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Convergence of the Binomial-Tree Option Price to Black-Sholes Option Price

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Abstract

The goal of this paper was to show that, with increasing n-steps in the binomial-tree model, it would lead to a convergence of the exact value of an option price as calculated in the Black-Scholes Formula. In this study we have taken assumptions of certain variables and used them among both the binomial-tree method and the Black-Scholes formula. We have shown that when we take $n = 1000$ in the binomial-tree we get an option price very similar and close to that of the calculated option price of Black-Scholes.

1.Introduction

Options, a type of derivative product with underlying securities, is a large and important part of the world of finance. R. Ward presents the workings of options in his book; Ordinary options can be bought or sold, and the options are divided into calls and puts. Buying a put or a call, or selling a put or a call are all ways of carrying out a strategy, whether the buyer (or seller) is hedging, speculating or arbitraging. Sometimes, options are combined with buying or selling the underlying stock or index itself. Options carry value and that needs to be quantified.[1] The black scholes model is to be explored in this paper.

R. Ward further details how buying a call option provides the owner the right to buy the underlying asset at a certain strike price. Buying a put option provides the owner the right to sell the underlying asset at a certain strike price. Taking a selling position on these options, on the other hand, obligates the owner of the contract to sell the underlying asset at the strike price if it is a call, or to buy the underlying asset if it is a put - but there is the possibility that the owner of the buying sides of the contracts will not exercise the given right of the option in

[1] *Options and Options Trading: A Simplified Course That Takes You from Coin Tosses to Black-Scholes,* R. Ward, 2004 ch:1

question. In this study we will exclusively research European-style options and not American. What differs them from one another is that European-options can only be exercised at the expiration of the contract, while American ones can be exercised at any time during the length of the contracts which makes them harder to evaluate. The Black-Scholes model does not include the possibility of early exercising of the contracts and that outlines this paper to exclude American options. Furthermore, the model does not include calculations of any occurring dividend of the underlying.[2]

1.1 Black-Scholes European Option Pricing

During the 1970's Fischer Black, Myron Scholes, and Robert Merton had achieved a major breakthrough, which was developing the Black-Scholes-Merton formula (Black-Scholes formula). Black-Scholes formula (B.S) is a mathematical model for pricing an option contract that is described to be one of the best ways to determine the price of options.[3] The B.S model only requires five input variables to determine the option price, those variables being:

- S_0= Initial stock price
- K= Strike price
- r= Risk-free interest rate
- T= Time until contract maturity
- σ= Standard deviation of log returns (Volatility)

S_0, K, r and T are all constants given from the option contract, however the volatility σ can either be calculated using historical information (historical volatility) to find what the

[2] *Options and Options Trading: A Simplified Course That Takes You from Coin Tosses to Black-Scholes,* R. Ward, 2004 ch:1
[3] Investopedia, https://www.investopedia.com/terms/b/blackscholes.asp, accessed May 1st 2021

volatility was, or it can be calculated by future prediction (implied volatility) to predict what the volatility of a certain stock will be. The B.S formula to calculate the option prices are:

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

where d_1 and d_2 are:

$$d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

The $N(x)$ function is a cumulative probability distribution function for standard normal distribution. It is a probability that a standard normal distributed random variable $Z \sim \phi(0,1)$ is less than a number x, i.e $P(Z < x)$.

The B.S model is not perfect and therefore some assumptions need to be met before the model can be used, those assumptions being:[4]

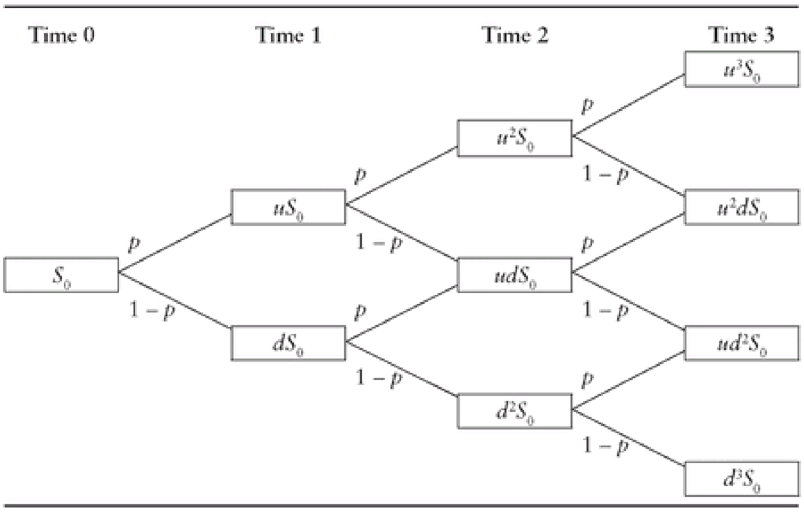
- Lognormal Distribution - Stock price follows geometric brownian motion
- No Dividends
- Exercise On Expiration Date - The option can only be exercised at maturity
- Risk-Free Interest Rate
- Normal Distribution - Stock returns are normally distributed meaning the volatility of the market is constant throughout
- No Arbitrage

[4] CFI, https://corporatefinanceinstitute.com/resources/knowledge/trading-investing/black-scholes-merton-model/, accessed May 1st 2021

1.2 Binomial-Tree European Option Pricing

The Binomial-Tree approach to pricing options is the simplest method is modeling different paths that the stock price can take. Either the stock price can increase or decrease through this method the option prices can also be calculated. As can be seen in figure 1 the binomial tree begins with the original starting stock price S_0 and for every time increment Δt the stock price is changed by either uS_0 or dS_0, where u and d are the up and down factors respectively. Every step up multiplies S_0 by u and inversely every step down multiples S_0 by d. The probability $p$refers to the probability that the stock price will increase or go 'up', and $1-p$ refers to the stock price decreasing or going 'down'.

Figure 1: Binomial-Tree



The up-down factors and the probability are mathematically defined as:

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{e^{r\sqrt{\Delta t}} - d}{u - d}$$

Where all the variables are:

- S_0 - Initial stock price
- p - Risk-Neutral Probability
- r - Risk-free interest rate
- Δt - Time Increment
- u - Assumed or observed factor of size on upward movement of underlying
- d - Assumed or observed factor of size on downward movement of underlying

At each time increment Δt there exists an option price f, these are labeled as $f, f_u, f_d, f_{uu}, f_{dd}$ etc. depending on the tree step. The binomial-tree model can have n-step many paths where the larger n becomes the more accurate the original stock option price would be. To calculate the original option price f at $t = 0$ we would need to calculate the value of f at every point in time in the binomial tree starting from $t = T$, where T is at maturity. The general formula for calculating the option price f for a one step binomial-tree is:

$$f = e^{-r\Delta t}(pf_u + (1 - p)f_d)$$

However, following this formula for every path of the stock price and backtracking towards $t = 0$ will give you the same results as using the general formula for a n-step binomial-tree.

1.3 Research Question

The Binomial-Tree and Black-Scholes methods for calculating the option price are two of the most popular ways to do so. The aim of this paper is to show how both methods can be used to get the same option price. By letting the number of steps in the binomial tree get very large, a convergence of an option price will be found, and that price will equal the black-scholes option price.

2.European Option Pricing

All calculations conducted in this paper will be done using a python code program and will assume the following values of the variables:

- $S_0 = 100$
- $K = 100$
- $r = 0.1$
- $T = 1$year
- $\Delta t = \frac{T}{n}$, where n is the number of steps in the binomial-tree, $n \in Z$
- $\sigma = 0.3, x$(variable)

All the variables will be constant, except for the volatility where it will be both constant and variable to show how it affects the final option price. Due to these options being European options, this means that the option can only be exercised at maturity, which is the assumption of B.S and the Binomial-Tree.

2.1 Black-Scholes Put and Call Option Pricing

Using the assumed variables and calculating the option price using the B.S call and put formulas, we get:

$$c = 16.7341$$

$$p = 7.2178$$

Having the volatility σvariable and changing, gives the the following table of values:

σ:	0.4	0.5	0.6	0.7	0.8
Call Price (c):	20.3184	23.9267	27.5200	31.0767	34.5821
Put Price (p):	10.8022	14.4104	18.0037	21.5604	25.0658

As can be seen in the table, the volatility plays a big role in the calculation of the option price, the higher the volatility becomes the bigger the payoff as the risk also increases.

2.2 Binomial-Tree Put and Call Option Pricing

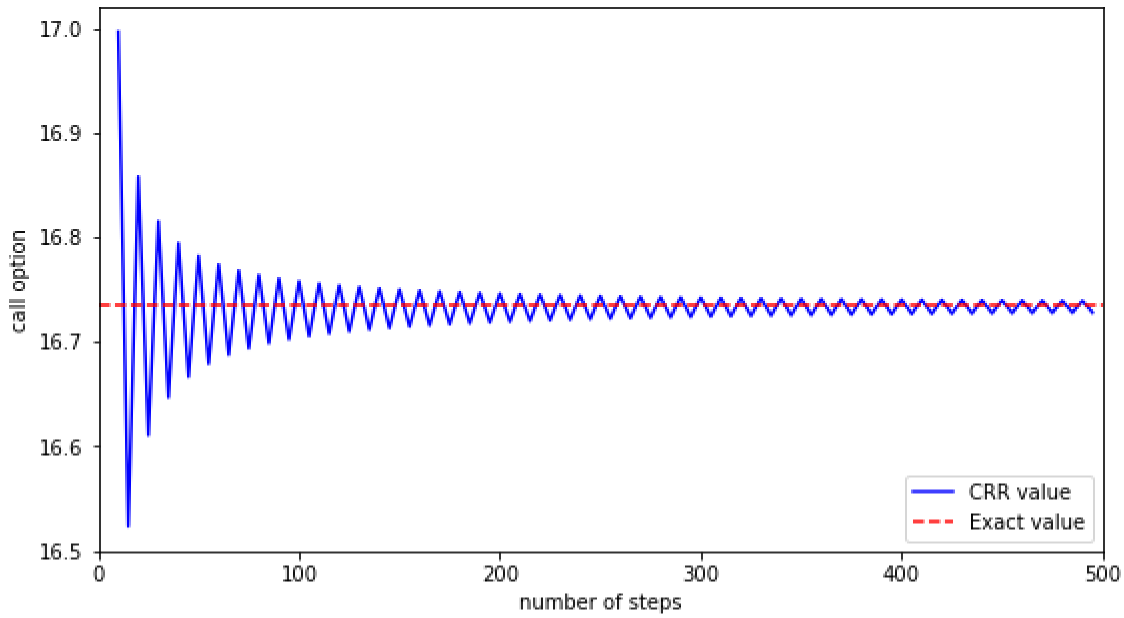
Using the assumed variables and calculating the option price using the Binomial-Tree method, we get:

n (number of steps):	2	50	100	500	1000
Call Price (c):	18.9382	16.7820	16.7578	16.7388	16.7364
Put Price (p):	9.4219	7.2658	7.2415	7.2225	7.2202

3.Discussion

As can be seen from calculating both the B.S and Binomial-Tree option pricing, that as the n-steps of the binomial-tree increase, the option price gets closer and closer to the B.S option price. In other words the option price converges to the B.S option price. This property of the binomial-tree an be modelled in a graph to show its convergence:

Figure 2: Binomial-Tree Option Price Convergence to Black-Scholes Option Price



As can be seen in the figure 2, as the number of steps increases, the binomial-tree option price (CRR price) converges towards the exact value (B.S). The reason the binomial-tree option price converges as such is because the larger the n-step value, the most accurate the option price would be. As the time increment Δtgets smaller and smaller, which if taken to infinity would most likely be equal to the exact value of the B.S option price.

There are benefits to using both the Binomial-Tree method and the B.S method, however as can be seen in this case the B.S option price can be gotten through a simple formula and minimal computational power. However, the binomial-tree method requires a significant amount more time and computational power as one would need to calculate the option price for every step in the binomial-tree and for large values of n, that would take significant more time.

4.References

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