



# **Middle School Challenge '17**

*Mathematical Puzzle Programs*



## Middle School Challenge '17

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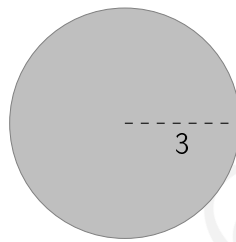
## Middle School Challenge '17

# Which Witch?

## Main Puzzle 1

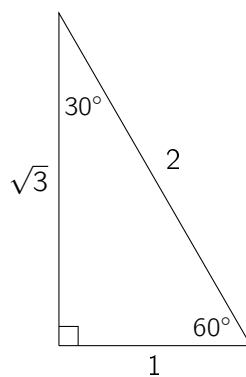
Grendel the Witch is doing what witches do: brewing potions!

In order to work efficiently, Grendel wants to use three cauldrons at the same time. Viewed from the top-down, each cauldron is perfectly circular, with a radius of 3 feet.



In order to stir each potion successfully, Grendel must be equally close to all three cauldrons used. **How can Grendel position the cauldrons so that she's as close as possible to all three? How far would she be from the center of each cauldron in this configuration?** She's a frail old thing, so don't be afraid to squeeze her into a small space if necessary...

By the way, some of Grendel's *mathematics* may help here. She has a feeling that the following triangle is important to this particular puzzle. It is drawn below such that the shortest side has length 1, but you may need to scale things a bit...



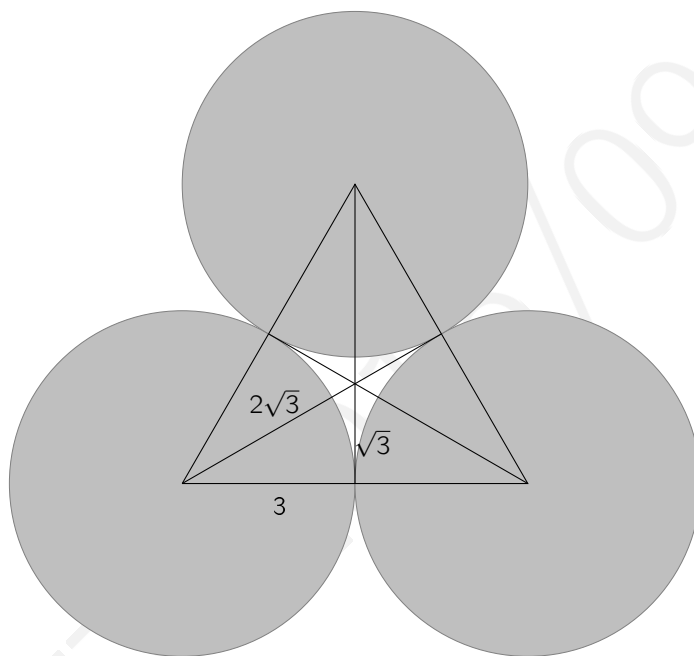


# Middle School Challenge '17

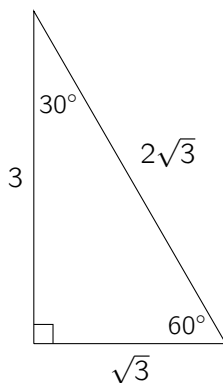
## Which Witch?

### Solution - Main Puzzle 1

The solution,  $2\sqrt{3}$  feet, is shown by the following diagram.



Each small triangle is 30-60-90 as they were dissected from an equilateral 60-60-60 triangle. The appropriate side lengths are obtained by scaling the triangle given in the puzzle by a factor of  $\sqrt{3}$ .



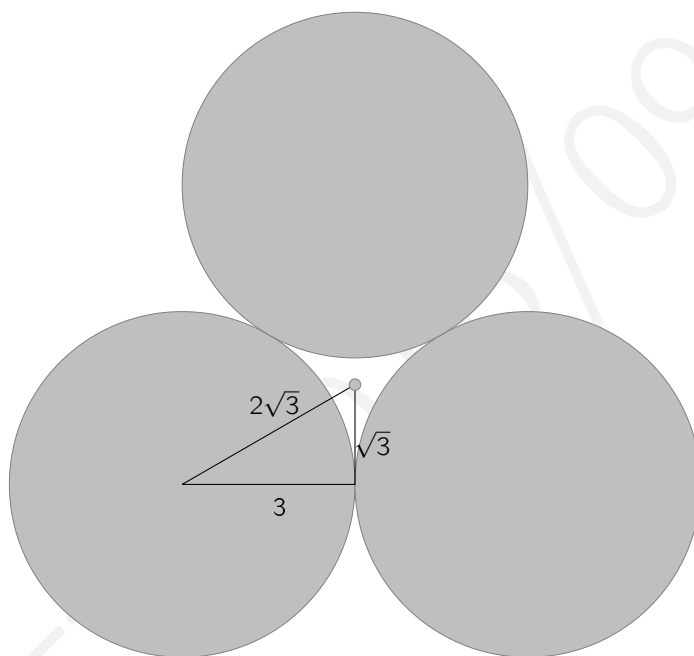


## Middle School Challenge '17

# Which Witch?

## Bonus Puzzle 1

Now Grendel has bewitched a broom to do the hard work of stirring her cauldrons for her, as shown below.



Magicked brooms are rather finicky however, and unlike Grendel won't work in too tight a space. So, **can you figure out the area of the small space between the three cauldrons?** If it helps, the area of a circle is  $\pi r^2$  where  $r$  is the radius of the circle, and the area of a triangle is  $\frac{1}{2}bh$  where  $b$  is the length of its base and  $h$  is the height of the triangle measured perpendicularly to the base.

**Mark your answer below.**

- ☐  $2\pi\sqrt{3}$
- ☐  $9\sqrt{3} - \frac{9}{2}\pi$
- ☐  $4\sqrt{6} + 3\pi$
- ☐  $6\pi\sqrt{6}$
- ☐  $\pi\sqrt{3} + \frac{9\sqrt{3}}{4}$

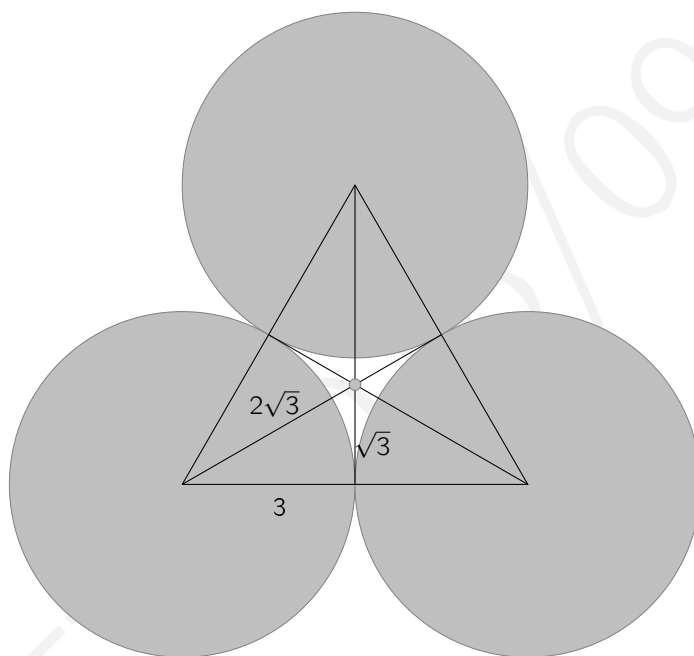


Middle School Challenge '17

# Which Witch?

## Solution - Bonus Puzzle 1

The solution may be obtained by counting the triangles and circle wedges from the solution to the Main Puzzle:



Each small triangle has area  $A_T = \frac{1}{2}(3)(\sqrt{3}) = \frac{3\sqrt{3}}{2}$ , so the total area in the large triangle is  $6A_T = 9\sqrt{3}$ .

Meanwhile, each gray circle has area  $A_C = \pi(3)^2 = 9\pi$ .

Since the large triangle contains six circle wedges, each of which (witch?) is  $\frac{1}{12}$  a circle, the area containing the broom is  $A = 6A_T - \frac{1}{2}A_C = 9\sqrt{3} - \frac{9}{2}\pi$ .



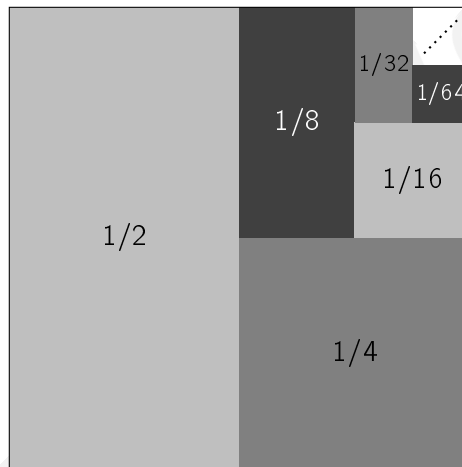
## Middle School Challenge '17

# Fear the Hungry Dead

## Main Puzzle 2

The insatiable Zom Bea eats anything in her path. Last year at Count Calcula's party she ate ALL of the Count's famous meat pie! She started by eating half of the pie... then she ate half of the remaining pie! Actually, she continued to have more and more servings, each time eating half of the remaining pie.

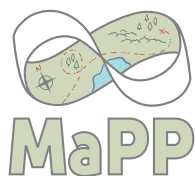
Here's a picture to give you the idea. (Hey, what kind of monster bakes a square pie, though?)



Since there was no pie left for anyone else, the Count has given Zom some stipulations for the amount she is allowed to eat this year. (Otherwise, guess who isn't getting invited to any future parties!) The Count has made it clear: "Zom's first serving must be no larger than  $\frac{1}{4}$  the entire pie, and any future serving cannot be larger than  $\frac{1}{4}$  the most recent serving!"

Zom is terrified that this delicious meat pie will be her last, so maybe you can help her out. **What fraction of the pie is she allowed to eat? Make sure you draw a picture which explains why.** Bear in mind, she feels like she could eat servings of the delectable pie *forever*.

Oh, and Zom doesn't know what shape the pie will be this year, but does it matter?

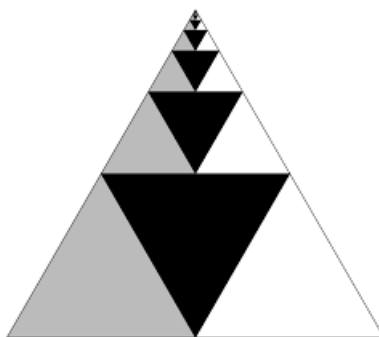


## Middle School Challenge '17

# Fear the Hungry Dead

## Solution - Main Puzzle 2

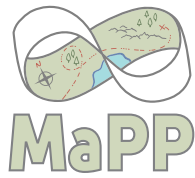
The answer is  $\frac{1}{3}$ . Here's a handy image which explains why: each gray triangle represents  $\frac{1}{4}$  of the previous serving, and each row contains  $\frac{1}{3}$  gray triangles.



This image also works (as do many others); similar to the above triangle, each gray square represents  $\frac{1}{4}$  of the previous serving, and each L-shape contains  $\frac{1}{3}$  gray squares.







## Middle School Challenge '17

# Fear the Hungry Dead

## Bonus Puzzle 2

After quickly gobbling up her third of the pie, Zom Bea realized that Count Calcula hadn't counted on her eating all her servings at once!

"Very well!" said the Count. "I see that you will never be satisfied. So, I will continue to bake you my meat pies. In 2017, I will bake you  $\frac{1}{17}$  of a pie. In 2018, I will bake you  $\frac{1}{18}$  of a pie. And so on! Are you happy now?"

Zom's not sure, exactly. Since zombies and vampires never die, how many total meat pies can Zom count on enjoying over the years to come?

**Mark your answer below.**

- ☐ Less than 1
- ☐ At least 1, but less than 2
- ☐ At least 2, but less than 10
- ☐ At least 10, but less than 100
- ☐ At least 100



## Middle School Challenge '17

# Fear the Hungry Dead

## Solution - Bonus Puzzle 2

The Count is offering up what mathematicians call a “harmonic series” of meat pies:

$$\sum_{i=17}^{\infty} \frac{1}{i} = \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \dots$$

Actually, unlike in the Main Puzzle, this infinite sum is just that: infinite! Here’s a nice way to see that this sum cannot be finite.

$$\begin{aligned} \sum_{i=17}^{\infty} \frac{1}{i} &= \left( \frac{1}{17} + \dots + \frac{1}{32} \right) + \left( \frac{1}{33} + \dots + \frac{1}{64} \right) + \dots \\ &\geq \left( \frac{1}{32} + \dots + \frac{1}{32} \right) + \left( \frac{1}{64} + \dots + \frac{1}{64} \right) + \dots \\ &= \frac{16}{32} + \frac{32}{64} + \dots \\ &= \frac{1}{2} + \frac{1}{2} + \dots \\ &= \infty \end{aligned}$$

So, it might take a while, but Zom has at least 100 pies ahead of her to enjoy.



## Middle School Challenge '17

# Mummy Madness

### Main Puzzle 3

Halloween is just no fun for Marvin. Everyone else gets to dress up in costumes, but Marvin is tired of wearing the same mummy outfit that he wears everyday. This year is going to be different though! Marvin has decided to enter Count Calcula's Annual Costume Contest and he is determined to win.

Since good \*always\* defeats evil, Marvin has decided that the best costume will be a superhero! Obviously in order to pull this off, Marvin needs a super belt.

"On my belt, I'm going to have five super symbols representing: strength, heart, persistence, benevolence, and the power to FLY (duh!). I want lines connecting each super symbol to each of the other super symbols, but I do NOT want those lines to cross! They can go around or behind on the back of the belt, but I don't want the lines crossing over each other anywhere! Can you help me??"

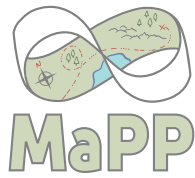
It looks like Marvin is tired of having all of his mummy wrapping crossing over itself..

"Amp'd squad! I'm giving you an inner tube and some markers to work with. If you can draw on that tube and show me how to make my belt work, I'm POSITIVE I can defeat Count Calcula in the costume contest! I am so positive that I'll even reward you with a puzzle piece if you help me!"



#### Challenge Overview

- \* Draw the 5 super symbols on the inner tube with a total of 10 lines connecting each symbol to all of the other symbols in such a way that no two lines cross over each other.
- \* When you think you've got it, go and present your solution to Marvin in **Challenge Room X!**



## Middle School Challenge '17

# Mummy Madness

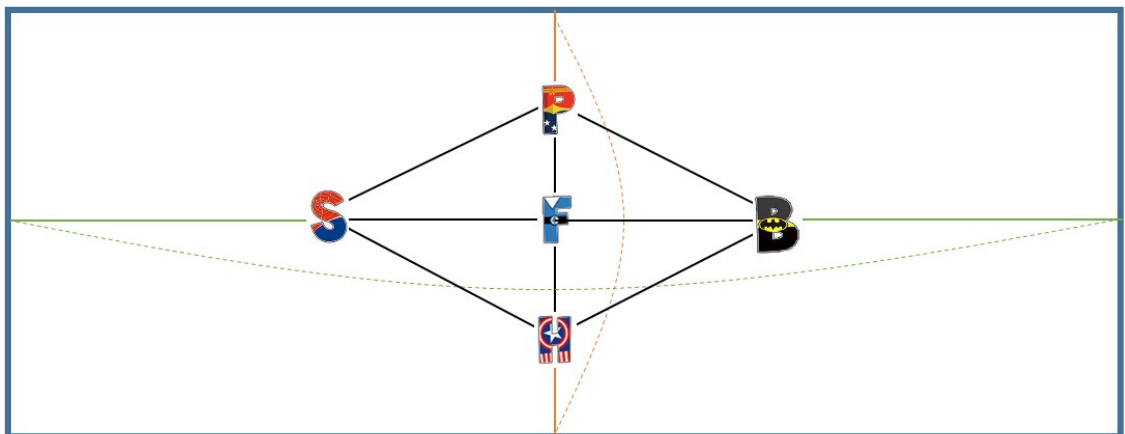
### Solution - Main Puzzle 3

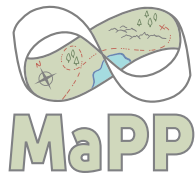
#### Hints

- \* Help the students to try drawing four of the symbols on a piece of paper in such a way that none of the lines cross. Once they've done that, see if they can draw the same picture on the inner tube, but have one of the lines wrap around the inside or back of the inner tube.

#### Solution

This problem is equivalent to asking one to embed a complete graph on 5 vertices ( $K_5$ ) on the torus (or donut). A solution can be drawn on a two-dimensional rectangle where the four corners are understood to represent the same point and the sides across from each other can be wrapped around and pasted together to form the torus. There are multiple solutions to this problem, one is included here. In the picture solution below, the orange and green edges wrap around the inside or “back” of the torus respectively.





## Middle School Challenge '17

# Mummy Madness

### Bonus Puzzle 3

Thanks to your help, Marvin's got the coolest super belt in town!

**"Thanks so much, Amp'd Squad! Count Calcula doesn't stand a chance against me in the costume contest now! I was thinking though.. it might be nice if my belt also had a symbol to represent invisibility! Can y'all find a way to make me a new belt with invisibility added to it?? I'd be so grateful and would definitely reward you with another puzzle piece!"**

Can you make Marvin a new belt on your inner tube with lines connecting each of the six symbols to all of the other symbols in such a way that no two lines cross? If you can figure out how to help Marvin with this, there is NO WAY Count Calcula can defeat him in the costume contest!



#### Challenge Overview

- \* Draw the 6 super symbols on the inner tube with a total of 50 lines connecting each symbol to all of the other symbols in such a way that no two lines cross over each other.

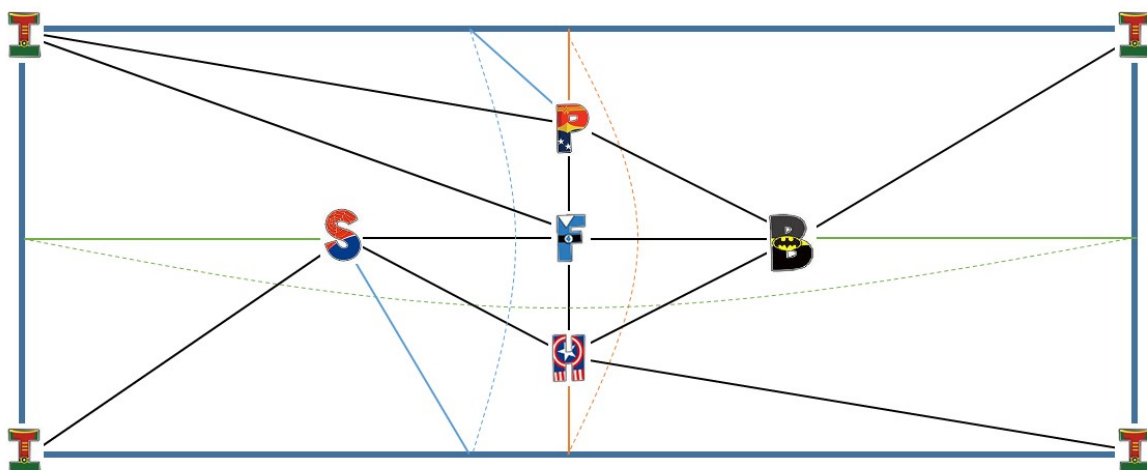


Middle School Challenge '17

# Mummy Madness

## Solution - Bonus Puzzle 3

This problem is equivalent to asking one to embed a complete graph on 6 vertices ( $K_6$ ) on the torus (or donut). A solution can also be drawn on a two-dimensional representation of the torus. Since each of the corners represents the same point, a convenient way to draw the solution is to place one of the super powers (invisibility) on the corners. One such solution is included below.





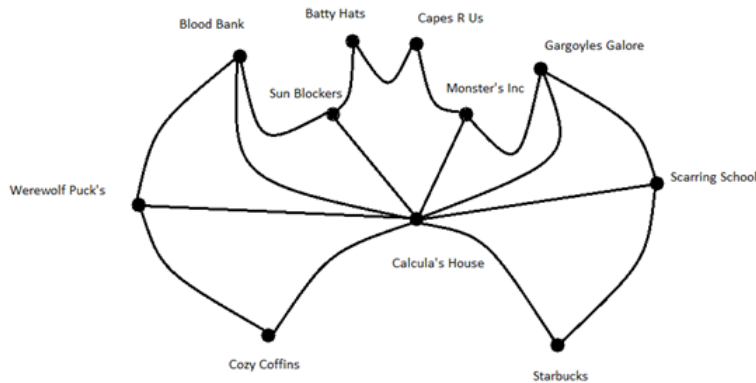
## Middle School Challenge '17

# Count on It

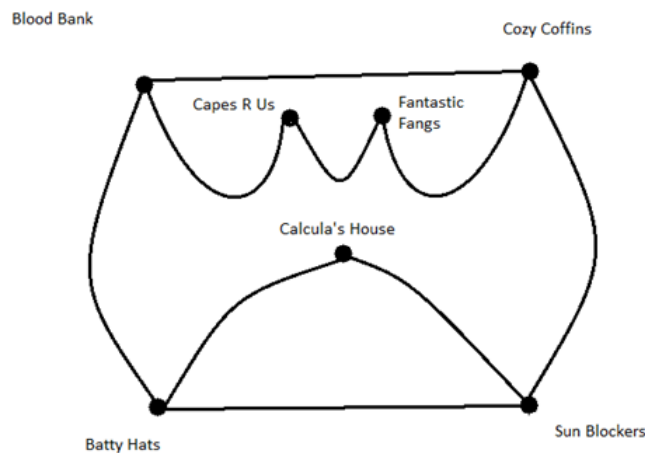
### Main Puzzle 4

Count Calcula is making plans to build a new part of Transylvania. The cost to construct a building is 100 dragon teeth per road leading to that building. He wants you to determine the cost of the three plans below.

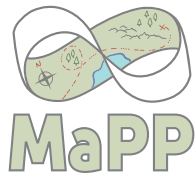
Plan A)



Plan B)



Plan C) Igor misplaced the blueprints to Plan C but says there are 75 roads.



## Middle School Challenge '17

# Count on It

## Solution - Main Puzzle 4

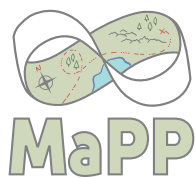
This problem is essentially the degree-sum formula for graph theory. The degree of a building is the number of roads which lead to it. Adding up the degrees of every building will actually count the number of road twice. Why? Every road leads to exactly two buildings, meaning every road contributes exactly 2 to the sum of degrees.

With this in mind we see Plan A has 17 roads, so the sum of degrees of every building is 34 meaning it will cost 3400 dragon teeth to construct Plan A.

Plan B has 9 roads, so 18 is the sum of degrees meaning it will cost 1800 dragon teeth.

Plan C has 75 roads, so 150 is the degree-sum meaning it will cost 15,000 dragon teeth.





## Middle School Challenge '17

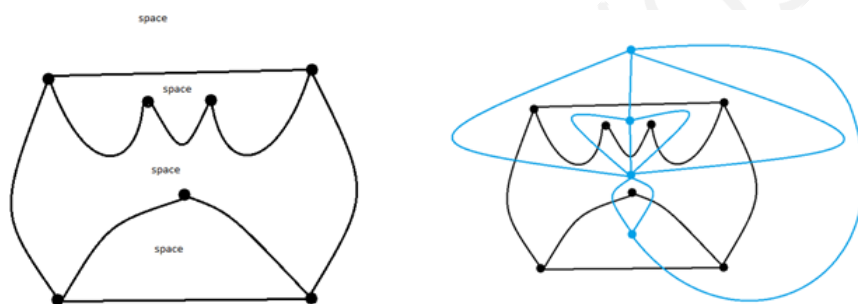
# Count on It

## Bonus Puzzle 4

A ghost city will be built around the new addition to Transylvania and follows a few rules which make haunting easier. The rules are as follows:

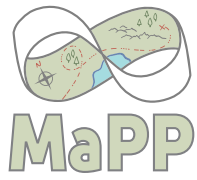
- 1) Every space in Transylvania must have exactly one ghost building.
- 2) A ghost road is made if it joins two ghost buildings by crossing over a road in Transylvania.

Below is the ghost city of Transylvania Plan B:



If Transylvania Plan D has 33 roads what is the cost of constructing the ghost city?

If the length of a space is the number of roads the space touches what is the sum of lengths of spaces in Plan D?



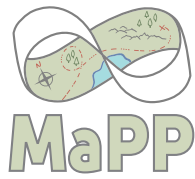
Middle School Challenge '17

## Count on It

### Solution - Bonus Puzzle 4

Cost: 6600 dragon teeth. Length-sum: 66

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## Middle School Challenge '17

# Ghostly Charm

### Main Puzzle 5

Goolia the Ghost was wondering around the Wayward Tavern yesterday when she saw three witches, Flo, Vi, and Ru, seated around a cauldron with the following cards:



Flo told the other witches that she was going to select one card and give it special powers. She told Vi the suit (Bats, Black Cats, Pumpkins, or Candy Corn) and told Ru the rank (A, 2, 3, 4, 5, or 6). Goolia then heard Vi make the following comment:

*Vi: Double double toil and trouble! I don't know what card Flo picked, but I know that Ru doesn't know either!*

The witches aren't the only ones interested in knowing what card Flo selected though..

**"Amp'd Squad! I may be a ghost, but I want to know \*witch\* card has special powers! I know that what Vi said narrows down the possibilities, but I can't quite figure out how! If you can help me figure it out, I'll reward you with a puzzle piece!"**

#### Challenge Overview

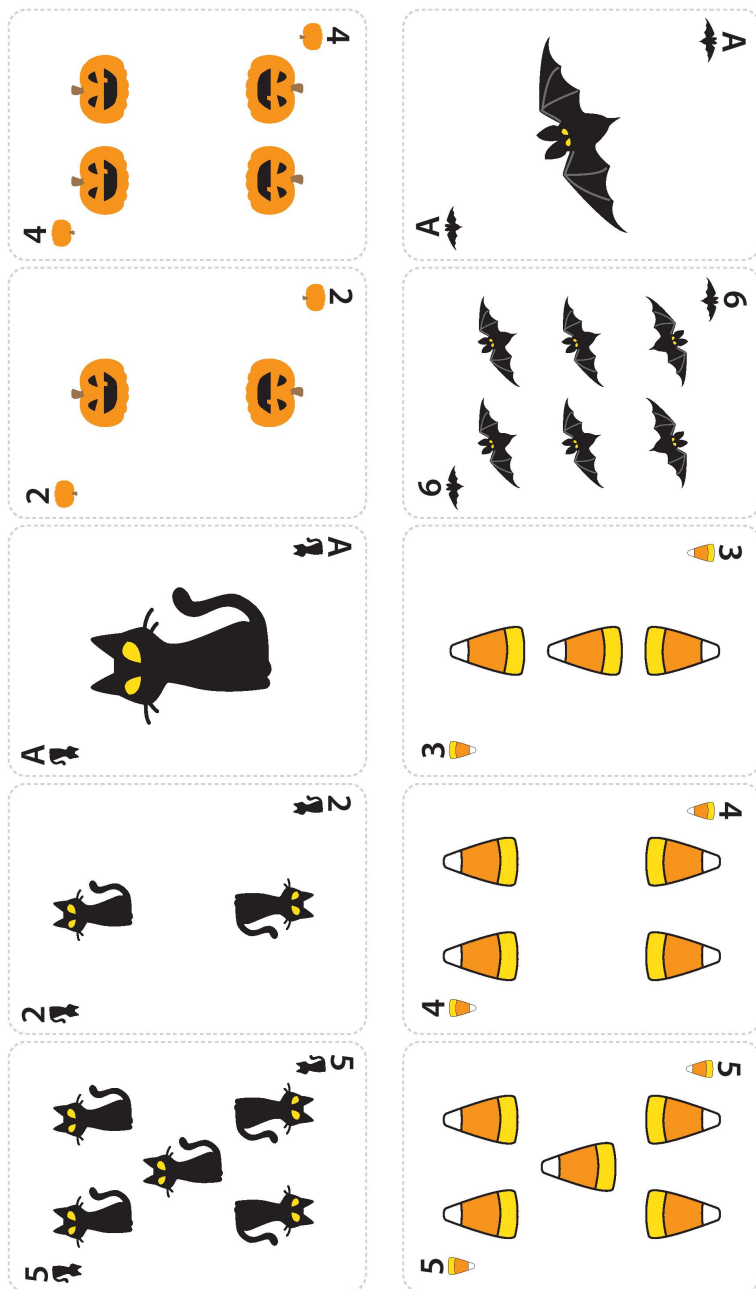
- \* Follow the logic of the witches' conversation to determine the remaining possibilities of Flo's card.
- \* When you think you've got it, go and present your solution to Goolia in **Challenge Room X**! Make sure you can explain to her how you figured out the answer!



Middle School Challenge '17

# Ghostly Charm

Cards





## Middle School Challenge '17

# Ghostly Charm

## Solution - Main Puzzle 5

### Hints

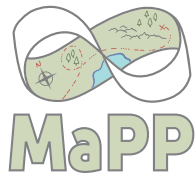
- \* Think about what cards Flo could have chosen that would allow for either Vi or Ru to know what the card is right away.
- \* Think about suits instead of individual cards.

### Solution

Flo's card could be any of the Cats or Pumpkins! Let's see how to get there.

Vi's comment lets us know that Vi (who was told the suit) does not know what the card is just based off of the suit alone (this makes sense since all four suits have more than two cards) and also that she knows Ru does not know the card either. This tells us that the suit cannot be Bats or Candy Corn.

We know this is true since if Vi had been told the suit were Candy Corn, then it is possible that the 3 of Candy Corn was the card Flo chose. Since there is only one 3, that means that Ru would have been given 3 as the rank and would, therefore, know which card Flo had chosen. Since Vi is claiming that this situation can't happen, we know the suit cannot be Candy Corn. A similar argument works for Bats since there is only one 6 in the set of cards.



Middle School Challenge '17

## Ghostly Charm

### Bonus Puzzle 5

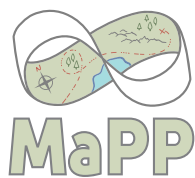
Now that Goolia knows which suits Flo's card could be, she listens to more of their conversation!

*Ru: Muahaha!! Fire burn, and cauldron bubble! I didn't know what card Flo picked, but after listening to what Vi has to say, I have figured it out!*

*Vi: Fair is foul and foul is fair! In that case, I now know too!*

**"Amp'd Squad! I am so close to knowing which card has special powers!! With your help, I know we can figure out which card Flo chose! I'll gladly provide a puzzle piece for your troubles!"**

With the clues Goolia overheard, can you help her figure out which card has the special powers??



## Middle School Challenge '17

# Ghostly Charm

## Solution - Bonus Puzzle 5

Hints:

- \* Try ruling out certain cards. From the first puzzle, we know that the suit is either Cats or Pumpkins. If Ru is able to figure out which card Flo chose only knowing the rank and two choices for the suit, what cards can't be possibilities?
- \* Another approach could be guess and check: pick a card and then go back through Vi and Ru's conversation. See if all their statements are true, assuming that Flo's card is the same one you picked.

Solution:

The answer is the 4 of Pumpkins! Let's see how to get there.

We know from the first part of the puzzle that Flo's card is either Cats or Pumpkins.

The next clue we're given is that Ru (who was told the rank) didn't know what card was chosen, but now that she knows the suit isn't Bats or Candy Corn, she knows what card Flo chose. This tells us that the card cannot have rank 2 since both of the remaining suits (Cats and Pumpkins) have a 2. If the rank Ru was given was a 2, then she would not be able to figure out Flo's card with only the information that the suit is Cats or Pumpkins.

So, from the first two clues, we know that the card is either the 4 of Pumpkins, Ace of Cats, or 5 of Cats.

Lastly, from the third clue, we know that once the card has been narrowed down to three possibilities, Vi is able to figure out the card as well. Since Vi only knows the suit of the card that means the card has to be the 4 of Pumpkins! If Vi had been told that the suit was Cats instead, she would not know whether or not the card was the Ace or 5 of Cats.