Exercise - 1 / Question - 4

(1) Single data point: $\boxed{p(x_n \mid \theta)}$

(2) Likelihood: $\boxed{L(\theta) = p(X \mid \theta) = \prod_{n=1}^{N} p(x_n \mid \theta)}$

(3) Log - likelihood:

$$E(\theta) = \ln(L(\theta)) = \ln\left(\prod_{n=1}^{N} p(x_n \mid \theta)\right)$$

$$= \ln\left(p(x_1 \mid \theta) \cdot p(x_2 \mid \theta) \cdot \ldots p(x_N \mid \theta)\right)$$

Apply

Rule of log:
$\log(a \cdot b) = \log a + \log b$

Get

$$= \ln(p(x_1 \mid \theta)) + \ln(p(x_2 \mid \theta)) + \ldots + \ln(p(x_N \mid \theta))$$

$$= \boxed{\sum_{n=1}^{N} \ln(p(x_n \mid \theta))}$$

$$\boxed{E(\theta) = \ln(L(\theta)) = \ln\left(\prod_{n=1}^{N} p(x_n \mid \theta)\right) = \sum_{n=1}^{N} \ln(p(x_n \mid \theta))}$$

(4) Maximize Log-likelihood, i.e. take derivative and set to zero:

$$\frac{\partial E(\theta)}{\partial \theta} = \frac{\partial}{\partial \theta} \sum_{n=1}^{N} \ln(p(x_n \mid \theta))$$

$$= \frac{\partial}{\partial \theta}\left[\ln(p(x_1 \mid \theta)) + \ln(p(x_2 \mid \theta)) + \ldots + \ln(p(x_N \mid \theta))\right]$$

$$= \frac{\partial}{\partial\theta} \ln(p(x_1|\theta)) + \frac{\partial}{\partial\theta} \ln(p(x_2|\theta)) + \dots + \frac{\partial}{\partial\theta} \ln(p(x_N|\theta))$$

Rule of derivative:

$$\frac{d}{dx} \ln[f(x)] = \frac{1}{f(x)} f'(x)$$

Apply

?

Get

$$= \frac{1}{p(x_1|\theta)} \cdot \frac{\partial}{\partial\theta} p(x_1|\theta) + \dots + \frac{1}{p(x_N|\theta)} \frac{\partial}{\partial\theta} p(x_N|\theta)$$

$$= \boxed{\sum_{n=1}^{N} \frac{\frac{\partial}{\partial\theta} p(x_n|\theta)}{p(x_n|\theta)} \overset{!}{=} 0}$$

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1 $p(x|\theta) = \theta^2 . x . \exp(-\theta x) \, f(x)$ where $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

!! As we know all data points $x_1, \dots, x_N > 0$, $f(x)$ will take value 1 for all data points $x_1, \dots, x_N$.

$$= \theta^2 . x . \exp(-\theta x) . 1$$

3 $E(\theta) = \ln(L(\theta)) = \ln\left(\prod_{n=1}^{N} p(x_n|\theta)\right) = \sum_{n=1}^{N} \ln(p(x_n|\theta))$

!! As we know $p(x|\theta)$ from 1, let's put it there.

$$= \sum_{n=1}^{N} \ln(\theta^2 . x_n . \exp(-\theta x_n))$$

④ $$\frac{\partial E(\theta)}{\partial \theta} = \frac{\partial}{\partial \theta}\left[\sum_{n=1}^{N} \ln\left(\theta^2 \cdot x_n \cdot \exp(-\theta x_n)\right)\right]$$

Apply & Get

Rule of log:

$\log(a.b) = \log a + \log b$

$$= \frac{\partial}{\partial \theta}\left[\sum_{n=1}^{N} \ln\theta^2 + \ln x_n + \ln\left(\exp(-\theta x_n)\right)\right]$$

$$= \frac{\partial}{\partial \theta}\left[\sum_{n=1}^{N} \ln\theta^2 + \sum_{n=1}^{N} \ln x_n + \sum_{n=1}^{N} \ln\left(\exp(-\theta x_n)\right)\right]$$

Apply

Rule of log:

$\log_b M^k = k \log_b M$

Apply

Rule of log:

$\log_b b^k = k$

Get

Get

$$= \frac{\partial}{\partial \theta}\left[\sum_{n=1}^{N} 2\ln\theta + \sum_{n=1}^{N} \ln x_n + \sum_{n=1}^{N} -\theta x_n\right]$$

$$= \frac{\partial}{\partial \theta}\left[2N\ln\theta + \sum_{n=1}^{N} \ln x_n + \left(\sum_{n=1}^{N} -x_n\right)\cdot\theta\right]$$

$$= \frac{\partial(2N\ln\theta)}{\partial\theta} + \frac{\partial}{\partial\theta}\sum_{n=1}^{N} \ln x_n - \sum_{n=1}^{N} x_n \cdot \frac{\partial(\theta)}{\partial\theta}$$

$$= 2N\cdot\frac{1}{\theta} + 0 - \sum_{n=1}^{N} x_n \cdot 1$$

$$= \frac{2N}{\theta} + 0 - \sum_{n=1}^{N} x_n \overset{!}{=} 0 \quad ***$$

$$\boxed{\frac{2N}{\theta} \quad \sum_{n=1}^{N} x_n} \Rightarrow \boxed{\theta = \frac{2N}{\sum_{n=1}^{N} x_n}}$$