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Geometry and Differential Operator

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The memo is written for understanding approximations when the field-aligned coordinates are used for the conventional turbulence ordering ($k_{\|} \ll k_{\perp}$). Special cares must be taken in order to keep $m \neq 0$ for $n = 0$ modes. Otherwise, in the (m, n) phase space, the $n = 0$ and $m > 0$ modes should be suppressed to keep turbulence ordering ($k_{\|} \ll k_{\perp}$) consistent for $n = 0$ modes.

I. Geometry

In a axisymmetric toroidal system, the magnetic field can be expressed as

$$\mathbf{B} = I(\psi)\nabla\zeta + \nabla\zeta \times \nabla\psi,$$

where ψ is the poloidal flux, θ is the poloidal angle-like coordinate, and ζ is the toroidal angle. Here, $I(\psi) = RB_t$. The two important geometrical parameters are: the curvature, κ, and the local pitch, $\nu(\psi, \theta)$,

$$\nu(\psi, \theta) = I(\psi)\mathcal{J}/R^2.$$

The local pitch $\nu(\psi, \theta)$ is related to the MHD safety q by $\hat{q}(\psi) = 2\pi^{-1} \oint \nu(\psi, \theta)d\theta$ in the closed flux surface region, and $\hat{q}(\psi) = 2\pi^{-1} \int_{inboard}^{outboard} \nu(\psi, \theta)d\theta$ in the scrape-off-layer. Here $\mathcal{J} = (\nabla\psi \times \nabla\theta \cdot \nabla\zeta)^{-1}$ is the coordinate Jacobian, R is the major radius, and Z is the vertical position.

II. Geometry and Differential Operators

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A. Differential Operators

For such an axisymmetric equilibrium the metric coefficients are only functions of ψ and θ. Three spatial differential operators appear in the equations given as: $\mathbf{v_E} \cdot \nabla_\perp$, $\nabla_\parallel$ and $\nabla_\perp^2$.

$$
\begin{aligned}
\nabla_\parallel &= \mathbf{b_0} \cdot \nabla = \frac{1}{\mathcal{J}B}\frac{\partial}{\partial\theta} + \frac{I}{BR^2}\frac{\partial}{\partial\zeta} = \frac{B_p}{hB}\frac{\partial}{\partial\theta} + \frac{B_t}{RB}\frac{\partial}{\partial\zeta}, \quad (1)\\
\mathcal{J}\nabla^2 &= \frac{\partial}{\partial\psi}\left(\mathcal{J}J_{11}\frac{\partial}{\partial\psi}\right) + \frac{\partial}{\partial\psi}\left(\mathcal{J}J_{12}\frac{\partial}{\partial\theta}\right)\\
&+ \frac{\partial}{\partial\theta}\left(\mathcal{J}J_{22}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(\mathcal{J}J_{12}\frac{\partial}{\partial\psi}\right)\\
&+ \frac{1}{R^2}\frac{\partial^2}{\partial\zeta^2}. \quad (2)\\
\nabla_\parallel^2 &= \mathbf{b}_0 \cdot \nabla(\mathbf{b}_0 \cdot \nabla) = \frac{1}{\mathcal{J}B}\frac{\partial}{\partial\theta}\left(\frac{1}{\mathcal{J}B}\frac{\partial}{\partial\theta}\right) + \frac{1}{\mathcal{J}B}\frac{\partial}{\partial\theta}\left(\frac{B_t}{RB}\frac{\partial}{\partial\zeta}\right) \quad (3)\\
&+ \frac{B_t}{\mathcal{J}RB^2}\frac{\partial^2}{\partial\theta\partial\zeta} + \left(\frac{B_t}{\mathcal{J}RB}\right)^2\frac{\partial^2}{\partial\zeta^2}, \quad (4)\\
\nabla_\perp^2\Phi &= -\nabla\cdot[\mathbf{b}\times(\mathbf{b}\times\nabla\Phi)] = \nabla^2\Phi - (\nabla\cdot\mathbf{b})(\mathbf{b}\cdot\nabla\Phi) - \nabla_\parallel^2\Phi \quad (5)
\end{aligned}
$$

where the coordinate Jacobian and metric coefficients are defined as following:

$$
\begin{aligned}
\mathcal{J} &= \nabla\psi\times\nabla\theta\cdot\nabla\zeta = \frac{h}{B_p}, \quad (6)\\
h &= \sqrt{Z_\theta^2 + R_\theta^2}, \quad (7)\\
J_{11} &= |\nabla\psi|^2 = \frac{R^2}{\mathcal{J}^2}(Z_\theta^2 + R_\theta^2), \quad (8)\\
J_{12} &= J_{21} = \nabla\psi\cdot\nabla\theta = -\frac{R^2}{\mathcal{J}^2}(Z_\theta Z_\psi + R_\psi R_\theta), \quad (9)\\
J_{13} &= J_{31} = 0, \quad (10)\\
J_{22} &= |\nabla\theta|^2 = \frac{R^2}{\mathcal{J}^2}(Z_\psi^2 + R_\psi^2), \quad (11)\\
J_{23} &= J_{32} = 0, \quad (12)\\
J_{33} &= |\nabla\zeta|^2 = \frac{1}{R^2}. \quad (13)
\end{aligned}
$$

B. Concentric circular cross section inside the separatrix without the SOL

For concentric circular cross section inside the separatrix without the SOL, the differential operators are reduced to:

$$
\begin{aligned}
R &= R_0 + r cos\theta, \quad (14)\\
Z &= r sin\theta, \quad (15)
\end{aligned}
$$

$$
\begin{aligned}
B_t &= \frac{B_{t0}R_0}{R}, && (16)\\
B_p &= \frac{1}{R}\frac{\partial\psi}{\partial r}, && (17)\\
R_\psi &= \frac{cos\theta}{RB_p}, && (18)\\
R_\theta &= -rsin\theta, && (19)\\
Z_\psi &= \frac{sin\theta}{RB_p}, && (20)\\
Z_\theta &= rcos\theta, && (21)\\
\mathcal{J} &= \frac{r}{B_p}, && (22)\\
h &= r, && (23)\\
J_{11} &= |\nabla\psi|^2 = r^2B_p^2, && (24)\\
J_{12} &= J_{21} = \nabla\psi\cdot\nabla\theta = 0, && (25)\\
J_{13} &= J_{31} = 0, && (26)\\
J_{22} &= |\nabla\theta|^2 = \frac{1}{r^2}, && (27)\\
J_{23} &= J_{32} = 0, && (28)\\
J_{33} &= |\nabla\zeta|^2 = \frac{1}{R^2}, && (29)\\
\nabla^2 &\simeq \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{R^2}\frac{\partial^2}{\partial\zeta^2} && (30)
\end{aligned}
$$

C. Field-aligned coordinates with θ as the coordinate along the field line

A suitable coordinate mapping between field-aligned ballooning coordinates (x, y, z) and the usual flux coordinates (ψ, θ, ζ) is

$$\begin{aligned} x &= \psi - \psi_s, \\ y &= \theta, \\ z &= \zeta - \int_{\theta_0}^{\theta} \nu(x, y) dy. \end{aligned} \tag{31}$$

as shown in Fig. 1. The covering area given by the square ABCD in the usual flux coordinates is the same as the parallelogram ABEF in the field-aligned coordinates. The magnetic separatrix is denoted by $\psi = \psi_s$. In this choice of coordinates, x is a flux surface label, y, the poloidal angle, is also the coordinate along the field line, and z is a field line label within the flux surface.

The coordinate Jacobian and metric coefficients are defined as following:

$$\mathcal{J} = \nabla\psi \times \nabla\theta \cdot \nabla\zeta = \frac{h}{B_p}, \tag{32}$$

$$h = \sqrt{Z_\theta^2 + R_\theta^2}, \tag{33}$$

$$\mathcal{J}_{11} = |\nabla x|^2 = \frac{R^2}{\mathcal{J}^2}(Z_\theta^2 + R_\theta^2), \tag{34}$$

$$\mathcal{J}_{12} = \mathcal{J}_{21} = \nabla x \cdot \nabla y = -\frac{R^2}{\mathcal{J}^2}(Z_\theta Z_\psi + R_\psi R_\theta), \tag{35}$$

$$\mathcal{J}_{22} = |\nabla y|^2 = \frac{R^2}{\mathcal{J}^2}(Z_\psi^2 + R_\psi^2), \tag{36}$$

$$\mathcal{J}_{13} = \mathcal{J}_{31} = \nabla x \cdot \nabla z = -\nu \nabla x \cdot \nabla y - |\nabla x|^2 \left(\int_{y_0}^{y} \frac{\partial \nu(x, y)}{\partial \psi} dy \right) = -|\nabla x|^2 I_s, \tag{37}$$

$$\mathcal{J}_{23} = \mathcal{J}_{32} = \nabla y \cdot \nabla z = -\nu |\nabla y|^2 - \nu \nabla x \cdot \nabla y \left(\int_{y_0}^{y} \frac{\partial \nu(x, y)}{\partial \psi} dy \right), \tag{38}$$

$$\mathcal{J}_{33} = |\nabla z|^2 = \left| \nabla\zeta - \nu \nabla\theta - \nabla\psi \left(\int_{y_0}^{y} \frac{\partial \nu(x, y)}{\partial \psi} dy \right) \right|^2, \tag{39}$$

$$I_s = \frac{\mathcal{J}_{12}}{|\nabla\psi|^2} \nu(x, y) + \left(\int_{y_0}^{y} \frac{\partial \nu(x, y)}{\partial \psi} dy \right). \tag{40}$$

Here h is the local minor radius, I_s is the integrated local shear, and y_0 is an arbitrary integration parameter, which, depending on the choice of Jacobian, determines the location where $I_s = 0$. The disadvantage of this choice of coordinates is that the Jacobian diverges near the X-point as $B_p \to 0$ and its effect spreads over the entire flux surafces near the separatrix as the results of coordinate transform z. Therefore a better set of coordinates is needed for X-point divertor geometry. The derivatives are obtained from the chain rule as follows:

$$\frac{d}{d\psi} = \frac{\partial}{\partial x} - \left(\int_{y_0}^{y} \frac{\partial \nu(x,y)}{\partial \psi} dy\right) \frac{\partial}{\partial z}, \tag{41}$$

$$\frac{d}{d\theta} = \frac{\partial}{\partial y} - \nu(x,y)\frac{\partial}{\partial z}, \tag{42}$$

$$\frac{d}{d\zeta} = \frac{\partial}{\partial z}. \tag{43}$$

In the field-aligned ballooning coordinates, the parallel differential operator is simple, involving only one coordinate y

$$\partial_{\parallel}^{0} = \mathbf{b}_0 \cdot \nabla_{\parallel} = \left(\frac{B_p}{hB}\right)\frac{\partial}{\partial y}. \tag{44}$$

which requires a few grid points. The total axisymmetric drift operator becomes

The perturbed $\mathbf{E}\times\mathbf{B}$ drift operator becomes

$$\begin{aligned} \delta \mathbf{v_E} \cdot \nabla_{\perp} &= \frac{c}{BB_{\parallel}^{*}}\left\{-\frac{I}{J}\frac{\partial\langle\delta\phi\rangle}{\partial\theta} + B_p^2 \frac{\partial\langle\delta\phi\rangle}{\partial z}\right\}\frac{\partial}{\partial\psi} \\ &+ \frac{c}{BB_{\parallel}^{*}}\left\{\frac{I}{\mathcal{J}}\frac{\partial\langle\delta\phi\rangle}{\partial\psi} + \frac{\mathcal{J}_{12}}{R^2}\frac{\partial\langle\delta\phi\rangle}{\partial z}\right\}\frac{\partial}{\partial\theta} \\ &- \frac{c}{BB_{\parallel}^{*}}\left\{B_p^2\frac{\partial\langle\delta\phi\rangle}{\partial\psi} + \frac{\mathcal{J}_{12}}{R^2}\frac{\partial\langle\delta\phi\rangle}{\partial\theta}\right\}\frac{\partial}{\partial z}, \end{aligned} \tag{45}$$

when the conventional turbulence ordering ($k_{\parallel} \ll k_{\perp}$) is used, the perturbed $\mathbf{E}\times\mathbf{B}$ drift operator can be further reduced to a simple form

$$\delta \mathbf{v_E} \cdot \nabla_{\perp} = \frac{cB}{B_{\parallel}^{*}}\left(\frac{\partial\langle\delta\phi\rangle}{\partial z}\frac{\partial}{\partial x} - \frac{\partial\langle\delta\phi\rangle}{\partial x}\frac{\partial}{\partial z}\right) \tag{46}$$

where $\partial/\partial\theta \simeq -\nu\partial/\partial z$ is used. In the perturbed $\mathbf{E}\times\mathbf{B}$ drift operator the poloidal and radial derivatives are written in the usual flux (ψ,θ,ζ) coordinates in order to have various options for valid discretizations. The general Laplacian operator for potential is

$$\begin{aligned} \mathcal{J}\nabla^2\Phi &= \frac{\partial}{\partial x}\left(\mathcal{J}\mathcal{J}_{11}\frac{\partial\Phi}{\partial x} + \mathcal{J}\mathcal{J}_{12}\frac{\partial\Phi}{\partial y} + \mathcal{J}\mathcal{J}_{13}\frac{\partial\Phi}{\partial z}\right) \\ &+ \frac{\partial}{\partial y}\left(\mathcal{J}\mathcal{J}_{21}\frac{\partial\Phi}{\partial x} + \mathcal{J}\mathcal{J}_{22}\frac{\partial\Phi}{\partial y} + \mathcal{J}\mathcal{J}_{23}\frac{\partial\Phi}{\partial z}\right) \\ &+ \frac{\partial}{\partial z}\left(\mathcal{J}\mathcal{J}_{31}\frac{\partial\Phi}{\partial x} + \mathcal{J}\mathcal{J}_{32}\frac{\partial\Phi}{\partial y} + \mathcal{J}\mathcal{J}_{33}\frac{\partial\Phi}{\partial z}\right). \end{aligned} \tag{47}$$

The general perpendicular Laplacian operator for potential is

$$\mathcal{J}\nabla_{\perp}^2\Phi = \frac{\partial}{\partial x}\left(\mathcal{J}\mathcal{J}_{11}\frac{\partial\Phi}{\partial x} + \mathcal{J}\mathcal{J}_{12}\frac{\partial\Phi}{\partial y} + \mathcal{J}\mathcal{J}_{13}\frac{\partial\Phi}{\partial z}\right)$$

$$
\begin{aligned}
&+ \quad \frac{\partial}{\partial y}\left(\mathcal{J}\mathcal{J}_{21}\frac{\partial \Phi}{\partial x} + \mathcal{J}\mathcal{J}_{22}\frac{\partial \Phi}{\partial y} + \mathcal{J}\mathcal{J}_{23}\frac{\partial \Phi}{\partial z}\right) \\
&+ \quad \frac{\partial}{\partial z}\left(\mathcal{J}\mathcal{J}_{31}\frac{\partial \Phi}{\partial x} + \mathcal{J}\mathcal{J}_{32}\frac{\partial \Phi}{\partial y} + \mathcal{J}\mathcal{J}_{33}\frac{\partial \Phi}{\partial z}\right) \\
&- \quad \left(\frac{B_p}{hB}\right)\frac{\partial}{\partial y}\left[\left(\frac{B_p}{hB}\right)\frac{\partial \Phi}{\partial y}\right] \\
&- \quad \left(\frac{B_p}{hB}\right)^2 \frac{\partial \ln B}{\partial y}\frac{\partial \Phi}{\partial y}.
\end{aligned} \tag{48}
$$

The general perpendicular Laplacian operator for axisymmetric potential $\Phi_0(x, y)$ is

$$
\begin{aligned}
\mathcal{J}\nabla_\perp^2 \Phi_0 \quad = \quad & \frac{\partial}{\partial x}\left(\mathcal{J}\mathcal{J}_{11}\frac{\partial \Phi_0}{\partial x} + \mathcal{J}\mathcal{J}_{12}\frac{\partial \Phi_0}{\partial y}\right) \\
+ \quad & \frac{\partial}{\partial y}\left(\mathcal{J}\mathcal{J}_{21}\frac{\partial \Phi_0}{\partial x} + \mathcal{J}\mathcal{J}_{22}\frac{\partial \Phi_0}{\partial y}\right) \\
- \quad & \left(\frac{B_p}{hB}\right)\frac{\partial}{\partial y}\left[\left(\frac{B_p}{hB}\right)\frac{\partial \Phi_0}{\partial y}\right] \\
- \quad & \left(\frac{B_p}{hB}\right)^2 \frac{\partial \ln B}{\partial y}\frac{\partial \Phi}{\partial y}.
\end{aligned} \tag{49}
$$

For the perturbed potential $\delta\phi$, we can drop the $\partial/\partial y$ terms in Eq. (69) due to the elongated nature of the turbulence ($k_\parallel/k_\perp \ll 1$). The general perpendicular Laplacian operator for perturbed potential $\delta\phi$ reduces to

$$
\begin{aligned}
\mathcal{J}\nabla_\perp^2 \delta\phi \quad = \quad & \frac{\partial}{\partial x}\left(\mathcal{J}\mathcal{J}_{11}\frac{\partial \delta\phi}{\partial x} + \mathcal{J}\mathcal{J}_{13}\frac{\partial \delta\phi}{\partial z}\right) \\
+ \quad & \frac{\partial}{\partial z}\left(\mathcal{J}\mathcal{J}_{31}\frac{\partial \delta\phi}{\partial x} + \mathcal{J}\mathcal{J}_{33}\frac{\partial \delta\phi}{\partial z}\right).
\end{aligned} \tag{50}
$$

If the non-split potential Φ is a preferred option, the gyrokinetic Poisson equation (18) and the general perpendicular Laplacian operator Eq. (69) have to be used. Then the assumption $k_\parallel/k_\perp \ll 1$ is not used to simplify the perpendicular Laplacian operator.