Tensor Calc, Multilinear Algebra and Diff. Geom.

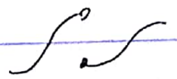
30.1.13

Lecture 1: Topological spaces

@ coarsest level: spacetime is a set

Not enough to talk about continuity of maps.

In classical physics, no jumps.



weakest structure that can be established on a set which allows a def. of continuity: a topology.

Def. Let M be a set.
A topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ (power set) satisfying

(i) $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$

(ii) $U \in \mathcal{O}, V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$

(iii) $U_\alpha \in \mathcal{O} \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{O}$

α - index set. can be uncountable.

Terminology

M set

$\mathcal{O}$ topology

$(M, \mathcal{O})$ topological space (a set w/ additional structure)

$U \in \mathcal{O} \Leftrightarrow$ call $U \subseteq M$ an open set

$M \setminus A \Leftrightarrow$ call $A \subset M$ a closed set

△ open $\not\Rightarrow$ closed
($\neq$)

2. Continuous Maps

$f : M \longrightarrow N$

The answer to the q. of whether a map is cont. depends (by def.) on which topology is chosen on M and N.

Example. (1) $M = \{1, 2, 3\}$

(a) $\mathcal{O}_1 = \{\emptyset, \{1,2,3\}\}$ is a topo.

(b) $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1,2,3\}\}$ Not a topo.

(2) M any set.

(a) $M_{chaotic} := \{\emptyset, M\}$

(b) $M_{discrete} := \mathcal{P}(M)$

2.a & 2.b are utterly useless

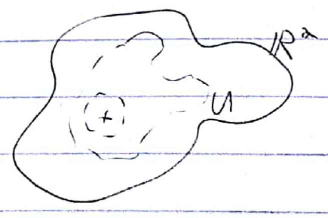
(3) $M = \mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$

Def. $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathbb{R}^d)$

2 steps: soft ball

(α) $B_r(p) := \{(q_1, q_2, \ldots q_d) \mid \sum (q_i - p_i)^2 < r\}$

(β) $U \in \mathcal{O}_{standard}$
$\Updownarrow$
$\forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U$

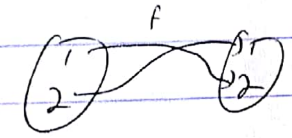


Def. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topo. spaces.
Then, a map $f: M \to N$ is called continuous
(wrt $\mathcal{O}_N$ and $\mathcal{O}_M$) if

$$\forall\, V \in \mathcal{O}_N : \quad \mathrm{preim}_f(V) \in \mathcal{O}_M$$

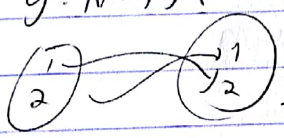
Mnemonic: "A map is cont. iff the preimage of (all) open sets are open sets."

Example (a) $M = \{1, 2, \cancel{3}\}$ $\quad \mathcal{O}_M = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
$N = \{1, 2\}$ $\quad \mathcal{O}_N = \{\emptyset, \{1,2\}\}$



$\mathrm{preim}(\emptyset) = \emptyset \in \mathcal{O}_M$
$\mathrm{preim}(\{1,2\}) = M \in \mathcal{O}_M$
$\Rightarrow f$ is cont.

(b) $g: N \to M$



$\mathrm{preim}(\{1\}) = \{1\} \notin \mathcal{O}_N$!
$\Rightarrow$ Not cont

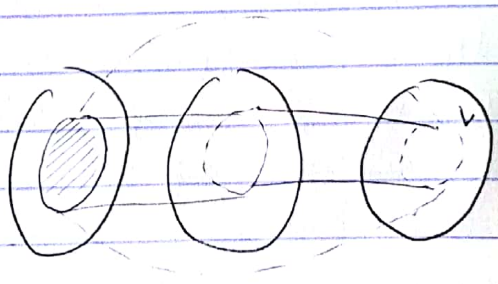
Same map, different topologies.

3. Composition of continuous maps

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "g\circ f"] & N \arrow[r] & P \end{tikzcd}$$

$g \circ f : M \to P$

Key Thm. $\begin{matrix} f \text{ cont.} \\ g \text{ cont.} \end{matrix} \Rightarrow g\circ f$ cont.



Proof. Let $V \in \mathcal{O}_P$

$$\begin{aligned} \mathrm{preim}_{g\circ f}(V) &= \{ m \in M \mid (g\circ f)(m) \in V \} \\ &= \{ m \in M \mid f(m) \in \mathrm{preim}_g(V) \} \\ &= \underbrace{\mathrm{preim}_f(\underbrace{\mathrm{preim}_g(V)}_{\in \mathcal{O}_N})}_{\mathcal{O}_M} \end{aligned}$$

4. Inherting a topology

There are many useful ways to inherent a topo. from same given topo. space (s).

Important for space time physicists :

$$S \subseteq M \;\; \mathcal{O}_M$$

Q. can one construct on S a topo from $\mathcal{O}_M$ on M?

Yes. Def. $\mathcal{O}|_S \subset \mathcal{P}(S)$ "subset topo."

$$\mathcal{O}|_S := \{ U \cap S \mid U \in \mathcal{O}_M \}$$

claim : $\mathcal{O}|_S$ is a topo.

(i) $\emptyset = \emptyset \cap S \Rightarrow \emptyset \in \mathcal{O}|_S$

$M = M \cap S \Rightarrow \cancel{M}\, S \in \mathcal{O}|_S$

(ii) $A, B \in S \Rightarrow \exists\, \hat{A}, \hat{B} \in \mathcal{O}_M$

$$\left.\begin{array}{l} A = \hat{A} \cap S \\ B = \hat{B} \cap S \end{array}\right. \Rightarrow A \cap B = (\hat{A} \cap S) \cap (\hat{B} \cap S)$$

$$= (\underbrace{\hat{A} \cap \hat{B}}_{\in \mathcal{O}_M}) \cap S$$

$$\Rightarrow A \cap B \in \mathcal{O}|_S$$

(iii)

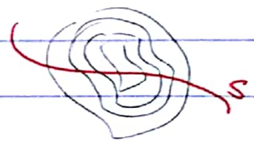
use of this specific way to inherit a topo from a superset :

$$\begin{tikzcd} S \subseteq M \arrow[r, "f"] \arrow[r, bend left, "f|_S"] & N \end{tikzcd} \qquad \mathcal{O}|_S \qquad \underbrace{\mathcal{O}_M \qquad \mathcal{O}_N}_{\text{cont}}$$

$$\Downarrow$$

$f|_S : S \to N$ is cont.

$$\begin{tikzcd} S \subseteq \mathbb{R}^2 \arrow[r, "T"] \arrow[r, bend left, "T|_S"] & \mathbb{R} \end{tikzcd}$$



Tensor Calculus, Multilinear Algebra, and Diff Geom.

Lecture 2: Manifolds

Topological spaces: there are so many of them that mathematicians cannot even classify them.

For spacetime physics, we may focus on topo. spaces $(M, \mathcal{O})$ that can be charted, analogously to how the surface of the earth is charted.

1. Topological Manifolds

Def. A topo. space $(M, \mathcal{O})$ is called a d-dimensional topo. manifold if

$$\forall p \in M : \exists \underset{\hat{p}}{U} \in \mathcal{O} : \exists x : U \to x(U) \subseteq \mathbb{R}^d$$

(1) x is invertible

$x^{-1} : x(U) \to U$

(2) x is continuous

U is equip. w/ $\mathcal{O}|_U$

$\mathbb{R}^d$ is equip. w/ $\mathcal{O}_{std}$

(3) x^{-1} is continuous

Examples

(i)

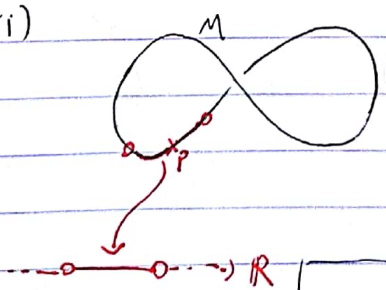
$M \subseteq \mathbb{R}^3$

this is a 2-dim. top. manifold.

Terminology: (U, x) is

- called a chart of $(M, \mathcal{O})$
- $\mathcal{A} = \{(U_\alpha, x_\alpha) \mid \alpha \in A\}$ an atlas if $\bigcup U_\alpha = M$

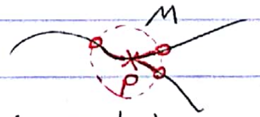
(ii)



$\subseteq (\mathbb{R}^3, \mathcal{O}_{st})$

$\mathcal{O}_{st}|_M$

(iii)



$\subseteq (\mathbb{R}^2, \mathcal{O}_{st})$

$(M, \mathcal{O}_{st}|_M)$ topo. space

But fails to be topo. Mani.

- $x : U \to x(U) \subseteq \mathbb{R}^d$ chart map

$x(p) = (x^1(p), x^2(p), \ldots, x^d(p))$

$x^i : U \to \mathbb{R}$

(U, x) chart

$$\left.\begin{array}{l} x^1 : U \to \mathbb{R} \\ \vdots \\ x^d : U \to \mathbb{R} \end{array}\right\} \Leftrightarrow x : U \to \mathbb{R}^d$$

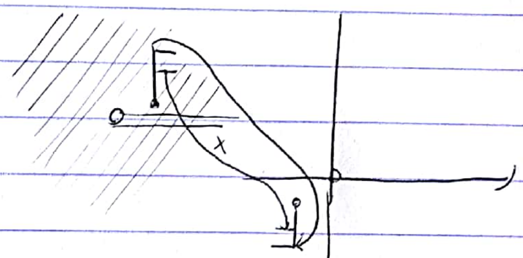
called coordinate maps

Example: $M = \mathbb{R}^2$, $\mathcal{O}_{st}$

$U = \mathbb{R}^2 \setminus \{(a,0)\}$, $a \in \mathbb{R}^+$

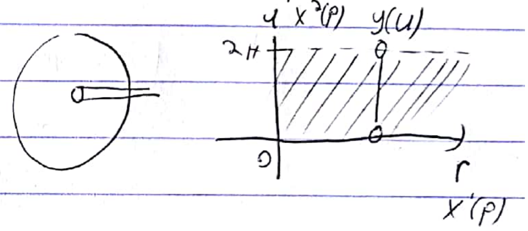
$x : U \to \mathbb{R}$

$(m,n) \longmapsto (-m,-n)$



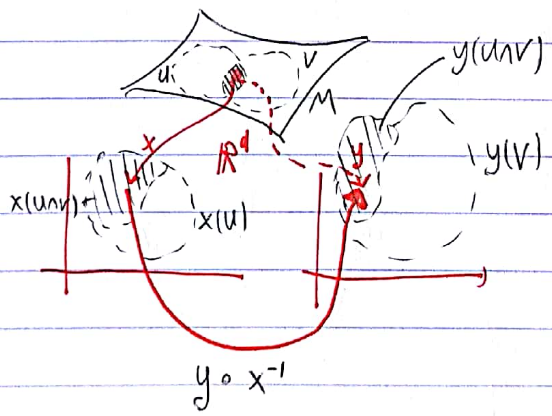
another chart map on U:

$(m,n) \overset{y}{\longmapsto} \left(\sqrt{m^2+n^2}, \arctan\left(\frac{n}{m}\right)\right)$



3. Chart transition maps

Imagine two charts (U, x) and (V, y) with overlapping regions. $U \cap V \neq \emptyset$



Formally:

$$\begin{tikzcd} & U \cap V \arrow[dl, "x"'] \arrow[dr, "y"] & \\ \mathbb{R}^d \supseteq x(U \cap V) \arrow[rr, "y \circ x^{-1}"'] & & y(U \cap V) \subseteq \mathbb{R}^d \end{tikzcd}$$

called the chart transition map.

Informally: contains the instructions how to glue together the charts of the atlas.

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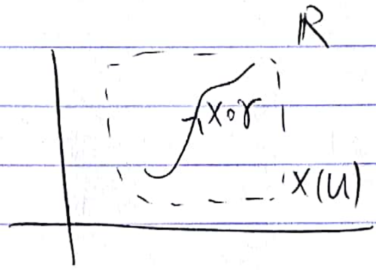
4. Manifold philosophy

Often it is desireable (or indeed the only way) to define properties like continuity of real world objects ("$\mathbb{R} \xrightarrow{\gamma} M$") by judging suitable conditions not on the real world object but on a chart representative of the r-w object.

Advantages:

$$\begin{tikzcd} \gamma : \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & U \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \end{tikzcd}$$



Disadvantage: Maybe ill-defined (because an arbitrarily chosen chart is employed). We need to make sure that the defined property does not change if we afford another "fantasy" (i.e, chart).

Formally:

$$\begin{tikzcd} & y(U) \\ \mathbb{R} \arrow[ur, "y\circ\gamma"] \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma \text{ cont.}"'] & U \arrow[u, dashed, "y"'] \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \end{tikzcd}$$

$$y\circ\gamma = (y\circ x^{-1})\circ x\circ\gamma$$
$$= y\circ(x^{-1}\circ x)\circ\gamma$$

Def: $x\circ\gamma : \mathbb{R} \to \mathbb{R}^d$ diff.

$\Updownarrow$

γ diff