



# **Middle School Challenge '17**

*Mathematical Puzzle Programs*



## Middle School Challenge '17

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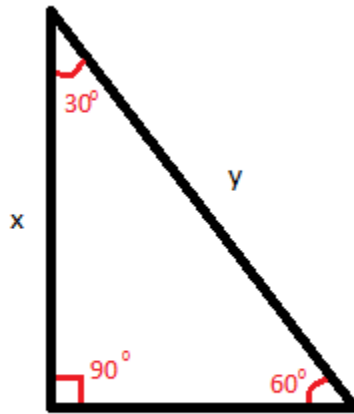
# Which Witch?

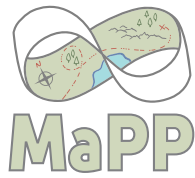
## Main Puzzle 1

Grendel the Witch is cooking up a storm! She is brewing potions in several cauldrons.

Grendel needs to make at least three potions at a time, so she needs to use at least 3 cauldrons at once. If the cauldrons have a radius of 3 feet and Grendel must be equally close to all cauldrons used, what is the shortest distance Grendel can be from the center of a cauldron?

There is a special triangle which might help you answer this puzzle: the 30-60-90 triangle. Let  $x$  be the side opposite of the  $60^\circ$  angle. Let  $y$  be the side opposite the  $90^\circ$  angle. It must be that  $y = \frac{2}{\sqrt{3}}x$ .



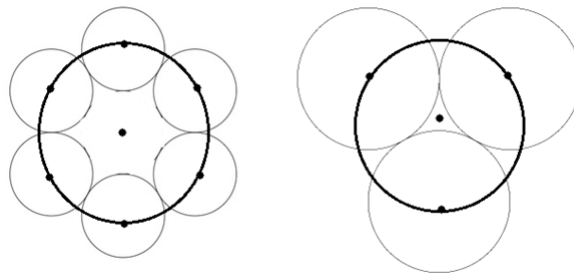


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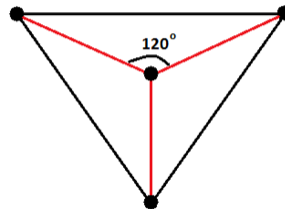
# Which Witch?

### Solution - Main Puzzle 1

Since Grendel is equally close to the center of every cauldron used, these centers lay on a big circle with Grendel as the center. The distance to Grendel is minimized by making no gaps between any two cauldrons. As we use fewer and fewer cauldrons the radius of this “big circle” is minimized. This means using 3 cauldrons minimizes the distance from Grendel to a cauldron.

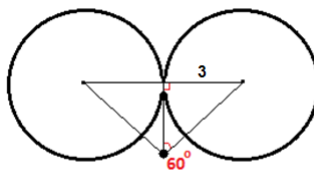


To determine how close Grendel is in the circle to the right we note the triangle formed by connecting the three circle centers and note Grendel is equally close to them.



The black lines are equal length and the red lines are equal length so we have 3 equivalent isosceles triangles within an equilateral triangle. So the three angles at the center all equal to one another and sum up to  $360^\circ$ . This means each of these angles is  $120^\circ$ .

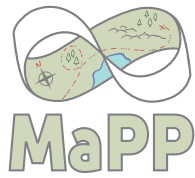
Now let's look at just the “top two” circle centers, as well as Grendel.



By drawing upward from Grendel we will cut the segment between the two circle centers in half. However, what we've really done is form a 30, 60, 90 triangle whose side opposite to the  $60^\circ$  angle is 3. The hypotenuse

of this triangle happens to be the distance we are trying to find so all we must do now is use the formula we were given at the start:  $y = \frac{2}{\sqrt{3}}x$ . (Where  $y$  is the hypotenuse and  $x$  is the side opposite to the  $60^\circ$  angle)

Plugging in 3 for  $x$  yields  $y = \frac{6}{\sqrt{3}} = 2\sqrt{3}$ .

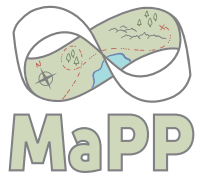


## Middle School Challenge '17

# Which Witch?

## Bonus Puzzle 1

Grendel decided she's working too hard so she got her brooms to do the work for her. Grendel set up several brooms and cauldrons in her basement. Every broom is equally close to 3 cauldrons and is  $\frac{2}{\sqrt{3}}$  feet away from them. What is the maximum number of brooms a single cauldron can be  $\frac{2}{\sqrt{3}}$  feet away from? What shape do all the brooms surrounding one cauldron form?



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## Which Witch?

Solution - Bonus Puzzle 1

6, a hexagon.

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## Middle School Challenge '17

# Fear the Hungry Dead

## Main Puzzle 2

The insatiable Carli Ivor eats anything in her path. Last year at Count Calcula's party she ate ALL of the Counts famous meat pie! She started by eating half of the pie. Then she ate half of the remaining pie. She continued to have more servings, each time eating half of the remaining pie. A picture of what she ate last year is given below.

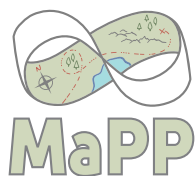
This year, the Count gave Carli some stipulations to the amount she could eat. If she eats more than her allotment she will not be invited to anymore parties! Carli is terrified she will not be able to eat anymore of the delicious meat pie, so she sent along a request.

..\*grow!\*...\*moaning\*.."Ahem, sorry, something was caught in my throat. It must have been who, I mean what, I ate earlier. Count Calcula gave me some odd rules to eat by and I need help understanding them! At first, I am allowed to eat one fourth of the entire meat pie! The Count then said if I was still hungry, I could eat a fourth of the portion that I just ate! Then if I was STILL hungry I could eat a fourth of the most recent portion! He said I could continue in this manner until I was full.

Well, since I will not get full this poses a problem. Just tell me the most I am allowed to eat in order to satisfy the Counts restrictions!

Carli Ivor has a puzzle piece if you can answer her question.

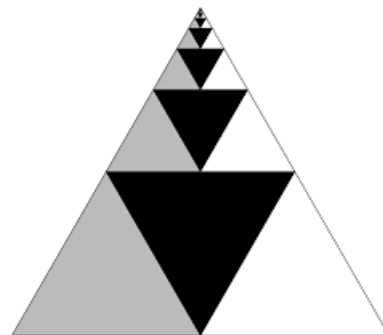
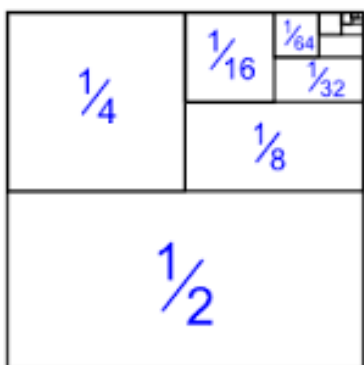




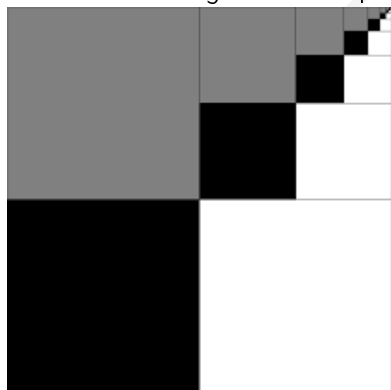
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# Fear the Hungry Dead

Solution - Main Puzzle 2



The following are two separate pictures to show that the answer is  $\frac{1}{3}$





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## Fear the Hungry Dead

### Bonus Puzzle 2

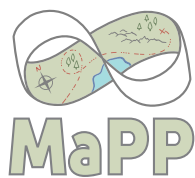
Carli was pleased with your previous answer, but Count Calcula is devious, and may yet stop Carli Ivor from eating his pies.

Carli Ivor's calculations are as follows  $s_1 = \frac{1}{5} = .2$

$$s_2 = \frac{1}{5} + \frac{1}{5^2} = \frac{1}{5} + \frac{1}{25} = .2 + .04 = .24$$

$$s_3 = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} = .2 + .04 + .008 = .248$$

She did a few more calculations and realized that the answer was becoming close to .25 and she concluded that if she continued forever, that one fourth was the most she would be able to eat (which is correct!).



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# Fear the Hungry Dead

## Solution - Bonus Puzzle 2

This is a well known series called a geometric series. If  $|r| < 1$ , then  $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ . In particular, if  $a = \frac{1}{x}$  and  $r = \frac{1}{x}$ , then  $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} (\frac{1}{x})(\frac{1}{x})^{n-1} = \frac{\frac{1}{x}}{1-\frac{1}{x}} = \frac{1}{x-1}$ .

We expect students to plug in the first few terms of several different series, and take an educated guess for the solution. Try  $x=6, 7$ , and  $8$ !



## Middle School Challenge '17

# Mummy Madness

### Main Puzzle 3

Halloween is just no fun for Marvin. Everyone else gets to dress up in costumes, but Marvin is tired of wearing the same mummy outfit that he wears everyday. This year is going to be different though! Marvin has decided to enter Count Calcula's Annual Costume Contest and he is determined to win.

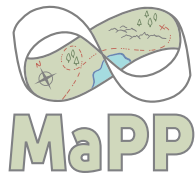
Since good \*always\* defeats evil, Marvin has decided that the best costume will be a superhero! Obviously in order to pull this off, Marvin needs a super belt.

On my belt, I'm going to have five super symbols representing: strength, heart, persistence, benevolence, and the power to FLY (duh!). I want lines connecting each super symbol to each of the other super symbols, but I do NOT want those lines to cross! They can go around or behind on the back of the belt, but I don't want the lines crossing over each other anywhere! Can you help me?

It looks like Marvin is tired of having all of his mummy wrapping crossing over itself..

And squad! I'm giving you an inner tube and some markers to work with. If you can draw on that tube and show me how to make my belt work, I'm POSITIVE I can defeat Count Calcula in the costume contest! I am so positive that I'll even reward you with a puzzle piece if you help me!

Draw the 5 super symbols *P, S, F, B, H* on the inner tube with a total of 10 lines connecting each symbol to all of the other symbols in such a way that no two lines cross over each other.



## Middle School Challenge '17

# Mummy Madness

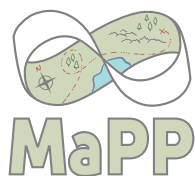
## Solution - Main Puzzle 3

### Hints

\* Help the students to try drawing four of the symbols on a piece of paper in such a way that none of the lines cross. Once they've done that, see if they can draw the same picture on the inner tube, but have one of the lines wrap around the inside or back of the inner tube.

### Solution

This problem is equivalent to asking one to embed a complete graph on 5 vertices  $K_5$  on the torus (or donut). A solution can be drawn on a two-dimensional rectangle where the four corners are understood to represent the same point and the sides across from each other can be wrapped around and pasted together to form the torus. There are multiple solutions to this problem, one is included here. In the picture solution below, the orange and green edges wrap around the inside or "back" of the torus respectively.



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# Mummy Madness

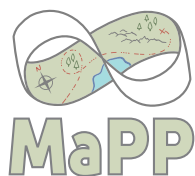
## Bonus Puzzle 3

Thanks to your help, Marvin's got the coolest super belt in town!

"Thanks so much, Amp's Squad! Count Calcula doesn't stand a chance against me in the costume contest now! I was thinking though... it might be nice if my belt also had a symbol to represent invisibility! Can you all find a way to make me a new belt with invisibility added to it?? I'd be so grateful and would definitely reward you with another puzzle piece!"

Can you make Marvin a new belt on your inner tube with lines connecting each of the six symbols to all of the other symbols in such a way that no two lines cross? If you can figure out how to help Marvin with this, there is NO WAY Count Calcula can defeat him in the costume contest!

Draw the 6 super symbols  $F, H, I, S, P, B$  on the inner tube with a total of 15 lines connecting each symbol to all of the other symbols in such a way that no two lines cross over each other.



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# **Mummy Madness**

## **Solution - Bonus Puzzle 3**

This problem is equivalent to asking one to embed a complete graph on 6 vertices ( $K_6$ ) on the torus (or donut). A solution can also be drawn on a two-dimensional representation of the torus. Since each of the corners represents the same point, a convenient way to draw the solution is to place one of the super powers (invisibility) on the corners. One such solution is included below.



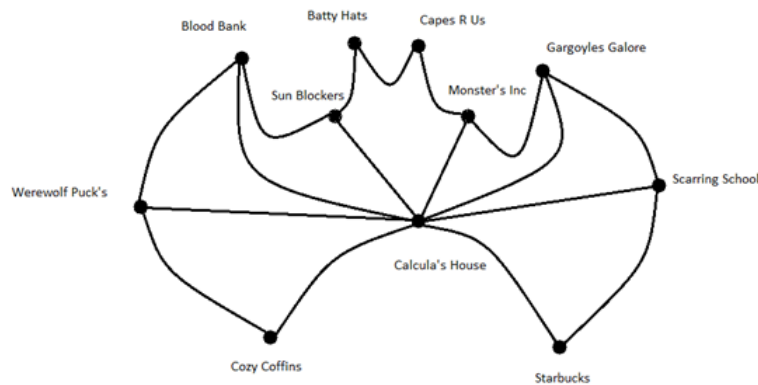
## Middle School Challenge '17

# Count on It

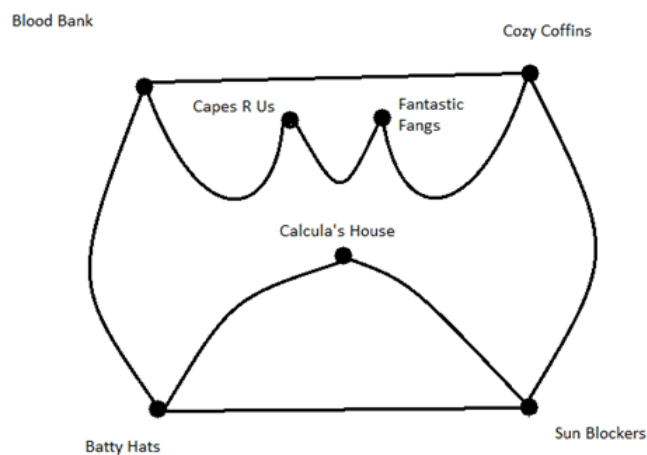
### Main Puzzle 4

Count Calcula is making plans to build a new part of Transylvania. The cost to construct a building is 100 dragon teeth per road leading to that building. He wants you to determine the cost of the three plans below.

Plan A)

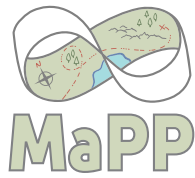


Plan B)



Plan C) Igor misplaced the blueprints to Plan C but says there are 75 roads.





## Middle School Challenge '17

# Count on It

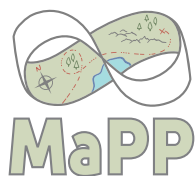
## Solution - Main Puzzle 4

This problem is essentially the degree-sum formula for graph theory. The degree of a building is the number of roads which lead to it. Adding up the degrees of every building will actually count the number of road twice. Why? Every road leads to exactly two buildings, meaning every road contributes exactly 2 to the sum of degrees.

With this in mind we see Plan A has 17 roads, so the sum of degrees of every building is 34 meaning it will cost 3400 dragon teeth to construct Plan A.

Plan B has 9 roads, so 18 is the sum of degrees meaning it will cost 1800 dragon teeth.

Plan C has 75 roads, so 150 is the degree-sum meaning it will cost 15,000 dragon teeth.



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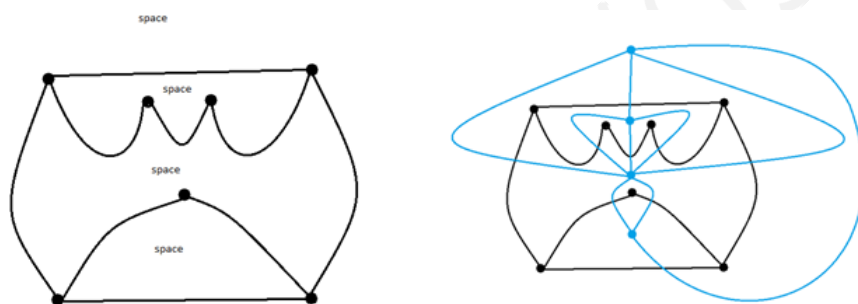
# Count on It

## Bonus Puzzle 4

A ghost city will be built around the new addition to Transylvania and follows a few rules which make haunting easier. The rules are as follows:

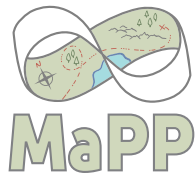
- 1) Every space in Transylvania must have exactly one ghost building.
- 2) A ghost road is made if it joins two ghost buildings by crossing over a road in Transylvania.

Below is the ghost city of Transylvania Plan B:



If Transylvania Plan D has 33 roads what is the cost of constructing the ghost city?

If the length of a space is the number of roads the space touches what is the sum of lengths of spaces in Plan D?



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## Count on It

### Solution - Bonus Puzzle 4

Cost: 6600 dragon teeth. Length-sum: 66

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## Middle School Challenge '17

# Mummy Madness

### Main Puzzle 5

Goolia the Ghost was wondering around the Wayward Tavern yesterday when she saw three witches, Flo, Vi, and Ru, seated around a cauldron with the following cards:

Flo told the other witches that she was going to select one card and give it special powers. She told Vi the suit (Bats, Black Cats, Pumpkins, or Candy Corn) and told Ru the rank (A, 2, 3, 4, 5, or 6). Goolia then heard Vi make the following comment:

Vi: Double double toil and trouble! I don't know what card Flo picked, but I know that Ru doesn't know either!

The witches aren't the only ones interested in knowing what card Flo selected though..

â€œI'm Squad! I may be a ghost, but I want to know \*witch\* card has special powers! I know that what Vi said narrows down the possibilities, but I can't quite figure out how! If you can help me figure it out, I'll reward you with a puzzle piece!â€œ

Follow the logic of the witches' conversation to determine the remaining possibilities of Flo's card.



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# Mummy Madness

### Solution - Main Puzzle 5

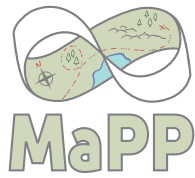
#### Hints

- \* Think about what cards Flo could have chosen that would allow for either Vi or Ru to know what the card is right away.
- \* Think about suits instead of individual cards.

#### Solution

Flo's card could be any of the Cats or Pumpkins! Let's see how to get there. Vi's comment lets us know that Vi (who was told the suit) does not know what the card is just based off of the suit alone (this makes sense since all four suits have more than two cards) and also that she knows Ru does not know the card either. This tells us that the suit cannot be Bats or Candy Corn.

We know this is true since if Vi had been told the suit were Candy Corn, then it is possible that the 3 of Candy Corn was the card Flo chose. Since there is only one 3, that means that Ru would have been given 3 as the rank and would, therefore, know which card Flo had chosen. Since Vi is claiming that this situation can't happen, we know the suit cannot be Candy Corn. A similar argument works for Bats since there is only one 6 in the set of cards.

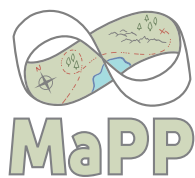


**Middle School Challenge '17**

# **Mummy Madness**

## **Bonus Puzzle 5**

Now that Goolia knows which suits Flo's card could be, she listens to more of their conversation! Ru: Muahaha!! Fire burn, and cauldron bubble! I didn't know what card Flo picked, but after listening to what Vi has to say, I have figured it out! Vi: Fair is foul and foul is fair! In that case, I now know too! "Squad! I am so close to knowing which card has special powers!! With your help, I know we can figure out which card Flo chose! I'll gladly provide a puzzle piece for your troubles! With the clues Goolia overheard, can you help her figure out which card has the special powers??"



## Middle School Challenge '17

# Mummy Madness

## Solution - Bonus Puzzle 5

Hints:

\* Try ruling out certain cards. From the first puzzle, we know that the suit is either Cats or Pumpkins. If Ru is able to figure out which card Flo chose only knowing the rank and two choices for the suit, what cards can't be possibilities?

\* Another approach could be guess and check: pick a card and then go back through Vi and Ru's conversation. See if all their statements are true, assuming that Flo's card is the same one you picked.

Solution:

The answer is the 4 of Pumpkins! Let's see how to get there.

We know from the first part of the puzzle that Flo's card is either Cats or Pumpkins.

The next clue we're given is that Ru (who was told the rank) didn't know what card was chosen, but now that she knows the suit isn't Bats or Candy Corn, she knows what card Flo chose. This tells us that the card cannot have rank 2 since both of the remaining suits (Cats and Pumpkins) have a 2. If the rank Ru was given was a 2, then she would not be able to figure out Flo's card with only the information that the suit is Cats or Pumpkins.

So, from the first two clues, we know that the card is either the 4 of Pumpkins, 1 of Cats, or 5 of Cats.

Lastly, from the third clue, we know that once the card has been narrowed down to three possibilities, Vi is able to figure out the card as well. Since Vi only knows the suit of the card that means the card has to be the 4 of Pumpkins! If Vi had been told that the suit was Cats instead, she would not know whether or not the card was the 1 or 5 of Cats.