Free Resolutions of Matroid Ideals

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ABSTRACT. Several properties of ideals and the varieties they represent have been reduced to combinatorial problems. Building on the framework of the theory of combinatorial commutative algebra layed down by Miller and Sturmfels in [7] we study the way that the facet ideal of a matroid relates to the geometry of the matroid polytope via it's cellular resolution. The facet ideal has been studied in [6], but we state the correspondence with the Bergman complex of the matroid as shown by Ardila in unpublished notes [1]. We then propose analogous theorems for Lagrangian matroids.

1 Introduction

Matroids are classical combinatorial objects that are families of subsets satisfying symmetric exchange axioms. Recently, they have begun to appear in the context of commuative algebra. Since matroids are essentially simplicial complexes, we can draw from much of the theory about square free monomial ideals. Sturmfels and Miller in [7] have layed much of the ground work for what we attempt to do in this paper. Trygve, Roksvold, and Verdure define the facet ideal of a matroid M, which is the Stanley-Reisner ideal of the Alexander dual of the matroid dual of M. We take a simpler route for now, and attempt to explicitly state the correspondence shown by Ardila between the resolution of the facet ideal of a matroid and the Bergman complex of the matroid.

The main properties of the Bergman complex were shown by Ardila and Klivens in [2]. The topology of this complex is well known. The fact that it appears in the cellular resolution of the matroid polytope is encouraging, as it connects the Möbius function, a well known invariant of the matroid, to the facet ideal. We hope to generalize the notion of facet ideal and Bergman complex as well as invarients such as the Móbius function and Betti numbers to other coxeter matroids in future work.

2 Preliminaries

In this section we will define some of the basic objects of study and necessary background material from combinatorics and commutative algebra. For a thorough discussion of combinatorial commutative algebra and in particular cellular resolutions, we refer the reader to [7], as we will use thses definitions and notation.

2.1 Simplicial Homology

We begin with simplicial homology as it is the most basic, and closely related to cellular homology. With the latter we have to be careful of orientations.

Let $\Bbbk$ be a field and $S = \Bbbk[\mathbf{x}]$ the polynomial ring over $\Bbbk$ in n indeterminates $\mathbf{x} = x_1, \ldots, x_n$.

Definition. *[7] A* ***monomial*** *in $\Bbbk[\mathbf{x}]$ is a product $\mathbf{x^a} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{N}^n$ of nonnegative integers. An ideal $I \subseteq \Bbbk[\mathbf{x}]$ is called a* ***monomial ideal*** *if it is generated by monomials.*

As a vector space over $\Bbbk$, the polynomial ring S is a direct sum

$$S = \bigoplus_{\mathbf{a} \in \mathbf{N}^n} S_{\mathbf{a}},$$

where $S_{\mathbf{a}} = \Bbbk\{\mathbf{x^a}\}$ is the vector subspace of S spanned by the monomial $\mathbf{x^a}$. Since the product $S_{\mathbf{a}} \cdot S_{\mathbf{b}}$ of graded pieces equals the graded piece $S_{\mathbf{a+b}}$ in degree $\mathbf{a}+\mathbf{b}$, we say that S is an $\mathbf{N}^n$-graded $\Bbbk$-algebra.

Definition. *[7] A* ***simplicial complex*** *$\triangle$ on the vertex set $\{1, \ldots, n\}$ is a collection of subsets called faces or simplices such that if $\sigma \in \triangle$ is a face and $\tau \subseteq \sigma$, then $\tau \in \triangle$. A simplex $\sigma \in \triangle$ of cardinality $|\sigma| = i+1$ has dimension i and is called an i-face of $\triangle$. The dimension $\dim(\triangle)$ of $\triangle$ is the maximum of the dimensions of its faces, or it is $-\infty$ if $\triangle = \{\}$ is the void complex, which has no faces.*

Let $\triangle$ be a simplicial complex on $\{1, \ldots, n\}$. Let $F_i(\triangle)$ be the set of i-dimensional faces of $\triangle$, and let $\Bbbk^{F_i(\triangle)}$ be a vector space over $\Bbbk$ whose basis elements e_σ correspond to i-faces $\sigma \in F_i(\triangle)$.

Definition. *The* ***reduced chain complex*** *of $\triangle$ over $\Bbbk$ is the complex $\widetilde{C}.(\triangle;\Bbbk)$:*

$$0 \longleftarrow \Bbbk^{F_{-1}(\triangle)} \xleftarrow{\partial_0} \cdots \longleftarrow \Bbbk^{F_{i1}(\triangle)} \xleftarrow{\partial_i} \Bbbk^{F_i(\triangle)} \longleftarrow \cdots \xleftarrow{\partial_{n-1}} \Bbbk^{F_{n1}(\triangle)} \longleftarrow 0.$$

The boundary maps ∂_i are defined by setting $\operatorname{sgn}(j,\sigma) = (-1)^{r-1}$ if j is the r^{th} element of the set $\sigma \subseteq \{1,\ldots,n\}$, written in increasing order, and

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} \operatorname{sgn}(j,\sigma) e_{\sigma \setminus j}.$$

If $i < -1$ or $i > n-1$, then $\Bbbk^{F_i(\triangle)} = 0$ and $\partial_i = 0$ by definition. We can see that $\partial_i \circ \partial_{i+1} = 0$, i.e, the image of the $(i+1)^{st}$ boundary map ∂_{i+1} lies inside the kernel of the i^{th} boundary map ∂_i.

Definition. *The $\Bbbk$-vector space*

$$\widetilde{H}_i(\triangle;\Bbbk) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$$

in ***homological degree*** *i is the i^{th}* ***reduced homology*** *of $\triangle$ over k.*

Moreover, $\widetilde{H}_{n-1}(\triangle;\Bbbk) = \ker(\partial_{n-1})$, and when $\triangle$ is not the irrelevant complex $\{\varnothing\}$, we get also $\widetilde{H}_i(\triangle;\Bbbk) = 0$ for $i < 0$ or $i > n-1$. The irrelevant complex $\triangle = \{\varnothing\}$ has homology only in homological degree -1, where $\widetilde{H}_{-1}(\triangle;\Bbbk) \cong \Bbbk$. The dimension of the zeroth reduced homology $\widetilde{H}_0(\triangle;\Bbbk)$ as a $\Bbbk$-vector space is one less than the number of connected components of $\triangle$. Elements of $\ker(\partial_i)$ arecalled ***i-cycles*** and elements of $\operatorname{im}(\partial_{i+1})$ are called ***i-boundaries***.

2.2 Free Resolutions and Betti Numbers

A ***free S-module*** of finite rank is a direct sum $F \cong S^r$ of copies of S, for some nonnegative integer r. In our case, F will be $\mathbf{N}^n$-graded, which means that $F \cong S(-\mathbf{a}_1) \oplus \cdots \oplus S(-\mathbf{a}_r)$ for some vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r \subset \mathbf{N}^n$. A sequence

$$\mathcal{F}.: \quad 0 \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \cdots \longleftarrow F_{\ell-1} \xleftarrow{\phi_\ell} `F_\ell \longleftarrow 0$$

of maps of free S-modules is a **complex** if $\phi_i \circ \phi_{i+1} = 0$ for all i. The complex is **exact** in homological degree i if $\ker(\phi_i) = \mathrm{im}(\phi_{i+1})$. When the free modules F_i are $\mathbf{N}^n$-graded, we require that each homomorphism ϕ_i be degree-preserving, so that it takes elements in F_i of degree $\mathbf{a} \in \mathbf{N}^n$ to degree $\mathbf{a}$ elements in F_{i-1}.

Definition. *A complex $\mathcal{F}.$ is a **free resolution** of a module M over $S = \Bbbk[x_1, \ldots, x_n]$ if $\mathcal{F}.$ is exact everywhere except in homological degree 0, where $M = F_0/\mathrm{im}(\phi_1)$. The image in F_i of the homomorphism ϕ_{i+1} is the i^{th} **syzygy module** of M. The length of the resolution is the greatest homological degree of a nonzero module in the resolution.*

Definition. *A **monomial matrix** is an array of scalar entries λ_{qp} whose columns are labeled by source degrees $\mathbf{a}_p$, whose rows are labeled by target degrees $\mathbf{a}_q$, and whose entry $\lambda_{qp} \in \Bbbk$ is zero unless $\mathbf{a}_p \succeq \mathbf{a}_q$. A monomial matrix is **minimal** if $\lambda_{qp} = 0$ when $\mathbf{a}_p = \mathbf{a}_q$. A homomorphism of free modules, or a complex of such, is **minimal** if it can be written down with minimal monomial matrices.*

Definition. *If the complex $\mathcal{F}.$ is a minimal free resolution of a finitely generated $\mathbf{N}^n$-graded module M and $F_i = \bigoplus_{\mathbf{a} \in \mathbf{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$, then the i^{th} **Betti number** of M in degree $\mathbf{a}$ is the invariant $\beta_{i,\mathbf{a}} = \beta_{i,\mathbf{a}}(M)$.*

Now, for an important lemma that is useful in practice when computing Betti numbers.

Lemma 1. *1.32 (Sturmfels and Miller) The i^{th} Betti number of an $\mathbf{N}^n$-graded module M in degree $\mathbf{a}$ equals the vector space dimension $\dim_{\Bbbk} \mathrm{Tor}_i^S(\Bbbk, M)_{\mathbf{a}}$.*

2.3 Cellular Resolutions

Similar to simplicial complexes, we can conider a collection of polytopes in the following way.

Definition. *A **polyhedral cell complex** X is a finite collection of convex polytopes called faces of X, satisfying two properties:*

- *If $\mathcal{P}$ is a polytope in X and $\mathcal{F}$ is a face of $\mathcal{P}$, then $\mathcal{F}$ is in X.*
- *If $\mathcal{P}$ and $\mathcal{Q}$ are in X, then $\mathcal{P} \cap \mathcal{Q}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$.*

*A **labeled cell complex** X has its r vertices have labels that are vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r$ in $\mathbf{N}^n$. The label on an arbitrary face F of X is the exponent $\mathbf{a}_F$ on the least common multiple $\mathrm{lcm}(\mathbf{x}^{\mathbf{a}_i} \mid i \in F)$ of the monomial labels $\mathbf{x}^{\mathbf{a}_i}$ on vertices in F.*

Definition. *Let X be a labeled cell complex. The **cellular monomial matrix** supported on X uses the reduced chain complex of X for scalar entries, with $\varnothing$ in homological degree 0. Row and column* *labels are those on the corresponding faces of X. The* ***cellular free complex*** $\mathcal{F}_X$ *supported on X is the complex of* $\mathbf{N}^n$*-graded free S-modules represented by the cellular monomial matrix supported on X. The free complex* $\mathcal{F}_X$ *is a* ***cellular resolution*** *if it is acyclic.*

The label on the empty face $\varnothing \in X$ is $0 \in \mathbf{N}^n$, which is the exponent on $1 \in S$. It is also possible to write down the boudary chain ∂ of $\mathcal{F}_X$ without using monomial matrices:

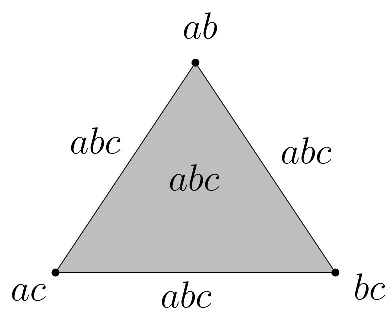
$$\mathcal{F}_X = \bigoplus_{F \in X} S(-\mathbf{a}_F), \quad \partial(F) \quad = \quad \sum_{\text{facets } G \text{ of } F} \operatorname{sgn}(G,F)\mathbf{x}^{\mathbf{a}_F - \mathbf{a}_G} G.$$

The sign for (G,F) equals ± 1 and is part of the data in the boundary map of the chain complex of X.

Proposition 2. *(Sturmfels and Miller) [7] The cellular free complex* $\mathcal{F}_X$ *supported on X is a cellular resolution if and only if* $X_{\preceq \mathbf{b}}$ *is acyclic over* $\Bbbk$ *for all* $\mathbf{b} \in \mathbf{N}^n$*. When* $\mathcal{F}_X$ *is itself acyclic, it a free resolution of* S/I *where* $I = \langle \mathbf{x}^{\mathbf{a}_v} \mid v \text{ is a vertex} \rangle$ *is generated by the monomials with exponents that are vertex vectors on X.*

Similar to the simplicial case, if X is a polyhedral cell complex then $\widetilde{H}.(X;\Bbbk)$ denotes the homology of the reduced chain complex $\widetilde{C}.(X;\Bbbk)$

Example 3. *Consider the following labeled cellular complex X for the monomial ideal* $I = \langle ab, ac, bc \rangle$ *in* $S = \Bbbk[a,b,c]$.



Then $X_{\preceq \mathbf{b}}$ is the whole triangle for $\mathbf{b} = 111$. We can see it is just a single vertex for the degrees 110, 011, and 101, and empty for lower degrees. Therefore, $X_{\preceq \mathbf{b}}$ is acyclic for all $\mathbf{b}$ and we obtain the following nonminimal free resolution

$$0 \longleftarrow S^1 \longleftarrow S^3 \longleftarrow s^3 \longleftarrow S \longleftarrow 0$$

Theorem 4. *(Sturmfels and Miller) [7] If* $\mathcal{F}_X$ *is a cellular resolution of the monomial quotient S/I, then the Betti numbers of I can be calculated for* $i \geq 1$ *as*

$$\beta_{i,\mathbf{b}}(I) \; = \; \dim_{\Bbbk} \widetilde{H}_{i-1}(X_{\prec \mathbf{b}};\Bbbk).$$

For each $\sigma \subseteq \{1,\ldots,n\}$, dene the restriction of $\triangle$ to σ by

$$\triangle|_\sigma = \{\tau \in \triangle \mid \tau \subseteq \sigma\}$$

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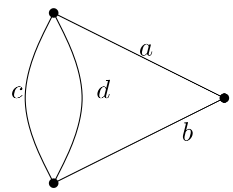
2.4 Matroids

Now we introduce the first main combinatorial object of our study. There are several ways of defining matroids. Examining the connections between these equivalent ways of characterizing a matroid is in itself an enlightening activity. However we will follow [8] and use our earlier definition of simplicial complexes.

We'll first start with a simple motivating example:

Example 5. *Consider the set of the edges E of the graph below:*



Let $E = \{edges\ of\ G\}$ and $\mathcal{I}$ be the collection of edge sets that that do not contain cycles. So

$$\mathcal{I} = \{a,b,c,d,ab,ad,ac,cb,db\}$$

where $ab = \{a,b\}$ for the sake of clarity. Maximal elements of $\mathcal{I}$ are just the spanning trees of G.

It is clear that if $A \in \mathcal{I}$ and $B \subset A$ then B does not contain any cycles and therefore $B \in \mathcal{I}$ as well. Furthermore, if we interstect all the elements of $\mathcal{I}$ with some $S \subseteq E$ then all the elements will be the edge sets without cycles on the connected components of $E \setminus S$. These maximal elements will be spanning trees on these components, meaning they'll all have the same cardinality. $(E,\mathcal{I})$ is an instance of a *matroid*.

Let's formalize this concept in such a way that we preserve the important properties described above:

Definition. *A* **matroid** *is a pair $(E,\mathcal{I})$ where $\mathcal{I}$ is a pure simplicial complex on a ground set E.*

By *pure* we simply mean that in any subcomplex on $\mathcal{I}$ induced by a subset $S \subseteq E$ the maximal elements all have the same cardinality. Elements of $\mathcal{I}$ are called the **independent sets** of the matroid. The maximal independent sets are called the **bases**, alluding to the notion of a basis of a vector space. Since $\mathcal{I}$ is pure, if we restrict the matroid to a subset $S \subseteq E$ we have a submatroid $M|S$.

A matroid is actually determined by it's bases. So we have an equivalent definition conditioning only on the set of bases.

The symmetric difference of two sets is denoted $A\Delta B = (A \setminus B) \cup (B \setminus A)$

Definition. *A set system $(E,\mathcal{F})$ satisfies the* **symmetric exchange property** *if for every $A,B \in \mathcal{F}$ and $a \in A\Delta B$ there exists $b \in A\Delta B$ such that*

$$A\Delta\{a,b\} \in \mathcal{F}$$

A matroid is a set system $(E,\mathcal{B})$ where $\mathcal{B}$ satisfies the symmetric exchange property and where every element of $\mathcal{B}$ has the same cardinality.

See [4] for a treatment of matroids in terms of exchange properties.

The **circuits** of a matroid are the minimal *dependent sets* (which are simply all the elements of $2^E \setminus \mathcal{I}$.) A **loop** in a matroid is an element $e \in E$ such that $\{e\}$ is not independent. Loops in a graph would correspond to loops in the matroid on that graph. Conversely a **coloop** of a matroid is an element that belongs to *every* base.

There is a set function $r : 2^E \to \mathbf{N}$ associated with a matroid called the **rank function**. It is defined as

$$r(A) := \max |J| \text{ where } J \text{ is an independent set and } J \subset A$$

It is called the rank function because it generalizes the notion of the rank of a matrix. The rank function is a non decreasing **submodular** set function, that is

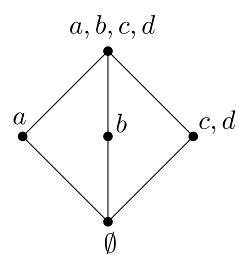
$$r(A) + r(B) \geq r(A \cap B) + r(A \cup B)$$

and

$$\text{if } B \subset B \quad \text{then} \quad r(B) \leq r(A)$$

The **restriction** of a matroid to a subset S of E is denoted $M \mid S$ and is a matroid on S whose independent sets are those of M that are contained in S.

A **k-flat** of a matroid are all the maximal sets $A \subset E$ such that $r(A) = k$. The *closure* of a set A is denoted $\overline{A}$ and is defined as the smallest flat containing A in the matroid. Given this closure operator, we actually have a lattice of flats associated with each matroid, denoted $L(M)$, where the grading is the rank of each flat. For example 5 $L(M)$ is



Now it makes sense to define the Möbius function on a matroid. Let

$$\mu(M) = \mu(L_M) = (-1)^{r(M)} \mu_{L_M}(\hat{0}, \hat{1})$$

We will see how this invariant appears later in the context of the resolution of an ideal associated with the matroid.

3 Other Objects Associated with Matroids

3.1 Matroid Polytopes

One very natural direction to take when dealing with matroid is try and represent it geometrically. Here we don't mean representation in the sense of trying to determine if a matroid M is isomorphic to the matroid induced by a collection of vectors in a vector space. Not every matroid is representable in this way.

However, viewing the matroid as just a family of sets satisfying the above axioms, we can represent it by the characteristic vectors of it's bases.

Definition. *For a subset $A \subseteq S$ the incidence vector of A is*

$$e_A = \sum_{i \in A} e_i$$

where $e_A \in \mathbf{R}^S$.

These vectors allow us to identify set systems with subsets of vertices on the $n-$cube.

Definition. *For a matroid $M = (E, \mathcal{B})$*

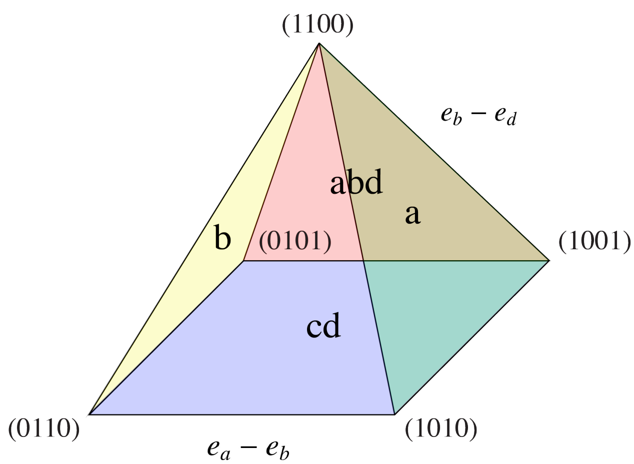
$$\mathcal{P}_M = \operatorname{conv}\{e_B,\ B \in \mathcal{B}\}$$

is the **independence matroid polytope.**

For the rest of this paper we will simply refer to this construction as the *matroid polytope*. We can equivalently define the matroid polytope as the base polytope of the rank function:

$$\mathcal{P}_M = \{x \in \mathbf{R}^E \ | \ \langle e_A, x\rangle < r(A) \ \ A \subset E, \ \text{ and } \ \langle e_E, x\rangle = r(M)\}$$

Below we have the matroid polytope of the matroid in example 5



Remark. *As indicated in the figure, each edge of a matroid polytope is parallel to a root in the type A root system. Matroid polytopes are special types of* **generalized permutohedra***, which are in* *bijection with base polytopes of submodular functions. This larger class of polytopes shares the property that it's edge directions are parallel to roots, and that it's normal fan is refined by the Braid arrangement. Given this connection, we hope to see the results regarding the ideals discussed below extended to this larger class of polytopes.*

3.2 Bergman Complex of a Matroid

While the Bergman complex was initially defined in the context of algebraic geometry, and then *tropical varieties*, Sturmfels defined the Bergman complex for a matroid, which Ardila and Klivens studied in [2]. tWe reproduce their definition here.

For $\omega \in \mathbf{R}^E$ and a matroid M on E of rank r, the weight of a base B is defined as $\omega \cdot e_B$. The bases of minimum $\omega-$weight of a matroid are denoted M_ω and form a sub-matroid.

Definition. *The* **Bergman fan** *of a matroid M on a set E is the set of vectors*

$$\widetilde{\mathcal{B}}(M) := \{\omega \in \mathbf{R}^E \ : \ M_\omega \text{ has no loops } \}$$

and the **Bergman complex** *of M is*

$$\mathcal{B}(M) := \{\omega \in S^{n-2} \ : \ M_\omega \text{ has no loops } \}$$

The main theorem relates the Bergman complex of a matroid M to the lattice of flats $L(M)$. First we need to recall that the order complex of a poset $\triangle(P)$ is a simplicial complex on the elements of P where faces are chains in P. Ardila and Klivens show that

Theorem 6. *The Bergman complex of M is a realization of* $\triangle(L(M) - \{\hat{0}, \hat{1}\})$

which has the corollary

Corollary 7. *The Bergman complex* $\mathcal{B}(M)$ *is homotopy equivalent to a wedge of* $\hat{\mu}(M)$ $(r(M)-2)$*-dimensional spheres.*

We are ommiting some details here, but these results, in part, follow from the fact that a matroid polytope is a generalized permutohedron and has edge directions $e_i - e_j$. A face of P_M is maxized by all vectors in $\mathbf{R}^E$ with the same ordering on the coefficients. That is, they are in the same *weight class*, or in the same face of the Braid arrangement, which we mentioned earlier refines the normal fan of P_M.

4 Free Resolutions of Matroids

First we define a certain ideal associated to a matroid. Then we begin by reproducing here a theorem and proof supplied to us by Federico Ardila showing that a matroid ideal can resolve itself through the matroid polytope, and that the Betti numbers can be related to the Bergman complex associated to the matroid.

Definition. *Let* $\triangle$ *be a simplicial complex. Then the* ***facet ideal*** *of of* $\triangle$ *is*

$$\mathcal{F}(\triangle) = \langle \mathbf{x}^\sigma \mid \sigma \text{ is a facet of } \triangle \rangle$$

The facet ideal is a square-free monomial ideal. In particular, when the simplicial complex is a matroid, the generators of the facet ideal are the basis elements of the matroid.

Let $\mathcal{P}_M$ be the matroid polytope of M a matroid with

- Vertex B labeled $a_B = e_B$ for B a base of M.
- Face F labeled $a_F = \sum b$ not a coloop in $M(F)e_n$

We have that

$$\mathbf{x}^{a_F} = \mathrm{lcm}_{B \in F}\, \mathbf{x}^{\mathbf{a}_B}$$

and

$$\mathcal{F}_{\mathcal{P}_M} := \text{cellular free complex on } \mathcal{P}_M = \bigoplus_{F \text{ a face}} S(-\mathbf{a}_F)$$

and the associated facet ideal

$$B_M = \left\langle \prod_{b \in B} x_b \;\middle|\; B \text{ a basis for } M \right\rangle$$

Proposition 8. *(Ardila) $\mathcal{F}_{\mathcal{P}_M}$ is a cellular resolution for B_M.*

Proof. Let $\mathbf{b} \in \mathbf{N}^n$ and $S = \mathrm{supp}(\mathbf{b})$. For $\mathbf{a}, \mathbf{b} \in \mathbf{N}^n$ let $\mathbf{a} \preceq \mathbf{b}$ if $a_i \leq b_i$ for all i.

$$\begin{aligned}(\mathcal{P}_M)_{\preceq \mathbf{b}} &= \{\text{faces with labels} \preceq \mathbf{b}\ \} \\ &= \{\text{faces with labels} \subseteq S\} \\ &= \{\text{faces of } \mathcal{P}_M \text{ on the face } e_S \text{ of the simplex}\} \\ &= \mathcal{P}_{M|S}\end{aligned}$$

Where $\mathcal{P}_{M|S}$ denotes the matroid polytope on M restricted to S. Since $\mathcal{P}_{M|S}$ is a polytope, it is acyclic.

Therefore, with repect to the matroid in our motivating example 5 , we have the free resolution

$$0 \longleftarrow S \longleftarrow S^5 \longleftarrow S^8 \longleftarrow S^5 \longleftarrow S \longleftarrow 0$$

given by the faces of the matroid polytope. □

Theorem 9. *Let $\beta_{i,\mathbf{b}}$ denote the Betti number of B_M in degree $\mathbf{b}$. Then*

$$\beta_{i,\mathbf{b}}(M) = \begin{cases} (-1)^{r(S)}\mu(M|S) & i = |S| - c(S) - r(S) + 1 \\ 0 & \textit{otherwise.} \end{cases}$$

where $S = \mathrm{supp}(\mathbf{b})$.

Proof. By lemma 1 we have that $\beta_{i,\mathbf{b}}(M) = \dim_{\Bbbk} \mathrm{Tor}_i^S(\Bbbk, B_M)_{\mathbf{b}}$. As before let $S = \mathrm{supp}(\mathbf{b})$.

But when we restrict $(\mathcal{F}_{\mathcal{P}_M} \otimes \Bbbk)$ to elements of degree $\mathbf{b}$ we are looking at faces supported on S. That is we are looking at $P_{M|S}$, and futhermore, all faces of exactly degree $\mathrm{supp}(\mathbf{b})$ are the minors of $M \mid S$ that are loopless (any matroid supported on anything less than the full set necessarily contains a loop). Therefore $(\mathcal{F}_{\mathcal{P}_M} \otimes \mathbb{k})_{\mathbf{b}}$ is a complex on the faces of $\mathcal{P}_{M|S}$ labeled x^S which is isomorphic to the chain complex for the homology of the Bergman complex of $M|S$.

So we have

$$\beta_{i,\mathbf{b}}(M) = \dim_{\mathbb{k}} \widetilde{H}_{\dim \mathcal{P}_{M|S}-1-i}(\mathcal{B}(M|S);\mathbb{k})$$

By corollary 7 we know that $\mathcal{B}(M|S)$ is a wedge of $\mu(M \mid S)$ $(r(M \mid S)-2)$ spheres. Therefore it only has $r(M \mid S)-2$ dimensional holes. So when

$$\dim \mathcal{P}_{M|S} - 1 - i = r(M \mid S) - 2.$$

We know that $P_{M|S} = |S| - c(M \mid S)$, where $c(M)$ is the number of connected components of a matroid M. So we have non-trivial holes when

$$i = |S| - c(M \mid S) - r(M \mid S) + 1$$

□

5 Lagrangian Matroids- Future Directions

Here, we tentatively define another type of matroid and propose an analogue of the Bergman complex. First, let $[n] = \{1,2,3\ldots,n\}$ = and $[n]^* = \{1^*,2^*,3^*,\ldots,n^*\}$. Let $J = [n] \cup [n]^*$. A set $K \subset J$ is *admissible* if when $i \in J$ then $i^* \notin J$. Admissible sets are basically in bijection with pairs of disjoint subsets of $[n]$, or *orthants*. Let J_k be the set of $k-$adminssible sets.

Definition. *A family of sets $\mathcal{F} \subset J_n$ are the bases of a* **Lagrangian** *matroid if $\mathcal{F}$ satisfies the symmetric exchange axiom.*

In [3] Booth, Moreira, and Pinto show that the notion of **circuit** and **loop** are well defined for Lagrangian Matroids. In fact they are the same as in the type A matoids discussed above. They also state that these matroids, which are symplectic matroids of rank n, as described in [5], and are equivalent to the Δ matroids of Bouchet. Whatever the case, since we have loops, and from [4] we have a general theory of ω maximal/minimal bases of a coxeter matroid, we propose the following definition

Definition. *The* **Bergman fan** *of a Lagrangian matroid M on $J = [n] \cup [n]^*$ is the set of vectors*

$$\widetilde{\mathcal{B}}_L(M) := \{\omega \in \mathbf{R}^n \ : \ M_\omega \ \textit{has no loops}\}$$

and the **Bergman complex** *of M is*

$$\mathcal{B}_L(M) := \{\ \omega \in S^{n-2} \ : \ M_\omega \ \textit{has no loops}\}$$

Note that now ω is going to have a weight class that corresponds to a face of the type B hyperplane arrangment.

6 Concluding Remarks

All of the above is related to *type A* matroids in the language of Coxeter matroids. We have seen through Ardilas proof that the classic invariant of a matroid, the mobius function, can be used to describe the $\mathbf{N}^n$-graded Betti numbers associated to the facet ideal. This is due to the above stated correspondence of the Bergman complex to the cellular resolution of the matroid polytope.

The question is now, how much of this can be transported to the theory of Coxeter matroids. For example, does the facet ideal of Lagrangian matroid have a cellular resolution via it's matroid polytope that can be characterized nicely by the analogue of the Bergman complex. It remains to see what can be shown about this *type B* Bergman complex, and whether it can be extended to all symplect matroids.

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