Institut für Strömungsmechanik und Umweltphysik im Bauwesen

Assignment 3

Finite Volumes - Transport of Heat and Water Vapor

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Information

- working groups of 2-3 people
- all codes have to be submitted via e-mail to krishna@hydromech.uni-hannover.de
- the present document has to be signed and handed in
- tasks marked with a * are optional
- add comments to all of your codes
- all plots have to have labeled axes and legends

Declaration of Independence

I hereby declare that the present assignment was worked on independently without help from a third party. No codes or other results were taken from other homework.

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1 Problem: Moisture in a wall

At the inner side of an exterior wall the temperature is higher than at its outer side. If water vapor diffuses (gas phase) from a warm room into an exterior wall, it can precipitate inside the wall into liquid phase, because cold air can hold less water vapor than warm air. This leads to moisture inside the wall and should be avoided. Therefore, we want to investigate with a numerical simulation when this phenomenon occurs.

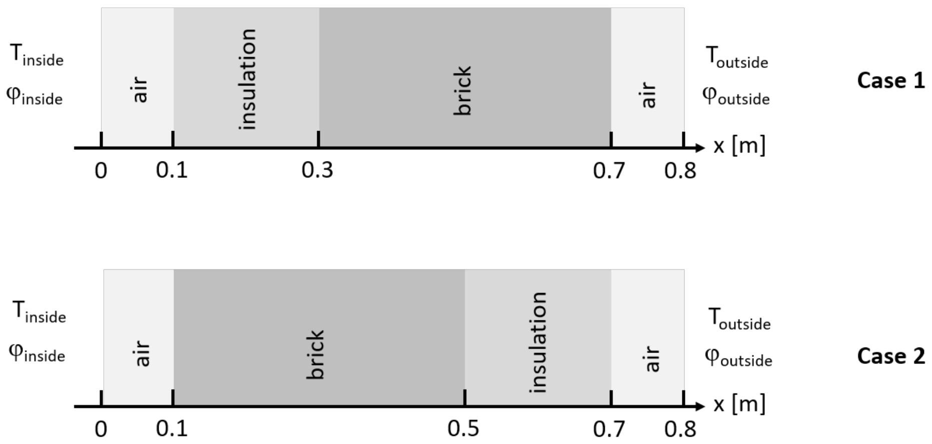

Figure 1: Composition of the wall. Case 1: inside insulation. Case 2: outside insulation.

The wall consists of an insulation layer and brick (see Fig. 1). The material parameters are given in Tab. 1. At the inner side at $x = 0\,\mathrm{m}$ the temperature is $T_{inside} = 25\,°\mathrm{C}$ and the relative air humidity 60 %. At the outer side at $x = L\,\mathrm{m}$ the temperature is $T_{outside} = 5\,°\mathrm{C}$ and the relative air humidity 40 %.

	heat conductivity λ in (W/(m·K))	diffusion resistivity μ in (-)	density ρ in (kg/m^3)
air	0.026	1	1.3
brick	1	5	1600
insulation	0.05	3	100

specific heat capacity in all materials: c_p**=1000** $\frac{J}{kg\cdot K}$

Table 1: Material parameters

1.1 Temperature profile

Fourier's law connects heat current density H in (W/m^2) and temperature T in (°C) or (K) via heat conductivity λ in (W/(m·K)):

$$H = -\lambda \frac{\partial T}{\partial x} \tag{1}$$

The stationary heat transport equation with spatially variable heat conductivity $\lambda(x)$ is

$$\boxed{-\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right) = 0.} \tag{2}$$

The boundary conditions are

$$T|_{x=0} = T_L \tag{3}$$

$$T|_{x=L} = T_R. \tag{4}$$

Time dependent heat transport is described by

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) \tag{5}$$

with mass density ρ and specific heat capacity c_p.

1.2 Moisture profile

The mass current density J in (kg/m^{2}s) of water vapor can be written in terms of the water vapor diffusion coefficient in air δ in (kg/(m·s·Pa)), the dimensionless diffusion resistivity of the material $\mu(x)$ and the partial pressure of water vapor p in (Pa):

$$J = -\frac{\delta}{\mu(x)} \frac{\partial p}{\partial x} \tag{6}$$

This leads to the stationary transport equation for water vapor with spatially variable diffusion resistivity $\mu(x)$:

$$-\frac{\partial J}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\delta}{\mu(x)} \frac{\partial p}{\partial x}\right) = 0 \tag{7}$$

The water vapor diffusion coefficient in air δ is spatially constant and can therefore be eliminated from this equation:

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu(x)} \frac{\partial p}{\partial x}\right) = 0 \tag{8}$$

The boundary conditions are

$$p|_{x=0} = p_L \tag{9}$$

$$p|_{x=L} = p_R. \tag{10}$$

1.3 Dew point

For a given temperature θ in (°C) the partial pressure of water vapor under saturated conditions can be calculated with the following empirical equation according to DIN-Norm [1]

$$\boxed{p_{sat} = a\left(b + \frac{\theta}{100\,°\mathrm{C}}\right)^n.} \tag{11}$$

The coefficients for a temperature range of $0\,°\mathrm{C} \leq \theta \leq 30\,°\mathrm{C}$ are $a = 288.68\,\mathrm{Pa}$, $b = 1.098$ and $n = 8.02$.

If the partial pressure of the water vapor is larger than the partial pressure under saturated conditions, the water precipitates into the liquid phase.

1.4 Relative air humidity

The relation between relative air humidity φ and partial pressure of the water vapor p is

$$\boxed{\varphi\left(\theta\right) = \frac{p}{p_{sat}\left(\theta\right)}.} \tag{12}$$

2 Discretization with finite volumes

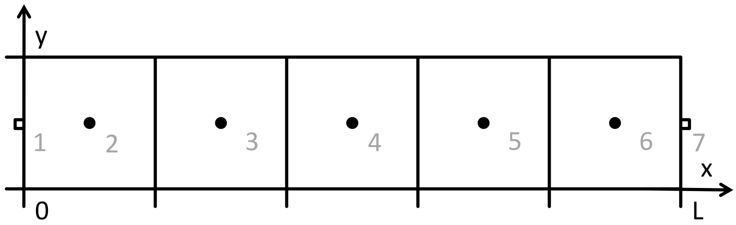

Figure 2: Example for a finite volume discretization of a domain of length L. The node numbers i are in grey. The black points mark the centers of the volume elements. The small squares mark the centers of the boundaries for which boundary conditions are given. The distance between inner nodes is Δx. The distance between an inner and a boundary node is $\Delta x/2$.

We consider a one-dimensional equidistant grid with continuously increasing numbers for the volume elements (see Fig. 2). Integration over one volume element V_i of the stationary heat conduction equation (2) leads to

$$-\int_{V_i} \frac{\partial H(x)}{\partial x} dx = \int_{V_i} \frac{\partial}{\partial x} \left(\lambda(x) \frac{\partial T}{\partial x} \right) dx. \tag{13}$$

The volume V_i with the central point x_i is limited by the boundaries at positions $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$. We assume that the extent in y and z direction is constant for all volume elements. The integral can be simplified to:

$$-\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial H(x)}{\partial x} dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x} \left(\lambda(x) \frac{\partial T}{\partial x} \right) dx. \tag{14}$$

Approximating the variables as constant per volume element

$$T_i = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} T(x)\, dx \tag{15}$$

with the length of the volume element and the distance between two nodes $\Delta x = \left| x_{i+\frac{1}{2}} - x_{i+\frac{1}{2}} \right|$ leads to

$$\boxed{- H|_{x=x_{i+\frac{1}{2}}} + H|_{x=x_{i-\frac{1}{2}}} = \left(\lambda(x) \frac{\partial T}{\partial x}\bigg|_{x=x_{i+\frac{1}{2}}} - \lambda(x) \frac{\partial T}{\partial x}\bigg|_{x=x_{i-\frac{1}{2}}} \right).} \tag{16}$$

The fluxes over the element interfaces into the neighbouring cells are approximated by central finite differences with $\Delta x/2$:

$$\boxed{H|_{x=x_{i+\frac{1}{2}}} = -\lambda(x) \frac{\partial T}{\partial x}\bigg|_{x=x_{i+\frac{1}{2}}} \approx -\lambda_{i+\frac{1}{2}} \frac{T_{i+1} - T_i}{\Delta x}.} \tag{17}$$

If $\lambda(x)$ is a continuous function, then $\lambda_{i+\frac{1}{2}} = \lambda\left(x_{i+\frac{1}{2}}\right)$. Usually, the heat conductivity λ is known only at the central points of the volume elements. Then, an average value can be used for $\lambda_{i+\frac{1}{2}}$.

For the spatial discretization we get

$$\boxed{\lambda_{i+\frac{1}{2}} \frac{T_{i+1} - T_i}{\Delta x} - \lambda_{i-\frac{1}{2}} \frac{T_i - T_{i-1}}{\Delta x} = 0.} \tag{18}$$

2.1 Interpolation

To calculate the fluxes between cells an average conductivity is needed. The arithmetic mean of the conductivities of volume elements i and $i+1$ is

$$\lambda_{i+\frac{1}{2}} = \frac{1}{2} \left(\lambda_i + \lambda_{i+1} \right). \tag{19}$$

Alternatively, the harmonic mean can be used:

$$\lambda_{i+\frac{1}{2}} = 2 \left(\frac{1}{\lambda_i} + \frac{1}{\lambda_{i+1}} \right)^{-1}. \tag{20}$$

3 Tasks

3.1 Interpolation and conductivities

1. Calculate the **harmonic and the arithmetic** means of the heat conductivity λ for a wall that consists of 20 cm insulation and 20 cm brick.

$$\bar{\lambda}_h = 2\left(\frac{1}{\lambda_{in}} + \frac{1}{\lambda_b}\right)^{-1} = 2\left(\frac{1}{0{,}05} + 1\right)^{-1} \frac{W}{m \cdot K} = 0{,}095 \frac{W}{m \cdot K}$$

$$\bar{\lambda}_a = \frac{1}{2}\left(\lambda_{in} + \lambda_b\right) = \frac{1}{2}\left(0{,}05 + 1\right) \frac{W}{m \cdot K} = 0{,}525 \frac{W}{m \cdot K}$$

2. Calculate the corresponding heat current density through the wall when the outside temperature is 20°C and the inside temperature is 30°C.

$$H = -\lambda \frac{dT}{dx} \approx -\lambda \frac{T_i - T_o}{\Delta x}$$

$$\Delta x = 2 \cdot 20cm = 40\,cm = 0{,}4m$$

$$T_i - T_o = 30°C - 20°C = 10°C = 283{,}15\,K$$

$$H_h = \lambda_h \frac{283{,}15\,K}{0{,}4\,m} = 67{,}25 \frac{W}{m^2}$$

$$H_a = \lambda_a \frac{283{,}15\,K}{0{,}4\,m} = 371{,}63 \frac{W}{m^2}$$

3. Write the functions `lambda = f_lambda1(x)` and `lambda = f_lambda2(x)` for case 1 and 2, respectively (see Fig. 1). These functions shall return a vector containing the heat conductivity λ at different values of x. `x` shall be a vector and its values shall be in the range between 0 and L.

4. Write the functions `inv_mue = f_mue1(x)` and `inv_mue = f_mue2(x)` for case 1 and 2, respectively (see Fig. 1). These functions shall return a vector containing the inverse of the diffusion resistivity μ at different values of x. `x` shall be a vector and its values shall be in the range between 0 and L.

5. Write the functions `rho = f_rho1(x)` and `rho = f_rho2(x)` for case 1 and 2, respectively (see Fig. 1). These functions shall return a vector containing the density ρ at different values of x. `x` shall be a vector and its values shall be in the range between 0 and L.

3.2 Numerical Scheme

1. What type of boundary conditions would you use to simulate the heat and the moisture transport for cases 1 and 2 (see Fig. 1)?

Dirichlet Boundary condition

2. Assuming you discretize your domain into 5 elements according to Fig. 2, write down your finite volume discretization of the heat transport for each element. Include the boundary nodes 1 and 7 with given values T_{inside} and $T_{outside}$.

$$\times \; \boxed{1 \mid 2 \mid 3 \mid 4 \mid 5} \; \times \qquad \lambda_{i+\frac{1}{2}} T_{i+1} - \left(\lambda_{i+\frac{1}{2}} + \lambda_{i-\frac{1}{2}}\right) T_i + \lambda_{i-\frac{1}{2}} T_{i-1} = 0$$

$$T_1 = T_{inside}$$

$$\lambda_{2+\frac{1}{2}} T_3 - \left(\lambda_{2+\frac{1}{2}} + 2\lambda_{2-\frac{1}{2}}\right) T_2 + 2\lambda_{2-\frac{1}{2}} T_1 = 0$$

$$\lambda_{3+\frac{1}{2}} T_4 - \left(\lambda_{3+\frac{1}{2}} + \lambda_{3-\frac{1}{2}}\right) T_3 + \lambda_{3-\frac{1}{2}} T_2 = 0$$

$$\lambda_{4+\frac{1}{2}} T_5 - \left(\lambda_{4+\frac{1}{2}} + \lambda_{4-\frac{1}{2}}\right) T_4 + \lambda_{4-\frac{1}{2}} T_3 = 0$$

$$\lambda_{5+\frac{1}{2}} T_6 - \left(\lambda_{5+\frac{1}{2}} + \lambda_{5-\frac{1}{2}}\right) T_5 + \lambda_{5-\frac{1}{2}} T_4 = 0$$

$$2\lambda_{6+\frac{1}{2}} T_7 - \left(2\lambda_{6+\frac{1}{2}} + \lambda_{6-\frac{1}{2}}\right) T_6 + \lambda_{6-\frac{1}{2}} T_5 = 0$$

$$T_7 = T_{outside}$$

3. Now, build your system of equations $\mathbf{A_T} \cdot \vec{T} = \vec{rhs}$. Fill out Matrix $\mathbf{A_T}$ and right hand side vector $\vec{rhs}$. Include the boundary conditions.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\lambda_{2-\frac{1}{2}} & -(\lambda_{2+\frac{1}{2}} + 2\lambda_{2-\frac{1}{2}}) & \lambda_{2+\frac{1}{2}} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{3-\frac{1}{2}} & -(\lambda_{3+\frac{1}{2}} + \lambda_{3-\frac{1}{2}}) & \lambda_{3+\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{4-\frac{1}{2}} & -(\lambda_{4-\frac{1}{2}} + \lambda_{4+\frac{1}{2}}) & \lambda_{4+\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{5-\frac{1}{2}} & -(\lambda_{5-\frac{1}{2}} + \lambda_{5+\frac{1}{2}}) & \lambda_{5+\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{6-\frac{1}{2}} & -(2\lambda_{6+\frac{1}{2}} + \lambda_{6-\frac{1}{2}}) & 2\lambda_{6+\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{A_T \rightarrow \text{see below}} \cdot \vec{T} = \begin{bmatrix} T_{inside} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ T_{outside} \end{bmatrix} \qquad (21)$$

$$A_T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\lambda_{2-\frac{1}{2}} & -(\lambda_{2+\frac{1}{2}} + 2\lambda_{2-\frac{1}{2}}) & \lambda_{2+\frac{1}{2}} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{3-\frac{1}{2}} & -(\lambda_{3+\frac{1}{2}} + \lambda_{3-\frac{1}{2}}) & \lambda_{3+\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{4-\frac{1}{2}} & -(\lambda_{4-\frac{1}{2}} + \lambda_{4+\frac{1}{2}}) & \lambda_{4+\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{5-\frac{1}{2}} & -(\lambda_{5-\frac{1}{2}} + \lambda_{4+\frac{1}{2}}) & \lambda_{5+\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{6-\frac{1}{2}} & -(2\lambda_{6+\frac{1}{2}} + \lambda_{6-\frac{1}{2}}) & 2\lambda_{6+\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\frac{1}{\mu_{2-\frac{1}{2}}} & -\left(\frac{1}{\mu_{2+\frac{1}{2}}} + \frac{2}{\mu_{2-\frac{1}{2}}}\right) & \frac{1}{\mu_{2+\frac{1}{2}}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\mu_{3-\frac{1}{2}}} & -\left(\frac{1}{\mu_{3+\frac{1}{2}}} + \frac{1}{\mu_{3-\frac{1}{2}}}\right) & \frac{1}{\mu_{3+\frac{1}{2}}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\mu_{4-\frac{1}{2}}} & -\left(\frac{1}{\mu_{4-\frac{1}{2}}} + \frac{1}{\mu_{4+\frac{1}{2}}}\right) & \frac{1}{\mu_{4+\frac{1}{2}}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_{5-\frac{1}{2}}} & -\left(\frac{1}{\mu_{5-\frac{1}{2}}} + \frac{1}{\mu_{4+\frac{1}{2}}}\right) & \frac{1}{\mu_{5+\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu_{6-\frac{1}{2}}} & -\left(\frac{2}{\mu_{6+\frac{1}{2}}} + \frac{1}{\mu_{6-\frac{1}{2}}}\right) & \frac{2}{\mu_{6+\frac{1}{2}}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$p_{inside} = \varphi(T_{inside}) \cdot p_{sat}(inside)$

$p_{outside} = \varphi(T_{outside}) \cdot p_{sat}(T_{outside})$

4. Analogously, build your system of equations $\mathbf{A_p} \cdot \vec{p} = \vec{rhs}$ for the water vapor transport. Fill out Matrix $\mathbf{A_p}$ and right hand side vector $\vec{rhs}$. Include the boundary conditions.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\frac{1}{\mu_{2-\frac12}} & -\left(\frac{1}{\mu_{2+\frac12}}+\frac{2}{\mu_{2-\frac12}}\right) & \frac{1}{\mu_{2+\frac12}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\mu_{3-\frac12}} & -\left(\frac{1}{\mu_{3+\frac12}}+\frac{1}{\mu_{3-\frac12}}\right) & \frac{1}{\mu_{3+\frac12}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\mu_{4-\frac12}} & -\left(\frac{1}{\mu_{4-\frac12}}+\frac{1}{\mu_{4+\frac12}}\right) & \frac{1}{\mu_{4+\frac12}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_{5-\frac12}} & -\left(\frac{1}{\mu_{5-\frac12}}+\frac{1}{\mu_{4+\frac12}}\right) & \frac{1}{\mu_{5+\frac12}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu_{6-\frac12}} & -\left(\frac{2}{\mu_{6+\frac12}}+\frac{1}{\mu_{6-\frac12}}\right) & \frac{2}{\mu_{6+\frac12}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{A_p \text{ see above}} \cdot \vec{p} = \begin{bmatrix} p_{inside} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p_{outside} \end{bmatrix} \tag{22}$$

3.3 Implementation in Matlab

1. Write the function `[A, x] = matrix(@f, L, N)` that generates the matrix $\mathbf{A} \in \mathbb{R}^{N\times N}$ for either your heat or your water vapor transport.

	@f	function handle to one of your functions `f_lambda1`,`f_mue1`, ...
With input values ...	L	length of the domain
	N	number of elements

	A	system matrix
With output values ...	x	vector containing the locations of the elements center points and of the boundary nodes

2. Write the script `wall.m` which calculates and plots the profiles of temperature, moisture and saturated partial pressure. Follow the instructions below:

   - use your function `matrix` to build $\mathbf{A_T}$ and $\mathbf{A_p}$
   - calculate the temperature and the pressure distribution
   - calculate p_{sat} along your domain
   - calculate φ along your domain
   - plot one figure containing subplots for the temperature, p_{sat} and φ . Highlight the areas of precipitation in your plots.

3. Use your script `wall.m` to simulate the cases 1 and 2 (see Fig. 1). Discretize your domain into 80 elements. Save the results in the figures `case1.png` and `case2.png`.

4. Compare the results. Which wall composition would you choose to insulate your house and why?

We would choose Case 2, as in this case $\varphi < 1$ and therefore water does not precipate to liquid phase, which would cause mold.

In case 2 φ exceeds 1 and therefore leads to mold.

3.4 Time dependent implementation

[In this section, only the heat transport equation (equation 5) is considered.]

1. Develop a numerical scheme to simulate time dependent heat transport (see equation 5). Use implicit time integration thus the discretized system reads $\mathbf{A_T}\vec{T}^n = \vec{T}^{n-1}$. Write down the decretization equation for element i and timestep n.

see below

Time Dependent Transport Equation

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right)$$

$$\Rightarrow \int_{V_i} \rho c_p \frac{\partial T}{\partial t} dV = \int_{V_i} \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) dV$$

$$\Rightarrow \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho c_p \frac{\partial T}{\partial t} dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) dx$$

central difference $\Rightarrow$ for spatial discretization

$$\rho_i c_p \frac{\partial T}{\partial t} \Delta x = \lambda \frac{\partial T}{\partial x}\bigg|_{i+\frac{1}{2}} - \lambda \frac{\partial T}{\partial x}\bigg|_{i-\frac{1}{2}}$$

$$= \lambda_{i+\frac{1}{2}} \frac{T_{i+1} - T_i}{\Delta x} - \lambda_{i-\frac{1}{2}} \frac{T_i - T_{i-1}}{\Delta x}$$

implicit time integration for $i = 3,4,5$

$$\rho_i c_p \frac{T_i^n - T_i^{n-1}}{\Delta t} \Delta x = \lambda_{i+\frac{1}{2}} \frac{T_{i+1}^n - T_i^n}{\Delta x} - \lambda_{i-\frac{1}{2}} \frac{T_i^n - T_{i-1}^n}{\Delta x}$$

$$\Leftrightarrow T_i^n - T_i^{n-1} = \frac{\Delta t}{\Delta x^2} \frac{1}{\rho c_p}\left(\lambda_{i+\frac{1}{2}}\left(T_{i+1}^n - T_i^n\right) - \lambda_{i-\frac{1}{2}}\left(T_i^n - T_{i-1}^n\right)\right)$$

$$\Leftrightarrow -T_i^{n-1} = \frac{\Delta t}{\Delta x^2} \frac{1}{\rho_i c_p}\left(T_{i+1}^n\left(\lambda_{i+\frac{1}{2}}\right) - T_i^n\left(\lambda_{i+\frac{1}{2}} + \lambda_{i-\frac{1}{2}} + \frac{\Delta x^2 \rho_i c_p}{\Delta t}\right) + T_{i-1}^n \lambda_{i-\frac{1}{2}}\right)$$

$$\Leftrightarrow T_i^{n-1} = \frac{\Delta t}{\Delta x^2} \frac{1}{\rho_i c_p}\left(-\lambda_{i+\frac{1}{2}} T_{i+1}^n + \left(\lambda_{i+\frac{1}{2}} + \lambda_{i-\frac{1}{2}} + \frac{\Delta x^2 \rho_i c_p}{\Delta t}\right) T_i^n - \lambda_{i-\frac{1}{2}} T_{i-1}^n\right)$$

$$\Leftrightarrow T_i^{n-1} = -\left(\frac{\Delta t \, \lambda_{i-\frac{1}{2}}}{\rho_i c_p \Delta x^2}\right) T_{i-1}^n + \left(1 + \frac{\Delta t}{\rho_i c_p \Delta x^2}\left(\lambda_{i+\frac{1}{2}} + \lambda_{i-\frac{1}{2}}\right)\right) T_i^n - \left(\frac{\Delta t \, \lambda_{i+\frac{1}{2}}}{\rho_i c_p \Delta x^2}\right) T_{i+1}^n$$

$$T_2^{n-1} = \frac{\Delta t}{\Delta x^2 \rho_2 c_p}\left(-\lambda_{1+\frac{1}{2}} T_3^n + \left(\lambda_{2+\frac{1}{2}} + 2\lambda_{2-\frac{1}{2}} + \frac{\Delta x^2 \rho_2 c_p}{\Delta t}\right) T_2^n - 2\lambda_{2-\frac{1}{2}} T_1^n\right)$$

$$T_6^{n-1} = \frac{\Delta t}{\Delta x^2 \rho_6 c_p}\left(-2\lambda_{6+\frac{1}{2}} T_7^n + \left(2\lambda_{6+\frac{1}{2}} + \lambda_{6-\frac{1}{2}} + \frac{\Delta x^2 \rho_6 c_p}{\Delta t}\right) T_6^n + \lambda_{6-\frac{1}{2}} T_5^n\right)$$

$$T_1^{n-1} = T_{inside}$$

$$T_7^{n-1} = T_{outside}$$

$$\Longrightarrow \quad A_{T_{ij}} = -\frac{\Delta t}{\Delta X}\frac{1}{\rho_i c_p} A_{T_{ij}}^{\text{without time}} + \mathbb{1}_{ij} \qquad \text{for } i,j \in [2, N-1]$$

$$a = -\frac{\Delta t}{\Delta X^2 c_p}$$

$$A_T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
a\frac{2}{\rho_2}\lambda_{2-\frac{1}{2}} & -a\left(\lambda_{2+\frac{1}{2}} + 2\lambda_{2-\frac{1}{2}}\right)\frac{1}{\rho_2}+1 & a\lambda_{2+\frac{1}{2}}\cdot\frac{1}{\rho_2} & 0 & 0 & 0 & 0 \\
0 & a\lambda_{3-\frac{1}{2}}\frac{1}{\rho_3} & -a\frac{\left(\lambda_{3+\frac{1}{2}} + \lambda_{3-\frac{1}{2}}\right)}{\rho_3}+1 & a\lambda_{3+\frac{1}{2}}/\rho_3 & 0 & 0 & 0 \\
0 & 0 & a\frac{\lambda_{4-\frac{1}{2}}}{\rho_4} & -a\frac{\left(\lambda_{4-\frac{1}{2}} + \lambda_{4+\frac{1}{2}}\right)}{\rho_4}+1 & a\lambda_{4+\frac{1}{2}}/\rho_4 & 0 & 0 \\
0 & 0 & 0 & a\frac{\lambda_{5-\frac{1}{2}}}{\rho_5} & -a\frac{\left(\lambda_{5-\frac{1}{2}} + \lambda_{4+\frac{1}{2}}\right)}{\rho_5}+1 & a\lambda_{5+\frac{1}{2}}/\rho_5 & 0 \\
0 & 0 & 0 & 0 & a\frac{\lambda_{6-\frac{1}{2}}}{\rho_6} & -a\frac{\left(2\lambda_{6+\frac{1}{2}} + \lambda_{6-\frac{1}{2}}\right)}{\rho_6}+1 & a2\lambda_{6+\frac{1}{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

2. How does our matrix $\mathbf{A_T}$ change?

$$\mathbf{A_T} = \begin{bmatrix} \text{see above} \end{bmatrix} \tag{23}$$

3. * Write the script `walltransient.m` to implement your time dependent scheme.

   - Use case 1 as your domain.
   - As an initial state, calculate the stationary solution with the inside temperature equal to the outside temperature $T_{inside} = T_{outside} = 20°C$.
   - Now set the right boundary temperature to $-5°C$.

   At what time the temperature in the brick falls below 15°C? At what point does the temperature in the brick falls below 15°C?