Linear Systems and Control - Week 7

Controller Design - Full State Feedback Controller

Motivation for Controller Design

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- Any/all eigenvalue(s) of matrix A is/are non-negative
- Any/all pole(s) of transfer function is/are non-negative
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Solution:

- Check the pre-requisites of controller (if pre-requisites full-filled then goto next step)
- Design a suitable controller and
- Integrate/connect the controller with the system.

Types of Controller

There are 3 types of techniques to design controllers which are:

- Full-state feedback controller or state feedback controller
- Observer-based state feedback controller
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In today lecture, we will study the design of full-state feedback controller and its pre-requisites.

State Feedback Controller Pre Reqs

There are 2 pre-requisites before we can proceed to design of full state feedback controller:

- Matrix C must be equal to identity and matrix D must be equal to zero (or absent)
- The system must pass controllability test.

Let us talk about controllability test now.

Pre-req 2: Controllability Test

A system is controllable or it passes controllability test if the following crietria is satisfied:

- First, determine the order of the system and call it $\boldsymbol{n}$.
- Second, using $\boldsymbol{n}$, construct matrix $\boldsymbol{P}$ follows:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A^2B} & \dots & \boldsymbol{A^{n-1}B} \end{bmatrix} \tag{1}$$

- Third, compute rank of matrix $\boldsymbol{P}$
- Finally, check if rank of matrix $\boldsymbol{P}$ is equal to $\boldsymbol{n}$ or not.

If $\boldsymbol{rank(P) = n}$, then the system is controllable and we can proceed to design of controller, otherwise STOP. No controller can be designed.

Rank of matrix

Rank: The number of linearly independent rows or columns of a matrix.

To determine rank, we need to convert a matrix into row-echoelon form

Find the rank of a matrix using normal form,

$$\mathbf{P} = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

Solution:

Reduce the matrix to echelon form,

$$\begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{pmatrix} ---> \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank of matrix

If a matrix is square, then we can determine its rank from determinant also.

If determinant of a square matrix is non-zero, then its rank is full (equal to the order).

Forexample

$$\begin{aligned} \boldsymbol{P} &= \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} \end{bmatrix} \\ \boldsymbol{det(P)} &= \mathbf{(1)(3) - (2)(4)} \\ &= \mathbf{3 - 8} \\ &= \mathbf{-5} \end{aligned}$$

As determinant of matrix $\boldsymbol{P}$ is $\mathbf{-5}$, which is non-zero, hence rank of matrix $\boldsymbol{P}$ is 2.

Common mistakes in exam papers

Remember: the pre-requisite say construct matrix $\boldsymbol{P}$ and check rank of matrix $\boldsymbol{P}$.

Donot check rank of all matrices - especially matrix $\boldsymbol{A}$.

Size of matrix $\boldsymbol{A}$ tells us about $\boldsymbol{n}$ only

Example

Consider a system having the following state space model:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Check the following:

- Do we need a controller?
- If we need a controller, identify which controller to design
- Design that controller and place the eigenvalues at $(-3, -5)$.

Solution - Do we need a controller

First, we check stability of this system. The eigenvalues of this system can be obtained from $\boldsymbol{det(\lambda I - A) = 0}$

Solution - Do we need a controller

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$$\begin{aligned} det(\lambda I - A) &= det \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \\ &= det \begin{bmatrix} \lambda - 2 & -3 \\ 0 & \lambda - 5 \end{bmatrix} \\ &= (\lambda - 2)(\lambda - 5) - (0)(-3) \\ &= (\lambda - 2)(\lambda - 5) - (0) \\ &= (\lambda - 2)(\lambda - 5) \end{aligned}$$

The eigenvalues of matrix A are at 2 and 5, which indicates it is an unstable system.

Solution - Which controller to design

Now, which controller to choose?

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

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$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

As matrix C is identity matrix, we proceed to design of full state feedback controller and check the second pre-requisite.

Solution - Pre requisite 2

Let us compute now pre-requisite number $\mathbf{2}$ which is the controllability test.

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$$\boldsymbol{P = \begin{bmatrix} B & AB \end{bmatrix}}$$

$$\boldsymbol{P = \begin{bmatrix} 1 & 8 \\ 2 & 10 \end{bmatrix}}$$

$$\boldsymbol{det(P) = -6}$$

As determinant $\boldsymbol{P}$ is non-zero, so $\boldsymbol{rank(P) = 2}$, and it passes controllability test.

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$$\boldsymbol{det(P) = -6}$$

As determinant $\boldsymbol{P}$ is non-zero, so $\boldsymbol{rank(P) = 2}$, and it passes controllability test.

Let us proceed to design of controller now.

Solution - Generalized Steps

To design controller, the steps are as follows:

- Construct matrix $\boldsymbol{K}$ whose size is transpose the size of $\boldsymbol{B}$
- Populate matrix $\boldsymbol{K}$ with elements starting from $\boldsymbol{k_1}$, $\boldsymbol{k_2}$ and so on
- Pre-multiply $\boldsymbol{B}$ with $\boldsymbol{K}$ to obtain $\boldsymbol{BK}$, and then compute $\boldsymbol{det(sI-(A-BK))}$
- Obtain the desired characteristic equation and compare coefficients to obtain the values of $\boldsymbol{k_1}$, $\boldsymbol{k_2}$, $\boldsymbol{k_3}$ and so on

Solution of Controller Design

$$\boldsymbol{K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}}$$

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$$A - BK = \begin{bmatrix} 2 - k_1 & 3 - k_2 \\ 0 - 2k_1 & 5 - 2k_2 \end{bmatrix}$$

$$sI - (A - BK) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 - k_1 & 3 - k_2 \\ 0 - 2k_1 & 5 - 2k_2 \end{bmatrix}$$

$$sI - (A - BK) = \begin{bmatrix} s - 2 + k_1 & -3 + k_2 \\ 2k_1 & 2k_2 + s - 5 \end{bmatrix}$$

Solution of Controller Design

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Solution of Controller Design

$$sI - (A - BK) = \begin{bmatrix} s - 2 + k_1 & -3 + k_2 \\ 2k_1 & 2k_2 + s - 5 \end{bmatrix}$$

$$det(sI - (A - BK)) = s^2 + (k_1 + 2k_2 - 7)s + (k_1 - 4k_2 + 10)$$

Now lets compare it with desired characteristic equation:

$$(s + 3)(s + 5) = s^2 + 8s + 15$$

Compare coefficients to obtain values of k_1 and k_2.

MATLAB code

MATLAB code for designing state feedback controller

```
A=[2 3; 0 5 ];

B=[1; 2];

P=[B A*B]

rank(P)

desiredegn=[-3 -5];

K=place(A,B,desiredegn)
```