Mälardalens universitet

Seminar Report

Roulette wheels in Las Vegas casinos

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Abstract

This paper addresses a set of problems about probabilities and estimates, and walks the reader through the solution process. First, the problem formulation will show what is to be investigated and any information we have to begin with. After applying learnt methods we will see what knowledge or truths are uncovered.

Key words: Maximum Likelihood Estimator, Binomial distribution, Roulette in Las Vegas.

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1. Problem Formulation

Roulette wheels in Las Vegas casinos have 38 numbered pockets. Every time the ball is rolled, it is equally likely to land in any one of them. Marie Louise (an eastern European princess widely believed to be the most beautiful woman in the world), spent the whole night of her birthday, January 16th, in the casino of the Bellagio Hotel, playing roulette, always betting $50 on a *split* between numbers 13 and 16 (the probability of winning such bet is 1/19, and it pays $17 per $1 bet). Indeed it was her lucky day, for she won 25 times on her 25th birthday.

1.1 Inquiry: A

State a statistical model for the number *k* of wins in *n* plays, and the assumptions that validate the model.

1.2 Inquiry: B

Compute the probability of winning 25 times with a *split*, in 450 bets. note: Stirling's approximation for the factorial of a positive integer: $n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$ for large *n*, where $e = exp(1)$.

1.3 Inquiry: C

Find the maximum likelihood estimate of the number *n* of times that Marie Lousie played that night. (note: It was a long night.)

1.4 Inquiry: D

Compute the maximum likelihood estimate of the amount of money that Marie Louise lost that night (note: She did not have to pawn her sapphire earrings).

1.5 Inquiry: E

Describe how you arrived at the answers you gave above, the assumptions you made or the approximations you used, and state the lessons learned.

2. Results

2.1 Solution: A

The statistical model that best describes how many times Marie Louise won out of how many times she played follows a binomial distribution. According to Wackerly et al. (2008), a binomial experiment possesses the following properties:

1. The experiment consists of a fixed number, n, of identical trials.
2. Each trial results in one of two outcomes: success, S, or failure, F.
3. The probability of success on a single trial is equal to some value p and remains the same from trial to trial. The probability of a failure is equal to $q = (1 - p)$.
4. The trials are independent.
5. The random variable of interest is Y, the number of successes observed during the n trials.

From the formulated problem we can verify that:

1. Marie Louise betted n times, and each bet was identical: a split between 13 and 16.
2. We can classify winning such a bet as a success and losing as a failure.
3. The probability of winning such a bet is equal to $p = \frac{1}{19}$ and remains the same for every try. Additionally, the probability of failure equals to $q = 1 - \frac{1}{19} = \frac{18}{19}$.
4. Every trial is independent since previous trials do not affect future ones.
5. We are interested in the random variable Y, the number of times she won out of n trials.

Furthermore, the problem provides the following information:

- She won 25 times, i.e. $Y = 25$.
- We do not know how many times she played, i.e. the variable n is unknown.
- She was betting $50 each round, and winning a bet on a split pays $17 per $1 betted. This yields a total net profit of $\$17 * \$50 - \$50 = \$800$ per won bet.
- The model is of discrete nature, i.e. non-negative natural numbers, $\mathbb{N}^{+}$.

In other words, binomial experiments agree to an unchanging probability, a number of attempts and successes, and independent trials. These are the assumptions made that allow us to use the binomial probability mass function as our statistical model. Hence, the following formula represents the problem at hand:

$$Y \sim Bin(n, p) \Rightarrow p(y) = C_y^n \cdot p^n \cdot (1 - p)^{n-y} \tag{1}$$

Where $C_y^n = \frac{n!}{y!(n-y)!}$ is the binomial coefficient.

2.2 Solution: B

Provided in *Inquiry: B*, *Stirling's approximation* given by:

$$n! \approx \sqrt{2\pi n} \cdot (\frac{n}{e})^n \tag{2}$$

is an approximation that facilitates the evaluation for large factorials.

If we take $n = 450$, with equation (2), the approximation will be:

$$450! \approx \sqrt{2\pi \cdot 450} \cdot (\frac{450}{e})^{450}$$

With the aforementioned statistical model in (1), the probability of winning 25 bets (i.e. $y = 25$) with a split, out of a total of 450 games, can be calculated as follows: (with $p = \frac{1}{19}$)

$$C_{25}^{450} \cdot (\frac{1}{19})^{25} \cdot (\frac{18}{19})^{425} \approx 0.0787 = 7.87\%$$

Where

$$C_{25}^{450} = \frac{450!}{25! \cdot (450-25)!} = \frac{450!}{25! \cdot 425!}$$

The expected value of successes for a binomial distribution, as explained by Wackerly et al. (2008), is given by $E[Y] = np$. With $n = 450$ and $p = \frac{1}{19}$, $E[Y] = 450 \cdot \frac{1}{19} \approx 23.68$. This means that Marie Louise could be considered slightly lucky as she won 25 out of 450 times played.

2.3 Solution: C

In essence, probability theory states that the probability of success and number of successes should reflect an expected total number of rounds played. This is portrayed as follows:

$$probability\ of\ success\ = \frac{successful\ outcomes}{total\ outcomes} \Leftrightarrow p\ = \frac{y}{n}.$$

With the parameters $p\ = \frac{1}{19}$ and $y\ =\ 25$ provided, we have:

$$\frac{1}{19} = \frac{25}{n} \Rightarrow n\ =\ 25\ \cdot\ 19 \Rightarrow n\ =\ 475$$

We can argue for or against this result by finding the maximum likelihood estimate (MLE) of the number n of times that Marie Lousie played that night. So in order to find the MLE of n, we need to produce a likelihood function $L(n)$ and maximize it with respect to n. This will then provide an estimation on how many games Marie Louise should have played in order to win exactly 25 bets. Wackerly et al. (2008) define the likelihood function for a binomial distribution with $y\ =\ 25$ and $p\ = \frac{1}{19}$ as follows:

$$L(n)\ =\ L(y_1, y_2,\ ...,\ y_n|n)\ = C^n_{25}\ \cdot\ (\frac{1}{19})^{25}\ \cdot\ (\frac{18}{19})^{n-25} \tag{3}$$

With $C^n_{25}\ = \frac{n!}{25!(n-25)!}$ being the binomial coefficient.

We can observe from (3) that the function is monotonically increasing for positive values of n, thus we can take the natural logarithm of both sides and applying properties of logarithms we obtain the following log-likelihood function:

$$ln[L(n)]\ =\ ln(n!)\ -\ ln(25!)\ -\ ln[(n\ -\ 25)!]\ +\ 25ln(\frac{1}{19})\ +\ (n\ -\ 25)ln(\frac{18}{19})$$

Now, since the function is monotonically increasing, the values of n that maximizes $L(n)$ are the same that maximizes $ln[L(n)]$. And because $ln[L(n)]$ is differentiable, we can apply the derivative test with respect to n to determine the maximum of the function. Accordingly, we need to solve:

$$\frac{d}{dn}\ [ln(n!)\ -\ ln[(n\ -\ 25)!]\ +\ n\ \cdot\ ln(\frac{18}{19})]\ =\ 0. \tag{4}$$

Derivative terms that do not contain n are already omitted as their derivatives with respect to n equals zero. Appendix 1 shows how to simplify equation (4) using Stirling's approximation. The resulting equation is:

$$\frac{1}{2n} + ln(n) - \frac{1}{2(n-25)} - ln(n - 25) + ln(\frac{18}{19}) = 0. \qquad (5)$$

We have obtained an approximation to the solution for equation (5) by plotting the function in MATLAB, as shown below:

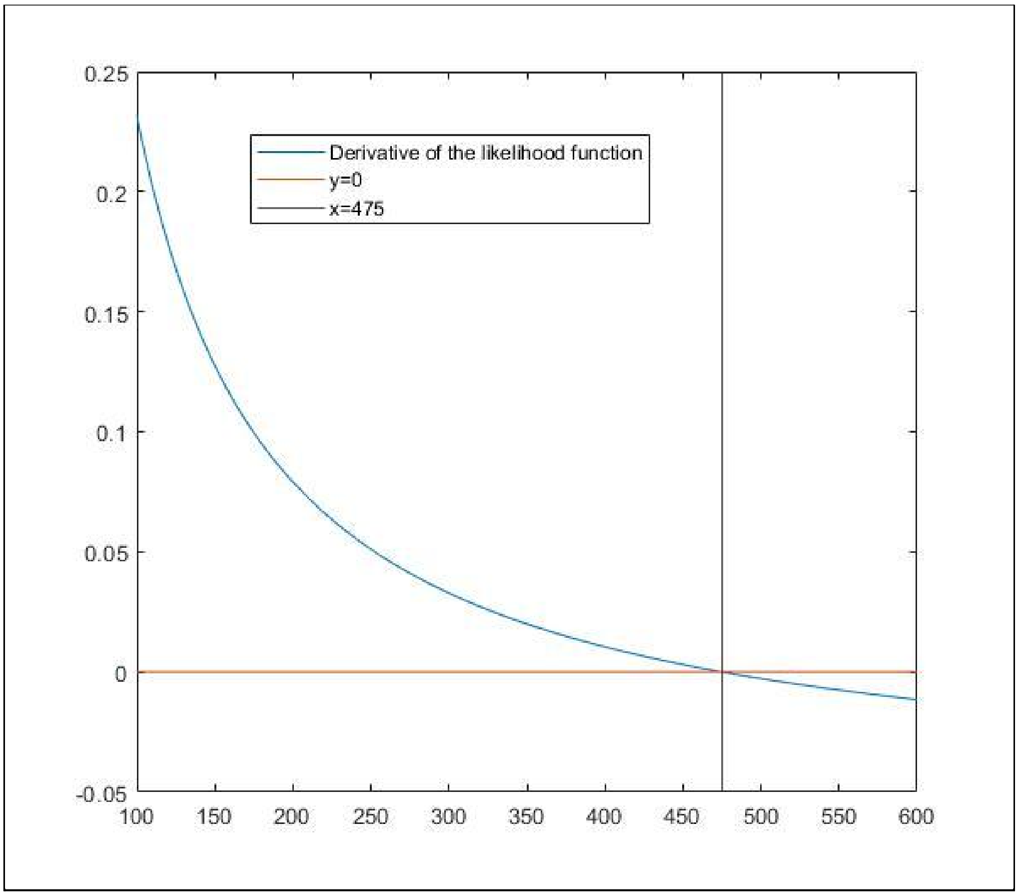

Figure 1: Plot of the derivative of $ln[L(n)]$, $y = 0$ *and* $x = 475$.

Direct observation shows that $n = 475$ is an approximated solution to equation (5), which agrees with the initial reasoning of this subsection. The MATLAB code to reproduce this plot can be found in Appendix 2.

2.4 Solution: D

As estimated by the method of maximum likelihood above, Marie Louise played an approximate total of 475 times. Knowing that:

- she won exactly 25 times, i.e. $y = 25$
- she betted \$50 each round
- and that winning a bet on a split pays \$17 per \$1 betted

We can also estimate the amount of money she lost that night. The amount is calculated as follows:

$$\$17 \cdot \$50 \cdot y - \$50 \cdot y - \$50 \cdot (n - y) \Leftrightarrow \$800 \cdot y - \$50 \cdot n + \$50 \cdot y$$

$$\Rightarrow \$850 \cdot y - \$50 \cdot n$$

With $y = 25$ and $n = 475$ we obtain:

$$\$850 \cdot 25 - \$50 \cdot 475 =- \$2.500$$

This is easily explained by the fact that she:

- Wins \$800 each successful round,
- Loses \$50 each unsuccessful round.
- Wins 25 times,
- Loses $475 - 25 = 450$ times,

The amount lost is $\$2500$. Indeed, Marie Louise, the princess of Wales, would arguably not need to pawn her sapphire earrings.

2.5 Solution: E

The assumptions solutions are clarified under each respective subsection, and Appendix 1 details the algebra that lead to the obtained results.

We learned that regardless if you are the princess of Wales, or any other member of society, the games at the casino provide chances of winning and odds of losing the money gambled. The odds do not change, the rounds are identical and independently distributed. So after a long night at the casino and winning the bet 25 times, the method of maximum likelihood implies there were 475 rounds of blackjack played. Strictly following a predetermined strategy made Marie Louise lose \$2500. But during the night, it is probable that she was, for some time(s), building up some profits or losses. It is possible that she was profitable at some point, but kept playing, and then went into negative territory. The odds *do* favor the casino as per the general business model, and eventually she ended up with a loss on the evening of her birthday.

Marie Louise won 25 bets at the casino, and each time yielding $850, that sure must be a rewarding feeling. We are inclined to believe she had a good time, perhaps with good company as well.

3. References

[1] D. D. Wackerly, W. Mendenhall, and R. L. Scheaffer. *Mathematical Statistics with Applications, 7th Edition*. Thomson Learning, 2008.

4. Appendix 1

The following appendix comprises substantial mathematical derivation of equation (5) that is not explicitly dealt with in the text and outlines very detailed information.

From equation (3):

$$L(n) = C^{n}_{25} \cdot (\frac{1}{19})^{25} \cdot (\frac{18}{19})^{n-25} = \frac{n!}{25!(n-25)!} \cdot (\frac{1}{19})^{25} \cdot (\frac{18}{19})^{n-25}$$

Taking natural logarithms on both sides:

$$\Rightarrow ln[L(n)] = ln[\frac{n!}{25!(n-25)!} \cdot (\frac{1}{19})^{25} \cdot (\frac{18}{19})^{n-25}]$$

Applying logarithm properties to the right side:

$$= ln(n!) - [ln(25!) + ln[(n-25)!]] + ln(\frac{1}{19})^{25} + ln(\frac{18}{19})^{(n-25)}$$

$$= ln(n!) - ln(25!) - ln[(n-25)!] + 25ln(\frac{1}{19}) + (n-25)ln(\frac{18}{19})$$

Taking the derivative with respect to n:

$$\Rightarrow \frac{d}{dn}[ln(n!) - ln(25!) - ln[(n-25)!] + 25ln(\frac{1}{19}) + (n-25)ln(\frac{18}{19})]$$

$$= \frac{d}{dn}[ln(n!)] - \frac{d}{dn}[ln(25!)] - \frac{d}{dn}[ln[(n-25)!]] + \frac{d}{dn}[25ln(\frac{1}{19})] + \frac{d}{dn}[n \cdot ln(\frac{18}{19})]$$

$$- \frac{d}{dn}[25ln(\frac{18}{19})]$$

$$= \frac{d}{dn}[ln(n!)] - 0 - \frac{d}{dn}[ln[(n-25)!]] + 0 + \frac{d}{dn}[n \cdot ln(\frac{18}{19})] - 0$$

$$= \frac{d}{dn}[ln(n!)] - \frac{d}{dn}[ln[(n-25)!]] + \frac{d}{dn}[n \cdot ln(\frac{18}{19})]$$

Applying Stirling's approximation from equation (2):

$$\approx \frac{d}{dn}[ln(\sqrt{2\pi n} \cdot (\frac{n}{e})^n)] - \frac{d}{dn}[ln[(\sqrt{2\pi(n - 25)} \cdot (\frac{n-25}{e})^{n-25})!]])]$$

$$+ \frac{d}{dn}[\sqrt{2\pi n} \cdot (\frac{n}{e})^n \cdot ln(\frac{18}{19})]$$

$$\approx \frac{d}{dn}[ln(\sqrt{2\pi n} \cdot (\frac{n}{e})^n) - ln[(\sqrt{2\pi(n - 25)} \cdot (\frac{n-25}{e})^{n-25}] + n \cdot ln(\frac{18}{19})]$$

$$= \frac{d}{dn}[ln(\sqrt{2\pi}) + ln(\sqrt{n}) + ln((\frac{n}{e})^n) - ln(\sqrt{2\pi}) - ln(\sqrt{n - 25}) - ln((\frac{n-25}{e})^{n-25})$$

$$+ n \cdot ln(\frac{18}{19})]$$

$$= \frac{d}{dn}[ln(\sqrt{2\pi}) + ln(\sqrt{n}) + n \cdot ln(\frac{n}{e}) - ln(\sqrt{2\pi}) - ln(\sqrt{n - 25})$$

$$- (n - 25) \cdot ln((\frac{n-25}{e})^{n-25} + n \cdot ln(\frac{18}{19})]$$

$$= \frac{1}{2n} + ln(n) - \frac{1}{2n-50} - ln(n - 25) + ln(\frac{18}{19}) \quad (5)$$

5. Appendix 2

The following appendix comprises the MATLAB code to reproduce figure 1:

```
f=@(x) [(1./(2.*x))+log(x)-(1./(2.*x-50))-log(x-25)+log(18/19);
x=linspace(100,600,500000);
y1=f(x);
y2=0*x;
plot(x,y1,x,y2)
hold on
xline(475)
legend('Derivative of the likelihood function','y=0','x=475')
```