Table 12.19

	Price option 1	Price option 2	Price option 3	
First	25	30	40	Period 1
Business	50	40	45	
Economy	50	53	65	
First	22	45	50	Period 2
Business	45	55	75	
Economy	50	60	80	
First	45	60	75	Period 3
Business	20	40	50	
Economy	55	60	75	

Use the actual demands that resulted from the prices you set in period 1 to rerun the model at the beginning of period 2 to set price levels for period 2 and provisional price levels for period 3.

Repeat this procedure with a rerun at the beginning of period 3. Give the final operational solution.

Contrast this solution to one obtained at the beginning of period 1 by pricing to maximise yield based on expected demands.

12.25 Car rental 1

A small ('cut price') car rental company, renting one type of car, has depots in Glasgow, Manchester, Birmingham and Plymouth. There is an estimated demand for each day of the week except Sunday when the company is closed. These estimates are given in Table 12.20. It is not necessary to meet all demand.

Table 12.20

	Glasgow	Manchester	Birmingham	Plymouth
Monday	100	250	95	160
Tuesday	150	143	195	99
Wednesday	135	80	242	55
Thursday	83	225	111	96
Friday	120	210	70	115
Saturday	230	98	124	80

Cars can be rented for one, two or three days and returned to either the depot from which rented or another depot at the start of the next morning. For example, a 2-day rental on Thursday means that the car has to be returned on Saturday morning; a 3-day rental on Friday means that the car has to be returned on Tuesday morning. A 1-day rental on Saturday means that the car has to be returned on Monday morning and a 2-day rental on Tuesday morning.

Table 12.21

From	To			
	Glasgow	Manchester	Birmingham	Plymouth
Glasgow	60	20	10	10
Manchester	15	55	25	5
Birmingham	15	20	54	11
Plymouth	8	12	27	53

Table 12.22

From	To			
	Glasgow	Manchester	Birmingham	Plymouth
Glasgow	–	20	30	50
Manchester	20	–	15	35
Birmingham	30	15	–	25
Plymouth	50	35	25	–

The rental period is independent of the origin and destination. From past data, the company knows the distribution of rental periods: 55% of cars are hired for one day, 20% for two days and 25% for three days. The current estimates of percentages of cars hired from each depot and returned to a given depot (independent of day) are given in Table 12.21.

The marginal cost, to the company, of renting out a car (‘wear and tear’, administration etc.) is estimated as follows:

1-Day hire	£20
2-Day hire	£25
3-Day hire	£30

The ‘opportunity cost’ (interest on capital, storage, servicing, etc.) of owning a car is £15 per week.

It is possible to transfer undamaged cars from one depot to another depot, irrespective of distance. Cars cannot be rented out during the day in which they are transferred. The costs (£), per car, of transfer are given in Table 12.22.

Ten percent of cars returned by customers are damaged. When this happens, the customer is charged an excess of £100 (irrespective of the amount of damage that the company completely covers by its insurance). In addition, the car has to be transferred to a repair depot, where it will be repaired the following day. The cost of transferring a damaged car is the same as transferring an undamaged one (except when the repair depot is the current depot, when it is zero). Again the transfer of a damaged car takes a day, unless it is already at a repair depot. Having arrived at a repair depot, all types of repair (or replacement) take a day.

Only two of the depots have repair capacity. These are (cars/day) as follows:

Manchester	12
Birmingham	20

Having been repaired, the car is available for rental at the depot the next day or may be transferred to another depot (taking a day). Thus, a car that is returned damaged on a Wednesday morning is transferred to a repair depot (if not the current depot) during Wednesday, repaired on Thursday and is available for hire at the repair depot on Friday morning.

The rental price depends on the number of days for which the car is hired and whether it is returned to the same depot or not. The prices are given in Table 12.23 (in £).

Table 12.23

	Return to Same Depot	Return to Another Depot
1-Day hire	50	70
2-Day hire	70	100
3-Day hire	120	150

There is a discount of £20 for hiring on a Saturday so long as the car is returned on Monday morning. This is regarded as a 1-day hire.

For simplicity, we assume the following at the beginning of each day:

1. Customers return cars that are due that day
2. Damaged cars are sent to the repair depot
3. Cars that were transferred from other depots arrive
4. Transfers are sent out
5. Cars are rented out
6. If it is a repair depot, then the repaired cars are available for rental.

In order to maximise weekly profit, the company wants a 'steady state' solution in which the same expected number will be located at the same depot on the same day of subsequent weeks.

How many cars should the company own and where should they be located at the start of each day?

This is a case where the integrality of the cars is not worth modelling. Rounded fractional solutions are acceptable

12.26 Car rental 2

In the light of the solution to the problem stated in Section 12.25, the company wants to consider where it might be most worthwhile to expand repair capacity. The weekly fixed costs, given below, include interest payments on the necessary loans for expansion.

The options are as follows:

1. Expand repair capacity at Birmingham by 5 cars per day at a fixed cost per week of £18 000.
2. Further expand repair capacity at Birmingham by 5 cars per day at a fixed cost per week of £8000.
3. Expand repair capacity at Manchester by 5 cars per day at a fixed cost per week of £20 000.
4. Further expand repair capacity at Manchester by 5 cars per day at a fixed cost per week of £5000.
5. Create repair capacity at Plymouth of 5 cars per day at a fixed cost per week of £19 000.

If any of these options is chosen, it must be carried out in its entirety, that is, there can be no partial expansion. Also, a further expansion at a depot can be carried out only if the first expansion is also carried out, so for example option (2) at Birmingham cannot be chosen unless option (1) is also chosen. If option (2) is chosen, thereby also choosing option (1), these count as two options. Similar stipulations apply regarding the expansions at Manchester. At most three of the options can be carried out.

12.27 Lost baggage distribution

A small company with six vans has a contract with a number of airlines to pick up lost or delayed baggage, belonging to customers in the London area, from Heathrow airport at 6 p.m. each evening. The contract stipulates that each customer must have their baggage delivered by 8 p.m. The company requires a model, which they can solve quickly each evening, to advise them what is the minimum number of vans they need to use and to which customers each van should deliver and in what order. There is no practical capacity limitation on each van. All baggage that needs to be delivered in a two-hour period can be accommodated in a van. Having ascertained the minimum number of vans needed, a solution is then sought, which minimises the maximum time taken by any van.

On a particular evening, the places where deliveries need to be made and the times to travel between them (in minutes) are given in Table 12.24. No allowance

Up to six planes can be flown:

$$n \leq 6.$$

13.24.3 Objective

$$\text{Maximize} \quad \sum_{i,c,h} Q_i r_{1ich} + \sum_{i,j,c,h} Q_i Q_j r_{2ijch} + \sum_{i,j,k,c,h} Q_i Q_j Q_k r_{3ijkch} - 50n$$

(measuring in £1000) where Q_i is the probability of scenario i.

This model has 1200 constraints, one bound and 982 variables, of which 117 are 0–1 integer and one general integer.

In defining the data, it is desirable that the demands in the scenarios cover the *extremes* as well as the most likely cases. The recommended sales will not exceed those of the most extreme scenario in the solution to this model. In this example, it can be seen that final demands (known with hindsight) exceed those of all scenarios in a few cases. As a result, the solution will not result in sales to meet all of these demands.

Models for subsequent weeks (with recourse) are obtained from this model by fixing prices and sales for weeks that have elapsed.

13.25 Car rental 1

We model this problem as a deterministic linear programme. There would be advantage to be gained from modelling it as a stochastic programme. In order to do this, we would need to obtain data to quantify the uncertain demand.

13.25.1 Indices

i, j = Glasgow, Manchester, Birmingham, Plymouth
t = Monday, Tuesday, Wednesday, Thursday, Friday, Saturday
k = 1, 2, 3 (days hired)

Although this is a ‘dynamic’ problem, we seek a steady-state solution and can therefore regard the set of days as ‘circular’, that is, Monday of a week ‘follows’ after the subsequent Saturday; that is, if $t =$ Monday then $t - 1 =$ Saturday.

13.25.2 Given data

D_{it} = estimated rental demand at depot i on day t as given in Table 12.19
P_{ij} = proportion of cars rented at depot i to be returned to depot j as given in Table 12.21
C_{ij} = cost of transfer of a car from depot i to depot j as given in Table 12.22
Q_k = proportion of cars hired for k days

R$_i$ = repair capacity of depot i
RCA$_k$ = rental cost for k days with return to same depot as given in Table 12.23
RCB$_k$ = rental cost for k days with return to another depot as given in Table 12.23
RCC = rental cost for Saturday with return to same depot on Monday
RCD = rental cost for Saturday with return to another depot on Monday
CS$_k$ = marginal cost to company of k day hire of a car for all i, t

13.25.3 Variables

n = total number of cars owned
nu_{it} = number of undamaged cars at depot i at the beginning of day t for all i, t
nd_{it} = number of damaged cars at depot i at the beginning of day t for all i, t
tr_{it} = number of cars rented out from depot i at beginning of day t for all i, t
eu_{it} = undamaged cars left at depot i during day t for all i, t
ed_{it} = damaged cars left at depot i at the beginning of day t for all i, t
tu_{ijt} = number of undamaged cars at depot i at the beginning of day t, to be transferred to depot j for all i, j, t
td_{ijt} = number of damaged cars at depot i at the beginning of day t, transferred to depot t for all i, j, t
rp_{it} = number of damaged cars to be repaired at depot i on day for all i, t

13.25.4 Constraints

Total number of undamaged cars into depot i on day t

$$\sum_{jk} 0.9 P_{ji} Q_k tr_{jt-k} + \sum_j tu_{jit-1} + rp_{it-1} + eu_{it-1} = nu_{it} \quad \text{for all } i, t.$$

Total number of damaged cars into depot i on day t

$$\sum_{jk} 0.1 P_{ji} Q_k tr_{jt-k} + \sum_j td_{jit-1} + ed_{it-1} = nd_{it} \quad \text{for all } i, t.$$

Total number of undamaged cars out of depot i on day t

$$tr_{it} + \sum_j tu_{ijt} + rp_{it-1} + eu_{it} = nu_{it} \quad \text{for all } i, t.$$

Total number of damaged cars out of depot i on day t

$$rp_{it-1} + \sum_j td_{ijt} + ed_{it} = nd_{it} \quad \text{for all } i, t.$$

Repair capacity of depot i on all days

$$rp_{it} \leq R_i \quad \text{for all } i, t.$$

Demand at depot i on day t

$$tr_{it} \leq D_{it} \quad \text{for all } i, t.$$

Total number of cars equals number hired out from all depots on Monday for 3 days, plus those on Tuesday for 2 or 3 days, plus all damaged and undamaged cars in depots at the beginning of Wednesday.

$$\sum_i \left(0.25tr_{i1} + 0.45tr_{i2} + nu_{i2} + nd_{i2}\right) = n \quad \text{for all } i.$$

13.25.5 Objective

Note that £10 has been added to the profit on each rented car to reflect the surcharge of £100 charged on the 10% of cars that are returned damaged.

$$\begin{aligned}
\text{Profit} = & \sum_{itk, t \neq \text{SATURDAY}} P_{ii} Q_k (RCA_k - CS_k + 10) tr_{it} \\
& + \sum_{ijtk, t \neq \text{SATURDAY}} P_{ij} Q_k (RCB_k - CS_k + 10) tr_{it} \\
& + \sum_{i\text{SATURDAY}} P_{ii} Q_1 (RCC - CS_1 + 10) tr_{it} \\
& + \sum_{ij\text{SATURDAY}} P_{ij} Q_1 (RCD - CS + 10) tr_{it} \\
& + \sum_{i\text{SATURDAY}k} P_{ii} Q_k (RCA_k - CS_k + 10) tr_{it} + \sum_{i\text{SATURDAY}k} P_{ij} Q_k (RCB_k - CS_k \\
& + 10) tr_{it} \\
& - \sum_{ijt} C_{ij} tu_{ijt} - \sum_{ijt} C_{ij} td_{ijt} - 15n.
\end{aligned}$$

This model has a total of 84 constraints and 337 variables.

It is not necessary to constrain the w_{ijkl} to be non-negative. They can be regarded as 'free' variables. Avoiding these two stipulations helps the model to solve more easily.

13.26 Car rental 2

We introduce the following 0–1 integer variables with the following interpretations:

δ_{B1} $= 1$, if the Birmingham repair capacity is expanded by 5 cars per day.
δ_{B2} $= 1$, if the Birmingham repair capacity is expanded by a further 5 cars per day.
δ_{M1} $= 1$, if the Manchester repair capacity is expanded by 5 cars per day.
δ_{M2} $= 1$, if the Manchester repair capacity is expanded by a further 5 cars per day.
δ_{P} $= 1$, if the Plymouth repair capacity is expanded by 5 cars per day.

The following expression is added to the objective function:

$$18\,000\delta_{\mathrm{B1}} + 8\,000\delta_{\mathrm{B2}} + 20\,000\delta_{\mathrm{M1}} + 5\,000\delta_{\mathrm{M2}} + 19\,000\delta_{\mathrm{P}}$$

together with the following extra constraints:

$$\delta_{\mathrm{B1}} \geq \delta_{\mathrm{B2}}, \quad \delta_{\mathrm{M1}} \geq \delta_{\mathrm{M2}}, \quad \delta_{\mathrm{B1}} + \delta_{\mathrm{B2}} + \delta_{\mathrm{M1}} + \delta_{\mathrm{M2}} + \delta_{\mathrm{P}} \leq 3$$

and the repair capacity constraints for Birmingham, Manchester and Plymouth have $5\delta_{\mathrm{B1}} + 5\delta_{\mathrm{B2}}, 5\delta_{\mathrm{M1}} + 5\delta_{\mathrm{M2}}$ and $5\delta_P$, respectively added.

13.27 Lost baggage distribution

We formulate this problem as an integer programming model. It is an extension of the travelling salesman problem, discussed in section 9.5. It can be regarded as a (simple) case of the vehicle routing problem. In the problem here, there are no vehicle capacites and no time windows for the delivery to each location (apart from the overall 2 hours guarantee). Nevertheless, it is potentially a very difficult problem to solve, for more than a modest number of locations. Also it can give rise to a very large model. There are a number of different ways of formulating a model, which vary greatly in their size and difficuly of solution. We suggest a model that is practicable in both size and computational tractibility. Although the problem is ‘symmetric’ in the sense that the distance from X to Y is the same as that from Y to X, we extend the asymmetric formulation of the travelling salesman problem in order to give the ‘time’ to return to Heathrow, after a van has completed all its deliveries, as 0. In this way, only the total time taken to reach the last delivery is counted within the 2 hours time limit. We are therefore seeking a ‘Hamiltoniam path’ (as opposed to a circuit) for each van, starting from Heathrow.

13.27.1 Variables

All variables are 0–1 and integer

$x_{ijk} = 1$, iff van k visits, and goes directly from location i to j for all i, j, and k.

$y_{ik} = 1$, iff location i is visited by van k for all i, k.

$\delta_k = 1$, iff van k is used.

14.25 Car rental

The numbers of cars in all aspects of the following solution have been rounded from the fractional answers that result from the linear programming model.

The company should own 624 cars and pursue the following policies that will result in a weekly profit of £122 398.

The estimated number of undamaged cars in each depot at the beginning of each day will be as follows (there will also be hired out cars not in any depot).

	Glasgow	Manchester	Birmingham	Plymouth
Monday	68	99	145	46
Tuesday	66	94	154	39
Wednesday	70	100	125	47
Thursday	69	115	117	44
Friday	71	102	126	44
Saturday	65	96	154	73

The estimated number of damaged cars in each depot at the beginning of each day will be as follows.

	Glasgow	Manchester	Birmingham	Plymouth
Monday	11	12	20	6
Tuesday	7	12	20	4
Wednesday	8	12	20	6
Thursday	9	12	20	5
Friday	11	12	20	5
Saturday	7	12	22	3

Of the undamaged cars, the following should be rented out each day, for the periods and destinations in the proportions given in the statement of the problem.

	Glasgow	Manchester	Birmingham	Plymouth
Monday	68	99	95	46
Tuesday	66	94	154	39
Wednesday	70	80	125	47
Thursday	69	115	111	44
Friday	71	102	70	0
Saturday	65	93	124	73

No transfers of undamaged cars should be made but the following transfers of damaged cars should be made (arriving the following day).

Glasgow to Manchester

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
3	2	3	2	3	2

Glasgow to Birmingham

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
5	5	3	4	9	5

Plymouth to Birmingham

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
5	3	6	5	5	3

The two repair depots of Manchester and Birmingham are fully occupied on all six days repairing 12 and 20 cars, respectively, each day. Repair capacity is clearly a limiting factor on the operation of the company. This is reflected by the high-shadow prices on the repair capacity constraints, which vary between £617 and £646 per car per day in both repairing depots. A plan to increase repair capacity forms the subject of problem 12.27. The solution above was obtained in 153 iterations.

14.26 Car rental 2

Only the Birmingham repair capacity should be increased, using both expansion options, to expand to 22 cars per day. Repair capacities in all depots are again fully used.

This allows the company to expand its fleet to 895 cars, resulting in a new weekly profit of £135 511. All demands still cannot be fully met.

This model solved in 5 nodes.

14.27 Lost baggage distribution

Two vans are needed. Solving the model, defined in Section 13.27 (a relaxation of the final model), leads to the ‘solution’ given in Figure 14.10. This required 2263 nodes to solve.