CHAPTER 7: DIAGONALIZATION AND QUADRATIC FORMS

7.1 Orthogonal Matrices

1. **(a)** $AA^T=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}=I$ and $A^TA=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}=I$ therefore A is an orthogonal matrix; $A^{-1}=A^T=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

(b) $AA^T=\begin{bmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}=I$ and $A^TA=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}=I$ therefore A is an orthogonal matrix; $A^{-1}=A^T=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}$

3. **(a)** $\|\mathbf{r}_1\|=\sqrt{0^2+1^2+\left(\frac{1}{\sqrt2}\right)^2}=\sqrt{\frac{3}{2}}\neq 1$ so the matrix is not orthogonal.

(b) $AA^T=\begin{bmatrix}-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&-\frac{2}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\\frac{1}{\sqrt6}&-\frac{2}{\sqrt6}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\end{bmatrix}=I$ and $A^TA=\begin{bmatrix}-\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\\frac{1}{\sqrt6}&-\frac{2}{\sqrt6}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&-\frac{2}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\end{bmatrix}=I$

therefore A is an orthogonal matrix; $A^{-1}=A^T=\begin{bmatrix}-\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\\frac{1}{\sqrt6}&-\frac{2}{\sqrt6}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\end{bmatrix}$

5. $A^TA=\begin{bmatrix}\frac45&-\frac{9}{25}&\frac{12}{25}\\0&\frac45&\frac35\\-\frac35&-\frac{12}{25}&\frac{16}{25}\end{bmatrix}\begin{bmatrix}\frac45&0&-\frac35\\-\frac{9}{25}&\frac45&-\frac{12}{25}\\\frac{12}{25}&\frac35&\frac{16}{25}\end{bmatrix}=I$;

row vectors of A, $\mathbf{r}_1=\begin{bmatrix}\frac45&0&-\frac35\end{bmatrix}$, $\mathbf{r}_2=\begin{bmatrix}-\frac{9}{25}&\frac45&-\frac{12}{25}\end{bmatrix}$, $\mathbf{r}_3=\begin{bmatrix}\frac{12}{25}&\frac35&\frac{16}{25}\end{bmatrix}$, form an orthonormal set since $\mathbf{r}_1\cdot\mathbf{r}_2=\mathbf{r}_1\cdot\mathbf{r}_3=\mathbf{r}_2\cdot\mathbf{r}_3=0$ and $\|\mathbf{r}_1\|=\|\mathbf{r}_2\|=\|\mathbf{r}_3\|=1$;

column vectors of A, $\mathbf{c}_1=\begin{bmatrix}\frac45\\-\frac{9}{25}\\\frac{12}{25}\end{bmatrix}$, $\mathbf{c}_2=\begin{bmatrix}0\\\frac45\\\frac35\end{bmatrix}$, $\mathbf{c}_3=\begin{bmatrix}-\frac35\\-\frac{12}{25}\\\frac{16}{25}\end{bmatrix}$, form an orthonormal set since

$\mathbf{c}_1\cdot\mathbf{c}_2=\mathbf{c}_1\cdot\mathbf{c}_3=\mathbf{c}_2\cdot\mathbf{c}_3=0$ and $\|\mathbf{c}_1\|=\|\mathbf{c}_2\|=\|\mathbf{c}_3\|=1$.

7. $T_A(\mathbf{x}) = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{23}{5} \\ \frac{18}{25} \\ \frac{101}{25} \end{bmatrix}$; $\|T_A(\mathbf{x})\| = \sqrt{\frac{529}{25} + \frac{324}{625} + \frac{10201}{625}} = \sqrt{38}$ equals $\|\mathbf{x}\| = \sqrt{4+9+25} = \sqrt{38}$

9. Yes, by inspection, the column vectors in each of these matrices form orthonormal sets. By Theorem 7.1.1, these matrices are orthogonal.

11. Let $A = \begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 2(a^2+b^2) & 0 \\ 0 & 2(a^2+b^2) \end{bmatrix}$, so a and b must satisfy $a^2 + b^2 = \frac{1}{2}$.

13. **(a)** Formula (4) in Section 7.1 yields the transition matrix $P = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$; since P is orthogonal, $P^{-1} = P^T$ therefore $\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -1+3\sqrt{3} \\ 3+\sqrt{3} \end{bmatrix}$

(b) Using the matrix P we obtained in part (a), $\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} - \sqrt{3} \\ 1 + \frac{5}{2}\sqrt{3} \end{bmatrix}$

15. **(a)** Following the method of Example 6 in Section 7.1 (also see Table 7 in Section 8.6), we use the orthogonal matrix $P = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to obtain

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ 5 \end{bmatrix}$$

(b) Using the matrix P we obtained in part (a), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \\ -3 \end{bmatrix}$

17. **(a)** We follow the method of Example 6 in Section 7.1, with the appropriate orthogonal matrix obtained from Table 7 in Section 8.6: $P = \begin{bmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{5\sqrt{3}}{2} \\ 2 \\ -\frac{\sqrt{3}}{2} + \frac{5}{2} \end{bmatrix}$$

(b) Using the matrix P we obtained in part (a), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{3\sqrt{3}}{2} \\ 6 \\ -\frac{\sqrt{3}}{2} - \frac{3}{2} \end{bmatrix}$

19. If $B = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is the standard basis for R^3 and $B' = \{\mathbf{u}_1',\ \mathbf{u}_2',\ \mathbf{u}_3'\}$, then $[\mathbf{u}_1']_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

$[\mathbf{u}_2']_B = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$, and $[\mathbf{u}_3']_B = \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$, so the transition matrix from B' to B is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

and $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$.

21. **(a)** Rotations about the origin, reflections about any line through the origin, and any combination of these are rigid operators.

(b) Rotations about the origin, dilations, contractions, reflections about lines through the origin, and combinations of these are angle preserving.

(c) All rigid operators on R^2 are angle preserving. Dilations and contractions are angle preserving operators that are not rigid.

23. **(a)** Denoting $\mathbf{p}_1 = p_1(x) = \frac{1}{\sqrt{3}}$, $\mathbf{p}_2 = p_2(x) = \frac{1}{\sqrt{2}}x$, and $\mathbf{p}_3 = p_3(x) = \sqrt{\frac{3}{2}}x^2 - \sqrt{\frac{2}{3}}$ we have

$$\langle \mathbf{p}, \mathbf{p}_1 \rangle = p(-1)p_1(-1) + p(0)p_1(0) + p(1)p_1(1) = (1)\left(\tfrac{1}{\sqrt{3}}\right) + (1)\left(\tfrac{1}{\sqrt{3}}\right) + (3)\left(\tfrac{1}{\sqrt{3}}\right) = \tfrac{5}{\sqrt{3}}$$

$$\langle \mathbf{p}, \mathbf{p}_2 \rangle = p(-1)p_2(-1) + p(0)p_2(0) + p(1)p_2(1) = (1)\left(\tfrac{-1}{\sqrt{2}}\right) + (1)(0) + (3)\left(\tfrac{1}{\sqrt{2}}\right) = \sqrt{2}$$

$$\langle \mathbf{p}, \mathbf{p}_3 \rangle = p(-1)p_3(-1) + p(0)p_3(0) + p(1)p_3(1) = (1)\left(\tfrac{1}{\sqrt{6}}\right) + (1)\left(-\tfrac{2}{\sqrt{6}}\right) + (3)\left(\tfrac{1}{\sqrt{6}}\right) = \tfrac{\sqrt{2}}{\sqrt{3}}$$

$$\langle \mathbf{q}, \mathbf{p}_1 \rangle = q(-1)p_1(-1) + q(0)p_1(0) + q(1)p_1(1) = (-3)\left(\tfrac{1}{\sqrt{3}}\right) + (0)\left(\tfrac{1}{\sqrt{3}}\right) + (1)\left(\tfrac{1}{\sqrt{3}}\right) = -\tfrac{2}{\sqrt{3}}$$

$$\langle \mathbf{q}, \mathbf{p}_2 \rangle = q(-1)p_2(-1) + q(0)p_2(0) + q(1)p_2(1) = (-3)\left(\tfrac{-1}{\sqrt{2}}\right) + (0)(0) + (1)\left(\tfrac{1}{\sqrt{2}}\right) = 2\sqrt{2}$$

$$\langle \mathbf{q}, \mathbf{p}_3 \rangle = q(-1)p_3(-1) + q(0)p_3(0) + q(1)p_3(1) = (-3)\left(\tfrac{1}{\sqrt{6}}\right) + (0)\left(-\tfrac{2}{\sqrt{6}}\right) + (1)\left(\tfrac{1}{\sqrt{6}}\right) = -\tfrac{\sqrt{2}}{\sqrt{3}}$$

$$(\mathbf{p})_S = (\langle \mathbf{p}, \mathbf{p}_1, \mathbf{p}, \mathbf{p}_2, \mathbf{p}, \mathbf{p}_3 \rangle) = \left(\tfrac{5}{\sqrt{3}}, \sqrt{2}, \tfrac{\sqrt{2}}{\sqrt{3}}\right)$$

$$(\mathbf{q})_S = (\langle \mathbf{q}, \mathbf{p}_1, \mathbf{q}, \mathbf{p}_2, \mathbf{q}, \mathbf{p}_3 \rangle) = \left(-\tfrac{2}{\sqrt{3}}, 2\sqrt{2}, -\tfrac{\sqrt{2}}{\sqrt{3}}\right)$$

(b) $\|\mathbf{p}\| = \sqrt{\left(\frac{5}{\sqrt{3}}\right)^2 + \left(\sqrt{2}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} = \sqrt{\frac{25}{3} + 2 + \frac{2}{3}} = \sqrt{11}$

$$d(\mathbf{p},\mathbf{q}) = \sqrt{\left(\frac{5}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)^2 + \left(\sqrt{2} - 2\sqrt{2}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\right)^2} = \sqrt{\frac{49}{3} + 2 + \frac{8}{3}} = \sqrt{21}$$

$$\mathbf{p},\mathbf{q} = \left(\frac{5}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{3}}\right) + \left(\sqrt{2}\right)\left(2\sqrt{2}\right) + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)\left(-\frac{\sqrt{2}}{\sqrt{3}}\right) = -\frac{10}{3} + 4 - \frac{2}{3} = 0$$

25. We have $A^T = \left(I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T\right)^T = I_n^T - \frac{2}{\mathbf{x}^T\mathbf{x}}\left(\mathbf{x}\mathbf{x}^T\right)^T = I_n^T - \frac{2}{\mathbf{x}^T\mathbf{x}}\left(\mathbf{x}^T\right)^T\mathbf{x}^T = I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T = A$ therefore

$$A^TA = AA^T = \left(I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T\right)\left(I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T\right) = I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T + \frac{4}{\left(\mathbf{x}^T\mathbf{x}\right)^2}\mathbf{x}\mathbf{x}^T\mathbf{x}\mathbf{x}^T$$

$$= I_n - \frac{4}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T + \frac{4\left(\mathbf{x}^T\mathbf{x}\right)}{\left(\mathbf{x}^T\mathbf{x}\right)^2}\mathbf{x}\mathbf{x}^T = I_n - \frac{4}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T + \frac{4}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T = I_n$$

27. (a) Multiplication by $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation through θ.

In this case, $\det(A) = \cos^2\theta + \sin^2\theta = 1$.

The determinant of $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ is $\det(A) = -\cos^2\theta - \sin^2\theta = -1$.

We can express this matrix as a product $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Multiplying by $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ is a reflection about the x-axis followed by a rotation through θ.

(b) By Formula (6) of Section 4.9, multiplication by $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ is a reflection about the line through the origin that makes the angle $\frac{\theta}{2}$ with the positive x-axis.

29. Let A and B be 3×3 standard matrices of two rotations in R^3: T_A and T_B, respectively.
The result stated in this Exercise implies that A and B are both orthogonal and $\det(A) = \det(B) = 1$.
The product AB is a standard matrix of the composition of these rotations $T_A \circ T_B$.
By part (c) of Theorem 7.1.2, AB is an orthogonal matrix.
Furthermore, by Theorem 2.3.4, $\det(AB) = \det(A)\det(B) = 1$.
We conclude that $T_A \circ T_B$ is a rotation in R^3.
(One can show by induction that a composition of more than two rotations in R^3 is also a rotation.)

True-False Exercises

(a) False. Only square matrices can be orthogonal.

(b) False. The row and column vectors are not unit vectors.

(c) False. Only square matrices can be orthogonal. (The statement would be true if $m = n$.)

(d) False. The column vectors must form an orthonormal set.

(e) True. Since $A^T A = I$ for an orthogonal matrix *A*, *A* must be invertible (and $A^{-1} = A^T$).

(f) True. A product of orthogonal matrices is orthogonal, so A^2 is orthogonal; furthermore, $\det\left(A^2\right) = (\det A)^2 = (\pm 1)^2 = 1$.

(g) True. Since $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for an orthogonal matrix.

(h) True. This follows from Theorem 7.1.3.

7.2 Orthogonal Diagonalization

1. $\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{vmatrix} = \lambda^2 - 5\lambda$

The characteristic equation is $\lambda^2 - 5\lambda = 0$ and the eigenvalues are λ = 0 and λ = 5.
Both eigenspaces are one-dimensional.

3. $\begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3)$

The characteristic equation is $\lambda^3 - 3\lambda^2 = 0$ and the eigenvalues are λ = 3 and λ = 0.
The eigenspace for λ = 3 is one-dimensional; the eigenspace for λ = 0 is two-dimensional.

5. $\begin{vmatrix} \lambda-4 & -4 & 0 & 0 \\ -4 & \lambda-4 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^4 - 8\lambda^3 = \lambda^3(\lambda - 8)$

The characteristic equation is $\lambda^4 - 8\lambda^3 = 0$ and the eigenvalues are λ = 0 and λ = 8.
The eigenspace for λ = 0 is three-dimensional; the eigenspace for λ = 8 is one-dimensional.

7. $\det(\lambda I - A) = \begin{vmatrix} \lambda-6 & -2\sqrt{3} \\ -2\sqrt{3} & \lambda-7 \end{vmatrix} = \lambda^2 - 13\lambda + 30 = (\lambda-3)(\lambda-10)$ therefore A has eigenvalues 3 and 10.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & \frac{2}{\sqrt{3}} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{2}{\sqrt{3}}t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -2 \\ \sqrt{3} \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $10I - A$ is $\begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 10$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{\sqrt{3}}{2}t$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} \sqrt{3} \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A:

$P = \begin{bmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{bmatrix}$. We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}$.

9. Cofactor expansion along the second row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & 0 & 36 \\ 0 & \lambda+3 & 0 \\ 36 & 0 & \lambda+23 \end{vmatrix}$

$= (\lambda+3)\begin{vmatrix} \lambda+2 & 36 \\ 36 & \lambda+23 \end{vmatrix} = (\lambda-25)(\lambda+3)(\lambda+50)$ therefore A has eigenvalues 25, -3, and -50.

The reduced row echelon form of $25I - A$ is $\begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_1 = 25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{4}{3}t$, $x_2 = 0$, $x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_2 = -3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = t$, $x_3 = 0$. A vector $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-50I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_3 = -50$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{3}{4}t$, $x_2 = 0$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors; the basis $\{\mathbf{p}_2\}$ is already orthonormal. This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$.

We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$.

11. $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda-3)^2$ therefore A has eigenvalues 3 and 0.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_1 = \lambda_2 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - t$, $x_2 = s$, $x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and

$\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis

for this eigenspace: $\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, then proceed

to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_3 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ orthogonally diagonalizes A resulting in

$$P^{-1}AP = P^T AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

13. $\det(\lambda I - A) = \begin{vmatrix} \lambda+7 & -24 & 0 & 0 \\ -24 & \lambda-7 & 0 & 0 \\ 0 & 0 & \lambda+7 & -24 \\ 0 & 0 & -24 & \lambda-7 \end{vmatrix} = (\lambda+25)^2(\lambda-25)^2$ therefore A has eigenvalues -25 and 25.

The reduced row echelon form of $-25I - A$ is $\begin{bmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = \lambda_2 = -25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = -\frac{4}{3}s$, $x_2 = s$, $x_3 = -\frac{4}{3}t$, $x_4 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $25I - A$ is $\begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = \lambda_4 = 25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = \frac{3}{4}s$, $x_2 = s$, $x_3 = \frac{3}{4}t$, $x_4 = t$. Vectors $\mathbf{p}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{p}_4 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$ form a basis for this eigenspace.

Applying the Gram-Schmidt process to the two bases $\{\mathbf{p}_1, \mathbf{p}_2\}$, $\{\mathbf{p}_3, \mathbf{p}_4\}$ amounts to simply normalizing the vectors since the four vectors are already orthogonal. This yields the columns of a matrix P that orthogonally diagonalizes A: $P=\begin{bmatrix}-\frac{4}{5}&0&\frac{3}{5}&0\\ \frac{3}{5}&0&\frac{4}{5}&0\\ 0&-\frac{4}{5}&0&\frac{3}{5}\\ 0&\frac{3}{5}&0&\frac{4}{5}\end{bmatrix}$.

We have $P^{-1}AP=P^TAP=\begin{bmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\lambda_3&0\\0&0&0&\lambda_4\end{bmatrix}=\begin{bmatrix}-25&0&0&0\\0&-25&0&0\\0&0&25&0\\0&0&0&25\end{bmatrix}$.

15. $\det(\lambda I-A)=\begin{vmatrix}\lambda-3&-1\\-1&\lambda-3\end{vmatrix}=(\lambda-2)(\lambda-4)$ therefore A has eigenvalues 2 and 4.

The reduced row echelon form of $2I-A$ is $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=2$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1=-t$, $x_2=t$. A vector $\mathbf{p}_1=\begin{bmatrix}-1\\1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I-A$ is $\begin{bmatrix}1&-1\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2=4$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1=t$, $x_2=t$. A vector $\mathbf{p}_2=\begin{bmatrix}1\\1\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A:

$P=\begin{bmatrix}-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$. We have $P^{-1}AP=P^TAP=\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}=\begin{bmatrix}2&0\\0&4\end{bmatrix}$.

Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix}3&1\\1&3\end{bmatrix}=(2)\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}+(4)\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=(2)\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}+(4)\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{bmatrix}.$$

17. $\det(\lambda I-A)=\begin{vmatrix}\lambda+3&-1&-2\\-1&\lambda+3&-2\\-2&-2&\lambda\end{vmatrix}=(\lambda+4)^2(\lambda-2)$ therefore A has eigenvalues -4 and 2.

The reduced row echelon form of $-4I-A$ is $\begin{bmatrix}1&1&2\\0&0&0\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=-4$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1=-s-2t$, $x_2=s$, $x_3=t$. Vectors $\mathbf{p}_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\mathbf{p}_2=\begin{bmatrix}-2\\0\\1\end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace:

$\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{2}t$, $x_2 = \frac{1}{2}t$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$ orthogonally diagonalizes A resulting in $P^TAP = D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix} = (-4)\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + (-4)\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} + (2)\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= (-4)\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + (-4)\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} + (2)\begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

19. The three vectors are orthogonal, and they can be made into orthonormal vectors by a simple normalization. Forming the columns of a matrix P in this way we obtain an orthogonal matrix

$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$. When the diagonal matrix D contains the corresponding eigenvalues on its main diagonal, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, then Formula (2) in Section 7.2 yields $PDP^T = A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$.

21. Yes. The Gram-Schmidt process will ensure that columns of P corresponding to the same eigenvalue are an orthonormal set. Since eigenvectors from distinct eigenvalues are orthogonal, this means that P will be an orthogonal matrix. Then since A is orthogonally diagonalizable, it must be symmetric.

23. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = \lambda^2 - 2 = (\lambda-\sqrt{2})(\lambda+\sqrt{2})$ therefore A has eigenvalues $\pm\sqrt{2}$.

A is symmetric, so by Theorem 7.2.2(b), eigenvectors from different eigenspaces are orthogonal.

The reduced row echelon form of $\sqrt{2}I - A$ is $\begin{bmatrix} 1 & 1-\sqrt{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \sqrt{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = (\sqrt{2}-1)t$, $x_2 = t$. A vector $\begin{bmatrix} \sqrt{2}-1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-\sqrt{2}I - A$ is $\begin{bmatrix} 1 & 1+\sqrt{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -\sqrt{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = (-\sqrt{2}-1)t$, $x_2 = t$. A vector $\begin{bmatrix} -\sqrt{2}-1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Unit eigenvectors chosen from two different eigenspaces will meet our desired condition. For instance, let $\mathbf{u}_1 = \begin{bmatrix} \frac{\sqrt{2}-1}{4-2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} \frac{-\sqrt{2}-1}{4+2\sqrt{2}} \\ \frac{1}{4+2\sqrt{2}} \end{bmatrix}$.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda+1)(\lambda-3)$ therefore A has eigenvalues -1 and 3.

A is symmetric, so by Theorem 7.2.2(b), eigenvectors from different eigenspaces are orthogonal.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Unit eigenvectors chosen from two different eigenspaces will meet our desired condition. For instance, let $\mathbf{u}_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

25. $A^T A$ is a symmetric $n \times n$ matrix since $(A^T A)^T = A^T (A^T)^T = A^T A$. By Theorem 7.2.1 it has an orthonormal set of n eigenvectors.

29. By Theorem 7.1.3(b), if A is an orthogonal $n\times n$ matrix then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n. Since the eigenvalues of a symmetric matrix must be real numbers, for every such eigenvalue λ and a corresponding eigenvector $\mathbf{x}$ we have $\|\mathbf{x}\| = \|A\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ hence the only possible eigenvalues for an orthogonal symmetric matrix are 1 and –1.

True-False Exercises

(a) True. For any square matrix *A*, both AA^T and A^TA are symmetric, hence orthogonally diagonalizable.

(b) True. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are from distinct eigenspaces of a symmetric matrix, they are orthogonal, so $\|\mathbf{v}_1+\mathbf{v}_2\|^2 = \langle \mathbf{v}_1+\mathbf{v}_2, \mathbf{v}_1+\mathbf{v}_2\rangle = \langle \mathbf{v}_1, \mathbf{v}_1\rangle + \langle 2\mathbf{v}_1, \mathbf{v}_2\rangle + \langle \mathbf{v}_2, \mathbf{v}_2\rangle = \|\mathbf{v}_1\|^2 + 0 + \|\mathbf{v}_2\|^2$.

(c) False. An orthogonal matrix is not necessarily symmetric.

(d) True. By Theorem 1.7.4, if A is an invertible symmetric matrix then A^{-1} is also symmetric.

(e) True. By Theorem 7.1.3(b), if A is an orthogonal $n\times n$ matrix then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n. For every eigenvalue λ and a corresponding eigenvector $\mathbf{x}$ we have $\|\mathbf{x}\| = \|A\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ hence $|\lambda| = 1$.

(f) True. If *A* is an $n \times n$ orthogonally diagonalizable matrix, then *A* has an orthonormal set of *n* eigenvectors, which form a basis for R^n.

(g) True. This follows from part (a) of Theorem 7.2.2.

7.3 Quadratic Forms

1. **(a)** $3x_1^2 + 7x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(b) $4x_1^2 - 9x_2^2 - 6x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(c) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -4 & \frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

3. $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 5y^2 - 6xy$

5. $Q = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the characteristic polynomial of the matrix *A* is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$, so the eigenvalues of A are $\lambda = 3,\ 1$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. In terms of the new variables, we have

$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T \left(P^T AP\right)\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 3y_1^2 + y_2^2$.

7. $Q = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -2 & 0 \\ -2 & \lambda-4 & 2 \\ 0 & 2 & \lambda-5 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 39\lambda - 28 = (\lambda-1)(\lambda-4)(\lambda-7)$$

so the eigenvalues of A are 1, 4, and 7.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -2t$, $x_2 = 2t$, $x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = \frac{1}{2}t$, $x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $7I - A$ is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 7$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{1}{2}t$, $x_2 = -t$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the three bases amounts to simply normalizing the vectors. Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. In terms of the new variables, we have

$$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T \left(P^T AP\right)\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 4y_2^2 + 7y_3^2 .$$

9. **(a)** $2x^2 + xy + x - 6y + 2 = 0$ can be expressed as $\begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -6 \end{bmatrix}}_{K} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{(2)}_{f} = 0$

(b) $y^2 + 7x - 8y - 5 = 0$ can be expressed as $\begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} 7 & -8 \end{bmatrix}}_{K} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{(-5)}_{f} = 0$

11. **(a)** $2x^2 + 5y^2 = 20$ is $\frac{x^2}{10} + \frac{y^2}{4} = 1$ which is an equation of an ellipse.

(b) $x^2 - y^2 - 8 = 0$ is $x^2 - y^2 = 8$ or $\frac{x^2}{8} - \frac{y^2}{8} = 1$ which is an equation of a hyperbola.

(c) $7y^2 - 2x = 0$ is $x = \frac{7}{2}y^2$ which is an equation of a parabola.

(d) $x^2 + y^2 - 25 = 0$ is $x^2 + y^2 = 25$ which is an equation of a circle.

13. We can rewrite the given equation in the matrix form $\mathbf{x}^T A\mathbf{x} = -8$ with $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$.

The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 2 \\ 2 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda + 2)$ so A has eigenvalues 3 and -2.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -2t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-2I - A$ is $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{1}{2}t$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors.

This yields the columns of a matrix P that orthogonally diagonalizes A - of the two possibilities, $\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ we choose the latter, i.e., $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$, since its determinant is 1 so that the substitution $\mathbf{x} = P\mathbf{x}'$ performs a rotation of axes. In the rotated coordinates, the equation of the conic becomes $\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = -8$, i.e., $3y'^2 - 2x'^2 = 8$; this equation represents a hyperbola.

Solving $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we conclude that the angle of rotation is $\theta = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \approx 63.4°$.

15. We can rewrite the given equation in the matrix form $\mathbf{x}^T A\mathbf{x} = 15$ with $A = \begin{bmatrix} 11 & 12 \\ 12 & 4 \end{bmatrix}$.

The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 11 & -12 \\ -12 & \lambda - 4 \end{vmatrix} = (\lambda - 20)(\lambda + 5)$ so A has eigenvalues 20 and -5.

The reduced row echelon form of $20I - A$ is $\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 20$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{4}{3}t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-5I - A$ is $\begin{bmatrix} 1 & \frac{3}{4} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -5$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{3}{4}t$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors.

This yields the columns of a matrix P that orthogonally diagonalizes A - of the two possibilities, $\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$

and $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ we choose the former, i.e., $P = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$, since its determinant is 1 so that the substitution $\mathbf{x} = P\mathbf{x}'$ performs a rotation of axes. In the rotated coordinates, the equation of the conic becomes $\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 15$, i.e., $4x'^2 - y'^2 = 3$; this equation represents a hyperbola.

Solving $P = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we conclude that the angle of rotation is $\theta = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.9°$.

17. All matrices in this exercise are diagonal, therefore by Theorem 5.1.2, their eigenvalues are the entries on the main diagonal. We use Theorem 7.3.2 (including the remark below it).

(a) positive definite **(b)** negative definite **(c)** indefinite

(d) positive semidefinite **(e)** negative semidefinite

19. For all $(x_1, x_2) \neq (0,0)$, we clearly have $x_1^2 + x_2^2 > 0$ therefore the form is positive definite (an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which are $\lambda_1 = \lambda_2 = 1$ then use Theorem 7.3.2).

21. For all $(x_1, x_2) \neq (0,0)$, we clearly have $(x_1 - x_2)^2 \geq 0$, but cannot claim $(x_1 - x_2)^2 > 0$ when $x_1 = x_2$ therefore the form is positive semidefinite

(an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ which are $\lambda = 2$ and $\lambda = 0$ then use the remark under Theorem 7.3.2).

23. Clearly, the form $x_1^2 - x_2^2$ has both positive and negative values (e.g., $3^2 - 1^2 > 0$ and $2^2 - 4^2 < 0$) therefore this quadratic form is indefinite (an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ which are $\lambda = -1$ and $\lambda = 1$ then use Theorem 7.3.2).

25. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 7)$; since both eigenvalues $\lambda = 3$ and $\lambda = 7$ are positive, by Theorem 7.3.2, A is positive definite.

Determinant of the first principal submatrix of A is $\det([5]) = 5 > 0$.

Determinant of the second principal submatrix of A is $\det(A) = 21 > 0$.

By Theorem 7.3.4, A is positive definite.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 1 & 0 \\ 1 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = (\lambda - 1)(\lambda - 3)(\lambda - 5)$; since all three eigenvalues $\lambda = 1$, $\lambda = 3$, and $\lambda = 5$ are positive, by Theorem 7.3.2, A is positive definite.

Determinant of the first principal submatrix of A is $\det([2]) = 2 > 0$.

Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) = 3 > 0$.

Determinant of the third principal submatrix of A is $\det(A) = 15 > 0$.

By Theorem 7.3.4, A is positive definite.

27. **(a)** Determinant of the first principal submatrix of A is $\det([3]) = 3 > 0$.

Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\right) = -4 < 0$.

Determinant of the third principal submatrix of A is $\det(A) = -19 < 0$.

By Theorem 7.3.4(c), A is indefinite.

(b) Determinant of the first principal submatrix of A is $\det([-3]) = -3 < 0$.

Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}\right) = 5 > 0$.

Determinant of the third principal submatrix of A is $\det(A) = -25 < 0$.

By Theorem 7.3.4(b), A is negative definite.

29. The quadratic form $Q = 5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$ can be expressed in matrix notation as $Q = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & k \end{bmatrix}$. The determinants of the principal submatrices of A are $\det([5]) = 5$, $\det\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = 1$, and $\det A = k - 2$. Thus Q is positive definite if and only if $k > 2$.

31. **(a)** We assume A is symmetric so that $\mathbf{x}^T A\mathbf{y} = \left(\mathbf{x}^T A\mathbf{y}\right)^T = \mathbf{y}^T A^T\mathbf{x} = \mathbf{y}^T A\mathbf{x}$. Therefore

$$T(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y})^T A(\mathbf{x}+\mathbf{y}) = \mathbf{x}^T A\mathbf{x} + \mathbf{y}^T A\mathbf{x} + \mathbf{x}^T A\mathbf{y} + \mathbf{y}^T A\mathbf{y} = T(\mathbf{x}) + 2\mathbf{x}^T A\mathbf{y} + T(\mathbf{y}).$$

(b) $T(c\mathbf{x}) = (c\mathbf{x})^T A(c\mathbf{x}) = c^2\left(\mathbf{x}^T A\mathbf{x}\right) = c^2 T(\mathbf{x})$

33. **(a)** For each $i = 1, \ldots, n$ we have

$$\begin{aligned}
(x_i - \bar{x})^2 &= x_i^2 - 2x_i\bar{x} + \bar{x}^2 \\
&= x_i^2 - 2x_i \frac{1}{n}\sum_{j=1}^{n} x_j + \frac{1}{n^2}\left(\sum_{j=1}^{n} x_j\right)^2 \\
&= x_i^2 - \frac{2}{n}\sum_{j=1}^{n} x_i x_j + \frac{1}{n^2}\left(\sum_{j=1}^{n} x_j^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} x_j x_k\right)
\end{aligned}$$

Thus in the quadratic form $s_x^2 = \frac{1}{n-1}[(x_1-\overline{x})^2+(x_2-\overline{x})^2+\cdots+(x_n-\overline{x})^2]$ the coefficient of x_i^2 is $\frac{1}{n-1}\left[1-\frac{2}{n}+\frac{1}{n^2}n\right]=\frac{1}{n}$, and the coefficient of x_ix_j for $i \neq j$ is $\frac{1}{n-1}\left[-\frac{2}{n}-\frac{2}{n}+\frac{2}{n^2}n\right]=-\frac{2}{n(n-1)}$. It follows that $s_x^2 = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} \frac{1}{n} & -\frac{1}{n(n-1)} & \cdots & -\frac{1}{n(n-1)} \\ -\frac{1}{n(n-1)} & \frac{1}{n} & \cdots & -\frac{1}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n(n-1)} & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} \end{bmatrix}$.

(b) We have $s_x^2 = \frac{1}{n-1}[(x_1-\overline{x})^2+(x_2-\overline{x})^2+\cdots+(x_n-\overline{x})^2]\geq 0$, and $s_x^2=0$ if and only if $x_1=\overline{x}$, $x_2=\overline{x}$, ..., $x_n=\overline{x}$, i.e., if and only if $x_1=x_2=\cdots=x_n$. Thus s_x^2 is a positive semidefinite form.

35. The eigenvalues of A must be positive and equal to each other. That is, A must have a positive eigenvalue of multiplicity 2.

37. If A is an $n\times n$ symmetric matrix such that its eigenvalues λ_1, ..., λ_n are all nonnegative, then by Theorem 7.3.1 there exists a change of variable $\mathbf{y}=P\mathbf{x}$ for which $\mathbf{x}^T A\mathbf{x}=\lambda_1 y_1^2+\cdots+\lambda_n y_n^2$. The right hand side is always nonnegative, consequently $\mathbf{x}^T A\mathbf{x}\geq 0$ for all $\mathbf{x}$ in R^n.

True-False Exercises

(a) True. This follows from part (a) of Theorem 7.3.2 and from the margin note next to Definition 1.

(b) False. The term $4x_1x_2x_3$ cannot be included.

(c) True. One can rewrite $(x_1-3x_2)^2 = x_1^2-6x_1x_2+9x_2^2$.

(d) True. None of the eigenvalues will be 0.

(e) False. A symmetric matrix can also be positive semidefinite or negative semidefinite.

(f) True.

(g) True. $\mathbf{x}\cdot\mathbf{x} = x_1^2+x_2^2+\cdots+x_n^2$

(h) True. Eigenvalues of A^{-1} are reciprocals of eigenvalues of A. Therefore if all eigenvalues of A are positive, the same is true for all eigenvalues of A^{-1}.

(i) True.

(j) True. This follows from part (a) of Theorem 7.3.4.

(k) True.

(l) False. If $c < 0$, $\mathbf{x}^T A\mathbf{x} = c$ has no graph.

7.4 Optimization Using Quadratic Forms

1. We express the quadratic form in the matrix notation $z = 5x^2 - y^2 = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & 0 \\ 0 & \lambda + 1 \end{vmatrix} = (\lambda - 5)(\lambda + 1)$ therefore the eigenvalues of A are $\lambda = 5$ and $\lambda = -1$.

The reduced row echelon form of $5I - A$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 5$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = t$, $y = 0$. A vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = 0$, $y = t$. A vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

We conclude that the constrained extrema are

- constrained maximum: $z = 5$ at $(x, y) = (\pm 1, 0)$;
- constrained minimum: $z = -1$ at $(x, y) = (0, \pm 1)$.

3. We express the quadratic form in the matrix notation $z = 3x^2 + 7y^2 = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 \\ 0 & \lambda - 7 \end{vmatrix} = (\lambda - 3)(\lambda - 7)$ therefore the eigenvalues of A are $\lambda = 3$ and $\lambda = 7$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = t$, $y = 0$. A vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

The reduced row echelon form of $7I - A$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 7$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = 0$, $y = t$. A vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

We conclude that the constrained extrema are

- constrained maximum: $z = 7$ at $(x, y) = (0, \pm 1)$;
- constrained minimum: $z = 3$ at $(x, y) = (\pm 1, 0)$.

5. We express the quadratic form in the matrix notation

$$w = 9x^2 + 4y^2 + 3z^2 = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-9 & 0 & 0 \\ 0 & \lambda-4 & 0 \\ 0 & 0 & \lambda-3 \end{vmatrix} = (\lambda-3)(\lambda-4)(\lambda-9)$ therefore the eigenvalues of A are

$\lambda = 3$, $\lambda = 4$, and $\lambda = 9$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$

consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = 0$, $y = 0$, $z = t$. A vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace - this

vector is already normalized.

The reduced row echelon form of $9I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 9$

consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = t$, $y = 0$, $z = 0$. A vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace

eigenspace - this vector is already normalized.

We conclude that the constrained extrema are

- constrained maximum: $w = 9$ at $(x,y,z) = (\pm 1,0,0)$;
- constrained minimum: $w = 3$ at $(x,y,z) = (0,0,\pm 1)$.

7. The constraint $4x^2 + 8y^2 = 16$ can be rewritten as $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$. We define new variables x_1 and y_1 by $x = 2x_1$ and $y = \sqrt{2}y_1$. Our problem can now be reformulated to find maximum and minimum value of

$2\sqrt{2}x_1y_1 = \begin{bmatrix} x_1 & y_1 \end{bmatrix}\underbrace{\begin{bmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ subject to the constraint $x_1^2 + y_1^2 = 1$. We have

$\det(\lambda I - A) = \begin{vmatrix} \lambda & -\sqrt{2} \\ -\sqrt{2} & \lambda \end{vmatrix} = \lambda^2 - 2 = (\lambda - \sqrt{2})(\lambda + \sqrt{2})$ thus A has eigenvalues $\pm\sqrt{2}$.

The reduced row echelon form of $\sqrt{2}I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \sqrt{2}$

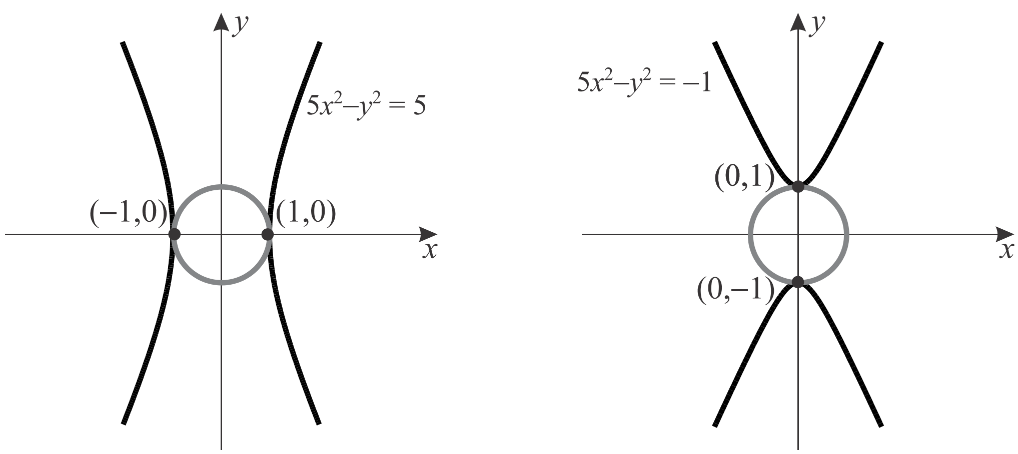
consists of vectors $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ where $x_1 = t$, $y_1 = t$. A vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. In terms of the original variables, this corresponds to $x = 2x_1 = \sqrt{2}$ and $y = \sqrt{2}y_1 = 1$.

The reduced row echelon form of $-\sqrt{2}I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -\sqrt{2}$ consists of vectors $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ where $x_1 = -t$, $y_1 = t$. A vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. In terms of the original variables, this corresponds to $x = 2x_1 = -\sqrt{2}$ and $y = \sqrt{2}y_1 = 1$.

We conclude that the constrained extrema are

- constrained maximum value: $\sqrt{2}$ at $(x,y) = (\sqrt{2},1)$ and $(x,y) = (-\sqrt{2},-1)$;
- constrained minimum value: $-\sqrt{2}$ at $(x,y) = (-\sqrt{2},1)$ and $(x,y) = (\sqrt{2},-1)$.

9. The following illustration indicates positions of constrained extrema consistent with the solution that was obtained for Exercise 1.



11. **(a)** The first partial derivatives of $f(x,y)$ are $f_x(x,y) = 4y - 4x^3$ and $f_y(x,y) = 4x - 4y^3$.
Since $f_x(0,0) = f_y(0,0) = 0$, $f_x(1,1) = f_y(1,1) = 0$, and $f_x(-1,-1) = f_y(-1,-1) = 0$, f has critical points at $(0,0)$, $(1,1)$, and $(-1,-1)$.

(b) The second partial derivatives of $f(x,y)$ are $f_{xx}(x,y) = -12x^2$, $f_{xy}(x,y) = 4$, and $f_{yy}(x,y) = -12y^2$ therefore the Hessian matrix of f is $H(x,y) = \begin{bmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{bmatrix}$.

$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda & -4 \\ -4 & \lambda \end{vmatrix} = (\lambda - 4)(\lambda + 4)$ so $H(0,0)$ has eigenvalues -4 and 4; since $H(0,0)$ is indefinite, f has a saddle point at $(0,0)$;

$\det(\lambda I - H(1,1)) = \begin{vmatrix} \lambda + 12 & -4 \\ -4 & \lambda + 12 \end{vmatrix} = (\lambda + 8)(\lambda + 16)$ so $H(1,1)$ has eigenvalues -8 and -16; since $H(1,1)$ is negative definite, f has a relative maximum at $(1,1)$;

$\det(\lambda I - H(-1,-1)) = \begin{vmatrix} \lambda + 12 & -4 \\ -4 & \lambda + 12 \end{vmatrix} = (\lambda + 8)(\lambda + 16)$ so $H(-1,-1)$ has eigenvalues -8 and -16; since $H(-1,-1)$ is negative definite, f has a relative maximum at $(-1,-1)$

13. The first partial derivatives of *f* are $f_x(x,\ y) = 3x^2 - 3y$ and $f_y(x,\ y) = -3x - 3y^2$. To find the critical points we set f_x and f_y equal to zero. This yields the equations $y = x^2$ and $x = -y^2$. From this we conclude that $y = y^4$ and so *y* = 0 or *y* = 1. The corresponding values of *x* are *x* = 0 and *x* = –1 respectively. Thus there are two critical points: (0, 0) and (–1, 1).

The Hessian matrix is $H(x,\ y) = \begin{bmatrix} f_{xx}(x,\ y) & f_{xy}(x,\ y) \\ f_{yx}(x,\ y) & f_{yy}(x,\ y) \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & -6y \end{bmatrix}$.

$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda & 3 \\ 3 & \lambda \end{vmatrix} = (\lambda - 3)(\lambda + 3)$ so $H(0,0)$ has eigenvalues -3 and 3; since $H(0,0)$ is indefinite, f has a saddle point at $(0,0)$;

$\det(\lambda I - H(-1,1)) = \begin{vmatrix} \lambda + 6 & 3 \\ 3 & \lambda + 6 \end{vmatrix} = (\lambda + 3)(\lambda + 9)$ so $H(-1,1)$ has eigenvalues -3 and -9; since $H(-1,1)$ is negative definite, f has a relative maximum at $(-1,1)$.

15. The first partial derivatives of *f* are $f_x(x,\ y) = 2x - 2xy$ and $f_y(x,\ y) = 4y - x^2$. To find the critical points we set f_x and f_y equal to zero. This yields the equations $2x(1-y) = 0$ and $y = \frac{1}{4}x^2$. From the first, we conclude that $x = 0$ or $y = 1$. Thus there are three critical points: (0, 0), (2, 1), and (–2, 1).

The Hessian matrix is $H(x,\ y) = \begin{bmatrix} f_{xx}(x,\ y) & f_{xy}(x,\ y) \\ f_{yx}(x,\ y) & f_{yy}(x,\ y) \end{bmatrix} = \begin{bmatrix} 2 - 2y & -2x \\ -2x & 4 \end{bmatrix}$.

$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 4)$ so $H(0,0)$ has eigenvalues 2 and 4; since $H(0,0)$ is positive definite, f has a relative minimum at $(0,0)$.

$\det(\lambda I - H(2,1)) = \begin{vmatrix} \lambda & 4 \\ 4 & \lambda - 4 \end{vmatrix} = \lambda^2 - 4\lambda - 16$ so the eigenvalues of $H(2,1)$ are $2 \pm 2\sqrt{5}$. One of these is positive and one is negative; thus this matrix is indefinite and f has a saddle point at (2, 1).

$\det(\lambda I - H(-2,1)) = \begin{vmatrix} \lambda & -4 \\ -4 & \lambda - 4 \end{vmatrix} = \lambda^2 - 4\lambda - 16$ so the eigenvalues of $H(-2,1)$ are $2 \pm 2\sqrt{5}$. One of these is positive and one is negative; thus this matrix is indefinite and f has a saddle point at $(-2,1)$.

17. The problem is to maximize $z = 4xy$ subject to $x^2 + 25y^2 = 25$, or $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$.

Let $x = 5x_1$ and $y = y_1$, so that the problem is to maximize $z = 20x_1y_1$ subject to $\|(x_1,\ y_1)\| = 1$.

Write $z = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

$$\begin{vmatrix} \lambda & -10 \\ -10 & \lambda \end{vmatrix} = \lambda^2 - 100 = (\lambda + 10)(\lambda - 10).$$

The largest eigenvalue of A is $\lambda = 10$ which has positive unit eigenvector $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Thus the maximum value of $z = 20\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 10$ which occurs when $x = 5x_1 = \frac{5}{\sqrt{2}}$ and $y = y_1 = \frac{1}{\sqrt{2}}$, which are the coordinates of one of the corner points of the rectangle.

19. **(a)** The first partial derivatives of $f(x,y)$ are $f_x(x,y) = 4x^3$ and $f_y(x,y) = 4y^3$.

Since $f_x(0,0) = f_y(0,0) = 0$, f has a critical point at $(0,0)$.

The second partial derivatives of $f(x,y)$ are $f_{xx}(x,y) = 12x^2$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2$. We have $f_{xx}(0,0)\,f_{yy}(0,0) - f_{xy}^2(0,0) = 0$ therefore the second derivative test is inconclusive.

The first partial derivatives of $g(x,y)$ are $g_x(x,y) = 4x^3$ and $g_y(x,y) = -4y^3$.

Since $g_x(0,0) = g_y(0,0) = 0$, g has a critical point at $(0,0)$.

The second partial derivatives of $g(x,y)$ are $g_{xx}(x,y) = 12x^2$, $g_{xy}(x,y) = 0$, and $g_{yy}(x,y) = -12y^2$. We have $g_{xx}(0,0)g_{yy}(0,0) - g_{xy}^2(0,0) = 0$ therefore the second derivative test is inconclusive.

(b) Clearly, for all $(x,y) \neq (0,0)$, $f(x,y) > f(0,0) = 0$ therefore f has a relative minimum at $(0,0)$.

For all $x \neq 0$, $g(x,0) > g(0,0) = 0$; however, for all $y \neq 0$, $g(0,y) < g(0,0) = 0$ - consequently, g has a saddle point at $(0,0)$.

21. If $\mathbf{x}$ is a unit eigenvector corresponding to λ, then $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda(\mathbf{x}^T\mathbf{x}) = \lambda(1) = \lambda.$

True-False Exercises

(a) False. If the only critical point of the quadratic form is a saddle point, then it will have neither a maximum nor a minimum value.

(b) True. This follows from part (b) of Theorem 7.4.1.

(c) True.

(d) False. The second derivative test is inconclusive in this case.

(e) True. If det(A) < 0, then A will have a negative eigenvalue.

7.5 Hermitian, Unitary, and Normal Matrices

1. $\overline{A} = \begin{bmatrix} -2i & 1+i \\ 4 & 3-i \\ 5-i & 0 \end{bmatrix}$ therefore $A^* = \overline{A}^T = \begin{bmatrix} -2i & 4 & 5-i \\ 1+i & 3-i & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & i & 2-3i \\ -i & -3 & 1 \\ 2+3i & 1 & 2 \end{bmatrix}$

5. **(a)** $(A)_{13} = 2-3i$ does not equal $(A^*)_{13} = 2+3i$

(b) $(A)_{22} = i$ does not equal $(A^*)_{22} = -i$

7. $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -2+3i \\ -2-3i & \lambda+1 \end{vmatrix} = \lambda^2 - 2\lambda - 16 = \left(\lambda - \left(1+\sqrt{17}\right)\right)\left(\lambda - \left(1-\sqrt{17}\right)\right)$ so A has real eigenvalues $1+\sqrt{17}$ and $1-\sqrt{17}$.

For the eigenvalue $\lambda = 1+\sqrt{17}$, the augmented matrix of the homogeneous system $\left(\left(1+\sqrt{17}\right)I - A\right)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -2+\sqrt{17} & -2+3i & 0 \\ -2-3i & 2+\sqrt{17} & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + \frac{2+\sqrt{17}}{13}(-2+3i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = \frac{2+\sqrt{17}}{13}(2-3i)t$, $x_2 = t$. The vector

$\mathbf{v}_1 = \begin{bmatrix} \frac{2+\sqrt{17}}{13}(2-3i) \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 1+\sqrt{17}$.

For the eigenvalue $\lambda = 1-\sqrt{17}$, the augmented matrix of the homogeneous system

$\left(\left(1-\sqrt{17}\right)I-A\right)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix} -2-\sqrt{17} & -2+3i & 0 \\ -2-3i & 2-\sqrt{17} & 0 \end{bmatrix}$.

As before, this yields $x_1+\frac{2-\sqrt{17}}{13}(-2+3i)x_2=0$. The general solution of this equation (and, consequently, of the entire system) is $x_1=\frac{2-\sqrt{17}}{13}(2-3i)t$, $x_2=t$. The vector $\mathbf{v}_2=\begin{bmatrix} \frac{2-\sqrt{17}}{13}(2-3i) \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda=1-\sqrt{17}$.

We have

$\mathbf{v}_1\cdot\mathbf{v}_2=\left(\frac{2+\sqrt{17}}{13}(2-3i)\right)\left(\overline{\frac{2-\sqrt{17}}{13}(2-3i)}\right)+(1)\left(\overline{1}\right)=\left(\frac{2+\sqrt{17}}{13}(2-3i)\right)\left(\frac{2-\sqrt{17}}{13}(2+3i)\right)+(1)(1)=$
$\frac{(2+\sqrt{17})(2-\sqrt{17})}{13^2}(2-3i)(2+3i)+1=\frac{4-17}{13^2}(4+9)+1=-1+1=0$ therefore the eigenvectors from different eigenspaces are orthogonal.

9. The following computations show that the row vectors of A are orthonormal:

$\|\mathbf{r}_1\|=\sqrt{\left|\frac{3}{5}\right|^2+\left|\frac{4}{5}i\right|^2}=\sqrt{\frac{9}{25}+\frac{16}{25}}=1$; $\|\mathbf{r}_2\|=\sqrt{\left|-\frac{4}{5}\right|^2+\left|\frac{3}{5}i\right|^2}=\sqrt{\frac{16}{25}+\frac{9}{25}}=1$;

$\mathbf{r}_1\cdot\mathbf{r}_2=\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)+\left(\frac{4}{5}i\right)\left(-\frac{3}{5}i\right)=-\frac{12}{5}+\frac{12}{5}=0$

By Theorem 7.5.3, A is unitary, and $A^{-1}=A^*=\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5}i & -\frac{3}{5}i \end{bmatrix}$.

11. The following computations show that the column vectors of A are orthonormal:

$\|\mathbf{c}_1\|=\sqrt{\left|\frac{1}{2\sqrt{2}}\left(\sqrt{3}+i\right)\right|^2+\left|\frac{1}{2\sqrt{2}}\left(1+i\sqrt{3}\right)\right|^2}=\sqrt{\frac{4}{8}+\frac{4}{8}}=1$;

$\|\mathbf{c}_2\|=\sqrt{\left|\frac{1}{2\sqrt{2}}\left(1-i\sqrt{3}\right)\right|^2+\left|\frac{1}{2\sqrt{2}}\left(i-\sqrt{3}\right)\right|^2}=\sqrt{\frac{4}{8}+\frac{4}{8}}=1$;

$\mathbf{c}_1\cdot\mathbf{c}_2=\frac{1}{2\sqrt{2}}\left(\sqrt{3}+i\right)\frac{1}{2\sqrt{2}}\left(1+i\sqrt{3}\right)+\frac{1}{2\sqrt{2}}\left(1+i\sqrt{3}\right)\frac{1}{2\sqrt{2}}\left(-i-\sqrt{3}\right)=0$

By Theorem 7.5.3, A is unitary, therefore $A^{-1}=A^*=\begin{bmatrix} \frac{1}{2\sqrt{2}}\left(\sqrt{3}-i\right) & \frac{1}{2\sqrt{2}}\left(1-i\sqrt{3}\right) \\ \frac{1}{2\sqrt{2}}\left(1+i\sqrt{3}\right) & \frac{1}{2\sqrt{2}}\left(-i-\sqrt{3}\right) \end{bmatrix}$.

13. $\det(\lambda I-A)=\begin{vmatrix} \lambda-4 & -1+i \\ -1-i & \lambda-5 \end{vmatrix}=(\lambda-3)(\lambda-6)$ thus A has eigenvalues $\lambda=3$ and $\lambda=6$.

The reduced row echelon form of $3I-A$ is $\begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda=3$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x=(-1+i)t$, $y=t$. A vector $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $6I-A$ is $\begin{bmatrix} 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda = 6$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = \left(\frac{1}{2} - \frac{1}{2}i\right)t$, $y = t$. A vector $\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases amounts to simply normalizing the respective vectors. Therefore A is unitarily diagonalized by $P = \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$. Since P is unitary, $P^{-1} = P^* = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$. It follows that $P^{-1}AP = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix} \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$.

15. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 6 & -2-2i \\ -2+2i & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 8)$ thus A has eigenvalues $\lambda = 2$ and $\lambda = 8$.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & \frac{1}{2} + \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = \left(-\frac{1}{2} - \frac{1}{2}i\right)t$, $y = t$. A vector $\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $8I - A$ is $\begin{bmatrix} 1 & -1-i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 8$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = (1+i)t$, $y = t$. A vector $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases amounts to simply normalizing the respective vectors. Therefore A is unitarily diagonalized by $P = \begin{bmatrix} \frac{-1-i}{\sqrt{6}} & \frac{1+i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$. Since P is unitary, $P^{-1} = P^* = \begin{bmatrix} \frac{-1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$. It follows that $P^{-1}AP = \begin{bmatrix} \frac{-1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & 2+2i \\ 2-1i & 4 \end{bmatrix} \begin{bmatrix} \frac{-1-i}{\sqrt{6}} & \frac{1+i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$.

17. The characteristic polynomial of A is $(\lambda - 5)(\lambda^2 + \lambda - 2) = (\lambda + 2)(\lambda - 1)(\lambda - 5)$; thus the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = 5$. The augmented matrix of the system $(-2I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -1 & 1-i & 0 \\ 0 & 1+i & -2 & 0 \end{bmatrix}$, which can be reduced to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1+i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1-i \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = -2$, and $\mathbf{p}_1 = \begin{bmatrix} 0 \\ \frac{1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ is a unit eigenvector. Similar computations show that

$\mathbf{p}_2 = \begin{bmatrix} 0 \\ \frac{-1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$ is a unit eigenvector corresponding to $\lambda_2 = 1,$ and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a unit eigenvector corresponding to $\lambda_3 = 5.$ The vectors $\{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3\}$ form an orthogonal set, and the unitary matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3]$ diagonalizes the matrix A:

$$P^*AP = \begin{bmatrix} 0 & \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ \frac{1-i}{\sqrt{3}} & \frac{-1+i}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

19. $A = \begin{bmatrix} 0 & i & 2-3i \\ i & 0 & 1 \\ -2-3i & -1 & 4i \end{bmatrix}$

21. **(a)** $(-A)_{12} = -i$ does not equal $(A^*)_{12} = i$;

also, $(-A)_{13} = -2+3i$ does not equal $(A^*)_{13} = 2-3i$

(b) $(-A)_{11} = -1$ does not equal $(A^*)_{11} = 1$;

also, $(-A)_{13} = -3+5i$ does not equal $(A^*)_{13} = -3-5i$ and

$(-A)_{23} = i$ does not equal $(A^*)_{23} = -i$.

23. $\det(\lambda I - A) - \begin{bmatrix} \lambda & 1-i \\ -1+i & \lambda - i \end{bmatrix} = \lambda^2 - i\lambda + 2 = (\lambda - 2i)(\lambda + i)$; thus the eigenvalues of A, $\lambda = 2i$ and $\lambda = -i$, are pure imaginary numbers.

25. $A^* = \begin{bmatrix} 1-2i & 2-i & -2+i \\ 2-i & 1-i & i \\ -2+i & i & 1-i \end{bmatrix}$; we have $AA^* = A^*A = \begin{bmatrix} 15 & 8 & -8 \\ 8 & 8 & -7 \\ -8 & -7 & 8 \end{bmatrix}$

27. **(a)** If $B = \frac{1}{2}(A + A^*)$, then $B^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}(A^* + A^{**}) = \frac{1}{2}(A^* + A) = B$. Similarly, $C^* = C$.

(b) We have $B + iC = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = A$ and $B - iC = \frac{1}{2}(A + A^*) - \frac{1}{2}(A - A^*) = A^*$.

(c) $AA^* = (B + iC)(B - iC) = B^2 - iBC + iCB + C^2$ and $A^*A = B^2 + iBC - iCB + C^2$.

Thus $AA^* = A^*A$ if and only if $-iBC + iCB = iBC - iCB$, or $2iCB = 2iBC$.

Thus A is normal if and only if B and C commute i.e., $CB = BC$.

31. $A\mathbf{x}=\begin{bmatrix}\frac{7}{5}+\frac{11}{5}i\\ -\frac{1}{5}+\frac{2}{5}i\end{bmatrix}$; $\quad \|A\mathbf{x}\|=\sqrt{\left|\frac{7}{5}+\frac{11}{5}i\right|^2+\left|-\frac{1}{5}+\frac{2}{5}i\right|^2}=\sqrt{\frac{49}{25}+\frac{121}{25}+\frac{1}{25}+\frac{4}{25}}=\sqrt{7}$ equals

$\|\mathbf{x}\|=\sqrt{|1+i|^2+|2-i|^2}=\sqrt{1+1+4+1}=\sqrt{7}$ which verifies part (b);

$A\mathbf{y}=\begin{bmatrix}\frac{7}{5}+\frac{4}{5}i\\ -\frac{1}{5}+\frac{3}{5}i\end{bmatrix}$; $\quad A\mathbf{x}\cdot A\mathbf{y}=\left(\frac{7}{5}+\frac{11}{5}i\right)\overline{\left(\frac{7}{5}+\frac{4}{5}i\right)}+\left(-\frac{1}{5}+\frac{2}{5}i\right)\overline{\left(-\frac{1}{5}+\frac{3}{5}i\right)}$

$=\left(\frac{7}{5}+\frac{11}{5}i\right)\left(\frac{7}{5}-\frac{4}{5}i\right)+\left(-\frac{1}{5}+\frac{2}{5}i\right)\left(-\frac{1}{5}-\frac{3}{5}i\right)=\left(\frac{93}{25}+\frac{49}{25}i\right)+\left(\frac{7}{25}+\frac{1}{25}i\right)=4+2i$ equals

$\mathbf{x}\cdot\mathbf{y}=(1+i)\overline{(1)}+(2-i)\overline{(1-i)}=(1+i)(1)+(2-i)(1+i)=(1+i)+(3+i)=4+2i$ which verifies part (c).

33. $A^*=\begin{bmatrix}\bar{a}&0&0\\0&0&\bar{b}\\0&\bar{c}&0\end{bmatrix}$; $AA^*=\begin{bmatrix}a\bar{a}&0&0\\0&c\bar{c}&0\\0&0&b\bar{b}\end{bmatrix}=\begin{bmatrix}|a|^2&0&0\\0&|c|^2&0\\0&0&|b|^2\end{bmatrix}$; $A^*A=\begin{bmatrix}a\bar{a}&0&0\\0&b\bar{b}&0\\0&0&c\bar{c}\end{bmatrix}=\begin{bmatrix}|a|^2&0&0\\0&|b|^2&0\\0&0&|c|^2\end{bmatrix}$

A is normal if and only if $|b|=|c|$.

35. $A=\begin{bmatrix}\frac{1}{\sqrt{2}}&-\frac{i}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{bmatrix}$ is both Hermitian and unitary.

37. Part (a): $\left(A^*\right)^* \underset{\text{Def.1}}{=} \left(\overline{\overline{A}^T}\right)^T \underset{\text{Th. 5.3.2(b)}}{=} \left(\overline{\overline{A}}^T\right)^T \underset{\text{Th. 5.3.2(a)}}{=} \left(A^T\right)^T=A$

Part (e): $(AB)^* \underset{\text{Def.1}}{=} \left(\overline{AB}\right)^T \underset{\text{Th. 5.3.2(c)}}{=} \left(\bar{A}\ \bar{B}\right)^T=\left(\bar{B}\right)^T\left(\bar{A}\right)^T \underset{\text{Def.1}}{=} B^*A^*$

39. If A is unitary, then $A^{-1}=A^*$ and so $(A^*)^{-1}=\left(A^{-1}\right)^*=\left(A^*\right)^*$; thus A^* is also unitary.

41. A unitary matrix A has the property that $\|A\mathbf{x}\|=\|\mathbf{x}\|$ for all x in C^n. Thus if A is unitary and $A\mathbf{x}=\lambda\mathbf{x}$ where $\mathbf{x}\neq\mathbf{0}$, we must have $|\lambda|\|\mathbf{x}\|=\|A\mathbf{x}\|=\|\mathbf{x}\|$ and so $|\lambda|=1$.

43. If $H=I-2\mathbf{u}\mathbf{u}^*$, then $H^*=(I-2\mathbf{u}\mathbf{u}^*)^*=I^*-2\mathbf{u}^{**}\mathbf{u}^*=I-2\mathbf{u}\mathbf{u}^*=H$; thus H is Hermitian.

$HH^*=(I-2\mathbf{u}\mathbf{u}^*)(I-2\mathbf{u}\mathbf{u}^*)=I-2\mathbf{u}\mathbf{u}^*-2\mathbf{u}\mathbf{u}^*+4\mathbf{u}\mathbf{u}^*\mathbf{u}\mathbf{u}^*=I-4\mathbf{u}\mathbf{u}^*+4\mathbf{u}\|\mathbf{u}\|^2\mathbf{u}^*=I$

so H is unitary.

45. **(a)** This result can be obtained by mathematical induction.

(b) $\det\left(A^*\right)=\det\left(\left(\bar{A}\right)^T\right)=\det\left(\bar{A}\right)=\overline{\det(A)}$.

True-False Exercises

(a) False. Denoting $A=\begin{bmatrix}0&i\\i&2\end{bmatrix}$, we observe that $(A)_{12}=i$ does not equal $\left(A^*\right)_{12}=-i$.

(b) False. For $\mathbf{r}_1 = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} 0 & -\frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \end{bmatrix}$,

$\mathbf{r}_1 \cdot \mathbf{r}_2 = -\frac{i}{\sqrt{2}}\left(\overline{0}\right) + \frac{i}{\sqrt{6}}\left(\overline{-\frac{i}{\sqrt{6}}}\right) + \frac{i}{\sqrt{3}}\left(\overline{\frac{i}{\sqrt{3}}}\right) = 0 + \left(\frac{i}{\sqrt{6}}\right)^2 - \left(\frac{i}{\sqrt{3}}\right)^2 = -\frac{1}{6} + \frac{1}{3} = \frac{1}{6} \neq 0$

thus the row vectors do not form an orthonormal set and the matrix is not unitary by Theorem 7.5.3.

(c) True. If A is unitary, so $A^{-1} = A^*$, then $(A^*)^{-1} = A = (A^*)^*$.

(d) False. Normal matrices that are not Hermitian are also unitarily diagonalizable.

(e) False. If A is skew-Hermitian, then $(A^2)^* = (A^*)(A^*) = (-A)(-A) = A^2 \neq -A^2$.

Chapter 7 Supplementary Exercises

1. **(a)** For $A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$, $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $A^{-1} = A^T = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$.

(b) For $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$, $A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so $A^{-1} = A^T = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$.

3. Since A is symmetric, there exists an orthogonal matrix P such that $P^T AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$.

Since A is positive definite, all $\lambda's$ must be positive. Let us form a diagonal matrix

$C = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}$. Then $A = PDP^T = PCC^T P^T = (PC)(PC)^T$. The matrix $(PC)^T$ is nonsingular

(it is a transpose of a product of two nonsingular matrices), therefore it generates an inner product on R^n:

$$\langle \mathbf{u}, \mathbf{v} \rangle = (PC)^T \mathbf{u} \cdot (PC)^T \mathbf{v} = \mathbf{u}^T \left(PCC^T P^T\right) \mathbf{v} = \mathbf{u}^T A \mathbf{v}$$

5. The characteristic equation of A is $\lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 2)(\lambda - 1)$, so the eigenvalues are $\lambda = 0, 2, 1$.

Orthogonal bases for the eigenspaces are $\lambda = 0$: $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$; $\lambda = 2$: $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$; $\lambda = 1$: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Thus $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ orthogonally diagonalizes A, and $P^T AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7. In matrix form, the quadratic form is $\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - 5\lambda + \frac{7}{4} = 0$ which has solutions $\lambda = \frac{5 \pm 3\sqrt{2}}{2}$ or $\lambda \approx 4.62, 0.38$. Since both eigenvalues of A are positive, the quadratic form is positive definite.

9. **(a)** $y - x^2 = 0$ or $y = x^2$ represents a parabola.

(b) $3x - 11y^2 = 0$ or $x = \frac{11}{3}y^2$ represents a parabola.

11. Partitioning U into columns we can write $U = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \ldots \mid \mathbf{u}_n]$. The given product can be rewritten in partitioned form as well:

$$A = U \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \ldots \mid \mathbf{u}_n] \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix} = [z_1\mathbf{u}_1 \mid z_2\mathbf{u}_2 \mid \ldots \mid z_n\mathbf{u}_n]$$

By Theorem 7.5.3, the columns of U form an orthonormal set. Therefore, columns of A must also be orthonormal: $(z_i\mathbf{u}_i) \cdot (z_j\mathbf{u}_j) = \left(z_i \overline{z_j}\right)\left(\mathbf{u}_i \cdot \mathbf{u}_j\right) = 0$ for all $i \neq j$ and $\| z_i\mathbf{u}_i \| = | z_i | \| \mathbf{u}_i \| = 1$ for all i.

By Theorem 7.5.3, A is a unitary matrix.

13. Partitioning the given matrix into columns $A = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3]$, we must find $\mathbf{u}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that

$\mathbf{u}_1 \cdot \mathbf{u}_2 = \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{6}} + \frac{c}{\sqrt{3}} = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = -\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{6}} + \frac{c}{\sqrt{3}} = 0$, and $\| \mathbf{u}_1 \|^2 = a^2 + b^2 + c^2 = 1$.

Subtracting the second equation from the first one yields $a = 0$. Therefore $c = -\frac{\sqrt{3}}{\sqrt{6}} b = -\frac{b}{\sqrt{2}}$.

Substituting into $\| \mathbf{u}_1 \|^2 = 1$ we obtain $b^2 + \frac{b^2}{2} = 1$ so that $b^2 = \frac{2}{3}$.

There are two possible solutions:

- $a = 0$, $b = \sqrt{\frac{2}{3}}$, $c = -\frac{1}{\sqrt{3}}$ and
- $a = 0$, $b = -\sqrt{\frac{2}{3}}$, $c = \frac{1}{\sqrt{3}}$.