Aalto University
School of Electrical Engineering

ELEC-E8103 Modelling, Estimation and Dynamic Systems

Physical Modeling

Quan Zhou,
Department of Electrical Engineering and Automation
Aalto University, School of Electrical Engineering
Email: quan.zhou@aalto.fi

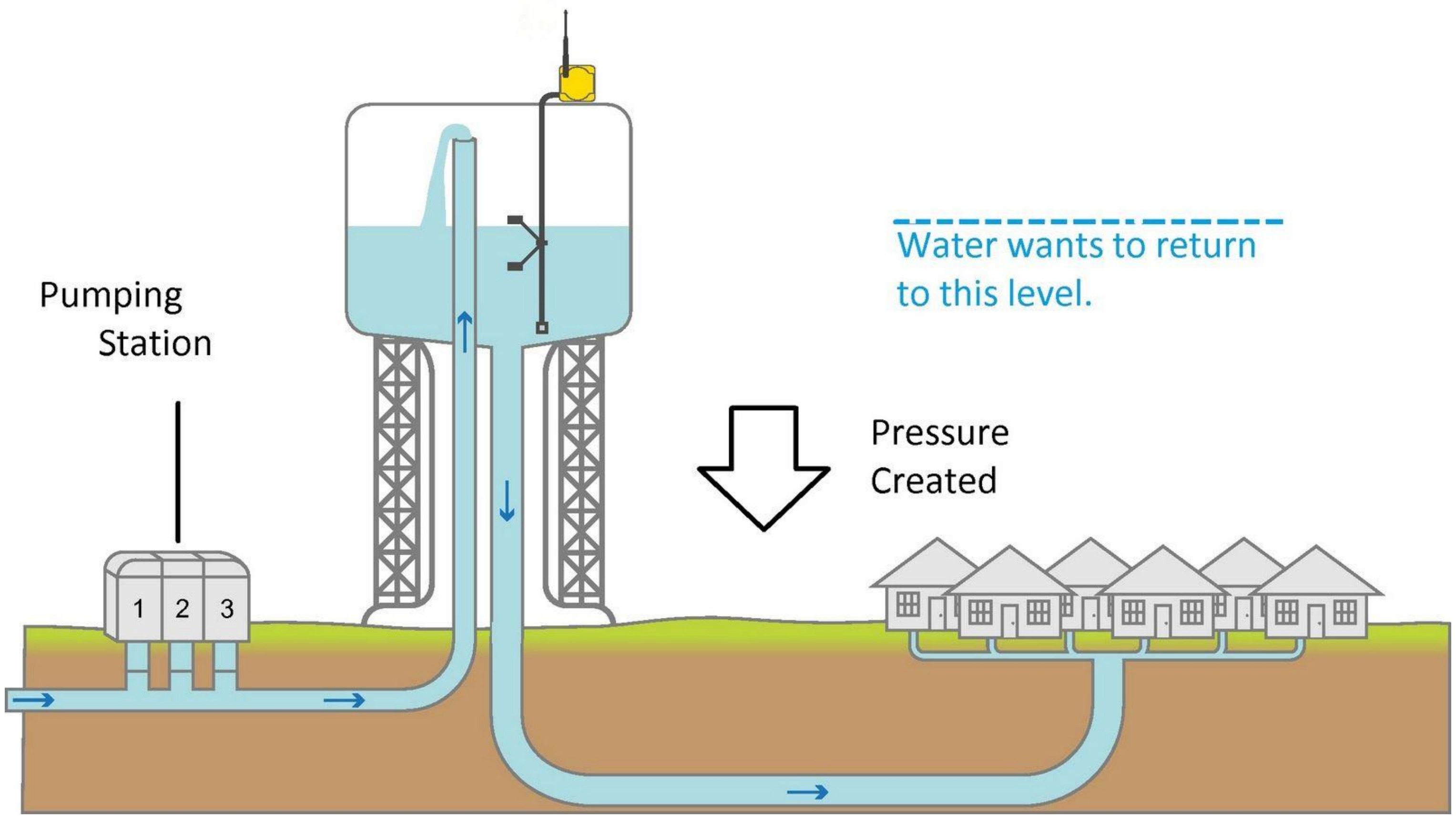
Learning Goals

Course Learning Outcomes

- Select proper modeling approach for specific problems,
- Formulate mathematical models of physical systems,
- Construct models of systems using modeling tools such as MATLAB and Simulink,
- Estimate the parameters of linear and nonlinear static systems from measurement data,
- Identify the models of linear dynamic systems from measurement data

- Structuring the problem
- Ordinary differential equation
- State space construction
- Linearization
- Phases of modelling

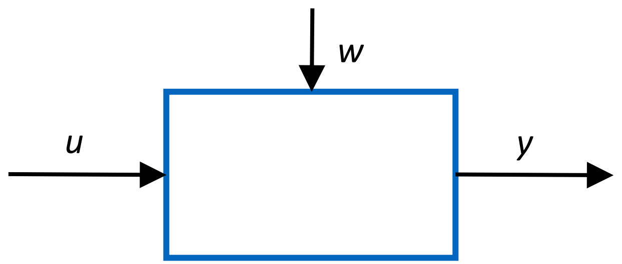
Structuring the problem



Questions

- What signals are of interest (outputs)?
- What are the important quantities? Are they...
  - ...time-varying?
  - ...internal variables?
  - ...constants?
- Which variables affect other variables?
  - Is the relationship static vs. dynamic?

Variable, constants and parameters

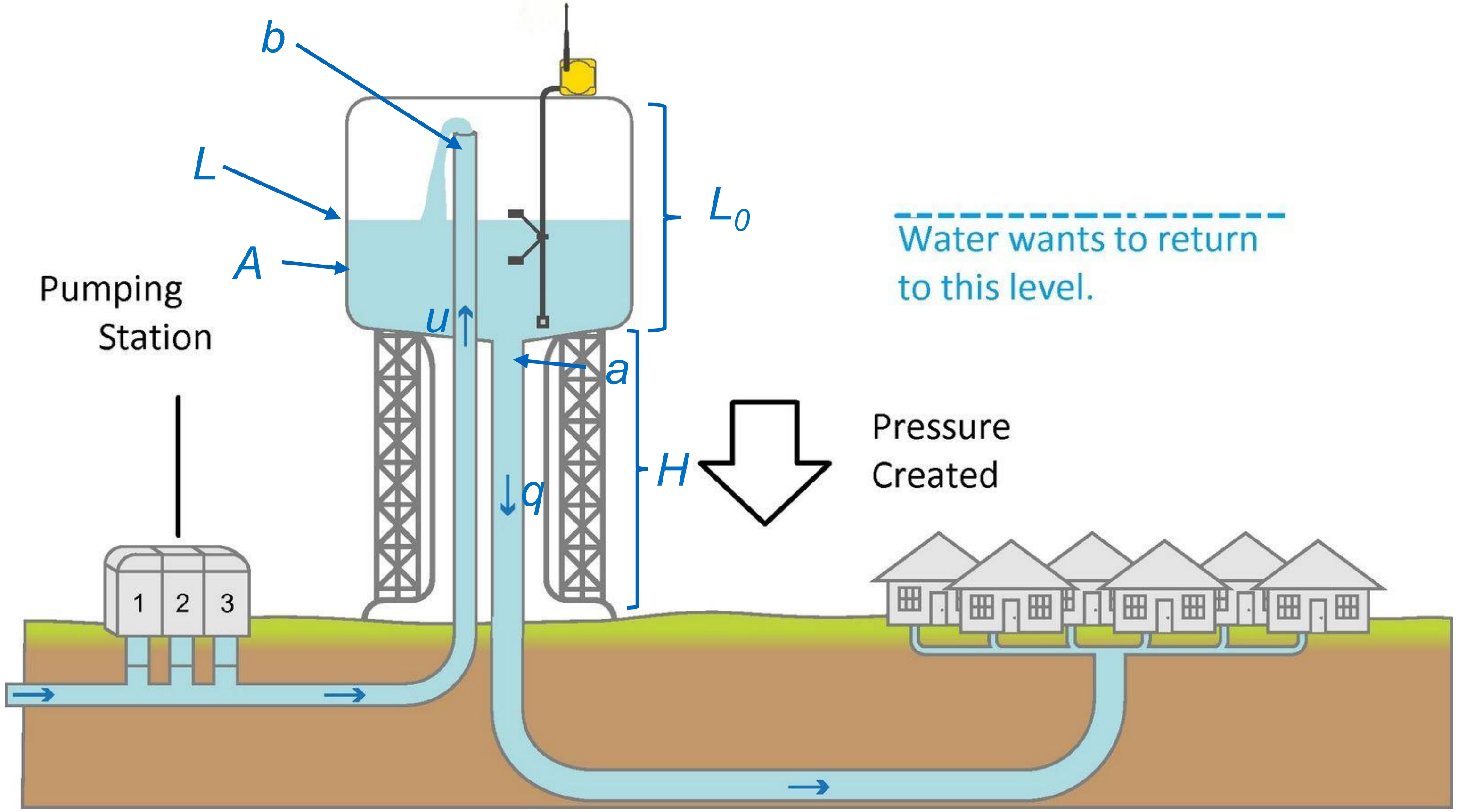


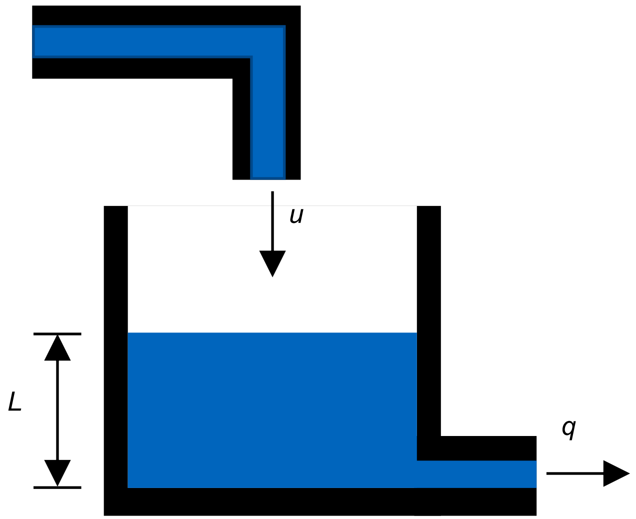
- Input, *u*
  - We can control or influence
- Disturbance, *w*
  - We cannot control or influence
- Output, *y*
  - Our primary interests
- Vector form:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \begin{bmatrix} w(t) \\ w_2(t) \\ \vdots \\ w_r(t) \end{bmatrix}, \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

- Other notations
  - Constant
    - A quantity does not vary
  - System parameter
    - A constant given by the system
  - Design parameter
    - A constant we can vary to give system different properties
  - Variable or signal
    - A quantity in the model varies with time
  - External signal
    - A variable affects the system and not influences by other variables
  - Internal variables
    - A variable in the system besides input or output
- Remark
  - Input *u* and disturbance *w* behave the same for a model or in a simulation. It is not important to distinguish them during the modeling/simulation process, but important for the actual problem.

Examples: Flow System



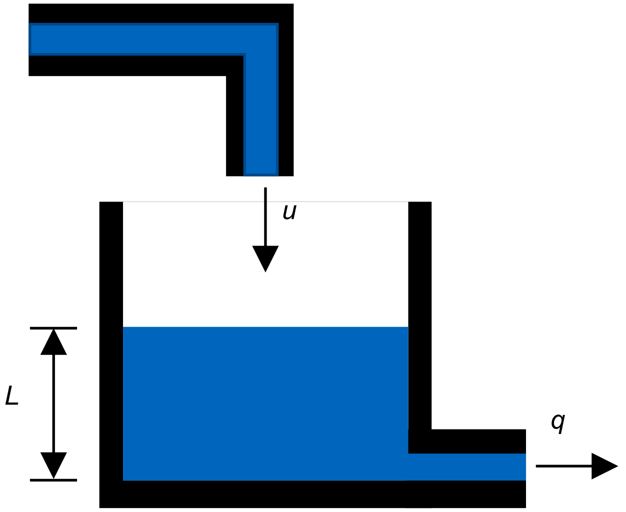


Cross section: A
Outflow hole size: a
Liquid level: L
inflow rate: u
Outflow rate: q
Gravitational constant: g
Density of water: d

- What are?
  - Input
  - Output
  - Disturbance
  - Constant
  - System parameter
- What are?
  - Design parameter
  - Signal or variable
  - External variable
  - Internal variable

Block diagram

- Logical decomposition of the functions of systems based on information flow
- The selection of input(s) and output(s) (of a block) depends on what are of interests
- Important for analyzing large systems



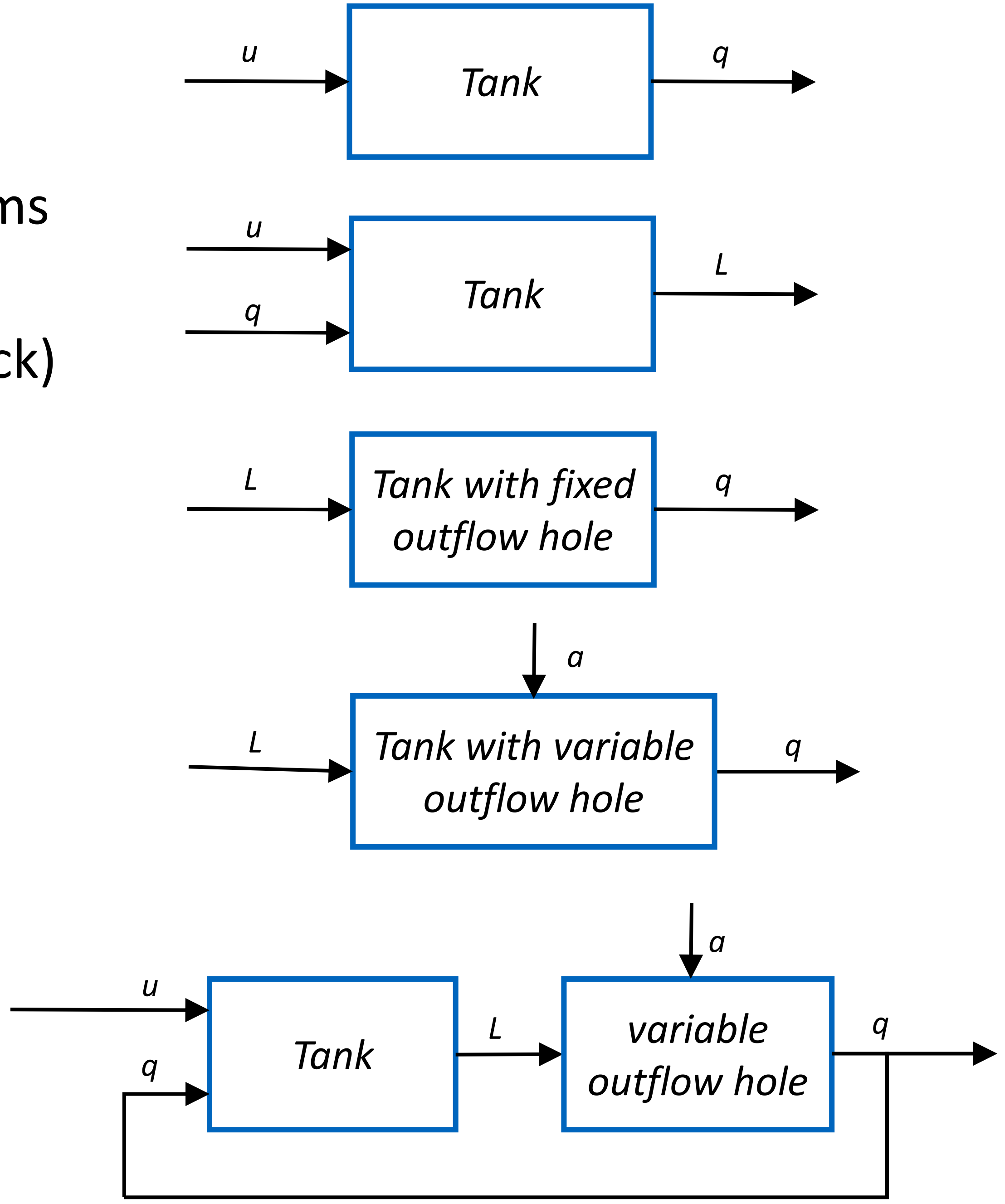
Cross section: A

Outflow hole size: a

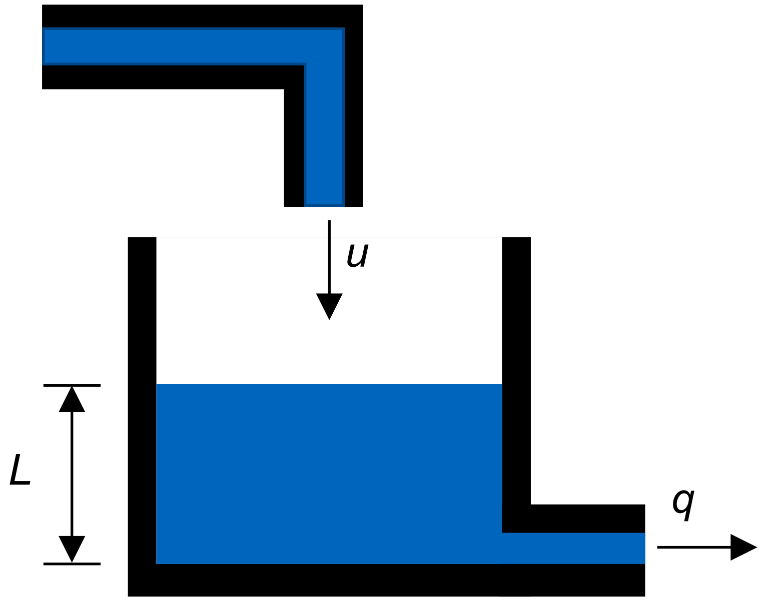
Liquid level: L

inflow rate: u

Outflow rate: q



Examples: Flow System…



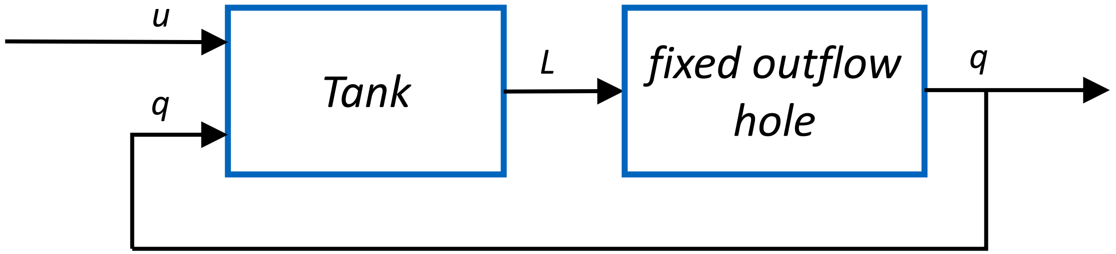
Cross section: *A*

Outflow hole size: *a*

Liquid level: *L*

inflow rate: *u*

Outflow rate: *q*

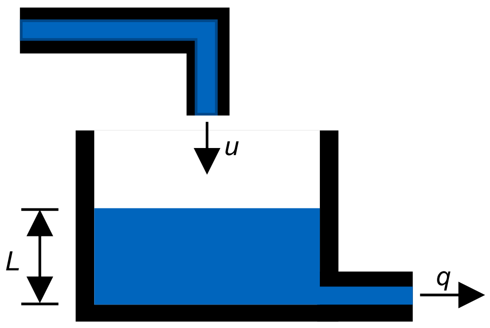


- If we are interested in the outflow rate q. The basic equations:

$$q(t) = a\sqrt{2gL(t)}$$

$$\frac{d}{dt}L(t) = -\frac{q(t)}{A} + \frac{1}{A}u(t)$$
$$= -\frac{a\sqrt{2g}}{A}\sqrt{L(t)} + \frac{1}{A}u(t)$$

Example: flow system…



$$\frac{d}{dt}L(t) = -\frac{a\sqrt{2g}}{A}\cdot\sqrt{L(t)} + \frac{1}{A}u(t)$$
$$q(t) = a\sqrt{2g}\cdot\sqrt{L(t)}$$

Can we generalize it to?

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

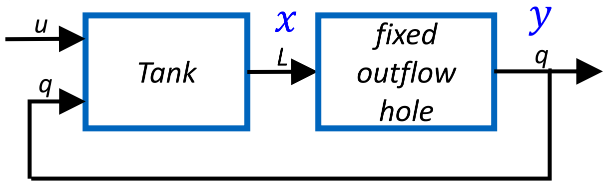
ODE with three variables: *u*, *y*, *x*

If we are interested in the outflow rate:

input → $u(t) = u(t),$

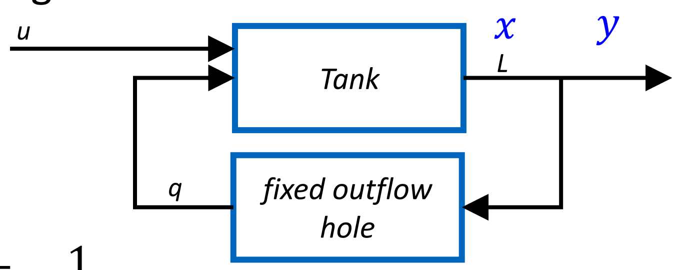
Internal state → $x(t) = L(t),$

output → $y(t) = q(t)$



$$f(x(t), u(t)) = -\frac{a\sqrt{2g}}{A}\cdot\sqrt{x(t)} + \frac{1}{A}u(t)$$
$$h\big(x(t), u(t)\big) = a\sqrt{2g}\cdot\sqrt{x(t)}$$

If we are only interested in the height:



$$x(t) = L(t)$$
$$y(t) = x(t)$$

$$f(x(t), u(t)) = -\frac{a\sqrt{2g}}{A}\cdot\sqrt{x(t)} + \frac{1}{A}u(t)$$
$$h\big(x(t), u(t)\big) = x(t)$$

In both cases, the system is written in the same form.

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How to do it in MATLAB

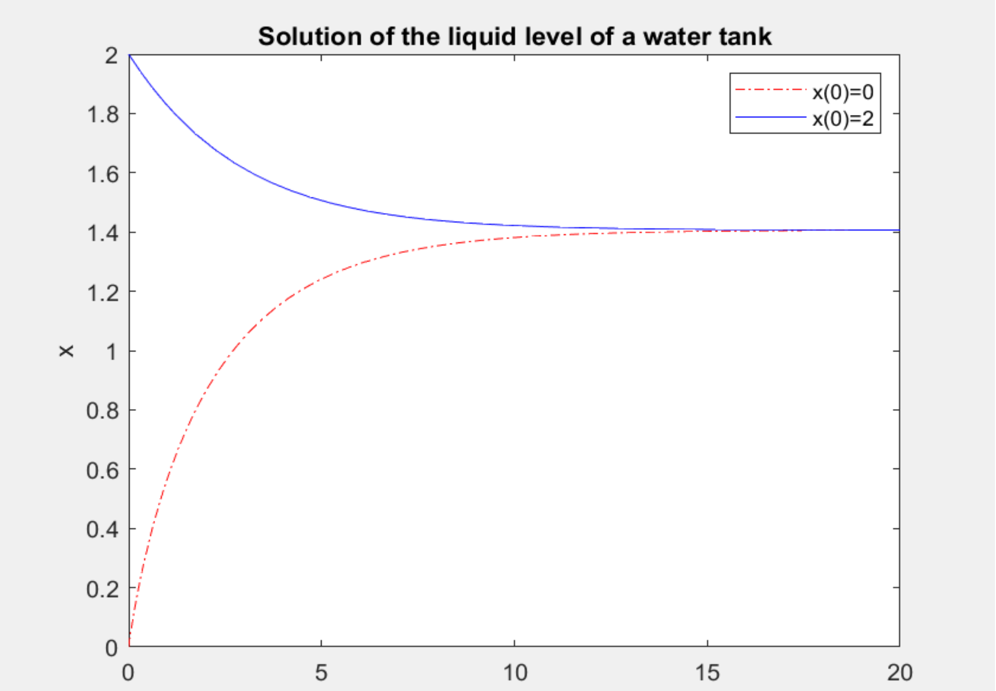
$$f(x(t),u(t)) = -\frac{a\sqrt{2g}}{A}\cdot\sqrt{x(t)} + \frac{1}{A}u(t)$$

$$h\big(x(t),u(t)\big) = x(t)$$

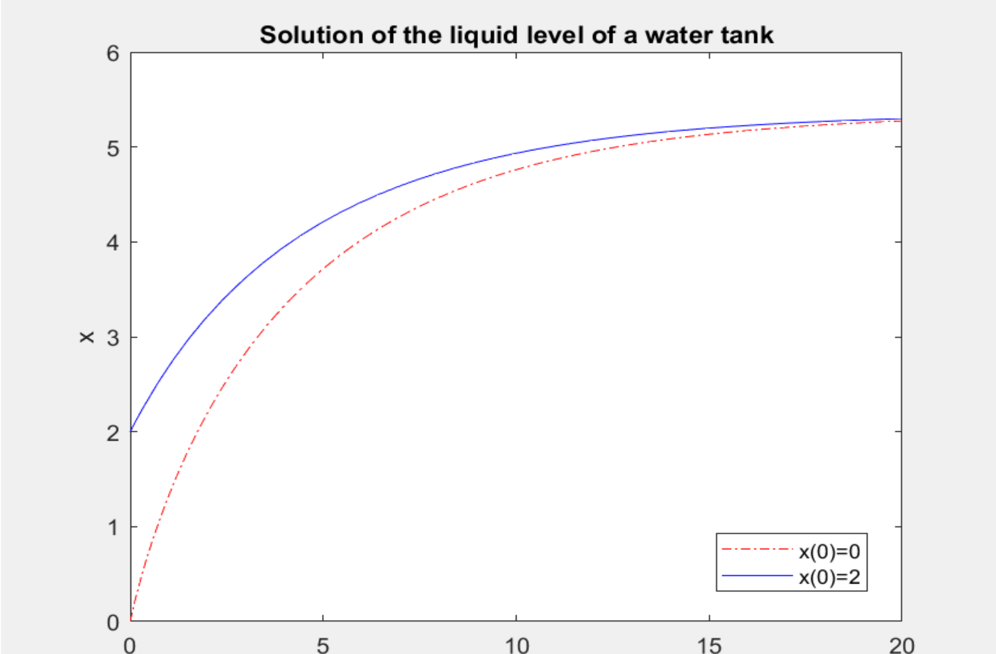
```
%% Ljung 1994 case 3: flow system
u = 1;
A = 1;
g = 9.8;
a = 0.2;

model = @(t,x) -a*sqrt(2*g)/A*sqrt(x(1)) + 1/A*u+0.05;

options = odeset('RelTol',1e-4,'AbsTol',1e-6);
timeSpan = [0 20];
initCond = 0;
[T1,X1] = ode45(model,timeSpan, initCond,options);
initCond = 2;
[T2,X2] = ode45(model,timeSpan, initCond,options);

Y1 = X1; % we are only interested in the height
Y2 = X2;

clf
plot(T1,Y1,'r-.'); hold on; plot(T2,Y2,'b-');
title('Solution of the liquid level of a water tank');
xlabel('time');
ylabel('x');
legend('x(0)=0','x(0)=2')
```

Anonymous function

$x(0) = 0$



$u = 1$

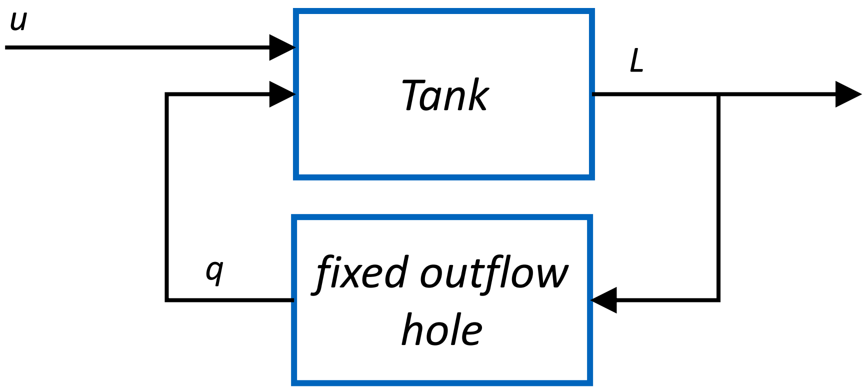


$u = 2$

rsity
lectrical

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

Alternative model using *q* state variable



- How about using q as the internal variable?
- The basic equation:

$$q(t) = a\sqrt{2gL(t)}$$

So

$$L(t) = \frac{1}{2a^2g} q(t)^2$$

from

$$\frac{d}{dt} L(t) = -\frac{q(t)}{A} + \frac{1}{A} u(t)$$

we have

$$\frac{2q(t)}{2a^2g} \frac{d}{dt} q(t) = -\frac{q(t)}{A} + \frac{1}{A} u(t)$$
$$\frac{d}{dt} q(t) = -\frac{a^2g}{A} + \frac{a^2g}{A} q(t)^{-1} u(t)$$

- Let x be internal state q, u as input, y as output, a general form is:

$$\dot{x}(t) = -\frac{a^2g}{A} + \frac{a^2g}{A} x(t)^{-1} u(t)$$
$$y(t) = \frac{1}{2a^2g} x(t)^2$$

How does this alternative model compare with the previous formulation?

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

Differential Equation System

- How about a high order differential equation?

$$g(y^{(n)}(t), y^{(n-1)}(t), \dots, y(t), u^{(m)}(t), u^{(m-1)}(t), \dots, u(t)) = 0$$

- We can use a system of 1st order differential equations

By introducing internal variables: $x_1(t), \dots, x_n(t)$, or in vector form:

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad n \left\{ \begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ \dot{x}_2(t) &= f_2(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ &\vdots \\ \dot{x}_n(t) &= f_n(\underbrace{x_1(t), \dots, x_n(t)}_{n}, \underbrace{u_1(t), \dots, u_m(t)}_{m}) \end{aligned} \right.$$

So that $\dot{x}(t) = f(x(t), u(t))$,

$$\text{where } f(x, u) = \begin{bmatrix} f_1(x, u) \\ \vdots \\ f_n(x, u) \end{bmatrix}$$

The output can be calculated from internal variables and input:

$$y(t) = h(x(t), u(t)) \quad p \left\{ \begin{aligned} y_1(t) &= h_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ y_2(t) &= h_2(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ &\vdots \\ y_p(t) &= h_p(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \end{aligned} \right.$$

State Space Models

Continuous time

$$\dot{x}(t) = f\big(x(t), u(t)\big)$$
$$y(t) = h\big(x(t), u(t)\big)$$

$u(t)$: input, an m-dimensional column vector

$y(t)$: output, a p-dimensional column vector

$x(t)$: state, an n-dimensional column vector

The model is said to be *n*th order.

If the function $f(x, u)$ is continuously differentiable and if $u(t)$ is a piecewise continuous function, then a unique solution exists for $t \geq t_0$ with $x(t_0) = x_0$.

Discrete time

$$x(t_{k+1}) = f(x(t_k), u(t_k))$$
$$y(t_{k+1}) = h(x(t_k), u(t_k))$$
$$k = 0,1,2, \ldots$$

$u(t_k)$: input at t_k, an m-dimensional column vector

$y(t_k)$: output at t_k, a p-dimensional column vector

$x(t_k)$: state at t_k, an n-dimensional column vector

The model is said to be *n*th order.

A unique solution exists for $t \geq t_0$ with initial value $x(t_0) = x_0$.

The models are linear if $f(x, u)$ and $h(x, u)$ are **linear function** of x and u, we have:

$$\dot{x} = Ax + Bu \quad \text{or} \quad x_{k+1} = Ax_k + Bu_k$$
$$y = Cx + Du \quad \text{or} \quad y_{k+1} = Cx_k + Du_k$$

where the dimensions of the matrix are, $A: n \times n, B: n \times m, C: p \times n, D: p \times m$

Mass-Spring-Damper System

$$u \longrightarrow \boxed{\textit{Mass-Spring-Damper}} \longrightarrow y$$

质量弹簧阻尼器

$$f = m\ddot{y}$$
$$f = ky$$
$$f = b\dot{y}$$

displacement

$$m\ddot{y} + b\dot{y} + ky = u$$

mass   damper   spring   force

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$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Example: Mass-Spring-Damper System

To study the dynamics of the system under force input

System

$$m\ddot{y} + b\dot{y} + ky = u$$

Let

$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = \dot{y}(t) \end{cases}$$

We have

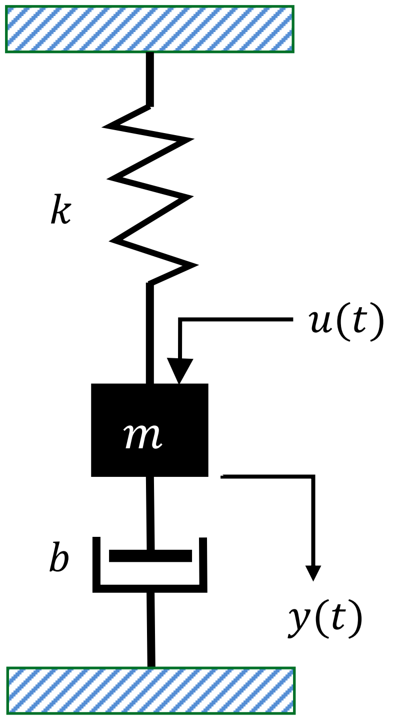
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$
$$= -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

So

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



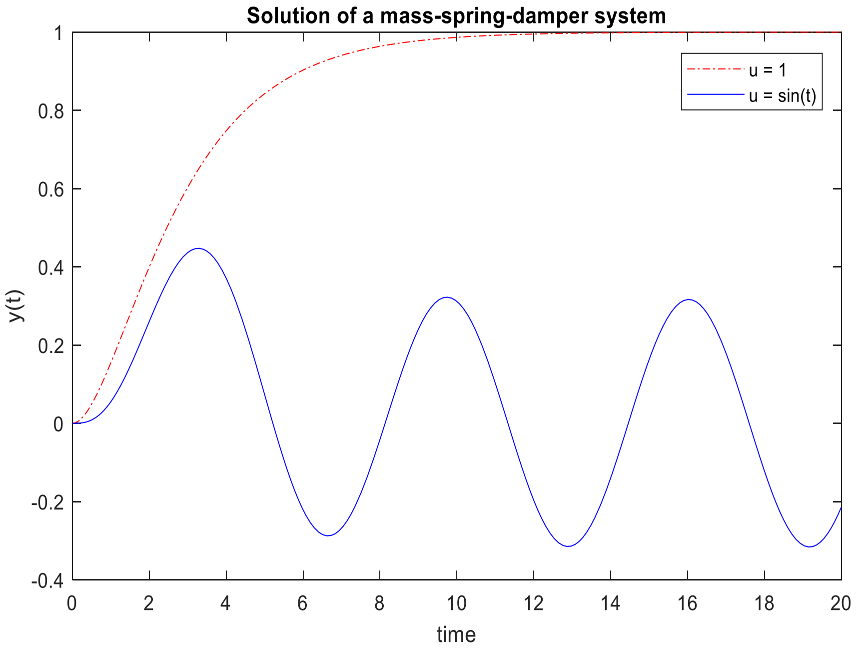
$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0$$

MATLAB implementation

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$C = [1 \quad 0], \qquad D = 0$$



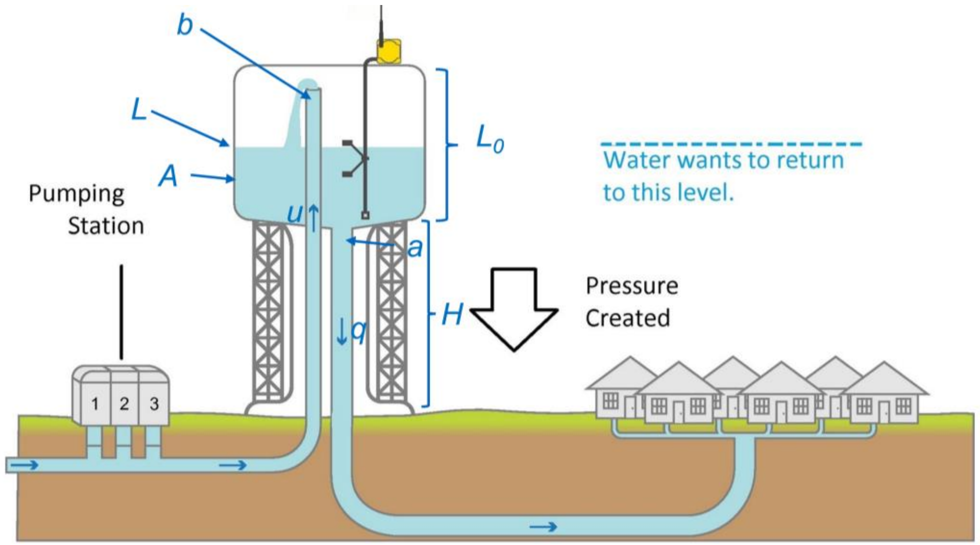

```
%% Mass-Spring-Damper System
clear all;
clf;
k = 1;
m = 2;
b = 3;

A = [ 0 1; -k/m -b/m];
B = [ 0 1/m]';
C = [1 0];
D = 0;

u = 1;
system = @(t,x) A*x + B*u;
output = @(x) C*x + D*u;

options = odeset('RelTol',1e-4,'AbsTol',1e-6);

timeSpan = [0 20];
initCond = [0 0]';
[T1,X1] = ode45(system,timeSpan, initCond,options);
Y1 = output(X1');

u = @(t) sin(t);
system = @(t,x) A*x + B*u(t); % u(t) = sin(t)
output = @(t,x) C*x + D*u(t);
[T2,X2] = ode45(system,timeSpan,initCond,options);
Y2 = output(T2', X2');

plot(T1,Y1,'r-.'); hold on; plot(T2,Y2,'b-');
title('Solution of a mass-spring-damper system');
xlabel('time');
ylabel('y(t)');
legend('u = 1','u = sin(t)')
```

You can also use `lsim` in Matlab

Question: what are the steps to formulate a differential equation model for a physical system?

Phase I: Structuring the Problem

- What signals are of interest (outputs)?
- Important quantities? Are they...
  - ...constants?
  - ...time-varying?
  - ...internal variables?
  - ...external variables?
  - ...inputs?
  - ...disturbance?
- Which variables affect other variables?
  - Is the relationship static vs. dynamic?

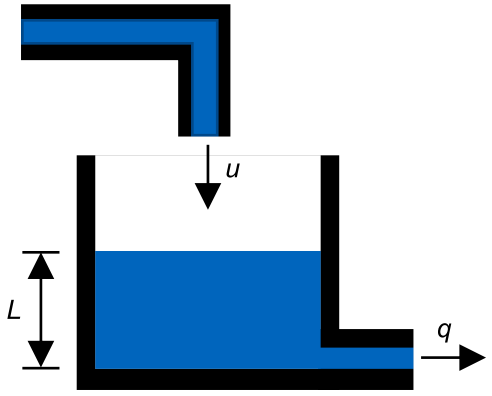


Phase II: Setting up the basic equations

- Quantify the relationships between variables using...
  - ...first principles (e.g. Ohm's law, F = ma)
  - ...curves fitted to data
  - ...known curve shape, but with exact values unknown

- **Conservation laws**: relate to quantities of the same kind
  - Power in – Power out = Stored energy per unit time
  - Input flow rate – Output flow rate = Stored volume per unit time
  - Kirchoff law: sum of currents at a junction = 0

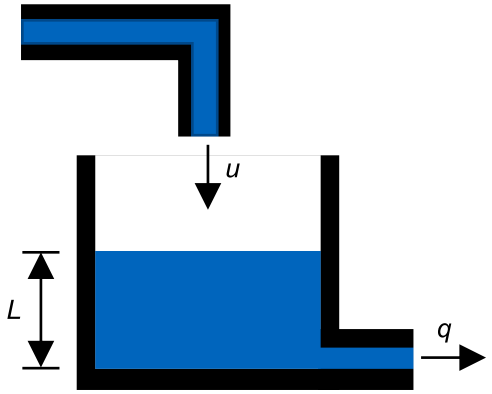
- **Constitutive relationships**: relate quantities of different kinds
  - Force – pressure
  - Young's modulus
  - Friction law
  - ...



$$q(t) = a\sqrt{2gL(t)}$$

Phase III: Forming the state-space model

1. Choose state variables
   – Stored quantities: Position of point mass (potential), velocity of point mass (kinetic), charge of capacitor (electric), current through inductor (magnetic field), temperature (thermal), tank level (volume)



2. Write **time derivative** of **each state variable** as a function of **state** and **inputs**

$$q(t) = a\sqrt{2gL(t)}$$

$$\frac{d}{dt}L(t) = -\frac{q(t)}{A} + \frac{1}{A}u(t)$$
$$= \frac{a\sqrt{2g}}{A}\sqrt{L(t)} + \frac{1}{A}u(t)$$

3. Express **outputs** as functions of **state** and **inputs**

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$
or
$$x(t_{k+1}) = f(x(t_k), u(t_k))$$
$$y(t_{k+1}) = h(x(t_k), u(t_k))$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
or
$$x_{k+1} = Ax_k + Bu_k$$
$$y_{k+1} = Cx_k + Du_k$$

Linearization

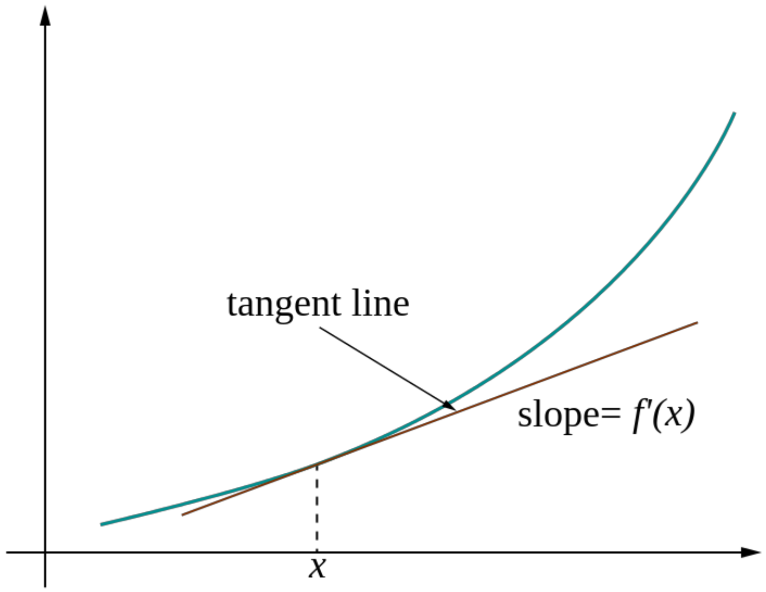
$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

A non-linear system can be linearized around a solution (x_0, u_0) to have:

$$\Delta\dot{x} = A\Delta x + B\Delta u$$
$$\Delta y = C\Delta x + D\Delta u$$

Where A, B, C, D are partial derivative (Jacobian) of f and h around (x_0, u_0), and

$$\Delta x = x - x_0$$
$$\Delta y = y - y_0$$
$$\Delta u = u - u_0$$



Justification: Taylor's theorem

If $x = x_0$ is a stationary solution of the system corresponding to u_0

If the f has continuous partial derivatives around (x_0, u_0), we can expand f at (x_0, u_0) using Taylor series

$$\dot{x} = f(x, u) = f(x_0, u_0) + \left(\frac{\partial f}{\partial x} \cdot (x - x_0) + \frac{\partial f}{\partial u}(u - u_0)\right) + \cdots$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$ are partial derivative of f at the point (x_0, u_0).

If we takes only the first order term, we get the linearized version of non-linear function at the neighborhood of (x_0, u_0)

Recall: Predator and Prey

The first species preys on the second

Population: $N_i(t)$
Birth rate: λ_i
Mortality rate: $\mu_i(N_1, N_2)$

Mortality model for the 1st species

$$\mu_1(N_1, N_2) = \gamma_1 - \alpha_1 N_2$$

Mortality model for the 2nd species

$$\mu_2(N_1, N_2) = \gamma_2 + \alpha_2 N_1$$

Predator and Prey: nonlinear model

- The first species preys on the second
  - Population: $N_i(t)$
  - Birth rate: λ_i
  - Mortality rate: $\mu_i(N_1, N_2)$
- Mortality model for the 1st species

$$\mu_1(N_1, N_2) = \gamma_1 - \alpha_1 N_2$$

- Mortality model for the 2nd species

$$\mu_2(N_1, N_2) = \gamma_2 + \alpha_2 N_1$$

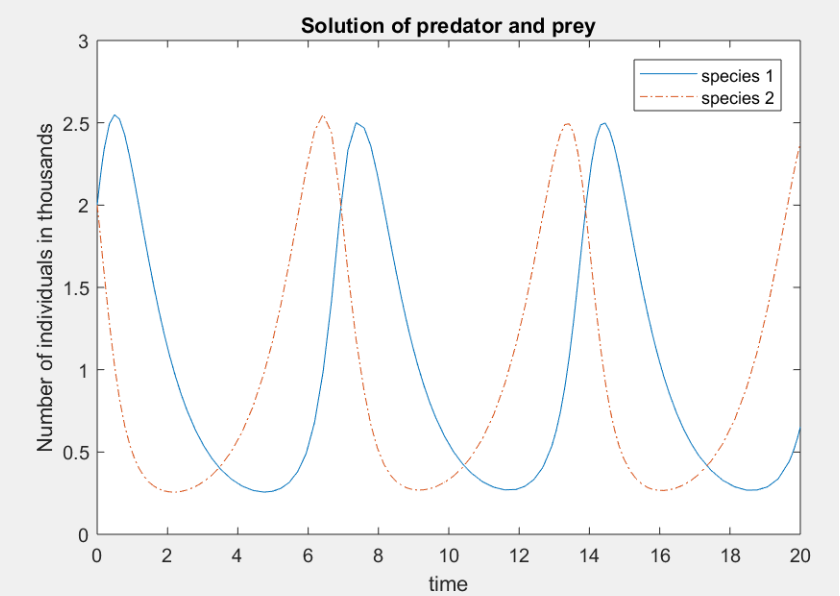
Mortality rate factor due to aging, accidents

Mortality rate factor due to the other specie

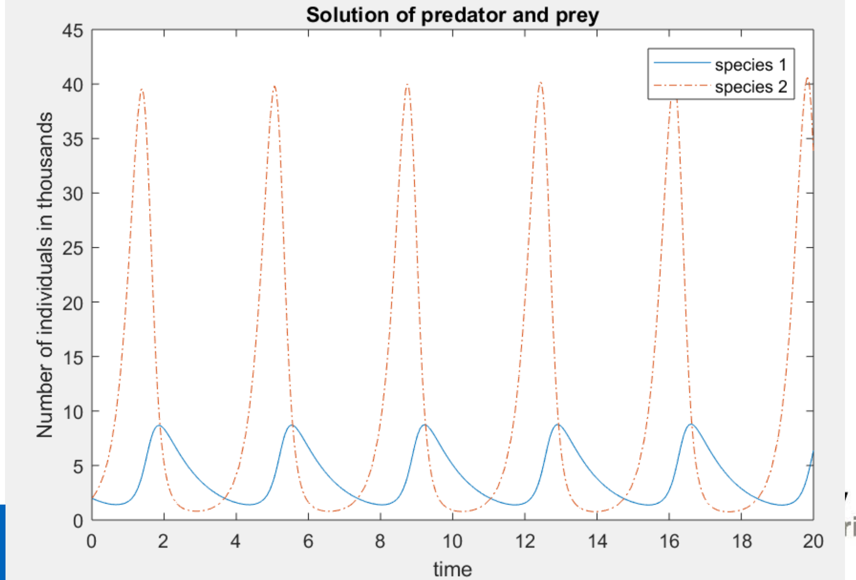
- System model

$$\frac{d}{dt} N_1(t) = \big(\lambda_1 - \mu_1(N_1, N_2)\big) N_1(t)$$
$$= (\lambda_1 - \gamma_1) N_1(t) + \alpha_1 N_1(t) N_2(t)$$
$$\frac{d}{dt} N_2(t) = \big(\lambda_2 - \mu_2(N_1, N_2)\big) N_2(t)$$
$$= (\lambda_2 - \gamma_2) N_2(t) - \alpha_2 N_1(t) N_2(t)$$

$(\lambda_1 = 1, \lambda_2 = 2), (\gamma_1 = 2, \gamma_2 = 1)(\alpha_1 = \alpha_2 = 1)$



$(\lambda_1 = 1, \lambda_2 = 5), (\gamma_1 = 2, \gamma_2 = 2)\ (\alpha_1 = 0.1, \alpha_2 = 1)$

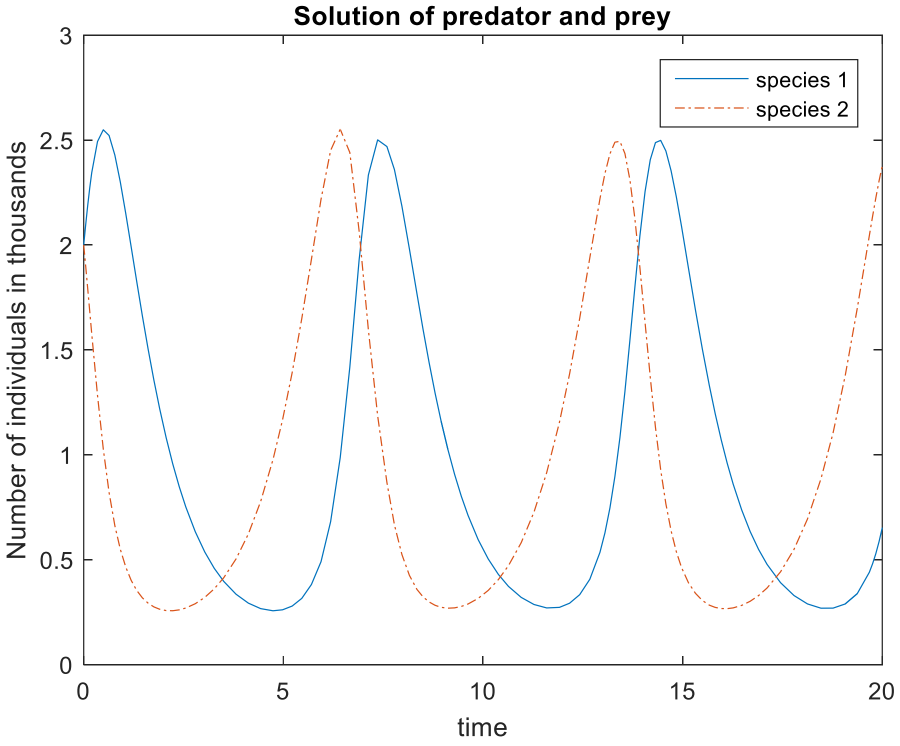


Describe a real-world phenomena, see more in Lotka–Volterra equations

Nonlinear version of predator and prey

```
%% Ljung case 2: Predator and Prey
lambda = [1 2];
gamma = [2 1];
alpha = [1 1];

model2 = @(t,y) [(lambda(1)-gamma(1))*y(1) + alpha(1)*y(1)*y(2);...
                 (lambda(2)-gamma(2))*y(2) - alpha(2)*y(1)*y(2)];

N1 = 2;
N2 = 2;
% N1 = (lambda(2)-gamma(2))/alpha(2);
% N2 = (gamma(1)-lambda(1))/alpha(1);

options = odeset('RelTol',1e-3,'AbsTol',[1e-6 1e-6]);
timeSpan = [0 20];
initCond = [N1 N2];
[t,y] = ode45(model2,timeSpan,initCond,options);
plot(t,y(:,1),'-',t,y(:,2),'-.')
title('Solution of predator and prey');
xlabel('time');
ylabel('Number of individuals in thousands');
legend('species 1','species 2')
```

$$\frac{d}{dt}N_1(t) = (\lambda_1 - \gamma_1)N_1(t) + \alpha_1 N_1(t)N_2(t)$$
$$\frac{d}{dt}N_2(t) = (\lambda_2 - \gamma_2)N_2(t) - \alpha_2 N_1(t)N_2(t)$$



Predator and Prey: linear model

$$\dot{x} = f(x_0, u_0) + \left(\frac{\partial f}{\partial x} \cdot (x - x_0) + \frac{\partial f}{\partial u}(u - u_0)\right)$$

- The model of the predator and prey

$$\frac{d}{dt}\underbrace{\begin{matrix} N_1(t) \\ N_2(t) \end{matrix}}_{x} \begin{matrix} = \\ = \end{matrix} \underbrace{\begin{matrix} (\lambda_1 - \gamma_1)N_1(t) + \alpha_1 N_1(t)N_2(t) \\ (\lambda_2 - \gamma_2)N_2(t) - \alpha_2 N_1(t)N_2(t) \end{matrix}}_{f}$$

- Stationary solution (by setting the derivative to 0): $\boxed{N_1^* = \frac{\lambda_2 - \gamma_2}{\alpha_2},\ N_2^* = \frac{\gamma_1 - \lambda_1}{\alpha_1}}$
- Linearization around the stationary point (partial derivative against N_1 and N_2)

$$\frac{d}{dN_1}\left((\lambda_1 - \gamma_1)N_1 + \alpha_1 N_1 N_2\right) = \lambda_1 - \gamma_1 + \alpha_1 N_2 = 0$$

$$\frac{d}{dN_2}\left((\lambda_1 - \gamma_1)N_1 + \alpha_1 N_1 N_2\right) = \alpha_1 N_1 = \frac{\alpha_1}{\alpha_2}(\lambda_2 - \gamma_2)$$

$$\frac{d}{dN_1}\left((\lambda_2 - \gamma_2)N_2 - \alpha_2 N_1 N_2\right) = -\alpha_2 N_2 = -\frac{\alpha_2}{\alpha_1}(\gamma_1 - \lambda_1)$$

$$\frac{d}{dN_2}\left((\lambda_2 - \gamma_2)N_2 - \alpha_2 N_1 N_2\right) = \lambda_2 - \gamma_2 - \alpha_2 N_1 = 0$$

- The linearized solution is at the stationary point N_1^*, N_2^*

$$\frac{d}{dt}\begin{bmatrix} \Delta N_1(t) \\ \Delta N_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\alpha_1}{\alpha_2}(\lambda_2 - \gamma_2) \\ -\frac{\alpha_2}{\alpha_1}(\gamma_1 - \lambda_1) & 0 \end{bmatrix}\begin{bmatrix} \Delta N_1(t) \\ \Delta N_2(t) \end{bmatrix}$$

$$0 = (\lambda_1 - \gamma_1)N_1 + \alpha_1 N_1 N_2$$
$$0 = (\lambda_2 - \gamma_2)N_2 - \alpha_2 N_1 N_2$$

$$0 = (\lambda_1 - \gamma_1 + \alpha_1 N_2)N_1$$
$$0 = (\lambda_2 - \gamma_2 - \alpha_2 N_1)N_2$$

Solutions:

$$0 = N_1$$
$$0 = N_2$$

or

$$0 = \lambda_1 - \gamma_1 + \alpha_1 N_2$$
$$0 = \lambda_2 - \gamma_2 - \alpha_2 N_1$$

$$(\gamma_1 - \lambda_1)/\alpha_1 = N_2$$
$$(\lambda_2 - \gamma_2)/\alpha_2 = N_1$$

compare to

$$\Delta\dot{x} = A\Delta x + B\Delta u$$
$$\Delta y = C\Delta x + D\Delta u$$

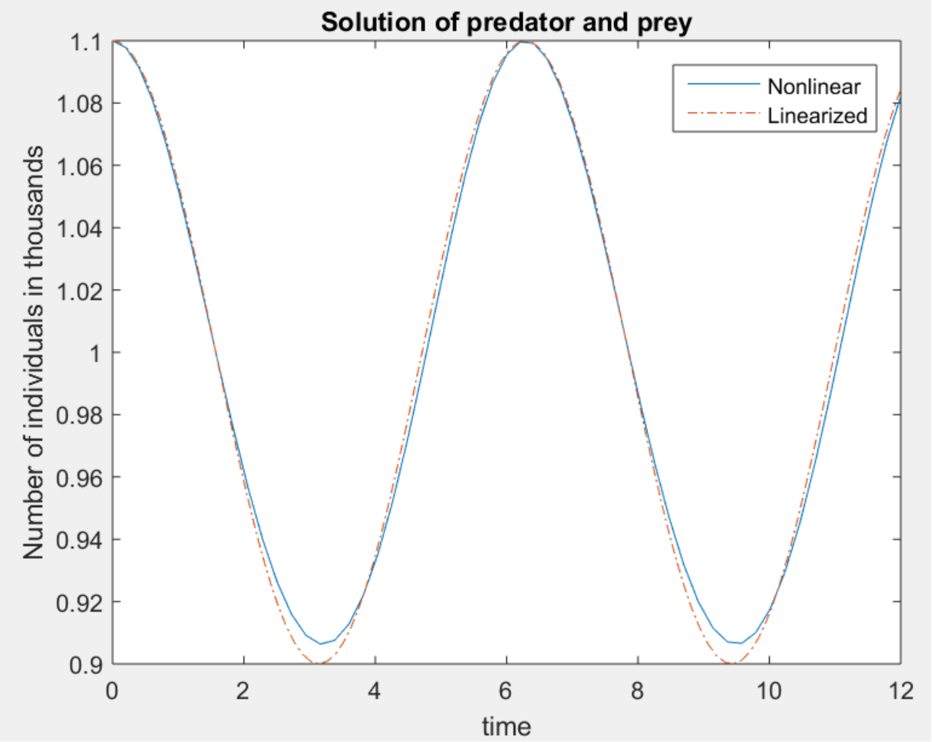
$$\Delta\dot{x} = A\Delta x + B\Delta u$$
$$\Delta y = C\Delta x + D\Delta u$$

Linearized predator and prey

```
%% Linearized model
lambda = [1 2];
gamma = [2 1];
alpha = [1 1];

A = [ 0 alpha(1)/alpha(2)*(lambda(2)-gamma(2)); ...
      -alpha(2)/alpha(1)*(gamma(1)-lambda(1)) 0];

model3 = @(t,y) A*y;

options = odeset('RelTol',1e-4,'AbsTol',[1e-6 1e-6]);
N1 = (lambda(2)-gamma(2))/alpha(2);
N2 = (gamma(1)-lambda(1))/alpha(1);
dN = [0.1 0];
timeSpan = [0 12];
initCond = [dN(1) dN(2)];
[t3,y3] = ode45(model3,timeSpan,initCond,options);
initCond = [N1+dN(1),N2+dN(2)];
[t2,y2] = ode45(model2,timeSpan,initCond,options);

plot(t2,y2(:,1),'-',t3,y3(:,1)+N1,'-.')
title('Solution of predator and prey');
xlabel('time');
ylabel('Number of individuals in thousands');
legend('Nonlinear','Linearized')
```

stationary condition

$$\frac{d}{dt}\begin{bmatrix}\Delta N_1(t)\\ \Delta N_2(t)\end{bmatrix} = \begin{bmatrix} 0 & \frac{\alpha_1}{\alpha_2}(\lambda_2-\gamma_2) \\ -\frac{\alpha_2}{\alpha_1}(\gamma_1-\lambda_1) & 0\end{bmatrix}\begin{bmatrix}\Delta N_1(t)\\ \Delta N_2(t)\end{bmatrix}$$



Recap: Laplace Transform

- A frequency domain representation of continuous time signal. Definition:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

$$f(t) = 0 \text{ for } t \leq 0;$$

$$f(t) \text{ is piecewise continuous for } t \geq 0$$

- Similar to Fourier Transform, where

$$s = \sigma + i\omega$$

- Inverse transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} F(s)e^{st}d\omega$$

- In practices, the inverse form can be calculated by e.g. Partial-Fraction Expansion

Important properties:

$$af(t) + bg(t) \quad \leftrightarrow \quad a\,\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

$$\boxed{\frac{df}{dt} \quad \leftrightarrow \quad sF(s) - f(0)}$$

$$f(t - \tau) \quad \leftrightarrow \quad F(s)e^{-s\tau}$$

$$f(t)e^{-at} \quad \leftrightarrow \quad F(s + a)$$

$$tf(t) \quad \leftrightarrow \quad -\frac{dF(s)}{ds}$$

$$\boxed{\int_0^t f(\tau)d\tau \quad \leftrightarrow \quad \frac{1}{s}F(s)}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

Transfer Function

- Consider a differential equation

$$a_0\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\ddot{x} + b_1\dot{x} + b_2 x$$

- Take Laplace transform of both side, we get

$$(a_0 s^3 + a_1 s^2 + a_2 s + a_3)Y(s) = (b_0 s^2 + b_1 s + b_2)X(s)$$

- Then we have

$$\frac{Y(s)}{X(s)} = \frac{b_0 s^2 + b_1 s + b_2}{a_0 s^3 + a_1 s^2 + a_2 s + a_3} = G(s)$$

- Where $G(s)$ is called transfer function, and

$$Y(s) = G(s)X(s)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Transfer Function and State-Space Representation

- If $x(0) = 0$, take Laplace Transform, we get

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

- Or

$$(sI - A)X(s) = BU(s)$$
$$X(s) = (sI - A)^{-1}BU(s)$$

- And

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

- So

$$G(s) = C(sI - A)^{-1}B + D$$

Example

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Problem

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0$$

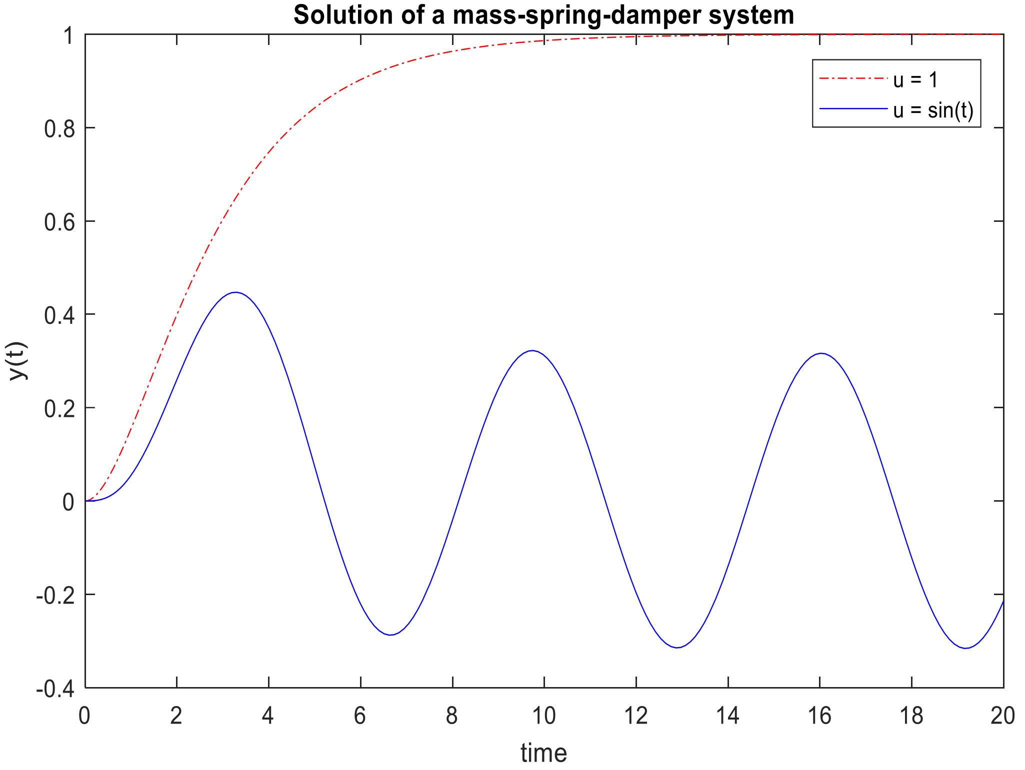
$$G(s) = C(sI - A)^{-1}B + D$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$
$$= \frac{1}{ms^2 + bs + k}$$

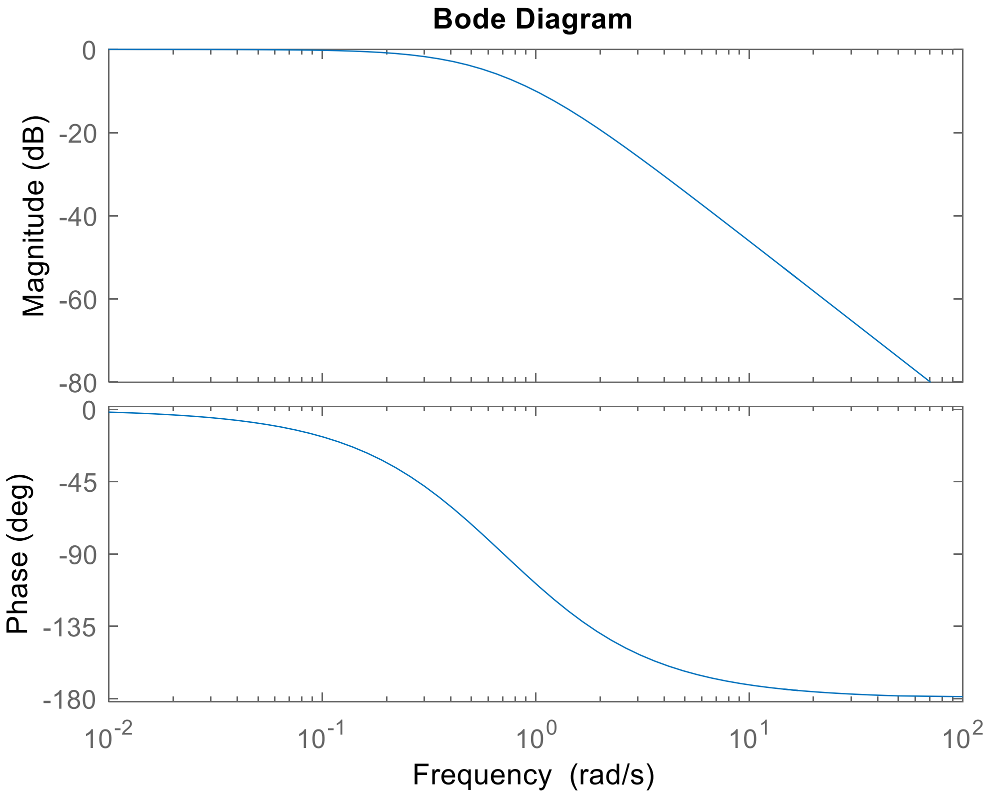
In Matlab

```matlab
%% state-space to transfer function
m = 2;
b = 3;
k = 1;

A = [ 0 1; -k/m -b/m];
B = [ 0 1/m]';
C = [1 0];
D = 0;

sys = ss(A,B,C,D);
[num,den] = ss2tf(A,B,C,D,1);

H = tf(num,den);


%% bode plot

bode(H)

```

```
H =

        1
  ---------------
  2 s^2 + 3 s + 1

Continuous-time transfer function.
```

Time response and frequency response





Phases of physical modeling

Phase I: Problem is structured

- Divide into subsystems, find cause-and-effect and interactions, draw block diagram

Phase II: Basic equations are formulated

- Relationships between variables and constants

Phase III: The state-space is formed

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$
or
$$x(t_{k+1}) = f(x(t_k), u(t_k))$$
$$y(t_{k+1}) = h(x(t_k), u(t_k))$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
or
$$x_{k+1} = Ax_k + Bu_k$$
$$y_{k+1} = Cx_k + Du_k$$

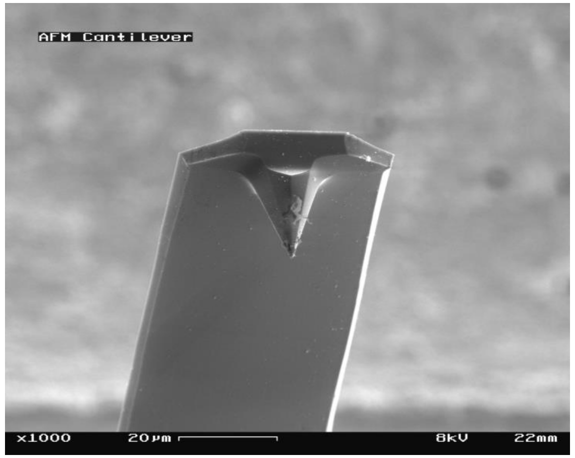
Number of state variables

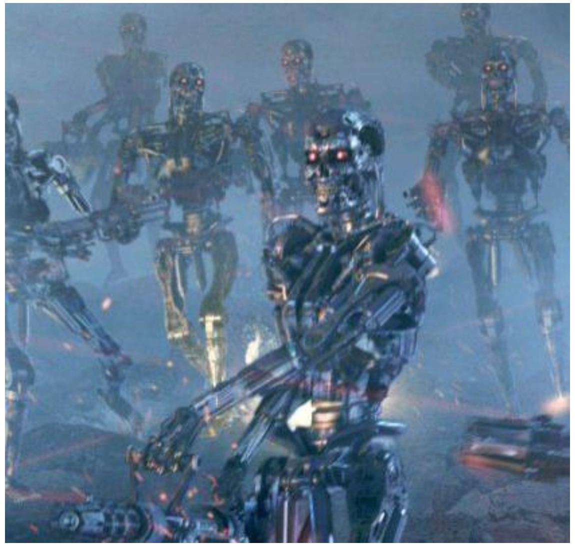
- Enough state variables? **Easy to check**
  - Can time derivatives be expressed only using state variables and inputs?

- Too many state variables (redundant)? **Difficult to check**
  - Linear systems: rank of a certain matrix can tell if states are completely redundant
  - Linear systems: we can also approximate high number of states with a lower number of states ➔ **Model order reduction methods**
  - Nonlinear systems?
  - However, redundant states will not affect simulation results, just wastes computations

Neglecting small effects

- Most physical laws are approximations, but the error might be acceptable
  - Gases are not ideal, flows are not laminar, springs do not have linear force to elongation etc, objects are deforming, etc.
- Small effects can be neglected
  - Air drag neglected at low velocities, adhesion force may be neglected in large body interaction
- Intuition, experience, back of an envelope calculations, and rules-of-thumb!





Separation of time constants

- Often time constants have different orders of magnitude. Which time scale is of interest?
  - Nuclear reactor control rods *t* ~ few seconds (neglect since fuel burnout *t* ~ months)
- Focus on timescale relevant to intended model use
  - Faster subsystems approximated with static relationships
    - Motion dynamics can be ignored for high level planning
  - Slower subsystems approximated with constants
    - Ambient temperature can often be regard as constant, degrading of the system is normally not taken into account
- We achieve...
  - Lower order model
  - Quick numerical solution

Aggregation of state variables

- *Merge several similar variables into* one *state variable*

- Usually by taking average:
  - Motion of the four wheels vs. the motion of the car
  - Air density is uniform in a room vs. density/pressure/temperature different at each point (partial differential equations)
  - Population: World vs. Country vs. City
  - Investments: Total vs. Private/Government vs. Per economic sector

Summary

- Structuring of the problem
- Ordinary differential equation
- State space construction
- Linearization
- Phases of modelling

Readings

- Ljung & Glad, Modeling of Dynamic System 1994, Ch. 3.1-3.6. Selective reading Ch. 4.
- Laplace transform: Franklin, Powell, Emani-Naeini, Feedback Control of Dynamic Systems, 2010, Ch. 3.1; Ogata, Modern Control Engineering, 2010, Ch. 2
- Other references for integral transforms:
  - any good engineering math books, e.g. Kreyszig 2011