Class 1 : PWF 2021 - QML 11/29/2021

1901: Black body radiation → Planck law
1927: double slit experiments 1905: Einstein quantization EM

Postulates of Quantum Mechanism

Postulate 1 [State Vector]:

Associated with any isolated physical system is a complex vector space with an inner product (Hilbert space), known as the state space of the system.

The system is described by the state vector.

Hilbert space $\mathcal{H} - \mathbb{C}^d$

state vector $|\psi\rangle$ $(d \times 1)$ complex vector with a norm $= 1$

$$|\langle\psi|\psi\rangle|^2 = 1$$

Postulate 2 [Evolution]:

The evolution of a closed system is described by a unitary transformation.

$|\psi\rangle$ at time t

$|\psi'\rangle$ — " t' with $t' > t$

$$\Rightarrow |\psi'\rangle = U(t';t)|\psi\rangle \quad (*)$$

$U(t';t)\,U^\dagger(t';t) = \mathbb{1}$
$U^\dagger(t';t)\,U(t';t) = \mathbb{1}$

[Schrödinger's Equation] The time evolution of the state of a closed quantum system is described by Scrödinger's eq:

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle \quad (*)$$

H is the Hamiltonian of the systems.

for any $|x\rangle$ $\langle x|H|x\rangle$ is real.

if H is independent of t $\quad U(t) = e^{-itH} \quad (*)$

Postulate 3 [Quantum Measurements]:

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators.

The index m refers to the measurement outcome.

If the state of the system was $|\psi\rangle$, immediately after measurement the probability that outcome m occurs is

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

and the state of the system post-measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{p(m)}}$$

The operators $\{M_m\}$ satisfy $\sum_m M_m^\dagger M_m = \mathbb{1}$ (completness)

$$1 = \sum_m p(m) = \sum_m \langle\psi| M_m^\dagger M_m |\psi\rangle = \langle\psi| \underbrace{\sum_m M_m^\dagger M_m}_{\mathbb{1}} |\psi\rangle$$

$$= \langle\psi|\mathbb{1}|\psi\rangle = 1$$

A special type of measurements: Projective measurements.

Described by an observables: Hermitian operator M

$$M = \sum_m m\, \underline{P_m}$$

$$P_m P_{m'}^\dagger = P_m^\dagger P_{m'} = P_m \delta_{m,m'}$$

$$E[M] = \underline{\sum_m p_m\, m} = \sum_m m \langle\psi|P_m|\psi\rangle = \underline{\langle\psi|M|\psi\rangle}$$

Usually when we measure QC = Quantum Computing, we measure in the eigenbasis of Z.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1\, \underline{|0\rangle\langle 0|} - 1\, \underline{|1\rangle\langle 1|}$$

multi-qubit Description ⟶ Evolution ⟶ measures

Postulate 4 [Composite System]:

The state space of a composite quantum system is the tensor product of the state space of the component physical systems:

e.g. n-qubit system:

each $\mathcal{H}_Q = \mathbb{C}^2$

whole system $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i = \mathcal{H}_Q^{\otimes n}$

$\{|\psi_i\rangle\}$ $\quad |\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle$

$\{|0\rangle, |1\rangle\}$

$\{|0\rangle\otimes|0\rangle, |0\rangle\otimes|1\rangle, |1\rangle\otimes|0\rangle, |1\rangle\otimes|1\rangle\} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$\rightarrow \quad |\psi\rangle = \sum_i c_i \bigotimes_{i=1}^{n} |\psi_i\rangle \qquad (d \times 1)$

$\boxed{\text{Rephrase postulates of QM in terms of the density operator:}}$

Density operator

1) $\{|\psi_i\rangle, p_i\}$ $\quad \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

2) ρ' at time t' $\quad \rho' = U\rho U^\dagger$

3) $p(m) = \mathrm{Tr}[M_m^\dagger M_m \rho] = \mathrm{Tr}[P_m \rho]$

$$\rho_m = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}[M_m^\dagger M_m \rho]}$$

Properties :
- $Tr[\rho] = 1$
- $\rho \geq 0 \quad \forall |x\rangle \quad \langle x|\rho|x\rangle \geq 0$

- Pure States $\quad \rho = |\psi\rangle\langle\psi| \quad Tr[\rho^2] = 1 \quad \rho^2 = \rho$
- Mixed states $\quad Tr[\rho^2] \leq 1$

QM $\rightarrow$ QC