Probability exercises - Problem Set 6 - Continuous Random Variables and Random Vectors

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For this Problem Set I added the notation **!**. It means that an exercise is more difficult than the others in the Problem Set. **!!**, as you might imagine, is translated to *only for try-harders.*

Exercise 1. *Let X be a random continuous variable with $f_X(x) = \frac{4}{3x^2}$ density function, for $1 \le x \le 4$. Compute the following probabilities.*

1. $P(1 < x < 3)$, $P(1 \le X < 3)$, $P(1.5 < X < 3)$.

$$P(1 < x < 3) = P(1 \le X < 3) = \int_1^3 \frac{4}{3x^2} dx = \left[\frac{-4}{3x}\right]_1^3 = \frac{4}{3} - \frac{4}{9} = \frac{8}{9}$$

$$P(1.5 < X < 3) = \int_{1.5}^3 \frac{4}{3x^2} dx = \left[\frac{-4}{3x}\right]_{1.5}^3 = \frac{4}{9}$$

2. $P(X \le 1)$, $P(X < 1)$, $P(0 \le X \le 2 \mid 1 \le X \le 3)$.

$$P(X \le 1) = P(X < 1) = 0$$

$$P(0 \le X \le 2 \mid 1 \le X \le 3) = \frac{P(1 \le X \le 2)}{P(1 \le X \le 3)} =$$

3. ***!*** $P(X \in \mathbb{N}) = 0$.

4. ***!!*** $P(X \in \mathbb{Q}) = 0$.

For the last two exercises you might want to check the concept of countable set.

Exercise 2. *Let X be a uniform random variable in $[0, 1]$. Compute the following probabilities:*

- $P(X = 1)$, $P(X = 0)$, $P(X = 0.5)$, $P(X \in [0, 1])$, $P(X \in [0, 0.5])$, $P(X \leq 0.5)$, *and* $P(X \geq 1)$

$$P(X = 1) = P(X = 0) = P(X = 0.5) = 0$$
$$P(X \in [0, 1]) = 1, \; P(X \in [0, 0.5]) = P(X \leq 0.5) = \frac{1}{2}$$

$$P(X \geq 1) = 0$$

- *What is the density function of $f_X(x)$ (in the lectures), and $F_X(x)$ (you must calculate it)? Show how you can go from one to another (Show the operations).*

$$F_X(x) = \begin{cases} 0 & \textit{if } x < 1 \\ x & \textit{if } x \in [0, 1] \\ 1 & \textit{if } x > 1 \end{cases}$$

Exercise 3. *! The support of a certain continuous random variable is the set $[1, 5]$. Its density function is proportional to x^2.*

1. *Write its density function.*

$$f_X(x) = \begin{cases} cx^2 & \textit{if } x \in [1, 5] \\ 0 & \textit{Otherwise} \end{cases}$$

   With $c = \frac{3}{124}$.

2. *What is the probability of* $(1, 3]$

$$P(X \in (1, 3]) = \int_1^3 cx^2 dx = \frac{13}{62}$$

Exercise 4. *Find the density function of the continuous random variable that has as distribution function:*

$$F_X(x) = \begin{cases} 0 & \textit{if } x < 0 \\ 1-(1+3x)e^{-3x} & \textit{if } x \geq 0 \end{cases}$$

Is this function continuous? Is it right-continuous? Yes, yes.

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 9xe^{-3x} & \text{if } x \geq 0 \end{cases}$$

Exercise 5. *Let X and Y be two discrete independent random variables with support in the set $\{0, 1, 2\}$ such that:*

$$P(X = 0) = 0.3, \quad P(X = 1) = 0.5, \quad P(Y = 0) = 0.6, \quad P(Y = 1) = 0.1$$

1. *Build a table that shows $P(X = x, Y = y)$ for every (x, y) in the common support.*

	X	0.3	0.5	0.2
Y		**0**	**1**	**2**
0.6	**0**	0.18	0.3	0.12
0.1	**1**	0.03	0.05	0.02
0.7	**2**	0.21	0.35	0.14

2. *Find $P(X = Y)$.*

$$P(X = Y) = 0.18 + 0.05 + 0.14$$

3. *Find $P(X + Y = 2)$.*

$$P(X + Y = 2) = 0.12 + 0.21 + 0.05$$

Exercise 6. *The joint density of X and Y is $f(x, y) = \lambda^3 xe^{-\lambda(x+y)}$ for $x > 0$ and $y > 0$ (0 otherwise).*

1. *Find the marginal densities and show that X and Y are independent.*

$$f_X(x) = \int_0^\infty \lambda^3 xe^{-\lambda(x+y)} dy = \lambda^2 xe^{-\lambda(x+y)}\Big]_0^\infty = \lambda^2 xe^{-\lambda x}$$

$$f_Y(y) = \int_0^\infty \lambda^3 x e^{-\lambda(x+y)} dx = \lambda e^{-\lambda y}$$

You just need to check that the product matches the density function.

2. *! Find $P(X \leq a, Y \leq b)$ for every a and b positive numbers. The result might be an integral of x and y that you don't know how to calculate, just indicate the integral.*

$$P(X \leq a, Y \leq b) = \int_0^a \int_0^b \lambda^3 x e^{-\lambda(x+y)} dy dx$$

3. *Find $P(X \leq a)$ for $a > 0$.*

$$P(X \leq a, Y \leq b) = \int_0^a \int_0^\infty \lambda^3 x e^{-\lambda(x+y)} dy dx$$

Exercise 7. *Let (X, Y) a random vector with joint density function*

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(y-x)^2}, & \textit{if } -1 < x < 1,\ -\infty < y < \infty \\ 0, & \textit{otherwise} \end{cases}$$

1. *! Find $f_X(x)$.*[1]

$$f_X(x) = \begin{cases} 0 & \textit{if } |x| > 1 \\ \frac{1}{2} & \textit{if } |x| < 1 \end{cases}$$

2. *Find $f_{Y|X}(y)$.*

$$f_{Y|X}(y) = \frac{f_{XY}(x,y)}{f_X(x)}$$

[1] You might want to use some integral calculator on the internet...