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Section :- C

Subject :- Calculus

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Q4) a) If $u = \ln\left(\frac{x^2+y^2}{x+y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

Sol:-

$$u = \ln\left(\frac{x^2+y^2}{x+y}\right)$$

$$= x\frac{\partial}{\partial x}\left(\ln\left(\frac{x^2+y^2}{x+y}\right)\right) + y\frac{\partial}{\partial y}\left(\ln\left(\frac{x^2+y^2}{x+y}\right)\right)$$

$$= x\left\{\frac{x+y}{(x^2+y^2)}\left[\frac{(x+y)(2x)-(x^2+y^2)}{(x+y)^2}\right]\right\} + y\left[\frac{x+y}{(x^2+y^2)}\left[\frac{(x+y)(2y)-(x^2+y^2)(1)}{(x+y)^2}\right]\right]$$

$$= \frac{1(x+y)}{(x+y)^2(x^2+y^2)}\left[x(2x^2+2y-x^2-y^2)+y(2xy+2y^2-x^2-y^2)\right]$$

$$= \frac{1}{(x+y)(x^2+y^2)}\left[2x^3+2x^2y-x^3-xy^2+2xy^2+2y^3-x^2y-y^3\right]$$

$$= \frac{(x^3+y^3+x^2y+xy^2)}{x(x^2+y^2)+y(x^2+y^2)}$$

$$= \frac{x^3+y^3+x^2y+xy^2}{x^3+xy^2+x^2y+y^3}$$

$$= \boxed{1}$$ Hence proved.

Q1)a) Define derivatives of a function ...... daily life examples.

Ans:-

Derivatives:-

Definition:-

The derivative of a function of f(x) ~~which~~ with respect to "x" is the function "f" where value of "x" is:-

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{\Delta x}$$

One variable:-

$$f'(x) = \lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Two variable:-

$$f'(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta x}$$

$$f''(x,y) = \lim_{\Delta y \to 0} \frac{f(x+\Delta x 2y) - f(x,y)}{\Delta x}$$

Application:-

There can be and are many Application of derivative, some are given below:

① Light
② Mass transfer
③ Medical
④ Astronomy etc

(P.T.O)

Examples from daily life:

There can be many examples in our daily life such as;

① Speed of a bike with respect to time is example of derivative.

② Distance traveled by a person with respect to time is example of derivative.

③ Growth of a person with respect to time is example of derivative.

Q1)b) Evaluate $\int \sqrt{\frac{1+x}{1-x}}\,dx$

Solution:-

$$= \int \sqrt{\frac{1+x}{1-x}}\,dx$$

multiplying and dividing $(\sqrt{1+x})$.

$$= \int \sqrt{\frac{1+x}{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}$$

$$= \int \frac{(\sqrt{1+x})^2}{(\sqrt{1-x})(\sqrt{1+x})}$$

R.W

$$(\sqrt{1+x})^2 = \left((1+x)^{1/x}\right)^x$$
$$= 1+x$$

$$= \int \frac{1+x}{\sqrt{1^2-x^2}}$$

$$\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}\frac{x}{1} - \frac{1}{2}\int -2x(1-x^2)^{-1/2}\,dx$$

$$= \sin^{-1}x - \frac{1}{2}\left[\frac{(1-x^2)^{-1/2+1}}{-\frac{1}{2}+1}\right] + C$$

$$= \sin^{-1}x - \frac{1}{2}\,\frac{(1-x^2)^{+1/2}}{1/2} + C$$

$$\boxed{= \sin^{-1}x - \sqrt{1-x^2} + C}$$ Ans

Q5) a) Evaluate $\lim_{x \to 1} \frac{1 - x + \ln x}{1 - \sqrt{2x - x^2}}$

Soln

$$= \lim_{x \to 1} \frac{(1 - x + \ln x)}{1 - \sqrt{2x - x^2}}$$

By rationlization

$$= \lim_{x \to 1} \frac{(1 - x + \ln x)}{(1 - \sqrt{2x - x^2})} \times \frac{1 + \sqrt{2x - x^2}}{1 + \sqrt{2x - x^2}}$$

$$= \lim_{x \to 1} \frac{(1 - x + \ln x)(1 + \sqrt{2x - x^2})}{1 - 2x - x}$$

applying limit we get;

$$= \frac{(1 - 1 + \ln(1))(1 + \sqrt{2(1) - 1})}{\cancel{1} - 2(1) - \cancel{x}}$$

$$= \frac{(0)(1 + 1)}{-2}$$

$$= 0 \cdot \frac{2}{-2}$$

$$\boxed{= 0}$$ Ans.

Q2) a) Evaluate $\int_0^4 \int_{y/2}^2 e^{x^2}\,dx\,dy$

<u>Sol:-</u>

$$\int_0^4 \int_{y/2}^2 e^{x^2}\,dx\,dy$$

$$= \int_0^4 \left(\int_{y/2}^2 e^{x^2}\,dx \right) dy$$

$$= \int_0^4 \left(e^{x^2}(2x) \Big|_{y/2}^2 \right) dy$$

$$= \int_0^4 \left(2(x) e^{x^2} \Big|_{y/2}^2 \right) dy$$

$$= \int_0^4 \left([2(2)e^{(2)^2}] - [2(y/2)\,e^{y^2/4}] \right) dy$$

$$= \int_0^4 [4e^4 - ye^{y^2/4}]\,dy$$

Now evaluate the 2^{nd} integral.

$$= \int_0^4 (4e^4 - ye^{y^2/4})\,dy$$

$$= 4\int_0^4 e^4\,dy - \int_0^4 y \cdot e^{y^2/4}$$

$$= 4e^4 - \frac{1}{2}\int_0^4 2y \cdot e^{y^2/4}\,dy$$

By multiplying and dividing by 2

$$= 4e^4 - \frac{1}{2}\int_0^4 e^{y^2/4}\,2y\,dy$$

(P.T.O)

using exponential formula.

$$\int e^{f(x)} f'(x) = e^{f(x)} + C$$

$$= 4e^4 - \frac{1}{2}\left[e^{y^2/4}\right]_0^4$$

$$= 4e^4 - \frac{1}{2}\left[e^{(4)^2/4} - e^{(0)^2/4}\right]$$

$$\Rightarrow 4e^4 - \frac{1}{2}\left[e^{16/4} - e^0\right]$$

$$\Rightarrow 4e^4 - \frac{1}{2}\left[e^4 - 1\right]$$

$$\boxed{= 4e^4 - \frac{e^4 - 1}{2}}$$ Ans.

Q2) b) Evaluate for extreme values of
$f(x,y) = (x+1)(y+1)(x+y+1)$.

$f(x) = \frac{\partial}{\partial x} (x+1)(y+1)(x+y+1)$

$= \frac{\partial}{\partial x} (xy+x+y+1)(x+y+1)$

$= \frac{\partial}{\partial x} (x^2y + xy^2 + xy + x^2 + xy + x + yx + y^2 + y$
$+ x + y + 1)$

$= \frac{\partial}{\partial x} (x^2y + xy^2 + 3xy + x^2 + 2x + y^2 + 2y + 1)$

Differentiate

$= 2xy + y^2 + 3y + 2x + 2 + 0 + 0 + 0$

$f(x) = 2xy + y^2 + 3y + 2x + 2$

$f(y) = \frac{\partial}{\partial y} (x^2y + xy^2 + 3xy + x^2 + 2x + y^2 + 2y + 1)$

$= x^2 + 2xy + 3x + 0 + 0 + 2y + 2 + 0$
$f(y) = x^2 + 2xy + 3x + 2y + 2$

We have
$f(x) = 2xy + y^2 + 3y + 2x + 2$, $f(y) = x^2 + 2xy + 3x +$
$2y + 2$

finding values of x and y

$(x,y) = (0,-2)$ and $(x,y) = (-3/8, -5)$

$f_{xx} = 2y+2$, $f_{yy} = 2x+2$, $f_{xy} = 2x+2y+3$

→ at $(x,y) = (0,-2)$

we have $f_{xx} = -2$, $f_{yy} = +2$, $f_{xy} = -1$

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(2) - (-1)^2$$
$$= -4 - 1 = -5$$

As $f_{xx} = -2 < 0$ and $f_{xx}f_{yy}\ f_{xy}^2 = -5 < 0$

so,

$f(x,y)$ has saddle point at $(0,-2)$

→ at $(x,y) = (-3/8, -5)$

we have $f_{xx} = -8$, $f_{yy} = 5/4$, $f_{xy} = -31/4$

$$f_{xx}f_{yy} - f_{xy}^2 = (-8)(5/4) - (-31/4)^2$$

$$= \frac{-1121}{16} = -70.06$$

$f_{xx}f_{yy} - f_{xy}^2 < 0$

$\boxed{\text{So, } f(x,y) \text{ has saddle point at } (-3/8, -5).}$ - Ans

Q3) b) If $u = x^2+y^2+z^2$ then compute

$\Delta \vec{\nabla} u$.

Sol:

$$u = x^2+y^2+z^2$$

let

$$u = f(x,y,z)$$

we know that gradient of function is;

$$\text{Grad}\{f(x,y,z)\} \cdot \vec{\nabla}\{f(x,y,z)\}$$

where As,

$$\vec{\nabla}\{f(x,y,z)\} = (i^2/_{2x} + j^2/_{2y} + k^2/_{2z}) \; f(x,y,z)$$

In this case $f(x,y,z) = u$

So,

$$\vec{\nabla}\{f(x,y,z)\} = (i\,d/dx + j\,d/dy + k\,d/dz)\,(x^2+y^2+z^2)$$

$$= i(2x+0+0) + \hat{j}(0+2y+0) + k(0+0+2z)$$

$$= (2x)i + (2y)\hat{j} + (2z)\hat{k}$$

So,

$$\boxed{\vec{\nabla} u = 2xi + 2yj + 2zk}\ \text{Ans.}$$

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Q5) b) If $f(x) = x^4 - \frac{1}{3}x^3$ then ~~complete~~ compute extreme value

Sol

$$f(x) = x^4 - \frac{1}{3}x^3$$

$$f(x) = 4x^3 - \frac{1}{3}\,3x^2$$

$$f'(x) = 4x^3 - x^2 = 0$$

$$4x^3 - x^2 = 0$$
$$x^2(4x-1) = 0$$

$$x^2 = 0 \qquad\qquad 4x - 1 = 0$$
$$x = 0 \qquad\qquad 4x = 1$$
$$x = \frac{1}{4}$$

$$x = +\frac{1}{2}$$

$$f''(x) = 12x^2 - 2x$$

at $x = 0$ and $x = +\frac{1}{2}$

$$f'(0) = 0 \qquad\qquad f'(\tfrac{1}{2}) = 3 - 1$$
$$= 2$$ minima

$$f'(-\tfrac{1}{2}) = 3 + 1$$
$$= 4$$ maxima

$$f(1/2) = 1/16 - 1/3 \cdot (1/2)^3$$

$$= \frac{1}{16} - \frac{1}{3(8)}$$

$$= \frac{1}{16} - \frac{1}{24}$$

$$= \frac{8}{24 \times 16}$$

$$\boxed{f(1/2) = 1/48}$$

$$f(-1/2) = 1/16 - 1/3(-1/8)$$

$$= 1/16 + 1/24$$

$$= \frac{40}{16 \times 24}$$

$$= \frac{10}{4 \times 24}$$

$$\boxed{f(-1/2) = 5/48}$$

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Q3) a) $V = \rho^m \Rightarrow \rho^2 \Rightarrow x^2+y^2+z^2$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial v^2}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)\rho^{m-2}$$

Solt

$$m = 2$$

$$V = \rho^2 = x^2+y^2+z^2$$

L.H.S

$$\frac{\partial v}{\partial x} = 2x+0+0$$

$$= 2x$$

$$\frac{\partial^2 V}{\partial x^2} = 2$$

How similarity

$$\frac{d^2 V}{d y^2} = 2$$

and then

$$\frac{\partial^2 V}{\partial z} = 2$$

$$L.H.S = 2+2+2 = 6$$

R.H.S

$$m(m+1)\rho^{m-2}$$

we considering $m = 2$

$$= 2(2+1)\rho^{2-2}$$

$$= 2(3) = 6$$

$= 6(1)$

$= 6$

So L.H.S = R.H.S

Q4) b) Sol Improper function as here we are long division

$$\int \frac{x^4}{x^4+2x^2+1}\,dx = \int \frac{-2x^2-1}{x^4+2x^2+1}\,dx$$

$$= \int 1\,dx - \int \frac{(-2x^2-1)}{(x^4+2x^2+1)}\,dx$$

$$= x + \int \frac{-2x^2+1}{x^4+2x^2+1}\,dx$$

$$= x - \int \frac{2}{x^2+1} - 1/(x^2+1)^2\,dx$$

Now integration by parts

$$\int \frac{1}{x^2+1}\,dx \qquad u = \frac{1}{x^2+1}$$

Then

$$\int \frac{1}{x^2+1}\,dx = \frac{x+x}{2x^2+1} - \int \frac{2}{x^2+1}\,dx + \frac{a\tan x}{2}$$

$$= \frac{x+x}{2(x^2+1)} + \frac{a\tan x}{2} - 2\int \frac{1}{x^2+1}\,dx$$

As

$$\int \frac{1}{x^2+1} = a\tan x$$

Hence $= \dfrac{x+x}{2(x^2+1)} - \dfrac{3\,a\tan x}{2}$

$$\boxed{= \frac{x+x-3a\tan(x)(x^2+1)}{2(x^2+1)} + C} \quad \text{Ans}$$