# Generalized Permutohedra

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(joint with David Arcila, Julian Romero, and Federico Ardila)

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- We hope to show that Generalized permutohedra lie at the intersection of combinatorics, combinatorial optimization, Lie algebra, and Coxeter group theory.

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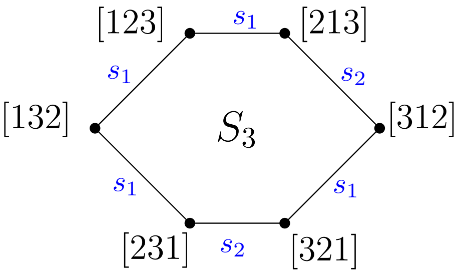
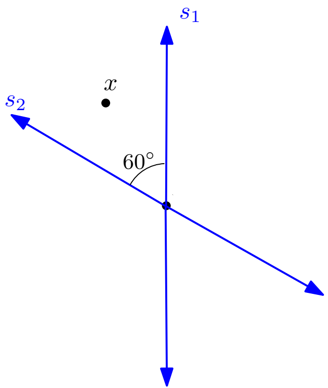



$s_1 = (12)$

$s_2 = (23)$

# $S_3$ as a geometric group action




Let $S_3$ act on a point in $\mathbf{R}^3$ by permuting coordinates ($S_n$ is interpreted as a subgroup of reflections in $GL(n)$)

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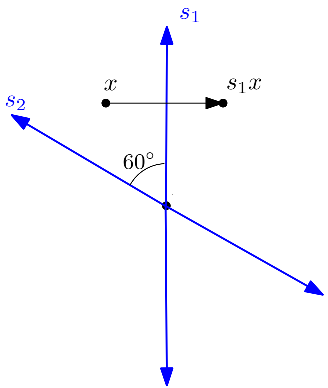

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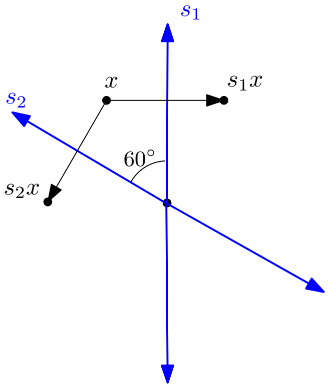

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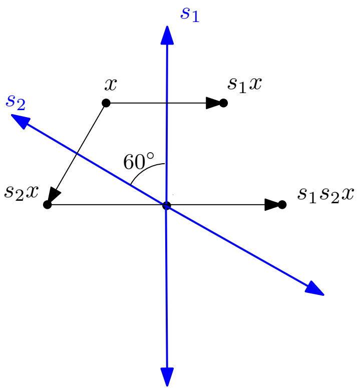

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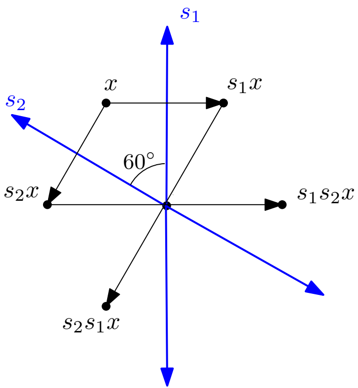

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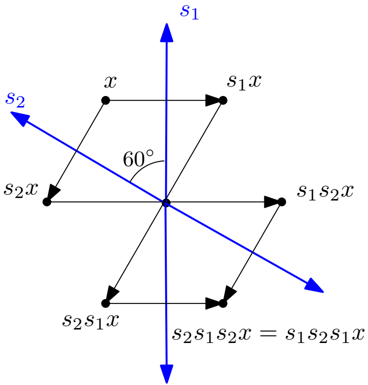
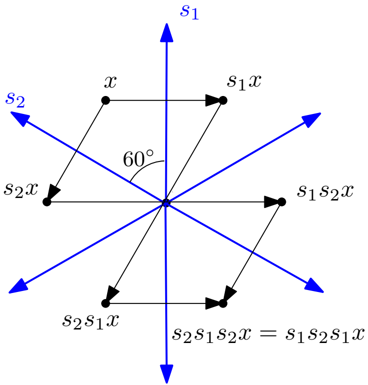



The last point has two representations under the $S_3$ group action.

# $S_3$ as a geometric group action




Note the 3rd hyperplane representing generated by $s_1, s_2$.

# The Braid Arrangement

This group action is equivalent to reflections across the hyperplane arrangement

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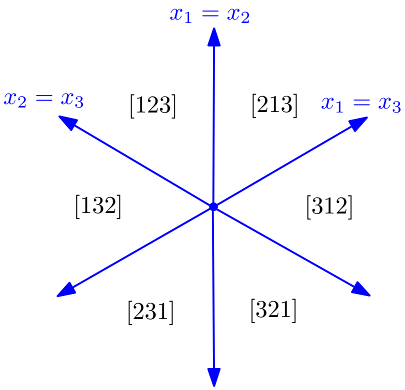

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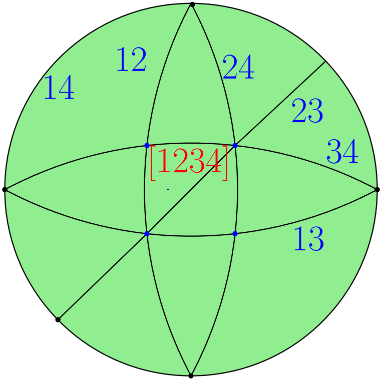

# Type $A$ root system

Introducing root systems ad hoc (Another **representation** of $S_n$):

- The set of $\pm$ normal vectors to the hyperplane arrangement of *a finite reflection group* form a **root system**

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$$\Pi_{n-1} := \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$$

The simple roots, that can generate all the other roots by reflection.

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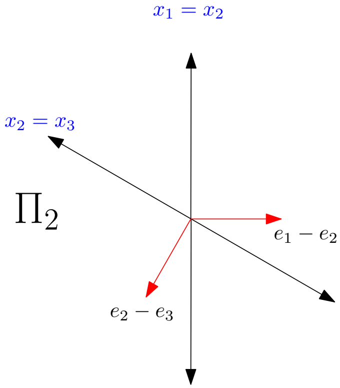

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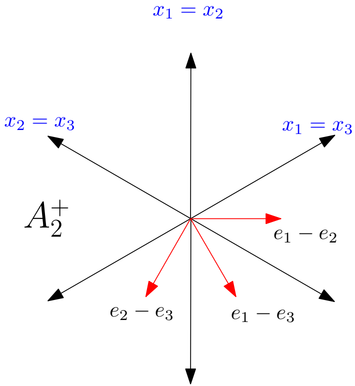
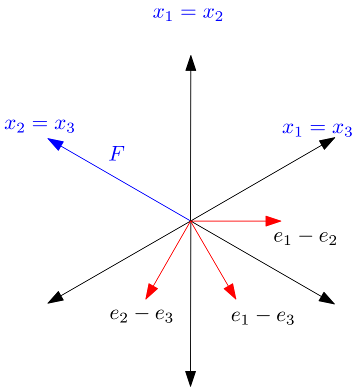


# Faces of $\mathcal{B}_n$

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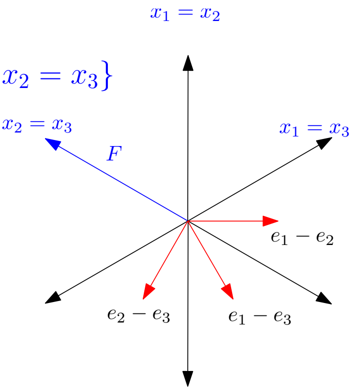


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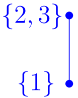


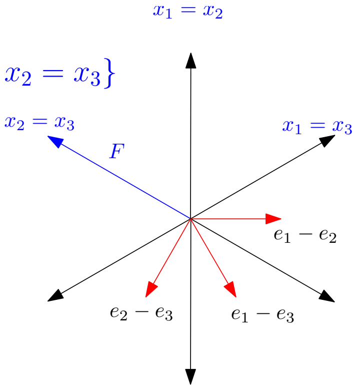


We can label faces of $\mathcal{B}_n$ with familiar combinatorial structures. Faces are described using roots!. These have natural interpretations as orderings on partitions of $[n]$

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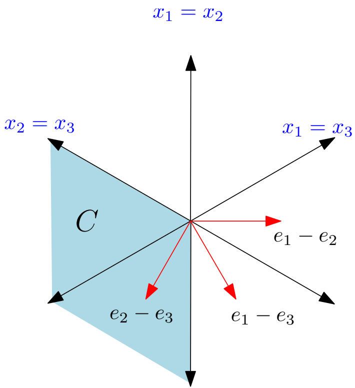


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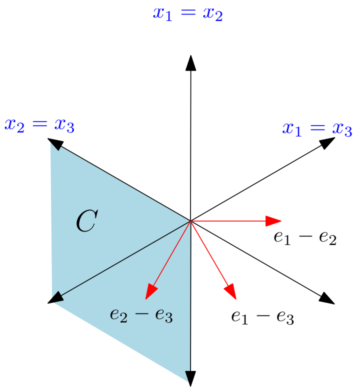


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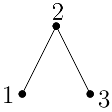
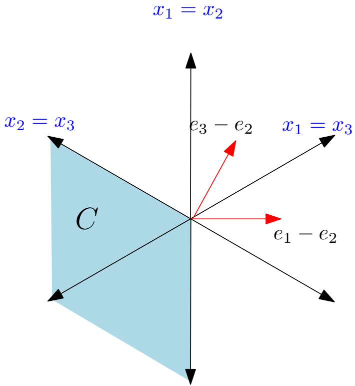
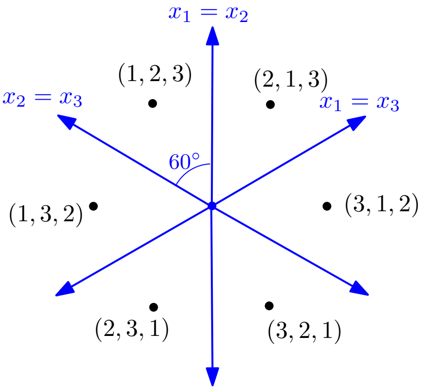



We can label faces of $\mathcal{B}_n$ with familiar combinatorial structures. **Braid cones** are formed by convex sets of faces of $\mathcal{B}_n$. They are in bijection with **(pre)posets** on $[n]$.

# Permutohedra $\Pi^A(x)$

The $A$ permutohedon, $\Pi^A(x)$ is the convex hull of the orbit of a general position point about $\mathcal{B}_n$. Like a map from $(\mathcal{B}_n)^C \to \{\text{polytopes}\}$



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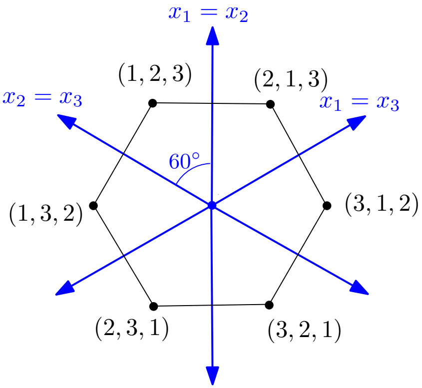

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- Vertices are in bijection with permutations in $S_n$.
- $k-$dimensional faces have $n-k$ dimensional normal cones. So they're in bijection with orderings on partitions of $[n]$ into $n-k$ parts.

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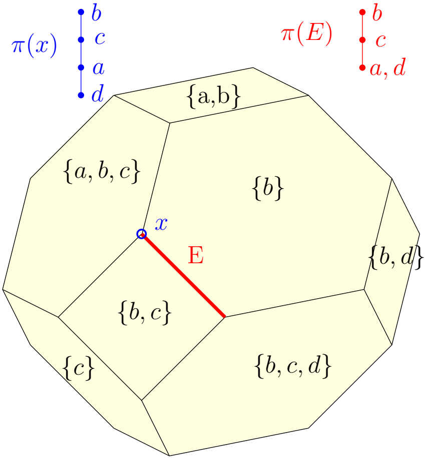
- Importantly! Facets are in bijection (and can thus be labeled by) subsets of $[n]$. A facet labeled $A \in 2^{[n]}$ is maximized by the vector
$$e_A = \sum_{i \in A} e_i$$
($e_A$ is in the normal cone).

# Describing the faces of $\Pi^A(x)$



# Deforming $\Pi^A(x)$

We therefore have a hyperplane description of any $\Pi^A(x)$:

$$\Pi^A(x) = \Pi^A(f) = \{\, y \quad | \, (e_A, y) \leq f(A) \quad \text{and} \quad (e_{[n]}, y) = f([n]) \,\}$$

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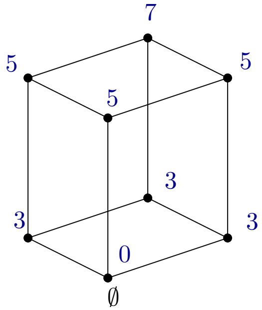
$f =$ 

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The **open cone of facet deformations** for $\Pi^A(f)$ is the polyhedral cone

$$(D^F)^\circ := \{\ g \in \mathbf{R}^{2^n} \quad \text{such that} \quad \mathcal{N}(\Pi^A(g)) = \mathcal{N}(\Pi^A(f))\}$$

I.e. all the set functions that preserve the normal fan.

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The closure of the **cone of facet deformations** for $\Pi^A(f)$ is the polyhedral cone

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This is in fact the cone of **submodular set functions**

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A **polymatroid** of a submodular function is

$$P_f := \{x \in \mathbf{R}^{[n]} \mid (e_A, x) \leq f(A) \text{ for all } A \subset [n]\}$$

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The **base polytope** of a submodular function is

$$B_f := \{x \in \mathbf{R}^{[n]} \mid \sum_{i=1}^{n} x_i = f([n])\} \bigcap P_f$$

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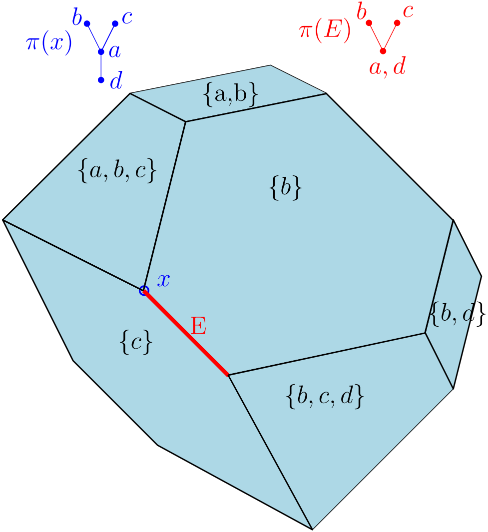
Since the normal fan is composed of **braid cones** the faces of generalized permutohedron are labeled by **preposets** or posets on partitions of $[n]$.

# Generalized type $A$ permutohedra



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The type $B_{n-1}$ root system includes $A_{n-1}$:

$$B_n := \{\pm e_i \pm e_j, e_i \mid \text{ for all } 1 \leq i \neq j \leq n \}$$

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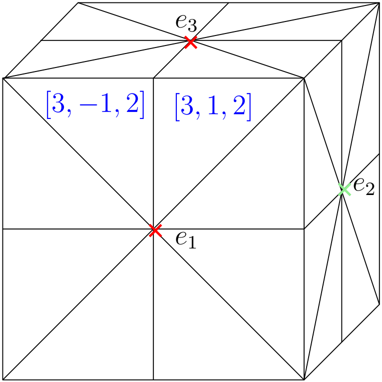
$B_3$



# The Signed (type $B$) Posets

We can use this type $B_n$ coxeter complex to define type $B$ cominatorial objects. As before lets look at cones:

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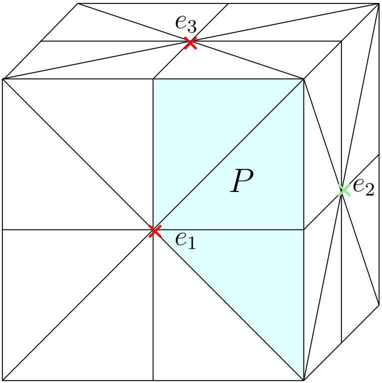

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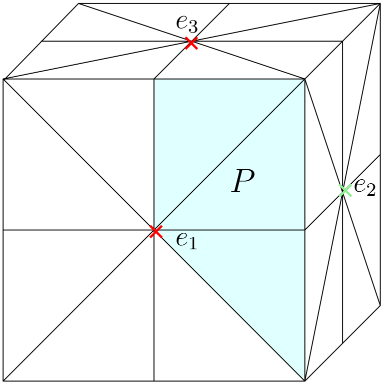
$$\mathrm{P} = \{x_2 \geq 0, \; x_1 \geq x_2 \; x_1 \geq x_3 \; x_2 \geq -x_3\}$$




The inequality description.

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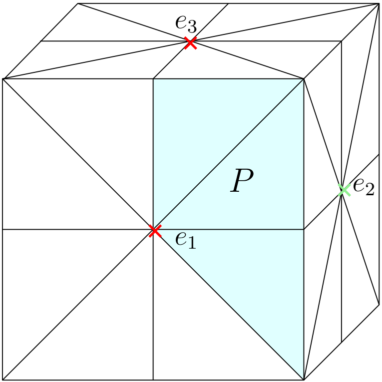


We actually only need a set of roots.

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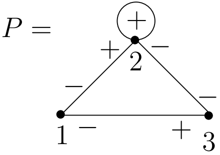
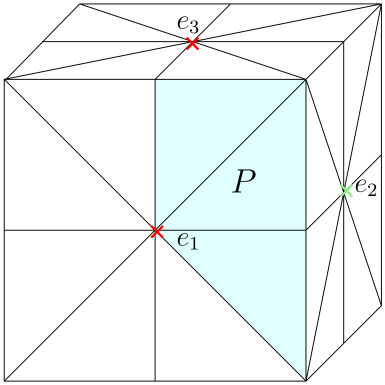
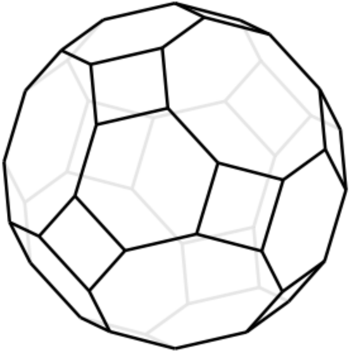



We can encode these root in a graph, as defined by Victor Reiner

# The Signed (type $B$) Permutohedron

Taking a point $x$ in the interior of a chamber of the type $B_n$ hyperplane arrangement, we let the hyperoctohedral group act on it and get $\Pi^B(x)$ :



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- Vertices $\rightarrow$ **signed permutations**
- Facets $\rightarrow$ **signed sets**.

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We generalize it as before, but using **signed set functions** $f : \{1, 0, -1\}^n \to \mathbf{R}$. We have the hyperplane description:

$$\Pi^B(x) = \Pi^B(f) = \{y \mid (e_A, y) \leq f(A) \quad \text{for all } e_A \in \{1, 0, -1\}^n\}$$

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These signed set functions are precicely the **bisubmodular set functions**.

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That is $\Pi^B(g)$ is a generalized type $B$ permutohedron if and only if

$$f(A) + f(B) \geq f(A \sqcap B) + f(A \sqcup B) \text{ for all } A, B \in \{1, 0, -1\}^n$$

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We can call these polytopes, **signed polymatroids**.

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We saw two examples of **classical reflection groups**: $A$ and $B$, or the symmetric group and the hyperoctohedral group.

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Exceptional irreducible root systems? Composite reflection groups?

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We saw that the Coxeter group underlying a finite system of reflections can use the geometry of the complex to define combinatorial-like structures.

The goal now is to take the framework of combinatorial theory and translate it into group theoretic and geometric terms that will make it as general as possible and reveal manageable algebraic characterizations.