HW02

EX1

Q1

注意到$\mathbf{v}_1$是最大奇异值σ_1的奇异向量，故有:

$$
|\mathbf{v}_1| = 1 \\
A\mathbf{v}_1 = \sigma_1\mathbf{u}_1
$$

可得:

$$
|\mathbf{u}_1^T A| = |\mathbf{u}_1^T \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T| = |\mathbf{u}_1^T \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T| = \sigma_1 |\mathbf{v}_1^T| = \sigma_1
$$

Q2

设$\mathbf{u}$为单位向量，将其写为:

$$
\mathbf{u} = \sum_{i=1}^{r} \alpha_i \mathbf{u}_i
$$

则有:

$$
\sum_{i=1}^{r} \alpha_i^2 = 1
$$

带入计算:

$$
\begin{aligned}
||\mathbf{u}^T A|| &= ||\sum_{i=1}^{r} \alpha_i \mathbf{u}_i \cdot \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i|| \\
&= ||\sum_{i=1}^{r} \alpha_i \sigma_i \mathbf{v_i}|| \\
&\leq \sigma_1 ||\sum_{i=1}^{r} \alpha_i \mathbf{v}_i|| \\
&= \sigma_1 \sqrt{\sum_{i=1}^{r} \alpha_i^2} \\
&= \sigma_1
\end{aligned}
$$

再由$||\mathbf{u}^T A|| \leq \sigma_1$，可得:

$$
||\mathbf{u}_1^T A|| = \sigma_1 = max_{||\mathbf{u}||=1} ||\mathbf{u}^T A||
$$

EX2

令$A = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T = \sum_{i=1}^{d} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$，其中$\sigma_{i+1} = \ldots = \sigma_d = 0$。

将x改写为:

$$\mathbf{x} = \sum_{i=1}^{d} \alpha_i \mathbf{v}_i$$

再令$B = A^T A$，则有：

$$B^k \mathbf{x} = (A^T A)^k \mathbf{x} = \sum_{i=1}^{d} \sigma_i^{2k} \alpha_i \mathbf{v}_i$$

并注意到以下事实：

$$|\alpha_i| = |x^T \mathbf{v}_i| \geq \delta$$

由$\sigma_2 < \frac{1}{2}\sigma_1$：

$$\begin{aligned}
||B^k \mathbf{x}||^2 &= ||\sum_{i=1}^{d} \sigma_i^{2k} \alpha_i \mathbf{v}_i||^2 \\
&= \sum_{i=1}^{d} \sigma_i^{4k} \alpha_i^2 \\
&\leq \sigma_1^{4k} \alpha_1^2 + \sigma_2^{4k}(1 - \alpha_1^2) \\
&< \sigma_1^{4k}[(\frac{1}{2})^{4k}(1 - \alpha_1^2) + \alpha_1^2]
\end{aligned}$$

结合$k = -\log_4 \epsilon\delta$：

$$\begin{aligned}
|\mathbf{w}^T \mathbf{v}_1| &> \frac{(B^k x)^T \mathbf{v}_1}{\sqrt{\sigma_1^{4k}((\frac{1}{2})^{4k}(1 - \alpha_1^2) + \alpha_1^2)}} \\
&= \frac{\sum_{i=1}^{d} \sigma_i^{2k} \alpha_i \mathbf{v}_i^T \cdot \mathbf{v}_1}{\sqrt{\sigma_1^{4k}((\frac{1}{2})^{4k}(1 - \alpha_1^2) + \alpha_1^2)}} \\
&= \frac{\sigma_1^{2k} \alpha_1}{\sqrt{\sigma_1^{4k}((\frac{1}{2})^{4k}(1 - \alpha_1^2) + \alpha_1^2)}} \\
&= \frac{1}{\sqrt{(\frac{1}{2})^{4k}(\frac{1}{\alpha_1^2} - 1) + 1}} \\
&= \frac{1}{\sqrt{(\frac{1}{2})^{-2\log_2 \epsilon\delta}(\frac{1}{\alpha_1^2} - 1) + 1}} \\
&= \frac{1}{\sqrt{(\epsilon\delta)^2(\frac{1}{\alpha_1^2} - 1) + 1}} \\
&\geq \frac{1}{\sqrt{(\epsilon\delta)^2(\frac{1}{\delta^2} - 1) + 1}} \\
&= \frac{1}{\sqrt{\epsilon^2(1 - \delta^2) + 1}} \\
&> \frac{1}{\sqrt{\epsilon^2 + 1}} \\
&\geq 1 - \frac{1}{2}\epsilon^2 \text{（伯努利不等式）}
\end{aligned}$$

当$\epsilon \in [0,2]$时，$\frac{1}{2}\epsilon^2 \le \epsilon$，即$1-\frac{1}{2}\epsilon^2 \ge 1-\epsilon$，题目得证。由题意可知，$\epsilon$恒为整数；再可知当$\epsilon > 2$时，$1-\epsilon$恒为负数，而模长显然恒为正数，题目显然得证。

综上所述，题目得证。

EX3

Q1

根据题意，$E[u_{ij}] = \frac{1}{2} \times (-1) + \frac{1}{2} \times 1 = 0, a$ 与u_{ij}独立

我们有：

$$
\begin{aligned}
E[b_j] &= E[\frac{1}{\sqrt{k}}\sum_{i=1}^{d} a_i u_{ij}] \\
&= \frac{1}{\sqrt{k}}\sum_{i=1}^{d} a_i E[u_{ij}] \\
&= \frac{1}{\sqrt{k}}\sum_{i=1}^{d} a_i \cdot 0 \\
&= 0
\end{aligned}
$$

Q2

根据题意中有关u_{ij}的独立性，显然有:

$$
E[u_{ij}^2] = 1
$$
$$
E[u_{ij}u_{lj}] = 0
$$

我们按照下面第二部分的方法来分解这个期望:

$$
\begin{aligned}
E[b_j^2] &= E[(\frac{1}{\sqrt{k}}\sum_{i=1}^{d} a_i u_{ij})^2] \\
&= \frac{1}{k}(\sum_{i=1}^{d} E[a_i^2 u_{ij}^2] + \sum_{i\neq l} E[a_i a_l u_{ij} u_{lj}]) \\
&= \frac{1}{k}(\sum_{i=1}^{d} E[a_i^2]E[u_{ij}^2] + \sum_{i\neq l} E[a_i a_l]E[u_{ij} u_{lj}]) \\
&= \frac{1}{k}(\sum_{i=1}^{d} E[a_i^2] + \sum_{i\neq l} E[a_i a_l]\cdot 0) \\
&= \frac{1}{k}\sum_{i=1}^{d} E[a_i^2] \\
&= \frac{1}{k}E[\sum_{i=1}^{d} a_i^2] \\
&= \frac{1}{k}||a||_2^2
\end{aligned}
$$

Q3

根据题意，我们有：

$$
\begin{aligned}
E[||f(a)||^2] &= E[\sum_{j=1}^{k}(\frac{1}{\sqrt{k}}\sum_{i=1}^{d}a_iu_{ij})^2)] \\
&= \sum_{j=1}^{k}E[b_j^2] \\
&= \frac{1}{k}\sum_{j=1}^{k}||a||_2^2 \\
&= ||a||_2^2
\end{aligned}
$$

EX4

设$k \in \mathbb{N}$，满足

$$k = \lceil \log_{0.4}(\delta) \rceil$$

将$\mathcal{A}$作为子过程k次，来提高成功率。

下面介绍$\mathcal{B}$的算法描述：

> for i = 1 to k *begin*
>
>   Query $\mathcal{A}$ on the input vertex x
>
>   if $\mathcal{A}$ output some $a_i \in \mathcal{P}$ with $d(x, a_i) \leq c \cdot r$ *begin*
>
>   output a_i
>
>   halt()
>
>   *end*
>
> *end*
>
> // If none of the k iterations output a point $a_i \in \mathcal{P}$ with $d(x, a_i) \leq c \cdot r$,
>
> output a point a_{k+1} in $\mathcal{P}$ randomly
>
> halt()

正确性和成功概率：$\mathcal{B}$未能输出$d(x, a_i) \leq c \cdot r$的点的概率就是k次迭代都未能输出这样的点的概率，就是0.4^k，根据选择的k，有$0.4^k \leq \delta$，故成功概率至少为$1 - \delta$

查询时间：查询时间是$\mathcal{A}$的k倍，因此时间为$kT_{\mathcal{A}} = \lceil \log_{0.4}(\delta) \rceil T_{\mathcal{A}}$

EX5

Q1

先证明：

$$E[\frac{(1+\alpha)^{X_n}}{\alpha}] = n + \frac{1}{\alpha}$$

由归纳法证明：

当$n=0$时，结论显然成立

当$n>0$时：

$$\begin{aligned}
E[\frac{(1+\alpha)^{X_{n+1}}}{\alpha}] &= \sum_{j=0}^{\infty} P(X_n=j)E[\frac{(1+\alpha)^{X_{n+1}}}{\alpha}|X_n=j] \\
&= \sum_{j=0}^{\infty} P(X_n=j)((1-\frac{1}{(1+\alpha)^j})\cdot\frac{(1+\alpha)^j}{\alpha} + \frac{1}{(1+\alpha)^j}\cdot\frac{(1+\alpha)^{j+1}}{\alpha}) \\
&= \sum_{j=0}^{\infty} P(X_n=j)(\frac{(1+\alpha)^j}{\alpha} - \frac{1}{\alpha} + \frac{1+\alpha}{\alpha}) \\
&= \sum_{j=0}^{\infty} P(X_n=j)(\frac{(1+\alpha)^j}{\alpha} + 1) \\
&= \sum_{j=0}^{\infty} P(X_n=j)\frac{(1+\alpha)^j}{\alpha} + 1 \\
&= E[\frac{(1+\alpha)^{X_n}}{\alpha}] + 1 \\
&= (n+1) + \frac{1}{\alpha}
\end{aligned}$$

由归纳法，得证

因此：

$$E[\frac{(1+\alpha)^{X_n}-1}{\alpha}] = n + \frac{1}{\alpha} - \frac{1}{\alpha} = n$$

由此可得：

$$E[(1+\alpha)^{X_n}] = \alpha n + 1$$

接下来求方差，显然有：

$$Var[\frac{(1+\alpha)^{X_n}}{\alpha}] = \frac{1}{\alpha^2}Var[(1+\alpha)^{X_n}]$$

先求平方期望：

$$
\begin{aligned}
E[(1+\alpha)^{2X_n}] &= \sum_{j=0}^{\infty} P(X_n=j)E[(1+\alpha)^{2X_{n+1}}|X_n=j] \\
&= \sum_{j=0}^{\infty} P(X_n=j)((1-\frac{1}{(1+\alpha)^j})\cdot(1+\alpha)^{2j}+\frac{1}{(1+\alpha)^j}\cdot(1+\alpha)^{2j+2} \\
&= \sum_{j=0}^{\infty} P(X_n=j)((1+\alpha)^{2j}+\alpha(\alpha+2)(1+\alpha)^j) \\
&= E[(1+\alpha)^{2X_n}]+\alpha(\alpha+2)E[(1+\alpha)^{X_n}] \\
&= E[(1+\alpha)^{2X_n}]+\alpha(\alpha+2)(\alpha n+1) \\
&= E[(1+\alpha)^{2X_0}]+\sum_{j=0}^{n-1}\alpha(\alpha+2)(\alpha j+1) \\
&= (\frac{\alpha^3}{2}+\alpha^2)n^2+(2\alpha-\frac{\alpha^3}{2})n+1
\end{aligned}
$$

由此计算方差：

$$
\begin{aligned}
Var[\frac{(1+\alpha)^{X_n}}{\alpha}] &= \frac{1}{\alpha^2}Var[(1+\alpha)^{X_n}] \\
&= \frac{1}{\alpha^2}(E[(1+\alpha)^{2X_n}]-E[(1+\alpha)^{X_n}]^2) \\
&= \frac{1}{\alpha^2}((\frac{\alpha^3}{2}+\alpha^2)n^2+(2\alpha-\frac{\alpha^3}{2})n+1-(\alpha^2n^2+2\alpha n+1)) \\
&= \frac{\alpha}{2}n^2-\frac{\alpha}{2}n \\
&< \frac{\alpha}{2}n^2
\end{aligned}
$$

Q2

算法描述：

> 1. 独立运行s次上述算法，得到n的所有估计值$\tilde{n}_1, \tilde{n}_1, \ldots,\ \tilde{n}_s$，其中$s \geq \frac{\alpha}{2\delta\epsilon^2}$
> 2. 输出$\tilde{n} = \frac{1}{s}\sum_{i=1}^{s}\tilde{n}_i$，

正确性：

$$E[\tilde{n}] = \frac{1}{s}\cdot s\cdot n = n$$

$$Var[\tilde{n}] = \frac{1}{s^2}\cdot s\cdot\frac{\alpha}{2}(n^2-n) \leq \frac{\alpha}{2s}n^2$$

由切比雪夫不等式可得：

$$P[|\tilde{n}-n| > \epsilon n] < \frac{Var[\tilde{n}]}{\epsilon^2 n^2} = \frac{\alpha}{2s\epsilon^2} \leq \delta$$

由此可知，该算法以至少$1-\delta$的概率，返回一个n的估计值$\tilde{n}$，满足$|\tilde{n}-n| \leq \epsilon n$

最坏空间：

$$s\log_2\log_{1+\alpha} n = O(\frac{1}{\delta\epsilon^2}\log\log n)$$

EX6

证明：设$Ham(\mathbf{x},\mathbf{y})=H$

$$\begin{aligned}\Pr[(U\mathbf{x})_i \neq (U\mathbf{y})_i] &= \Pr[\sum_{j=1}^{d} u_{ij}(x_j - y_j) \neq 0] \\ &= Pr[\sum_{x_j \neq y_j} u_{ij}(x_j - y_j)]\end{aligned}$$

注意到，这个求和式子一共有H项，在$mod 2$意义下，$x_j - y_j = 1$，因此，我们只需要在这H个u_{ij}中选奇数个为1，其余为0即可，这显然是一个二项分布：

$$\begin{aligned}\Pr[(U\mathbf{x})_i \neq (U\mathbf{y})_i] &= \Pr[\sum_{x_j \neq y_j} u_{ij}(x_j - y_j)] \\ &= \sum_{k \in odd} C_H^k p^k (1-p)^{H-k}\end{aligned}$$

下面我们就来求解这个组合求和式子，基于如下两个方程：

$$\sum_{k \in odd} C_H^k p^k (1-p)^{H-k} + \sum_{k \in even} C_H^k p^k (1-p)^{H-k} = 1$$
$$\sum_{k \in odd} C_H^k (-p)^k (1-p)^{H-k} + \sum_{k \in even} C_H^k (-p)^k (1-p)^{H-k} = (1-2p)^H$$

由此推出：

$$\Pr[(U\mathbf{x})_i \neq (U\mathbf{y})_i] = \sum_{k \in odd} C_H^k p^k (1-p)^{H-k} = \frac{1}{2}(1-(1-2p)^{Ham(\mathbf{x},\mathbf{y})})$$