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ELEC-E8103 Modelling, Estimation and Dynamic Systems

Analysis of Dynamic Systems

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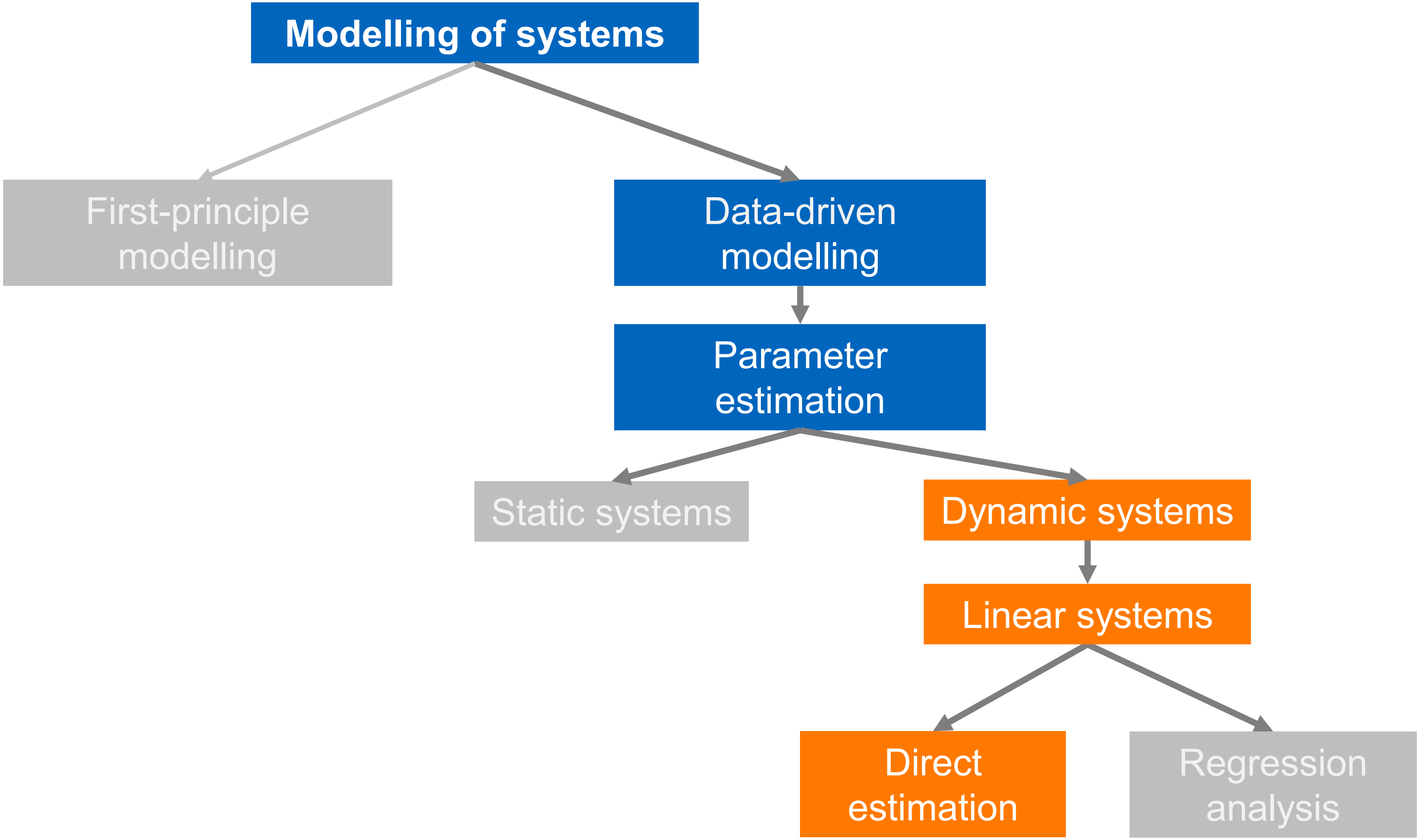
Overview



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Arrangement of lectures

- This week
  - Direct analysis of dynamic systems (today)
  - System identification (tomorrow)
- Next week
  - System identification toolbox

Learning goals

Course Learning Outcomes

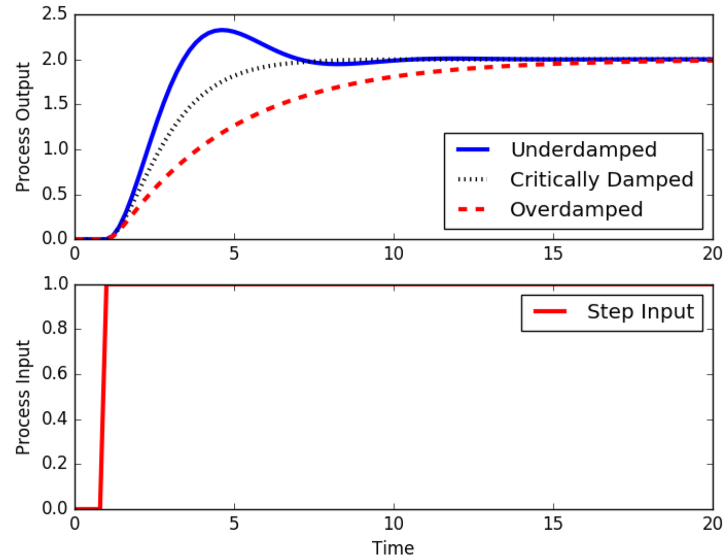
- Select proper modeling approach for specific practical problems,
- Formulate mathematical models of physical systems,
- Construct models of systems using modeling tools such as MATLAB and Simulink,
- Estimate the parameters of linear and nonlinear static systems from measurement data,
- Identify the models of linear dynamic systems from measurement data

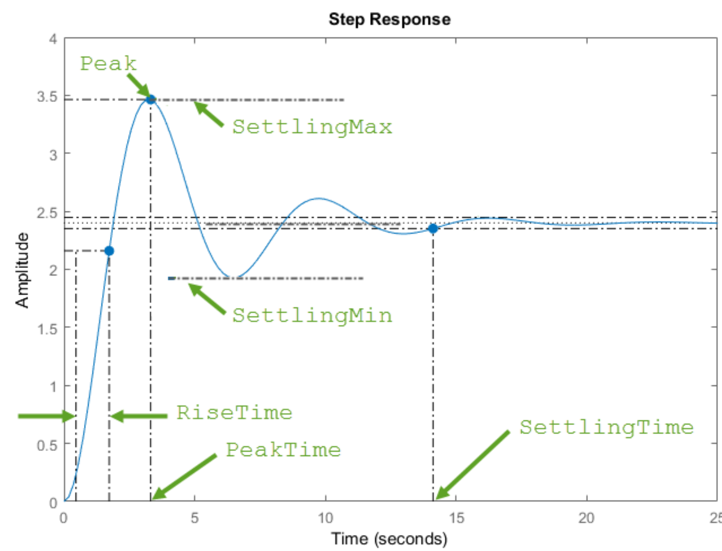
- Transient analysis
- Review the basics of stochastic process and correlation
- Correlation analysis
- Frequency analysis
- Spectral analysis

Transient analysis

Transient analysis

- Goal
  - To understand the relationship between variables and their time relationship
- Methods
  - Step response
  - Impulse response
- We get important information for modelling and simulation
  - The time delay and constants of the system
    - Rise time
      - Overdamped: 10%-90%,
      - Critically damped:5%- 95%,
      - Under damped:0%-100%
    - Settling time (to 5% or 2% of the final value)
    - Time constant (for 1st ordered system, 36.8% for decaying or 63.2% for rising signals)
  - The characteristics of the response
    - Oscillatory, under damped, over damped, …
    - Static gain, natural frequency, damping ratio





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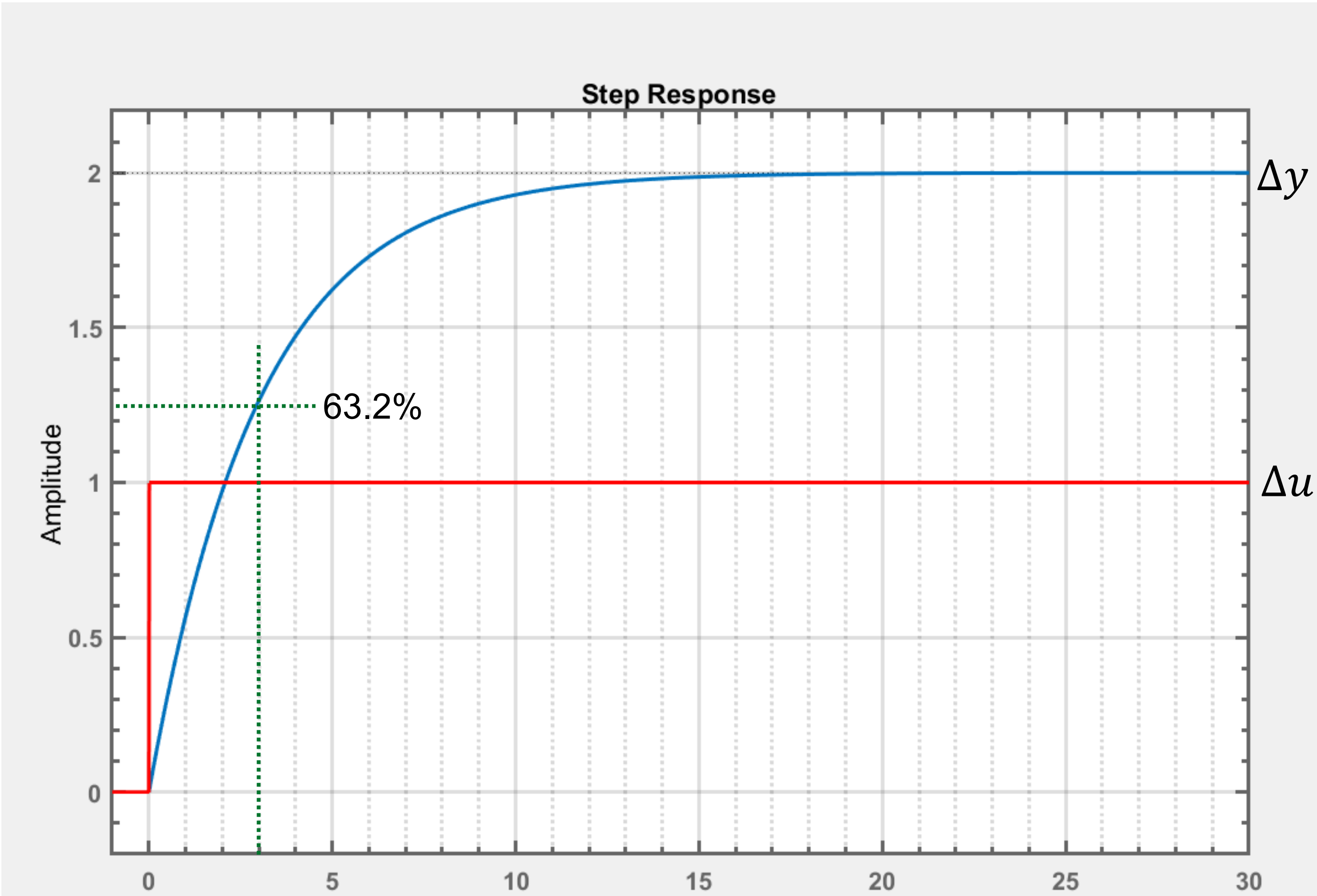
First order linear dynamic system



static gain

$$K = \frac{\Delta y}{\Delta u} = 2$$

time constant $\tau = 3$

$$G(s) = \frac{K}{\tau s + 1} = \frac{2}{3s + 1}$$

$$\tau \dot{y} + y = Ku$$

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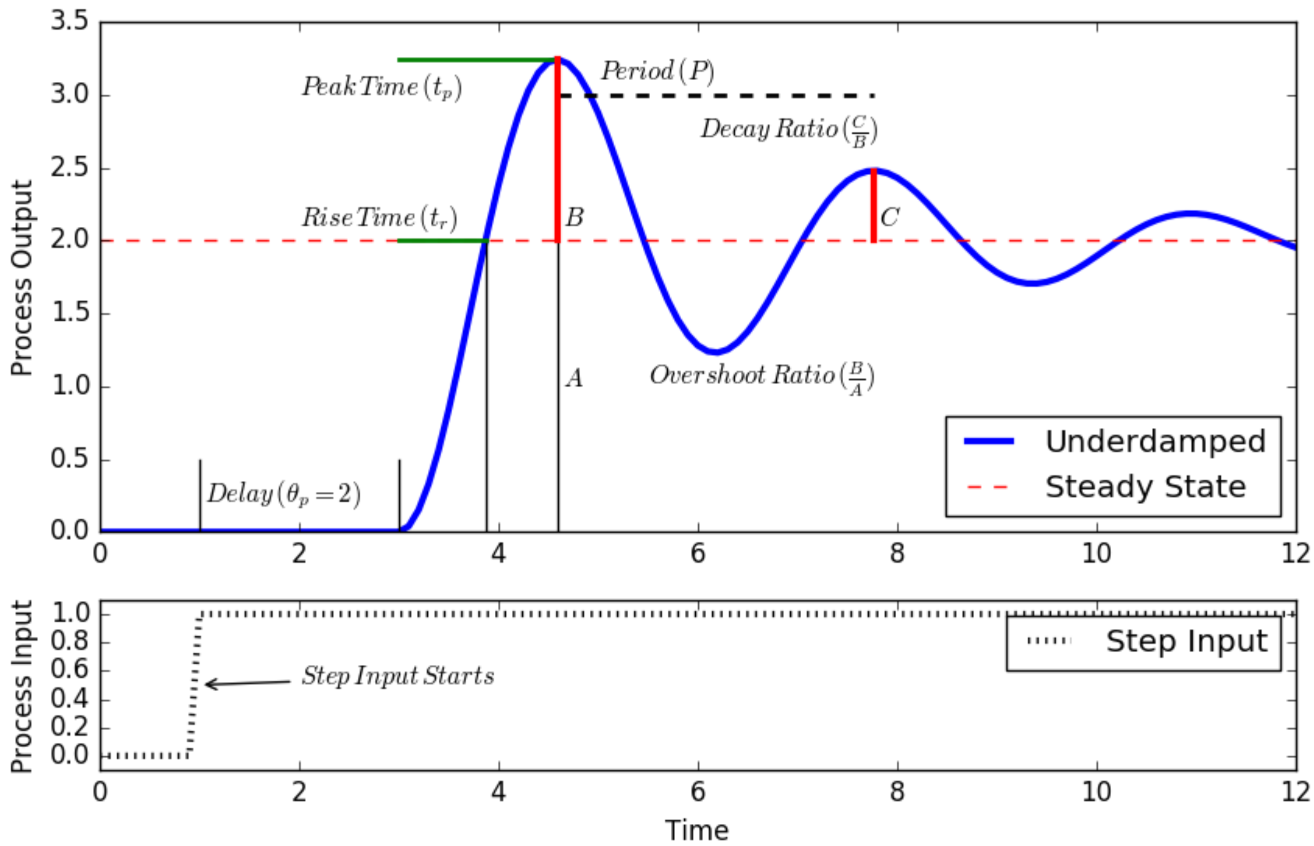
2nd order linear dynamic system

$$\tau_s^2 \frac{d^2y}{dt^2} + 2\zeta\tau_s \frac{dy}{dt} + y = K_p\, u\,(t - \theta_p)$$

K_p: Gain, ζ: damping coefficient, τ_s: 2nd order time constant = $1/\omega_n$, θ_p: delay



$$K_p = \frac{\Delta y}{\Delta u}$$

$$OS = B/A$$

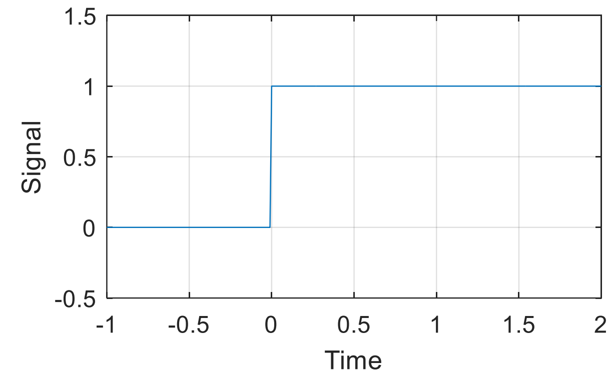
$$\zeta = \sqrt{\frac{(\ln(OS))^2}{\pi^2 + (\ln(OS))^2}}$$

$$\tau_s = \frac{\sqrt{1-\zeta^2}}{2\pi} P$$

Mathematical definition of input signals

- Step signal

$$u(t) = \begin{cases} a, t > 0 \\ 0, t \leq 0 \end{cases}$$



- Impulse signal

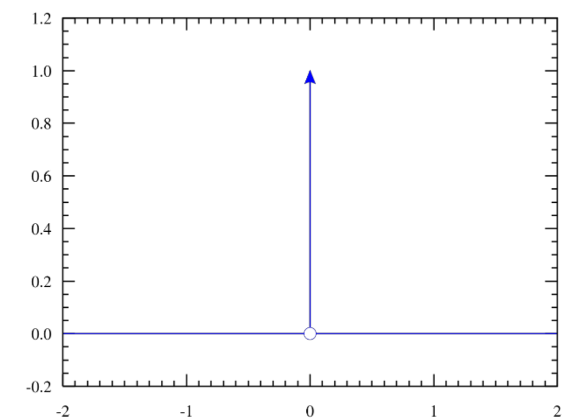
$$u(t) = \delta(t)$$

Dirac delta function, for continuous time system

$$u(t) = \begin{cases} +\infty, t = 0 \\ 0, t \neq 0 \end{cases}$$

Kronecker delta function, for discrete time system

$$u(t) = \begin{cases} 1, t = 0 \\ 0, t \neq 0 \end{cases}$$



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$$\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$$

Impulse response

- It describes the output of a LTI system when it is subjected to an impulse signal:

System

Time series model of the system

$$y(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)d\tau = g(t)$$

Response

If $u(t) = \delta(t)$

Input signal

for both continues or discrete cases.

Transient analysis

- **Goal**: using step/impulse response to estimate the transfer function of a system
- The response of a system

$$y(t) = \int_{\tau=-\infty}^{\infty} g(\tau)u(t-\tau)d\tau + v(t)$$

where $g(\tau)$ is the system and $v(t)$ is noise or disturbance.

- In discrete form

$$y(t) = \sum_{k=-\infty}^{+\infty} g(k)u(t-k) + v(t)$$

- Using impulse input

$$u(t) = \begin{cases} \alpha, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

- We have the response

$$y(t) = \alpha g(t) + v(t)$$

- So the estimated transfer function is

$$\hat{g}(t) = \frac{y(t)}{\alpha}$$

- We can also use a step input

$$u(t) = \begin{cases} \alpha, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$u(t-k) = 0$ if $t < k$

$$y(t) = \alpha \sum_{k=0}^{t} g(k) + v(t)$$

- So the estimated transfer function is

$$\hat{g}(t) = \frac{y(t) - y(t-1)}{\alpha}$$

- The estimation is subject to large error, but good for estimate system properties

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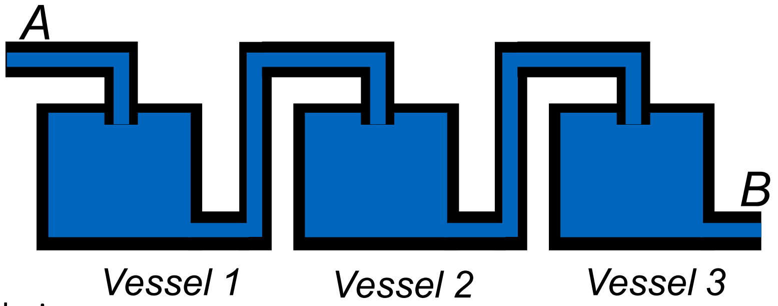
Example: Tank dynamics

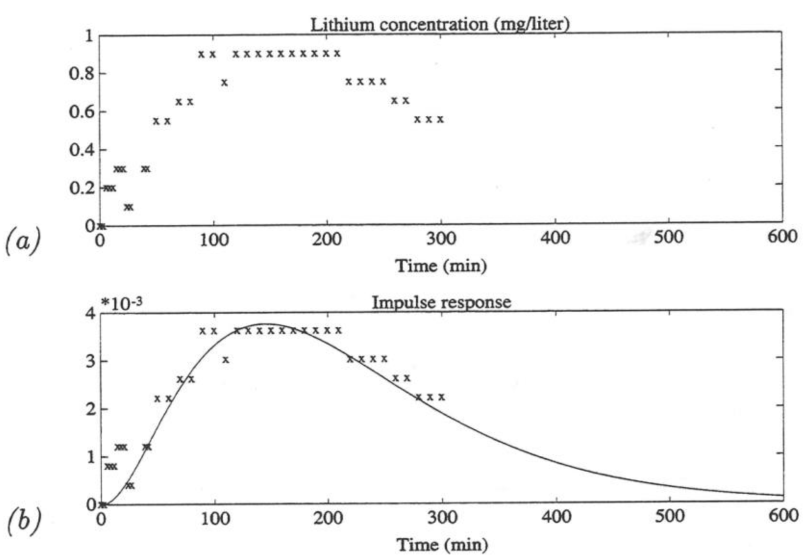


- System
  - Mixing tanks to smooth the concentration of a solution
- Goal
  - Investigate the dynamics of the system
- Method
  - Impulse response
  - Bucket of water with $2070\ g$ of radioactive lithium at point A
  - Measure the radioactivity at point B for 5 hours
- Other constant of the system
  - Flow rate: $8300\ liters/minute$
  - Tank volume: $600\ m^3$

- Analysis
  - $u(t) = u_0\delta(t)$
  - $u_0 = 2070\frac{grams}{min} = \left(\frac{2070}{8300}\right) * 10\text{^}3 \Rightarrow 250\ mg/L$
  - Model based on physical insight

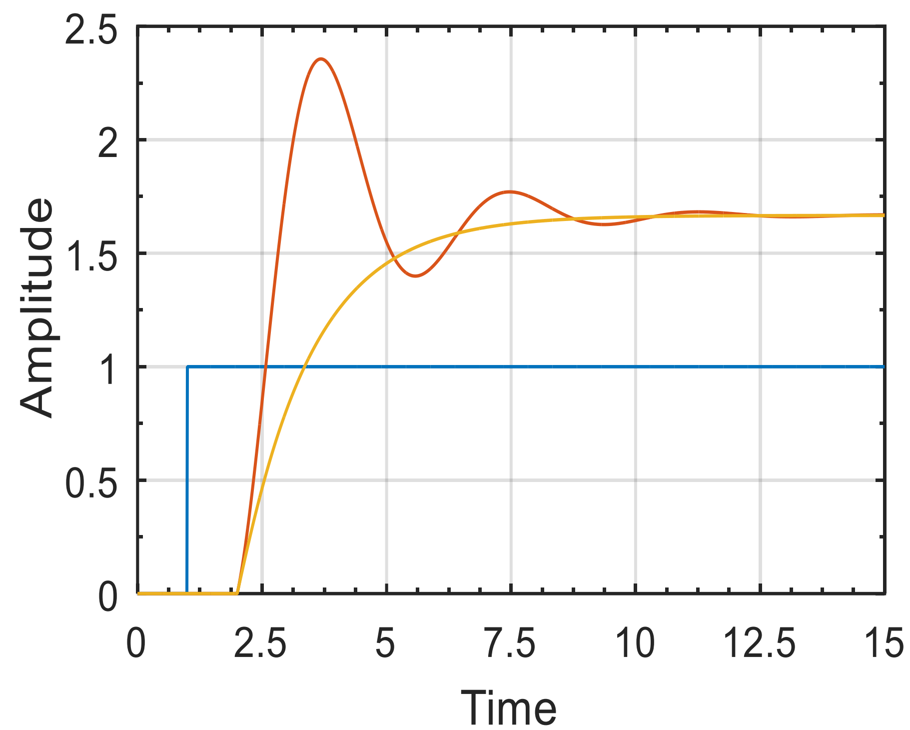
$$G(s) = \left(\frac{1}{s\tau + 1}\right)^3$$

  - 3 identical tank,
  - Everything poured in should leave, so the static gain = 1,



Summary

- Quick and easy insight
  - Order of the system
  - Time constant
  - Static gain
  - Time delay
  - Rise time,
  - natural frequency
  - Damping coefficient
- Widely used in industrial practices
- Information somehow limited
- Not sufficient for quantitative modeling



Math background:
Stochastic process and correlation

Random variable and PDF

- Random variable
  - The value varies and cannot be exactly predicted
  - e.g. static noise, the face of a throwed coin
- A random variable can be described by a probability density function (PDF) $p(x)$

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

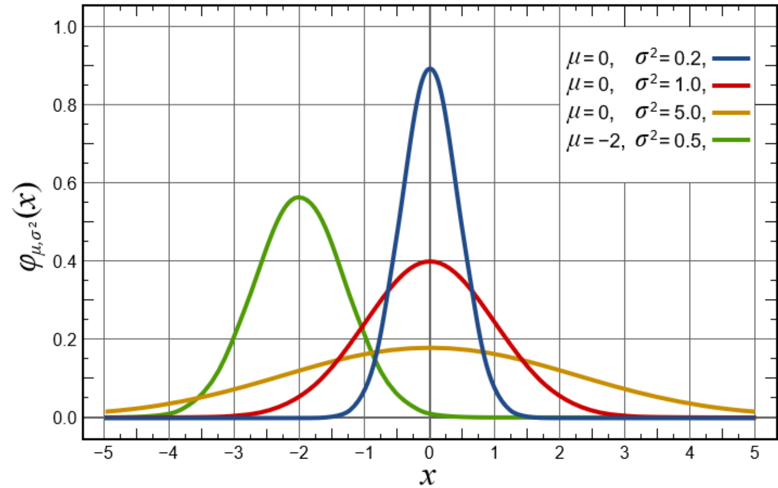
$$P(a \leq x \leq b) = \int_{a}^{b} p(x)dx$$

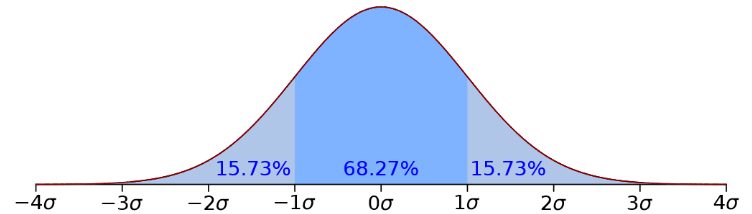
- For normal distribution

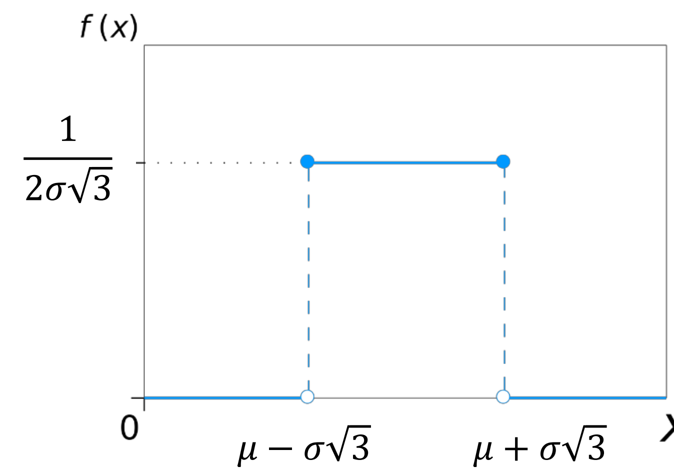
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

  - μ is the mean or expectation of the distribution (also median and mode)
  - σ is the standard deviation with variance of σ^2
- For uniform distribution

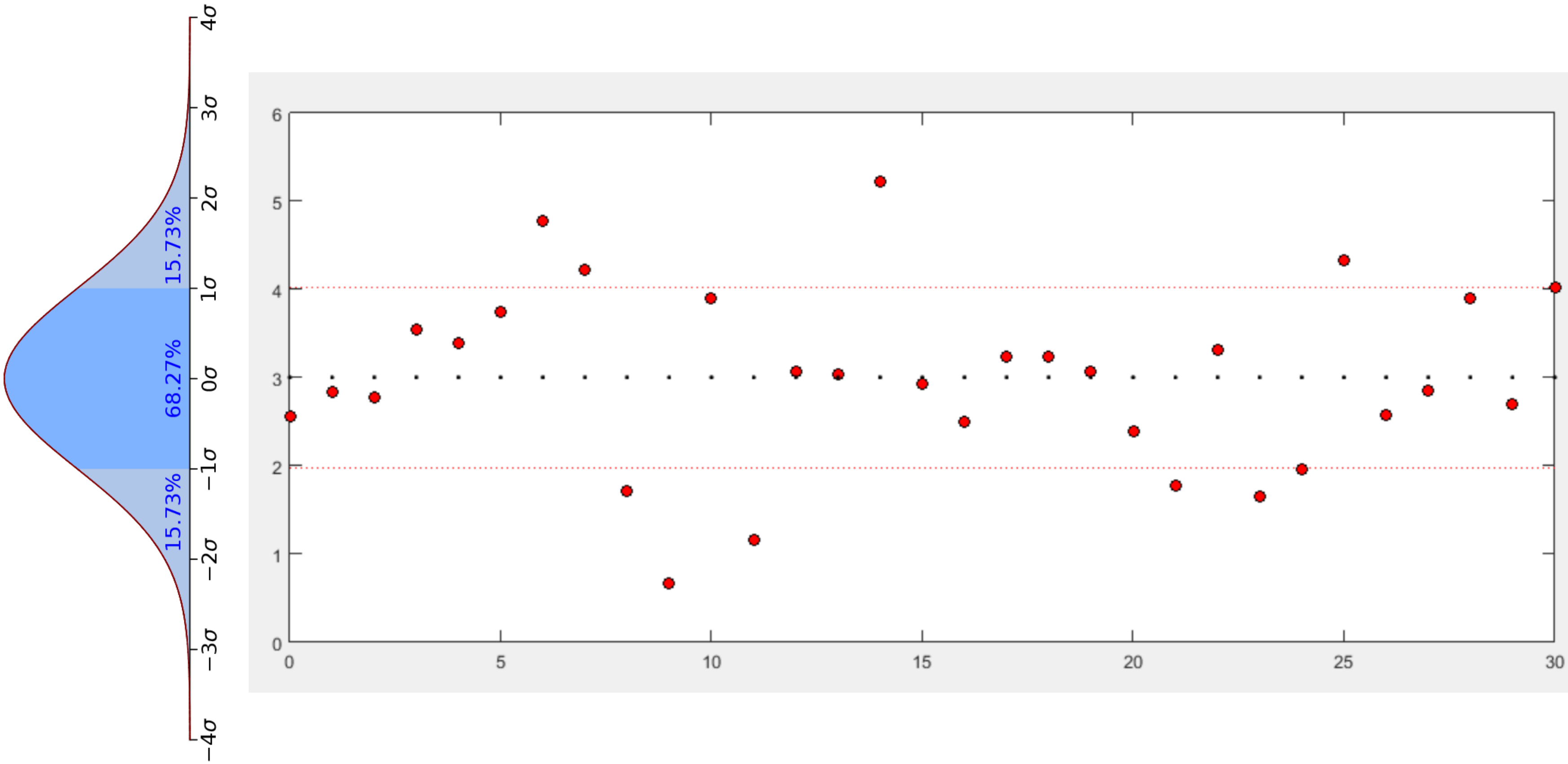
$$p(x) = \begin{cases} \frac{1}{2\sigma\sqrt{3}} & \text{for } -\sigma\sqrt{3} \leq x-\mu \leq \sigma\sqrt{3} \\ 0 & \text{otherwise} \end{cases}$$







Examples of PDF

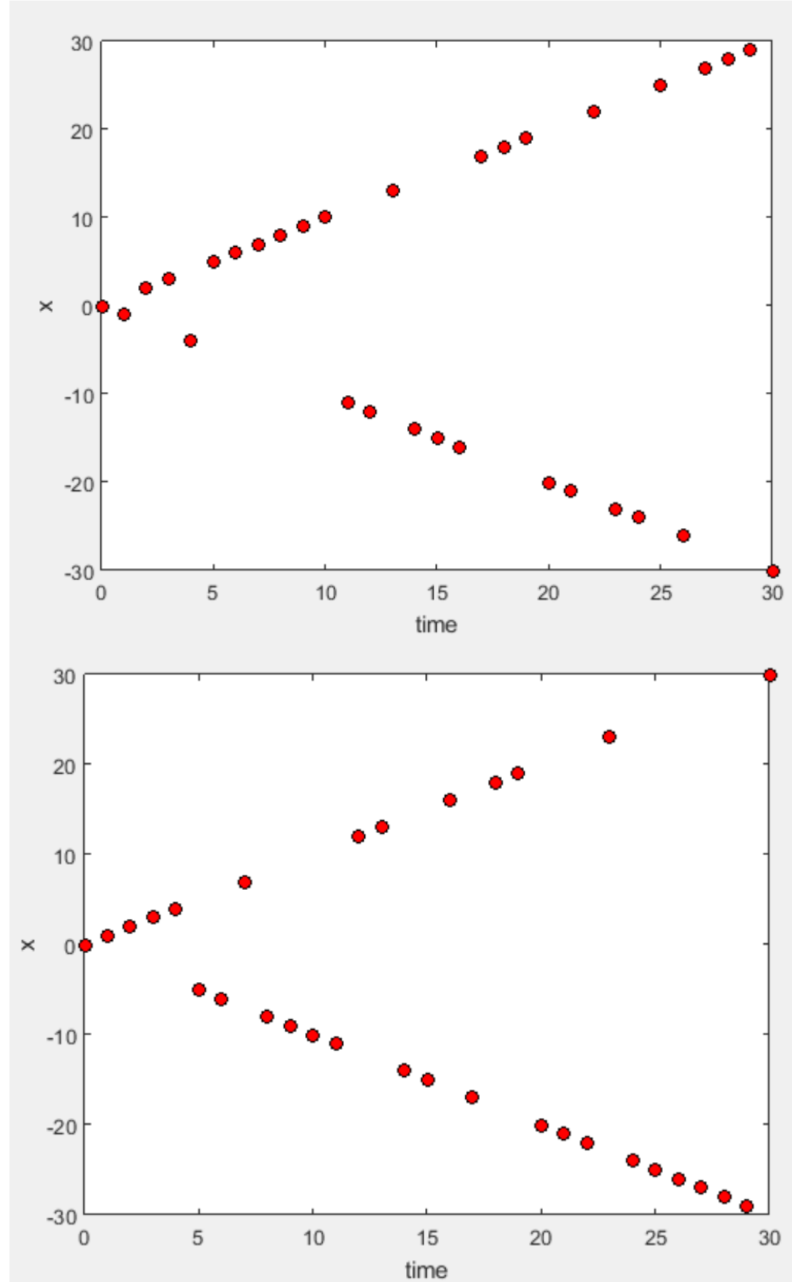


Stochastic process

- Stochastic process is a collection of random variables *varying* with time. Example:

$$\{x(t)\} = \begin{cases} t & P = 0.5 \\ -t & P = 0.5 \end{cases}$$

- Realization of stochastic process
  - Each realization can be a different sequence
- Stationary stochastic process
  - mean and variance do not change with time
- Ergodic stochastic process
  - time average of one sequence of events is the same as the ensemble average



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Mean and Variance

- Random variable
  - Mean or expectation

$$\mu_x = E[x] = \int_{-\infty}^{\infty} x\, p(x) dx$$

  - Variance

$$\sigma_x^2 = var(x) = E[(x - E[x])^2]$$

$$= \int_{-\infty}^{\infty} (x - E[x])^2 p(x) dx$$

$$= \int_{-\infty}^{\infty} x^2 p(x) dx - \int_{-\infty}^{\infty} 2E[x] x p(x) dx$$

$$+ \int_{-\infty}^{\infty} E[x]^2 p(x) dx$$

$$= E[x^2] - 2E[x]^2 + E[x]^2$$

$$= E[x^2] - E[x]^2$$

- Stochastic process
  - Mean or expectation

$$\mu_x = E[x(k)] = \int_{-\infty}^{\infty} x(k)\, p(x,k) dx$$

  - Variance

$$\sigma_x^2(k) = var\big(x(k)\big)$$

$$= E[(x(k) - E[x(k)])^2]$$

$$= E[x(k)^2] - E[x(k)]^2$$

- Stationary ergodic stochastic process

$$\mu_x = \bar{x} \approx \frac{1}{N} \sum_{k=1}^{N} x(k)$$

Estimation of variance

- We can estimate the variance with measured data

$$E[(x-\mu_x)^2] \approx \frac{1}{N}\sum_{k=1}^{N}(x(k)-\mu_x)^2$$

- When the expectation is unknown, we should calculate the mean first, so:

$$E[(x-\mu_x)^2] \approx \frac{1}{N}\sum_{k=1}^{N}(x(k)-\bar{x})^2$$

Covariance and correlation of random process

- Covariance measures of how much two random variables change together

$$\sigma_{xy} = cov(x, y) = E[(x - E[x])(y - E[y])]$$
$$= E\big[xy - xE[y] - yE[x] + E[x]E[y]\big]$$
$$= E[xy] - E[x]E[y] - E[x]E[y] + E[x]E[y]$$
$$= E[xy] - E[x]E[y]$$

- The variance is a special case

$$\sigma_x^2 = \sigma_{xx}$$

- If x and y are independent, i.e., uncorrelated, then $\sigma_{xy} = 0$

$$E[xy] = E[x]E[y]$$

- Pearson correlation coefficient

$$\rho_{xy} = corr(x, y) = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

- $\rho_{xy} \in [-1,1]$

  1: total positive correlation

  -1: total negative correlation

  0: not correlated

- Correlation without subtracting mean and scaled by variance

$$\rho_{xy} = corr(x, y) = E(xy)$$

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Covariance and correlation of stochastic process

- **Cross covariance** is a function that gives the covariance of the one process with the other at pairs of time points

$$\sigma_{xy}(t_1, t_2) = cov\big(x(t_1), y(t_2)\big) = E\Big[\Big(x(t_1) - E\big(x(t_1)\big)\Big)\Big(y(t_2) - E\big(y(t_2)\big)\Big)\Big]$$

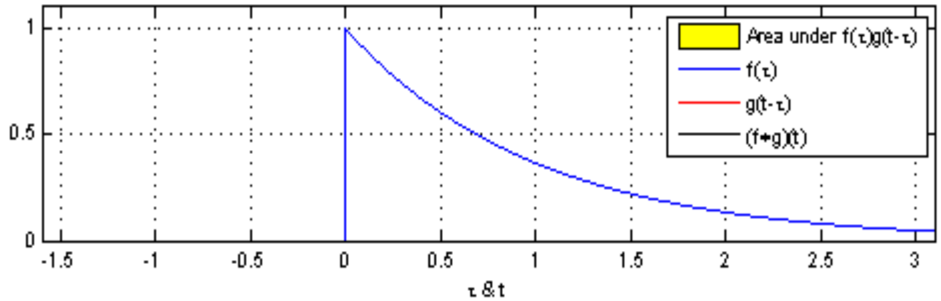
- The **cross correlation** is:

$$\rho_{xy}(t_1, t_2) = corr\big(x(t_1), y(t_2)\big) = E[x(t_1)y(t_2)]$$

- For stationary stochastic process, we introduce a variable $\tau = t_1 - t_2$

$$\sigma_{xy}(\tau) = cov\big(x(t), y(t-\tau)\big) = E\Big[\Big(x(t) - E\big(x(t)\big)\Big)\Big(y(t-\tau) - E\big(y(t-\tau)\big)\Big)\Big]$$
$$= E[x(t)y(t-\tau)] - E\big(x(t)\big)E\big(y(t-\tau)\big)$$

$$\rho_{xy}(\tau) = corr\big(x(t), y(t-\tau)\big) = E[x(t)y(t-\tau)]$$



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Estimation of correlation and covariance

- From previous slide

$$\rho_{xy}(\tau) = corr\big(x(t), y(t-\tau)\big) = E[x(t)y(t-\tau)]$$

$$\sigma_{xy}(\tau) = cov\big(x(t), y(t-\tau)\big) = E\Big[\Big(x(t) - E\big(x(t)\big)\Big)\Big(y(t-\tau) - E\big(y(t-\tau)\big)\Big)\Big]$$

- For stationary ergodic random process

$$\rho_{xy}(\kappa) = corr\big(x(k), y(k+\kappa)\big) \approx \frac{1}{N}\sum_{k=1}^{N} x(k)y(k+\kappa)$$

$$\sigma_{xy}(\kappa) = cov\big(x(k), y(k+\kappa)\big) \approx \frac{1}{N}\sum_{k=1}^{N} (x(k) - \bar{x})\,(y(k+\kappa) - \bar{y})$$

$$\hat{\rho}_{xy}(\kappa) = \frac{1}{N}\sum_{k=1}^{N} x(k)y(k+\kappa)$$

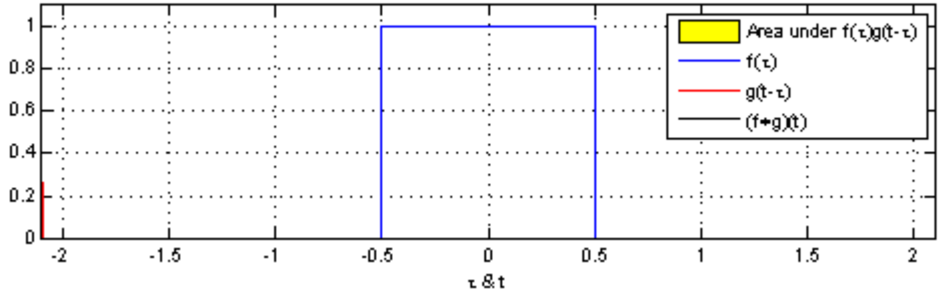
$$\hat{\sigma}_{xy}(\kappa) = \frac{1}{N}\sum_{k=1}^{N} (x(k) - \bar{x})\,(y(k+\kappa) - \bar{y})$$

Autocorrelation

$$\rho_{xx}(\kappa) = corr\big(x(k), x(k+\kappa)\big) = E[x(k)x(k+\kappa)]$$

- For white noise

$$\rho_{xx}(\kappa) = \begin{cases} \lambda, & \kappa = 0 \\ 0, & \kappa \neq 0 \end{cases}$$



Correlation analysis

Correlation analysis

$$\sigma_{xy}(\tau) = E[x(t)y(t-\tau)] - E\big(x(t)\big)E\big(y(t-\tau)\big)$$
$$\rho_{xy}(\tau) = E[x(t)y(t-\tau)]$$
$$\rho_{xx}(\tau) = E[x(k)x(k+\tau)]$$

Goal: estimate the model g_k from measurement $y(t)$ by correlation

$$y(t) = \sum_{k=0}^{\infty} g_k u(t-k) + v(t)$$

Let $\{u(t)\}$ be a realization of stochastic process with zero mean, and assuming $\{u(t)\}$and $\{v(t)\}$ are uncorrelated

How to calculate the model g_τ?

The cross-covariance function between y and u is

$\{u(t)\}$ has zero mean, so this term=0

$$\sigma_{yu}(\tau) = cov_{yu}(\tau)$$
$$= E[y(t)u(t-\tau)] - \underbrace{E\big(y(t)\big)E\big(u(t-\tau)\big)}_{=0}$$
$$= E\left[\sum_{k=0}^{\infty} g_k u(t-k)u(t-\tau)\right]$$
$$= \sum_{k=0}^{\infty} g_k E[u(t-k)u(t-\tau)]$$

$\tau' = \mathrm{t1} - \mathrm{t2} = t - k - \mathrm{t} + \tau = \tau - k$

$$= \sum_{k=0}^{\infty} g_k \sigma_{uu}(\tau - k)$$

Autocorrelation of input u

If the input is white noise

$$\sigma_{uu}(\tau) = \begin{cases} \lambda, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$$

We have

$$\sigma_{yu}(\tau) = \lambda g_\tau$$

We can estimate the system model g_τ from measured $y(t)$ data and $u(t)$!

$$\hat{\sigma}_{yu}(\tau) = \hat{\rho}_{yu}(\tau) = \frac{1}{N}\sum_{t=1}^{N} y(t)u(t-\tau)$$

Note: $\{y(t)\}$ $\{u(t)\}$ have zero mean

And the impulse response of the system can be calculated by

$$\hat{g}_\tau = \frac{1}{\lambda}\hat{\sigma}_{yu}(\tau)$$

Correlation analysis

- Assuming the input is uncorrelated with disturbances
- Correlation analysis gives a quick insight into time constants and time delays

cra

Estimate impulse response using prewhitened-based correlation analysis

collapse all in page

Syntax

```
ir=cra(data)
[ir,R,cl] = cra(data,M,na,plot)
```

Description

`ir=cra(data)` estimates the impulse response for the time-domain data, `data`.

`[ir,R,cl] = cra(data,M,na,plot)` estimates correlation/covariance information, `R`, and the 99% confidence level for the impulse response, `cl`.

Input Arguments

`data`	Input-output data. Specify `data` as an `iddata` object containing time-domain data only. `data` should contain data for a single-input, single-output experiment. For the multivariate case, apply `cra` to two signals at a time, or use `impulse`.
`M`	Number of lags for which the covariance/correlation functions are computed. `M` specifies the number of lags for which the covariance/correlation functions are computed. These are from `-M` to `M`, so that the length of `R` is `2M+1`. The impulse response is computed from `0` to `M`. **Default:** 20
`na`	Order of the AR model to which the input is fitted. For the prewhitening, the input is fitted to an AR model of order `na`. Use `na = 0` to obtain the covariance and correlation functions of the original data sequences. **Default:** 10
`plot`	Plot display control. Specify plot as one of the following integers: • 0 — No plots are displayed. • 1 — Plots the estimated impulse response with a 99% confidence region. • 2 — Plots all the covariance functions. **Default:** 1

Output Arguments

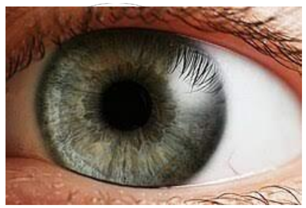
`ir`	Estimated impulse response. The first entry of `ir` corresponds to lag zero. (Negative lags are excluded in `ir`.)
`R`	Covariance/correlation information. • The first column of `R` contains the lag indices. • The second column contains the covariance function of the (possibly filtered) output. • The third column contains the covariance function of the (possibly prewhitened) input. • The fourth column contains the correlation function. The plots can be redisplayed by `cra(R)`.
`cl`	99 % significance level for the impulse response.

Frequency analysis

Frequency analysis

- For an unknown linear system with transfer function $G(s)$, we can excite it with input
$$u(t) = u_0 \cos \omega t$$
- The output after possible transient fading away will be
$$y(t) = y_0 \cos(\omega t + \varphi) + v(t)$$
- where
$$y_0 = u_0 |G(i\omega)|$$
$$\varphi = \arg G(i\omega)$$
- If the system is driven by certain u_0 and $\omega = \omega_i$, and we measure the y_0 and φ, we can determine $G(i\omega)$ for certain $\omega = \omega_i$
- By driving the system respectively with different frequency, we can estimate the transfer function $G(i\omega)$

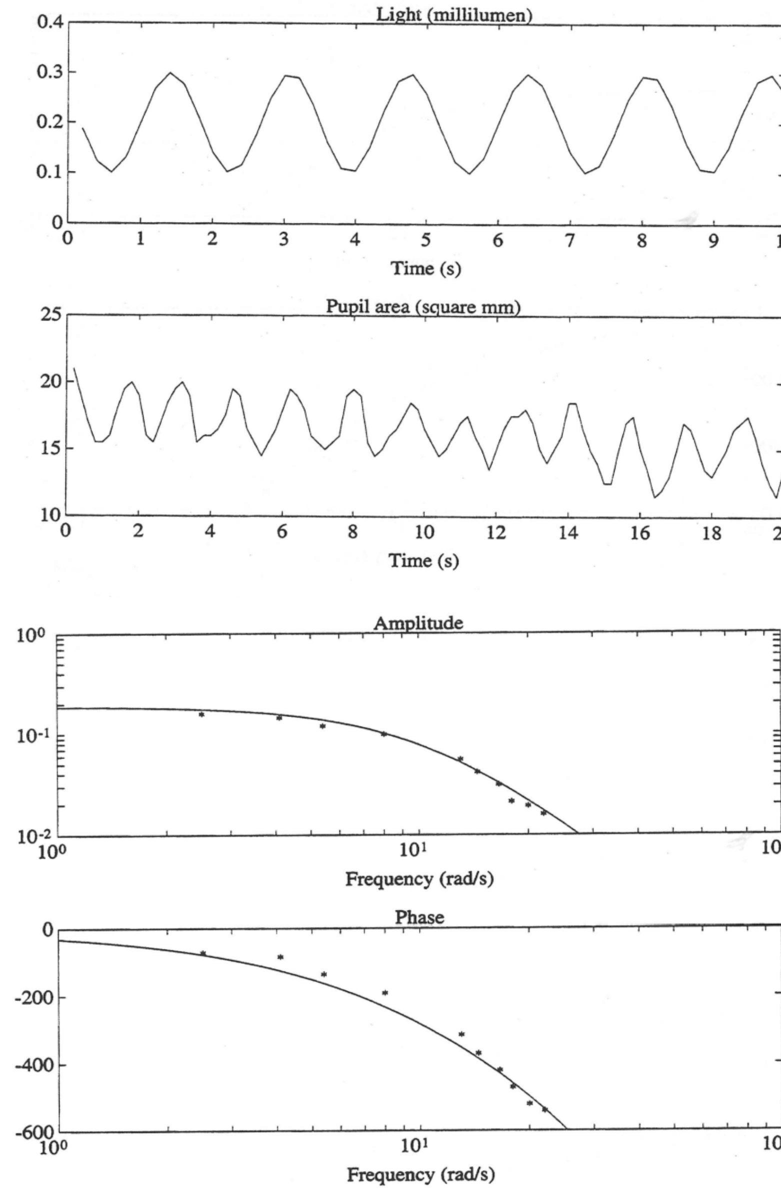
Example



- **Goal:** Test the reaction time of pupil
- **Method**:
  - Using a small light beam
    - Intensity varies as a sinusoid
  - Measure the area of the pupil
    - Using a wide infrared light beam
  - Multiple tests with different frequency
- The results is fit to a transfer function

$$G(s) = e^{-0.28s}\frac{0.19}{(1+0.09s)^3}$$

$\omega = 3.75\ rad/s$



Empirical transfer function estimate (ETFE)

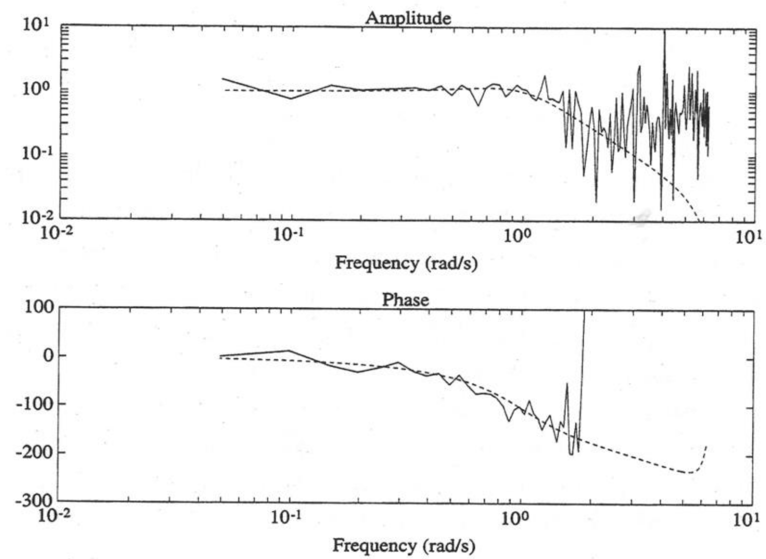
- **Goal:**
  - Expending frequency analysis to **multi frequency** input
- For a linear system can be described by a transfer function, if the input has finite energy, we have:

$$Y(\omega) = G(i\omega)U(\omega)$$

- The transfer function $G(i\omega)$ can be estimated by the **empirical transfer function estimate** (**ETFE**):

$$\hat{\hat{G}}_N(i\omega) = \frac{Y_N(\omega)}{U_N(\omega)}$$

- Where $Y_N(\omega)$ and $U_N(\omega)$ are calculated for different frequency intervals



etfe

Estimate empirical transfer functions and periodograms

Syntax

```
g = etfe(data)
g = etfe(data,M)
g = etfe(data,M,N)
```

Description

`g = etfe(data)` estimates a transfer function of the form:

$$y(t) = G(q)u(t) + v(t)$$

Spectral analysis

Signal Spectra

- Spectrum
  - Frequency content of a signal
  - The spectrum of a signal $w(t)$ is defined as

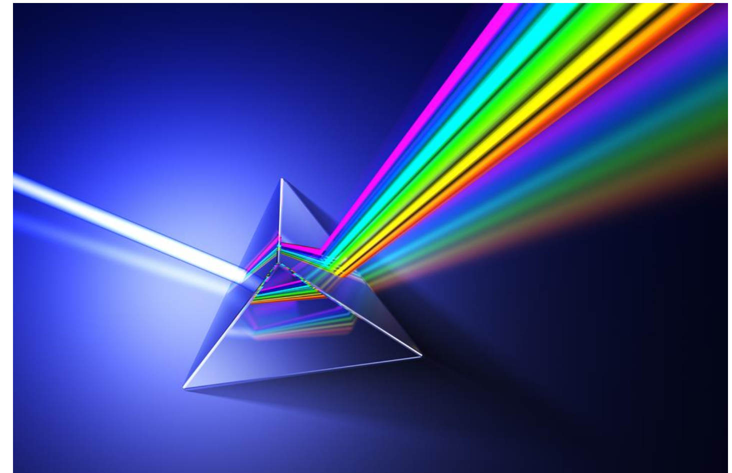
$$\Phi_w(\omega) = |W(\omega)|^2$$

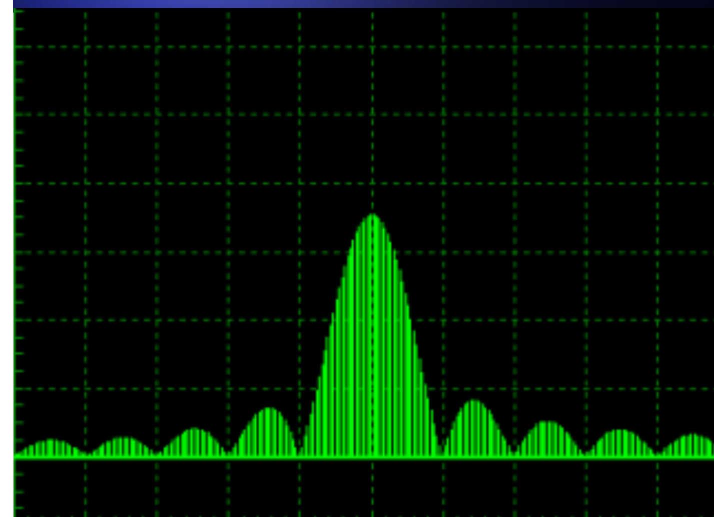
  - Where $W(\omega)$ is the Fourier transform of $w(t)$

$$W(\omega) = \int_{-\infty}^{\infty} w(t) e^{-i\omega t} dt$$

  - And the energy of $w(t)$ should be finite

$$\int_{-\infty}^{\infty} |w(t)| dt < \infty$$





Spectral density

- Measures the signal's energy (power) between frequency ω_1 and ω_2

$$P = \int_{\omega_1}^{\omega_2} \Phi_w(\omega) d\omega$$

Cross spectra

- The cross spectrum of two stationary process with zero mean is

$$\widehat{\Phi}_{yu}(\omega) = \sum_{k=-\gamma}^{\gamma} \hat{\sigma}_{yu}(k)e^{-i\omega k}$$

- Where

$$\hat{\sigma}_{yu}(k) = E[y(t+k)u(t)] = \frac{1}{N}\sum_{t=1}^{N} y(t+k)u(t)$$

- If y and u has the relationship

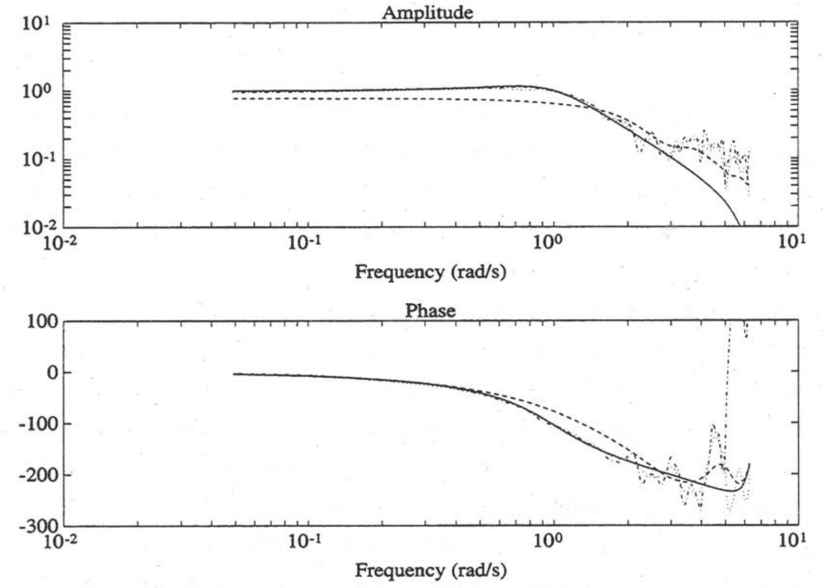
$$y(t) = G(t)u(t)$$

- Then

$$\widehat{\Phi}_{yu}(\omega) = G(i\omega)\widehat{\Phi}_{u}(\omega)$$

Spectral Analysis

- Estimate the model using spectral analysis of input and output
- Very common methods for analysis of signals and systems, if the system is linear
- Requires no specific input signal
- Does not work with feedback system



spa

Estimate frequency response with fixed frequency resolution using spectral analysis

Syntax

```
G = spa(data)
G = spa(data,winSize,freq)
G = spa(data,winSize,freq,MaxSize)
```

Summary

- Transient analysis
  - Impulse response
  - Step response
- Basics of stochastic process and correlation
  - PDF
  - Stochastic process
  - Mean and variance
  - Covariance
  - Correlation
- Correlation analysis
- Frequency analysis
- Spectral analysis

Readings

- Ch8.1-8.2, Ch 3.8, Ch 8.3-8.7, Ljung, Modeling of Dynamic Systems, 1994
- Wikipedia