Detecting Exoplanets and recovering Parameters

A short recap of my work in week 4

What did Tamanini and Danielski do?

1. Steal the important equations for the waveform of the strain h(t) from Cutler:

$$h_{I,II}(t) = \frac{\sqrt{3}}{2} A_{I,II}(t) \cos\left[\Psi_{obs}(t) + \Phi_{I,II}^{(p)}(t) + \Phi_D(t)\right]$$

2. Look at the Doppler-signal of a circumbinary exoplanet:

$$f_{obs}(t) = \left(1 + \frac{v_{\parallel}(t)}{c}\right) f_{GW} = \left(1 - \frac{K}{c}\cos\left(\frac{2\pi}{P}t + \varphi_0\right)\right) f_{GW}$$

3. Derive $\frac{\partial h_{I,II}}{\partial \lambda}(t)$ for the 9 par‘s of interest: $\ln(A)\,, f_0, \theta_S, \varphi_S, \theta_L, \varphi_L, K, P, \varphi_0$

4. Compute numerically the integral

$$\Gamma_{ij} = \frac{2}{S_n(f_0)} \sum_{\alpha=I,II} \int_0^{T_0} dt\, \frac{\partial h_\alpha}{\partial \lambda_i}(t) \frac{\partial h_\alpha}{\partial \lambda_j}(t) \;\; \Rightarrow \;\; \sigma_i^2 = \Sigma_{ii} = \left(\Gamma^{-1}\right)_{ii}$$

Deriving $\frac{\partial h_{I,II}}{\partial \lambda}(t)$ analytically

For the parameters of the exoplanet we can do this analytically, as the presence of an exoplanet only leads to a change in phase/frequency of the DWDs/GW:

$$\frac{\partial h_{I,II}}{\partial \lambda}(t) = -\frac{\sqrt{3}}{2} A_{I,II}(t)\left[2\pi \int_0^t \frac{\partial f}{\partial \lambda}(t')dt' + \frac{\partial \Phi_D}{\partial \lambda}(t)\right] \times \sin\left[2\pi \int_0^t f(t')dt' + \Phi_{I,II}^{(p)}(t) + \Phi_D(t)\right]$$

With $f(t) = \left(1 + \frac{v_{\parallel}(t)}{c}\right) f_{GW} = \left(1 - \frac{K}{c}\cos\left(\frac{2\pi}{P}t + \varphi_0\right)\right)(f_0 + f_1 t)$ -> analytical $\frac{\partial f}{\partial \lambda}(t)$ for $\lambda = \ln(A), f_0, K, P, \varphi_0$

```
dfK = D[f[K, P, t, ϕ], K]
leite ab
```

$$-\frac{\mathrm{Cos}\left[\frac{2\pi t}{P} + \phi\right] f_0}{c}$$

```
dfP = D[f[K, P, t, ϕ], P]
leite ab
```

$$-\frac{2 K \pi t\, \mathrm{Sin}\left[\frac{2\pi t}{P} + \phi\right] f_0}{c P^2}$$

```
dfϕ = D[f[K, P, t, ϕ], ϕ]
leite ab
```

$$\frac{K\, \mathrm{Sin}\left[\frac{2\pi t}{P} + \phi\right] f_0}{c}$$

And we can also do $2\pi \int_0^t \frac{\partial f}{\partial \lambda}(t')dt'$ analytically

```
Integrate[dfK, {t, 0, t'}]
```
integriere

$$-\frac{P\,\mathrm{Cos}\left[\phi + \frac{\pi t'}{P}\right]\,\mathrm{Sin}\left[\frac{\pi t'}{P}\right]\,f_0}{c\,\pi}$$

```
Integrate[dfP, {t, 0, t'}] // Simplify
```
integriere vereinfache

$$\frac{K\,f_0\,\left(P\,\left(\mathrm{Sin}[\phi] - \mathrm{Sin}\left[\phi + \frac{2\pi t'}{P}\right]\right) + 2\,\pi\,\mathrm{Cos}\left[\phi + \frac{2\pi t'}{P}\right]\,t'\right)}{2\,c\,P\,\pi}$$

```
Integrate[dfϕ, {t, 0, t'}] // Simplify
```
integriere vereinfache

$$\frac{K\,P\,\mathrm{Sin}\left[\frac{\pi t'}{P}\right]\,\mathrm{Sin}\left[\phi + \frac{\pi t'}{P}\right]\,f_0}{c\,\pi}$$

```
Integrate[f[K, P, t, ϕ], {t, 0, t'}] // Simplify
```
integriere vereinfache

$$f_0\left(-\frac{K\,P\,\mathrm{Cos}\left[\phi + \frac{\pi t'}{P}\right]\,\mathrm{Sin}\left[\frac{\pi t'}{P}\right]}{c\,\pi} + t'\right)$$

Regarding the position and angular momentum

The position and angular momentum of the binary source influences the strain non-trivially, see [3]:

$$h_I(t) = \frac{\sqrt{3}}{2} A_+ F_I^+(\theta_S, \phi_S, \psi_S)\cos(2\pi f t) + \frac{\sqrt{3}}{2} A_\times F_I^\times(\theta_S, \phi_S, \psi_S)\sin(2\pi f t)\,, \qquad (3.11)$$

$$F_I^+(\theta_S, \phi_S, \psi_S) = \tfrac{1}{2}(1+\cos^2\theta_S)\cos 2\phi_S \cos 2\psi_S - \cos\theta_S \sin 2\phi_S \sin 2\psi_S\,, \qquad (3.12a)$$

$$F_I^\times(\theta_S, \phi_S, \psi_S) = \tfrac{1}{2}(1+\cos^2\theta_S)\cos 2\phi_S \sin 2\psi_S + \cos\theta_S \sin 2\phi_S \cos 2\psi_S\,. \qquad (3.12b)$$

$$\cos\theta_S(t) = \tfrac{1}{2}\cos\bar\theta_S - \tfrac{\sqrt{3}}{2}\sin\bar\theta_S\cos(\bar\phi(t) - \bar\phi_S) \qquad (3.16)$$

$$\phi_S(t) = \alpha_1 + \pi/12 - \tan^{-1}\left[\frac{\sqrt{3}\cos\bar\theta_S + \sin\bar\theta_S\cos(\bar\phi(t)-\bar\phi_S)}{2\sin\bar\theta_S\cos(\bar\phi(t)-\bar\phi_S)}\right] \qquad (3.17)$$

$$\tan\psi_S(t) = \left(\hat{L}^a z_a - \hat{L}^a n_a z^b n_b\right) \Big/ \left(\epsilon_{abc} n^a \hat{L}^b z^c\right) \qquad (3.19)$$

$$\epsilon_{abc} n^a \hat{L}^b z^c = \tfrac{1}{2}\sin\bar\theta_L \sin\bar\theta_S \sin\left(\bar\phi_L - \bar\phi_S\right) - \tfrac{\sqrt{3}}{2}\cos\bar\phi(t)\left(\cos\bar\theta_L \sin\bar\theta_S \sin\bar\phi_S - \cos\bar\theta_S \sin\bar\theta_L \sin\bar\phi_L\right) - \tfrac{\sqrt{3}}{2}\sin\bar\phi(t)\left(\cos\bar\theta_S \sin\bar\theta_L \cos\bar\phi_L - \cos\bar\theta_L \sin\bar\theta_S \cos\bar\phi_S\right). \qquad (3.22)$$

$$\hat{L}^a z_a = \tfrac{1}{2}\cos\bar\theta_L - \tfrac{\sqrt{3}}{2}\sin\bar\theta_L\cos\left(\bar\phi(t) - \bar\phi_L\right) \qquad (3.20)$$

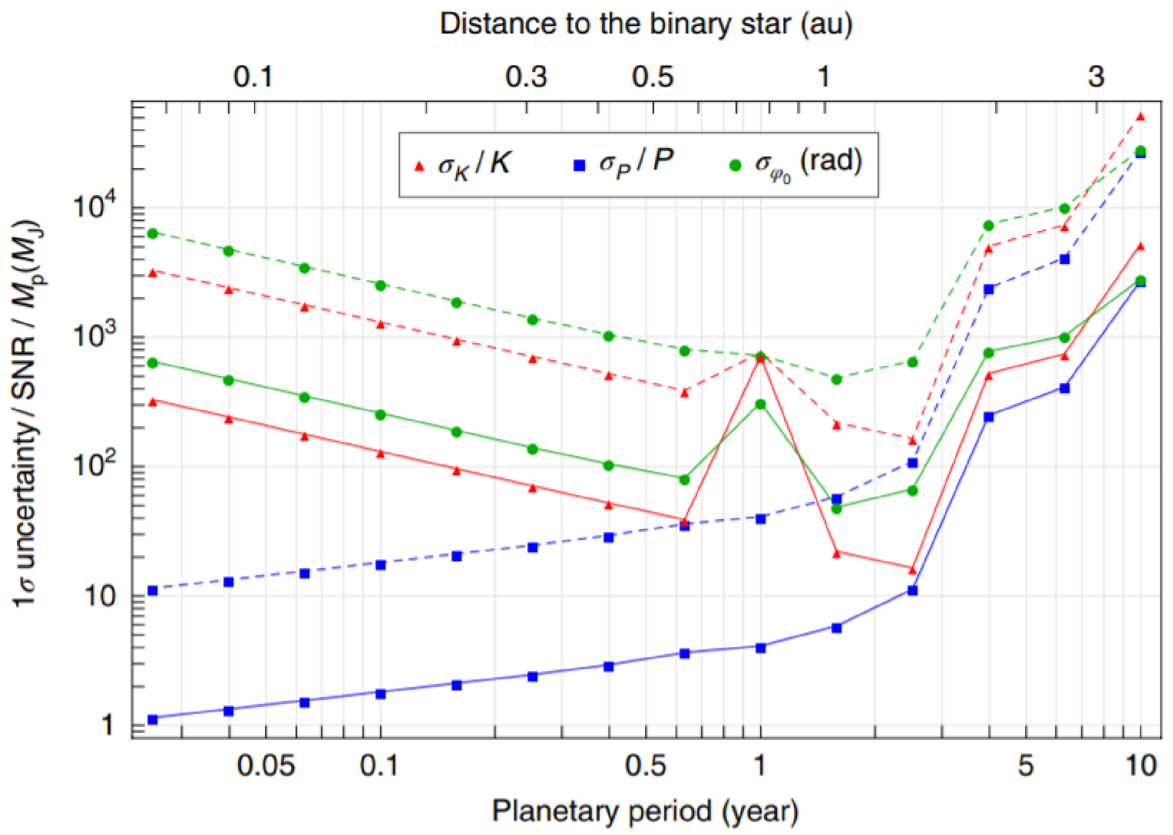
$$\hat{L}^a n_a = \cos\bar\theta_L\cos\bar\theta_S + \sin\bar\theta_L \sin\bar\theta_S \cos\left(\bar\phi_L - \bar\phi_S\right) \qquad (3.21)$$

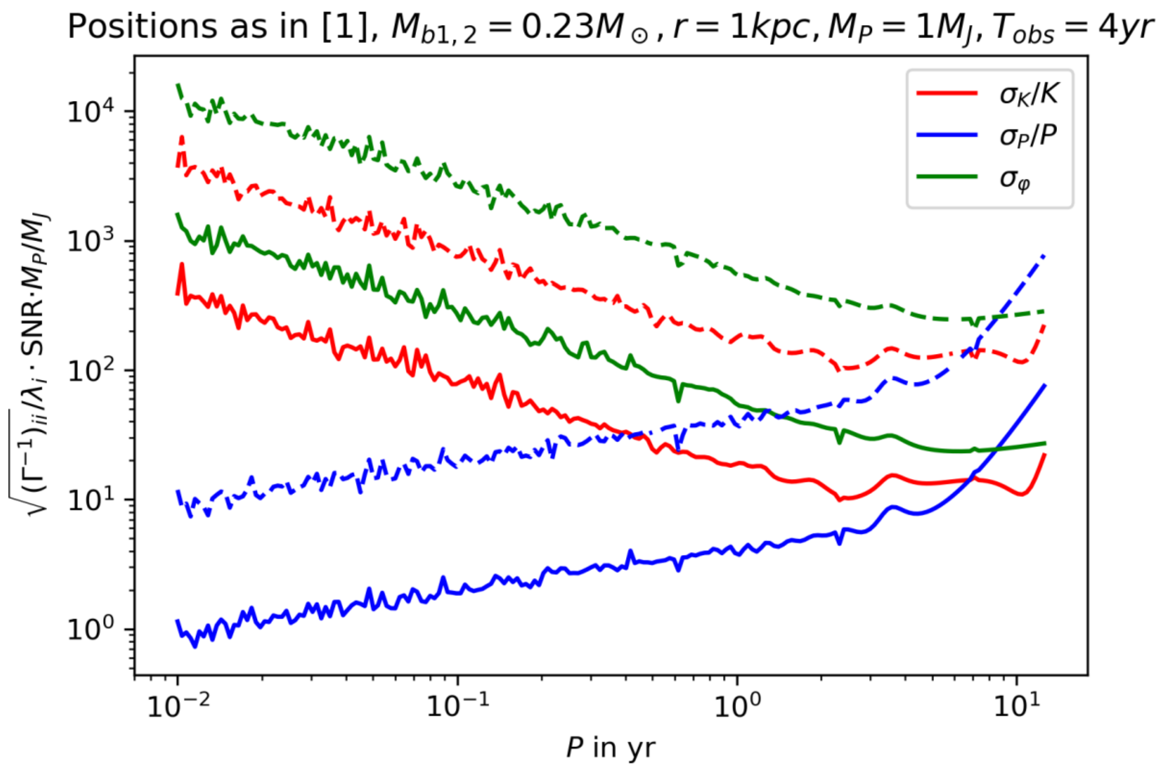
…so we do what Cutler proposes

Thus to evaluate the Fisher matrix (4.4), we need the derivatives of $A_\alpha(t)$ and $\chi_\alpha(t)$ with respect to the seven physical parameters $\ln \mathcal{A}$, φ_0, f_0, $\bar{\theta}_S$, $\bar{\phi}_S$, $\bar{\theta}_L$, $\bar{\phi}_L$. Clearly one might straightforwardly use the chain rule with Eqs. (3.15) and (3.32) to determine the partial derivatives of $A_\alpha(t)$ and $\chi_\alpha(t)$ with respect to the four angles $\bar{\theta}_S$, $\bar{\phi}_S$, $\bar{\theta}_L$,and $\bar{\phi}_L$, though the final expressions would be cumbersome. In our calculation, we preferred simply to take these derivatives numerically. The remaining partial derivatives are:

In our case we initialise a second class instance of our class binary(position + 1e-6) and compare it via (binary(position + 1e-6) – binary(position)) / 1e-6 (same as scipy.misc.derivative)

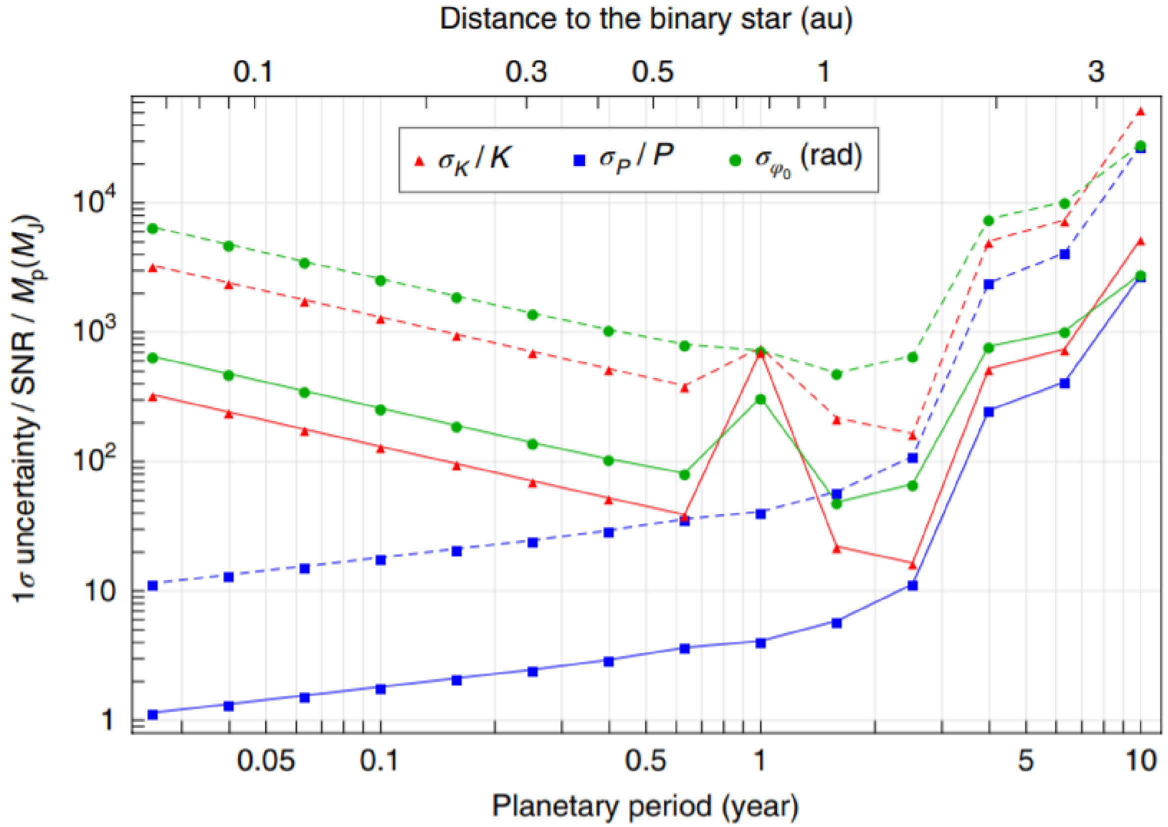
Computing the uncertainties approximately

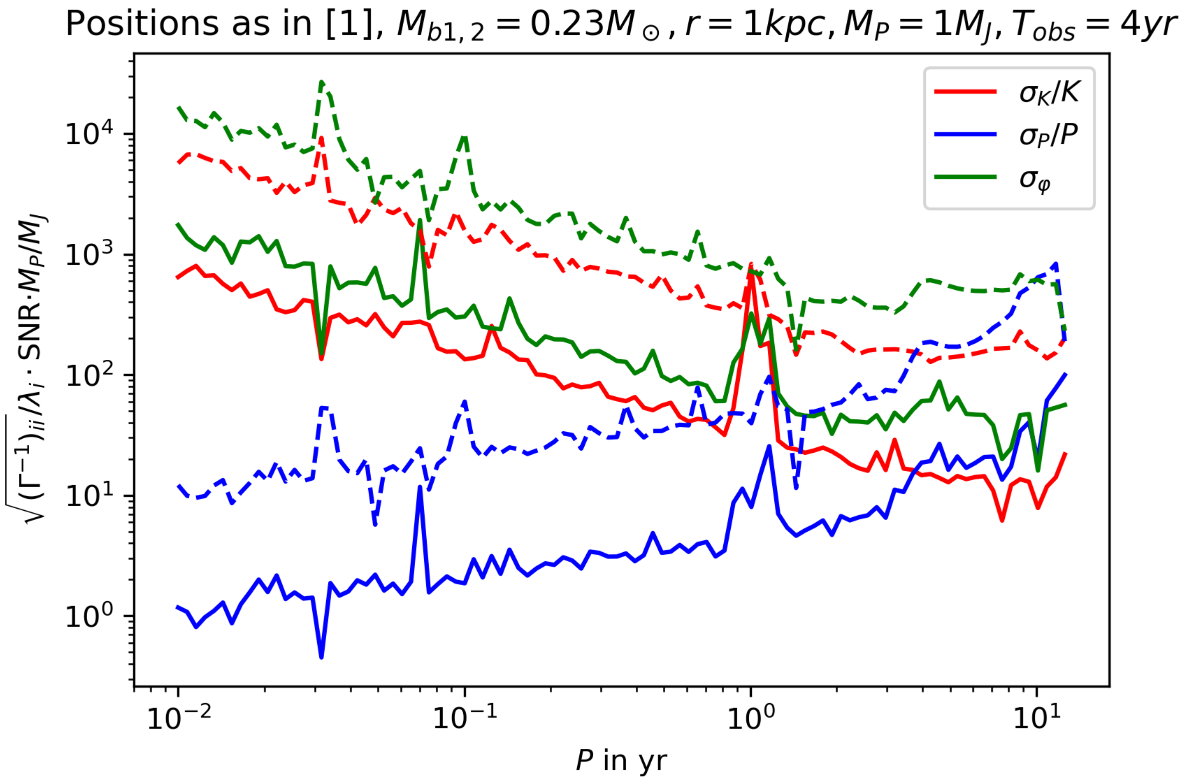




This calculation only included the lower diagonal block of the Fisher information

Computing the uncertainties over all par‘s



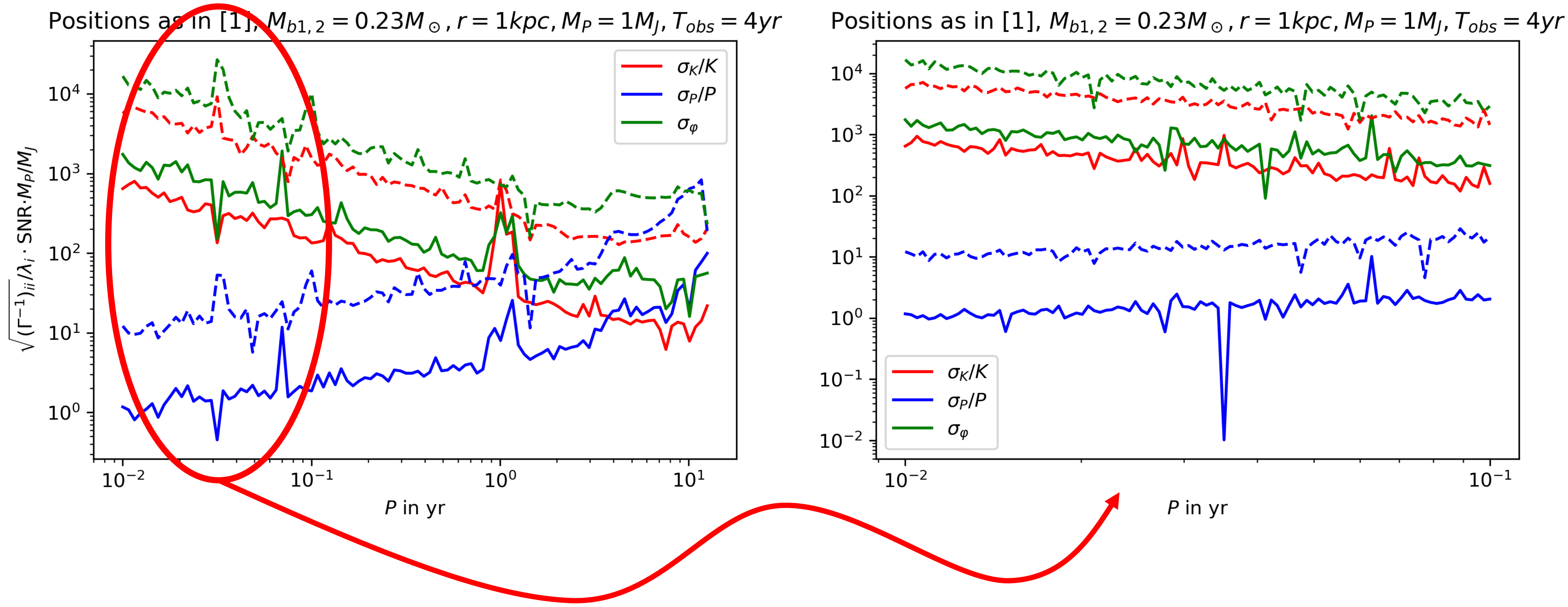


But we can do better by taking all relevant parameters into account -> degeneracy @ 1yr appears

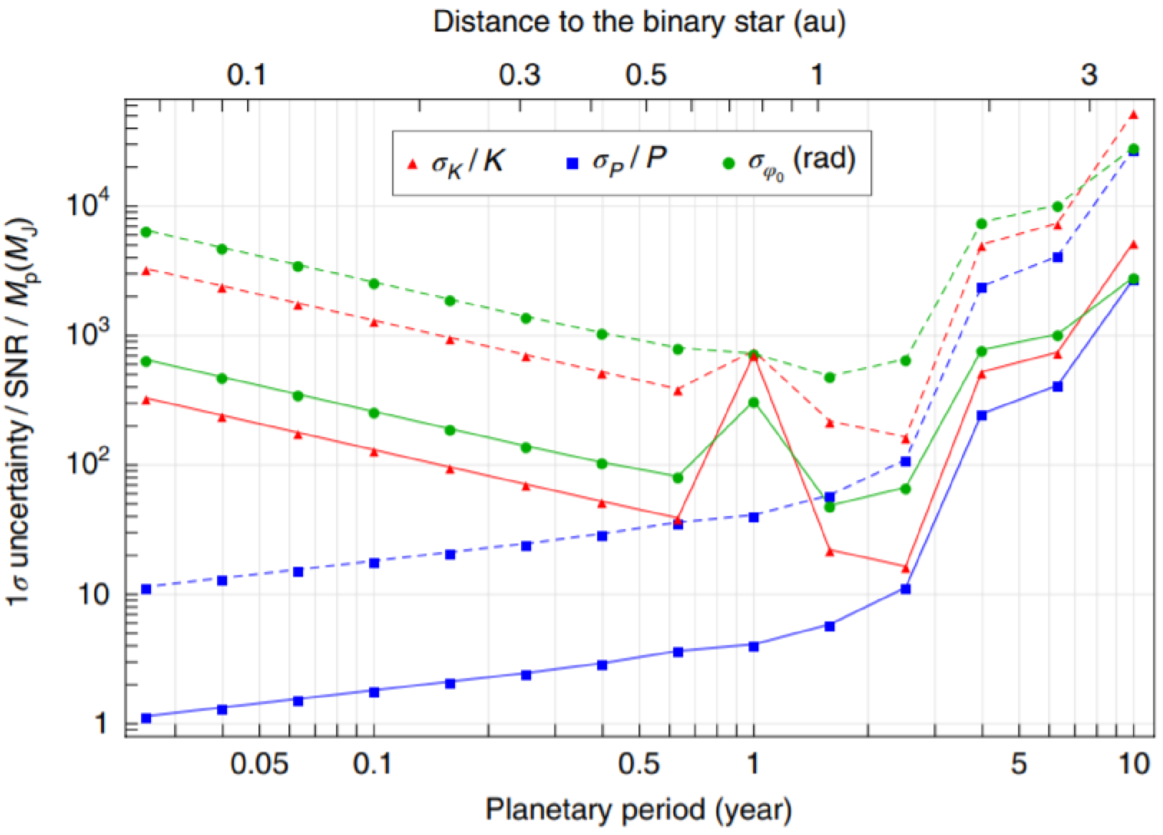
Problem 1: Why is it so fluctuating? Probably: Numerical derivation, integration, inversion, but WIP

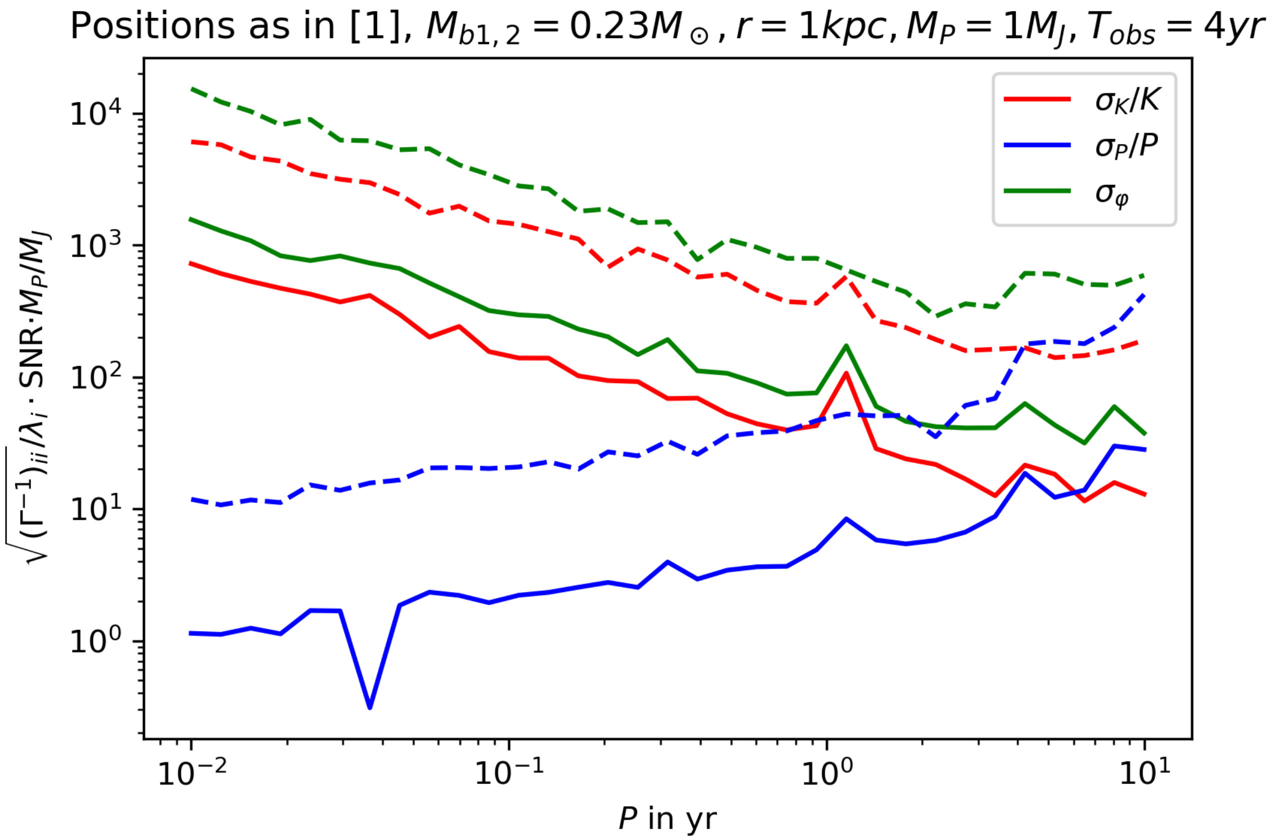
Problem 2: Where is the steep increase at P > 4 yr?

Some features get lost when zooming in



Using the geometric mean





Why is it so fluctuating?

- Currently per DWD we compute a 9x9 symmetric matrix $\Gamma_{ij} = \frac{2}{S_n(f_0)} \sum_{\alpha=I,II} \int_0^{T_0} dt\, \frac{\partial h_\alpha}{\partial \lambda_i}(t) \frac{\partial h_\alpha}{\partial \lambda_j}(t)$ so 45 integrals of an oscillating function with $f_{GW} \approx 1$ mHz and over $T_0 = 4$ yr, so $\sim 10^5$ periods

- Exchange: Accuracy in integration vs. computational cost ☹
  Right now, I tried to keep the computational cost low, trying to take 10^5 subintervals in scipy.integrate is too time intensive

Next milestones

- Understand the fluctuations a bit better and potentially fix them -> any ideas?
- Look at the parameter space for positions/SNR for the 25‘000 potential DWDs with SNR > 7 and calculate dependances of the planetary parameters -> we want a function:

`rel_uncertainty(pos, ang_mom, sepB, M, inc, sepP)`

which we can then use with respective priors in:

$$N_{\mathrm{bin}} = \int_0^z \int_0^\infty \int_{\nu_{\mathrm{ISCO}}}^{\nu} \frac{\mathrm{d}V}{\mathrm{d}z} \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}M} \mathcal{D}\left(M, q, \nu, z\right) \frac{\mathrm{d}\nu}{\dot{\nu}_{\mathrm{GW}}} \mathrm{d}M \mathrm{d}z, \tag{11}$$

-> In our case evaluations of `rel_uncertainty` on a grid would potentially be good enough

Next milestones

- Then performing a weighted integral as in [4], constraining f_{CBP} the fraction of circumbinary partners given N_{bin} detections via Bayesian inference:

$$N_{\mathrm{bin}} = \int_0^z \int_0^\infty \int_{\nu_{\mathrm{ISCO}}}^{\nu} \frac{\mathrm{d}V}{\mathrm{d}z} \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}M} \mathcal{D}\left(M, q, \nu, z\right) \frac{\mathrm{d}\nu}{\dot{\nu}_{\mathrm{GW}}} \mathrm{d}M \mathrm{d}z, \qquad (11)$$

- Then repeat the exercise with the strain and signal to noise of the IceGiant mission:

$$y_2^{\mathrm{GW}}(t) = \frac{\mu - 1}{2} \bar{\Psi}(t) - \mu \bar{\Psi}\left(t - \frac{\mu + 1}{2} T_2\right) + \frac{\mu + 1}{2} \bar{\Psi}(t - T_2), \qquad (1)$$

  to see if it could see the most promising exoplanet candidates/Jupiter-like planets

- Try to combine measurements of IceGiant and LISA

Code

You can find my code at

https://gitlab.ethz.ch/marcush/icegiantexoplanets.git

References

[1] Tamanini, N., Danielski, C. (2019). The gravitational-wave detection of exoplanets orbiting white dwarf binaries using LISA. *Nat Astron 3*, 858–866. https://doi.org/10.1038/s41550-019-0807-y

[2] Danielski, C., Korol, V., Tamanini, N., & Rossi, E.M. (2019). Circumbinary exoplanets and brown dwarfs with the Laser Interferometer Space Antenna. *Astronomy and Astrophysics, 632.*

[3] Cutler, C. (1998). Angular resolution of the LISA gravitational wave detector. *Physical Review D, 57*, 7089-7102.

[4] Soyuer, D., Zwick, L., D'Orazio, D., Saha, P. (2021). Searching for gravitational waves via Doppler tracking by future missions to Uranus and Neptune. *MNRAS: Letters*, 503, 1, L73-79. https://doi.org/10.1093/mnrasl/slab025

[5] Maggiore, M. (2008). *Gravitational Waves Volume 1: Theory and Experiments*. Oxford University Press