



Middle School Challenge '17

Mathematical Puzzle Programs



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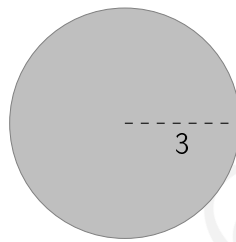
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Which Witch?

Main Puzzle 1

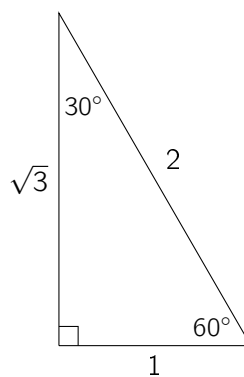
Grendel the Witch is doing what witches do: brewing potions!

In order to work efficiently, Grendel wants to use three cauldrons at the same time. Viewed from the top-down, each cauldron is perfectly circular, with a radius of 3 feet.



In order to stir each potion successfully, Grendel must be equally close to all three cauldrons used. **How can Grendel position the cauldrons so that she's as close as possible to all three? How far would she be from the center of each cauldron in this configuration?** She's a frail old thing, so don't be afraid to squeeze her into a small space if necessary...

By the way, some of Grendel's *mathematics* may help here. She has a feeling that the following triangle is important to this particular puzzle. It is drawn below such that the shortest side has length 1, but you may need to scale things a bit...



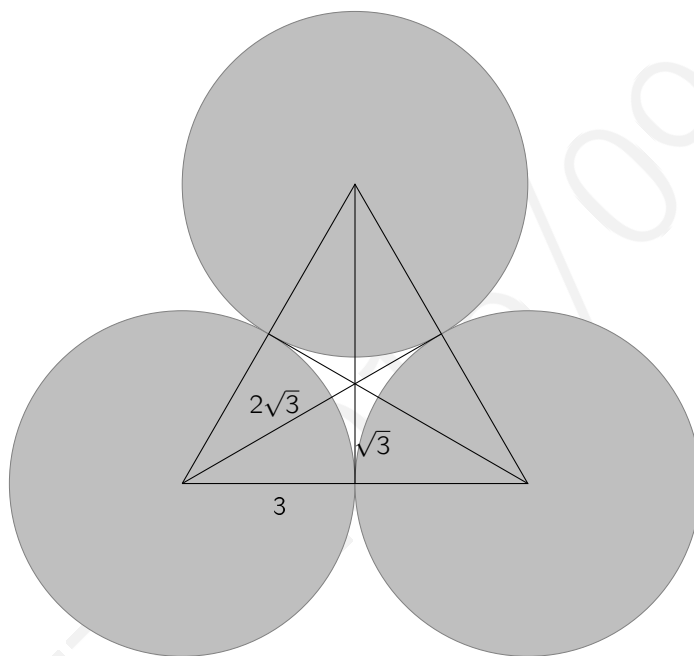


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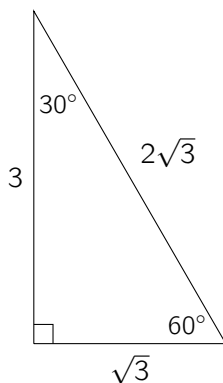
Which Witch?

Solution - Main Puzzle 1

The solution, $2\sqrt{3}$ feet, is shown by the following diagram.



Each small triangle is 30-60-90 as they were dissected from an equilateral 60-60-60 triangle. The appropriate side lengths are obtained by scaling the triangle given in the puzzle by a factor of $\sqrt{3}$.



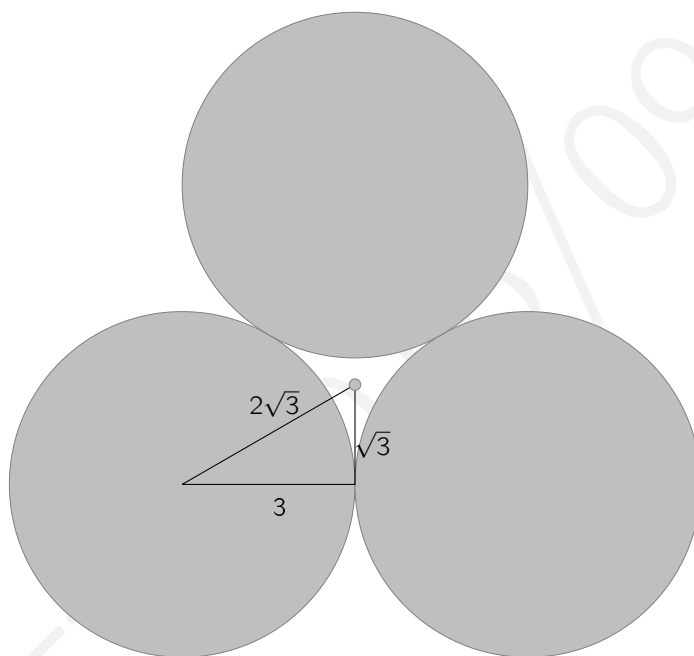


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Which Witch?

Bonus Puzzle 1

Now Grendel has bewitched a broom to do the hard work of stirring her cauldrons for her, as shown below.



Magicked brooms are rather finicky however, and unlike Grendel won't work in too tight a space. So, **can you figure out the area of the small space between the three cauldrons?** If it helps, the area of a circle is πr^2 where r is the radius of the circle, and the area of a triangle is $\frac{1}{2}bh$ where b is the length of its base and h is the height of the triangle measured perpendicularly to the base.

Mark your answer below.

- ☐ $2\pi\sqrt{3}$
- ☐ $9\sqrt{3} - \frac{9}{2}\pi$
- ☐ $4\sqrt{6} + 3\pi$
- ☐ $6\pi\sqrt{6}$
- ☐ $\pi\sqrt{3} + \frac{9\sqrt{3}}{4}$

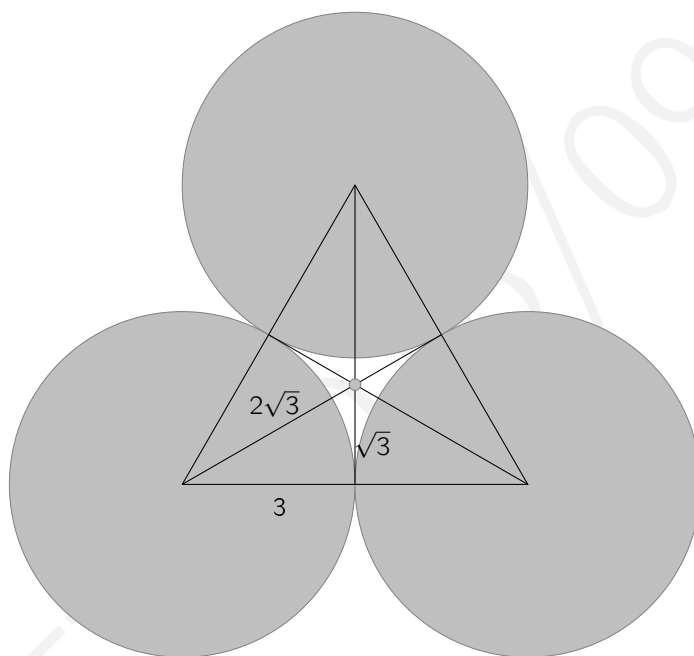


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Which Witch?

Solution - Bonus Puzzle 1

The solution may be obtained by counting the triangles and circle wedges from the solution to the Main Puzzle:



Each small triangle has area $A_T = \frac{1}{2}(3)(\sqrt{3}) = \frac{3\sqrt{3}}{2}$, so the total area in the large triangle is $6A_T = 9\sqrt{3}$.

Meanwhile, each gray circle has area $A_C = \pi(3)^2 = 9\pi$.

Since the large triangle contains six circle wedges, each of which (witch?) is $\frac{1}{12}$ a circle, the area containing the broom is $A = 6A_T - \frac{1}{2}A_C = 9\sqrt{3} - \frac{9}{2}\pi$.



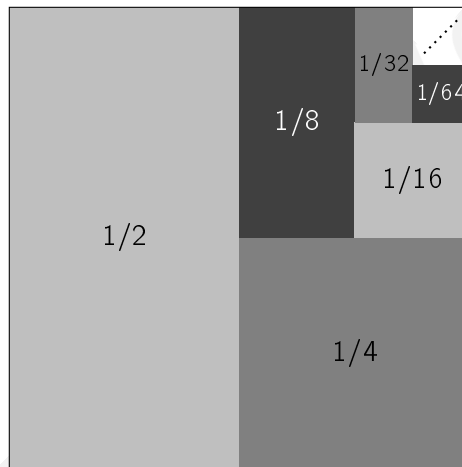
Middle School Challenge '17

Fear the Hungry Dead

Main Puzzle 2

The insatiable Zom Bea eats anything in her path. Last year at Count Calcula's party she ate ALL of the Count's famous meat pie! She started by eating half of the pie... then she ate half of the remaining pie! Actually, she continued to have more and more servings, each time eating half of the remaining pie.

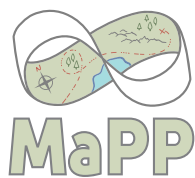
Here's a picture to give you the idea. (Hey, what kind of monster bakes a square pie, though?)



Since there was no pie left for anyone else, the Count has given Zom some stipulations for the amount she is allowed to eat this year. (Otherwise, guess who isn't getting invited to any future parties!) The Count has made it clear: "Zom's first serving must be no larger than $\frac{1}{4}$ the entire pie, and any future serving cannot be larger than $\frac{1}{4}$ the most recent serving!"

Zom is terrified that this delicious meat pie will be her last, so maybe you can help her out. **What fraction of the pie is she allowed to eat? Make sure you draw a picture which explains why.** Bear in mind, she feels like she could eat servings of the delectable pie *forever*.

Oh, and Zom doesn't know what shape the pie will be this year, but does it matter?

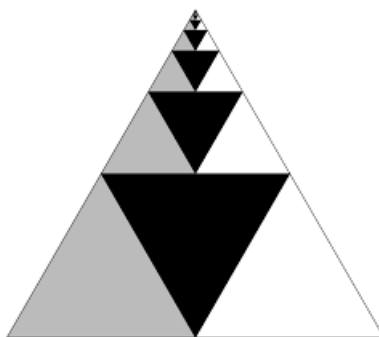


Middle School Challenge '17

Fear the Hungry Dead

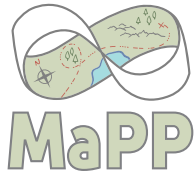
Solution - Main Puzzle 2

The answer is $\frac{1}{3}$. Here's a handy image which explains why: each gray triangle represents $\frac{1}{4}$ of the previous serving, and each row contains $\frac{1}{3}$ gray triangles.



This image also works (as do many others); similar to the above triangle, each gray square represents $\frac{1}{4}$ of the previous serving, and each L-shape contains $\frac{1}{3}$ gray squares.





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Fear the Hungry Dead

Bonus Puzzle 2

After quickly gobbling up her third of the pie, Zom Bea realized that Count Calcula hadn't counted on her eating all her servings at once!

"Very well!" said the Count. "I see that you will never be satisfied. So, I will continue to bake you my meat pies. In 2017, I will bake you $\frac{1}{17}$ of a pie. In 2018, I will bake you $\frac{1}{18}$ of a pie. And so on! Are you happy now?"

Zom's not sure, exactly. Since zombies and vampires never die, **how many total meat pies can Zom count on enjoying over the years to come?**

Mark your answer below.

- ☐ Less than 1
- ☐ At least 1, but less than 2
- ☐ At least 2, but less than 10
- ☐ At least 10, but less than 100
- ☐ At least 100



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Fear the Hungry Dead

Solution - Bonus Puzzle 2

The Count is offering up what mathematicians call a “harmonic series” of meat pies:

$$\sum_{i=17}^{\infty} \frac{1}{i} = \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \dots$$

Actually, unlike in the Main Puzzle, this infinite sum is just that: infinite! Here’s a nice way to see that this sum cannot be finite.

$$\begin{aligned} \sum_{i=17}^{\infty} \frac{1}{i} &= \left(\frac{1}{17} + \dots + \frac{1}{32} \right) + \left(\frac{1}{33} + \dots + \frac{1}{64} \right) + \dots \\ &\geq \left(\frac{1}{32} + \dots + \frac{1}{32} \right) + \left(\frac{1}{64} + \dots + \frac{1}{64} \right) + \dots \\ &= \frac{16}{32} + \frac{32}{64} + \dots \\ &= \frac{1}{2} + \frac{1}{2} + \dots \\ &= \infty \end{aligned}$$

So, it might take a while, but Zom has at least 100 pies ahead of her to enjoy.



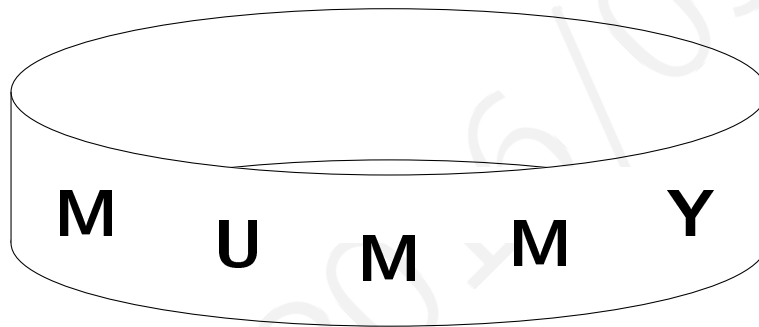
Middle School Challenge '17

Mummy Masquerade

Main Puzzle 3

Marvin Mummy has decided that this is finally the year he doesn't go as himself for Halloween. And he's figured out the costume that will finally make him the winner of Count Calcula's annual costume contest. "I'll dress up as *Mummy Man*, the undead superhero with the power to wrap up crime!"

His lack of imagination aside, Marvin has at least decided to put a little creativity into the design of his superbelt (the source of Mummy Man's powers, duh!).



"I want curves connecting each letter to each of the other letters, but I do NOT want those curves to cross! They can go around or even behind on the back of the belt, but I don't want the curve crossing another curve!"

Tape a strip of paper to make your own version of Mummy Man's belt, writing the letters M U M M Y anywhere you like. **How can you draw ten non-crossing curves which connect each pair of letters?** Remember, these curves are allowed to go around and behind the belt; actually, they'll have to!



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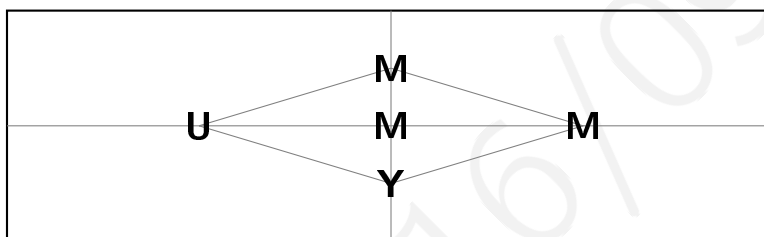
Mummy Masquerade

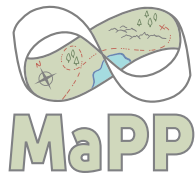
Solution - Main Puzzle 3

This problem is equivalent to asking one to embed a complete graph K_5 with 5 vertices onto the torus.

A solution can be drawn on a two-dimensional rectangle where points on opposite sides are identified (taped) together.

There are multiple solutions to this problem; here's one as an example.





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Mummy Masquerade

Bonus Puzzle 3

Thanks to your help, Marvin's got the coolest super belt in town!

"Awesome! But... could I do any better? I'd love a similar belt with the letters M A R V I N. Oh oh! Could I fit M O N S T E R? I got to know!"

What do you think? **What's the maximum number of letters which can be connected by non-crossing lines on Marvin's belt?**

Mark your answer below.

- ☐ 5
- ☐ 6
- ☐ 7
- ☐ 8
- ☐ 9 or more

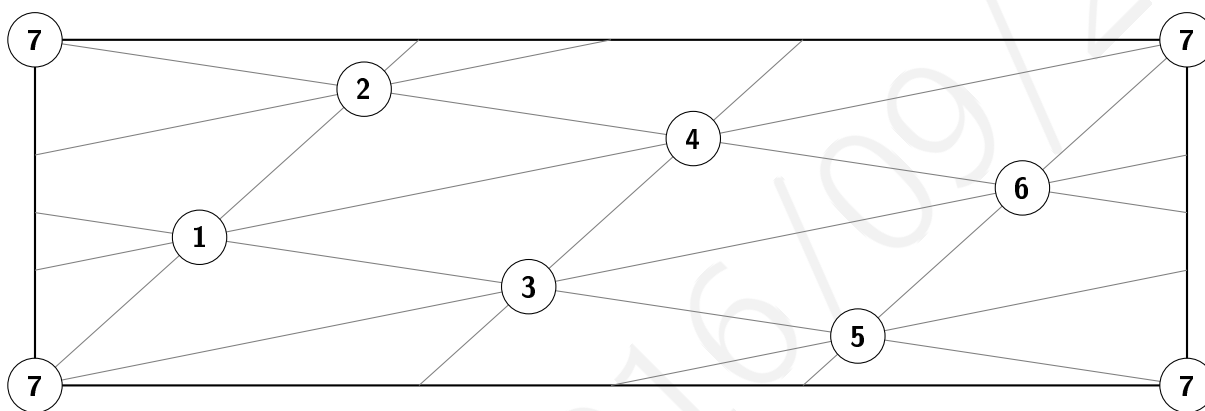


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Mummy Masquerade

Solution - Bonus Puzzle 3

The answer is 7 (although, the proof that 8 is impossible is non-trivial). Here's one such embedding:



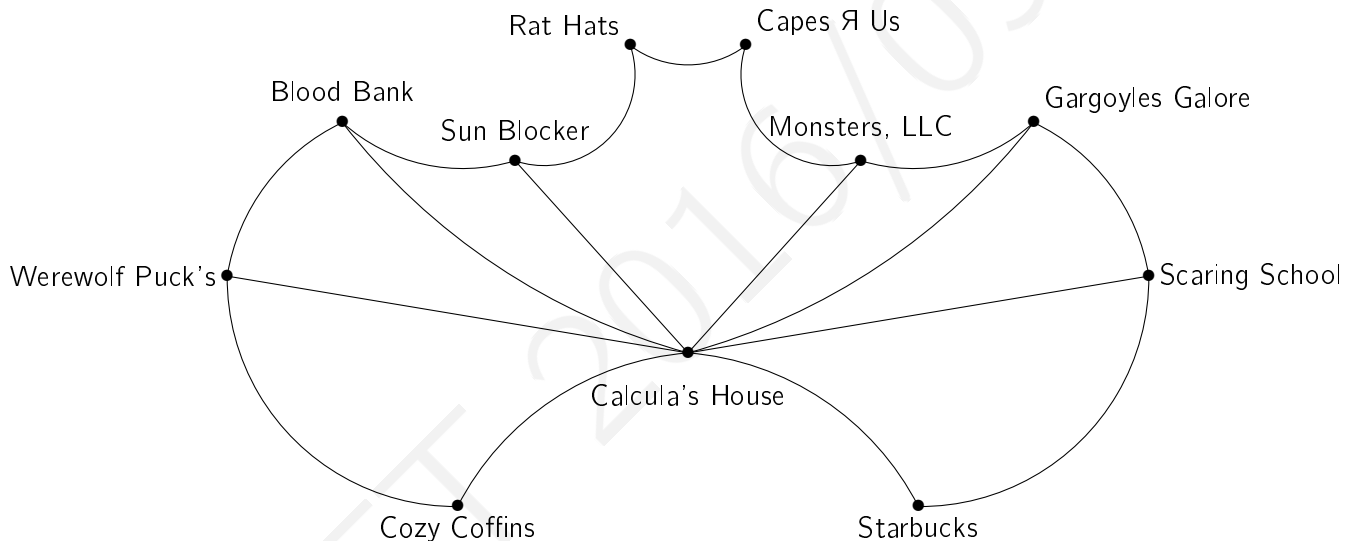


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Count on It

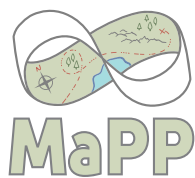
Main Puzzle 4

Count Calcula is planning construction for a new town called “Batty Borough” outside Transylvania. As everyone knows, the cost to construct a building is **100 dragon teeth** per road leading into that building. “Ack! Vat a pain in the neck! I wish someone would help me by *biting*, er, writing out the cost of my building plan!”



What would be the total cost of building Batty Borough?

Actually, the Count's ultimate goal is to renovate all of Transylvania itself, a monstrous metropolis with **57 buildings** and **73 roads**. He calculates that renovating each building would only cost **50 dragon teeth** per road leading into it, but he cannot seem to track down a roadmap similar to his plan for Batty Borough. All he knows is that each road begins and ends at a building. **Does the Count have enough information to calculate the cost of renovating Transylvania? If so, what would the renovation cost?**



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Count on It

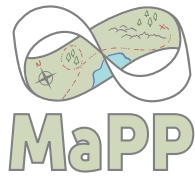
Solution - Main Puzzle 4

This problem is equivalent to the finding degree-sum formula in graph theory.

Rather than counting how many roads go into each building, and calculating each building's cost, it's much easier to simply count the total number of roads. Then, since each road leads into two buildings, we may multiply this number by two to calculate the total number of roads entering all buildings.

In particular, Batty Bourough has 17 roads, and therefore 34 road-ends. Since each building costs 100 teeth for each road end leading to it, the total cost to renovate Batty Bourough is 3400 teeth.

As for Transylvania, we can ignore the total number of buildings, and use the same process: 73 roads multiplied by 2 equals 146 road-ends, making the total cost to renovate Transylvania $146 \times 50 = 7300$ teeth.



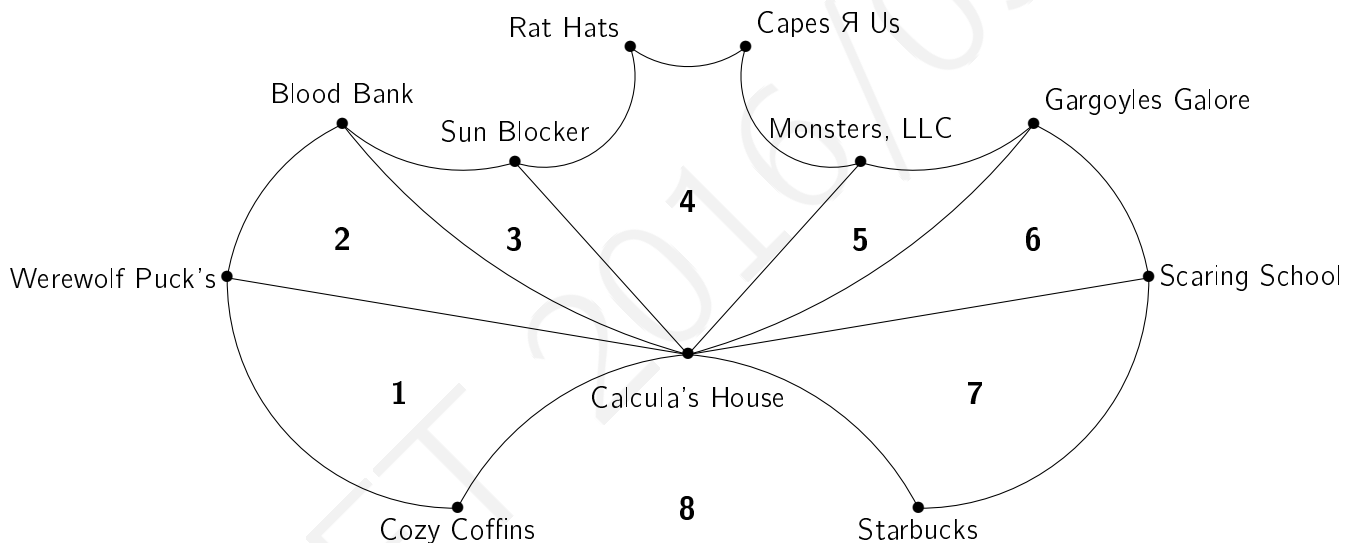
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Count on It

Bonus Puzzle 4

Count Calcula now has to face his scariest task yet: meeting with a certain Ms. Knope of the Transylvania Parks and Recreation office.

According to city code, there must be exactly one park in each area of the city which may be walked to without entering a building or crossing a road. For example, if Batty Borough had the same code, the Count would need to build 8 parks:



Remember, Transylvania has **57 buildings** and **73 roads**, and most importantly, no bridges (so roads cannot cross one another). Believe it or not, it's possible to predict for Ms. Knope how many parks should be in Transylvania. Can you figure out the pattern? (Here's a hint: try drawing some small towns and adding up the numbers of buildings, roads, and parks... do you see a pattern?)

How many parks should there be in Transylvania?

Mark your answer below.

- ☐ 11
- ☐ 18
- ☐ 27
- ☐ 39
- ☐ 56



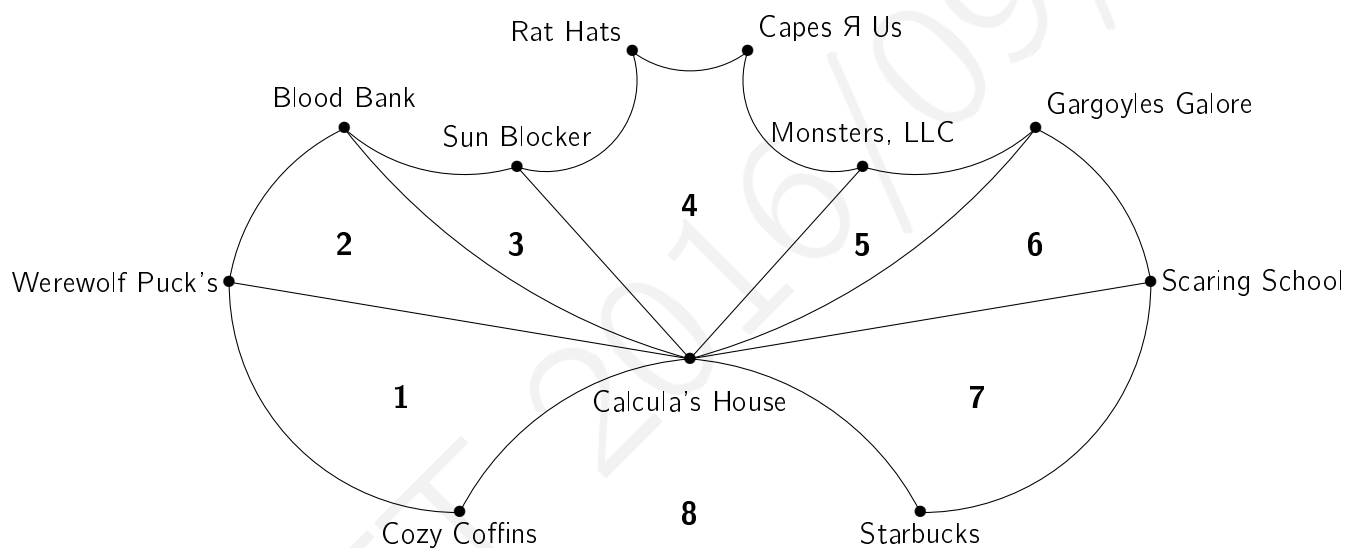
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Count on It

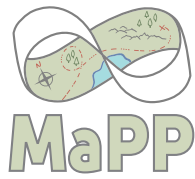
Solution - Bonus Puzzle 4

The pattern is called **Euler's formula**: for any such map where roads cannot cross, $B - R + P = 2$, where B is the number of buildings, R is the number of roads, and P is the number of parks.

You can see this in Batty Borough, where $11 - 17 + 8 = 2$.



So, in Transylvania, $57 - 73 + P = 2$ solves to $P = 18$.



Middle School Challenge '17

Ghostly Charm

Main Puzzle 5

Goolia the Ghost was wondering around the Wayward Tavern yesterday when she saw three witches, Flo, Vi, and Ru, seated around a cauldron with the following cards:



Flo told the other witches that she was going to select one card and give it special powers. She told Vi the suit (Bats, Black Cats, Pumpkins, or Candy Corn) and told Ru the rank (A, 2, 3, 4, 5, or 6). Goolia then heard Vi make the following comment:

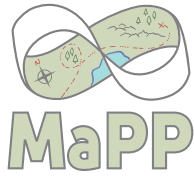
Vi: Double double toil and trouble! I don't know what card Flo picked, but I know that Ru doesn't know either!

The witches aren't the only ones interested in knowing what card Flo selected though..

"Amp'd Squad! I may be a ghost, but I want to know *witch* card has special powers! I know that what Vi said narrows down the possibilities, but I can't quite figure out how! If you can help me figure it out, I'll reward you with a puzzle piece!"

Challenge Overview

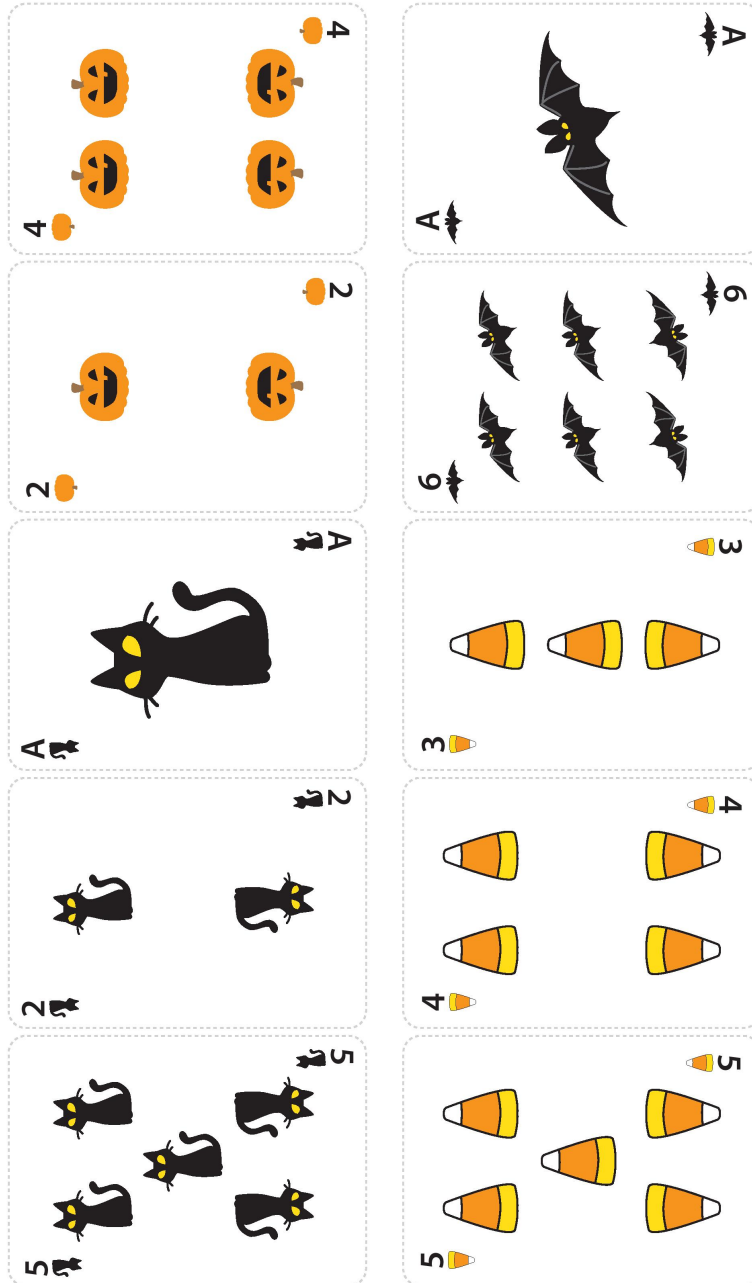
- * Follow the logic of the witches' conversation to determine the remaining possibilities of Flo's card.
- * When you think you've got it, go and present your solution to Goolia in **Challenge Room X**! Make sure you can explain to her how you figured out the answer!



Middle School Challenge '17

Ghostly Charm

Cards





Middle School Challenge '17

Ghostly Charm

Solution - Main Puzzle 5

Hints

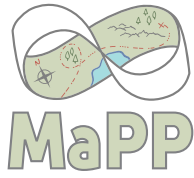
- * Think about what cards Flo could have chosen that would allow for either Vi or Ru to know what the card is right away.
- * Think about suits instead of individual cards.

Solution

Flo's card could be any of the Cats or Pumpkins! Let's see how to get there.

Vi's comment lets us know that Vi (who was told the suit) does not know what the card is just based off of the suit alone (this makes sense since all four suits have more than two cards) and also that she knows Ru does not know the card either. This tells us that the suit cannot be Bats or Candy Corn.

We know this is true since if Vi had been told the suit were Candy Corn, then it is possible that the 3 of Candy Corn was the card Flo chose. Since there is only one 3, that means that Ru would have been given 3 as the rank and would, therefore, know which card Flo had chosen. Since Vi is claiming that this situation can't happen, we know the suit cannot be Candy Corn. A similar argument works for Bats since there is only one 6 in the set of cards.



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Ghostly Charm

Bonus Puzzle 5

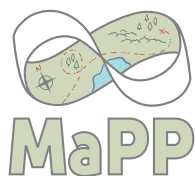
Now that Goolia knows which suits Flo's card could be, she listens to more of their conversation!

Ru: Muahaha!! Fire burn, and cauldron bubble! I didn't know what card Flo picked, but after listening to what Vi has to say, I have figured it out!

Vi: Fair is foul and foul is fair! In that case, I now know too!

"Amp'd Squad! I am so close to knowing which card has special powers!! With your help, I know we can figure out which card Flo chose! I'll gladly provide a puzzle piece for your troubles!"

With the clues Goolia overheard, can you help her figure out which card has the special powers??



Middle School Challenge '17

Ghostly Charm

Solution - Bonus Puzzle 5

Hints:

- * Try ruling out certain cards. From the first puzzle, we know that the suit is either Cats or Pumpkins. If Ru is able to figure out which card Flo chose only knowing the rank and two choices for the suit, what cards can't be possibilities?
- * Another approach could be guess and check: pick a card and then go back through Vi and Ru's conversation. See if all their statements are true, assuming that Flo's card is the same one you picked.

Solution:

The answer is the 4 of Pumpkins! Let's see how to get there.

We know from the first part of the puzzle that Flo's card is either Cats or Pumpkins.

The next clue we're given is that Ru (who was told the rank) didn't know what card was chosen, but now that she knows the suit isn't Bats or Candy Corn, she knows what card Flo chose. This tells us that the card cannot have rank 2 since both of the remaining suits (Cats and Pumpkins) have a 2. If the rank Ru was given was a 2, then she would not be able to figure out Flo's card with only the information that the suit is Cats or Pumpkins.

So, from the first two clues, we know that the card is either the 4 of Pumpkins, Ace of Cats, or 5 of Cats.

Lastly, from the third clue, we know that once the card has been narrowed down to three possibilities, Vi is able to figure out the card as well. Since Vi only knows the suit of the card that means the card has to be the 4 of Pumpkins! If Vi had been told that the suit was Cats instead, she would not know whether or not the card was the Ace or 5 of Cats.