Middle School Challenge '17

Mathematical Puzzle Programs

Version 2016/09/20

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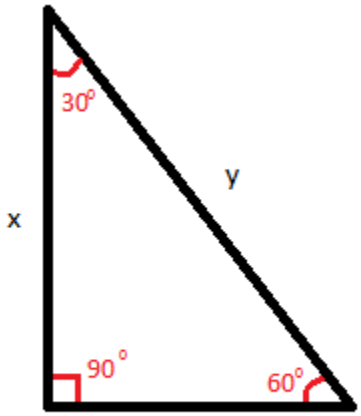
Which Witch?

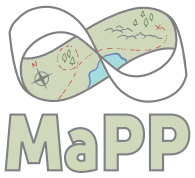
Main Puzzle 1

Grendel the Witch is cooking up a storm! She is brewing potions in several cauldrons.

Grendel needs to make at least three potions at a time, so she needs to use at least 3 cauldrons at once. If the cauldrons have a radius of 3 feet and Grendel must be equally close to all cauldrons used, what is the shortest distance Grendel can be from the center of a cauldron?

There is a special triangle which might help you answer this puzzle: the 30-60-90 triangle. Let x be the side opposite of the 60° angle. Let y be the side opposite the 90° angle. It must be that $y = \frac{2}{\sqrt{3}}x$.





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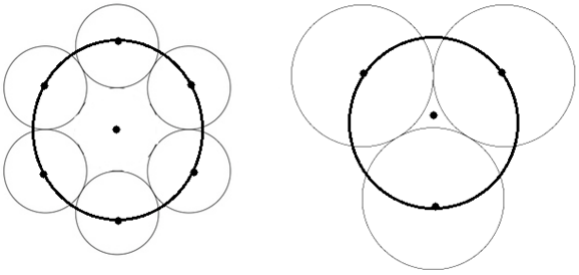
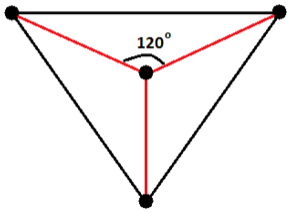
Which Witch?

Solution - Main Puzzle 1

Since Grendel is equally close to the center of every cauldron used, these centers lay on a big circle with Grendel as the center. The distance to Grendel is minimized by making no gaps between any two cauldrons. As we use fewer and fewer cauldrons the radius of this "big circle" is minimized. This means using 3 cauldrons minimizes the distance from Grendel to a cauldron.

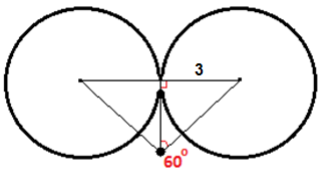


To determine how close Grendel is in the circle to the right we note the triangle formed by connecting the three circle centers and note Grendel is equally close to them.



The black lines are equal length and the red lines are equal length so we have 3 equivalent isosceles triangles within an equilateral triangle. So the three angles at the center all equal to one another and sum up to 360^o. This means each of these angles is 120^o.

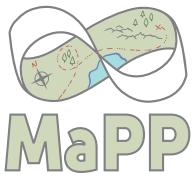
Now let's look at just the "top two" circle centers, as well as Grendel.



By drawing upward from Grendel we will cut the segment between the two circle centers in half. However, what we've really done is form a 30, 60, 90 triangle whose side opposite to the 60^o angle is 3. The hypotenuse

of this triangle happens to be the distance we are trying to find so all we must do now is use the formula we were given at the start: $y = \frac{2}{\sqrt{3}}x$. (Where y is the hypotenuse and x is the side opposite to the 60^o angle)

Plugging is 3 for x yields $y = \frac{6}{\sqrt{3}} = 2\sqrt{3}$.



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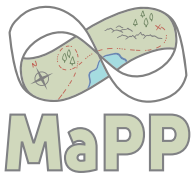
Which Witch?

Bonus Puzzle 1

Grendel decided she's working too hard so she got her brooms to do the work for her. Grendel set up several brooms and cauldrons in her basement. Every broom is equally close to 3 cauldrons and is $\frac{2}{\sqrt{3}}$ feet away from them. What is the maximum number of brooms a single cauldron can be $\frac{2}{\sqrt{3}}$ feet away from? What shape do all the brooms surrounding one cauldron form?



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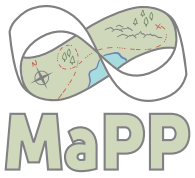
Which Witch?

Solution - Bonus Puzzle 1

6, a hexagon.



Middle School Challenge '17

Fear the Hungry Dead

Main Puzzle 2

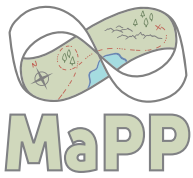
The insatiable Carli Ivor eats anything in her path. Last year at Count Calcula's party she ate ALL of the Counts famous meat pie! She started by eating half of the pie. Then she ate half of the remaining pie. She continued to have more servings, each time eating half of the remaining pie. A picture of what she ate last year is given below.

This year, the Count gave Carli some stipulations to the amount she could eat. If she eats more than her allotment she will not be invited to anymore parties! Carli is terrified she will not be able to eat anymore of the delicious meat pie, so she sent along a request.

..*growl*...*moaning*.."Ahem, sorry, something was caught in my throat. It must have been who, I mean what, I ate earlier. Count Calcula gave me some odd rules to eat by and I need help understanding them! At first, I am allowed to eat one fourth of the entire meat pie! The Count then said if I was still hungry, I could eat a fourth of the portion that I just ate! Then if I was STILL hungry I could eat a fourth of the most recent portion! He said I could continue in this manner until I was full.

Well, since I will not get full this poses a problem. Just tell me the most I am allowed to eat in order to satisfy the Counts restrictions!

Carli Ivor has a puzzle piece if you can answer her question.



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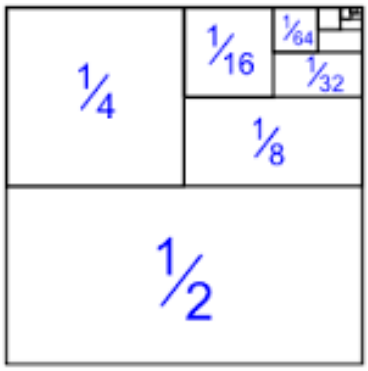
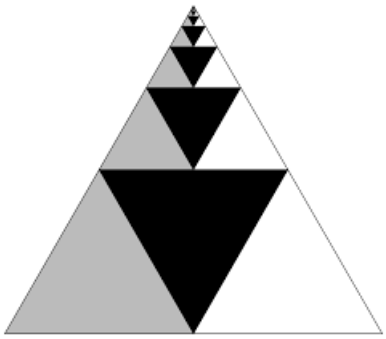
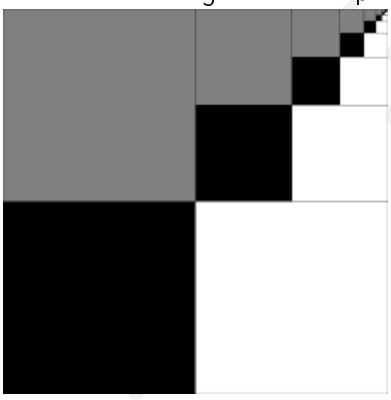
Fear the Hungry Dead

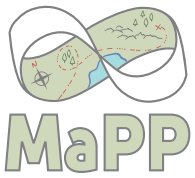
Solution - Main Puzzle 2





The following are two separate pictures to show that the answer is 1/3





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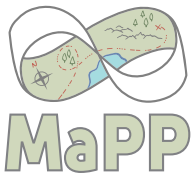
Fear the Hungry Dead

Bonus Puzzle 2

Carli was pleased with your previous answer, but Count Calcula is devious, and may yet stop Carli Ivor from eating his pies.

Carli Ivor's calculations are as follows $s_1 = \frac{1}{5} = .2$
$s_2 = \frac{1}{5} + \frac{1}{5^2} = \frac{1}{5} + \frac{1}{25} = .2 + .04 = .24$
$s_3 = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} = .2 + .04 + .008 = .248$

She did a few more calculations and realized that the answer was becoming close to .25 and she concluded that if she continued forever, that one fourth was the most she would be able to eat (which is correct!).



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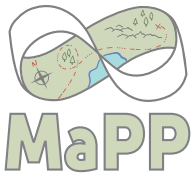
Fear the Hungry Dead

Solution - Bonus Puzzle 2

This is a well known series called a geometric series. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. In particular, if $a = \frac{1}{x}$ and $r = \frac{1}{x}$, then $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} (\frac{1}{x})(\frac{1}{x})^{n-1} = \frac{\frac{1}{x}}{1-\frac{1}{x}} = \frac{1}{x-1}$.

We expect students to plug in the first few terms of several different series, and take an educated guess for the solution. Try x=6,7,and 8!



Middle School Challenge '17

Mummy Madness

Main Puzzle 3

Halloween is just no fun for Marvin. Everyone else gets to dress up in costumes, but Marvin is tired of wearing the same mummy outfit that he wears everyday. This year is going to be different though! Marvin has decided to enter Count CalculaâĂŹs Annual Costume Contest and he is determined to win.

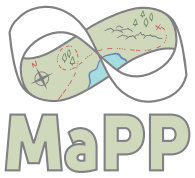
Since good *always* defeats evil, Marvin has decided that the best costume will be a superhero! Obviously in order to pull this off, Marvin needs a super belt.

âĂİOn my belt, IâĂŹm going to have five super symbols represent- ing: strength, heart, persistence, benevolence, and the power to FLY (duh!). I want lines connecting each super symbol to each of the other super symbols, but I do NOT want those lines to cross! They can go around or behind on the back of the belt, but I donâĂŹt want the lines crossing over each other anywhere! Can you help me??âĂİ

It looks like Marvin is tired of having all of his mummy wrapping crossing over itself..

âĂİAmpâĂŹd squad! IâĂŹm giving you an inner tube and some markers to work with. If you can draw on that tube and show me how to make my belt work, IâĂŹm POSITIVE I can defeat Count Calcula in the costume contest! I am so positive that IâĂŹll even reward you with a puzzle piece if you help me!âĂİ

Draw the 5 super symbols P, S, F, B, H on the inner tube with a total of 10 lines con- necting each symbol to all of the other symbols in such a way that no two lines cross over each other.



Middle School Challenge '17

Mummy Madness

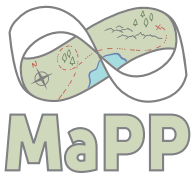
Solution - Main Puzzle 3

Hints

* Help the students to try drawing four of the symbols on a piece of paper in such a way that none of the lines cross. Once theyâĂŹve done that, see if they can draw the same picture on the inner tube, but have one of the lines wrap around the inside or back of the inner tube.

Solution

This problem is equivalent to asking one to embed a complete graph on 5 vertices K_5 on the torus (or donut). A solution can be drawn on a two- dimensional rectangle where the four corners are understood to represent the same point and the sides across from each other can be wrapped around and pasted together to form the torus. There are multiple solutions to this problem, one is included here. In the picture solution below, the orange and green edges wrap around the inside or âĂİbackâĂİ of the torus respectively.



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Mummy Madness

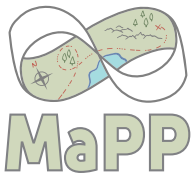
Bonus Puzzle 3

Thanks to your help, MarvinâĂŹs got the coolest super belt in town!

âĂİThanks so much, AmpâĂŹd Squad! Count Calcula doesnâĂŹt stand a chance against me in the costume contest now! I was thinking though.. it might be nice if my belt also had a symbol to represent invisibility! Can yâĂŹall find a way to make me a new belt with invis- ibility added to it?? IâĂŹd be so grateful and would definitely reward you with another puzzle piece!âĂİ

Can you make Marvin a new belt on your inner tube with lines connecting each of the six symbols to all of the other symbols in such a way that no two lines cross? If you can figure out how to help Marvin with this, there is NO WAY Count Calcula can defeat him in the costume contest!

Draw the 6 super symbols F, H, I, S, P, B on the inner tube with a total of 15 lines con- necting each symbol to all of the other symbols in such a way that no two lines cross over each other.



Middle School Challenge '17

Mummy Madness

Solution - Bonus Puzzle 3

This problem is equivalent to asking one to embed a complete graph on 6 vertices (K6) on the torus (or donut). A solution can also be drawn on a two- dimensional representation of the torus. Since each of the corners represents the same point, a convenient way to draw the solution is to place one of the super powers (invisibility) on the corners. One such solution is included below.

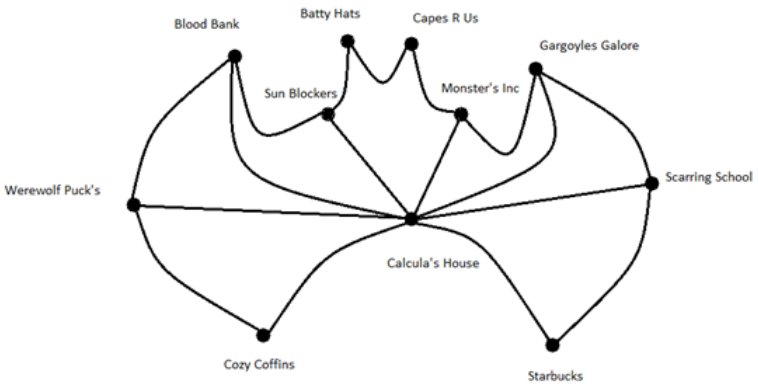
Middle School Challenge '17

Count on It

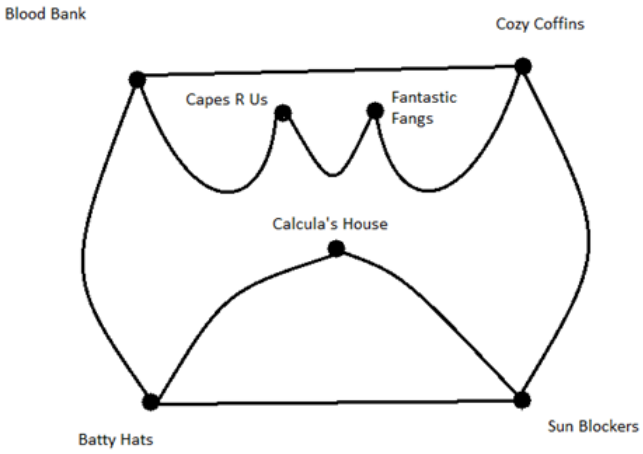
Main Puzzle 4

Count Calcula is making plans to build a new part of Transylvania. The cost to construct a building is 100 dragon teeth per road leading to that building. He wants you to determine the cost of the three plans below.

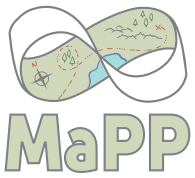
Plan A)



Plan B)



Plan C) Igor misplaced the blueprints to Plan C but says there are 75 roads.



Middle School Challenge '17

Count on It

Solution - Main Puzzle 4

This problem is essentially the degree-sum formula for graph theory. The degree of a building is the number of roads which lead to it. Adding up the degrees of every building will actually count the number of road twice. Why? Every road leads to exactly two buildings, meaning every road contributes exactly 2 to the sum of degrees.

With this in mind we see Plan A has 17 roads, so the sum of degrees of every building is 34 meaning it will cost 3400 dragon teeth to construct Plan A.

Plan B has 9 roads, so 18 is the sum of degrees meaning it will cost 1800 dragon teeth.

Plan C has 75 roads, so 150 is the degree-sum meaning it will cost 15,000 dragon teeth.

Middle School Challenge '17

Count on It

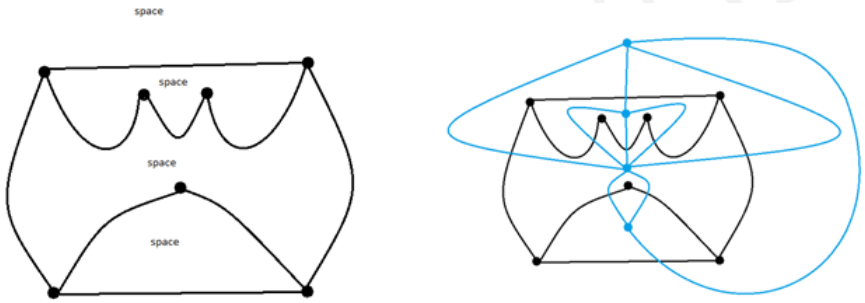
Bonus Puzzle 4

A ghost city will be built around the new addition to Transylvania and follows a few rules which make haunting easier. The rules are as follows:

1) Every space in Transylvania must have exactly one ghost building.

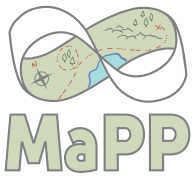
2) A ghost road is made if it joins two ghost buildings by crossing over a road in Transylvania.

Below is the ghost city of Transylvania Plan B:



If Transylvania Plan D has 33 roads what is the cost of constructing the ghost city?

If the length of a space is the number of roads the space touches what is the sum of lengths of spaces in Plan D?



Middle School Challenge '17

Count on It

Solution - Bonus Puzzle 4

Cost: 6600 dragon teeth. Length-sum: 66