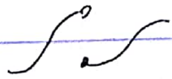
Tensor Calc, Multilinear Algebra and Diff. Geom.

30.1.18

Lecture 1: Topological spaces

@ coarsest level: spacetime is a set

Not enough to talk about continuity of maps.

In classical physics, no jumps. 

weakest structure that can be established on a set which allows a def. of continuity: a topology.

Def. Let M be a set.
A topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ (power set) satisfying

(i) $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$

(ii) $U \in \mathcal{O}, V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$

(iii) $U_\alpha \in \mathcal{O} \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{O}$

α - index set. can be uncountable.

Example. (1) $M = \{1,2,3\}$

(a) $\mathcal{O}_1 = \{\emptyset, \{1,2,3\}\}$ is a topo.

(b) $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1,2,3\}\}$ Not a topo.

(2) M any set.

(a) $M_{chaotic} := \{\emptyset, M\}$

(b) $M_{discrete} := \mathcal{P}(M)$

2.a & 2.b are utterly useless

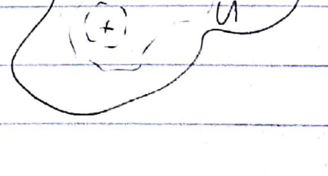
(3) $M = \mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$

Def. $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathbb{R})$

2 steps: soft ball

(α) $B_r(p) := \{(q_1, q_2, \ldots q_d) \mid \sum (q_i - p_i)^2 < r\}$

(β) $U \in \mathcal{O}_{standard}$
$\Updownarrow$
$\forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U$



Terminology

M set

$\mathcal{O}$ topology

$(M, \mathcal{O})$ topological space (a set w/ additional structure)

$U \in \mathcal{O} \Leftrightarrow$ call $U \subseteq M$ an open set

$M \setminus A \Leftrightarrow$ call $A \subset M$ a closed set

⚠ open $\not\Rightarrow$ closed
($\not\Leftarrow$)

2. Continuous Maps

$f : M \to N$

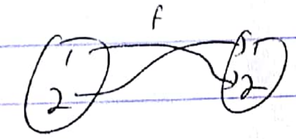
The answer to the q. of whether a map is cont. depends (by def.) on which topology is chosen on M and N.

Def. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topo. spaces.
Then, a map $f: M \to N$ is called <u>continuous</u>
(wrt $\mathcal{O}_N$ and $\mathcal{O}_M$) if

$$\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M$$

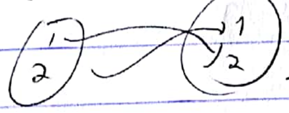
<u>Mnemonic</u>: "A map is cont. iff the preimage of (all) open sets are open sets."

<u>Example</u> (a) $M = \{1, 2\}$ $\quad \mathcal{O}_M = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
$N = \{1, 2\}$ $\quad \mathcal{O}_N = \{\emptyset, \{1,2\}\}$



$\text{preim}(\emptyset) = \emptyset \in \mathcal{O}_M$
$\text{preim}(\{1,2\}) = M \in \mathcal{O}_M$
$\Rightarrow f$ is cont.

(b) $g: N \to M$



$\text{Preim}(\{1\}) = \{1\} \notin \mathcal{O}_N$!
$\Rightarrow$ Not cont

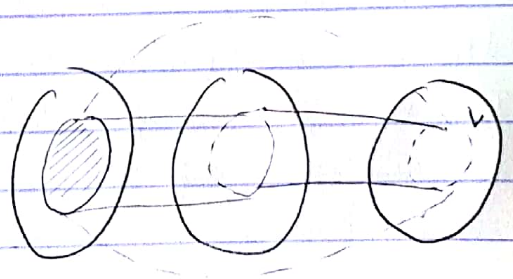
Same map, different topologies.

3. <u>Composition of continuous maps</u>

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "g \circ f"] & N \arrow[r] & P \end{tikzcd}$$

$g \circ f : M \to P$

<u>Key Thm</u>. $\begin{matrix} f \text{ cont.} \\ g \text{ cont.} \end{matrix} \Rightarrow g \circ f$ cont.



<u>Proof</u>. Let $V \in \mathcal{O}_P$

$$\begin{aligned} \text{preim}_{g\circ f}(V) &= \{m \in M \mid (g \circ f)(m) \in V\} \\ &= \{m \in M \mid f(m) \in \text{preim}_g(V)\} \\ &= \underbrace{\text{preim}_f(\underbrace{\text{preim}_g(V)}_{\in \mathcal{O}_N})}_{\mathcal{O}_M} \end{aligned}$$

4. Inherting a topology

There are many useful ways to inherent a topo. from some given topo. space (s).

Important for spacetime physicists :

$$S \subseteq M \quad \mathcal{O}_M$$

Q. can one constract on S a topo from $\mathcal{O}_M$ on M?

Yes. Def. $\mathcal{O}|_S \subset \mathcal{P}(S)$ "subset topo."

$$\mathcal{O}|_S := \{ U \cap S \mid U \in \mathcal{O}_M \}$$

claim: $\mathcal{O}|_S$ is a topo.

(i) $\emptyset = \emptyset \cap S \Rightarrow \emptyset \in \mathcal{O}|_S$

$M = M \cap S \Rightarrow \cancel{M}\ S \in \mathcal{O}|_S$

(ii) $A, B \in S \Rightarrow \exists\, \tilde{A}, \tilde{B} \in \mathcal{O}_M$

$$\left.\begin{array}{l} A = \tilde{A} \cap S \\ B = \tilde{B} \cap S \end{array}\right. \Rightarrow A \cap B = (\tilde{A} \cap S) \cap (\tilde{B} \cap S)$$

$$= \underbrace{(\tilde{A} \cap \tilde{B})}_{\in \mathcal{O}_M} \cap S$$

$$\Rightarrow A \cap B \in \mathcal{O}|_S$$

(iii)

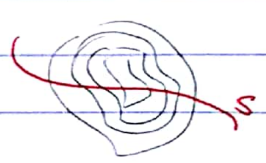
use of this specific way to inharit a topo from a superset :

$$\begin{tikzcd} S \subset M \arrow[r, "f"] \arrow[r, bend left, "f|_S"] & N \\ \mathcal{O}_M \arrow[u, squiggly] \arrow[r, phantom, "\underbrace{\qquad\qquad}_{\text{cont}}"] & \mathcal{O}_N \arrow[u, squiggly] \end{tikzcd} \qquad \mathcal{O}|_S \nearrow S$$

$$\Downarrow$$

$f|_S : S \to N$ is cont.

$$S \subset \mathbb{R}^2 \xrightarrow{T} \mathbb{R} \qquad T|_S$$



Tensor Calculus, Multilinear Algebra, and Diff Geom.

Lecture 2: Manifolds

Topological spaces: there are so many of them that mathematicians cannot even classify them.

For spacetime physics, we may focus on topo. spaces $(M, \mathcal{O})$ that can be charted, analogously to how the surface of the earth is charted.

1. Topological Manifolds

Def. A topo. space $(M, \mathcal{O})$ is called a d-dimensional topo. manifold if

$$\forall p \in M : \exists \underset{\underset{p}{\cap}}{U} \in \mathcal{O} : \exists x : U \to \cancel{\mathbb{R}^m} : x(U) \subseteq \mathbb{R}^d$$

(1) x is invertible

$x^{-1} : x(U) \to U$

(2) x is continuous

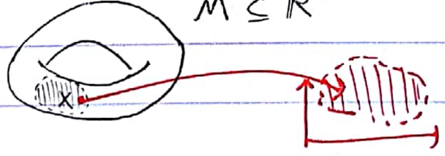
U is equip. w/ $\mathcal{O}|_U$

$\mathbb{R}^d$ is equip. w/ $\mathcal{O}_{std}$

(3) x^{-1} is continuous

Examples

(i) $M \subseteq \mathbb{R}^3$



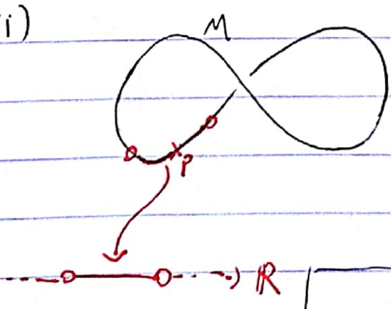
this is a 2-dim. top. manifold.

Terminology: (U, x) is

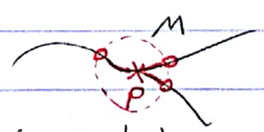
- called a chart of $(M, \mathcal{O})$
- $\mathcal{A} = \{(U_\alpha, x_\alpha) \mid \alpha \in A\}$ an atlas if $\bigcup U_\alpha = M$

(ii) $M \subseteq (\mathbb{R}^3, \mathcal{O}_{st})$

$\mathcal{O}_{st}|_M$



(iii) $M \subseteq (\mathbb{R}^2, \mathcal{O}_{st})$



$(M, \mathcal{O}_{st}|_M)$ topo. space

But fails to be topo. Mani.

- $x : U \to x(U) \subseteq \mathbb{R}^d$ chart map

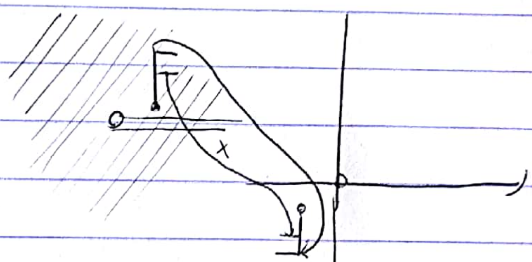
$x(p) = (x^1(p), x^2(p), \ldots, x^d(p))$

$x^i : U \to \mathbb{R}$

(U, x) chart

$$\left.\begin{array}{l} x^1 : U \to \mathbb{R} \\ \vdots \\ x^d : U \to \mathbb{R} \end{array}\right\} \Leftrightarrow x : U \to \mathbb{R}^d$$

called coordinate maps

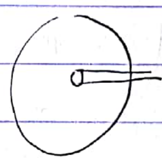


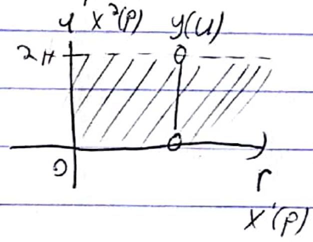
<u>Example</u>: $M = \mathbb{R}^2, \mathcal{O}_{st}$

$U = \mathbb{R}^2 \setminus \{(a, 0)\}, \quad a \in \mathbb{R}^+$

$x : U \to \mathbb{R}$

$(m, n) \longmapsto (-m, -n)$



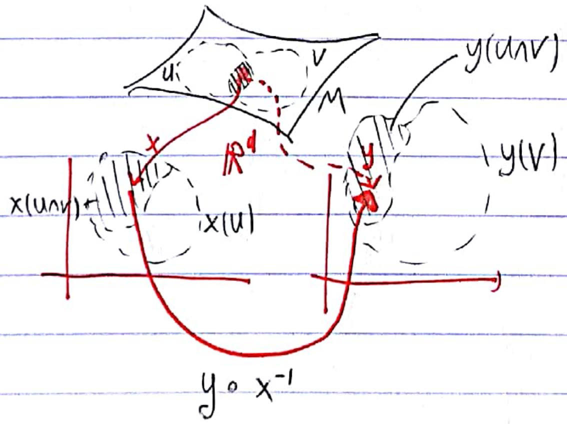


another chart map on U:

$(m, n) \overset{y}{\longmapsto} (\sqrt{m^2+n^2}, \arctan(\frac{n}{m}))$

3. <u>Chart transition maps</u>

Imagine two charts (U, x) and (V, y) with overlapping regions. $U \cap V \neq \emptyset$



<u>Formally</u>:

$$\begin{tikzcd} & U \cap V \arrow[dl, "x"'] \arrow[dr, "y"] & \\ \mathbb{R}^d \supseteq x(U \cap V) \arrow[rr, "y \circ x^{-1}"'] & & y(U \cap V) \subseteq \mathbb{R}^d \end{tikzcd}$$

called the chart transition map.

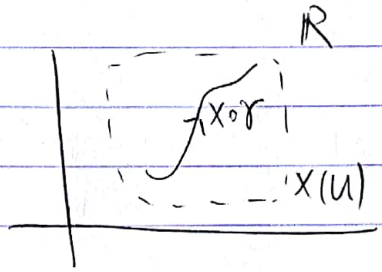
Informally: Contains the instructions how to glue together the charts of the atlas.

4. Manifold philosophy

Often it is desireable (or indeed the only way) to define properties like continuity of real world objects ("$\mathbb{R} \xrightarrow{\gamma} M$") by judging suitable conditions not on the real world object but on a chart representative of the r-w object.

Advantages:

$$\begin{tikzcd} \gamma: \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & U \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \end{tikzcd}$$



Disadvantage: Maybe ill-defined (because an arbitrarily chosen chart is employed). We need to make sure that the defined property does not change if we afford another "fantasy" (i.e., chart).

Formally:

$$\begin{tikzcd} & y(U) \\ \mathbb{R} \arrow[ur, "y \circ \gamma"] \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"', "\text{cont.}"'] & U \arrow[u, dashed, "y"'] \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \end{tikzcd}$$

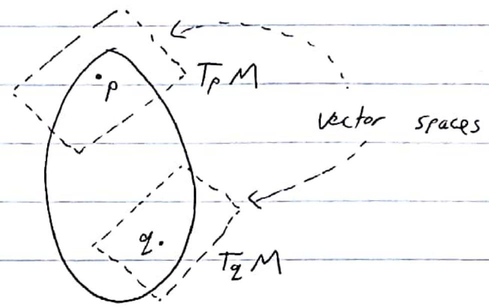
$$y \circ \gamma = (y \circ x^{-1}) \circ x \circ \gamma$$
$$= y \circ (x^{-1} \circ x) \circ \gamma$$

Def: $x \circ \gamma : \mathbb{R} \to \mathbb{R}^d$ diff.
$\Updownarrow$
γ diff

Tensor Calculus, Multilinear Algebra, and Diff. Geom. 11.4.18

Lecture 3: Multilinear Algebra

- We will not equip space (time) with a vector space structure.
  Do you know where $5 \cdot \text{paris} = ?$ $\text{paris} + \text{Vienna} = ?$
  However, the tangent spaces $T_p M$ do.



- Beneficial to first study vector spaces abstractly, for two reasons:
  (i) for construction of $T_p M$ one needs an intermediate vector space $C^\infty(M)$
  (ii) tensor techniques are most easily understood in an abstract setting.

1. Vector spaces

Def. A vector space $(V, +, \cdot)$ is
(i) a set V
(ii) $+ : V \times V \longrightarrow V$ "addition"
(iii) $\cdot : \mathbb{R} \times V \longrightarrow V$ "S-multiplication"

satisfying

C^+ $v + w = w + v$
A^+ $(u+v) + w = u + (v+w)$
N^+ $\exists\, 0 \in V : \forall v \in V : v + 0 = v$
I^+ $\forall v \in V : \exists\, (-v) \in V : v + (-v) = 0$
A $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$
D $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$
D $\lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)$
U $1 \cdot v = v$

Note that some of the op. are on the vector space while some are on $\mathbb{R}$.

Terminology: An element of a vector space is often referred to, informally, as a vector.

• $\mathbb{R}^3$ can be made into a vector space.

<u>Example.</u>

<u>def.</u> <u>set</u> $P := \{ p : (-1, +1) \longrightarrow \mathbb{R} \mid p(x) = \sum_{n=0}^{N} p_n \cdot x^n \}$, $p_n \in \mathbb{R}$

polynomials of (fixed) degree.

Is $\square$ a vector? No. $\square \in P$. But there is no $+, \cdot$ defined.

$\square(x) = x^2$

$+ : P \times P \longrightarrow P$

$(p, q) \longmapsto p + q$

where $(p \underset{\uparrow}{+} q)(x) := p(x) \underset{\uparrow}{+} q(x)$

op. over P op. over $\mathbb{R}$

$\cdot : \mathbb{R} \times P \to P$

$(\lambda, p) \longmapsto \lambda \cdot p$

where $(\lambda \cdot p)(x) := \lambda \cdot p(x)$

Again, is $\square$ a vector? Yes. But who cares? It's a bad Q.

$(P, +, \cdot)$ is a vector space.

$\square \in$

2. <u>Linear maps</u>

In topology, maps respected the "open set" structure.

These are the structure respecting maps between vector spaces:

<u>Def.</u> $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ vector spaces

then a map

$$\varphi : V \longrightarrow W$$

is called linear if

(i) $\varphi(v + \tilde{v}) = \varphi(v) + \varphi(\tilde{v})$

(ii) $\varphi(\lambda \cdot v) = \lambda \cdot \varphi(v)$

Example: Diff. operator

$$\delta : P \longrightarrow P$$
$$p \longmapsto \delta(p) := p'$$

Why is it linear?

(i) $\delta(p+q) = (p+q)' = p' + q' = \delta(p) + \delta(q)$

(ii) $\delta(\lambda \cdot p) = (\lambda \cdot p)' = \lambda \cdot p' = \lambda \cdot \delta(p)$

Notation: $\varphi : V \to W$ linear $\Leftrightarrow$: $\varphi : V \xrightarrow{\sim} W$

Theorem:

$$\begin{tikzcd} V \arrow[r, "\psi", "\sim"'] \arrow[rr, bend right, "\sim", "\varphi \circ \psi"'] & W \arrow[r, "\varphi", "\sim"'] & U \end{tikzcd}$$

- The composition of linear maps is linear.
- In order to truly understand what's going on you must not introduce a basis. It clouds everything.

Example: $\delta \circ \delta : P \xrightarrow{\sim} P$

3. Vector Space of Homomorphisims

fun fact: $(V, +, \cdot)$ $(W, +, \cdot)$ vector spaces

def. $\{\varphi : V \xrightarrow{\sim} W\}$ all linear maps
"$\mathrm{Hom}(V, W)$"

We can make this into a vector space

$$\oplus : \mathrm{Hom}(V,W) \times \mathrm{Hom}(V,W) \longrightarrow \mathrm{Hom}(V,W)$$
$$(\varphi, \psi) \longmapsto \varphi \oplus \psi$$
where $(\varphi \oplus \psi)(v) := \varphi(v) + \psi(v)$

$\odot$: ... similarly

$\mathrm{Hom}(V,W), \oplus, \odot$ is a vector space.

__Example:__ $\mathrm{Hom}(P,P)$ is a vector space.

$\delta \in \mathrm{Hom}(P,P)$

$\delta \circ \delta \in \mathrm{Hom}(P,P)$

$\underbrace{\delta \circ \dots \circ \delta}_{m} \in \mathrm{Hom}(P,P)$

$\Rightarrow \quad 5 \odot \delta \oplus \delta \circ \delta \in \mathrm{Hom}(P,P)$

4. __Dual Vector Space__

Heavily used special case:

$(V,+,\cdot)$ vector space:

__Def.__ $V^* := \{ \varphi : V \xrightarrow{\sim} \mathbb{R} \} = \mathrm{Hom}(V, \mathbb{R})$

If you know V and V^* you can construct all the rest.

$$\underbrace{(V^*, \oplus, \odot)}_{\text{dual vector space (to } V)} \text{ is a vector space.}$$

__Terminology:__ an element $\varphi \in V^*$ is called, informally, a covector.

__Example:__ $I : P \xrightarrow{\sim} \mathbb{R}$

i.e., $I \in P^*$

__def.__ $I(p) := \int_0^1 dx\, p(x)$

linear: $I(p+q) = \int_0^1 dx\, (p+q)(x) = \dots = I(p) + I(q)$

$I(\lambda p) = \lambda I(p)$

That is, $I = \int_0^1 dx$ is a covector. An element in P^*.

5. __Tensors__

__Def.__ Let $(V, +, \cdot)$ be a vector space.
An (r,s)- tensor, T, over V
is a multilinear map

$$T : \underbrace{V^* \times \ldots \times V^*}_{r} \times \underbrace{V \times \ldots \times V}_{s} \xrightarrow{\sim\}r+s} \mathbb{R}$$

~~new~~

__Example__ : T $(1,1)$-tensor

$\cancel{T(\varphi, v)}$

$$T(\varphi + \psi, v) = T(\varphi, v) + T(\psi, v)$$
$$T(\lambda\varphi, v) = \lambda T(\varphi, v)$$

$$T(\varphi, v+w) = T(\varphi, v) + T(\varphi, w)$$
$$T(\varphi, \lambda v) = \lambda T(\varphi, v)$$

__Example__ : ~~Th~~

$$T(\varphi + \psi, v+w) = T(\varphi, v+w) + T(\psi, v+w)$$
$$= T(\varphi, v) + T(\varphi, w) + T(\psi, v) + T(\cancel{\varphi}\psi, w)$$

__Excercise__ : Given $T : V^* \times V \xrightarrow{\sim} \mathbb{R}$ that is multilinear

def. $\phi_T : V \xrightarrow{\sim} (V^*)^* = V$ (if $\dim V < \infty$)

$$v \longmapsto \underbrace{T(\cdot, v)}_{V^* \xrightarrow{\sim} \mathbb{R}}$$

remember that
$V^* = \mathrm{Hom}(V, \mathbb{R})$

Given $\phi : V \xrightarrow{\sim} V$

can construct $T_\phi : V^* \times V \xrightarrow{\sim} \mathbb{R}$
$$(\varphi, v) \longmapsto \varphi(\phi(v))$$

$\Rightarrow$ given T : $T = T_{\phi_T}$

given ϕ : $\phi = \phi_{T_\phi}$ (wtf?)

Example: $g : P \times P \xrightarrow{\sim} \mathbb{R}$

$(p, q) \longmapsto \int_{-1}^{1} dx \, p(x) \, q(x)$ (inner product)

a $(0,2)$-tensor over P.

~~Info: $T \in \mathrm{Hom}($~~

6. Vectors and Covectors as tensors

Theorem. (inc. proof) $\varphi \in V^* \Leftrightarrow \varphi : V \xrightarrow{\sim} \mathbb{R} \Leftrightarrow \varphi$ is a $(0,1)$-tensor

"covector"

Theorem. $v \in V \; \cancel{\ldots} \; \underset{(\dim V < \infty)}{\overset{\sim}{=}} (V^*)^* \Leftrightarrow v : V^* \xrightarrow{\sim} \mathbb{R}$

$\Leftrightarrow v$ is $(1,0)$-tensor

All of this has been said without talking about a basis or of vectors as a collection of numbers. If you did not see the numbers then you did well.

7. Bases

Def. $(V, +, \cdot)$ vector space.

A subset $B \subset V$ is called a basis if

$\forall v \in V \; \exists! \text{ finite } \underset{\{f_1, \ldots, f_n\}}{F} \subset B : \underset{\text{unique}}{\exists!} \underbrace{v^1, v^2, \ldots, v^n}_{\in \mathbb{R}} : v = v^1 f_1 + \ldots + v^n f_n.$

There is another notion of a basis where F can be infinite. For that you need additional structure (a topology).

Def. If $\exists$ basis B with finitely many elements, say d many, then we call d the dim of the vector space. $d := \dim V$.

This is well defined.

Remark: $(V, +, \cdot)$ be a finite dim vector space. Having chosen a basis $e_1, \ldots, e_n$ of $(V, +, \cdot)$, we may uniquely associate

$$V \xrightarrow{\text{chosen basis}} (v^1, \dots, v^n)$$

called the components of V w.r.t the chosen basis

where $\quad v^1 e_1 + \dots + v^n e_n = V$

So, you may introduce a basis but it has a cost. Everything will depend on it now.

Basis for the dual space

choose basis $e_1, \dots, e_n$ for V

can choose basis $\epsilon^1, \dots, \epsilon^n$ for V^* with no relation to the basis of V.

However, more economical to require:

once $e_1, \dots, e_n$ on V has been chosen, then ~~required~~

$$\epsilon^a(e_b) = \delta_b^a = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

this uniquely determines choice of $\epsilon^1, \dots, \epsilon^n$ from the choice of $e_1, \dots, e_n$

Def. if a basis $\epsilon^1, \dots, \epsilon^n$ of V^* satisfie this, it is called the <u>dual basis</u> (of the dual space).

This way, we keep the number of choices down.

Example: P $\quad (N = 3)$

e_0, e_1, e_2, e_3 basis if

$$\left.\begin{aligned} e_0(x) &= 1 \\ e_1(x) &= x \\ e_2(x) &= x^2 \\ e_3(x) &= x^3 \end{aligned}\right\} e_a(x) = x^a$$

What is the dual basis?

$\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3$ dual basis $\quad \epsilon^a := \frac{1}{a!} \partial^a \Big|_{x=0}$

Proof: $\epsilon^a(e_b) = \delta_b^a$

8. Components of Tensors

and $e_1, \dots, e_n$ basis of V

Def. Let T be an (r,s)-tensor over a finite dim vector space V. Then define the ~~(r+s) dim V num~~ $(r+s)^{\dim V}$ many real numbers

$i_1, \dots, i_r, j_1, \dots, j_s \in \{1, \dots, \dim V\}$

$$\underbrace{T^{i_1, \dots, i_r}{}_{j_1, \dots, j_s}}_{\in \mathbb{R}} := T(\epsilon^{i_1}, \dots, \epsilon^{i_r}, e_{j_1}, \dots, e_{j_s})$$

Components of the tensor w.r.t the chosen basis

Why is this useful? Knowing the components one can reconstruct the tensor.

Example: T $(1,1)$-tensor

$$T^i{}_j := T(\epsilon^i, e_j)$$

to reconstruct,

$$T(\varphi, v) = T\Big(\sum_{i=1}^{\dim V} \underset{\mathbb{R}}{\underset{\cap}{\varphi_i}} \epsilon^i, \sum_{j=1}^{\dim V} \underset{\mathbb{R}}{\underset{\cap}{v^j}} e_j\Big)$$

$$= \sum_{i=1}^{\dim V} \sum_{j=1}^{\dim V} \varphi_i v^j \underbrace{T(\epsilon^i, e_j)}_{T^i{}_j}$$

$$=: \varphi_i v^j T^i{}_j$$

(fuck this notation)