Sobolev Spaces and Regularity of Elliptic PDE

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Abstract

Sobolev Spaces is a key point in the well-exposedness theory of Partial Differential Equation (PDE) Problem. In this project, we will give a detailed description of Sobolev Spaces, including the characterisation, the extension from bounded open set Ω to $\mathbf{R}^N$, the density and approximation by C_c^∞ or C^∞ functions, the trace operator which presents the boundary behaviors, and the significant Sobolev Inequality for embedding Sobolev Spaces into L^p or $C^{m,\alpha}$.

We mainly focus on the applications of Sobolev Spaces to Elliptic PDE Problem. We will introduce the variational problem in Sobolev Spaces, which gives a weak solution. And then introduce some results on Dirichlet Problem and Neumann Problem. We will see the importance of regularity problem in the Sobolev theory and present some regularity results in the end.

Key words: Sobolev Spaces, Elliptic PDE, Regularity.

Contents

1 Basic information about Sobolev Spaces

1.1 Definition and Notation

Definition 1. Let $\Omega \subset \mathbf{R}^N$ be an open subset. We introduce **Sobolev Spaces** $W^{1,p}(\Omega)$ for $1 \leq p \leq +\infty$ as following:

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \partial u \text{ exists in weak sense and belongs to } L^p(\Omega)\}$$

where ∂u refers to $\partial_i u$ for $\forall i \in 1, 2 \dots, N$. We say the derivative exists and belongs to $L^p(\Omega)$ is to say there exists $g_i \in L^p$ such that, for $\forall \phi \in C_c^\infty(\Omega)$, we have:

$$\int_\Omega u \cdot \partial_i \phi \, \mathrm{d}x = -\int_\Omega g_i \cdot \phi \, \mathrm{d}x, \quad \forall i \in \{1, 2, \dots, N\} \tag{1}$$

And we call g_i the weak derivative of u and often denoted by $\partial_i u$.

Intuitively, $W^{1,p}(\Omega)$ consists of L^p functions whose derivatives are also L^p functions. More generally, we could introduce $W^{m,p}$ for $m \in \mathbb{N}_+$ by the same approach. Before that, we introduce the notation of multiple index for derivative: Let $\alpha \in \mathbb{N}^N$, we use $\partial^\alpha u$ with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$ to denotes:

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}}$$

Of course we could say the multiple derivative $\partial^\alpha u$ exists in weak sense in L^p if $g_\alpha \in L^p$ verifies:

$$\int_\Omega u \cdot \partial^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g_\alpha \cdot \phi \, \mathrm{d}x \quad \forall \phi \in C_c^\infty(\Omega)$$

Definition 2. We define **Sobolev Spaces** $W^{m,p}(\Omega)$ as

$$W^{m,p} = \{u(x) : \partial^\alpha u \text{ exists in weak sense and in } L^p(\Omega) \text{ for } \forall |\alpha| \leq m\}$$

Remark 1. Sometimes the test functions ϕ in (1) are taken from $C_c^1(\Omega)$, due to the density of $C_c^\infty(\Omega)$ in $C_c^1(\Omega)$, we shall notice these two definitions are equivalent.

Remark 2. In theory of Distribution, we regard $u \in L^p(\Omega)$ as a distribution T_u in $\mathcal{D}'(\Omega)$, thus $\partial^\alpha u$ always exists in sense of distribution. Recall that T_u and $\partial^\alpha T_u$ is defined as: for $\forall \varphi \in C_c^\infty(\Omega)$:

$$T_u(\varphi) = \int_\Omega u \cdot \varphi \, \mathrm{d}x$$
$$(\partial^\alpha T_u)\varphi = (-1)^\alpha T_u(\partial^\alpha \varphi)$$

From this point of view, we say a function $u \in W^{m,p}$ is to say $\exists g_\alpha \in L^p$ such that $T_{g_\alpha} = \partial^\alpha T_u$

Remark 3. For case p=2, we use H^m to denote $W^{m,p}$. We shall see in later sections that H^m is a Hilbert Space with scalar product $\langle u, v \rangle_m$ as:

$$\langle u, v \rangle_m = \sum_{|\alpha| \leq m} \langle \partial^\alpha u, \partial^\alpha v \rangle \tag{2}$$

with $\langle u, v \rangle$ the usual scalar product in L^2

Remark 4. We equip Sobolev Spaces $W^{m,p}$ with norm $\| \cdot \|_{m,p}$ defined by

$$\|u\|_{m,p} = \sum_{|\alpha \leq m|} \|\partial^\alpha u\|_p \tag{3}$$

Example 1. Take $N = 1, \Omega = (-1, 1)$, then function $u(x) = |x|$ belongs to $W^{1,p}$ for $\forall p \in [1, +\infty]$ but does not belong to $W^{2,p}$.

Proof. Firsty, it is obvious that $u \in L^p$. We could verify that

$$u'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \quad \text{in weak sense}$$

As by integration by parts, for $\forall \varphi(x) \in C_c^\infty(\Omega)$, we have:

$$\begin{aligned} \int_{-1}^{1} u(x)\varphi'(x) \, \mathrm{d}x &= \int_{0}^{1} x\varphi'(x) \, \mathrm{d}x + \int_{-1}^{0} -x\varphi'(x) \, \mathrm{d}x \\ &= -\int_{0}^{1} \varphi(x) \, \mathrm{d}x + \int_{-1}^{0} \varphi(x) \, \mathrm{d}x \\ &= -\int_{-1}^{1} u'(x)\varphi(x) \, \mathrm{d}x \end{aligned}$$

Thus u' exists and belongs to L^p, but when we try to calculate u'' in weak sense:

$$\begin{aligned} \int_{-1}^{1} u(x)\varphi''(x) \, \mathrm{d}x &= -\int_{-1}^{1} u'(x)\varphi'(x) \, \mathrm{d}x \\ &= -\int_{0}^{1} \varphi'(x) \, \mathrm{d}x + \int_{-1}^{0} \varphi'(x) \, \mathrm{d}x \\ &= 2\varphi(0) \end{aligned}$$

One could easily verify that there does not exist a function $g \in L^p$ such that

$$\int_{-1}^{1} g(x)\varphi(x) \, \mathrm{d}x = 2\varphi(0) \qquad \forall \varphi \in C_c^\infty(\Omega)$$

□

1.2 Completeness

Now we investigate the properties of $(W^{m,p}, \|\cdot\|_{m,p})$ and wish to characterise Sobolev Spaces. Firstly we have the following observations:

Theorem 1. *Suppose $\Omega \subset \mathbf{R}^N$, $1 \leq p \leq +\infty$. Then Sobolev Spaces $(W^{m,p}, \|\cdot\|_{m,p})$ is complete. Moreover, $(H^m, \langle \cdot, \cdot \rangle_m)$ is a Hilbert Space.*

Proof. Now suppose $(u_n)_{n\in\mathbb{N}}$ is a Cauthy Sequence in $W^{m,p}$, i.e. $\forall \varepsilon > 0$, when n, k is sufficiently large we have

$$\begin{aligned} &\|u_n - u_k\|_{m,p} < \varepsilon \\ \Longrightarrow \ &\|\partial^\alpha u_n - \partial^\alpha u_k\|_p < \varepsilon \quad \text{for } \forall |\alpha| \leq m \end{aligned}$$

Thus $\{\partial^\alpha u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(L^p(\Omega), \|\cdot\|_p)$. Due to the fact that $(L^p(\Omega), \|\cdot\|_p)$ is a Banach Space, we deduce that $\partial^\alpha u_n$ has a limit in $L^p(\Omega)$, denoted as f_α. Let u be the limit of u_n. To show u is still in $W^{1,p}$, we prove now $\partial^\alpha u = f_\alpha \in L^p$. We have for $\varphi \in C_c^\infty(\Omega)$:

$$\begin{aligned} &\int_\Omega u_n \cdot \partial^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \partial^\alpha u_n \cdot \varphi \, \mathrm{d}x \\ \Longrightarrow \ &\lim_{n\to\infty} \int_\Omega u_n \cdot \partial^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \lim_{n\to\infty} \int_\Omega \partial^\alpha u_n \cdot \varphi \, \mathrm{d}x \end{aligned}$$

By Dominated Convergence Theorem, we could interchange the limit and the integral:

$$\implies \int_\Omega \lim_{n\to\infty} u_n \cdot \partial^\alpha \varphi \ \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \lim_{n\to\infty} \partial^\alpha u_n \cdot \varphi \ \mathrm{d}x$$
$$\implies \int_\Omega u \cdot \partial^\alpha \varphi \ \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f_\alpha \cdot \varphi \ \mathrm{d}x$$

Thus now $u \in W^{m,p}(\Omega)$ with $\partial^\alpha u = f_\alpha \in L^p(\Omega)$, that implies $W^{m,p}$ is complete.

For Space $(H^m, \langle\cdot,\cdot\rangle_m)$, we notice that the norm induced by its scalar product $\langle\cdot,\cdot\rangle_m$ is equivalent to the $W^{m,2}$ norm, hence it is complete. It could be easily seen that $\langle\cdot,\cdot\rangle_m$ is symmetric and positive definite since the $\langle\cdot,\cdot\rangle$ in L^2 are symmetric and positive definite. □

Theorem 2. *For $1 < p \le +\infty$, The followings are equivalent:*

1. $u \in W^{1,p}(\Omega)$
2. $\exists\ C > 0$ *such that for* $\forall \varphi \in C_c^\infty(\Omega)$, *we have*
$$\int_\Omega u \cdot \partial\varphi(x) \ \mathrm{d}x \le C\|\varphi\|_{p'}$$
where p' is the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$
3. $\exists\ C > 0$ *such that for* $\forall h \in \mathbb{R}^N$, *let* $\omega = \{x \in \Omega : dist(x, \partial\Omega) > |h|\}$, *we have*
$$\|u(x) - u(x-h)\|_{L^p(\omega)} \le C|h|$$
we always use $\tau_h u(x)$ to denote $u(x+h)$, this is equivalent to $\|\tau_h u - u\|_p \le C|h|$

Proof. It is easy to see that 1. $\iff$ 2. by noticing that L^p and $L^{p'}$ are the dual space to each other.

$3 \implies 1, 2$: $\forall \varphi \in C_c^\infty(\omega)$, we have

$$\int_\omega [u(x) - u(x-h)]\varphi(x) \ \mathrm{d}x \le C|h|\|\varphi\|_{p'}$$
$$\iff \int_{\omega - h} u(x) \cdot \frac{[\varphi(x+h) - \varphi(x)]}{|h|} \le C\|\varphi\|_{p'}$$

We assume $h = |h| \cdot e$ with $e \in \mathbb{S}^{n-1}$ let $|h|$ goes to 0, by dominated convergence theorem we obtain

$$\int_\omega u(x) \cdot |e \cdot \nabla\varphi| \le C\|\varphi\|_{p'}$$

Choose e to be e_i and notice that when $|h|$ is very small, ω will approach Ω such that $\mathrm{supp}\varphi \subset \omega$. Thus

$$\int_\Omega u(x) \cdot \partial_i\varphi \ \mathrm{d}x \le C\|\varphi\|_{p'}$$

For every $\varphi \in C_c^\infty(\Omega)$. That is exactly Statement 2.

$1, 2 \implies 3$: $u \in W^{1,p}$, we have

$$\begin{aligned}\int_\omega |u(x) - u(x-h)|^p \ \mathrm{d}x &= \int_\omega |\int_0^1 h \cdot \nabla u(x - th) \ \mathrm{d}t|^p \ \mathrm{d}x \\ &\le |h|^p \int_0^1 \int_\omega |\nabla u(x-th)|^p \ \mathrm{d}x \ \mathrm{d}t \\ &\le |h|^p \int_0^1 \int_\Omega |\nabla u|^p \ \mathrm{d}x \ \mathrm{d}t \le |h|^p \int_\Omega |\nabla u|^p \ \mathrm{d}x\end{aligned}$$

Hence, $\|\tau_h u - u\|_p \le |h| \cdot \|\nabla u\|_p$. We get Statement 3. □

1.3 Density

Proposition 1. If $\Omega = \mathbf{R}^N$, then $C_c^\infty(\mathbf{R}^N)$ is dense in $W^{1,p}(\mathbf{R}^N)$ for $\forall\ 1 \le p < \infty$

Proof. We choose a sequence of mollifiers $\rho_n \in C_c^\infty(\mathbf{R}^N)$, let $\xi_n \in C_c^\infty(\mathbf{R}^N)$ and $0 \le \xi_n \le 1$ with $\xi_n(x) = 1, \forall |x| \le n$, and $\partial \xi_n$ be bounded by $C > 0$. For every $u \in W^{1,p}$, we consider $u_n = \xi_n \cdot (\rho_n * u)$, Then $u_n \in C_c^\infty \subset W^{1,p}$. It is easy to verify that

$$\partial(\rho_n * u) = \rho_n * \partial u = (\partial \rho_n) * u$$

Now ρ_n are mollifiers and both u and ∂u belongs to L^p space, thus $\rho_n * u \to u$ in L^p and $\rho_n * \partial u \to \partial u$ in L^p. Suppose $\int_{|x|>n_0} |u(x)|^p \le \varepsilon^p$ and $\|\rho_{n_0} * u - u\|_p \le \varepsilon$, then for $\forall n \ge n_0$:

$$\begin{aligned}
\|u_n - u\|_p &\le \|(1-\xi_n)\cdot u\|_p + \|\xi_n \cdot (\rho_n * u - u)\|_p \\
&\le \Big(\int_{|x|>n} |u(x)|^p \ \mathrm{d}x\Big)^{1/p} + \|\rho_{n_0} * u - u\|_p \le 2\varepsilon \\
\|\partial u_n - \partial u\|_p &\le \|\partial \xi_n \cdot (\rho_n * u)\|_p + \|\xi \cdot (\rho_n * \partial u - \partial u)\|_p \\
&\le C\varepsilon + \varepsilon
\end{aligned}$$

Thus $u_n \to u$ in $W^{1,p}$ norm. □

Remark 5. It is totally the same to prove $C_c^\infty(\mathbf{R}^N)$ is dense in $W^{m,p}$ for $\forall\ 1 \le p < \infty$.

But when we consider the case that Ω is a bounded open subset, $C_c^\infty(\Omega)$ is not necessarily dense in $W^{1,p}(\Omega)$. Intuitively, in $\mathbf{R}^N$, $u \in W^{1,p}$ automatically satisfies $\lim_{|x|\to\infty} u(x) = 0$ in some sense. But that is not the case for Ω bounded. For example:

Example 2. $\Omega = (0,1)$ in $\mathbf{R}$, $u(x) = 1$ in Ω, then $\partial u = 0$ a.s. For any $g \in C_c^\infty(\Omega)$ we have the Poincare's Inequality:

$$\|g\|_{L^p(\Omega)} \le C \cdot \|\nabla g\|_{L^p(\Omega)} \tag{4}$$

The proof is easy for $\Omega = (0,1)$ and $g \in C_c^\infty(\Omega)$:

$$\begin{aligned}
\int_\Omega |g(x)|^p \ \mathrm{d}x &= \int_\Omega |g(x) - g(0)|^p \ \mathrm{d}x \\
&= \int_\Omega \Big| \int_0^x g'(t) \ \mathrm{d}t\Big|^p \ \mathrm{d}x \\
&\le \int_\Omega \int_0^1 |g'(t)|^p \ \mathrm{d}x \ \mathrm{d}t \le |\Omega| \cdot \|\partial g\|_p^p
\end{aligned}$$

Thus by Poincare's inequality we never could find $g_n \in C_c^\infty(\Omega)$ such that $\|g_n\|_{L^p} \to \|u\|_{L^p} > 0$ but $\|\partial g_n\|_{L^p} \to \|\partial u\|_{L^p} = 0$

Remark 6. We use $W_0^{m,p}(\Omega)$ and $H_0^m(\Omega)$ to denote the closure of $C_c^\infty(\Omega)$ in the respective spaces.

Remark 7. The Poincare's Inequality holds for every function $u \in W_0^{1,p}(\Omega)$ when Ω is a bounded open set and $1 \le p < +\infty$. We shall see it in the later section.

To deal with this, firstly we have the following theorem, whose idea and proof all come from the $\Omega = \mathbf{R}^N$ case. The proof is the same as the $\mathbf{R}^N$ case.

Theorem 3. *Let $\Omega \subset \mathbf{R}^N$ be an open subset, then there exists $u_n \in C_c^\infty(\Omega)$ such that, take $\forall\ \omega \subset\subset \Omega$ be a compact subset, we have:*

$$u_n \to u \ \text{in} \ L^p(\Omega)$$
$$\partial u_n|_\omega \to \partial u|_\omega \ \text{in} \ L^p(\omega)$$

2 Extension of $W^{1,p}(\Omega)$

2.1 Extension to $\mathbf{R}^N$

To deal with the problems of differences between Ω to be a bounded open set and $\mathbf{R}^N$. We consider the question of extension. That is, for Ω bounded open subset and $u \in W^{1,p}(\Omega)$, does there exist $\tilde{u} \in W^{1,p}(\mathbf{R}^N)$ such that $\tilde{u}|_\Omega = u$ a.e. Fortunately, such an extension exists provided Ω is smooth enough.

Before everything begins we introduce some notation:

$$\mathbf{R}^N_+ := \{x \in \mathbf{R}^N : x = (x_1, \ldots, x_N) \text{ with } x_N > 0\}$$
$$Q := \{x \in \mathbf{R}^N : |x_i| < 1 \quad \forall i = 1, 2, \ldots, N\}$$
$$Q_+ := \mathbf{R}^N_+ \cap Q = \{x = (x'_N, x_N) \in Q : x_N > 0\}$$

Theorem 4. *Let $\Omega \subset \mathbf{R}^N$ be an open subset of class C^1, which means $\partial\Omega$ is a C^1 curve or hyperplane. For $\forall\ 1 \leq p \leq +\infty$, there exists a linear continuous operator $T : W^{1,p}(\Omega) \to W^{1,p}(\mathbf{R}^N)$ such that for $\forall\ u \in W^{1,p}(\Omega)$, Tu is an extension of u.*

The proof consists of sevral part:

1. We prove the extension from a half cube Q_+ to Q
2. We show the extension from Q_+ to $\mathbf{R}^N$
3. By Partition of Unity, we show the extension from Ω to $\mathbf{R}^N$

We shall give a brief proof to the extension, for more details please refer to *Functional Analysis, Sobolev Spaces and PDE* by Haim Brezis[Bré11], pages 272-277.

Proof. Step 1. The extension from Q_+ to Q. Consider

$$u^*(x) = \begin{cases} u(x) & x_N > 0 \\ u(x'_N, -x_N) & x_N < 0 \end{cases} \tag{5}$$

We could verify that $u^* \in W^{1,p}(Q)$ with ∂u^*. The only difficult part is that the test functions in Q is different from Q_+. Especially for neighbourhood of $\{x_N = 0\}$:

$$\int_Q u^* \frac{\partial \phi}{\partial x_i} = \int_{Q+} u \frac{\partial \psi}{\partial x_i}, \quad \psi = \Big(\phi(x'_N, x_N) + \phi(x'_N, -x_N)\Big) \tag{6}$$

To Deal with this, we consider $\eta \in C_c^\infty(0,1)$, then $\eta(x_N) \cdot \psi(x)$ is a test function in Q_+. Take $\eta_n \in C_c^\infty$ and $\eta_n(x) \to 1$ for $x \in (0,1)$. We could prove the existence of $\partial_i u^*$ for $i = 1, 2, ..., N-1$. For $\partial_N u^*$, notice that $\psi(x'_N, 0) = 0$, we could find η_n such that $\int_{Q+} u \cdot \partial(\eta_n)\psi \to 0$. Then we deduce the conclusion.

Step 2. From Q_+ to $\mathbf{R}^N$

The extension is easily achieved by using Step 1 several times: we extend Q to each direction as what we have done to extend Q_+ to Q. Then we have an open set Ω containing Q and extension $Pu \in W^{1,p}(\Omega)$. Choose $\chi \in C_c^\infty(\Omega)$ and $\chi = 1$ in Q. Then $\chi \cdot Pu \in W^{1,p}(\mathbf{R}^N)$ is an extension of u.

Step 3. Partition of Unity and "Flatten"

For a general Ω with $\Gamma = \partial\Omega\ C^1$ and bounded. By Partition of Unity, we could find U_i a family of open sets covering Γ. Then "flatten" if by map $H_i : Q \to U_i$ with $H_i(Q_+) = \Omega \cap U_i$. So we could extend it as $Q_+ \to Q$. See details in [Bré11]. □

Proposition 2. An immediate conclusion from Theorem 4 is the density of $C^\infty(\bar{\Omega})$ in $W^{1,p}(\Omega)$, where $C^\infty(\bar{\Omega})$ means it is C^∞ in a neighbourhood of $\bar{\Omega}$: $\forall u \in W^{1,p}(\Omega), \exists\ (u_n) \in C^\infty(\bar{\Omega})$ such that

$$u_n \to u \text{ in } W^{1,p}(\Omega) \tag{7}$$

Proof. Since $C_c^\infty(\mathbf{R}^N)$ is dense in $W^{1,p}(\mathbf{R}^N)$, we could extend u to $Pu \in W^{1,p}(\mathbf{R}^N)$, then there exists $u_n \in C_c^\infty(\mathbf{R}^N)$ such that:

$$\|u_n - Pu\|_{W^{1,p}(\mathbf{R}^N)} \to 0 \tag{8}$$

We consider restriction to Ω, then $u_n|_\Omega \in C^\infty(\bar{\Omega})$, $Pu|_\Omega = u$, we have

$$\|u_n|_\Omega - u\|_{W^{1,p}(\Omega)} \le \|u_n - P_u\|_{W^{1,p}(\mathbf{R}^N)} \to 0 \tag{9}$$

That completes the proof. □

2.2 Trace Operator

Now we introduce the concept of "Trace". For most PDE problems, we consider boundary condition. So our goal is to use Sobolev Spaces to deduce some well-posedness of solution to PDE problem. A problem is whether we could give a well definition of a function $u \in W^{1,p}(\Omega)$ on the boundary $\partial\Omega$. As we all know that $u \in L^p(\Omega)$ does not imply anything on the boundary. Luckily, we have trace operator who well defines the value of u on the boundary.

Theorem 5. *Assume $\Omega \subset \mathbf{R}^N$ is bounded open of class C^1, then there exists a linear continuous map:*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that:

1. *$Tu = u|_{\partial\Omega}$, if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$*
2. *$\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$*
   for each $u \in W^{1,p}(\Omega)$, with constant C depending only on p and Ω.

We call Tu the trace of u. We will not present the proof of the theorem here, ones who are interested could refer to *Partial Differential Equations* by Evans Lawrence [Eva22], page 258. An important result of trace is related to $W_0^{1,p}$:

Theorem 6 (Trace Zero)**.** *Assume $\Omega \subset \mathbf{R}^N$ is a bounded open subset of class C^1. For $u \in W^{1,p}(\Omega)$, we have*

$$u \in W_0^{1,p} \iff Tu = 0 \tag{10}$$

Proof. The sufficiency is trivial, Recall that $W_0^{1,p}$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,p}$. If $(u_n)_n$ is a sequence of functions in $C_c^\infty(\Omega)$, we have $Tu_n = 0$.

$$\|Tu_n - Tu\|_{L^p} \le C\|u_n - u\|_{W^{1,p}(\Omega)} \to 0$$

That implies $Tu = 0$ in $L^p(\partial\Omega)$. The other side of the proof is relatively hard, one could refer to Evans' book [Eva22] pages 259-260. □

Remark 8. From Theorem 6 we could see immediately that $W_0^{1,p(\Omega)} \neq W^{1,p}(\Omega)$ for any bounded open $\Omega \subset \mathbf{R}^N$, as we take $u \in C^1(\bar{\Omega})$ with $u|_{\partial\Omega} \neq 0$, then $u \in W^{1,p}(\Omega)$ but $u \notin W_0^{1,p}(\Omega)$.

3 Sobolev Inequality

This section is to present some results of Sobolev Inequality. The inequality will show us whether we could embed Sobolev Spaces $W^{m,p}$ to a better space such as L^q Spaces for some q or Holder Spaces $C^{0,\alpha}$. That kind of embedding indicates some regularity of Sobolev Spaces and will have important applications to PDE theory of well-posedness.

3.1 Sobolev Embedding

Theorem 7 (Sobolev, Gagliardo, Nirenberg, Morrey)**.** *Let $\Omega \subset \mathbf{R}^N$ be of class C^1, $u \in W^{1,p}(\Omega)$. We have the follwing Sobolev embedding theorems and Sobolev inequalities:*

1. *if $p < N$, then $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-p}$. And we have the inquality in case $\Omega = \mathbf{R}^N$:*

$$\|u\|_{p^*} \leq C \cdot \|\nabla u\|_p \quad for\ \forall u \in W^{1,p}(\Omega) \tag{11}$$

2. *if $p = N$, then $W^{1,p}(\Omega) \subset L^q$ for $\forall q \in [p, +\infty)$.*

3. *if $p > N$, then $W^{1,p}(\Omega) \subset L^\infty \cap C^{0,\alpha}$ with $\alpha = 1 - N/p$. And we have the inequality:*

$$|u(x) - u(y)| \leq C \cdot |x - y|^\alpha \cdot \|\nabla u\|_p \quad for\ \forall u \in W^{1,p}(\Omega) \tag{12}$$

Proof. **1. For $p < N$:**

We only need to focus on the proof of $\Omega = \mathbf{R}^N$ case, for general case we could use the extension of $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbf{R}^N)$. Now by the density of C_c^∞, we consider $u \in C_c^\infty$ and $p = 1$ case. We have:

$$|u(x)| = |\int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}\ \mathrm{d}t| \leq \int_{-\infty}^{+\infty} |\frac{\partial u}{\partial x_i}|\ \mathrm{d}t \tag{13}$$

We denote $x_i' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbf{R}^{N-1}$, let

$$f_i(x_i') = \int_{-\infty}^{+\infty} |\frac{\partial u}{\partial x_i}|\ \mathrm{d}t, \quad \text{then} \quad \|f_i\|_{L^1(\mathbf{R}^{N-1})} = \|\partial_i u\|_{L^1(\mathbf{R}^N)}$$

we compute the L^{p^*} norm of u:

$$\|u\|_{\frac{N}{N-1}} \leq (\int_{\mathbf{R}^N} |\prod_{i=1}^N f_i(x_i')|^{\frac{1}{N-1}}\ \mathrm{d}x)^{\frac{N-1}{N}} \tag{14}$$

To proceed, We introduce the lemma: if $g_i \in L^1(\mathbf{R}^{N-1})$, then we have

$$\|\prod_{i=1}^N g_i(x_i')\|_{L^1(\mathbf{R}^N)} \leq \prod_{i=1}^N \|g_i\|_{L^{N-1}(\mathbf{R}^{N-1})} = \prod_{i=1}^N \|g_i^{N-1}\|_{L^1(\mathbf{R}^{N-1})}^{\frac{1}{N-1}} \tag{15}$$

The lemma is proved easily by induction on N with Holder's inequality. You may verify that by yourself. We apply this lemma to equation (14):

$$\|u\|_{\frac{N}{N-1}} \leq (\prod_{i=1}^N \|f_i\|_{L^1(\mathbf{R}^{N-1})})^{\frac{1}{N}} \leq \max_{i=1,2,\ldots,n} \|\partial_i u\|_{L^1(\mathbf{R}^N)} = \|\nabla u\|_{L^1}$$

That completes the proof of Sobolev inequality in case of $p = 1 < N$. For general $p < N$, we just need to apply the Soblolev Inequality of p=1 case:

$$\|u^m\|_{\frac{N}{N-1}} \leq \|\nabla u^m\|_{L^1} = \|m \cdot u^{m-1} \cdot \nabla u\|_{L^1} \leq C \cdot \|\nabla u\|_p \|u^{m-1}\|_{p'} \tag{16}$$

For $m \in [1, +\infty)$ such that the norm is finite. Choose m such that $mN/(N-1) = (m-1)p'$, we have $m = (N-1)p/(N-p)$. By substituting m in inequality (16), we deduce our desired inequality.

Furthermore, Sobolev Inequality shows that the identity map: $W^{1,p} \to L^{p^*}$ is a continuous injection.

□

2. For $p = N$**:**

We omit the detailed proof here, ones who are interested in details may refer to Haim Brezis' book [Bré11], page 281. We mention here that $W^{1,p}(\Omega) \subset L^q(\Omega)$ for all $q \in [p, +\infty)$, but in general it is false that $W^{1,p}(\Omega) \subset L^\infty(\Omega)$. As the example following shows:

Example 3. For $\Omega = B(0, \frac{1}{e}) \subset \mathbf{R}^2$, the ball centered with radius $1/e$, we consider $u(x) \in W^{1,2}(\Omega)$:

$$u(x) = \log|\log(r)|, \quad \text{where } r = |x| = \sqrt{\sum x_i^2} \tag{17}$$

Obviously $u \notin L^\infty(\Omega)$. We have $\nabla u(x) = \frac{x}{r^2|\log(r)|} \in L^2(\Omega)$, since

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x = 2\pi \cdot \int_0^{1/e} \frac{1}{r^2|\log(r)|^2} r dr = 2\pi \int_0^{1/e} \frac{1}{r\log^2(r)} \, \mathrm{d}r = 2\pi < \infty \tag{18}$$

And u is in $L^2(\Omega)$, we could verify that

$$\int_\Omega |u|^2 \, \mathrm{d}x = 2\pi \int_0^{1/e} r \cdot (\log|\log(r)|)^2 \, \mathrm{d}r < \infty \tag{19}$$

Thus $u \in W^{1,2}(\Omega)$, by Trace Zero theorem we could even say that $u \in W_0^{1,2}(\Omega)$. But $u \notin L^\infty$.

3. For $p > N$**:**

Proof. We firstly prove that $W^{1,p} \subset C^{0,\alpha}$ with continuous injection. Let $u \in W^{1,p}$, Without loss of generality, we prove the holder continuity of u:

$$u(x) - u(0) = \int_0^1 x \cdot \nabla u(tx) \, \mathrm{d}t \leq |x| \int_0^1 |\nabla u(tx)| \, \mathrm{d}t \tag{20}$$

We consider x in $B(0, r)$, a neighbourhood of 0. Let ω_N denote the volume of unit ball in $\mathbf{R}^N$, $\bar{u}$ denote the mean value of u in $B(0, r)$:

$$\bar{u} = \frac{1}{r^N \omega_N} \int_{B(0,r)} u(x) \, \mathrm{d}x$$

Then

$$\begin{aligned}
|\bar{u} - u(0)| &\leq \frac{1}{r^N \omega_N} \int_{B(0,r)} |u(x) - u(0)| \, \mathrm{d}x \leq \frac{1}{r^{N-1}\omega_N} \int_{B(0,r)} \int_0^1 |\nabla u(tx)| \, \mathrm{d}t \, \mathrm{d}x \\
&\leq \frac{1}{r^{N-1}\omega_N} \int_0^1 \frac{1}{t^N} \int_{B(0,tr)} |\nabla u(y)| \, \mathrm{d}y \, \mathrm{d}t \\
&\leq \frac{1}{r^{N-1}\omega_N} \int_0^1 \frac{1}{t^N} \Big(\int_\Omega |\nabla u(y)|^p \, \mathrm{d}y \Big)^{1/p} \Big(\int_\Omega 1_{B(0,tr)}(x) \, \mathrm{d}x \Big)^{1/p'} \, \mathrm{d}t \\
&= r^{1-N/p} \cdot \omega_N^{-1/p} \cdot \|\nabla u\|_p \cdot \frac{1}{1 - N/p} = C \cdot r^{1-N/p} \|\nabla u\|_p
\end{aligned}$$

The same conclusion holds if we do the translation, then we could deduce that

$$|\bar{u} - u(x)| \leq C \cdot r^{1-N/p} \cdot \|\nabla u\|_p \tag{21}$$

Hence, for any $x, y \in \mathbf{R}^N$,we pick $2r = |x-y|$, we could find a ball with radius r covering x, y. Apply the inequality (21):

$$|u(x) - u(y)| \leq C \cdot |r|^{1-N/p} \|\nabla u\|_p \tag{22}$$

For continuous injection to L^∞, we take $r = 1$, the inequality (21) implies:

$$u(x) \leq \frac{1}{\omega_N} \left| \int_{B(x,1)} u(y) \, \mathrm{d}y \right| + C\|\nabla u\|_p \overset{Holder}{\leq} C\|u\|_{W^{1,p}(\mathbf{R}^N)} \tag{23}$$

for $\forall x \in \mathbf{R}^N$. That is $\|u\|_{L^\infty} \leq C\|u\|_{W^{1,p}}$ □

Proposition 3. More generally, for Sobolev Spaces $W^{m,p}$, one could verify the following conclusions by applying the Sobolev Inequality in case of $W^{1,p}$:

1. if $p < N/m$, we have $W^{m,p}(\mathbf{R}^N) \subset L^q(\mathbf{R}^N)$ with $q = Np/(N - mp)$.
2. if $p = N/m$, we have $W^{m,p}(\mathbf{R}^N) \subset L^q(\mathbf{R}^N)$ for $\forall q \in [p, +\infty)$
3. if $p > N/m$, we have $W^{m,p}(\mathbf{R}^N) \subset L^\infty \cap C^{k,\alpha}$, with $k \in \mathbb{N}$, $0 < \alpha < 1$, $k + \alpha = m - N/p$.

Remark 9. From the proposition we could see, if $D^\beta u$ always exists and in L^p for all $\beta \in \mathbb{N}^N$, then $u \in \cap_{m=1}^\infty W^{m,p} \subset C^\infty(\mathbf{R}^N)$, which gives out a claim on the regularity of u.

Theorem 8 (Rellich-Kondrachov)**.** *If Ω is a bounded open set in $\mathbf{R}^N$ with a C^1 boundary. Then the following injections of $W^{1,p}(\Omega)$ are **compact**:*

1. *$p < N$, $W^{1,p} \subset L^q$ with $q \in [p, p^*)$*
2. *$p = N$, $W^{1,p} \subset L^q$ with $q \in [p, +\infty)$*
3. *$p > N$, $W^{1,p} \subset C(\bar{\Omega})$*

Remark 10. The compact injection shows, if we have $u_n \to u$ in $W^{1,p}(\Omega)$, then we could find a subsequence u_{n_k}, such that $u_{n_k} \to u$ in L^q or $C(\bar{\Omega})$.

Theorem 9 (Poincare's Inequality)**.** *If Ω is a bounded open set, then for $u \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$. We have the following inequality:*

$$\|u\|_p \leq C \cdot \|\nabla u\|_p \tag{24}$$

An immediate result is that the norm $\|\nabla u\|_p$ is equivalent to $\|u\|_{W^{1,p}}$ in $W_0^{1,p}(\Omega)$

And a new characterisation of $W_0^{1,p}(\Omega)$ is given by, for all $\varphi \in C^\infty(\bar{\Omega})$:

$$\int_\Omega u \cdot \partial\varphi \, \mathrm{d}x \leq C \cdot \|\varphi\|_{p'}$$

Remark 11. Notice that the difference between this characterisation and characterisation of $W^{1,p}$ in section1.2 is that we choose φ from different test function spaces.

4 Application to 2^{nd} Order Elliptic PDE

Now we turn to 2^{nd} Order Elliptic PDE. We will see that Sobolev Spaces are great spaces for us to find a solution. In this Chapter, I will introduce how elliptic PDEs are solved in Sobolev Spaces then focus on the Regularity Problem.

4.1 General Method

Now we focus on some elliptic PDE. Our goal is to determine the well-exposedness of PDEs: existence, uniqueness and regularity of solution. For example, we consider the following PDEs with different boundary conditions:

$$\text{Dirichlet Problem} \begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = 0 \quad \text{in } \partial\Omega \end{cases} \tag{25}$$

$$\text{Neumann Problem} \begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega \end{cases} \tag{26}$$

A classical solution to PDE (25) is a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and a classical solution to PDE (26) is a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Take Dirichlet Problem as an example, the space $C^2(\Omega) \cap C(\bar{\Omega})$ is not a space with good property. To use some functional analysis tools to deduce the existence and uniqueness, we need a better space, which is Sobolev Spaces. And the corresponding solution is called a weak solution, which is a solution to the variational problem:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that:}$$
$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega u \cdot v = \int f \cdot v \qquad \forall v \in H_0^1 \tag{27}$$

The Method has 4 steps in general:

Step 1. Show that every classical solution must be a solution to the variational formula.

Step 2. Show that the existence and uniqueness of weak solution.

Step 3. Show that the weak solution has regular property.

Step 4. Show that under regular condition that the weak solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a classical solution.

The Step 1 could be easily achieved by divergence theorem: if $v \in C^1(\bar{\Omega})$, then

$$\int_\Omega -\Delta u \cdot v \, \mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v \, \mathrm{d}S \tag{28}$$

Now assume u is a classical solution of Dirichlet Problem (25). If $v \in H_0^1(\Omega)$, by Trace Operater we have that $v = 0$ in $\partial\Omega$ in sense of trace. Thus the last term in formula (28) equals to 0. We then deduce the formula (27). $u = 0$ in $\partial\Omega$ shows $u \in H_0^1(\Omega)$, which completes the proof that u is a solution to the variational problem.

One should notice that we shall carefully verify the variational formula, as for Neumann Problem (26), we could verify the corresponding variational problem is:

$$\text{Find } u \in H^1(\Omega) \text{ such that:}$$
$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega u \cdot v = \int f \cdot v \qquad \forall v \in H^1 \tag{29}$$

Remark 12. We could see that in variational problem it is natural to consider in H^1 since we need the integral $\langle \nabla u, \nabla v \rangle_{L^2}$ makes sense. And H^1 is a Hilbert Space.

Remark 13. We could see that the only difference of variational problem of Dirichlet condition (27) and Neumann condition (29) is the space they are working on. For Dirichlet condition it is in H_0^1, for Neumann condition is in H^1. Although $H_0^1 \subset H^1$, the difference may lead to totally different solution.

The Step 2 follows from Lax-Milgram Theorem. As we have $L : v \to \int_\Omega fv \, \mathrm{d}x$ is a linear continuous functional on H_0^1. We define $a(u, v)$, a bilinear form on H_0^1 as:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega u \cdot v \tag{30}$$

if we could verify that

- a is continuous, that is, $|a(u, v)| \leq C\|u\|_{H_0^1}\|v\|_{H_0^1}$
- a is coercive, that is, $a(u, u) \geq C\|u\|_{H_0^1}$ for some constant $C > 0$

Then we could apply Lax-Milgram Theorem to deduce that there exists a unique solution $u \in H_0^1$ such that $a(u, v) = L(v)$ for $\forall v \in H_0^1$. Hence u is our desired weak solution.

Remark 14. To apply Lax-Milgram theorem, we must work in a Hilbert Space, thus we need to transfer the original PDE to a variational problem. More generally, we could find variational problem with bilinear form $a(u, v)$ coercive and continuous for elliptic PDE.

Consider a more general elliptic PDE:

$$\begin{cases} \sum -D_j(a_{ij}(x)D_i u(x)) + \sum k \cdot u = f \\ \text{Dirichlet Condition or Neumann Condition} \end{cases} \tag{31}$$

With $\sum a_{ij}\xi_i\xi_j \geq c \cdot \|\xi\|^2$, for some $c > 0$ and $\forall \xi \in \mathbf{R}^N$. $k(x) \geq \lambda > 0$. The variational problem is

$$\text{Find } u \in H_0^1(\Omega)(\text{or } H^1) \text{ such that:}$$
$$\int_\Omega \sum a_{ij}(x)\partial_i u \partial_j v + \int_\Omega k \cdot uv = \int f \cdot v \qquad \forall v \in H_0^1(\text{or } H^1) \tag{32}$$

If a_{ij} is positive definite and bounded, $k \geq \lambda > 0$, then one could verify that we are able to apply the Lax-Milgram Theorem to this variational problem and deduce a unique solution.

The Step 3 is the most challenging part and we shall see some results in next section.

The Step 4 follows the same way of Step 1, provided we have u is a weak solution with C^2 continuity. Take $v \in C_c^\infty(\Omega) \subset H_0^1(\Omega)$, we could conclude that $-\Delta u + u = f$ for almost every $x \in \Omega$. And trace zero theorem shows $u \in H_0^1(\Omega) \implies u = 0$ in $\partial\Omega$. Hence, we recover a classical solution from a weak solutions.

Remark 15. We see that the Dirichlet boundary condition comes directly from the fact $u \in H_0^1$. As for Neumann boundary condition, we have a weak solution $u \in H^1$, and still have $-\Delta u + u = f$ by taking $v \in C_c^\infty$. Now apply formula (28) to the variational formula (29) for $v \in C^1(\bar{\Omega}) \subset H^1$, we get:

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}x = 0 \qquad \text{for } \forall v \in C^1(\bar{\Omega}) \tag{33}$$

That implies $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

4.2 Regularity

In this section we present some important regularity result. Firstly we still focus on the Dirichlet Problem (25) and Neumann Problem (26).

Theorem 10. *Assume u satisfies the variational problem:*

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega u \cdot v \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x \qquad \text{for } \forall v \in H_0^1(\Omega) \tag{34}$$

- *If Ω is of class C^1 and $f \in L^2$, then $u \in H^2(\Omega)$. And $\|u\|_{H^2} \le C \cdot \|f\|_{L^2}$*
- *If Ω is of class C^{m+1} and $f \in H^m$, then $u \in H^{m+2}(\Omega)$. And $\|u\|_{H^{m+2}} \le C \cdot \|f\|_{H^m}$*

Moreover, we have the same conclusion for Neumann problem.

Proof. Let $\Omega \subset \mathbf{R}^N$, for $N = 1$ case, formula (34) is exactly the definition of $\partial^2 u = f - u \in L^2$. But for $N > 1$ case, the conclusion is not that obvious. Firstly we prove the case $f \in L^2$, $\Omega = \mathbf{R}^N$. It suffices to prove $Du \in H^1$. By characterisation of H^1 (Theorem 2) , we just need to show

$$\|\tau_h \nabla u - \nabla u\|_p \le C \cdot |h| \qquad for \ \forall h \in \mathbf{R}^N$$

Consider $D_h u = [u(x+h) - u(x)]/|h| \in H_0^1$, Notice that we have the following fact:

$$\int_\Omega u \cdot D_h v = \int_\Omega u(x) \cdot \frac{v(x+h) - v(x)}{|h|} \, \mathrm{d}x = \int_\Omega \frac{u(x-h) - u(x)}{|h|} \cdot v(x) \, \mathrm{d}x = \int_\Omega D_{-h} u \cdot v$$

Thus, take $v = D_{-h}(D_h u)$ in variation formula (34), we get

$$\|D_h u\|_{H^1}^2 = \int_\Omega |\nabla D_h u|^2 + \int_\Omega |D_h u|^2 = \int_\Omega f \cdot D_{-h} D_h u \le C \cdot \|f\|_{L^2} \cdot \|D_h u\|_{H^1} \tag{35}$$

$$\iff \|D_h \nabla u\|_{L^2} \le \|D_h u\|_{H^1} \le C \cdot \|f\|_{L^2}$$

That shows $\nabla u \in H^1(\mathbf{R}^N)$ with $\|\nabla u\|_{H^1} \le C \cdot \|f\|_{L^2}$. Notice here we could do the translation because $\Omega = \mathbf{R}^N$ and thus $D_h u \in H^1(\mathbf{R}^N)$. If Ω is an open bounded set, that translation may cause $D_h u \notin H^1(\Omega)$.

Secondly we consider the case $\mathbf{R}_+^N$. The half space of $\mathbf{R}^N$. Let $\Gamma = \partial \mathbf{R}^N$. We could conclude in the same way for $h \parallel \Gamma$:

$$\|D_h u\|_{H^1} \le C \cdot \|f\|_{L^2}.$$

Thus $\partial_i \partial_j u \in L^2$ for $i = 1, 2, \ldots, N-1$; $j = 1, 2, \ldots, N$. The only term remaining unknown is $\partial_N^2 u$. But take $v \in C_c^\infty$, we could verify that, in weak sense,

$$\partial_N^2 u = -\sum_{i=1}^{N-1} \partial_i^2 u + u - f \in L^2 \tag{36}$$

At last we consider Ω bounded set with C^1 boundary. We use the technique of partition of unity and "flatten" of boundary. One would verify the conclusion by extension of Sobolev Spaces and the same way of $\mathbf{R}^N$and $\mathbf{R}_+^N$.

For $f \in H^m$ case, we prove by induction on m. By conclusion of $f \in L^2$ case, we only need to verify the following formula, then deduce that $Du \in H^2(\Omega)$

$$\int_\Omega \nabla(Du) \cdot \nabla \phi + \int_\Omega Du \cdot \phi = \int_\Omega Df \cdot \phi \qquad for \ \forall \phi \in H_0^1(\Omega) \tag{37}$$

We only need to take $v = D\phi \in C_c^\infty$ in formula (34). And utilise the density of C_c^∞ in $H_0^1(\Omega)$ $\square$

Remark 16. For a general elliptic PDE:

$$\int_\Omega \sum a_{ij} \cdot \partial_i u \partial_j v + \int_\Omega \sum b_i \cdot \partial_i u \cdot v + \int_\Omega c \cdot uv = \int_\Omega f \cdot v \tag{38}$$

Under condition that $a_{ij} \in C^{m+1}, b_i \in C^m, f \in H^m$ and $\partial\Omega$ is of class C^m, we have the same conclusion that the solution $u \in H^m$

Remark 17. If f is C^∞, the regularity agrument shows the solution $u \in C^\infty$. Since by Sobolev embedding theorem, we have $u \in \cap_{m=1}^{\infty} H^m \subset C^\infty(\Omega)$

Apart from the above theorem, there are more results about regularity in elliptic PDE thoery recently who may require less conditions on $a_{ij}(x)$. We introduce some of them here. For details of proof one may refer to Chapter 4 of *Elliptic Partial Differential Equations, Second Condition* by Qing Han and FangHua Lin.[HL11]

Theorem 11 (local boundedness)**.** *Suppose $a_{ij} \in L^\infty(B_1)$ and $c, f \in L^q(B_1)$ for $q > N/2$, where $B_1 \subset \mathbf{R}^N$ is the unit ball. And*

$$\begin{cases} \sum a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \textit{for some } \lambda > 0, \forall \xi \in \mathbf{R}^N \\ \|a_{ij}\|_{L^\infty} + \|c\|_{L^q} \leq \Lambda \quad \textit{for some } \Lambda > 0 \end{cases} \tag{39}$$

Then for any solution $u \in H^1(B_1)$ such that:

$$\int_{B_1} a_{ij}\partial_i u \partial_j v + \int_{B_1} c \cdot uv = \int_{B_1} f \cdot v \quad \textit{for } \forall v \in H_0^1(B_1)$$

We have $u \in L^\infty_{loc}(B_1)$.

Theorem 12 (Holder's Continuity, De Giorgi)**.** *We denote $Lu = \sum D_i(a_{ij}(x)\partial_j u(x))$, where a_{ij} satisfies:*

$$\lambda \cdot |\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq \Lambda \cdot |\xi|^2 \tag{40}$$

For some $\lambda, \Lambda > 0$, then for weak solution of $Lu = 0$ in B_1, we have $u \in L^\infty_{loc} \cap C^{0,\alpha}_{loc}$ for some $0 < \alpha < 1$. Moreover, we have the inequality:

$$\|u\|_{L^\infty(B_{1/2})} + \|u\|_{C^{0,\alpha}(B_{1/2})} \leq C \cdot \|u\|_{L^2(B_1)} \tag{41}$$

Theorem 13 (Schauder's estimate)**.** *If Ω is bounded and of class $C^{m+2,\alpha}$, and f, a_{ij} be $C^{m,\alpha}$, then we have the weak solution u to $Lu = f$ belongs to $C^{m+2,\alpha}$ and*

$$\|u\|_{C^{m+2,\alpha}} \leq C \cdot \|f\|_{C^{m,\alpha}} \tag{42}$$

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