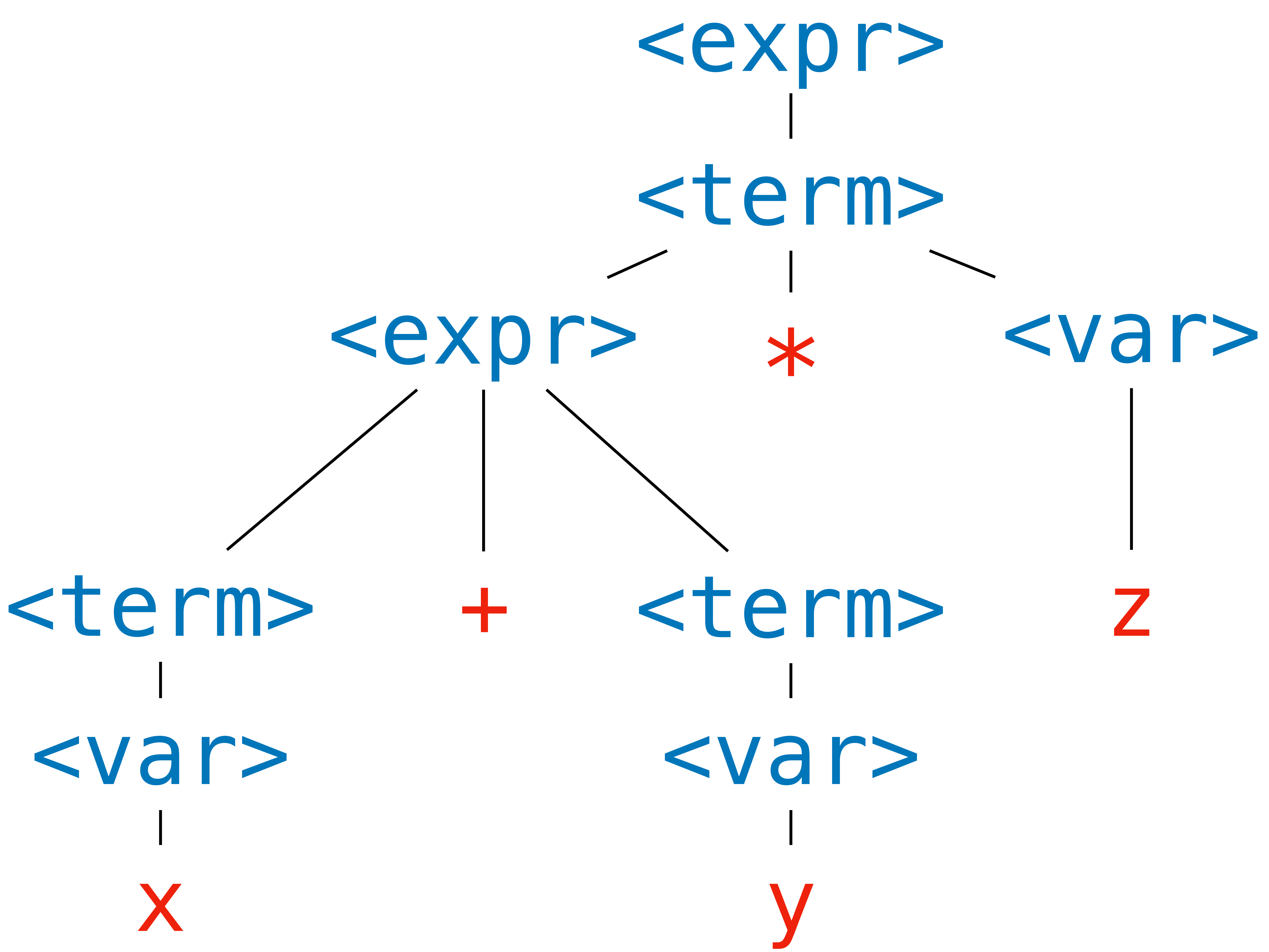
Answer

```
<expr> ::= <expr> + <term>
         | <term>
<term> ::= <term> * <var>
         | <var>
<var>  ::= x | y | z
```

x + y * z

The Issue of Parentheses Returns

```
<expr> ::= <expr> + <term>
         | <term>
<term> ::= <term> * <var>
         | <var>
<var>  ::= x | y | z
```



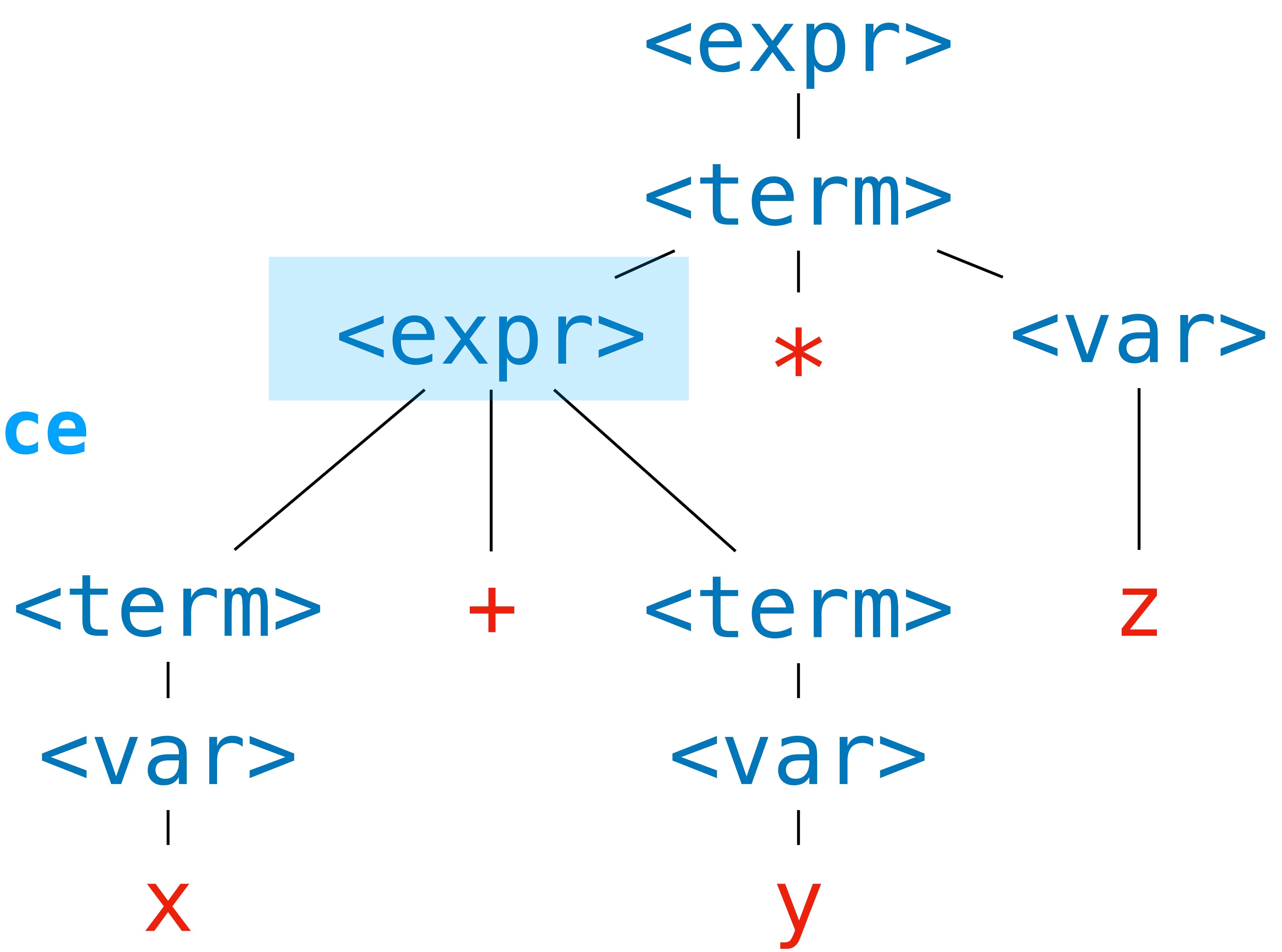
"multiply the sum of x and y with z"

Question. *Can we derive this parse tree?*

The Issue of Parentheses Returns

```
<expr> ::= <expr> + <term>
         | <term>
<term> ::= <term> * <var>
         | <var>
<var>  ::= x | y | z
```

No, we need to introduce parentheses again.



"multiply the sum of x and y with z"

Question. *Can we derive this parse tree?*

Dealing with Parentheses

```
<expr> ::= <expr> + <term>
         | <term>
<term> ::= <term> * <var>
         | <pars>
<pars> ::= <var> | ( <expr> )
<var>  ::= x | y | z
```

We further factor out the part of the rule for parentheses. Note that any expression can appear in the parentheses.

(This is a circular, or mutually recursive, definition.)

Example

```
<expr> ::= <expr> + <term>
         | <term>
<term> ::= <term> * <var>
         | <pars>
<pars> ::= <var> | ( <expr> )
<var>  ::= x | y | z
```

(x + y) * z

Other Considerations

There's a lot left to make a working grammar:

- » actual values (e.g., **(1 + 23) * 4**)
- » variable names (e.g, **valid_var + 33**)
- » multiple operations with the same precedence (e.g. **1 + 3 - 2**)
- » multiple operations with different associativity (?)

Other Considerations

There's a lot left to make a working grammar:

- actual values (e.g., **(1 + 23) * 4**)
- variable names (e.g, **valid_var + 33**)
- multiple operations with the same precedence (e.g. **1 + 3 - 2**)
- multiple operations with different associativity (?)

This is what we will be doing when we build our interpreters.

Summary

To avoid ambiguity, we make choices beforehand about the fixity, associativity and precedence.

Determining ambiguity can be tricky, but usually possible for simple grammars.

We make the grammars of programming languages unambiguous so that we don't make unspoken assumptions about the users input.