SRKL schemes for option pricing under the Heston model

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Contents

Motivation and Research Question

Motivation:

- Explore impact of numerical methods for option pricing under the Heston model.

Research Question:

- *Can the valuation of financial options be enhanced through the integration of SRKL numerical methods to solve the SDEs of the Heston model?*

Methods:

- Monte Carlo and SRKL schemes; Euler-Maruyama and Midpoint.

Objectives:

- Assess convergence and consistency.
- Enhance option pricing with SRKL schemes.

Option Pricing

Definition (Risk–Neutral pricing)

- Let $\mathbb{P}(\omega)$ be the true probability of outcomes $\omega \in \Omega$.
- Let $\mathbb{Q}(\omega)$ be the adjusted probability removing the effect of risk.

Under the risk-neutral probability $\mathbb{Q}$

$$V_0 = e^{-rT} E_{\mathbb{Q}}[V_T].$$

European & Asian option payoffs

$$h_c = (X - K)^+, \quad \text{for calls.} \quad h_p = (K - X)^+, \quad \text{for puts.}$$

K is the strike price, and X equals to S_T for European options, and A or G for arithmetic or geometric Asian options

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}, \qquad G = \left(\prod_{i=1}^{n} S_{t_i}\right)^{n^{-1}}$$

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Black–Scholes & Heston Model

Definition (Black–Scholes Model)

The underlying asset follows a geometric Brownian motion:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t. \tag{1}$$

Theorem (Geometric Brownian Motion)

The solution to equation (1) *takes the form* $S_t = S_0 \exp((r - \sigma^2/2)t + \sigma W_t)$.

The prices for the calls and puts are:

$$C(S_0, t) = S_0 N(d_1) - Ke^{-r(T-t)} N(d_2), \qquad d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$P(S_0, t) = Ke^{-r(T-t)} N(-d_2) - S_0 N(-d_1), \quad d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

Definition (Heston Model)

The *Heston model* with given initial value is given by

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{V_t}S_t\mathrm{d}W_t^1, \tag{2a}$$

$$\mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \sigma\sqrt{V_t}\mathrm{d}W_t^2. \tag{2b}$$

The Wiener processes, W_t^1 and W_t^2 are correlated, and in particular

$$E\left[\mathrm{d}W_t^1\mathrm{d}W_t^2\right] = \rho\min(s,t), \quad \rho \in [-1,1].$$

Theorem

The system (2) *is equivalent to the system*

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sqrt{V_t}S_t\,\mathrm{d}W_t^1,$$

$$\mathrm{d}V_t = \kappa(\theta - V_t)\,\mathrm{d}t + \sigma\sqrt{V_t}\left[\rho\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2\right].$$

Monte Carlo & Variance Reduction Techniques

Algorithm 2: Monte Carlo simulation with Antithetic Variate

Input: Initial stock price S_0, variance σ, risk-free interest rate r, time horizon in years T, number of time steps N, number of simulations M

Output: Antithetic estimate of stock price path

$\mathrm{d}t \leftarrow T/N$;

generate a $M \times N$ standard normal random variable: $z \sim \mathcal{N}(0,1)$;

simulate a $M \times N$ Brownian motion and the negative for the antithetic path;

$dW^1 \leftarrow \sqrt{\mathrm{d}t} \times z_1$;

$dW^2 \leftarrow -dW^1$;

initiate two zero $M \times N$ matrices to hold the values of S_1 and S_2;

assign starting price;

$S_1(:,1) \leftarrow S_0$;

$S_2(:,1) \leftarrow S_0$;

generate stock price path and antithetic path;

for $i \leftarrow 1$ **to** N **do**

$$S_1(:,i+1) \leftarrow S(:,i) \times \exp\left(\left(r - \tfrac{1}{2}V(:,i)\right)\mathrm{d}t + \sqrt{V(:,i)}\mathrm{d}W^1\right);$$

$$S_2(:,i+1) \leftarrow S(:,i) \times \exp\left(\left(r - \tfrac{1}{2}V(:,i)\right)\mathrm{d}t + \sqrt{V(:,i)}\mathrm{d}W^2\right);$$

end

calculate the expectations of each path;

$S_1 \leftarrow \sum_{i=1}^{M} S_1(:,N)$;

$S_2 \leftarrow \sum_{i=1}^{M} S_2(:,N)$;

calculate the antithetic estimate;

$S \leftarrow \frac{1}{2}(S_1 + S_2)$;

return S;

Stochastic Runge–Kutta Lawson Scheme

Discretization

Consider the interval $I = [t_0, T]$. We can create an ordered subset of I in N intervals such as $t_0 < t_1 < \cdots < t_N = T$, where $t_n = t_0 + nh$ for $n = 0, 1, \cdots, N$, and where $h = (T - t_0)/N$ is the step size.

SDEs for SRKL schemes

SRKL schemes are applicable to the following type of *stochastic differential equations*

$$\mathrm{d}\mathbf{X}(t) = \sum_{m=0}^{M} \left(A_m \mathbf{X}(t) + \mathbf{g}_m(t, \mathbf{X}(t))\right) \mathrm{d}W_m(t), \qquad \mathbf{X}(0) = \mathbf{x}_0, \tag{3}$$

where $W(0) = t$, $\mathbf{X}(t) \in \mathbb{R}^d$, the subsequent matrices A_k and A_l are constant and commute, and $W_1(t), \ldots, W_M(t)$ are independent Brownian motions.

Stochastic Runge–Kutta Lawson Scheme

General form of the SRKL scheme

The following general *SRKL* scheme was proposed by [2] Debrabant et al. (2020):

$$
\begin{aligned}
\mathbf{H}_i &= \mathbf{Y}_n + \sum_{j=1}^{s} \exp(-\Delta L_j^n) \sum_{m=0}^{M} Z_{ij}^{m,n} \tilde{g}_m(t_n + c_0^{n,j}, \exp(\Delta L_j^n)\mathbf{H}_j), \\
\mathbf{V}_n^{n+1} &= \mathbf{Y}_n + \sum_{i=1}^{s} \exp(-\Delta L_i^n) \sum_{m=0}^{M} z_i^{m,n} \tilde{g}_m(t_n + c_0^{n,i}, \exp(\Delta L_i^n)\mathbf{H}_i), \\
\mathbf{Y}_{n+1} &= \exp(\Delta L^n)\mathbf{V}_n^{n+1}.
\end{aligned}
\tag{4}
$$

Definition (Drift- and Full–Stochastic Schemes)

DSL schemes: $A_m = 0 \; \forall \; m > 0$, compute $\exp(L^n(t))$ once (SRKL Euler-Maruyama).

FSL schemes: contain at least one non-zero linear diffusion term in the operator $L^n(t)$, compute $\exp(L^n(t))$ at every step (SRKL Midpoint).

Stochastic Runge–Kutta Lawson scheme for the Heston model

In the Heston model we have:

Example

$$d = M = 2, \qquad A_0 = \begin{bmatrix} r & 0 \\ 0 & -\kappa \end{bmatrix}, \qquad A_1 = A_2 = \mathbf{0}_{2\times 2},$$

$$g_0(t, \mathbf{X}(t)) = \begin{bmatrix} 0 \\ \kappa\theta \end{bmatrix}, \qquad g_1(t, \mathbf{X}(t)) = \begin{bmatrix} \sqrt{V(t)}S(t) \\ \sigma\rho\sqrt{V(t)} \end{bmatrix}, \qquad g_2(t, \mathbf{X}(t)) = \begin{bmatrix} 0 \\ \sigma\sqrt{1-\rho^2}\sqrt{V(t)} \end{bmatrix}.$$

Coefficients suggested by [2]:

$$c_m^{n,i} = \sum_{j=1}^{s} Z_{ij}^{m,n}, \qquad \Delta W^{n,m} = c_m^n = \sum_{i=1}^{s} z_i^{m,n},$$

$$\Delta L_i^n = \begin{bmatrix} r & 0 \\ 0 & -\kappa \end{bmatrix} c_0^{n,i}, \qquad \Delta L^n = \begin{bmatrix} r & 0 \\ 0 & -\kappa \end{bmatrix} c_0^n.$$

SRKL for the Heston Model

Cont'd:

Example

$$\boldsymbol{H}_i = \boldsymbol{Y}_n + \sum_{j=1}^{s} e^{-\begin{bmatrix} r & 0 \\ 0 & -\kappa \end{bmatrix} c_0^{n,j}} \sum_{m=0}^{2} Z_{ij}^{m,n} \tilde{\boldsymbol{g}}_m\left(t_n + c_0^{n,j}, e^{-\begin{bmatrix} r & 0 \\ 0 & -\kappa \end{bmatrix} c_0^{n,j}} \boldsymbol{H}_j\right),$$

$$\boldsymbol{V}_{n+1}^n = \boldsymbol{Y}_n + \sum_{i=1}^{s} e^{-\begin{bmatrix} r & 0 \\ 0 & -\kappa \end{bmatrix} c_0^{n,i}} \sum_{m=0}^{2} z_i^{m,n} \tilde{\boldsymbol{g}}_m\left(t_n + c_0^{n,i}, e^{-\begin{bmatrix} r & 0 \\ 0 & -\kappa \end{bmatrix} c_0^{n,i}} \boldsymbol{H}_i\right),$$

$$\boldsymbol{Y}_{n+1} = e^{\begin{bmatrix} r & 0 \\ 0 & -\kappa \end{bmatrix} c_0^{n}} \boldsymbol{V}_{n+1}^n.$$

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Methodology

Methodology Steps

1 Understand the Heston model
2 Select Numerical Methods
3 Implement the Methods
3 Define Benchmark
4 Compute Option Prices
5 Measure Accuracy and Efficiency
6 Perform Sensitivity Test
7 Compare Results

Parameter inputs:

$$
\begin{aligned}
S_0 &= 80, & K &= 85, & \theta &= 0.05,\\
V_0 &= 0.04, & T &= 1, & \sigma &= 0.20,\\
r &= 0.05, & \kappa &= 1, & \rho &= -0.7.
\end{aligned}
$$

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Results (1/4): Heat Map

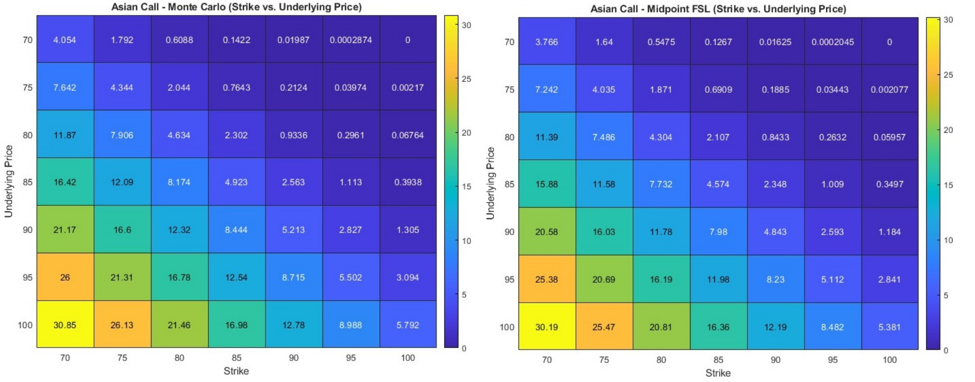

Figure: Heat map of arithmetic Asian call option prices under Heston model with Midpoint FSL & standard Monte Carlo method and varying initial price S_0 and strike price K.

Results (2/4): Convergences

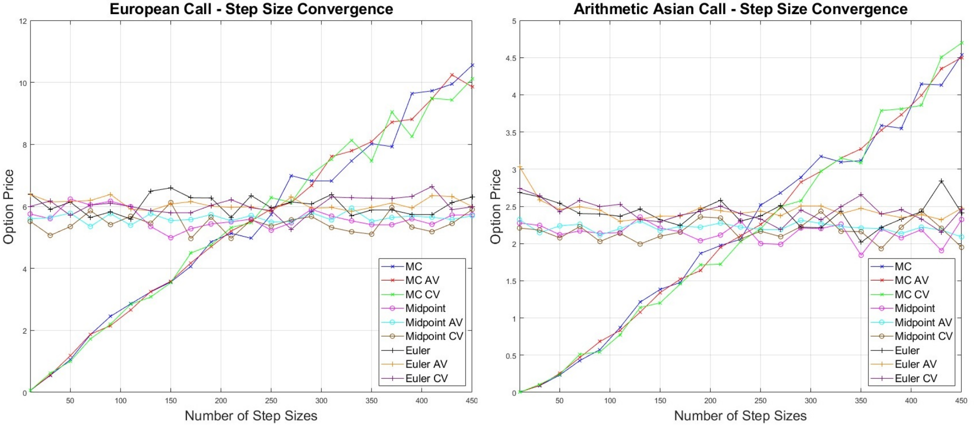

Figure: Relationship between step-size and convergence of European & arithmetic Asian call prices for different models, with the incorporation of variance reduction techniques.

Results (3/4): Table 1

Method	τ	C_{EU}	σ	$\sigma^2\tau$
Standard Monte Carlo	0.01780	5.98914	0.29084	0.001505
Antithetic Monte Carlo	0.03036	5.95960	0.20814	0.00131
Control Monte Carlo	0.01700	5.98914	0.29085	0.00143
Midpoint	0.27933	5.57079	0.28630	0.02290
Antithetic Midpoint	0.43846	5.53898	0.20496	0.01842
Control Midpoint	0.30401	5.57079	0.28629	0.02492
Euler	0.03231	5.98032	0.29129	0.00274
Antithetic Euler	0.05209	5.98366	0.20959	0.00228
Control Euler	0.02767	5.98032	0.29129	0.00234
Black–Scholes (Benchmark)	–	5.98824	–	–

Table: Price, standard error and efficiency of European call options under the Heston Model.

Results (4/4): Table 2

Method	τ	C_A	σ_A	$\sigma_A^2\tau$	C_G	σ_G	$\sigma^2\tau$
Antithetic Monte Carlo	0.05753	2.35835	0.09765	0.00055	2.24130	0.09385	0.00051
Control Monte Carlo	0.03029	2.30193	0.13550	0.00056	2.18983	0.13056	0.00052
Midpoint	0.25216	2.10707	0.13041	0.00429	2.00089	0.12543	0.00397
Antithetic Midpoint	0.55786	2.16689	0.09415	0.00495	2.05432	0.09031	0.00455
Control Midpoint	0.28627	2.10707	0.01065	0.0000	2.00089	0.12543	0.00450
Euler	0.04653	2.32112	0.13642	0.00087	2.20825	0.13145	0.00080
Antithetic Euler	0.12521	2.37718	0.09829	0.00121	2.25921	0.09446	0.00112
Control Euler	0.04261	2.32112	0.01066	0.00000	2.20824	0.13145	0.00092
Standard Monte Carlo (Benchmark)	0.03073	2.30193	0.13549	0.00056	2.18983	0.13057	0.00052

Table: Price, standard error and efficiency of Arithmetic and Geometric Asian call options under the Heston Model.

Conclussion

Summarized results:

- Midpoint: Under-pricing of call options.
- Midpoint: Less efficient than traditional methods.
- Euler: More accurate and efficient than Midpoint.
- Consistent no matter the step size.

Encountered Issues:

- Hard to implement.
- Unstable results.

Further research:

- FSL and DSL schemes comparison.
- Other derivatives, numerical methods, and models.
- Model fitting and portfolios.

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