Exercise Sheet 2 for the 'Quantum Machine Learning' Course

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This second set of exercises is aimed at introducing basic multi-qubit quantum circuit results that will be used throughout the course.

I. Tensor Product

The tensor product is a way of putting vector spaces together to form larger vector spaces.

Let $\mathcal{H}_A$ be the Hilbert space associated with a quantum system labeled with the letter "A", and let $\mathcal{H}_B$ be the Hilbert space associated with a quantum system labeled with the letter "B". Assume that $\mathcal{H}_A$ is of dimension d_A, while $\mathcal{H}_B$ is of dimension d_B. As mentioned in the previous exercise sheet, Hilbert spaces are complex vector spaces. The tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ is a Hilbert space that described the joint quantum system "$A+B$" that is in itself a complex vector space of dimension $d_A \times d_B$. If $\{|v_A\rangle\}$ and $\{|w_B\rangle\}$ are basis of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, then the space $\mathcal{H}_A \otimes \mathcal{H}_B$ has basis elements $\{|v_A\rangle \otimes |w_B\rangle\}$. Usually, we denote these simply as $\{|v_A w_B\rangle\}$.

By definition, the tensor product satisfies the following properties

- Let z be a scalar, let $|v\rangle \in \mathcal{H}_A$ and $|w\rangle \in \mathcal{H}_B$, then $z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z|w\rangle)$.
- Let $|v_1\rangle, |v_2\rangle \in \mathcal{H}_A$ and $|w\rangle \in \mathcal{H}_B$ then $(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$.
- Let $|v\rangle \in \mathcal{H}_A$ and $|w_1\rangle, |w_2\rangle \in \mathcal{H}_B$ then $|v\rangle \otimes (|w_1\rangle \otimes |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$.

Explicitly, the entries of the vector $|v\rangle \otimes |w\rangle$ are given by

$$(|v\rangle \otimes |w\rangle)_{ij} = v_i w_j. \tag{1}$$

The simplest operators that act on $\mathcal{H}_A \otimes \mathcal{H}_B$ are obtained from local operators acting on each Hilbert space. For instance, given operators $V : \mathcal{H}_A \to \mathcal{H}_A$ and $W : \mathcal{H}_B \to \mathcal{H}_B$, then, the tensor product operator $V \otimes W$ acts on the elements of $\mathcal{H}_A \otimes \mathcal{H}_B$ as

$$(V \otimes W)|v\rangle \otimes |w\rangle = V|v\rangle \otimes W|w\rangle\,. \tag{2}$$

Explicitly, let V and W be 2×2 matrices:

$$V = \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix}, \tag{3}$$

$$W = \begin{bmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{bmatrix} \tag{4}$$

then,

$$(V \otimes W) = \begin{bmatrix} v_{1,1}w_{1,1} & v_{1,1}w_{1,2} & v_{1,2}w_{1,1} & v_{1,2}w_{1,2} \\ v_{1,1}w_{2,1} & v_{1,1}w_{2,2} & v_{1,2}w_{2,1} & v_{1,2}w_{2,2} \\ v_{2,1}w_{1,1} & v_{2,1}w_{1,2} & v_{2,2}w_{1,1} & v_{2,2}w_{1,2} \\ v_{2,1}w_{2,1} & v_{2,1}w_{2,2} & v_{2,2}w_{2,1} & v_{2,2}w_{2,2} \end{bmatrix}. \tag{5}$$

More generally, an arbitrary operator $C : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$ is of the form

$$C = \sum_i c_i V_i \otimes W_i\,, \tag{6}$$

with $\{V_i\}$ and $\{W_i\}$ operators respectively acting on $\mathcal{H}_A$ and $\mathcal{H}_B$.

Exercise 1 (Tensor Product of Pauli matrices)**.** *Let X, Y and Z be Pauli matrices, and let I be the 2×2 identity matrix. Explicitly compute the 4×4 matrices $Z\otimes I$, $I \otimes Z$, $X \otimes X$.*

Exercise 2 (Transpose, conjugate and dagger of tensor products)**.** *Show that $(V \otimes W)^* = V^* \otimes W^*$, $(V \otimes W)^t = V^t \otimes W^t$, and $(V \otimes W)^\dagger = V^\dagger \otimes W^\dagger$.*

Exercise 3 (Tensor product of unitary matrices)**.** *Show that the tensor product of two unitary matrices, is unitary.*

Exercise 4 (Tensor product of Hermitian matrices)**.** *Show that the tensor product of two Hermitian matrices, is Hermitian.*

II. Composite systems

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems.

Let us consider a two qubit system, where one qubit is labeled with the letter "A", and the other labeled with the letter "B". Then, the computational basis of the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}^{\otimes 2} := \mathcal{H} \otimes \mathcal{H}$ (where $\mathcal{H}$ here denoted the Hilbert space of a single qubit) describing the composite two qubit system is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Exercise 5 (Vector representation of the basis)**.** *Using Eq. (1) and recalling that $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, find the vector representation of each element in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.*

Exercise 6 (n-qubit systems)**.** *How many elements are there in the basis of the Hilbert space $\mathcal{H}^{\otimes n}$ describing a composite n-qubit system?*

The general pure state of two qubit $|\psi\rangle \in \mathcal{H}^{\otimes 2}$ can be expressed as

$$|\psi\rangle = \sum_{ij=0,1} c_{ij}|ij\rangle \tag{7}$$

$$= c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle\,, \tag{8}$$

where $\sum_{ij=0,1}|c_{ij}|^2 = 1$.

Exercise 7 (Expectation value)**.** *Let $|\psi\rangle$ be the general state in Eq. (8). Compute the expectation value $\langle Z\otimes Z\rangle = \langle\psi|Z\otimes Z|\psi\rangle$.*

Similarly to measuring single qubit states, we can write the projective measurement operators for the computational basis as

$$M_{00} = |00\rangle\langle 00|\,, \quad M_{01} = |01\rangle\langle 01|\,, \tag{9}$$

$$M_{10} = |10\rangle\langle 10|\,, \quad M_{11} = |11\rangle\langle 11|\,. \tag{10}$$

So that

$$p(00) = \mathrm{Tr}[M_{00}\rho] = |\langle 00\,|\,\psi\rangle|^2 = |c_{00}|^2\,, \tag{11}$$

$$p(01) = \mathrm{Tr}[M_{01}\rho] = |\langle 01\,|\,\psi\rangle|^2 = |c_{01}|^2\,, \tag{12}$$

$$p(10) = \mathrm{Tr}[M_{10}\rho] = |\langle 10\,|\,\psi\rangle|^2 = |c_{10}|^2\,, \tag{13}$$

$$p(11) = \mathrm{Tr}[M_{11}\rho] = |\langle 11\,|\,\psi\rangle|^2 = |c_{11}|^2\,. \tag{14}$$

where $\rho = |\psi\rangle\langle\psi|$.

Recall from probability theory that given the joint probability of two discrete variables $p(x_i, y_j)$, the marginal probability $p(x_i) = \sum_j p(x_i, y_j)$. Use Eq. (8) to explicitly show that the probability of measuring just the first qubit in zero, or one, are

$$p_A(0) = p(00) + p(01)\,, \quad p_A(1) = p(10) + p(11)\,. \tag{15}$$

Performing a local measurement, e.g., measuring only the first qubit, and not measuring the second qubit can be obtained from the projective measurement operators

$$M_0^A = |0\rangle\langle 0| \otimes \mathbb{1}\,, \quad M_1^B = |1\rangle\langle 1| \otimes \mathbb{1}\,. \tag{16}$$

Exercise 8 (Marginal Probability)**.** *First, show that $I = |0\rangle\langle 0| + |1\rangle\langle 1|$, with I the 2×2 identity matrix. Then, how can we interpret the marginal probability as just performing a local measurement and not measuring the other qubit?*

Exercise 9 (Expectation value 2)**.** *Let $|\psi\rangle$ be the general state in Eq. (8). Show that the expectation value $\langle Z\otimes Z\rangle$ can be expressed as*

$$\langle Z\otimes Z\rangle = p(00) + p(11) - p(01) - p(10). \tag{17}$$

Given the state $\rho = |\psi\rangle\langle\psi|$ of a two-qubit composite system $\rho \in \mathcal{H}_{AB}$, sometimes we are interested in only the state of a *single qubit* without much regards for the second qubit. Here, there is a simple way in which we can describe the state of a single qubit via the so-called *reduced density matrix*. The reduced density matrix of qubit A is defined as

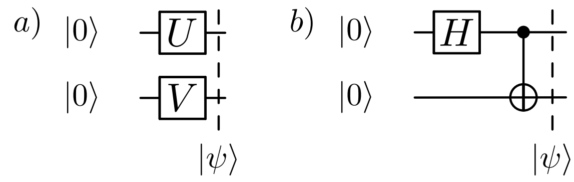

FIG. 1. **Quantum circuit examples**. a) Non-entangling quantum circuit. b) Entangling quantum circuit.

$$\rho_A = \mathrm{Tr}_B[\rho]\,, \tag{18}$$

where Tr_B is an operator map known as the partial trace over system B. Explicitly, the partial trace is computed as

$$\mathrm{Tr}_B\left[|\,v_1\rangle\langle v_2\,| \otimes |\,w_1\rangle\langle w_2\,|\right] = |\,v_1\rangle\langle v_2\,|\mathrm{Tr}\left[|\,w_1\rangle\langle w_2\,|\right]\,, \tag{19}$$

where here $|v_1\rangle$ and $|v_2\rangle$ are vectors of the Hilbert space describing qubit A, and $|w_1\rangle$ and $|w_2\rangle$ are vectors of the Hilbert space describing qubit B.

Exercise 10 (Partial Trace)**.** *Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ be a quantum state on two qubits. First, compute its density matrix $\rho = |\psi\rangle\langle\psi|$. Then, compute the reduced states ρ_A and ρ_B.*

III. Quantum Entanglement

Entanglement refers to non-classical correlations in quantum states. They are a key resource for quantum computation and one of, if not the, defining properties of quantum mechanics.

Consider a bipartite quantum system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The state $|\psi\rangle \in \mathcal{H}$ is said to be *separable (or non-entangled)* if $|\psi\rangle = |i\rangle\otimes|j\rangle$. Alternatively, the state $|\psi\rangle$ is said to be *entangled* if it cannot be written as a product form, that is, if $|\psi\rangle \neq |i\rangle\otimes|j\rangle$.

Exercise 11 (Local unitaries)**.** *Let $|\psi\rangle = |00\rangle$ be a separable state. Prove that an entangled state cannot be created by local single-qubit unitaries that are of the form $U\otimes V$ (see Fig. 1(a)).*

Exercise 12 (Reduced States of pure states)**.** *Show that the reduced state of any pure separable state is pure. That is, given a state $\rho = |\psi\rangle\langle\psi|$, with $|\psi\rangle = |v\rangle\otimes|w\rangle$, then $\rho_A^2 = \rho_A$ and $\rho_B^2 = \rho_B$.*

The previous exercise gives us an easy way to check if a bipartite pure quantum state is entangled or not. We simply compute the reduced states and check if they are pure. The latter can be achieved by computing their *purity*. The purity of a quantum state ρ is defined as $\mathcal{P}(\rho) = \mathrm{Tr}[\rho^2]$. For pure states $\rho = |\psi\rangle\langle\psi|$, $\mathcal{P}(\rho) = \mathrm{Tr}[\rho^2] = \mathrm{Tr}[\rho] = 1$.

Exercise 13 (Maximally entangled state)**.** *The state* $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ *is known as the Bell state, and it is known to be maximally entangled. Using the results of Exercise 10, compute* $\mathcal{P}(\rho_A)$ *and* $\mathcal{P}(\rho_B)$*. How can we interpret this result in terms of knowledge of the reduced state of a maximally entangled state?*

IV. Quantum Circuit Computation

A quantum computer is built from a quantum circuit containing wires and elementary quantum gates to carry around and manipulate the quantum information.

When working with quantum systems, we want to apply a general unitary U to an n-qubit state $|\psi\rangle$. However, usually we need to obtain a circuit description of U in terms of simple gates that we can implement in our quantum computer. For this purpose, it is very useful to determine *universal sets of quantum gates.* A set $\mathcal{S}$ of quantum gates is said to be universal if any unitary U can be express as a finite product of gates in $\mathcal{S}$. Mathematically, this translates onto

$$U = \prod_j W_j\,, \quad \text{where } W_j \in \mathcal{S}\,. \tag{20}$$

For instance, a famous universal gate set is composed of the gates

$$\mathcal{S} = \{R_x(\theta), R_y(\theta), R_z(\theta), P(\varphi), \text{CNOT}\}\,, \tag{21}$$

where $R_\mu(\theta)$ with with $\mu = x, y, z$ is a single qubit rotation about the μ-axis, $P(\varphi)$ is single-qubit gate known as the phase shift gate

$$P(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}, \tag{22}$$

and CNOT is two qubit gate called the *controlled-NOT*, or CNOT gate given by

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{23}$$

Exercise 14 (Gates are unitary)**.** *Prove that* $P(\varphi)$ *and CNOT are unitaries.*

Exercise 15 (CNOT action)**.** *Prove using matrix multiplication, that*

$$CNOT \cdot |00\rangle = |00\rangle \tag{24}$$
$$CNOT \cdot |01\rangle = |01\rangle \tag{25}$$
$$CNOT \cdot |10\rangle = |11\rangle \tag{26}$$
$$CNOT \cdot |11\rangle = |10\rangle \tag{27}$$

Exercise 16 (Bell Circuit)**.** *Show that the output state* $|\psi\rangle$ *of the circuit in Fig. 1(b) is the maximally entangled Bell state* $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$*. Here, H is the Hadamard gate defined as*

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{28}$$