Aalto University
School of Electrical
Engineering

ELEC-E8125 Reinforcement learning
Function approximation

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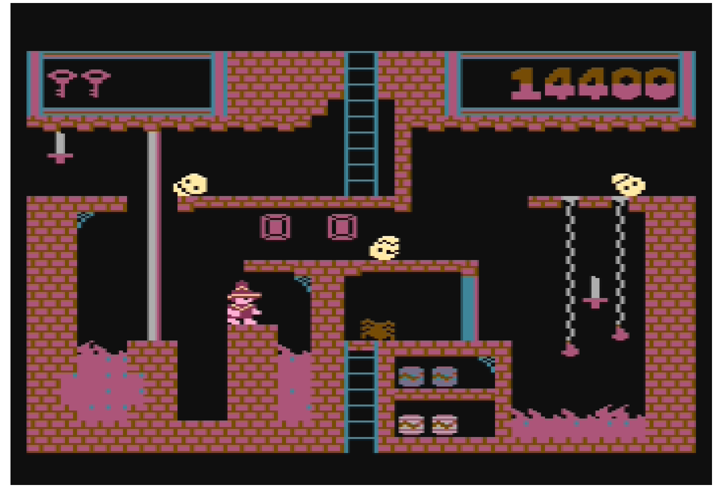
Today

- Function approximation for reinforcement learning

Learning goals

- Understand basis and limitations of value function approximation
- Understand incremental and batch approaches

Motivation

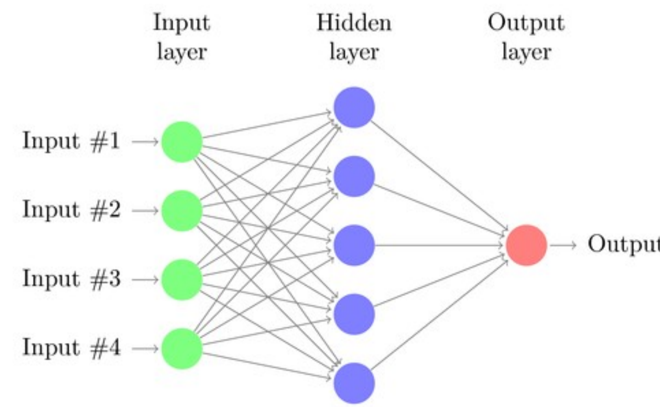


- How to solve problems with large state spaces?
- For example:
  - Backgammon: ~10^{20} states
  - Helicopter: continuous state space → infinite number of possible states https://www.youtube.com/watch?v=M-QUkgk3HyE

- Value of each state can not be stored in memory
- It is difficult to collect enough experience (too slow to learn each state independently)

Any other choices to represent V, Q?

Parametric regression models!



Neural network

Value function approximation

- Idea: Represent value function as a parametric approximation

vector

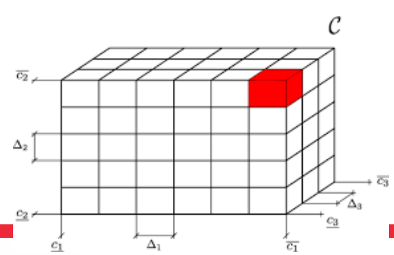
$$\hat{V}(s,\boldsymbol{\theta}), \hat{Q}(s,a,\boldsymbol{\theta})$$

- Function approximator types:
  - Generalized linear $\hat{V}(s,\boldsymbol{\theta})=\boldsymbol{\theta}^T\boldsymbol{\varphi}(s)$ $\hat{Q}(s,a,\boldsymbol{\theta})=\boldsymbol{\theta}^T\boldsymbol{\varphi}(s,a)$
  - General parametric (neural network)
  - Non-differentiable ones
    - e.g. decision tree, tiling

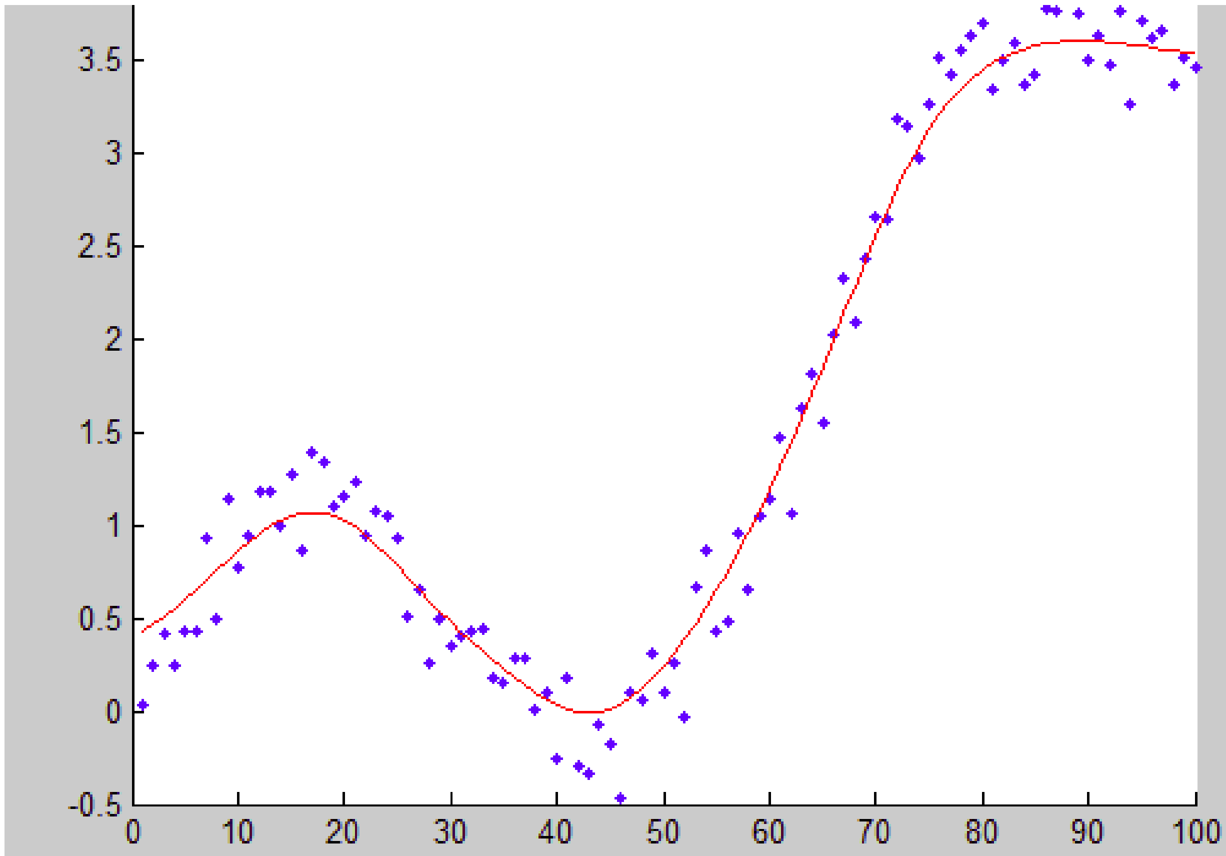
Features, for example
Radial basis function

$$\varphi_i(s)=e^{(s-s_i)^T\Sigma^{-1}(s-s_i)}$$

Tiling (grid)
Polynomial basis

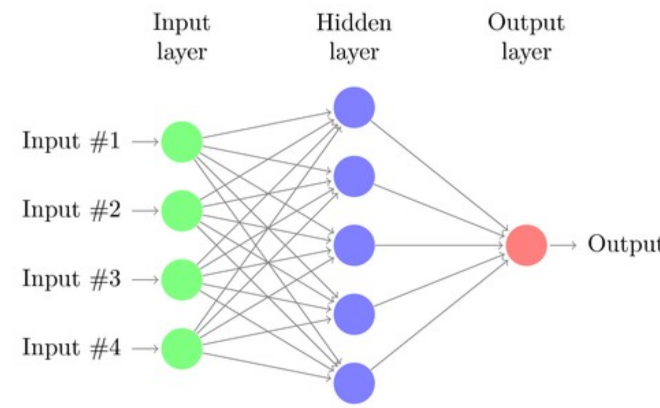


Example: Locally weighted regression



Yunyuongmok, 2014

Parametric regression models!



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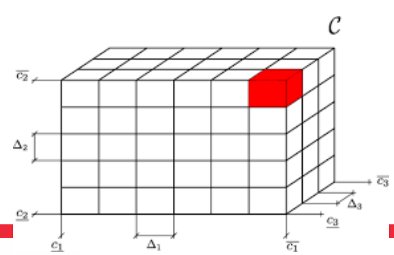

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How to *optimize* θ? Which criterion?

Stochastic gradient descent

- Idea: Minimize mean-squared error (MSE)

$$J(\boldsymbol{\theta}) = E\left[\left(V_{\pi}(s) - \hat{V}(s, \boldsymbol{\theta})\right)^2\right]$$

- Gradient descent update

Remember: $\boldsymbol{\theta}_{i+1} = \boldsymbol{\theta}_i + \Delta\boldsymbol{\theta}$

$$\Delta\boldsymbol{\theta} = -\frac{1}{2}\alpha \nabla_{\theta} J(\boldsymbol{\theta})$$

Let's simplify!

- *Stochastic gradient descent* samples update

$$\Delta\boldsymbol{\theta} = \alpha\left(V_{\pi}(s) - \hat{V}(s, \boldsymbol{\theta})\right)\nabla_{\theta}\hat{V}(s, \boldsymbol{\theta})$$

Where to get the target value?

Estimate state-value function

Incremental prediction

- MC:

$$\Delta \boldsymbol{\theta} = \alpha \left(G_t - \hat{V}(s_t, \boldsymbol{\theta}) \right) \nabla_{\theta} \hat{V}(s_t, \boldsymbol{\theta})$$

Remember discrete TD(0): $V(s_t) = V(s_t) + \alpha \left(r_t + \gamma V(s_{t+1}) - V(s_t) \right)$

- TD(0):

$$\Delta \boldsymbol{\theta} = \alpha \left(r_t + \gamma \hat{V}(s_{t+1}, \boldsymbol{\theta}) - \hat{V}(s_t, \boldsymbol{\theta}) \right) \nabla_{\theta} \hat{V}(s_t, \boldsymbol{\theta})$$

- TD(λ):

$$\Delta \boldsymbol{\theta} = \alpha \boldsymbol{E_t} \left(r_t + \gamma \hat{V}(s_{t+1}) - \hat{V}(s_t) \right)$$
$$\boldsymbol{E_t} = \gamma \lambda \boldsymbol{E_{t-1}} + \nabla_{\theta} \hat{V}(s_t, \boldsymbol{\theta})$$

vector

(Generalized) Linear function approximation

- Linear Monte-Carlo policy evaluation

$$\Delta\boldsymbol{\theta} = \alpha\left(G_t - \hat{V}(s_t, \boldsymbol{\theta})\right)\nabla_\theta \hat{V}(s_t, \boldsymbol{\theta}) \longleftarrow \text{What is the gradient?}$$

$$= \alpha\left(G_t - \hat{V}(s_t, \boldsymbol{\theta})\right)\boldsymbol{\varphi}(s_t)$$

  - Converges to local optimum

- Linear TD(0)

$$\Delta\boldsymbol{\theta} = \alpha\left(r_t + \gamma\hat{V}(s_{t+1}, \boldsymbol{\theta}) - \hat{V}(s_t, \boldsymbol{\theta})\right)\boldsymbol{\varphi}(s_t)$$

  - Converges on-policy to local optimum

- Linear TD(λ)

$$\boldsymbol{E}_t = \gamma\lambda\boldsymbol{E}_{t-1} + \boldsymbol{\varphi}(s_t)$$

Convergence of prediction - theoretical results

	Algorithm	Discrete	Linear	Non-linear
On-policy	MC	+	+	+
	TD(0)	+	+	-
	TD(λ)	+	+	-
Off-policy	MC	+	+	+
	TD(0)	+	-	-
	TD(λ)	+	-	-

Convergence not guaranteed

Gradient TD (non-linear GTD2, Maei 2009) converges off-policy with non-linear function approximation.

Incremental control

- Approach
  - Approximate policy evaluation for $\hat{Q}(s, a, \boldsymbol{\theta})$
  - ε-greedy policy improvement
- Policy evaluation for *Q* similar to *V*
  - MC, TD
- SARSA and Q-learning also possible

Approximation for action-value function

- Minimize MSE for $\hat{Q}(s, a, \boldsymbol{\theta})$
- MC

$$\Delta\boldsymbol{\theta} = \alpha\left(G_t - \hat{Q}(s_t, a_t, \boldsymbol{\theta})\right)\nabla_\theta \hat{Q}(s_t, a_t, \boldsymbol{\theta})$$

- TD(0) / SARSA

$$\Delta\boldsymbol{\theta} = \alpha\left(r_t + \gamma\hat{Q}(s_{t+1}, a_{t+1}, \boldsymbol{\theta}) - \hat{Q}(s_t, a_t, \boldsymbol{\theta})\right)\nabla_\theta \hat{Q}(s_t, a_t, \boldsymbol{\theta})$$

- TD(λ) / SARSA(λ)

$$\Delta\boldsymbol{\theta} = \alpha \boldsymbol{E}_t\left(r_t + \gamma\hat{Q}(s_{t+1}, a_{t+1}) - \hat{Q}(s_t, a_t)\right)$$

Convergence properties

Algorithm	Discrete	Linear	Non-linear
MC	+	(+)	-
SARSA	+	(+)	-
Q-learning	+	-	-

GQ(λ) (Maei & Sutton, 2010) convergent off-policy learning.

Batch prediction

- Sample efficiency important when few samples
- Batch methods find single best fit for given data
- One approach: Experience replay + stochastic gradient descent
  - Given data *D*, sample (state *s*, value *V(s)*) randomly and apply stochastic gradient descent update, repeat

$$\Delta \boldsymbol{\theta} = \alpha \left(V_{\pi}(s) - \hat{V}(s, \boldsymbol{\theta}) \right) \nabla_{\theta} \hat{V}(s, \boldsymbol{\theta})$$

  - Converges to least-squares solution

Not very efficient, convergence can take a long time.

Linear Least Squares for prediction

- With linear approximation, closed form solution available
- LSMC

$$E[\Delta \boldsymbol{\theta}] = \sum_{t=1}^{T} \alpha \left(G_t - \hat{V}(s_t, \boldsymbol{\theta}) \right) \boldsymbol{\varphi}(s_t) = 0$$

Solve!

$$\boldsymbol{\theta} = \left(\sum_{t=1}^{T} \boldsymbol{\varphi}(s_t) \boldsymbol{\varphi}(s_t)^T \right)^{-1} \sum_{t=1}^{T} \boldsymbol{\varphi}(s_t) G_t$$

- LSTD

$$\boldsymbol{\theta} = \left(\sum_{t=1}^{T} \boldsymbol{\varphi}(s_t) \left(\boldsymbol{\varphi}(s_t) - \gamma \boldsymbol{\varphi}(s_{t+1}) \right)^T \right)^{-1} \sum_{t=1}^{T} \boldsymbol{\varphi}(s_t) r_t$$

- LSTD(λ)

$$\boldsymbol{\theta} = \left(\sum_{t=1}^{T} \boldsymbol{E}_t \left(\boldsymbol{\varphi}(s_t) - \gamma \boldsymbol{\varphi}(s_{t+1}) \right)^T \right)^{-1} \sum_{t=1}^{T} \boldsymbol{E}_t r_t$$

LSTDQ + LSPI

- Off-policy batch evaluation: LSTDQ

$$\theta = \left(\sum_{t=1}^{T} \varphi(s_t, a_t) \left(\varphi(s_t, a_t) - \gamma \varphi(s_{t+1}, \pi(s_{t+1})) \right)^T \right)^{-1} \sum_{t=1}^{T} \varphi(s_t, a_t) r_t$$

- Update policy to greedy

$$\pi(s) = arg\,max_a \hat{Q}(s, a)$$

- Repeat until (approximate) convergence

Convergence of control

Algorithm	Discrete	Linear	Non-linear
MC control	+	(+)	-
SARSA	+	(+)	-
Q-learning	+	-	-
LSPI	+	(+)	

https://www.youtube.com/watch?v=V1eYniJ0Rnk

Example: Deep Q networks (Atari games, Mnih 2013, 2015)

- Learn $Q(s,a)$ directly from pixels
  - Inputs: image (pixels), joystick/button position
  - Output: value
- Reward = change in score
- Approximate $Q(s,a)$ using a deep neural network
- ε-greedy policy
- Experience replay, optimize Q-network in least squares sense using a stochastic gradient descent variant

Algorithm 1: deep Q-learning with experience replay.

Initialize replay memory D to capacity N

Initialize action-value function Q with random weights θ

Initialize target action-value function $\hat{Q}$ with weights $\theta^{-} = \theta$

For episode $= 1, M$ **do**

  Initialize sequence $s_1 = \{x_1\}$ and preprocessed sequence $\phi_1 = \phi(s_1)$

  For $t = 1$,T **do**

    With probability ε select a random action a_t

    otherwise select $a_t = \mathrm{argmax}_a Q(\phi(s_t), a; \theta)$

    Execute action a_t in emulator and observe reward r_t and image x_{t+1}

    Set $s_{t+1} = s_t, a_t, x_{t+1}$ and preprocess $\phi_{t+1} = \phi(s_{t+1})$

    Store transition $(\phi_t, a_t, r_t, \phi_{t+1})$ in D

    Sample random minibatch of transitions $(\phi_j, a_j, r_j, \phi_{j+1})$ from D

    Set $y_j = \begin{cases} r_j & \text{if episode terminates at step j+1} \\ r_j + \gamma \max_{a'} \hat{Q}(\phi_{j+1}, a'; \theta^{-}) & \text{otherwise} \end{cases}$

    Perform a gradient descent step on $\left(y_j - Q(\phi_j, a_j; \theta)\right)^2$ with respect to the network parameters θ

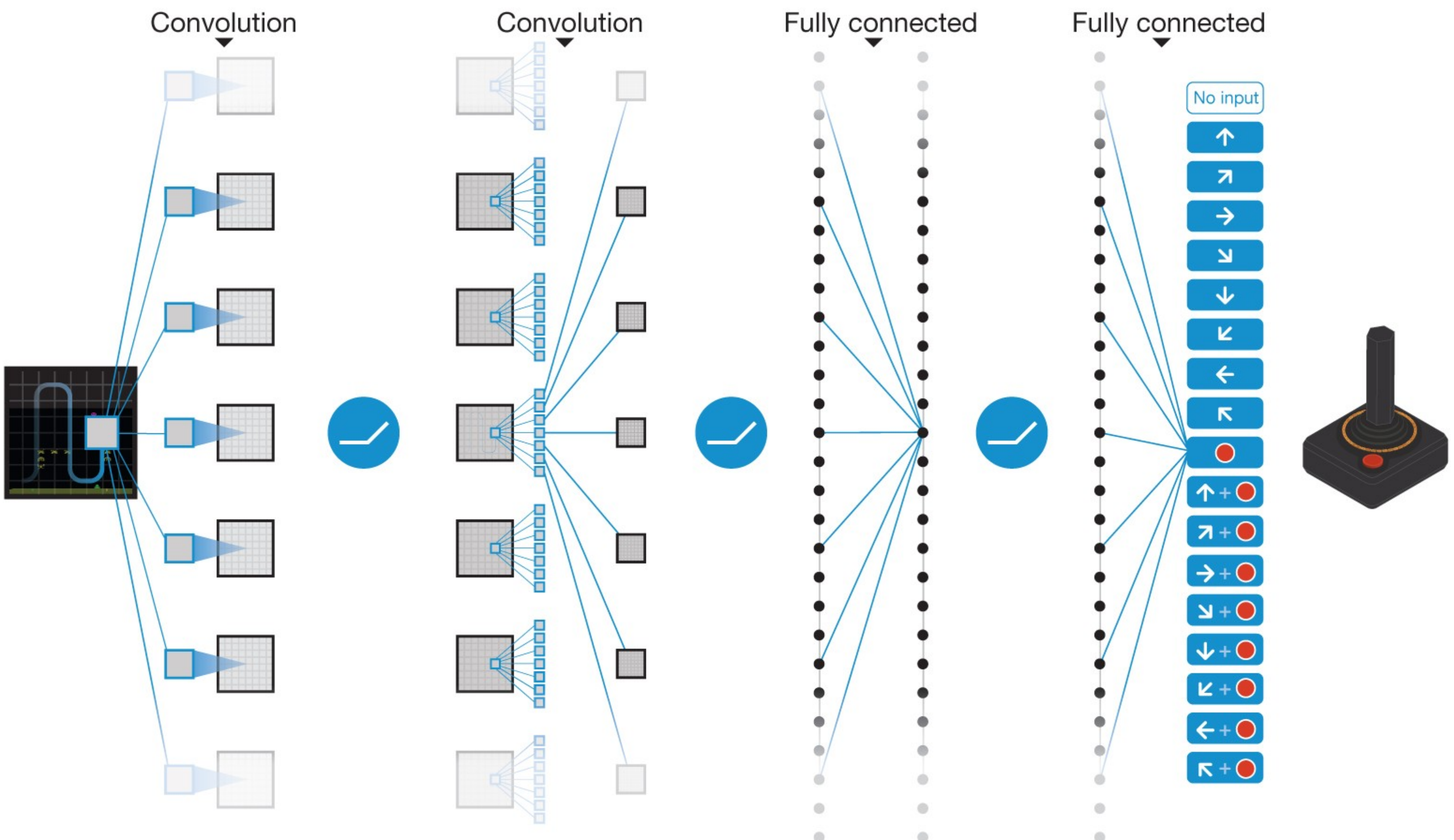
    Every C steps reset $\hat{Q} = Q$

  End For

End For

Mnih et al. Nature, 2015

Schematic illustration of the convolutional neural network



V Mnih *et al.* *Nature* **518**, 529-533 (2015) doi:10.1038/nature14236

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Summary

- Value function approximation for large and continuous state-spaces
- Convergence can be tricky especially for non-linear or off-policy cases

Still limited action spaces. We'll later take a look at optimal control that works in continuous action space.

Next: Policy gradient and actor-critic approaches

- Do we need value functions?
  - Can we parameterize and optimize a policy directly?

- Readings
  - Sutton & Barto Ch 13 - 13.3