Generalized Permutohedra

By Matthew Cadier Kim
Advised by Dr. Federico Ardila, Department of Mathematics

SAN FRANCISCO STATE UNIVERSITY

Introduction

Generalized Permutohedra are a family of polytopes whose normal fan is refined by the type A hyperplane arrangement, also known as the **Braid arrangement**. They are deformations of usual **permutohedra**: orbit polytopes of the type A reflection group.

This family of of polytopes can be characterised by: ☛polytopes whose normal fan is refined by the Braid arrangement, ☛Submodular Base polytopes. The relation to the **Symmetric group** means that we capture many sub families of polytopes that have type A symmetries such as matroid polytopes, associahedra, graphical zonotopes, Bruhat order polytopes, and nestohedra.

Preliminaries

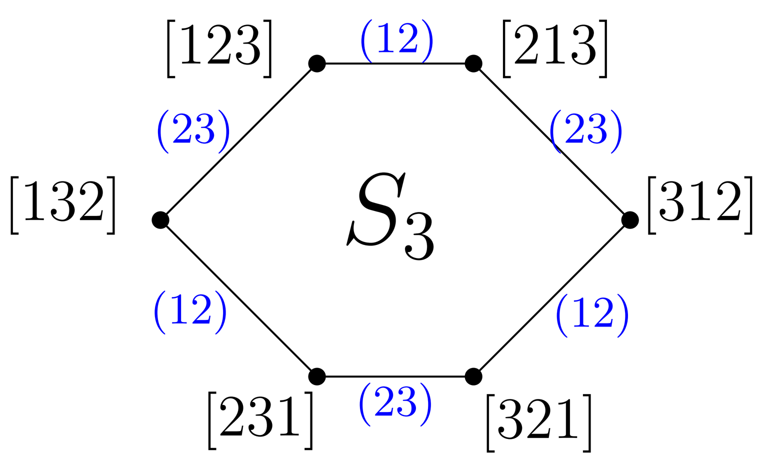
The Symmetric group

☞Consider the set $[n] = \{1, 2, \ldots, n\}$.

☞The **symmetric group** on $[n]$, denoted S_n is the set of **permutations** or orderings of the set $[n]$.

☞We will write $\pi \in S_n$ as $\pi = [\pi(1)\pi(2)\ldots\pi(n)]$.

In the figure we see that we can get all the permutations of S_n by just swaping adjacent numbers.



The Type A Root System

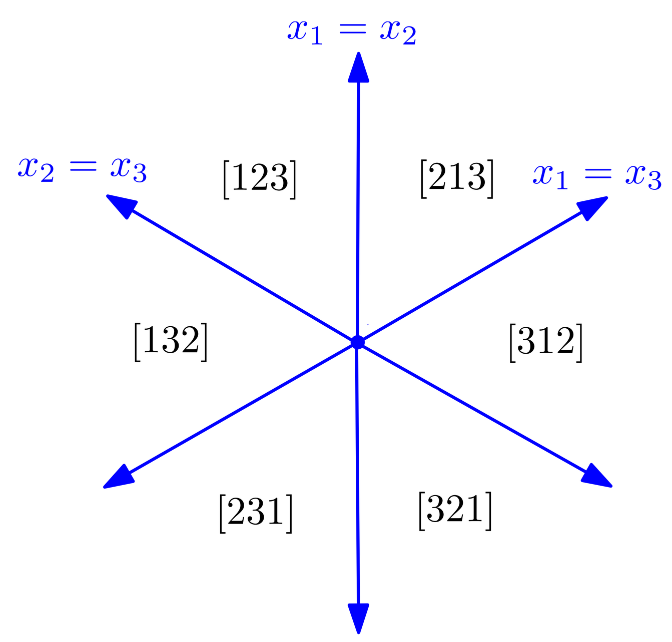


Figure 1: The Braid Arrangement $\mathcal{B}_3$

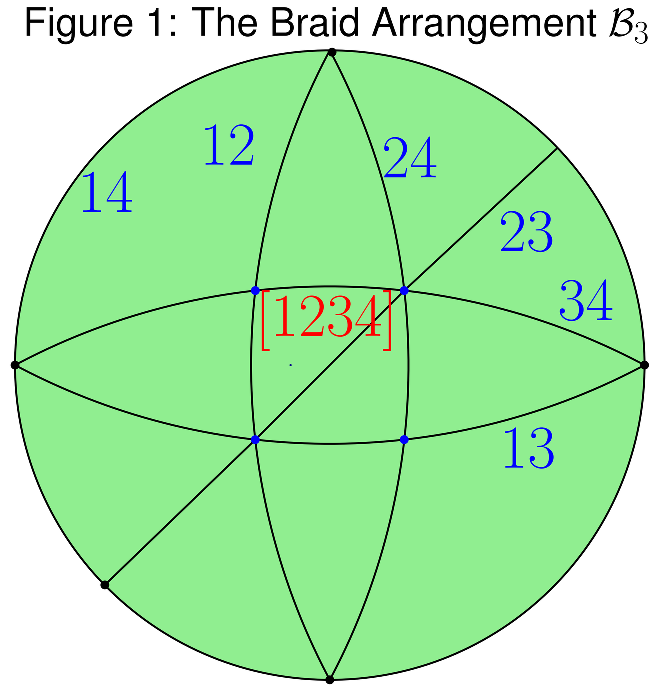


Figure 2: Projectivized $\mathcal{B}_4$

☞We translate permutations in S_n into geometric transformations. Switching elements ⇒ swapping coordinates in $\mathbb{R}^n$, i.e. orthogonal reflections across hyperplanes of the form $x_i = x_j$.

☞The type A root system consists of the vectors

$$A_{n-1} := \{e_i - e_j \mid i \neq j\}$$

The **Braid arrangement** is

$$\mathcal{B}_n = \{H_\alpha \mid \alpha \in A_{n-1}\}$$

Each **chamber** of the Braid arrangement corresponds to an element of S_n given by the ordering on the coordinates.

The Type A Orbit Polytopes

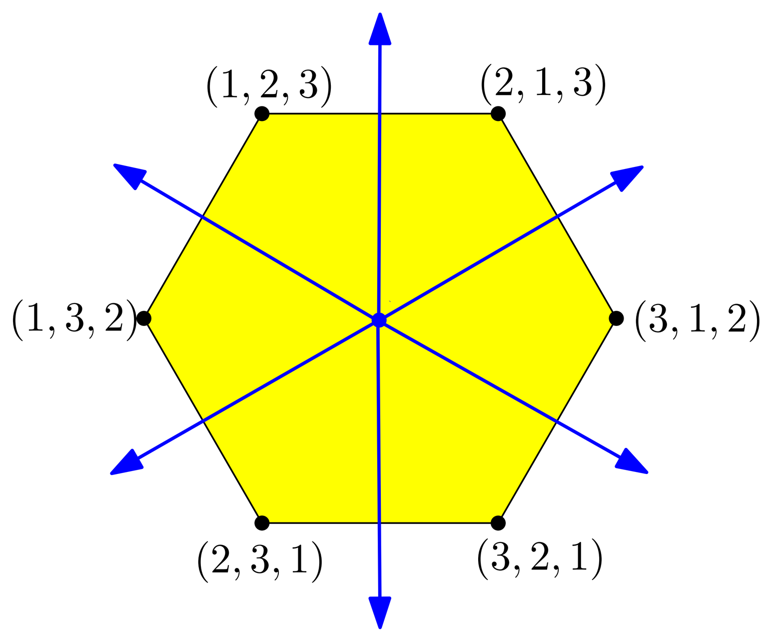


Figure 3: A standard P_3

Aka the Usual Permutohedron

☞Reflect the general position point $(x_1, x_2, \ldots, x_n)$ about $\mathcal{B}_n$.

☞This is the same as taking all the permutations of the coordinates.

☞These form the vertices of a **permutohedron** P_n.

☞It's normal fan is precisely $\mathcal{B}_n$.

This polytope has nice combinatorial descriptions:

Faces

■Vertices are in bijection with S_n.

■Edges are parallel to $e_i - e_j$ vectors and corresponds to $(i, j) \in S_n$.

■k dimensional faces are in bijection with $n - k$ ordered partitions of $[n]$.

■Facets (dim $n-2$) ⇔ subsets of $[n]$.

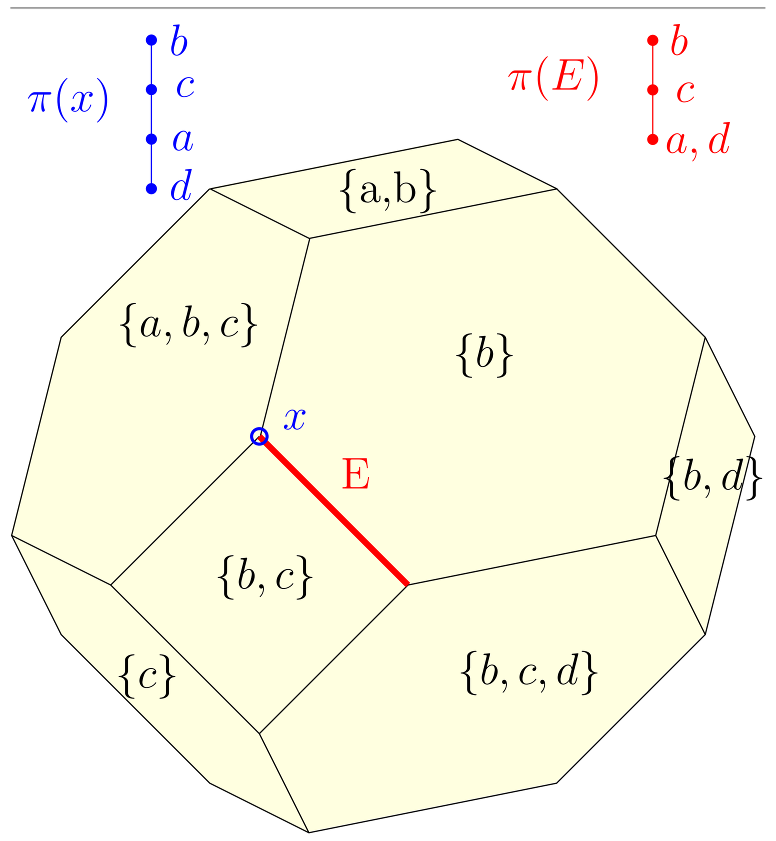


Figure 4: A P_4 permutohedron with faces labeled

The Cone of Facet Deformations

A facet of a permutohedron is maximized by the characteristic vector of label subset $e_A = \sum_{i \in A} e_i$. It's hyperplane description is given by a set function $f : 2^{[n]} \to \mathbb{R}^+$:

$$P(f) := \left\{ x \mid \sum_{i \in A} x_i \leq f(A) \quad \text{for all} \quad A \in 2^{[n]}, \quad \sum_i x = f([n]) \right\}$$

☞The set functions $g \in R^{2^n}$ such that the normal fan of $P(g)$ is the same as $P(f)$ form an open polyhedral cone.

☞The closure of this cone is the **facet deformation cone** C_P^F.

☞These set functions are precisely the positive valued **submodular set functions**.

Generalized Permutohedra

☞We deform a permutohedron P_n by parallel translates of the facets to obtain a **Generalized Permutohedron**

☞This is the equivalent to taking $P(g)$ for some g in the facet deformation cone C_P^F (a submodular function).

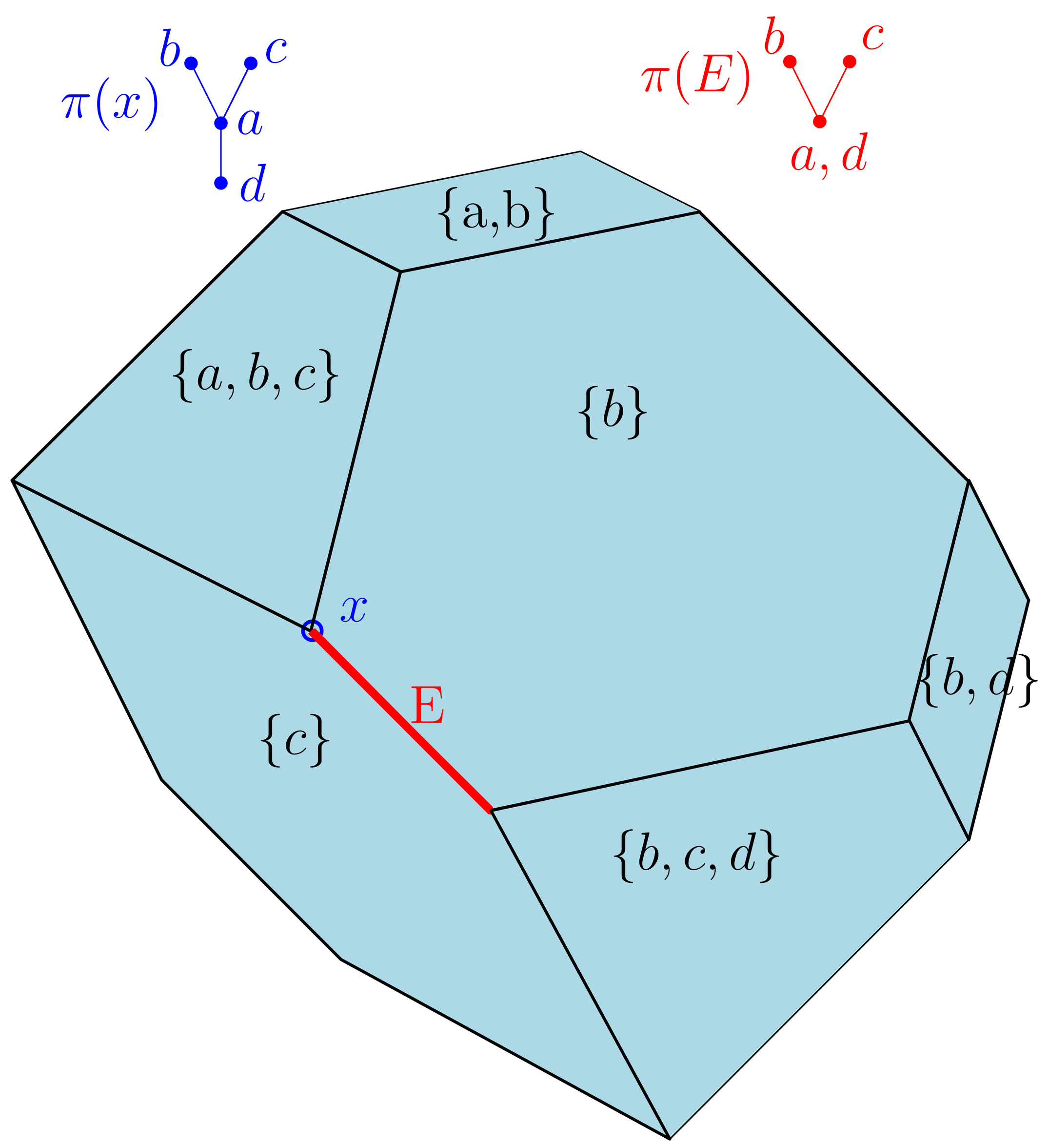


Figure 5: A generalized P_4 permutohedron

Faces

■Vertices are labeled by **posets**.

■Edges are still parallel to $e_i - e_j$ vectors and corresponds to $(i, j) \in S_n$.

■k dimensional faces are in bijection with **posets** on $n - k$ partitions of $[n]$, or **Preposets**.

■The normal fan of the whole polytope is refined by the Braid arrangement.

Future Work

Here we have generalized the type A permutohedron whose Coxeter group is the familiar S_n. However we would like to know what a generalized $W-$permutohedron will look like where W is an arbitrary **Coxeter group**. How we can describe, combinatorially, it's faces? It's normal cones? Solutions to optimization problems?

We are developing a theory of **Coxeter Posets** that will allow us to describe general Coxeter permutohedrons with the same clarity as in the type A case. Our geometric approach build on the new framework of the geometry of Coxeter groups developed as part of Coxeter Matroid theory.