PROJECT: SIGNED PERMUTAHEDRA

DAVID MAURICIO ARCILA
MATTHEW CADER KIM
JULIÁN ARIEL ROMERO BARBOSA

1. Introduction

Professor W. W. a brilliant chemist graduated from CalTech is passing through very difficult economical problems. This situation has lead him to do illegal stuff and now he is working on a meth lab for Mr. G., a very dangerous person. Professor W. W. needs to complete n different procedures in n distinct periods of time of the day to create the crystal. He has developed a technique that allows him to perform each procedures independently from the others but this technique only allows him to perform one procedure per period of time. In spite of this independence, each procedure comes with a deadline that guarantees a good percentage of purity of the crystal and he needs to satisfy the minimum requirements imposed by Mr. G. or his family will be in danger. As you may notice, he desperately needs an optimal way to create the crystal.
His lab assistant Jessee, has notice that it is possible to consider the deadlines as a vector $\mathbf{D} = (d_1, ..., d_n)$ where d_i is an integer between 1 and n and the percentage of purity lost due to delays as a vector $\mathbf{P} = (p_1, ..., p_n)$ where p_i is the non-negative real percentage of purity lost due a delay on the procedure i. Luckily, Jessee notice that the function they want to minimize is

$$c(\sigma) = \sum_{i=1}^{n} p_i \cdot \max\{\sigma_i - d_i, 0\}$$

where $\sigma \in \mathcal{S}_n$ represent a possible schedule they may choose.
Professor W.W. has always seen Jessee as a poor junkie that never do anything right and although the formulation of the problem was correct, he believes that this is not the best way to state the problem due to the presence of a non-linear function on it, he thinks for a while and then came up with a brilliant reformulation:

Definition 1. *A subset of procedures $X \subset [n]$ is called realizable if there is a schedule $\sigma \in \mathcal{S}_n$ such that every procedure in X is programmed before the deadline.*

Example 1.1. *Suppose that $D = (2, 3, 5, 6, 7, 1, 1)$ then the set $X = \{1, 2, 3, 6\}$ is realizable since the schedule $\sigma = (6, 1, 2, 3, 4, 5, 7)$ performs all the processes in X on* *time. The set $Y = \{6, 7\}$ is not realizable since the deadline of both procedures is in the same period of time.*

Note that for each schedule $\sigma \in \mathcal{S}_n$ we can associate it the realizable set $X_\sigma = \{i \in [n] : \ \sigma_i \leq d_i\}$ and that if $\hat{X}$ is a realizable set (with schedule $\hat{\sigma}$) that maximizes the function

$$P(X) := \sum_{i \in X} p_i$$

over all possible realizable sets X, then, for any $\tau \in \mathcal{S}_n$

$$c(\tau) = \sum_{i=1}^{n} p_i - \sum_{i \in X_\tau} p_i \geq \sum_{i=1}^{n} p_i - \sum_{i \in \hat{X}} p_i = c(\hat{\sigma}),$$

hence obtaining an optimal for Professor W.W.'s problem. Thus, we have proved:

Proposition 1.2. *Let $\mathcal{M}$ the class of all realizable sets, then*

$$\min_{\sigma \in \mathcal{S}_n} c(\sigma) = \sum_{i=1}^{n} p_i - \max_{X \in \mathcal{M}} P(X).$$

Realizable sets have a very rich structure as is shown in the two following propositions:

Proposition 1.3. *For each subset of procedures X and every $i \in [n]$ let us define the function $f(X, i) = |\{j \in X : \ d_j \leq i\}|$ (set $f(X, 0) = 0$ for every X). Then, X is realizable if and only if $f(X, i) \leq i$ for every $i \in [n]$.*

Proof. Suppose that σ is a schedule in which every procedure in X is on time. Let j_i be the i-th procedure in X to be completed. Note that $\sigma(j_i) \geq i$, since otherwise, we could not have completed $i - 1$ other jobs in X before j_i. On the other hand, $\sigma(j_i) \leq d_i$ by the definition of σ. Hence $d_{j_i} \geq i$ and $f(X, i) \leq i$.
Now, suppose that $f(X, i) \leq i$ for every integer $i \in [n]$. If we perform the procedures in X in increasing order of deadline, then we complete all tasks in X with deadlines i or less by the period of time i. In particular, for any $j \in X$, we perform task j on or before its deadlines d_i. Thus, X is realistic. □

Proposition 1.4. *Let $\mathcal{M}$ the class of all realizable sets, then*

(1) The empty set is in $\mathcal{M}$.
(2) If $X \in \mathcal{M}$ the every subset of X also is in $\mathcal{M}$.
(3) If X and Y are realizable with $|X| > |Y|$, then there is an procedure in x in $X \setminus Y$ such that $Y \cup \{x\}$ is realizable.

Proof. Only item *(3)* is not trivial. Let $\hat{i}$ be the largest integer such that $f(X, \hat{i}) \leq f(Y, \hat{i})$. This integer must exists because $f(X, 0) = 0 = f(Y, 0)$ and $f(X, n) = |X| > |Y| = f(Y, n)$. By definition of $\hat{i}$, there are more procedures with deadline $\hat{i} + 1$ in X

than in Y. Thus, we can choose a procedure x in $X \setminus Y$ with deadline $\hat{i}+1$.
Let $i \in [n]$. If $i \leq \hat{i}$, then $f(Y \cup \{x\}, i) = f(Y, i) \leq i$. On the other hand, if $i > \hat{i}$, then $f(Y \cup \{x\}, i) = f(Y, i) + 1 \leq f(X, i) < i$ by definition of $\hat{i}$ and because X is realizable. Then, it follows that $Y \cup \{x\}$ is realizable. □

After noticing this quite interesting property of the class of realizable sets, professor W.W. saw that he was facing a similar structure he encountered once in an Combinatorial Optimization course he took at CalTech and after some research he realizes that he has proved that $\mathcal{M}$ was a **Matroid**: a class of sets that satisfies *(1), (2) and (3)* of the proposition above, that is, an abstraction of the linear independence property over a set of vectors.
A great fact about Matroids is that problems like

$$\max_{X \in \mathcal{M}} \sum_{i \in X} p_i$$

can be seen as linear optimization problems over certain polyhedra. In particular:

Proposition 1.5. *For each set of procedures X let $e_X \in \mathbb{R}^n$ be the vector defined by*

$$(e_X)_i = \begin{cases} 1 & if\ i \in X \\ 0 & if\ i \notin X \end{cases}.$$

If r is the function defined by

$$r(A) = \{|X| : \ X \in \mathcal{M}, \ X \subset A\},$$

called the rank function of $\mathcal{M}$ then

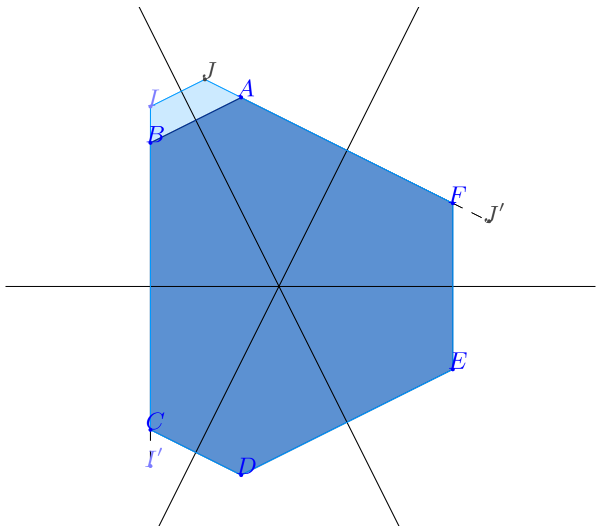
$$\max_{X \in \mathcal{M}} \sum_{i \in X} p_i = \max_{x \in P(r)} \boldsymbol{P} \cdot x$$

where

$$P(r) := \{x \in \mathbb{R}^n : \ e_X \cdot x \leq r(X), \ x_i \geq 0 \ i \in [n], \ X \subset [n]\}.$$

The polyhedron $P(r)$ is in fact the convex hull of all realizable vectors e_X and they are called Matroid Polytopes. The objective of this project is to study these polytopes and certain generalizations of them called Δ-Matroid Polytopes. These polytopes belongs to certain class of polytopes which are deformations of the permutohedron:
The *regular permutohedron* is formed by taking the convex hull of the points generated by permuting the coordinates of $(1, 2, \ldots, n)$. This is equivalent to reflecting the point about the set of hyperplanes $x_i = x_j$ for $i \neq j$. This is called the type A_n hyperplane arrangement. The *generalized permutohedron* is obtained from the *regular permutohedron* by moving each vertex so that directions of all edges are preserved, letting some of the edges degenerate into a single point. As an example of *generalized permutohedra* we can reflect an arbitrary point $(x_1, x_2, \ldots, x_n)$ about the type A_n hyperplane arrangement generating a permutohedron $P(x)$. We may notice that not all generalized permutohedra are generated in this way since if we do an appropriate

deformation of a $P(x)$ by moving one of the facets along certain direction, the orbit of the points of this translated facet may not be in the polytope as the following figure illustrates.



If $x \in \mathbb{R}^n$ is a point with $x_1 > x_2 > ... > x_n$ then there is a bijection between the faces of a permutohedron $P(x) = \{\sigma x : \ \sigma \in \mathcal{S}_n\}$ of dimension $n-k$ and ordered k-partitions of $[n]$. Given an ordered partition $S_1, ..., S_k$ of $[n]$ with $s_i = |S_i|$ we generate the face F_S as the convex hull of the vectors obtained by permuting the first s_1 coordinates of x in the S_1 positions, the next s_2 coordinates in the S_2 positions and so on. For example, let $x = (10, 8, 3, 2, 1)$, $S_1 = \{1\}$, $S_2 = \{2, 4\}$ and $S_3 = \{3, 5\}$. Then the points $(10, 8, 2, 3, 1)$, $(10, 8, 1, 3, 2)$, $(10, 3, 2, 8, 1)$, $(10, 3, 1, 8, 2)$ generate a face of dimension 2 in $P(x)$.
As a by-product of this bijection, every permutahedron $P(x)$ with the aforementioned condition has dimension $n-1$ and its facets are generated by 2-partitions of $[n]$ so that a facet description of it will be

$$P(x) = \{y \in \mathbb{R}^n : \sum_{i\in[n]} y_i = \sum_{i\in[n]} x_i; \ \sum_{i\in I} y_i \leq \sum_{i=1}^{|I|} x_i, \ I \subset [n]\}.$$

A very interesting fact is that every generalized permutahedron has a similar facet description:

Proposition 1.6. *Let P a generalized permutahedron (of type A). There exist coefficients z_I with $I \subset [n]$ such that*

$$P = \{y \in \mathbb{R}^n : \sum_{i\in[n]} y_i = z_{[n]}; \ \sum_{i\in I} y_i \leq z_I, \ I \subset [n]\}.$$

A question that naturally arises after we see this result is when does the reverse description holds? With this we mean, given a sequence of coefficients $\{z_I\}_{I\subset[n]}$, is the polytope

$$P(\{z_I\}) = \{y \in \mathbb{R}^n : \sum_{i\in[n]} y_i = z_{[n]};\ \sum_{i \in I} y_i \leq z_I,\ I \subset [n]\}$$

a generalized permutahedron? The answer lies in the following proposition:

Proposition 1.7. *$P(\{z_I\})$ is a generalized permutahedron if and only if the coefficients $\{z_I\}_{I\subset[n]}$ satisfy the supermodular property*

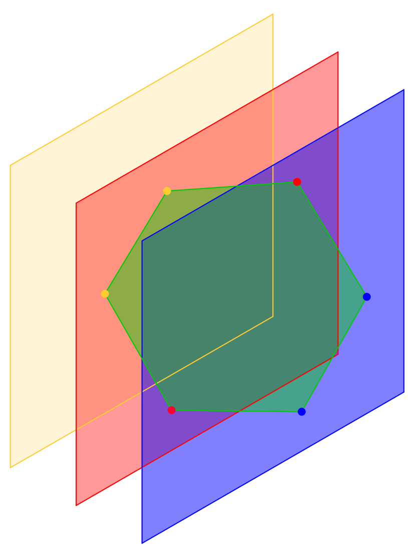
$$z_I + z_J \leq z_{I\cap J} + z_{I\cup J}.$$

Thus, there's a bijection between **submodular functions** (set functions $f : \mathcal{P}([n]) \to \mathbb{R}$ that satisfy the reverse inequality in Proposition 1.2) and generalized permutahedra.

When we add to our hyperplane arrangement the relations $x_i = 0$ and $x_i = -x_j$ for all $i \neq j$ and reflect the point $(1, 2, \ldots, n)$ across the hyperplanes of this larger set, we get the regular signed permutohedron, whose vertices are signed permutations of $(1, 2, \ldots, n)$. This larger hyperplane arrangement is the type BC_n arrangement. Again, allowing deformations preserving edge directions, we obtain a generalized signed permutohedron, and taking the orbit of any arbitrary point $(x_1, x_2, \ldots, x_n)$ about the BC_n hyperplane arrangement, we obtain examples of generalized signed permutohedra. Many results have been generated about the space of type A orbit polytopes. We would like to attempt to achieve similar results about the space of type BC orbit polytopes, and also the set of generalized signed permutohedra, or $\mathcal{BC}^n$.

2. Properties of the Permutohedron

2.1. Facial Structure of the permutohedron.



We are going to describe the facial structure of the permutohedron $P(a)$ where $a = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1 > \dots > \alpha_n$ as it is done in [5].

Lemma 2.1. *Let $P = conv(v_1, \dots, v_m) \subset \mathbb{R}^d$ be a polytope and let $F \subset P$ be a face. Then $F = conv(v_i : v_i \in F)$.*

Proof. See [5] $\square$

Lemma 2.2. *Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Suppose that $x_i > x_j$ and $y_i < y_j$ for some pair of indices $i \neq j$. Let $\overline{y}$ be the vector obtained from y by swapping y_i and y_j. Then*

$$\langle x, \overline{y} \rangle > \langle x, y \rangle$$

Proof. We have

$$\langle x, \overline{y} \rangle - \langle x, y \rangle = x_i y_j + x_j y_i - x_i y_i - x_j y_j = (x_i - x_j)(y_j - y_i) > 0$$ $\square$

Proposition 2.3. *Let $a = (\alpha_1, \dots, \alpha_n)$ be a point such that $\alpha_1 > \dots > \alpha_n$ and let $P = P(a)$ be it's orbit polytope. For a number $1 \leq k \leq n$, let $\mathcal{S}$ be a partition of $[n]$ into k pairwise disjoint non-empty subsets $S_1, \dots, S_k$. Let $s_i = |S_i|$ for $i = 1, \dots, k$, let $t_i = \sum_{j=1}^{i} s_j$ for $i = 1, \dots, k$ and let us define sets $A_1 = \{\alpha_j : 1 \leq j \leq s_1\}$ and $A_i = \{\alpha_j : t_{i-1} < j \leq t_i\}$ for $i = 2, \dots, k$.*

Let $F_{\mathcal{S}}$ be the convex hull of the points $b = \sigma(a), b = (\beta_1, \dots, \beta_n)$ such that $\{\beta_j : j \in S_i\} = A_i$ for all $i = 1, \dots, k$, that is permuting the first s_1 biggest numbers in the positions given by S_1, the next s_2 biggest numbers in the positions given by S_2 and so forth.

Then $F_{\mathcal{S}}$ is a face of P, dim $F_{\mathcal{S}} = n - k$ and for every face F of P we have $F = F_{\mathcal{S}}$ for some partition $\mathcal{S}$.

Proof. Lets describe first all the faces F of P containing a.

Let $c = (\gamma_1, \dots, \gamma_n)$ be a vector and λ be a number such that $\langle c, x \rangle \leq \lambda$ for all $x \in P$ and $\langle c, x \rangle = \lambda$ if and only if $x \in F$. Since $a \in F$, we have $\langle c, a \rangle = \lambda$. Lemma 2.2 implies that we must have $\gamma_1 \geq \dots \geq \gamma_n$, since if for some $i < j$ we had $\gamma_i < \gamma_j$, we would have obtainded $\langle c, \tau(a) \rangle > \langle c, a \rangle$ for the transposition τ that swaps α_i and α_j.

Let us split the sequence $\gamma_1 \geq \dots \geq \gamma_n$ into the subintervals $S_1, \dots, S_k$ for which de γ's do not change. Hence $S_1 = \{j : \gamma_j = \gamma_1\}$, $s_1 = t_1 = |S_1|$ and $S_i = \{j : \gamma_j = \gamma_{t_{i-1}+1}\}$, $s_i = |S_i|$ and $t_i = t_{i-1} + s_i$ for $i = 2, \dots, k$.

We observe that for $b = \sigma(a) = (\beta_1, \dots, \beta_n)$, we have $\langle b, c \rangle = \langle a, c \rangle$ if and only if $(\beta_1, \dots, \beta_{t_1})$ is a permutation of $(\alpha_1, \dots, \alpha_{t_1})$, $(\beta_{t_1} + 1, \dots, \beta_{t_2})$ is a permutation of $(\alpha_{t_1+1}, \dots, \alpha_2)$, and so forth. Applying Lemma 2.1, we conclude that $F = F_{\mathcal{S}}$ for the partition $\mathcal{S} = \{S_1, \dots, S_k\}$.

Vice versa, every vector $c = (\gamma_1, \dots, \gamma_n)$ with $\gamma_1 \geq \cdots \geq \gamma_n$ gives rise to a face $F_{\mathcal{S}}$ containing a, where $\mathcal{S} = \{S_1, \dots, S_k\}$ is the partition of $[n]$ into the subintervals on which γ's do not change.

Let σ be a permutation such that $\sigma(x) = y$, the action σ over $\mathbb{R}^n$ is an orthogonal transformation. As P is fixed by any permutation, then F is a face of P if and only if for some permutation σ, the set $\sigma(F)$ is a face of P containing a. If $\sigma(F)$ is $F_{\mathcal{S}}$ for the partition $\mathcal{S} = \{S_1, \dots, S_k\}$, then $F = F_{\mathcal{S}'}$ for $\mathcal{S}' = \{\sigma^{-1}(S_1), \dots, \sigma^{-1}(S_k)\}$.

Let $a_1 = (\alpha_1, \dots, \alpha_{s_1}) \in \mathbb{R}^{s_1}$, and let $a_i = (\alpha_{t_{i-1}+1}, \dots, \alpha_{s_i}) \in \mathbb{R}^{s_i}$ for $i = 2, \dots, k$. Geometrically, the face $F_{\mathcal{S}}$ is the direct product

$$F_{\mathcal{S}} = P(a_1) \times \cdots \times P(a_k)$$

of the permutation polytopes $P(a_i) \subset \mathbb{R}^{s_i}$, since all of them have different coordinates, then each $P(a_i)$has dimension $s_i - 1$. Therefore,

$$dimF_{\mathcal{S}} = \sum_{i=1}^{k} dimP(a_i) = \sum_{i=1}^{k} s_i - 1 = n - k$$

$\square$

Proposition 2.4. *The permutohedron $\prod_{n-1} = P(a)$ where $a = (n, n-1, \dots, 1)$ equals the minkowski sum*

$$\frac{n+1}{2}(1, \cdots, 1) + \sum_{i>j} [-\frac{e_i - e_j}{2}, \frac{e_i - e_j}{2}]$$

Proof. As proven in [6].

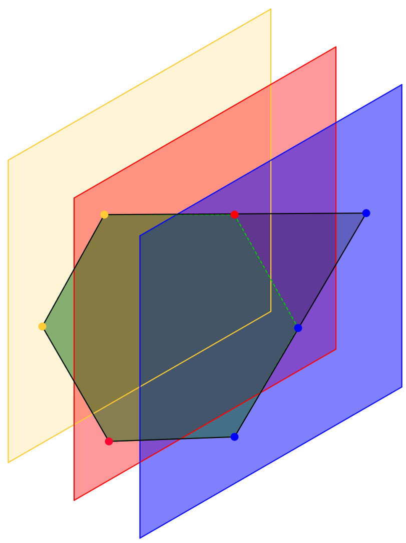
First see that that minkowski sum is invariant under the action of a transposition of the coordinates, the segments may change their direction but it is not important because each segment stays the same as a set, so is invariant under any permutation, now lets compute the points of the sum that maximize a linear functional c with $\gamma_1 \geq \cdots \geq \gamma_n$. A point maximizes the linear functional if it maximizes it in each segment of the minkowski sum, and in the segment $[-\frac{e_i - e_j}{2}, \frac{e_i - e_j}{2}]$ with $i > j$, the one that maximizes it is $\frac{e_i - e_j}{2}$, and adding each one of these points we get

$$\frac{n+1}{2}(1, \cdots, 1) + \sum_{i>j} \frac{e_i - e_j}{2}$$

$$= \frac{n+1}{2}(1, \cdots, 1) + \frac{n-1}{2}e_1 + \frac{n-2}{2}e_1 + \cdots + \frac{1}{2}e_{n-1} = (n, n-1, \dots, 1)$$

That means it is a point of the resulting polytope. If the order of the coordinates of c is different, it can be set in the right ordering applying the corresponding permutation, so the point maximizing it will be the preimage of $(n, n-1, \ldots, 1)$ in the permutation, and each point of the permutohedron maximize a linear functional whose coordinates satisfies the same ordering as it. If c has two equal coordinates, $\gamma_i = \gamma_j$ then all the segment $[-\frac{e_i - e_j}{2}, \frac{e_i - e_j}{2}]$ would maximize the linear functional, not giving a vertex, that means the vertices of the resulting polytope are the same as $\prod_{n-1}$. $\square$

3. Submodular Functions and Generalized Permutohedra



Definition 2. Generalized Permutohedra *are deformations of the usual permutohedron that preserve the edge directions. These are obtained by moving the facets of the usual permutohedron in a direction normal to their defining hyperplanes.*

An equivalent classification of generalized permutohedra are all polyhedra whose normal fan is refined by the Braid arrangement,

The Braid arrangement, denoted $\mathcal{B}_n$ is simply the type A_n hyperplane arrangement,

$$x_i = x_j \text{ for all } i \neq j$$

Each region created by the hyperplane arrangement corresponds to a permutation $\pi \in S_n$. They are in fact cones generated by $\{e_{I_1}, e_{I_2}, \ldots, e_{I_n}$ such that $I_1 \subsetneq \cdots \subsetneq I_n$ is a complete flag of sets where $I_j = I_{j-1} \cup \pi(j)$. If fact, when viewed as a fan in $\mathbb{R}_n^*$, the braid arrangement is exactly the normal fan of the usual permutohedron. Every face of $\mathcal{B}$ corresponds to an ordered composition $I = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_k$. Borrowing notation from [2] we denote by $\mathcal{B}_{S_1,\ldots S_k}$ the face of the braid arrangement containing all $\gamma \in \mathbb{R}_n *$ such that $y_i = y_j$ if $i, j \in S_a$ and the value of these equivalency classes are ordered according to the order on the subsets.

This relationship between the permutohedron and the Braid arrangement gives us the following theorem which will come into play when talking about generalized permutohedron.

For all $J \subset I$ we will write $e_J = \sum_{i \in J} e_i \in (\mathbb{R}^I)^*$. This is simply the characteristic vector of the set J.

Lemma 3.1. *If $A, B \subset I$ are comparable, then* $\operatorname{Cone}(e_A, e_B)$ *is contained in a face of $\mathcal{B}_I$. Equivalently, A, B determine a face of Π_I.*

Proof. WLOG assume that $A \subsetneq B$. Then $I = A \sqcup (B \setminus A) \sqcup (I \setminus B)$. This determines a $|I| - 3$ dimensional face of the permutehedron $Q \subset \Pi_I$. (if $B = I$, then this is a $|I| - 2$ dimensional face.) Therefore there must be a corresponding face $F \subset \mathcal{B}$ containing linear functionals maximizing Q. By definition, e_A maximizes on Q, as well as $e_{A \cup B \setminus A} = e_B$. This gives us the correct dimension and $F = \operatorname{Cone}(e_A, e_B, I)$, which contains $\operatorname{Cone}(e_A, e_B)$. □

Since each facet of the permutohedron corresponds to a non-empty subset of the I, the generalized permutohedra can be parametrized by a boolean function $z : 2^I \to \mathbb{R}$. In [3] this is referred to as the deformation cone $\mathcal{D}_n$. In [2] it is shown that these boolean functions are in fact submodular. And adapted proof is reproduced here. We start with some basic definitions.

Definition 3. *A* **boolean function** *is a function from the subsets of I to $\mathbb{R}$.*

They can equivalently be thought of as function from the poset B_n, the boolean lattice, to $\mathbb{R}$. A **submodular** function is a boolean function z such that

$$z(A \cap B) + z(A \cup B) \leq z(A) + z(B)$$

The **Base Polytope** of a boolean function z on 2^I is defined as

$$\mathcal{P}(z) := \{x \in \mathbb{R}^I \mid \sum_{i \in I} x_i = z(I) \text{ and } \sum_{i \in A} x_i \leq z(A) \text{ for all } A \subset I\}$$

Note that the submodular function on 2^I defining the regular permutoheron is simply the function

$$z(A) = |I| + (|I| - 1) + \cdots + (|I| - |A| + 1)$$

which gives us the hyperplane description of Π_I. Indeed any boolean function where $z(A) = z(B)$ if $|A| = |B|$ will give us the usual permutohedron.

Conjecture 3.2. *If F is a k dimensional face of $\mathcal{P}(z)$ where z is submodular, then k is contained in the intersection of $n - k$ defining hyperplanes $\bigcap_{J_i \in \mathcal{J}} H_{J_i}$ such that all the sets in $\mathcal{J}$ are comparable.*

We being a proof of this in the following theorem, however the reader may note that this is still a conjecture due to a lack of procedure that given any family of sets

$\mathcal{J}$ that define a k face, we can replace it with a family of comparable subsets of I without introducing linear dependencies at some step.

Theorem 3.3. *If z is submodular, then $\mathcal{P}(z)$ is a generalized permutohedron. Also, if P is as generalized permutohedron, then there is a unique submodular function z such that $P = \mathcal{P}(z)$*

Proof. To show that given a submodular function z, it's base polytope $\mathcal{P}(z)$ is a generalized permutohedron, we first show that any edge of $\mathcal{P}(z)$ has the direction $e_i - e_j$ for some pair (i, j). From this it follows that its facets are parallel to the corresponding facet of $\prod_{n-1}$

We will denote by $\mathcal{P}(z)_J$ the face of $\mathcal{P}(z)$ that is maximized by the linear functional e_J.

Now suppose that E is a one dimensional face of $\mathcal{P}(z)$ contained in the line L. L must be the intersection of $n-1$ defining hyperplanes of $\mathcal{P}(z)$. One of these hyperplanes is the ambient hyperplane $e_I(x) = z(I)$ of the base polyhedra. The other hyperplanes correspond to subsets of I. Let $\mathcal{J} = \{J_1, J_2, \dots J_{n-1}\}$ be the family of sets such that

$$L = \bigcap_{J_i \in \mathcal{J}} H_{J_1}$$

.

Suppose that J_1 and J_2 are incomparable, i.e. $J_1 \not\subset J_2$ and $J_2 \not\subset J_1$. Note that in the usual permutohedron, such faces would not intersect at all. Two facets are only adjacent if one is contained in the other. Therefore this polyhedron must be a deformation where some intermediary hyperplane has been pushed out or in. To observe this, take $v \in L$. Since $v \in \mathcal{P}(z)$, we have that

$$z(J_1 \cap J_2) + z(J_1 \cup J_2) \geq e_{J_1 \cap J_2} v + e_{J_1 \cup J_2} v = e_{J_1} v + e_{J_2} v = z(J_1) + z(J_2)$$

where the last equality come from the fact that $v \in \mathcal{P}(z)_{J_1} \cap \mathcal{P}(z)_{J_2}$. But since z is submodular, we also know that $z(J_1) + z(J_2) \geq z(J_1 \cap J_2) + z(J_1 \cup J_2)$ implying that $z(J_1) + z(J_2) = z(J_1 \cap J_2) + z(J_1 \cup J_2)$. Therefore $v \in \mathcal{P}(z)_{J_1 \cap J_2} \cap \mathcal{P}(z)_{J_1 \cup J_2}$.

Therefore v, and consequently, L is on all of the associated hyperplanes.

If $J_2 = I \setminus J_1$, (they are the set complements of each other) then the corresponding faces would be parallel in the base polyhedron, and would not intersect. Therefore for all incomparable pairs J_1, J_2, either $J_1 \cup J_2 \neq I$ or $J_1 \cap J_2 \neq \emptyset$.

If both $J_1 \cup J_2$ or $J_1 \cap J_2$ were already in $\mathcal{J}$ be as this would imply linear dependencies among the e_{J_i}. However it may be that one of them is already in $\mathcal{J}$. In this case we replace J_1 with whichever is not already in the set. We do not introduce linear dependencies since if $e_{J_1 \cap J_2}$ or $e_{J_1 \cup J_2}$ was in the span of the other e_{J_i} this would mean that there already was a linear dependence.

We conjectured in 3.2 that we can replace $\mathcal{J}$ with a family of $n-1$ comparable sets. Since there are no repetitions, this implies we have an almost complete flag of of $n-1$ sets, defining L:

$$I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{n-1} = I$$

meaining that at some point,$|I_i - I_{i-1}| = 2$. At all other steps, we're only adding one element. Then if $\{j,k\} = I_i - I_{i-1}$, the intersection will be the face where all of the coordinates in $I \setminus \{j,k\}$ have been fixed, but where x_j, x_k are variables. Thus the edge is parallel to the vector $e_i - e_j$.

Consider F a face of $\mathcal{B}_I$. Let $\lambda \in F$ and let $v \in V(\mathcal{P}(z))$ (if $\mathcal{P}(z)$ does not have vertices we first mod out the lineality space.) We know that all edges connected to v have the direction $e_i - e_j$. We orient adjacent edges towards v, that is an adjacent edge E has direction $e_i - e_j$ if $v_i > v_j$. From this it follows that v is λ maximal only if $\lambda \cdot e_i - e_j > 0$ or $\lambda_i > \lambda_j$. However, since the faces of $\mathcal{B}_n$ correspond to weak orderings on coordinates, if λ maximizes v then all $\gamma \in F$ do so as well. If a face $Q \subset \mathcal{P}(z)$ is λ-maximal, then so is $V(Q)$ and F is contained in the face of $\mathcal{N}(\mathcal{P}(z))$ maximizing Q. This shows that the braid arrangment refines $\mathcal{N}(\mathcal{P}(z))$ and that $\mathcal{P}(z)$ is a generalized permutohedron.

Now we consider P a generalized permutohedron. Since we already know that P is the base polytope $\mathcal{P}(z)$ of a boolean function $z : 2^I \to \mathbb{R}$, and that $\mathcal{N}(\mathcal{P}(z))$ is a subfan of $\mathcal{B}_n$ (cite[HopfMonoid] we wish to show that z is in fact submodular. Let $S := \operatorname{supp} \mathcal{N}(\mathcal{P}(z))$. If $e_A \notin S$, then $z(A) = \infty$ and the generalized permutohedron is unbounded in the direction e_A.

Assume that $e_A, e_B \in S$. Then we have that $z(A)$ and $z(B)$ are finite and $e_A, e_B \in S$. But this implies that $e_{A\cup B}$ and $e_{A\cap B}$ are supported by S. Since $A \cup B$ and $A \cap B$ are comparable, $\operatorname{Cone}(e_{A\cup B}, e_{A\cap B})$ is in a face of $\mathcal{B}$ and so is also contained in a face of the subfan $\mathcal{N}(\mathcal{P}(z))$. This implies that there is a face $Q \subset \mathcal{P}(z)$ maximized by $\operatorname{Cone}(e_{A\cup B}, e_{A\cap B})$, or that for $v \in Q$

$$z(A\cap B) + z(A \cup B) = e_{A\cap B}v + e_{A\cup B}v = e_A v + e_B v \leq z(A) + z(B)$$

and that z is submodular.

□

Example 3.4. *Let $P(z)$ be the generalized permutohedron generated by moving the facet defining hyperplanes associated to the subsets $I_1 = \{1,2,3\}$ and $I_2 = \{1,2,3,5\}$ of the standard permutohedron $P(a)$ with $a = (5,4,3,2,1) \in \mathbb{R}^5$ to the hyperplanes*

$$H_{I_1} = \{x \in \mathbb{R}^5 : \ e_{I_1}x = 13 = z(I_1)\},$$
$$H_{I_2} = \{x \in \mathbb{R}^5 : \ e_{I_2}x = 15 = z(I_2)\}.$$

Note that the points $p_1 = (4,5,4,0,2)$ and $p_2 = (4,5,4,1,1)$ lie in the intersection

$$P(z)_{J_1} \cap P(z)_{J_2} \cap P(z)_I = \{(4,5,4,a,2-a) : \ a \in (0,1)\} := D$$

where $J_1 = \{1,2\}$ and $J_2 = \{2,3\}$. Note that p_1 lies in the intersections of the hyperplanes H_{I_1} and H_{I_2} with $P(z)$ so that z must be a submodular function. Moreover, we have that $p_3 = (5,4,4,0,2) \in P(z)_{J_1} \cap P(z)_{I_1}$ with $I_1 = J_1 \cup J_2$ but $p_3 \notin P(z)_{J_2}$. In conclusion, we can't replace the intersection $P(z)_{J_1} \cap P(z)_{J_2}$ by $P(z)_{J_1} \cap P(z)_{J_1 \cup J_2}$, however

$$D = P(z)_{J_1 \cap J_2} \cap P(z)_{J_2} \cap P(z)_{J_1 \cup J_2} \cap P(z)_I$$

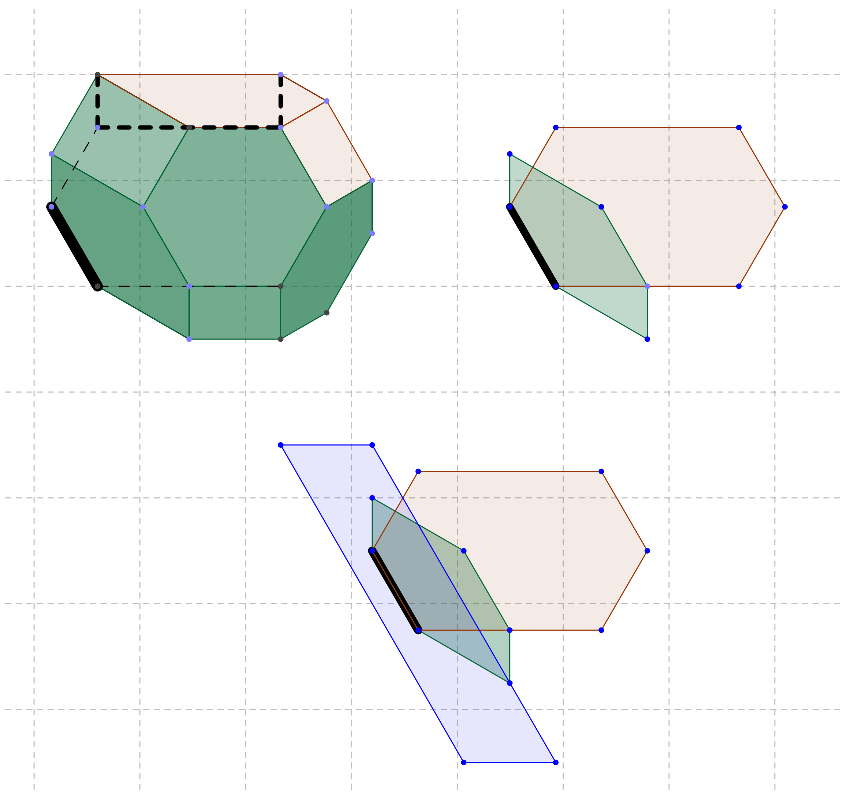


Figure. On the left we see the face $F = P(z)_{\{2\}} \cap P(z)_{\{1,2,3,4,5\}}$. On the right the faces $F \cap P(z)_{1,2}$ and $P(z)_{2,3}$. In the center we see the intersection of a projection of the hyperplane H_{I_1} with F.

4. Signed Permutohedron

Now lets consider orbit polytopes under the hyperplane arrangement BC_n

The type BC_n hyperplane arrangement is the root system obtained by adding few more planes to the type A reflection group:

Definition 4. *The type BC_n hyperplane arrangement in $\mathbb{R}^n$ is the set of all hyperplanes of the form*

$$x_i = x_j, \quad x_i = -x_j, \quad x_k = 0, \text{ for all } i \neq j$$

We may sometimes view this as fan in $\mathbb{R}_n^*$ of linear functionals. In particular, as in the case of type A permutohedra, this is equivalent to the normal fan of type BC permutohedra. The rays of this fan are precisely all the vectors in $\{-1, 0, 1\}^n$. Each non-zero ray corresponds to a facets of the BC orbit polytope, hence there are $3^n - 1$ facets.

Definition 5. *A* **Signed subset** *S of I is a pair of disjoint sets $S_+, S_- \subset I$. We define $|S| = |S_+| + |S_-|$. S is a proper signed subset if $|S| > 0$. Let $\mathcal{S}(I)$ denote the set of signed subsets of I. For two signed subsets S and T we say that $S \sqsubseteq T$ if $S_+ \subset T_+$ and $S_- \subset T_-$.*

We say $a \in S$ if $a \in S_-$ or $a \in S_+$.

We say $S = \emptyset$ if both $S_+ = \emptyset$ and $S_- = \emptyset$.

For S a signed subset we define $e_S = e_{S_+} - e_{S_-}$.

This might seem superfluous at the moment, however since there is a natural correspondence between signed subsets and facets of the type BC permutohedra, we utilize this notation for brevity.

4.1. **Facial structure of the signed permutohedron.**

Proposition 4.1. *Let $a = (\alpha_1, \dots, \alpha_n)$ be a point such that $\alpha_1 > \cdots > \alpha_n > 0$ and let $P = P_{BC}(a)$ be it's orbit polytope over the hyperplane arrangement BC_n. There is a bijection between the k dimensional faces and signed partitions of $[n]$ of the form $S_1, \dots, S_k, S_{k+1}$ where the first the first k parts are nonempty and $S_{k+1\,-} = \emptyset$, for $k = 1, \dots, n$.*

Proof. For a number $1 \leq k \leq n$, let $\mathcal{S}$ be a signed partition of $[n]$ into $k+1$ signed subsets $S_1, \dots, S_k, S_{k+1}$ with S_i non empty for $i = 1, \dots, k$, S_{k+1} possibly empty and $S_{k+1\,-} = \emptyset$. Let $s_i = |S_i|$ for $i = 1, \dots, k+1$, let $t_i = \sum_{j=1}^{i} s_j$ for $i = 1, \dots, k+1$ and let us define sets $A_1 = \{|\alpha_j| : 1 \leq j \leq s_1\}$ and $A_i = \{|\alpha_j| : t_{i-1} < j \leq t_i\}$ for $i = 2, \dots, k+1$.

Let $F_{\mathcal{S}}$ be the convex hull of the points $b = \sigma(a), b = (\beta_1, \dots, \beta_n)$ where σ is a signed permutation, such that $\{|\beta_j| : j \in S_i\} = A_i$ for all $i = 1, \dots, k+1$, $\beta_j > 0$ for all $j \in S_{i+}$ for $i = 1, \dots, k$ and $\beta_j < 0$ for all $j \in S_{i-}$ for $i = 1, \dots, k$, that is permuting the first s_1 biggest numbers in the positions given by S_1, and the ones in the positions S_{1+} being positive and the ones in the positions S_{1-} being negative, the next s_2 biggest numbers in the positions given by S_2, the ones in positions S_{2+} being positive and the ones in positions S_{2-} being negative, and so forth, and the last ones in positions given by S_{k+1} with any sign.

Then $F_{\mathcal{S}}$ is a face of P and for every face F of P we have $F = F_{\mathcal{S}}$ for some partition $\mathcal{S}$.

Lets describe first all the faces F of P containing a.

Let $c = (\gamma_1, \ldots, \gamma_n)$ be a vector and λ be a number such that $\langle c, x\rangle \leq \lambda$ for all $x \in P$ and $\langle c, x\rangle = \lambda$ if and only if $x \in F$. As $a \in F$, $\langle c, a\rangle = \lambda$. Lemma 2.2 implies that we must have $\gamma_1 \geq \cdots \geq \gamma_n \geq 0$, since if for some $i < j$ we had $\gamma_i < \gamma_j$, we would have obtainded $\langle c, \tau(a)\rangle > \langle c, a\rangle$ for the transposition τ that swaps α_i and α_j, and if for some i, $\gamma_i < 0$, then changing the sign of the i-th coordinate of a would increase the value of the functional.

Let us split the sequence $\gamma_1 \geq \cdots \geq \gamma_n \geq 0$ into the subintervals $S_1, \ldots, S_k, S_{k+1}$ for which de γ's do not change, leaving the ones that are equal to zero in the last set. Hence $S_1 = \{j : \gamma_j = \gamma_1\}$, $s_1 = t_1 = |S_1|$ and $S_i = \{j : \gamma_j = \gamma_{t_{i-1}+1}\}$, $s_i = |S_i|$ and $t_i = t_{i-1} + s_i$ for $i = 2, \ldots, k$, $S_{k+1} = \{j : \gamma_j = 0\}$, $s_{k+1} = |S_{k+1}|$

We observe that for $b = \sigma(a) = (\beta_1, \ldots, \beta_n)$, we have $\langle b, c\rangle = \langle a, c\rangle$ if and only if $\{\beta_i : i \in S_j\} = A_j$ for $j = 1, \ldots, k$. Applying Lemma 2.1, we conclude that $F = F_{\mathcal{S}}$ for the signed partition $S_1, \ldots, S_k, S_{k+1}$ identifying each subset with its signed subset where the $S_{i-} = \emptyset$, then we have $S_{i+} \neq \emptyset$ for $i = 1, \ldots k$.

Lets see the dimension of this face. Let $a_1 = (\alpha_1, \ldots, \alpha_{s_1}) \in \mathbb{R}^{s_1}$, and let $a_i = (\alpha_{t_{i-1}+1}, \ldots, \alpha_{s_i}) \in \mathbb{R}^{s_i}$ for $i = 2, \ldots, k+1$. Geometrically, the face $F_{\mathcal{S}}$ is the direct product

$$F_{\mathcal{S}} = P(a_1) \times \cdots \times P(a_k) \times P_{BC}(a_{k+1})$$

where $P(a)$ means the usual permutation polytopes discussed before, $P(a_i) \subset \mathbb{R}^{s_i}$, and $P_{BC}(a_{k+1}) \subset \mathbb{R}^{s_{k+1}}$, the orbit polytope over the type BC_n hyperplane arrangement. Since all of them have different coordinates, then each $P(a_i)$ has dimension $s_i - 1$, and $P_{BC}(a_{k+1})$ is full dimensional so it is of dimension s_{k+1}. Therefore,

$$dim F_{\mathcal{S}} = dim P_{BC}(a_{k+1}) + \sum_{i=1}^{k} dim P(a_i) = s_{k+1} + \sum_{i=1}^{k} s_i - 1 = n - k$$

Now lets see what happens with an arbitrary face. Let σ be a signed permutation such that $\sigma(x) = y$, σ is an orthogonal transformation over $\mathbb{R}^n$. As P is fixed by any signed permutation, then F is a face of P if and only if for some signed permutation σ, the set $\sigma(F)$ is a face of P containing a. If $\sigma(F)$ is $F_{\mathcal{S}}$ for the signed partition $S_1, \ldots, S_k, S_{k+1}$ where all the $S_i \neq \emptyset$ for $i = 1, \ldots k$ and $S_{i-} = \emptyset$ for $i = 1, \ldots k+1$, then $F = F_{\mathcal{S}'}$ for $\mathcal{S}'$ the signed partition $T_1, \ldots, T_k, T_{k+1}$ where $T_{i+} \cup T_{i-} = \{|j| : j \in \sigma^{-1}(S_{i+})\}$, T_{i-} has the elements that σ changes sign, and T_{i+} the others, for $i = 1, \ldots, k$, and $T_{k+1+} = \{|j| : j \in \sigma^{-1}(S_{k+1+})\}$.

$\square$

Proposition 4.2. *The signed permutohedron $P_{BC}(a)$ where $a = (n, n-1, \dots, 1)$ equals the minkowski sum*

$$\sum_{i=1}^{n}[-e_i, e_i] + \sum_{i>j}[-\frac{e_i - e_j}{2}, \frac{e_i - e_j}{2}] + [-\frac{e_i + e_j}{2}, \frac{e_i + e_j}{2}]$$

Proof. Observe that the minkowski sum is invariant under the action of any signed permutation, that is invariant under transposition, because if we take the transposition that swaps k and l, $[-e_k, e_k]$ becomes $[-e_l, e_l]$ and vice versa, the ones of type $[-\frac{e_i-e_j}{2}, \frac{e_i-e_j}{2}]$ give all of the same type possibly changing the order of the endpoints, and the ones of type $[-\frac{e_i+e_j}{2}, \frac{e_i+e_j}{2}]$ give all of the same type, and invariant under a change of signs, say k, $[-e_k, e_k]$ becomes $[e_k, -e_k]$ that is the same, $[-\frac{e_i-e_k}{2}, \frac{e_i-e_k}{2}]$ becomes $[-\frac{e_i+e_k}{2}, \frac{e_i+e_k}{2}]$, changes it's type, but the one that was $[-\frac{e_i+e_k}{2}, \frac{e_i+e_k}{2}]$ becomes $[-\frac{e_i-e_k}{2}, \frac{e_i-e_k}{2}]$.

Now lets see which points of the minkowski sum maximizes a linear functional $c = (\gamma_1, \dots, \gamma_n)$ with $\gamma_1 > \dots > \gamma_n > 0$. If $j = 1$, in the segment $[-e_1, e_1]$, e_1 maximizes the linear functional, in the segment $[-\frac{e_i-e_1}{2}, \frac{e_i-e_1}{2}]$, $-\frac{e_i-e_1}{2}$ maximizes the linear functional, and in the segment$[-\frac{e_i+e_1}{2}, \frac{e_i+e_1}{2}]$, $\frac{e_i+e_1}{2}$ maximizes the linear functional, then from the segments that contains e_1, we get the point $e_1 + \sum_{i=2}^{n} -\frac{e_i-e_1}{2} + \frac{e_i+e_1}{2} = ne_1$, the segments that contains e_2 but not e_1 would give that the point maximizing the linear functional in their sum is $e_2 + \sum_{i=3}^{n} -\frac{e_i-e_2}{2} + \frac{e_i+e_2}{2} = (n-1)e_2$, and so forth, then the point maximizing the linear functional c is $(n, n-1, \dots, 1)$. If there is a coordinate of c equal to zero, say $\gamma_i = 0$, then the whole segment $[-e_i, e_i]$ would maximize the functional, then the result would not be a vertex of polytope, as if there are two equal coordinates $\gamma_i = \gamma_j$, then the whole segment $[-\frac{e_i-e_1}{2}, \frac{e_i-e_1}{2}]$ would maximize it. Then there exists a signed permutation σ such that $\sigma(c)$ has positive coordinates and ordered in decreasing order, then the linear functional $\sigma(c)$ is maximized by $(n, n-1, \dots, 1)$, then c is maximized by $(\sigma^{-1}(n), \sigma^{-1}(n-1), \dots, \sigma^{-1}(1))$ that is a vertex of $P_{BC}(a)$.

Vice versa each vertex of $P_{BC}(a)$ is $\sigma(a)$ for a signed permutation, let c be a linear functional maximized by a, then $\sigma(c)$ is maximized by $\sigma(a)$, then the vertices of the signed permutahedron are the same as the vertices of the given minkowski sum. □

5. Bisubmodular Functions and Generalized Signed Permutohedra

We will now examine generalized type BC generalized permutohedra. As before, since signed subsets of I correspond to the facets of a type BC_I generalized permutohedra, we wish to characterize the maps from these signed subsets to $\mathbb{R}$ that give rise to proper generalized BC permutohedra.

Definition 6. Generalized type BC permutohedra *are defined in the same way as generalized type A permutohedra: they are all the deformations of the usual BC*

orbit polytope that preserves edge directions. Equivalently, they are polytopes whose normal fan is refined by the BC_n hyperplane arrangement.

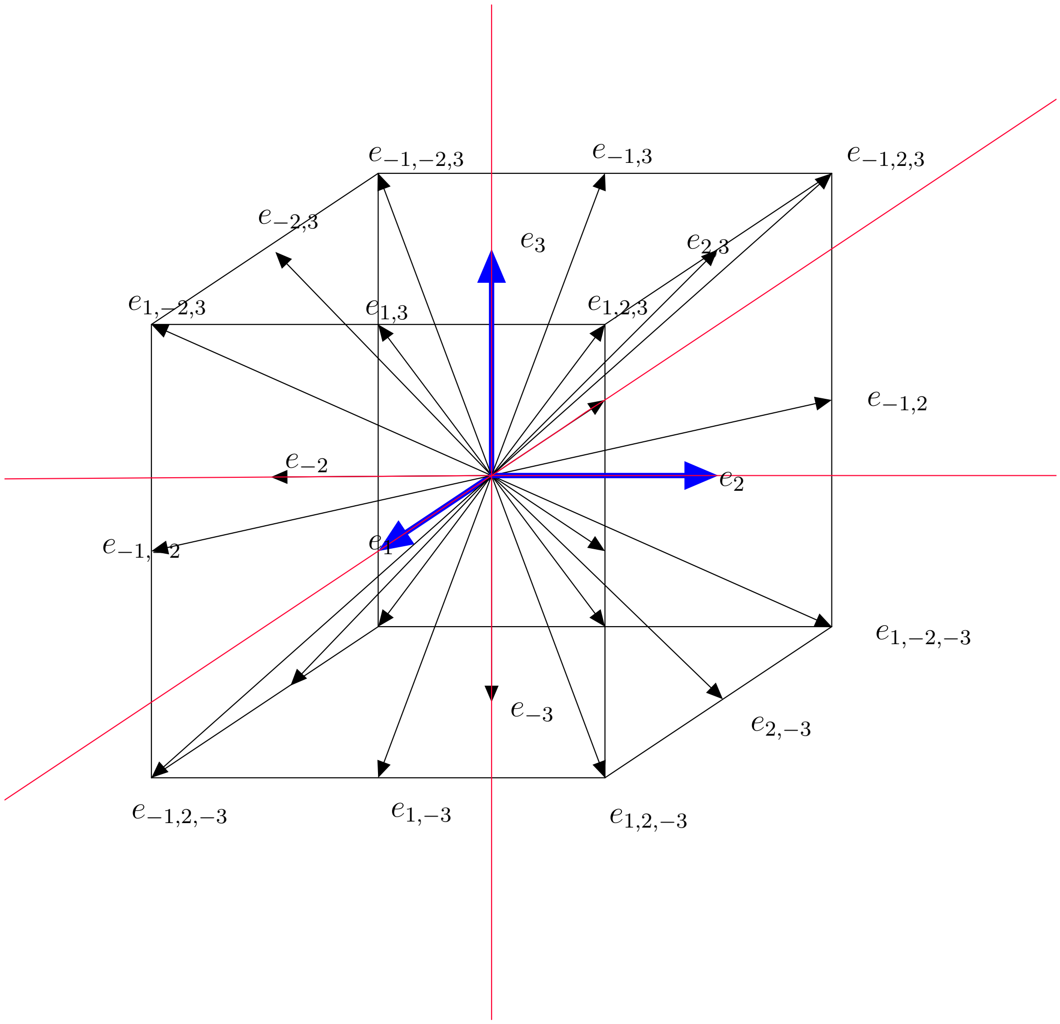


FIGURE 1. The rays generated by the BC_3 hyperplane arrangement. Each ray is labeled using subset notation

Just like we use submodular boolean functions to parametrize generalized type A permutohedra, we will show that bisubmodular functions parametrize the generalized BC polytopes.

We need to now define some operations on signed subsets that are analogous to intersection and union:

Definition 7. *The* **restricted union** *of two signed subsets S and T denoted $S \vee T = U$ is given by*

$$U_+ = (S_+ \cup T_+) \setminus (S_- \cup T_-) \quad \textit{and} \quad U_- = (S_- \cup T_-) \setminus (S_+ \cup T_+)$$

This definition assures us that the restricted union always exists.

Definition 8. *The* **restricted intersection** *of two signed subsets S and T denoted $S \wedge T = V$ is defined as*

$$V_+ = S_+ \cap T_+ \text{ and } V_- = S_- \cap T_-$$

Definition 9. *A function f from $\mathcal{S}(I)$ to $\mathbb{R} \cup \{\infty\}$ is called* bisubmodular *if it satisfies, for all signed subsets S,T,*

$$f(S) + f(T) \geq f(S \wedge T) + f(S \vee T)$$

Definition 10. *The* Base Polyheron *associated with $f : \mathcal{S}(I) \to \mathbb{R}$ is defined as*

$$\mathcal{P}(f) = \{x \in \mathbb{R}^n : e_S(x) \leq f(S) \text{ for all } S \in \mathcal{S}(I), \}$$

This construction above now looks very similar to the base polytope of a boolean function. However this Polyhedra can now be full dimensional. Note that the defining hyperplanes are all parallel to the type BC permutohedron. So given any generalized BC permutohderon polyhedron it can be expressed as $\mathcal{P}(f)$ for a $f : \mathcal{S}(I) \to \mathbb{R}$. However we have not conditioned f. We now wish to show that f is in fact bisubmodular if $\mathcal{P}(f)$ is a generalized BC permutohedra

Just like in the case of the type A permutohdron, we will speak interchangeably of signed subsets and the facets that they define. We'll denote by $\mathcal{P}(f)_S$, where S is a signed subset, the facet of $\mathcal{P}(f)$ that is maximized by e_S. Before we prove our main theorem, we'll establish a useful lemma reproduced from [7]:

Lemma 5.1. *If for $S, T \in \mathcal{S}(I)$, $x \in \mathcal{P}(f)_S \cap \mathcal{P}(f)_T$, then $x \in \mathcal{P}(f)_{S \vee T}$,*

$x \in \mathcal{P}(f)_{S \wedge T}$, and consequently if $U^+ = S+ \backslash T_-$ and $U_- = S_- \backslash T_-$, then $x \in \mathcal{P}(f)_U$ In other words, the restricted union and restricted intersection, and $S \setminus T$ of x-tight signed subsets are also x-tight signed subsets.

Proof. We use the linear functional notation:

$$\begin{aligned}
f(S \vee T) + f(S \wedge T) &\leq f(S) + f(T) \\
&= e_S(x) + e_T(x) \\
&= e_{S_+} - e_{S_-} + e_{T_+} - e_{T_-} \\
&= e_{(S_+ \cup T_+) \backslash (S_- \cup T_-)} - e_{(S_- \cup T_-) \backslash (S_+ \cup T_+)} + e_{S_+ \cap T_+} - e_{S_- \cap T_-} \\
&= e_{S \vee T} + e_{S \wedge T} \\
&\leq f(S \vee T) + f(S \wedge T)
\end{aligned}$$

where the last inequality comes from the fact that $x \in \mathcal{P}(f)$. Therefore

$$f(S \vee T) + f(S \wedge T) = f(S) + f(T)$$

and all these four subsets are $x-$tight, implying that x is also on the defining hyperplanes $H_{S\vee T}$ and $H_{S\wedge T}$.

Furthermore since S and $S\vee T$ are x-tight, then so is $S\wedge(S\vee T)$, but this is simply

$$S_+ \cap (S_+ \cup T_+) \setminus (S_- \cup T_-) = S_+ \setminus T_-$$

and

$$S_- \cap (S_- \cup T_-) \setminus (S_+ \cup T_+) = S_- \setminus T_+$$

Therefore $U = (S_+ \setminus T_-, S_- \setminus T_+)$ is also x-tight. □

Conjecture 5.2. *If E is an edge in in a bisubmodular polytope $\mathcal{P}(f)$ in $\mathbb{R}^n$ then it can be expressed as the intersection of $n-1$ defining hyperplanes H_{J_i} corresponding to a chain of strictly comparable signed sets $\mathcal{J} = \{S_1, S_2, \ldots, S_{n-1}\}$.*

The beginning of a proof of this is given in the proof of the following theorem. However as in the sub modular case, it remains to be shown that at each step of a swapping process to go from $\mathcal{J}$ an arbitrary edge defining family of signed sets to $\mathcal{J}'$ an almost complete flag of signed sets we do not introduce linear dependencies. We do not give a proper procedure that addresses cases where the restricted union or the restriceted intersection are already in $\mathcal{J}$.

Theorem 5.3. *If f is bisubmodular, $P(f)$ is a generalized signed permutohedron.*

Proof. Let f be bisubmodular, we first show that any edge of $P(f)$ has direction $e_i - e_j$ or $e_i + e_j$ for some pair (i,j), or has direction e_i for some i.

Now suppose E is a one dimensional face of $P(f)$. Then it must be the one-dimensional intersection L of $n-1$ defining hyperplanes of $P(f)$, each corresponding to some $S \in \mathcal{S}(I)$. Let $\mathcal{J} = \{S_1, S_2, \ldots S_{n-1}\}$ be the family of signed sets such that

$$L = \bigcap_{S\in\mathcal{J}} H_{S_i}$$

Suppose that $S, T \in \mathcal{J}$ are incomparable. That is neither $S \sqsubset T$ nor $T \sqsubset S$. Let $x \in \mathcal{P}(f)_S \cap \mathcal{P}(f)_T$. Then by lemma 5.1 we have that $S\vee T$, $S\wedge T$ and $U = (S_+ \setminus T_-, S_- \setminus T_+)$ are also x-tight. We want to pick a pair of these signed subsets that are strictly comparable. $S\wedge T$ is always guaranteed to be comparable to both S and T, but $S\vee T$ is not.

So if $S\wedge T = \emptyset$, then we should choose U. But it could be that $U = S$ and we have to choose $S\vee T$. However if $U = S$, then $S_+ \setminus T_- = S_+$ and $S_- \setminus T_+ = S_-$ implies that

$$S_+ \subset (S_+ \cup T_+) \setminus (S_- \cup T_-) \quad \text{and} \quad S_- \subset (S_- \cup T_-) \setminus (S_+ \cup T_+)$$

So our fall back $S\vee T$ is now strictly comparable to S.

The worst case scenario where $S \vee T = S \wedge T = U = \emptyset$ implies the corresponding linear functionals are the negatives of each other, and that the faces they define are parallel and cannot intersect in a line.

Therefore we can switch out signed subsets of $\mathcal{J}$ for comparable signed subsets while still defining the same intersection in a procedure that maintains linear independence as conjectured in 5.2 Let $\mathcal{J}' = \{T_1, T_2, \ldots, T_{n-1}\}$ be the result of such a substitution. This gives us an "almost complete" flag of $n-1$ signed subsets

$$T_1 \sqsubset T_2 \sqsubset \cdots \sqsubset T_{n-1}$$

where these are strict inclusions.

Therefore we consider three possibilities:

i $|T_{n-1}| = n-1$ in which case we've added one at every step, and $a \in I$ is the only remaining element not added. This implies that the L has direction e_a.

ii If $|T_{n-1}| = n$ then at some point we've added two elements $\{a, b\}$ to a signed subset in the chain. If they were both added to T_i^- or both added to T_i^+, then we get the equation $x_a + x_b = K$ or $-x_a - x_b = K$ and in both cases the direction is $e_a - e_b$.

iii Lastly if $|T_{n-1}| = n$ and we again added $\{a, b\}$ at some point where a was added to T_i^+ and b was added to T_i^- then we have the equation $x_a - x_b = K$ and the direction of L is $e_a + e_b$.

This shows that all edges of $\mathcal{P}(f)$ are the right direction. Let $v \in V(\mathcal{P}(f))$. Orient all the edges towards v. So if E is adjacent to v then we write its direction (depending on the what type of edge it is) as:

$$\begin{aligned} e_i - e_j &\quad \text{if } v_i > v_j \\ e_i &\quad \text{if } v_i > 0 \\ -e_i &\quad \text{if } v_i < 0 \\ e_i + e_j &\quad \text{if } v_i > -v_j \end{aligned}$$

Let $\gamma \in BC_n$. The vertex v maximizes γ if for every edge E adjacent to v with oriented direction e_S, we have that $\gamma \cdot e_S \geq 0$. However given all the possible orientations of edges, whether or not v is γ maximal now only depends on the face $F \subset BC_n$ containing γ. Therefore F is contained in the face $N \subset \mathcal{N}(\mathcal{P}(f))$ that maximizes v.

From this it follows that if $\gamma \in F \subset BC$ maximizes Q a face of $\mathcal{P}(f)$, then F is contained in $\mathcal{N}_Q(\mathcal{P}(f))$, the face of the normal fan that maximizes Q. This shows that $\mathcal{N}(\mathcal{P}(f))$ is refined by BC_n and that $\mathcal{P}(f)$ is a generalized BC permutohedra.

□

References

[1] F. Ardila, C. Benedetti, J, Doker. *Matroid Polytopes and their Volumes* preprint, 2011.

[2] M. Aguiar, F. Ardila, *The Hopf Monoid of Generalized Permutohedra* preprint 2011.
[3] J. Doker. *Geometry of Generalized Permutohedra* Doctroral Thesis, 2011.
[4] A. Postnikov, *Permutohedra, Associahedra, and Beyond* preprint? , 2005
[5] A. Barvinok, *A Course in Convexity* (Graduate studies in mathematics, ISSN 1065-7339 ; v. 54), 1963.
[6] Gunter M. Ziegler, *Lectures on Polytopes* (Graduate texts in mathematics, 152), 1998.
[7] A. Bouchet, W. H. Cunningham, *Delta-matroids, Jump Systems and Bisubmodular Polyhedra*, 1991