Qubit Introduction

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1 Introduction to Qubits

1.1 Computers and Computation

Models of Classical Computation

Classical computation can be implemented through various physical models:

- **Mechanical**: Curta calculator, Digi-comp I, Babbage's difference engine
- **Electrical**: Electronic circuits and transistor-based systems
- **Optical**: Photonic computing systems
- **Biological**: DNA and molecular computing

Conceptual Models

Key theoretical frameworks for computation include:

- Turing Machine (mathematical model of computation)
- Cellular Automata
- Von Neumann architecture (CPU, Memory, Bus, I/O)
- Logic-in-Memory computing paradigm

Quantum Computing Development

In the early 1980s, Richard Feynman proposed that simulating quantum systems requires quantum computers. This led to the fundamental question: what algorithms could provide a quantum advantage for real-world problems?

Two major milestones occurred in the mid-1990s:

1. Discovery of Shor's algorithm for factoring (addressing an important practical problem)
2. Development of quantum error-correcting codes by Shor and colleagues

Since then, research has focused on both the underlying physics and hardware development for quantum computers. Today, operating quantum computers with more than 100 qubits exist, with IBM's Condor exceeding 1000 qubits.

1.2 Data Representation: Classical vs. Quantum

Classical computers use bits (0 or 1), while quantum computers use quantum bits (qubits) that can exist in superposition: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$.

Quantum Parallelism

Unlike classical bits, quantum systems exhibit superposition, allowing exponential encoding of information. While N classical bits represent a single N-bit state, N qubits exist in a superposition of all 2^N possible states simultaneously.

For example, with $N = 3$ qubits:

$$|\psi\rangle = \sum_{i=0}^{7} c_i |i\rangle$$

where each c_i is a complex amplitude and $\sum_{i=0}^{7} |c_i|^2 = 1$.

This quantum parallelism enables quantum algorithms to process all 2^N computational paths concurrently, providing potential exponential speedup over classical approaches. However, measurement collapses this superposition to a single state, requiring careful algorithm design to extract useful information.

1.3 DiVincenzo Criteria for Quantum Computing

A viable qubit must satisfy the DiVincenzo Criteria:

1. Scalability of the physical system
2. Ability to initialize qubit states
3. Long coherence times (T_1, T_2, T_ϕ)
4. Universal quantum gate operations
5. High-fidelity quantum operations
6. Interconversion between stationary and flying qubits
7. Faithful transmission of flying qubits

1.4 Qubit Figure of Merits

Qubit Robustness and Coherence Time

Qubits lose information through interaction with the environment via:

- Energy relaxation (T_1): loss of energy to the environment
- Dephasing (T_ϕ): loss of phase coherence

The effective coherence time is $\frac{1}{T_2} = \frac{1}{T_\phi} + \frac{1}{2T_1}$, representing the average time a qubit remains coherent.

Gate Speed and Figure of Merit

The number of gates performable within qubit lifetime determines quantum advantage: $N_{\text{gates}} \sim t_{\text{gate}}/T_2$.

Gate Fidelity

Gate fidelity quantifies how accurately quantum gate operations are performed, measuring the closeness between ideal and actual operations under imperfections and noise. High-fidelity operations are crucial for reliable quantum computation and can be determined through Process Tomography or Randomized Benchmarking.

1.5 Bits vs. Qubits

Classical vs. Quantum Gates

Classical computers use Boolean logic gates (such as NOT and AND) to form a universal set of operations. Quantum computers similarly use quantum gates, but with fundamentally different properties:

- Classical gates: irreversible, map multiple inputs to outputs
- Quantum gates: unitary and reversible, preserve quantum information
- Single quantum gates (e.g., X-gate) can manipulate qubit states
- Universal quantum computation requires a combination of single and two-qubit gates

The X-gate example shows practical application: applying an X-gate to a qubit in state $|0\rangle$ transitions it to $|1\rangle$, achieved experimentally through a π-pulse envelope.

Circuits in Space vs. Circuits in Time

Classical logic operates as **circuits in space**:

- Example: NOT gate maps $0 \rightarrow 1$ and $1 \rightarrow 0$
- Horizontal axis represents space
- Input and output exist at different physical locations
- Measurements can be performed simultaneously on different wires

In contrast, quantum logic operates as **circuits in time**:

- Example: X gate (quantum NOT) acts on qubit states $|0\rangle$ and $|1\rangle$
- Horizontal axis represents time
- Input and output correspond to the same qubit before and after the operation
- State is sequentially updated through time evolution

Key distinction:

- Classical gates connect wires in space, allowing parallel independent operations
- Quantum gates are unitary operations that evolve the state of qubits in time
- Each gate application modifies and overwrites the qubit's state

Two-Qubit Gates

Two-qubit gates enable quantum entanglement and are essential for universal quantum computation.

CNOT Gate (Controlled-NOT): The CNOT gate operates on two qubits: a control qubit and a target qubit. It applies an X gate to the target if the control qubit is $|1\rangle$.

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Truth table:

Control	Target	Control	Target
$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$
$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$
$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$
$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$

For a control qubit in superposition $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and target in state $|0\rangle_B$:

$$|\psi\rangle_{\text{out}} = \text{CNOT}(|\psi\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

This creates an entangled Bell state, where the qubits are correlated regardless of measurement basis.

1.6 Qubit States

General Qubit State

Quantum mechanics tells us that any qubit system can exist in a superposition of states. The general state of a quantum bit is described by:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers, and the normalization constraint $\langle\psi|\psi\rangle = 1$ requires that:

$$|\alpha|^2 + |\beta|^2 = 1$$

The states $|0\rangle$ and $|1\rangle$ represent the computational basis as two-dimensional vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A general superposition state can be expressed as a weighted sum of basis states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Bra-Ket Notation and Inner Products

The transpose complex conjugate of a ket $|\psi\rangle$ is denoted as a bra:

$$|\psi\rangle^\dagger = \langle\psi| = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$$

where the † superscript indicates the Hermitian conjugate (transpose complex conjugate).
The inner product forms the bra-ket:

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$$

This represents the normalization condition ensuring the total probability is conserved.

Measurement of Qubits

When a qubit in superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is measured in the computational basis:

- The measurement is an active process where the apparatus interacts with the qubit
- The qubit state collapses to either $|0\rangle$ or $|1\rangle$
- The probability of measuring $|0\rangle$ is $P(0) = |\alpha|^2$
- The probability of measuring $|1\rangle$ is $P(1) = |\beta|^2$
- After measurement, the qubit is no longer in superposition
- Only partial information is extracted—you cannot access the full quantum state
- Information obtained depends on the chosen measurement basis

This measurement in the computational basis $\{|0\rangle, |1\rangle\}$ is called **Z-measurement**. Other measurement bases exist (X-basis, Y-basis), but the computational basis is most common in quantum computing systems.

The Born Rule

The probability of measuring state $|a\rangle$ from $|\psi\rangle$ is:

$$P(a) = |\langle a|\psi\rangle|^2$$

Common measurement bases include:

- Computational (Z-basis): $\{|0\rangle, |1\rangle\}$
- X-basis: $\{|+\rangle, |-\rangle\}$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- Y-basis: $\{|+i\rangle, |-i\rangle\}$ where $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

TODO:Bloch Sphere Representation

Any single-qubit pure state can be represented on the Bloch sphere:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$.
The Bloch vector is $\vec{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.
Key states: $|0\rangle \to (0, 0, 1)$, $|1\rangle \to (0, 0, -1)$, $|+\rangle \to (1, 0, 0)$, $|-\rangle \to (-1, 0, 0)$.

1.7 Single-Qubit Gates

Single-qubit gates are unitary transformations that evolve qubit states. A unitary matrix U satisfies $U^\dagger U = UU^\dagger = I$.

Pauli Gates

Pauli-X (NOT gate):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_x|0\rangle = |1\rangle, \quad \sigma_x|1\rangle = |0\rangle$$

Represents a π rotation around the x-axis.

Pauli-Z gate:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_z|0\rangle = |0\rangle, \quad \sigma_z|1\rangle = -|1\rangle$$

Represents a π rotation around the z-axis.

Pauli-Y gate:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\sigma_x\sigma_z$$

Represents a π rotation around the y-axis. All Pauli matrices satisfy $\sigma_i^2 = I$ and are Hermitian.

Commutation Relations

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \quad [\sigma_y, \sigma_z] = 2i\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y$$

Anticommutation Relations

$$\{\sigma_x, \sigma_y\} = 0, \quad \{\sigma_y, \sigma_z\} = 0, \quad \{\sigma_z, \sigma_x\} = 0$$

Rotation Gates

General rotation around axis n by angle θ:

$$R_n(\theta) = e^{-i\theta\sigma_n/2} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma_n$$

Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Creates superposition: $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$.

Phase Gate (S-gate)

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Adds a 90 phase shift: $S|+\rangle = |+i\rangle$, $S|-\rangle = |-i\rangle$.

1.8 Multiple Gate Applications

Sequential gate applications are computed via matrix multiplication (right to left in mathematical notation, left to right in circuits).

1.9 Density Matrix Formalism

Real quantum systems cannot be perfectly isolated and are described by the density matrix $\hat{\rho}$:

Pure States

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

Mixed States (Ensemble)

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

where $\sum_i p_i = 1$.

Properties

- Hermitian: $\hat{\rho} = \hat{\rho}^\dagger$
- Trace normalized: $\mathrm{tr}(\hat{\rho}) = 1$
- 2×2 matrix form: $\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$
- Diagonal elements represent probabilities
- Off-diagonal elements represent coherences (with $\rho_{12} = \rho_{21}^*$)

Pure vs. mixed states can be distinguished by $\mathrm{tr}(\hat{\rho}^2) = 1$ (pure) or $\mathrm{tr}(\hat{\rho}^2) < 1$ (mixed).