ETM540 - Homework#3 - Sensitivity Analysis

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Exercise:

After the recent forest fires, traditional supplier relationships were disrupted and the grocery retailer entire foods, needs to find new local suppliers. They have contacted Meredith Gray to buy as much organically grown ingredients as she could grow. She has three organic farms near Portland with the following characteristics:

```
FarmData=matrix(c(
  1,400,1500,2,600,2000,3,300,900),
  ncol = 3,byrow=2)
colnames(FarmData)<-c("Farm","Acreage","WaterAvail")
FarmData <-as.data.frame(FarmData)
attach(FarmData)
pander(FarmData, caption = "Farm Details")
```

Table 1: Farm Details

Farm	Acreage	WaterAvail
1	400	1500
2	600	2000
3	300	900

Three crops are needed. However the maximum acreage which can be grown of each crop is limited by amount of appropriate harvesting equipment available:

```
CropData=matrix(c(
  700,6,400,800,4,300,300,2,100),
  ncol = 3,byrow=2)
colnames(CropData)<-c("HarvestCap","WaterReq","Profit")
row.names(CropData) <- c("Corn","Beans","Wheat")
CropData <-as.data.frame(CropData)
attach(CropData)
pander(CropData, caption = "Crop Details")
```

Table 2: Crop Details

	HarvestCap	WaterReq	Profit
Corn	700	6	400
Beans	800	4	300
Wheat	300	2	100

Solution:

Mathemaical Representation of Meredith Gray's Crop Problem:

$$
\begin{split}
(\#x_{i,j} : Is\ Crop\ i\ in\ Farm\ j) \\
\text{Maximize } & \sum_j \sum_i P_i x_{i,j} \\
\text{subject to } & \sum_i x_{i,j} \leq Acreage_j \; \forall \; j \\
& \sum_i R_j x_{i,j} \leq WaterAvail_j \; \forall \; j \\
& \sum_j x_{i,j} \leq HarvestCap_i \; \forall \; i \\
& x_{i,j} \geq 0 \; \forall \; i,j
\end{split}
$$

Build Model with Maximizing Profit given constraints:

```
MaxCropModel <- MIPModel() %>%
  add_variable (x[i,j], i=1:3, j=1:3, type="continuous", lb=0) %>%
  set_objective (sum_expr(sum_expr(Profit[i]*x[i,j],i=1:3), j=1:3),"max") %>%
  add_constraint (sum_expr(x[i,j], i=1:3) <= Acreage[j], j=1:3) %>%
  add_constraint (sum_expr(WaterReq[i]*x[i,j], i=1:3) <= WaterAvail[j], j=1:3) %>%
  add_constraint (sum_expr(x[i,j], j=1:3) <= HarvestCap[i], i=1:3) %>%

  solve_model(with_ROI(solver = "glpk"))
MaxCropModel
```

```
## Status: optimal
## Objective value: 320000
```

```
CropTable =matrix(c((MaxCropModel$solution)), ncol = 3,byrow=3)
row.names(CropTable) <- c("Farm1", "Farm2","Farm3")
colnames(CropTable)<-c("Corn","Bean","Wheat")
pander(CropTable, caption = "Total Crop Produce Plan per Farm")
```

Table 3: Total Crop Produce Plan per Farm

	Corn	Bean	Wheat
Farm1	0	375	0
Farm2	200	200	0
Farm3	0	225	0

- As per above table, with given constraints the Maximum Profit = $320,000
- Table includes the amount of each crop to be grown per farm

Shadow Prices / Row Duals:

```
rdualCrop =matrix(c(
  get_row_duals(MaxCropModel)), ncol = 1)
dimnames(rdualCrop)<-list(c("Farm1Acreage", "Farm2Acreage", "Farm3Acreage",
                            "Farm1WaterAvail", "Farm2WaterAvai","Farm3WaterAvai",
                            "CornHarvCap","BeanHarvCap","WheatHarvCap"), c("Row Duals"))
rdualCrop <-as.data.frame(rdualCrop)
attach(rdualCrop)
pander(rdualCrop, caption = "Shadow Prices of Constrained Resources")
```

Table 4: Shadow Prices of Constrained Resources

	Row Duals
Farm1Acreage	0
Farm2Acreage	0
Farm3Acreage	0
Farm1WaterAvail	66.67
Farm2WaterAvai	66.67
Farm3WaterAvai	66.67
CornHarvCap	0
BeanHarvCap	33.33
WheatHarvCap	0

The shadow price for the Water Availability and Bean Harvest Capacity are more interesting:

Considering the first Row - This means that for Farm1 -> If we increase the Acreage by 1acre (unit change), there will be no affect of the Objective Function (Profit will stay the same)

Whereas, for in Farm 1 if we increase the Water Availability by 1 acre-ft (unit change), this will imply an increase in Profit by $66.67.

Similarly if we increase the Harvest Capacity for Beans by 1 acre (unit change), this will yield to an increase of $33.33 in our objective value.

The entire water availability for each farm is been consumed entirely by the optimal production plan. Increasing the quantity of water available to each farm may allow Meredith Gray to change her production plan and increase the profit. For increase in unit acre-ft of water availability to farm 1, farm2, farm3, will provide additional profit of $66.67/farm.

In addition, as per the optimal production plan, beans are grown upto their maximum harvest capacity, i.e 800 in this case. Hence, any further increase in the harvest capacity fof this particular crop will yield an increase in profit and change the entire production plan. For every unit increase in the beans harvest capacity implies $33.33 increase in our objective value.

Shadow Prices affect: Testing the Model with increase Water Availability =1 acre-ft/farm:

$$(\#x_{i,j} : Is\ Crop\ i\ in\ Farm\ j)$$

$$\begin{aligned} \text{Maximize } & \sum_i \sum_j P_i x_{i,j} \\ \text{subject to } & \sum_i x_{i,j} \leq Acreage_j \; \forall \; j \\ & \sum_i R_j x_{i,j} \leq (WaterAvail_j + 1) \;\; \forall \; j \end{aligned}$$

$$(\#Increase\ total\ water\ availability\ in\ each\ farm\ by\ 1 acre - ft)$$

$$\begin{aligned} & \sum_j x_{i,j} \leq HarvestCap_j \; \forall \; i \\ & x_{i,j} \geq 0 \; \forall \; i, j \end{aligned}$$

```
WaterAvailIncModel <- MIPModel() %>%
  add_variable (x[i,j], i=1:3, j=1:3, type="continuous", lb=0) %>%
  set_objective (sum_expr(sum_expr(Profit[i]*x[i,j],i=1:3), j=1:3),"max") %>%
  add_constraint (sum_expr(x[i,j], i=1:3) <= Acreage[j], j=1:3) %>%
  add_constraint (sum_expr(WaterReq[i]*x[i,j], i=1:3) <= (WaterAvail[j]+1), j=1:3) %>%
  add_constraint (sum_expr(x[i,j], j=1:3) <= HarvestCap[i], i=1:3) %>%

  solve_model(with_ROI(solver = "glpk"))
WaterAvailIncModel
```

```
## Status: optimal
## Objective value: 320200
```

```
WaterAvailInc_table =matrix(c((WaterAvailIncModel$solution)), ncol = 3,byrow=3)
row.names(WaterAvailInc_table) <- c("Farm1", "Farm2","Farm3")
colnames(WaterAvailInc_table)<-c("Corn","Bean","Wheat")
pander(WaterAvailInc_table,
       caption = "Crop Produce Plan with Additional Water Available =66.67 acre-ft/farm")
```

Table 5: Crop Produce Plan with Additional Water Available =66.67 acre-ft/farm

	Corn	Bean	Wheat
Farm1	0	375.2	0
Farm2	200.5	199.5	0
Farm3	0	225.2	0

As per above table, with given constraints the Maximum Profit = \$320,200. This can also be verified manually - the rduals for Water Availability for each farm = \$66.67. So when we increased the availability of water in all the farms, the profit would increase by *\$66.67+\$66.67+\$66.67 = \$200* compared to the previous production plan.

Max Profit with unit change in water avail/farm - Max Profit with existing constraints

= \$(320200 - 32000) = \$200

Considering the change in one constraint, the production plan has changed as seen in above table.

Shadow Price of Underutilzed Resources:

The shadow price on Acreage for each farm is zero. This means that even a large increase in Acrease for farm1 or farm2 or farm 3, would not affect the maximum profit or the optimal production plan. Essentially there is plenty of land availabile in each farm which is not been used, so having more would not enable any better profit plan.

Let's confirm this as well with a numerical example by increasing the Acreage of each farm by *5,000* acres.

$$
\begin{aligned}
\text{Maximize } & \sum_i \sum_j P_i x_{i,j} \\
\text{subject to } & \sum_i x_{i,j} \leq (Acreage_j + 100) \; \forall \, j \\
(\#Increase\ Acreage\ by\ 100acre\ /farm) & \\
& \sum_i R_j x_{i,j} \leq WaterAvail_j \;\; \forall \, j \\
& \sum_j x_{i,j} \leq HarvestCap_j \; \forall \, i \\
& x_{i,j} \geq 0 \; \forall \, i, j
\end{aligned}
$$

```
AcreageIncModel <- MIPModel() %>%
  add_variable (x[i,j], i=1:3, j=1:3, type="continuous", lb=0) %>%
  set_objective (sum_expr(sum_expr(Profit[i]*x[i,j],i=1:3), j=1:3),"max") %>%
  add_constraint (sum_expr(x[i,j], i=1:3) <= (Acreage[j]+100), j=1:3) %>%
  add_constraint (sum_expr(WaterReq[i]*x[i,j], i=1:3) <= WaterAvail[j], j=1:3) %>%
  add_constraint (sum_expr(x[i,j], j=1:3) <= HarvestCap[i], i=1:3) %>%

  solve_model(with_ROI(solver = "glpk"))
AcreageIncModel
```

```
## Status: optimal
## Objective value: 320000
```

```
AcreageInc_table =matrix(c((AcreageIncModel$solution)), ncol = 3,byrow=3)
row.names(AcreageInc_table) <- c("Farm1", "Farm2","Farm3")
colnames(AcreageInc_table)<-c("Corn","Bean","Wheat")
pander(AcreageInc_table, caption = "Crop Produce Plan with Additional Acreage= 100acre/farm")
```

Table 6: Crop Produce Plan with Additional Acreage= 100acre/farm

	Corn	Bean	Wheat
Farm1	0	375	0
Farm2	200	200	0
Farm3	0	225	0

As we anticipated, even though we increased each farm with 100 acres, there is no change in the Profit value. The production plan remained as is. This implies that with the existing constrains, there was already under utilized acres of land in each farm, so no matter how much we increase the land further, it won't change the optimal production plan.

Reduced Costs of Variables / cduals:

```
cdualCrop =matrix(c(get_column_duals(MaxCropModel)),ncol = 1)
dimnames(cdualCrop)<-list(c("Farm1Corn","Farm1Bean","Farm1Wheat",
                            "Farm2Corn","Farm2Bean","Farm2Wheat",
                            "Farm3Corn", "Farm3Bean", "Farm3Wheat"), c("Column Duals"))
cdualCrop <-as.data.frame(cdualCrop)
attach(cdualCrop)
pander(cdualCrop, caption = "Incorrect Reduced Cost Variables")
```

Table 7: Incorrect Reduced Cost Variables

	Column Duals
Farm1Corn	0
Farm1Bean	0
Farm1Wheat	-33.33
Farm2Corn	0
Farm2Bean	0
Farm2Wheat	-33.33
Farm3Corn	0
Farm3Bean	0
Farm3Wheat	-33.33

The above table provides the cduals / Reduced costs of each crop in farm1, farm2 and farm3 resp. If you notice the wheat grown in each farm yields a value of -33.33. This means that if we increase the growth of wheat by 1 unit in any farm, this will decrease our profit by $33.33. For the remaining crops - Corn and Beans, the value of the resources used in producing them equals that of the profit of each crop resp.

Let's check the value of Resources - For Corn in Farm 1 (i= 1 & j=1)

```
CornFarm1_res_used<-cbind(rdualCrop,c(1,0,0,6,0,0,1,1,1))
colnames(CornFarm1_res_used)<-c("Row Duals", "Corn in Farm1 Resources Used")
pander(CornFarm1_res_used, caption="Resources Used by Corn in Farm 1 & Shadow Prices")
```

Table 8: Resources Used by Corn in Farm 1 & Shadow Prices

	Row Duals	Corn in Farm1 Resources Used
Farm1Acreage	0	1
Farm2Acreage	0	0
Farm3Acreage	0	0
Farm1WaterAvail	66.67	6
Farm2WaterAvai	66.67	0
Farm3WaterAvai	66.67	0
CornHarvCap	0	1
BeanHarvCap	33.33	1
WheatHarvCap	0	1

Simply multiplying across and adding the value, we can see that the shadow cost of the resources used in growing corn in farm1 is:

$\$66.667 * 6 = \$400.02$ which is the same as the profit/corn.

Let's check Value of Resources - For Wheat in Farm 1 (i= 3 & j=1)

```
WheatFarm1_res_used<-cbind(rdualCrop,c(1,0,0,2,0,0,0,0,1))
colnames(WheatFarm1_res_used)<-c("Row Duals", "Wheat in Farm 1 Resources Used")
pander(WheatFarm1_res_used, caption="Resources Used by Wheat in Farm 1 & Shadow Prices")
```

Table 9: Resources Used by Wheat in Farm 1 & Shadow Prices

	Row Duals	Wheat in Farm 1 Resources Used
Farm1Acreage	0	1
Farm2Acreage	0	0
Farm3Acreage	0	0
Farm1WaterAvail	66.67	2
Farm2WaterAvai	66.67	0
Farm3WaterAvai	66.67	0
CornHarvCap	0	0
BeanHarvCap	33.33	0
WheatHarvCap	0	1

Multiplying across and adding the value, we can see that the shadow cost of the resources used in growing wheat in farm1 is:

$\$66.667 * 2 = \$133.34$ which is the same as the profit/wheat. Alas, the profit for each unit growth of wheat is just \$100 means that forcing the production of unit acre of wheat will decrease the production plan's profit by \$33.34. In other words, the impact on the objective function is $\$ - 33.34$ which is the same as the cdulas/reduced price entry of Tables.

Results when we grow wheat on atleast 1acre on land:

$$
\begin{aligned}
\text{Maximize } & \sum_i \sum_j P_i x_{i,j} \\
\text{subject to } & \sum_i x_{i,j} \leq Acreage_j \; \forall \; j \\
& \sum_i R_j x_{i,j} \leq WaterAvail_j \;\; \forall \; j \\
& \sum_j x_{i,j} \leq HarvestCap_j \; \forall \; i \\
& x_{i,j} \geq 0 \; \forall \; i \neq 3, j \neq 1 \\
& x_{3,1} \geq 1
\end{aligned}
$$

$(\#Plan\ to\ grow\ minimum\ 1acre\ wheat\ on\ farm1)$

```
WheatFarm1Model <- MIPModel() %>%
  add_variable (x[i,j], i=1:3, j=1:3, type="continuous", lb=0) %>%
  set_objective (sum_expr(sum_expr(Profit[i]*x[i,j],i=1:3), j=1:3),"max") %>%
  add_constraint (sum_expr(x[i,j], i=1:3) <= Acreage[j], j=1:3) %>%
  add_constraint (sum_expr(WaterReq[i]*x[i,j], i=1:3) <= WaterAvail[j], j=1:3) %>%
  add_constraint (sum_expr(x[i,j], j=1:3) <= HarvestCap[i], i=1:3) %>%
  add_constraint(x[3,1]>=1)%>%
```

```r
  solve_model(with_ROI(solver = "glpk"))
WheatFarm1Model
```

```
## Status: optimal
## Objective value: 319966.7
```

```r
WheatFarm1_table =matrix(c((WheatFarm1Model$solution)), ncol = 3,byrow=3)
row.names(WheatFarm1_table) <- c("Farm1", "Farm2","Farm3")
colnames(WheatFarm1_table)<-c("Corn","Bean","Wheat")
pander(WheatFarm1_table, caption = "Crop Produce Plan with atleast wheat 1acre in Farm1")
```

Table 10: Crop Produce Plan with atleast wheat 1acre in Farm1

	Corn	Bean	Wheat
Farm1	0	374.5	1
Farm2	199.7	200.5	0
Farm3	0	225	0

As we forced the production to grow wheat atleast on 1 acre, the objective value decreased to $319,966.7 (which is exactly $33.3 less than the original optimal solution $320,000). This confirms by the cduals reduced price value on wheat/farm.