Arbitrage Pricing Theory

A Comprehensive Study of Past & Future of *APT*

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Abstract

This paper features the *Arbitrage Pricing Theory (APT)* model within *Portfolio Theory*. First we describe and lay out the general formula of APT, followed by a literature review of the model and then we clarify its' ties to utility-based arguments and mean-variance efficiency. Before concluding with practical applications, empirical tests and corresponding potential errors are brought up.

Contents

1 Introduction

To understand the *Arbitrage Pricing Theory*, the concept of arbitrage must first be familiarized. In essence, arbitrage is the act of taking advantage of mispriced assets. With an appropriate course of action, risk free profits can be made.

An example to illustrate *statistical arbitrage* follows, which concerns a scenario with expected returns and beta factors, the latter being the exposure of the considered security to a certain macroeconomic factor such as inflation. An investor expects returns of 20 percent from a risky security, i.e. $E[R_A] = 20\%$, during some time period. He also knows that the expected return of a market index is 15 percent, meaning $E[R_{Ind}] = 15\%$. Moreover, the beta coefficient, or factor exposure level, to the inflation which affects the security is assumed to be 0.4, while the risk-free interest rate, r_f, is fixed at 5 percent. A long position in the undervalued risky security, combined with a short position in the market index yields a payoff per \$1, calculated as shown:

$$\$1 \cdot E[R_A] - \$1 \cdot ((1 - 0.4) \cdot r_f + 0.4 \cdot E[R_{Ind}])$$

Which evaluates to $0.20 - (0.03 + 0.06) = 0.11\$$. The result is a risk-free profit of 11 cents per dollar invested in the strategy. In a large scale, exploitation of a strategy like this would generate substantial profits, which certainly would attract profit-seeking businessmen- or women.

If an exploit as described above would be found by someone, many actors might flock to the opportunity. Would not the market get saturated and thereby potential profits dissipate? To answer this, first consider the following brief and simple example of arbitrage: an agent finds differences in prices of some product over two different locations, and wants to exploit this by buying the product at the cheaper location and thereafter selling it at the more expensive location, pocketing the difference minus any operational costs. We arrive at bringing forth a relevant perspective of the *Fundamental Theorem of Finance* which explains that price discrepancies, or arbitrage opportunities are quickly eliminated in the markets because the participants would pursue such profits en masse which diminishes the potential as prices adjust due to the market forces known as supply and demand. This is at least for practical or educational purposes a reasonable assumption since it makes financial modelling and calculations feasible or quantifiable. By extension, an widely recognized assumption in finance such as *efficient markets* have similar implications. The efficient markets condition takes for granted that a market has priced in relevant and public information, and an accompanying assumption is often that all investors have access to the same data. Manipulations such as these culminated in the APT, and different varieties of this are addressed in the literature review section.

2 Arbitrage Pricing Theory formula

According to [6], The model of APT spans a time frame of one period and the stochastic properties of returns are considered to be accordant with a factor structure as shown below:

$$r = \mu + \beta \cdot f + e \tag{1}$$

Conversely, a representation of APT is generally regarded as the expression below:

$$E[r_i] = r_f + \sum_{j=1}^{N} \beta_{i,j}[E(r_{M,j}) - r_f] \tag{2}$$

Wherein the summation addresses the multiple exposures that a security may be affected by. These are detailed in the later literature review section. Further, the associated risk premium of given factor j is depicted by the following mathematical component:

$$[E_{M,j} - r_f] \tag{3}$$

This concludes the brief overview of the mathematical details included in the *APT*.

3 Foreknowledge

APT frequently gets compared to a model called CAPM, *Capital Asset Pricing Model*, but that is not the case in this paper as to keep the focus narrow. Essentially though, the CAPM is a single factor case of the APT where the one factor is supposed to envelop all factors in the beta structure under one and the same mathematical component.

The APT formula's logic is derived from security pricing with Arrow-Debreu's model [4]. Within it, k securities are thought to cover all potential incurring states, and further, payoffs of respective assets are weighted in regard to the security's payoff. Subsequently, given the assumption of non-existent arbitrage opportunities, the present price of respective assets correspond to the weighted present prices of the security.

Instead of using payoffs and prices, the logic to APT stems from actual and expected returns. Conversely, though, it is thought that the unanticipated component of all assets' returns have a linear combination versus the security's component. Similarly, this applies for the anticipated components.

Simplified, Arrow-Debreu leads us from **Equation 1** to **Equation 2** by equating the term e to zero and then connecting the k amount of factors into k securities. Exact arbitrage is obtained and **Equation 2** holds, and the mathematical *law of large numbers* reinforces the theory.

In the next section, these authors' contributions are discussed further.

4 Literature Review

The literature used as reference for this report is all written by leading researchers in the area and are all well known papers. Furthermore, the report is mostly based on Huberman and Wang's paper *Arbitrage Pricing Theory* [3], which also cites most of the hereafter mentioned literature in this section. Moreover, the subsections to come detail some contributions by respective authors to the *APT*.

4.1 Huberman & Wang

This is the main literature used for this report and as such it should be the most thoroughly reviewed. This paper is a staff report written for the Federal Reserve Bank of New York, and it was written the year 2005 by Gur Huberman, a professor at Columbia University, together with staff member Zhenyu Wang at the aforementioned bank. Given their respective backgrounds, it is clear that the authors are not new to this field or to writing research papers. When looking at the online service *Google Scholar* it shows that this paper has 126 citations, which is a noteworthy amount considering the niche topic and the relatively recent release date. [3]

The authors' contributions to the topic includes:

- An introduced consideration of that deciding relevant factors through empirical tests is not entirely dependable. Namely, inclusion of factors in the APT relies on the user's free will, and that openness of choice necessitates a sharp proficiency, both in a practical and theoretical regard.
- A notion that factor decision carries more than just a single potential result, and that attained results relies on whether the actual present factors were accounted for in the APT modelling process by the user or not.
- A result from portfolio structure analysis which highlights a correlation regarding portfolio size and number of present influential factors. In other words, a portfolio with a large number of included assets should be impacted by a larger amount of factors than that of a small portfolio which likely is affected by a smaller amount of factors.
- Some identified key macroeconomic factors such as *inflation, interest-* & *exchange rate* and *gross domestic product.*

4.2 Black et. al.

The publication by Black and Scholes in 1972 is well-renowned and since its forthcoming the authors have become well-known. They have solid ties to the finance research world; the Black Scholes model for pricing options is one of the most widely known options pricing formulas. Furthermore, their works have been used as course literature for finance courses and is regarded as fundamental knowledge for students in finance. This report is also co-authored by some others, such as Michael C Jensen which is a famous economy professor from Harvard. When looking at Google Scholar we can see that this article has over 5000 citations, which clearly shows how trustworthy this report is. [1]

The authors' contributions to the topic includes:

- A solid comparative foundation for option pricing. This is a relevant tool for the APT since if we have attained a pricing for an asset or portfolio, financial derivatives such as options can be built upon them, then evaluation and comparisons can be made to see if results align or diverge. This is a common approach in practical applications such as *simulation.*
- While the Black-Scholes model does involve other mathematical compo-

nents and covers other areas of interest than APT, they do have a central, common assumption that is the *no-arbitrage* criteria which also relates to the *risk-neutral pricing*.

- The model is cited multiple times in the main publication [3] under observation in this paper. Including it and addressing it's relevance was thus prompted.

4.3 Arrow & Debreu

This is a study from 1954 written by Nobel Price awardees Kenneth Arrow and Gerard Debreu. They defined a Walrasian equilibrium in a competetive system regard. The model of Arrow-Debreu is an acknowledged result in economics since its debut, also known as equilibrium between aggregate supply and demand. [4]

This source contributes a fundamental basis within the foreknowledge section, being useful in the theoretical construction of the model. The importance of the authors' work is laid out in that section.

The source also delves deeper into details surrounding realism and generality while also discussing strengths and weaknesses.

4.4 Ross, 1976 & 1977

In Stephen A. Ross publication 1976, titled *The arbitrage theory of capital asset pricing*, he presented formally the model which this comprehensive study revolves. In conjunction with a later publication, Ross laid out theory which has come to substantial use academically for students during the years since. It has further served as a backbone to many extended and modified versions of the model, with which the purpose is unified; evaluating fair pricing of assets and portfolios. [6] [5]

5 Empirical tests and errors

As brought forth by Huberman & Wang [3], they found that practical applications derives from empirical data and theoretical knowledge that moreso pinpoints aspects of interwoven factor structure than solely grouped mean-variance efficient factor portfolios which complies with the linear structure in **Equation 1**.

In terms of validity of results, APT does inhibit risk of errors since there is a variety of potential factors to include, meaning that if the wrong metrics are considered, the estimated pricing of a capital asset will diverge from the actual, realistic price. For the purposes of this paper this section is limited, although the methods and errors are particularly important aspects when researching extensions or alterations of the APT model and as such it is necessary to bring up. With that being said, some aspects are delved into in the upcoming applications subsection.

6 Applications

Some applications for the APT according to Huberman & Wang [3] are:

- Asset allocation
- Computation of the cost of capital
- Performance evaluations of managed funds

Huberman & Wang [3] suggest that the APT works well for asset allocation since the structure of the definition given by **Equation 1** is intrinsically linked to mean-variance efficiency. This is because the APT is structured as we know using k factors, where f is a $k \times 1$ vector, this means that we have k assets spanning our efficient frontier. Therefore, we can construct a mean-variance efficient portfolio using only k assets. Having k for example, as the payoff functions of the traded securities makes asset allocation even easier, this is because when k is a small number the optimization problem will have a smaller dimension. However, the optimal portfolio constructed from the APT will not be mean-variance efficient if the factor structure is incorrect. That is to say, if one were to omit some important factors or include less important ones.

For performance evaluations you have to be careful when using APT according to Huberman & Wang [3], this is because the *Stochastic Discount Factor* apparent in the model is a random variable that can be negative. Thus, when applying APT to something like a derivative of an asset or on a trading strategy we will get a price contradiction. This means that especially when using an APT model with many factors, there is a high chance of pricing errors due to arbitrage. Furthermore, thanks to Black & Scholes ([1], as cited in Huberman & Wang [3]) we know that combinations of trading strategies on securities can mimic the payoffs of options, therefore it is suggested that we should be careful when interpreting excess returns on managed funds based on APT, since the returns are most likely from trades and not the portfolios themselves.

APT also works well for calculating the cost of capital since it is an asset pricing model. There have been many attempts for this purpose, for example, by Elton et. al. ([2], as cited in Huberman & Wang [3]). They used the APT to calculate the cost of capital for electrical utilities, and specify the factors as for example, *inflation rate, interest rates, the GDP growth rate and foreign currency exchange rates.* As we know, the APT formula does not specify which factors to use, which is why when using the model we will get different results depending on what factors we use. This is why in a lot of cases the CAPM is used for this purpose instead, having only one market beta factor that supposedly incorporates the relevant influences.

References

[1] Fischer Black, Michael C Jensen, Myron Scholes, et al. The capital asset pricing model: Some empirical tests. 1972.

[2] Edwin J Elton, Martin J Gruber, and Jianping Mei. *Cost of capital using arbitrage pricing theory: A case study of nine New York utilities.* Blackwell, 1994.

[3] Gur Huberman and Zhenyu Wang. Arbitrage pricing theory. 2005.

[4] G. Debreu K.J. Arrow. Existence of an equilibrium for a competetive economy. *Econometrica*, 1954.

[5] Stephen Ross. Return, risk and arbitrage. *Risk and Return in Finance*, 1977.

[6] Stephen A Ross. The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 1976.