Solution for problem 031601

Problem 031601: Check the scale factor of special conformal transformation: Eq. (4.16)

Solution: By using the properties of metric, we have

$$
\begin{aligned}
g_{\mu\nu}(x) &= \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta}(x') \\
&= \Lambda(x) \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g_{\alpha\beta}(x)
\end{aligned} \tag{1}
$$

then for the special conformal transformation

$$
x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2} \tag{2}
$$

one get

$$
\frac{\partial x'^{\alpha}}{\partial x^{\mu}} = \frac{\delta^{\alpha}_{\mu} - 2b^{\alpha} x_{\mu}}{h(x)} - \frac{\left(x^{\alpha} - b^{\alpha} x^2\right)\left(-2b_{\mu} + 2b^2 x_{\mu}\right)}{h^2(x)} \tag{3}
$$

where,

$$
h(x) = 1 - 2b \cdot x + b^2 x^2 \tag{4}
$$

So substitute (3) back to (1), we have

$$
\begin{aligned}
g_{\mu\nu}(x) = \Lambda(x) \Bigg(& g_{\alpha\beta}(x) \frac{1}{h^2(x)} \left(\delta^{\alpha}_{\mu} - 2b^{\alpha} x_{\mu}\right)\left(\delta^{\beta}_{\nu} - 2b^{\beta} x_{\nu}\right) \\
& - g_{\alpha\beta}(x) \frac{1}{h^3(x)} \left(\delta^{\alpha}_{\mu} - 2b^{\alpha} x_{\mu}\right)\left(x^{\beta} - b^{\beta} x^2\right)\left(-2b_{\nu} + 2b^2 x_{\nu}\right) \\
& - g_{\alpha\beta}(x) \frac{1}{h^3(x)} \left(\delta^{\beta}_{\nu} - 2b^{\beta} x_{\nu}\right)\left(x^{\alpha} - b^{\alpha} x^2\right)\left(-2b_{\mu} + 2b^2 x_{\mu}\right) \\
& + g_{\alpha\beta}(x) \frac{1}{h^4(x)} \left(x^{\alpha} - b^{\alpha} x^2\right)\left(x^{\beta} - b^{\beta} x^2\right)\left(-2b_{\mu} + 2b^2 x_{\mu}\right)\left(-2b_{\nu} + 2b^2 x_{\nu}\right) \Bigg) \\
= {} & \Lambda(x) \left(I + II + III + IV\right)
\end{aligned} \tag{5}
$$

where,

$$
\begin{aligned}
I &= g_{\alpha\beta}(x) \frac{1}{h^2(x)} \left(\delta^{\alpha}_{\mu} - 2b^{\alpha} x_{\mu}\right)\left(\delta^{\beta}_{\nu} - 2b^{\beta} x_{\nu}\right) \\
&= \frac{1}{h^2(x)} \left(g_{\mu\nu}(x) - 2b_{\mu} x_{\nu} - 2b_{\nu} x_{\mu} + 4b^2 x_{\mu} x_{\nu}\right) \\
\\
II &= -g_{\alpha\beta}(x) \frac{1}{h^3(x)} \left(\delta^{\alpha}_{\mu} - 2b^{\alpha} x_{\mu}\right)\left(x^{\beta} - b^{\beta} x^2\right)\left(-2b_{\nu} + 2b^2 x_{\nu}\right) \\
&= -\frac{1}{h^2(x)} \left(x_{\mu} - b_{\mu} x^2 - 2x_{\mu} b \cdot x + 2x_{\mu} x^2 b^2\right)\left(-2b_{\nu} + 2b^2 x_{\nu}\right)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
III &= -g_{\alpha\beta}(x)\frac{1}{h^3(x)}\left(\delta_\nu^\beta - 2b^\beta x_\nu\right)\left(x^\alpha - b^\alpha x^2\right)\left(-2b_\mu + 2b^2 x_\mu\right) \\
&= -\frac{1}{h^2(x)}\left(x_\nu - b_\nu x^2 - 2x_\nu b\cdot x + 2x_\nu x^2 b^2\right)\left(-2b_\mu + 2b^2 x_\mu\right) \\
IV &= g_{\alpha\beta}(x)\frac{1}{h^4(x)}\left(x^\alpha - b^\alpha x^2\right)\left(x^\beta - b^\beta x^2\right)\left(-2b_\mu + 2b^2 x_\mu\right)\left(-2b_\nu + 2b^2 x_\nu\right) \\
&= \frac{1}{h^3(x)}x^2\left(-2b_\mu + 2b^2 x_\mu\right)\left(-2b_\nu + 2b^2 x_\nu\right)
\end{aligned}
\tag{7}
$$

From these expressions, one can observe that

$$
II + \frac{1}{2}IV = \frac{1}{h^3(x)}(-x_\mu)\left(1 - 2b\cdot x + b^2x^2\right)\left(-2b_\nu + 2b^2x_\nu\right) = \frac{1}{h^2(x)}\left(2x_\mu b_\nu - 2b^2 x_\mu x_\nu\right) \tag{8}
$$

Similarly

$$
III + \frac{1}{2}IV = \frac{1}{h^2(x)}\left(2x_\nu b_\mu - 2b^2 x_\mu x_\nu\right) \tag{9}
$$

then, we have

$$
\begin{aligned}
I + II + III + IV = I &+ \left(II + \frac{1}{2}IV\right) + \left(III + \frac{1}{2}IV\right) \\
&= \frac{1}{h^2(x)}\left(g_{\mu\nu}(x) - 2b_\mu x_\nu - 2b_\nu x_\mu + 4b^2 x_\mu x_\nu\right) \\
&\quad + \frac{1}{h^2(x)}\left(2x_\mu b_\nu - 2b^2 x_\mu x_\nu\right) + \frac{1}{h^2(x)}\left(2x_\nu b_\mu - 2b^2 x_\mu x_\nu\right) \\
&= \frac{1}{h^2(x)}g_{\mu\nu}(x)
\end{aligned}
\tag{10}
$$

Now, turn back to (5), we have

$$
\Lambda(x) = h^2(x) = \left(1 - 2b\cdot x + b^2x^2\right)^2 \tag{11}
$$

That's indeed the result that we want to verify.