CHAPTER 5: EIGENVALUES AND EIGENVECTORS

5.1 Eigenvalues and Eigenvectors

1. $A\mathrm{x}=\begin{bmatrix}1&2\\3&2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}=-1\mathrm{x}$ therefore x is an eigenvector of A corresponding to the eigenvalue -1.

3. $A\mathrm{x}=\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}5\\10\\5\end{bmatrix}=5\mathrm{x}$ therefore x is an eigenvector of A corresponding to the eigenvalue 5.

5. **(a)** $\det(\lambda I-A)=\begin{vmatrix}\lambda-1&-4\\-2&\lambda-3\end{vmatrix}=(\lambda-1)(\lambda-3)-(-4)(-2)=\lambda^2-4\lambda-5=(\lambda-5)(\lambda+1)$.

The characteristic equation is $(\lambda-5)(\lambda+1)=0$. The eigenvalues are $\lambda=5$ and $\lambda=-1$.

The reduced row echelon form of $5I-A=\begin{bmatrix}4&-4\\-2&2\end{bmatrix}$ is $\begin{bmatrix}1&-1\\0&0\end{bmatrix}$. The general solution of $(5I-A)\mathrm{x}=0$ is $x_1=t$, $x_2=t$. In vector form, $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}t\\t\end{bmatrix}=t\begin{bmatrix}1\\1\end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=5$ is $\{(1,1)\}$.

The reduced row echelon form of $-1I-A=\begin{bmatrix}-2&-4\\-2&-4\end{bmatrix}$ is $\begin{bmatrix}1&2\\0&0\end{bmatrix}$. The general solution of $(-1I-A)\mathrm{x}=0$ is $x_1=-2t$, $x_2=t$. In vector form, $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2t\\t\end{bmatrix}=t\begin{bmatrix}-2\\1\end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=-1$ is $\{(-2,1)\}$.

(b) $\det(\lambda I-A)=\begin{vmatrix}\lambda+2&7\\-1&\lambda-2\end{vmatrix}=(\lambda+2)(\lambda-2)-(7)(-1)=\lambda^2+3$.

The characteristic equation is $\lambda^2+3=0$. There are no real eigenvalues.

(c) $\det(\lambda I-A)=\begin{vmatrix}\lambda-1&0\\0&\lambda-1\end{vmatrix}=(\lambda-1)^2$.

The characteristic equation is $(\lambda-1)^2=0$. The eigenvalue is $\lambda=1$.

The matrix $I-A=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is already in reduced row echelon form. The general solution of $(I-A)\mathrm{x}=0$ is $x_1=s$, $x_2=t$. In vector form, $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}s\\t\end{bmatrix}=s\begin{bmatrix}1\\0\end{bmatrix}+t\begin{bmatrix}0\\1\end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=1$ is $\{(1,0),(0,1)\}$.

(d) $\det(\lambda I-A)=\begin{vmatrix}\lambda-1&2\\0&\lambda-1\end{vmatrix}=(\lambda-1)^2$.

The characteristic equation is $(\lambda-1)^2=0$. The eigenvalue is $\lambda=1$.

The reduced row echelon form of $I-A=\begin{bmatrix}0&2\\0&0\end{bmatrix}$ is $\begin{bmatrix}0&1\\0&0\end{bmatrix}$. The general solution of $(I-A)\mathrm{x}=0$ is $x_1=t$, $x_2=0$. In vector form, $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}t\\0\end{bmatrix}=t\begin{bmatrix}1\\0\end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=1$ is $\{(1,0)\}$.

7. Cofactor expansion along the second column yields $\det(\lambda I-A)=\begin{vmatrix}\lambda-4&0&-1\\2&\lambda-1&0\\2&0&\lambda-1\end{vmatrix}$

$=(\lambda-1)\begin{vmatrix}\lambda-4&-1\\2&\lambda-1\end{vmatrix}=(\lambda-1)\left[(\lambda-4)(\lambda-1)-(-1)(2)\right]=(\lambda-1)(\lambda^2-5\lambda+6)$

$=(\lambda-1)(\lambda-2)(\lambda-3)$. The characteristic equation is $(\lambda-1)(\lambda-2)(\lambda-3)=0$.

The eigenvalues are $\lambda=1$, $\lambda=2$, and $\lambda=3$.

The reduced row echelon form of $I-A=\begin{bmatrix}-3&0&-1\\2&0&0\\2&0&0\end{bmatrix}$ is $\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$. The general solution of $(I-A)\mathrm{x}=0$ is $x_1=0$, $x_2=t$, $x_3=0$. In vector form, $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\t\\0\end{bmatrix}=t\begin{bmatrix}0\\1\\0\end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=1$ is $\{(0,1,0)\}$.

The reduced row echelon form of $2I-A=\begin{bmatrix}-2&0&-1\\2&1&0\\2&0&1\end{bmatrix}$ is $\begin{bmatrix}1&0&\frac{1}{2}\\0&1&-1\\0&0&0\end{bmatrix}$. The general solution of

$(2I-A)\mathrm{x}=0$ is $x_1=-\frac{1}{2}t$, $x_2=t$, $x_3=t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -\frac{1}{2}t \\ t \\ t \end{bmatrix}=t\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=2$ is $\{(-1,2,2)\}$ (scaled by a factor of 2 for convenience).

The reduced row echelon form of $3I-A=\begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(3I-A)\mathrm{x}=0$ is $x_1=-t$, $x_2=t$, $x_3=t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -t \\ t \\ t \end{bmatrix}=t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=3$ is $\{(-1,1,1)\}$.

9. Cofactor expansion along the second row yields $\det(\lambda I-A)=\begin{vmatrix} \lambda-6 & -3 & 8 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda+3 \end{vmatrix}$

$=(\lambda+2)\begin{vmatrix} \lambda-6 & 8 \\ -1 & \lambda+3 \end{vmatrix}=(\lambda+2)\left[(\lambda-6)(\lambda+3)-(8)(-1)\right]=(\lambda+2)\left(\lambda^2-3\lambda-10\right)$

$=(\lambda+2)(\lambda+2)(\lambda-5)$. The characteristic equation is $(\lambda+2)^2(\lambda-5)=0$.
The eigenvalues are $\lambda=-2$ and $\lambda=5$.

The reduced row echelon form of $-2I-A=\begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(-2I-A)\mathrm{x}=0$ is $x_1=t$, $x_2=0$, $x_3=t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} t \\ 0 \\ t \end{bmatrix}=t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=-2$ is $\{(1,0,1)\}$.

The reduced row echelon form of $5I-A=\begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(5I-A)\mathrm{x}=0$ is $x_1=8t$, $x_2=0$, $x_3=t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 8t \\ 0 \\ t \end{bmatrix}=t\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=5$ is $\{(8,0,1)\}$.

11. Cofactor expansion along the second column yields $\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 0 & 1 \\ 0 & \lambda-3 & 0 \\ -1 & 0 & \lambda-2 \end{vmatrix}$

$= (\lambda-3)\begin{vmatrix} \lambda-4 & 1 \\ -1 & \lambda-2 \end{vmatrix} = (\lambda-3)\left[(\lambda-4)(\lambda-2)-(1)(-1)\right] = (\lambda-3)\left(\lambda^2-6\lambda+9\right) = (\lambda-3)^3$.

The characteristic equation is $(\lambda-3)^3 = 0$. The eigenvalue is $\lambda = 3$.

The reduced row echelon form of $3I - A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of $(3I - A)\mathrm{x} = 0$ is $x_1 = t$, $x_2 = s$, $x_3 = t$. In vector form,

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 3$ is $\{(0,1,0),(1,0,1)\}$.

13. The matrix $\lambda I - A$ is lower triangular, therefore by Theorem 2.1.2 its determinant is the product of the entries on the main diagonal. Therefore the characteristic equation is $(\lambda-3)(\lambda-7)(\lambda-1) = 0$.

15. $T(x,y) = \begin{bmatrix} x+4y \\ 2x+3y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$; the standard matrix for the operator T is $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

The following results were obtained in the solution of Exercise 5(a). These statements apply to the matrix A, therefore they also apply to the associated operator T:

- the eigenvalues are $\lambda = 5$ and $\lambda = -1$,
- a basis for the eigenspace corresponding to $\lambda = 5$ is $\{(1,1)\}$,
- a basis for the eigenspace corresponding to $\lambda = -1$ is $\{(-2,1)\}$.

17. **(a)** The transformation D^2 maps any function $\mathbf{f} = f(x)$ in $C^\infty(-\infty,\infty)$ into its second derivative, i.e. $D^2(\mathbf{f}) = f''(x)$. From calculus, we have

$D^2(\mathbf{f}+\mathbf{g}) = \frac{d}{dx}\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}\left(f'(x)+g'(x)\right) = f''(x)+g''(x) = D^2(\mathbf{f})+D^2(\mathbf{g})$ and

$D^2(k\mathbf{f}) = \frac{d}{dx}\frac{d}{dx}\left(kf(x)\right) = \frac{d}{dx}\left(kf'(x)\right) = kf''(x) = kD^2(\mathbf{f})$. We conclude that D^2 is linear.

(b) Denote $\mathbf{f} = \sin\sqrt{\omega}x$ and $\mathbf{g} = \cos\sqrt{\omega}x$. We have

$D^2(\mathbf{f}) = \frac{d}{dx}\frac{d}{dx}\left(\sin\sqrt{\omega}x\right) = \frac{d}{dx}\left(\sqrt{\omega}\cos\sqrt{\omega}x\right) = \left(\sqrt{\omega}\right)\left(-\sqrt{\omega}\sin\sqrt{\omega}x\right) = -\omega\sin\sqrt{\omega}x = -\omega\mathbf{f}$ and

$D^2(\mathbf{g}) = \frac{d}{dx}\frac{d}{dx}\left(\cos\sqrt{\omega}x\right) = \frac{d}{dx}\left(-\sqrt{\omega}\sin\sqrt{\omega}x\right) = \left(-\sqrt{\omega}\right)\left(\sqrt{\omega}\cos\sqrt{\omega}x\right) = -\omega\cos\sqrt{\omega}x = -\omega\mathbf{g}$.

It follows that $\mathbf{f} = \sin\sqrt{\omega}x$ and $\mathbf{g} = \cos\sqrt{\omega}x$ are eigenvectors of D^2; $\lambda = -\omega$ is the eigenvalue associated with both of these eigenvectors.

19. **(a)** The reflection of any vector on the line $y = x$ is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,1)\}$.

The reflection of any vector perpendicular to the line $y = x$ (i.e., on the line $y = -x$) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(-1,1)\}$.

(b) The projection of any vector on the x-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0)\}$.

The projection of any vector perpendicular to the x-axis (i.e., on the y-axis) is the zero vector: an eigenvalue $\lambda = 0$ corresponds to the eigenspace $\text{span}\{(0,1)\}$.

(c) The result of the rotation through $90°$ of a nonzero vector is never a scalar multiple of the original vector. Consequently, this operator has no real eigenvalues.

(d) The result of the contraction of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^2.

(e) The result of the shear applied to any vector on the x-axis is the same vector whereas the result of the shear applied to a nonzero vector in any other direction is not a scalar multiple of the original vector. The only eigenvalue is $\lambda = 1$ and the corresponding eigenspace is $\text{span}\{(1,0)\}$.

21. **(a)** The reflection of any vector on the xy-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0,0),(0,1,0)\}$.

The reflection of any vector perpendicular to the xy-plane (i.e., on the z-axis) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(0,0,1)\}$.

(b) The projection of any vector on the xz-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0,0),(0,0,1)\}$.

The projection of any vector perpendicular to the xz-plane (i.e., on the y-axis) is the zero vector: an eigenvalue $\lambda = 0$ corresponds to the eigenspace $\text{span}\{(0,1,0)\}$.

(c) The result of the rotation applied to any vector on the x-axis is the same vector whereas the result of the rotation applied to a nonzero vector in any other direction is not a scalar multiple of the original vector. The only eigenvalue is $\lambda = 1$ and the corresponding eigenspace is $\text{span}\{(1,0,0)\}$.

(d) The result of the contraction of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^3.

23. A line through the origin in the direction of $\mathbf{x} \neq \mathbf{0}$ is invariant under A if and only if $\mathbf{x}$ is an eigenvector of A.

(a) $\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 1 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda-4)(\lambda-1)-(1)(-2) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$.

The characteristic equation is $(\lambda-2)(\lambda-3)=0$. The eigenvalues are $\lambda=2$ and $\lambda=3$.

The reduced row echelon form of $2I - A = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$. The general solution of $(2I-A)\mathrm{x}=0$ is $x=\frac{1}{2}t$, $y=t$. Therefore $y=2x$ is an equation of the corresponding invariant line.

The reduced row echelon form of $3I - A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(3I-A)\mathrm{x}=0$ is $x_1=t$, $x_2=t$. Therefore $y=x$ is an equation of the corresponding invariant line.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - (-1)(1) = \lambda^2 + 1$.

There are no real eigenvalues and no invariant lines.

25. **(a)** Since the degree of $p(\lambda)$ is 6, A is a 6×6 matrix (see Exercise 37).

(b) $p(0)\neq 0$, therefore 0 is not an eigenvalue of A. From parts (a) and (r) of Theorem 5.1.5, A is invertible.

(c) A has three eigenspaces since it has three distinct eigenvalues, each corresponding to an eigenspace.

27. Substituting the given eigenvectors $\mathbf{x}$ and the corresponding eigenvalues λ into $A\mathbf{x}=\lambda\mathbf{x}$ yields

$$A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } A\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can combine these three equations into a single equation $A\left[\begin{array}{r|r|r} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{r|r|r} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{array}\right]$.

Since the matrix $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible, we can multiply both sides on the right by its inverse,

$\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix}$, resulting in $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(Note that this exercise could also be solved by assigning nine unknown values to the elements of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then solving the system of nine equations in nine unknowns resulting from the equation $A\mathbf{x} = \lambda\mathbf{x}$.)

31. It follows from Exercise 28 that if the characteristic polynomial $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $p(\lambda) = \lambda^2 + c_1\lambda + c_2$ then $c_1 = -\text{tr}(A) = -a_{11} - a_{22}$ and $c_2 = \det(A) = a_{11}a_{22} - a_{12}a_{21}$. Therefore

$$\begin{aligned}
p(A) &= A^2 + c_1 A + c_2 I \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + (-a_{11} - a_{22})\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + (a_{11}a_{22} - a_{12}a_{21})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{bmatrix} + \begin{bmatrix} -a_{11}^2 - a_{22}a_{11} & -a_{11}a_{12} - a_{22}a_{12} \\ -a_{11}a_{21} - a_{22}a_{21} & -a_{11}a_{22} - a_{22}^2 \end{bmatrix} \\
&\qquad + \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

33. By Theorem 5.1.5(r), it follows from A being invertible that A cannot have a zero eigenvalue. Multiplying both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$ by A^{-1} on the left and applying Theorem 1.4.1 yields $A^{-1}A\mathbf{x} = \lambda A^{-1}\mathbf{x}$. Since $A^{-1}A = I$, dividing both sides of the equation by λ we obtain $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$. This shows that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} associated with an eigenvector $\mathbf{x}$.

35. Multiplying both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$ by the scalar s yields $s(A\mathbf{x}) = s(\lambda\mathbf{x})$. By Theorem 1.4.1, the equation can be rewritten as $(sA)\mathbf{x} = (s\lambda)\mathbf{x}$. This shows that $s\lambda$ is an eigenvalue of sA associated with the eigenvector $\mathbf{x}$.

True-False Exercises

(a) False. The vector $\mathbf{x}$ must be nonzero (without that requirement, $A\mathbf{x} = \lambda\mathbf{x}$ holds true for <u>all</u> $n \times n$ matrices A and <u>all</u> values λ by taking $\mathbf{x} = \mathbf{0}$).

(b) False. If λ is an eigenvalue of A then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ must have nontrivial solutions.

(c) True. Since $p(0) = 1 \neq 0$, zero is not an eigenvalue of A. By Theorem 5.1.5(r), we conclude that A is invertible.

(d) False. Every eigenspace must include the zero vector, which is not an eigenvector.

(e) False. E.g., the only eigenvalue of $A = 2I$ is 2. However, the reduced row echelon form of A is I, whose only eigenvalue is 1.

(f) False. By Theorem 5.1.5(h), the set of columns of A must be linearly dependent.

5.2 Diagonalization

1. $\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1$ does not equal $\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} = -2$ therefore, by Table 1 in Section 5.2, A and B are not similar matrices.

3. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (1)(1)(1) = 1$ does not equal $\begin{vmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 - 0 + 1\begin{vmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{vmatrix} = 0$ therefore, by Table 1 in Section 5.2, A and B are not similar matrices.

5. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 \\ -6 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1)$ therefore A has eigenvalues 1 and -1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{1}{3}t$, $x_2 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = 0$, $x_2 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors p_1 and p_2: $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the two columns can be interchanged.)

Calculating $P^{-1} = \frac{1}{(1)(1)-(0)(3)}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ and performing matrix multiplications we check that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

7. $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 0 & 2 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-3)^2$ thus A has eigenvalues 2 and 3 (with algebraic multiplicity 2).

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = 0$, $x_3 = 0$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = \lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -2t$, $x_2 = s$, $x_3 = t$. We can write $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ therefore vectors $\mathrm{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathrm{p}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. Note that the geometric multiplicity of this eigenvalue matches its algebraic multiplicity.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

To invert the matrix P, we can employ the procedure introduced in Section 1.5: since the reduced row echelon form of the matrix $\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$, we have $P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We check that $P^{-1}AP=\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&-2\\0&3&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}$.

9. **(a)** Cofactor expansion along the second column yields

$$\det(\lambda I-A)=\begin{vmatrix}\lambda-4&0&-1\\-2&\lambda-3&-2\\-1&0&\lambda-4\end{vmatrix}=(\lambda-3)\begin{vmatrix}\lambda-4&-1\\-1&\lambda-4\end{vmatrix}=(\lambda-3)\left[(\lambda-4)^2-1\right]=(\lambda-3)^2(\lambda-5)$$

therefore A has eigenvalues 3 (with algebraic multiplicity 2) and 5.

(b) The reduced row echelon form of $3I-A$ is $\begin{bmatrix}1&0&1\\0&0&0\\0&0&0\end{bmatrix}$, consequently $\operatorname{rank}(3I-A)=1$.

The reduced row echelon form of $5I-A$ is $\begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\end{bmatrix}$, consequently $\operatorname{rank}(5I-A)=2$.

(c) Based on part (b), the geometric multiplicities of the eigenvalues $\lambda=3$ and $\lambda=5$ are $3-1=2$ and $3-2=1$, respectively. Since these are equal to the corresponding algebraic multiplicities, by Theorem 5.2.4(b) A is diagonalizable.

11. Cofactor expansion along the second row yields

$$\det(\lambda I-A)=\begin{vmatrix}\lambda+1&-4&2\\3&\lambda-4&0\\3&-1&\lambda-3\end{vmatrix}=-3\begin{vmatrix}-4&2\\-1&\lambda-3\end{vmatrix}+(\lambda-4)\begin{vmatrix}\lambda+1&2\\3&\lambda-3\end{vmatrix}$$

$$=(-3)\left[(-4)(\lambda-3)-(2)(-1)\right]+(\lambda-4)\left[(\lambda+1)(\lambda-3)-(2)(3)\right]=\lambda^3-6\lambda^2+11\lambda-6.$$

Following the procedure described in Example 3 of Section 5.1, we determine that the only possibilities for integer solutions of the characteristic equation are ±1, ±2, ±3, and ±6.

Since $\det(1I-A)=0$, $\lambda-1$ must be a factor of the characteristic polynomial. Dividing $\lambda-1$ into $\lambda^3-6\lambda^2+11\lambda-6$ leads to $\det(\lambda I-A)=(\lambda-1)(\lambda^2-5\lambda+6)=(\lambda-1)(\lambda-2)(\lambda-3)$.

We conclude that the eigenvalues are 1, 2, and 3 - each of them has the algebraic multiplicity 1.

The reduced row echelon form of $1I-A$ is $\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=1$

contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1=t$, $x_2=t$, $x_3=t$. A vector $\mathrm{p}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$ forms a basis for this eigenspace.

This eigenvalue has geometric multiplicity 1.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{2}{3}t$, $x_2 = t$, $x_3 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has geometric multiplicity 1.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{4}t$, $x_2 = \frac{3}{4}t$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has geometric multiplicity 1.

Since for each eigenvalue the geometric multiplicity matches the algebraic multiplicity, by Theorem 5.2.4(b) A is diagonalizable.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

13. $\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ -3 & 0 & \lambda - 1 \end{vmatrix} = \lambda^2(\lambda - 1)$ so the eigenvalues are $\lambda = 0$ with the algebraic multiplicity 2 and $\lambda = 1$ with the algebraic multiplicity 1.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to

$\lambda_1 = \lambda_2 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{1}{3}t$, $x_2 = s$, $x_3 = t$. We can write

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$ therefore vectors $\mathrm{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathrm{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$ form a basis for this eigenspace. This eigenvalue has the geometric multiplicity 2.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = 0$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

This eigenvalue has the geometric multiplicity 1.

Since for each eigenvalue the geometric multiplicity matches the algebraic multiplicity, by Theorem 5.2.4(b) A is diagonalizable.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. (Note that this answer is not unique.)

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

15. **(a)** The degree of the characteristic polynomial of A is 3 therefore A is a 3×3 matrix. All three eigenspaces (for $\lambda = 1$, $\lambda = -3$, and $\lambda = 5$) must have dimension 1.

(b) The degree of the characteristic polynomial of A is 6 therefore A is a 6×6 matrix.
The possible dimensions of the eigenspace corresponding to $\lambda = 0$ are 1 or 2.
The dimension of the eigenspace corresponding to $\lambda = 1$ must be 1.
The possible dimensions of the eigenspace corresponding to $\lambda = 2$ are 1, 2, or 3.

17. $\det(\lambda I - A) = \begin{vmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1) - (-3)(-2) = \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3)$ therefore A has eigenvalues 2 and -3, each with the algebraic multiplicity 1.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$

consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{3}{2}t$, $x_2 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -3$

consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix $P = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and calculate $P^{-1} = \frac{1}{(3)(1)-(-1)(2)}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$ so that

$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} = D$.

Therefore $A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^{10} & 0 \\ 0 & (-3)^{10} \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1{,}024 & 0 \\ 0 & 59{,}049 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$

$= \begin{bmatrix} 24{,}234 & -34{,}815 \\ -23{,}210 & 35{,}839 \end{bmatrix}$.

19. To invert the matrix P, we can employ the procedure introduced in Section 1.5: since the reduced row echelon form of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & -5 & 1 \\ 0 & 1 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$, we have

$P^{-1} = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

We verify that $P^{-1}AP = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$ is a diagonal matrix therefore P diagonalizes A.

$$A^{11} = PD^{11}P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} (-2)^{11} & 0 & 0 \\ 0 & (-1)^{11} & 0 \\ 0 & 0 & 1^{11} \end{bmatrix}\begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} -2{,}048 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 10{,}237 & -2{,}047 \\ 0 & 1 & 0 \\ 0 & 10{,}245 & -2{,}048 \end{bmatrix}$$

21. Cofactor expansion along the first row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix}$

$$= (\lambda-3)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & \lambda-3 \end{vmatrix} = (\lambda-3)\big[(\lambda-2)(\lambda-3)-1\big]-(\lambda-3)$$

$$= (\lambda-3)\big[\lambda^2-5\lambda+6-1-1\big] = (\lambda-3)(\lambda^2-5\lambda+4) = (\lambda-1)(\lambda-3)(\lambda-4);$$

therefore, A has eigenvalues 1, 3, and 4, each with the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = 2t$, $x_3 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 3$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -t$, $x_2 = 0$, $x_3 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 4$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = -t$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix $P = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and find its inverse using the procedure introduced in Section 1.5.

Since the reduced row echelon form of the matrix $\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right]$$, we have $P^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

We conclude that $A^n = PD^nP^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

23. By inspection, both A and B have rank 1 (both matrices are in reduced row echelon form).

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and $PB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$.

Setting $AP = PB$ requires that $a = 0$, $b = a$, and $c = 0$.

For any value d, the matrix $P = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$ satisfies the equality $AP = PB$. However, $P = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$ has zero determinant therefore it is not invertible so that the similarity condition $B = P^{-1}AP$ cannot be met.

25. Since there exist matrices P and Q such that $B = P^{-1}AP$ and $C = Q^{-1}BQ$, we can write $C = Q^{-1}\left(P^{-1}AP\right)Q = \left(PQ\right)^{-1} A\left(PQ\right)$. Consequently, A is similar to C.

27. **(a)** The dimension of the eigenspace must be at least 1, but cannot exceed the algebraic multiplicity of the corresponding eigenvalue. Since the algebraic multiplicities of the eigenvalues $1, 3$, and 4 are 1, 2, and 3, respectively, we conclude that

- The dimension of the eigenspace corresponding to $\lambda = 1$ must be 1.
- The possible dimensions of the eigenspace corresponding to $\lambda = 3$ are 1 or 2.
- The possible dimensions of the eigenspace corresponding to $\lambda = 4$ are 1, 2, or 3.

(b) If A is diagonalizable then by Theorem 5.2.4(b) for each eigenvalue the dimension of the eigenspace must be equal to the algebraic multiplicity. Therefore

- The dimension of the eigenspace corresponding to $\lambda = 1$ must be 1.
- The dimension of the eigenspace corresponding to $\lambda = 3$ must be 2.
- The dimension of the eigenspace corresponding to $\lambda = 4$ must be 3.

(c) If the dimension of the eigenspace were smaller than 3 then by Theorem 4.6.2(a), a set of three vectors from that eigenspace would have to be linearly dependent. Consequently, for the set of the three vectors to be linearly independent, the eigenspace containing the set must be of dimension at least 3. This is only possible for the eigenspace corresponding to the eigenvalue $\lambda = 4$.

29. Using the result obtained in Exercise 30 of Section 5.1, we can take $P=\begin{bmatrix} -b & -b \\ a-\lambda_1 & a-\lambda_2 \end{bmatrix}$ where

$\lambda_1 = \frac{1}{2}\left[(a+d)+\sqrt{(a-d)^2+4bc}\right]$ and $\lambda_2 = \frac{1}{2}\left[(a+d)-\sqrt{(a-d)^2+4bc}\right]$.

31. $T(x_1,x_2)=\begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix for the operator T is $A=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

$\det(\lambda I - A)=\begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix}=\lambda^2-1=(\lambda-1)(\lambda+1)$; thus, A has eigenvalues 1 and -1, both with algebraic multiplicities 1.

The reduced row echelon form of $1I-A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1=-t$, $x_2=t$. A vector $\mathrm{p}_1=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I-A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2=-1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1=t$, $x_2=t$. A vector $\mathrm{p}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors p_1 and p_2: $P=\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the two columns can be interchanged.)

33. $T(x_1,x_2,x_3)=\begin{bmatrix} 3x_1 \\ x_2 \\ x_1-x_2 \end{bmatrix}=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix for T is $A=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

Since $\det(\lambda I - A)=\begin{vmatrix} \lambda-3 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ -1 & 1 & \lambda \end{vmatrix}=(\lambda-3)(\lambda-1)\lambda$, thus A has eigenvalues 0, 1, and 3, each with algebraic multiplicity 1.

The reduced row echelon form of $0I-A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=0$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = 0$, $x_3 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0$, $x_2 = -t$, $x_3 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 3t$, $x_2 = 0$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors p_1, p_2, and p_3: $P = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

True-False Exercises

(a) False. E.g., $A = I_2$ has only one eigenvalue $\lambda = 1$, but it is diagonalizable with $P = I_2$.

(b) True. This follows from Theorem 5.2.1.

(c) True. Multiplying $A = P^{-1}BP$ on the left by P yields $PA = BP$.

(d) False. The matrix P is not unique. For instance, interchanging two columns of P results in a different matrix which also diagonalizes A.

(e) True. Since A is invertible, we can take the inverse on both sides of the equality $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ obtaining $P^{-1}A^{-1}P = D^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_n \end{bmatrix}$.

Consequently, P diagonalizes both A and A^{-1}.

(f) True. We can transpose both sides of the equality $P^{-1}AP = D$ obtaining $P^T A^T \left(P^T\right)^{-1} = D^T = D$, i.e., $\left(\left(P^T\right)^{-1}\right)^{-1} A^T \left(P^T\right)^{-1} = D$. Consequently, $\left(P^T\right)^{-1}$ diagonalizes A^T.

(g) True. A basis for R^n must be a linearly independent set of n vectors, so by Theorem 5.2.1 A is diagonalizable.

(h) True. This follows from Theorem 5.2.2(b).

(i) True. From Theorem 5.1.5 we have $\det(A) = 0$. Since $\det\left(A^2\right) = \left(\det(A)\right)^2 = 0^2 = 0$, it follows from the same theorem that A^2 is singular.

5.3 Complex Vector Spaces

1. $\bar{\mathrm{u}} = \left(\overline{2-i}, \overline{4i}, \overline{1+i}\right) = (2+i, -4i, 1-i)$; $\operatorname{Re}(\mathrm{u}) = (2,0,1)$; $\operatorname{Im}(\mathrm{u}) = (-1,4,1)$;

$\|\mathrm{u}\| = \sqrt{|2-i|^2 + |4i|^2 + |1+i|^2} = \sqrt{\left(2^2 + (-1)^2\right) + \left(0^2 + 4^2\right) + \left(1^2 + 1^2\right)} = \sqrt{5+16+2} = \sqrt{23}$

3. **(a)** $\bar{\bar{\mathrm{u}}} = \overline{\overline{(3-4i, 2+i, -6i)}} = \overline{(3+4i, 2-i, 6i)} = (3-4i, 2+i, -6i) = \mathrm{u}$

(b) $\overline{k\mathrm{u}} = \overline{i(3-4i, 2+i, -6i)} = \overline{(4+3i, -1+2i, +6)} = (4-3i, -1-2i, 6)$

$\bar{k}\bar{\mathrm{u}} = \bar{i}\,\overline{(3-4i, 2+i, -6i)} = -i(3+4i, 2-i, 6i) = (4-3i, -1-2i, 6)$

(c) $\overline{\mathrm{u}+\mathrm{v}} = \overline{(4-3i, 4, 4-6i)} = (4+3i, 4, 4+6i)$

$\bar{\mathrm{u}} + \bar{\mathrm{v}} = (3+4i, 2-i, 6i) + (1-i, 2+i, 4) = (4+3i, 4, 4+6i)$

(d) $\overline{\mathrm{u}-\mathrm{v}} = \overline{(2-5i, 2i, -4-6i)} = (2+5i, -2i, -4+6i)$

$\bar{\mathrm{u}} - \bar{\mathrm{v}} = (3+4i, 2-i, 6i) - (1-i, 2+i, 4) = (2+5i, -2i, -4+6i)$

5. $i\mathbf{x} - 3\mathbf{v} = \bar{\mathbf{u}}$ can be rewritten as $i\mathbf{x} = 3\mathbf{v} + \bar{\mathbf{u}}$; multiplying both sides by $-i$ and using the fact that $(-i)(i) = 1$, we obtain $\mathrm{x} = (-i)(3\mathbf{v} + \bar{\mathrm{u}}) = (-i)\left[(3+3i, 6-3i, 12) + (3+4i, 2-i, 6i)\right]$ $= (-i)(6+7i, 8-4i, 12+6i) = (7-6i, -4-8i, 6-12i)$.

7. $\bar{A} = \begin{bmatrix} \overline{-5i} & \bar{4} \\ \overline{2-i} & \overline{1+5i} \end{bmatrix} = \begin{bmatrix} 5i & 4 \\ 2+i & 1-5i \end{bmatrix}$; $\operatorname{Re}(A) = \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$; $\operatorname{Im}(A) = \begin{bmatrix} -5 & 0 \\ -1 & 5 \end{bmatrix}$;

$\det(A) = (-5i)(1+5i) - (4)(2-i) = -5i + 25 - 8 + 4i = 17 - i$; $\operatorname{tr}(A) = -5i + (1+5i) = 1$

9. **(a)** $\overline{\overline{A}} = \begin{bmatrix} \overline{5i} & \overline{4} \\ \overline{2+i} & \overline{1-5i} \end{bmatrix} = \begin{bmatrix} -5i & 4 \\ 2-i & 1+5i \end{bmatrix} = A$

(b) $\overline{\left(A^T\right)} = \begin{bmatrix} \overline{-5i} & \overline{2-i} \\ \overline{4} & \overline{1+5i} \end{bmatrix} = \begin{bmatrix} 5i & 2+i \\ 4 & 1-5i \end{bmatrix}$; $\left(\overline{A}\right)^T = \begin{bmatrix} 5i & 4 \\ 2+i & 1-5i \end{bmatrix}^T = \begin{bmatrix} 5i & 2+i \\ 4 & 1-5i \end{bmatrix}$.

(c) From $AB = \begin{bmatrix} -5i & 4 \\ 2-i & 1+5i \end{bmatrix}\begin{bmatrix} 1-i \\ 2i \end{bmatrix} = \begin{bmatrix} (-5i)(1-i)+(4)(2i) \\ (2-i)(1-i)+(1+5i)(2i) \end{bmatrix}$

$= \begin{bmatrix} -5i-5+8i \\ 2-2i-i-1+2i-10 \end{bmatrix} = \begin{bmatrix} -5+3i \\ -9-i \end{bmatrix}$ we obtain $\overline{AB} = \begin{bmatrix} -5-3i \\ -9+i \end{bmatrix}$

$\overline{A}\,\overline{B} = \begin{bmatrix} 5i & 4 \\ 2+i & 1-5i \end{bmatrix}\begin{bmatrix} 1+i \\ 2i \end{bmatrix} = \begin{bmatrix} (5i)(1+i)+(4)(2i) \\ (2+i)(1+i)+(1-5i)(2i) \end{bmatrix}$

$= \begin{bmatrix} 5i-5-8i \\ 2+2i+i-1-2i-10 \end{bmatrix} = \begin{bmatrix} -5-3i \\ -9+i \end{bmatrix}$.

11. $\mathrm{u}\cdot\mathrm{v} = (i)(\overline{4}) + (2i)(\overline{-2i}) + (3)(\overline{1+i}) = (i)(4)+(2i)(2i)+(3)(1-i) = 4i-4+3-3i = -1+i$

$\mathrm{u}\cdot\mathrm{w} = (i)(\overline{2-i}) + (2i)(\overline{2i}) + (3)(\overline{5+3i}) = (i)(2+i)+(2i)(-2i)+(3)(5-3i)$

$= 2i-1+4+15-9i = 18-7i$

$\mathrm{v}\cdot\mathrm{w} = (4)(\overline{2-i}) + (-2i)(\overline{2i}) + (1+i)(\overline{5+3i}) = (4)(2+i)+(-2i)(-2i)+(1+i)(5-3i)$

$= 8+4i-4+5-3i+5i+3 = 12+6i$

Since both $\mathrm{u}^T\overline{\mathrm{v}} = \begin{bmatrix} i & 2i & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2i \\ 1-i \end{bmatrix} = \begin{bmatrix} -1+i \end{bmatrix}$ and $\overline{\mathrm{v}}^T\mathrm{u} = \begin{bmatrix} 4 & 2i & 1-i \end{bmatrix}\begin{bmatrix} i \\ 2i \\ 3 \end{bmatrix} = \begin{bmatrix} -1+i \end{bmatrix}$ are equal to $\mathrm{u}\cdot\mathrm{v} = -1+i$, Formula (5) holds.

(a) $\overline{\mathrm{v}\cdot\mathrm{u}} = \overline{(4)(\overline{i}\,)+(-2i)(\overline{2i})+(1+i)(\overline{3})} = \overline{(4)(-i)+(-2i)(-2i)+(1+i)(3)}$

$= \overline{-4i-4+3+3i} = \overline{-1-i} = -1+i = \mathrm{u}\cdot\mathrm{v}$

(b) $\mathrm{u}\cdot(\mathrm{v}+\mathrm{w}) = (i)(\overline{4+2-i}) + (2i)(\overline{-2i+2i}) + (3)(\overline{1+i+5+3i})$

$= (i)(6+i)+(2i)(0)+(3)(6-4i) = 6i-1+18-12i = 17-6i$ equals

$\mathrm{u}\cdot\mathrm{v}+\mathrm{u}\cdot\mathrm{w} = -1+i+18-7i = 17-6i$.

(c) $k(\mathrm{u}\cdot\mathrm{v}) = (2i)(-1+i) = -2-2i$ equals

$(k\mathrm{u})\cdot\mathrm{v} = (-2)(\overline{4}) + (-4)(\overline{-2i}) + (6i)(\overline{1+i})$

$= (-2)(4)+(-4)(2i)+(6i)(1-i) = -8-8i+6i+6 = -2-2i$.

13. $\mathbf{u}\cdot\overline{\mathbf{v}} = (i)(4)+(2i)(-2i)+(3)(1+i) = 4i+4+3+3i = 7+7i$

$$\overline{\mathbf{w}\cdot\mathbf{u}} = \overline{(2-i)(\overline{i})+(2i)(\overline{2i})+(5+3i)(\overline{3})} = \overline{(2-i)(-i)+(2i)(-2i)+(5+3i)(3)}$$

$$= \overline{-2i-1+4+15+9i} = \overline{18+7i} = 18-7i$$

$$\overline{(\mathbf{u}\cdot\overline{\mathbf{v}})-\overline{\mathbf{w}\cdot\mathbf{u}}} = \overline{7+7i-18+7i} = \overline{-11+14i} = -11-14i$$

15. $\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 5 \\ -1 & \lambda \end{vmatrix} = (\lambda-4)\lambda-(5)(-1) = \lambda^2-4\lambda+5$

Solving the characteristic equation $\lambda^2-4\lambda+5=0$ using the quadratic formula yields $\lambda = \frac{4\pm\sqrt{4^2-4(5)}}{2} = \frac{4\pm\sqrt{-4}}{2} = 2\pm i$ therefore A has eigenvalues $\lambda = 2+i$ and $\lambda = 2-i$.

For the eigenvalue $\lambda = 2+i$, the augmented matrix of the homogeneous system $((2+i)I-A)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix} -2+i & 5 & 0 \\ -1 & 2+i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $-x_1+(2+i)x_2=0$. The general solution of this equation (and, consequently, of the entire system) is $x_1=(2+i)t$, $x_2=t$. The vector $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda=2+i$.

According to Theorem 5.3.4, the vector $\begin{bmatrix} \overline{2+i} \\ \overline{1} \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 2-i$.

17. $\det(\lambda I - A) = \begin{vmatrix} \lambda-5 & 2 \\ -1 & \lambda-3 \end{vmatrix} = (\lambda-5)(\lambda-3)-(2)(-1) = \lambda^2-8\lambda+17$ Solving the characteristic equation $\lambda^2-8\lambda+17=0$ using the quadratic formula yields $\lambda = \frac{8\pm\sqrt{8^2-4(17)}}{2} = \frac{8\pm\sqrt{-4}}{2} = 4\pm i$ therefore A has eigenvalues $\lambda=4+i$ and $\lambda=4-i$.

For the eigenvalue $\lambda=4+i$, the augmented matrix of the homogeneous system $((4+i)I-A)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix} -1+i & 2 & 0 \\ -1 & 1+i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $-x_1+(1+i)x_2=0$. The general solution of this equation (and, consequently, of the entire system) is $x_1=(1+i)t$, $x_2=t$. The vector $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda=4+i$.

According to Theorem 5.3.4, the vector $\begin{bmatrix} \overline{1+i} \\ \overline{1} \end{bmatrix} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 4-i$.

19. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ implies $a = b = 1$. We have $|\lambda| = |1+i| = \sqrt{1+1} = \sqrt{2}$.

The angle inside the interval $(-\pi, \pi]$ from the positive x-axis to the ray that joins the origin to the point $(1,1)$ is $\phi = \frac{\pi}{4}$.

21. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$ implies $a = 1$ and $b = -\sqrt{3}$. We have $|\lambda| = |1-\sqrt{3}i| = \sqrt{1+3} = 2$.

The angle inside the interval $(-\pi, \pi]$ from the positive x-axis to the ray that joins the origin to the point $(1, -\sqrt{3})$ is $\phi = -\frac{\pi}{3}$.

23. $\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 5 \\ -4 & \lambda-7 \end{vmatrix} = (\lambda+1)(\lambda-7) - (5)(-4) = \lambda^2 - 6\lambda + 13$ Solving the characteristic equation $\lambda^2 - 6\lambda + 13 = 0$ using the quadratic formula yields $\lambda = \frac{6 \pm \sqrt{6^2 - 4(13)}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$ therefore A has eigenvalues $\lambda = 3+2i$ and $\lambda = 3-2i$.

For the eigenvalue $\lambda = 3-2i$, the augmented matrix of the homogeneous system $((3-2i)I - A)\mathrm{x} = 0$ is $\begin{bmatrix} 4-2i & 5 & 0 \\ -4 & -4-2i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) so it suffices to solve the equation corresponding to the second row, which yields $x_1 + (1 + \frac{1}{2}i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = (-1 - \frac{1}{2}i)t$, $x_2 = t$. Since $\begin{bmatrix} -2-i \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 3-2i$, it follows from Theorem 5.3.8 that the matrices $P = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$ satisfy $A = PCP^{-1}$.

25. $\det(\lambda I - A) = \begin{vmatrix} \lambda-8 & -6 \\ 3 & \lambda-2 \end{vmatrix} = (\lambda-8)(\lambda-2) - (-6)(3) = \lambda^2 - 10\lambda + 34$ Solving the characteristic equation $\lambda^2 - 10\lambda + 34 = 0$ using the quadratic formula yields $\lambda = \frac{10 \pm \sqrt{10^2 - 4(34)}}{2} = \frac{10 \pm \sqrt{-36}}{2} = 5 \pm 3i$ therefore A has eigenvalues $\lambda = 5+3i$ and $\lambda = 5-3i$.

For the eigenvalue $\lambda = 5-3i$, the augmented matrix of the homogeneous system $((5-3i)I - A)\mathrm{x} = 0$ is $\begin{bmatrix} -3-3i & -6 & 0 \\ 3 & 3-3i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in

Section 5.3) so it suffices to solve the equation corresponding to the second row, which yields $x_1 + (1-i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = (-1+i)t$, $x_2 = t$. Since $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 5-3i$, it follows from Theorem 5.3.8 that the matrices $P = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -3 \\ 3 & 5 \end{bmatrix}$ satisfy $A = PCP^{-1}$.

27. **(a)** Letting $k = a+bi$ we have $\mathbf{u}\cdot\mathbf{v} = (2i)(\overline{i}) + (i)(\overline{6i}) + (3i)(\overline{a+bi})$ $= (2i)(-i) + (i)(-6i) + (3i)(a-bi) = 2+6+3ai+3b = (8+3b)+(3a)i$. Setting this equal to zero yields $a = 0$ and $b = -\frac{8}{3}$ therefore the only complex scalar which satisfies our requirements is $k = -\frac{8}{3}i$.

(b) $\mathbf{u}\cdot\mathbf{v} = (k)(\overline{1}) + (k)(\overline{-1}) + (1+i)(\overline{1-i}) = (k)(1) + (k)(-1) + (1+i)(1+i) = 2i \neq 0;$ therefore, no complex scalar k satisfies our requirements.

True-False Exercises

(a) False. By Theorem 5.3.4, complex eigenvalues of a real matrix occur in conjugate pairs, so the total number of complex eigenvalues must be even. Consequently, in a 5×5 matrix at least one eigenvalue must be real.

(b) True. $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$ is the characteristic equation of a 2×2 complex matrix A.

(c) False. By Theorem 5.3.5, A has two complex conjugate eigenvalues if $\operatorname{tr}(A)^2 < 4\det(A)$.

(d) True. This follows from Theorem 5.3.4.

(e) False. E.g., $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ is symmetric, but its eigenvalue $\lambda = i$ is not real.

(f) False. (This would be true if we assumed $|\lambda| = 1$.)

5.4 Differential Equations

1. **(a)** We begin by diagonalizing the coefficient matrix of the system $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-3) - (-4)(-2) = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)$$

thus the eigenvalues of A are $\lambda = 5$ and $\lambda = -1$.

The reduced row echelon form of $5I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 5$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -2t$, $x_2 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$.

The substitution $\mathrm{y} = P\mathrm{u}$ yields the "diagonal system" $\mathrm{u}' = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\mathrm{u}$ consisting of equations $u_1' = 5u_1$ and $u_2' = -1u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1 e^{5x}$, $u_2 = c_2 e^{-x}$, i.e., $\mathrm{u} = \begin{bmatrix} c_1 e^{5x} \\ c_2 e^{-x} \end{bmatrix}$. From $\mathrm{y} = P\mathrm{u}$ we obtain the solution

$\mathrm{y} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{5x} \\ c_2 e^{-x} \end{bmatrix} = \begin{bmatrix} c_1 e^{5x} - 2c_2 e^{-x} \\ c_1 e^{5x} + c_2 e^{-x} \end{bmatrix}$; thus, $y_1 = c_1 e^{5x} - 2c_2 e^{-x}$ and $y_2 = c_1 e^{5x} + c_2 e^{-x}$.

(b) Substituting the initial conditions into the general solution obtained in part (a) yields a system

$$c_1 e^{5(0)} - 2c_2 e^{-0} = 0$$

$$c_1 e^{5(0)} + c_2 e^{-0} = 0$$

which can be rewritten as

$$\begin{array}{rcrcl} c_1 & - & 2c_2 & = & 0 \\ c_1 & + & c_2 & = & 0. \end{array}$$

The reduced row echelon form of this system's augmented matrix $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; therefore, $c_1 = 0$ and $c_2 = 0$.

The solution satisfying the given initial conditions can be expressed as $y_1 = 0$ and $y_2 = 0$.

3. **(a)** We begin by diagonalizing the coefficient matrix of the system $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

Cofactor expansion along the second column yields

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-4 & -1 \\ 2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\left[(\lambda-4)(\lambda-1)-(-1)(2)\right]$$

$$= (\lambda-1)\left(\lambda^2-5\lambda+6\right) = (\lambda-1)(\lambda-2)(\lambda-3).$$

The characteristic equation of A is $(\lambda-1)(\lambda-2)(\lambda-3)=0$. Thus, the eigenvalues of A are 1, 2, and 3 (each with the algebraic multiplicity 1).

The reduced row echelon form of $1I-A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda=1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1=0$, $x_2=t$, $x_3=0$. A vector $\mathrm{p}_1=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I-A$ is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda=2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1=-\frac{1}{2}t$, $x_2=t$, $x_3=t$. A vector $\mathrm{p}_2=\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I-A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda=3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1=-t$, $x_2=t$, $x_3=t$. A vector $\mathrm{p}_3=\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P=\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ diagonalizes A and $P^{-1}AP=D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = u_1$, $u_2' = 2u_2$, and $u_3' = 3u_3$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^x$, $u_2 = c_2e^{2x}$, $u_3 = c_3e^{3x}$ i.e., $\mathbf{u} = \begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution

$\mathbf{y} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix} = \begin{bmatrix} -c_2e^{2x} - c_3e^{3x} \\ c_1e^x + 2c_2e^{2x} + c_3e^{3x} \\ 2c_2e^{2x} + c_3e^{3x} \end{bmatrix}$ thus

$y_1 = -c_2e^{2x} - c_3e^{3x}$, $y_2 = c_1e^x + 2c_2e^{2x} + c_3e^{3x}$, and $y_3 = 2c_2e^{2x} + c_3e^{3x}$.

(b) Substituting the initial conditions into the general solution obtained in part (a) yields a system

$$-c_2e^{2(0)} - c_3e^{3(0)} = -1$$

$$c_1e^0 + 2c_2e^{2(0)} + c_3e^{3(0)} = 1$$

$$2c_2e^{2(0)} + c_3e^{3(0)} = 0$$

which can be rewritten as

$$\begin{array}{ccccccc} & - & c_2 & - & c_3 & = & -1 \\ c_1 & + & 2c_2 & + & c_3 & = & 1 \\ & & 2c_2 & + & c_3 & = & 0. \end{array}$$

The reduced row echelon form of this system's augmented matrix $\begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ is

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$; therefore, $c_1 = 1$, $c_2 = -1$, and $c_3 = 2$.

The solution satisfying the given initial conditions can be expressed as
$y_1 = e^{2x} - 2e^{3x}$, $y_2 = e^x - 2e^{2x} + 2e^{3x}$, and $y_3 = -2e^{2x} + 2e^{3x}$.

5. Assume $y = f(x)$ is a solution of $y' = ay$ so that $f'(x) = af(x)$.

We have $\frac{d}{dx}\left(f(x)e^{-ax}\right) = f'(x)e^{-ax} + f(x)(-a)e^{-ax} = af(x)e^{-ax} - af(x)e^{-ax} = 0$ for all x so there exists a constant c for which $f(x)e^{-ax}=c$, i.e., $f(x)=\frac{c}{e^{-ax}}=ce^{ax}$. We conclude that every solution of $y'=ay$ must have the form $f(x)=ce^{ax}$.

7. Substituting $y_1=y$ and $y_2=y'$ allows us to rewrite the equation $y''-y'-6y=0$ as $y_2'-y_2-6y_1=0$. Also, $y_2=y'=y_1'$ so we obtain the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 6y_1+y_2. \end{aligned}$$

The coefficient matrix of this system is $A=\begin{bmatrix}0&1\\6&1\end{bmatrix}$. The characteristic polynomial of A is

$\det(\lambda I-A)=\begin{vmatrix}\lambda&-1\\-6&\lambda-1\end{vmatrix}=\lambda(\lambda-1)-(-1)(-6)=\lambda^2-\lambda-6=(\lambda-3)(\lambda+2)$ thus the eigenvalues of A are $\lambda=3$ and $\lambda=-2$.

The reduced row echelon form of $3I-A$ is $\begin{bmatrix}1&-\frac{1}{3}\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=3$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1=\dfrac{1}{3}t$, $x_2=t$. A vector $\mathrm{p}_1=\begin{bmatrix}1\\3\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-2I-A$ is $\begin{bmatrix}1&\frac{1}{2}\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=-2$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1=-\frac{1}{2}t$, $x_2=t$. A vector $\mathrm{p}_2=\begin{bmatrix}-1\\2\end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P=\begin{bmatrix}1&-1\\3&2\end{bmatrix}$ diagonalizes A and $P^{-1}AP=D=\begin{bmatrix}3&0\\0&-2\end{bmatrix}$.

The substitution $\mathrm{y}=P\mathrm{u}$ yields the "diagonal system" $\mathrm{u}'=\begin{bmatrix}3&0\\0&-2\end{bmatrix}\mathrm{u}$ consisting of equations $u_1'=3u_1$ and $u_2'=-2u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1=c_1e^{3x}$, $u_2=c_2e^{-2x}$, i.e., $\mathrm{u}=\begin{bmatrix}c_1e^{3x}\\c_2e^{-2x}\end{bmatrix}$. From $\mathrm{y}=P\mathrm{u}$ we obtain the solution

$\mathrm{y}=\begin{bmatrix}1&-1\\3&2\end{bmatrix}\begin{bmatrix}c_1e^{3x}\\c_2e^{-2x}\end{bmatrix}=\begin{bmatrix}c_1e^{3x}-c_2e^{-2x}\\3c_1e^{3x}+2c_2e^{-2x}\end{bmatrix}$. Thus, $y_1=c_1e^{3x}-c_2e^{-2x}$ and $y_2=3c_1e^{3x}+2c_2e^{-2x}$.

We conclude that the original equation $y''-y'-6y=0$ has the solution $y=c_1e^{3x}-c_2e^{-2x}$.

9. Substituting $y_1=y$, $y_2=y'$, and $y_3=y''$ allows us to rewrite the equation $y'''-6y''+11y'-6y=0$ as $y_3'-6y_3+11y_2-6y_1=0$. With $y_2=y'=y_1'$ and $y_3=y''=y_2'$ we obtain the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= 6y_1 - 11y_2 + 6y_3. \end{aligned}$$

The coefficient matrix of this system is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$.

The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -6 & 11 & \lambda - 6 \end{vmatrix}$

$= \lambda \begin{vmatrix} \lambda & -1 \\ 11 & \lambda - 6 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 \\ -6 & \lambda - 6 \end{vmatrix} = \lambda\left[\lambda(\lambda - 6) + 11\right] - 6 = \lambda^3 - 6\lambda^2 + 11\lambda - 6$.

Following the procedure described in Example 3 of Section 5.1, we determine that the only possibilities for integer solutions of the characteristic equation are ± 1, ± 2, ± 3, and ± 6.

Since $\det(1I - A) = 0$, $\lambda - 1$ must be a factor of the characteristic polynomial. Dividing $\lambda - 1$ into $\lambda^3 - 6\lambda^2 + 11\lambda - 6$ leads to $\det(\lambda I - A) = (\lambda - 1)(\lambda^2 - 5\lambda + 6) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$.

We conclude that the eigenvalues are 1, 2, and 3 - each of them has the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$, $x_3 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{4}t$, $x_2 = \frac{1}{2}t$, $x_3 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 3$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{9}t$, $x_2 = \frac{1}{3}t$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

The substitution $\mathrm{y} = P\mathrm{u}$ yields the "diagonal system" $\mathrm{u}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathrm{u}$ consisting of equations $u_1' = u_1$, $u_2' = 2u_2$, and $u_3' = 3u_3$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^x$, $u_2 = c_2e^{2x}$, and $u_3 = c_3e^{3x}$, i.e., $\mathrm{u} = \begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix}$. From $\mathrm{y} = P\mathrm{u}$ we obtain the solution

$\mathrm{y} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix} = \begin{bmatrix} c_1e^x + c_2e^{2x} + c_3e^{3x} \\ c_1e^x + 2c_2e^{2x} + 3c_3e^{3x} \\ c_1e^x + 4c_2e^{2x} + 9c_3e^{3x} \end{bmatrix}$. Thus, $y_1 = c_1e^x + c_2e^{2x} + c_3e^{3x}$, $y_2 = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}$, and $y_3 = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$.

We conclude that the original equation $y''' - 6y'' + 11y' - 6y = 0$ has the solution

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

15. **(a)** Let y and z be functions in $C^\infty(-\infty,\infty)$ and let k be a real number. From calculus, we have

$L(y+z) = \frac{d^3}{dx^3}(y+z) - 2\frac{d^2}{dx^2}(y+z) - \frac{d}{dx}(y+z) + 2(y+z)$

$= y''' + z''' - 2y'' - 2z'' - y' - z' + 2y + 2z = L(y) + L(z)$ and

$L(ky) = \frac{d^3}{dx^3}(ky) - 2\frac{d^2}{dx^2}(ky) - \frac{d}{dx}(ky) + 2(ky) = ky''' - 2ky'' - ky' + 2ky = kL(y)$.

Therefore, L is a linear operator.

(b) Substituting $y_1 = y$, $y_2 = y'$, and $y_3 = y''$ we can rewrite $y''' - 2y' - y' + 2y = 0$ as the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= -2y_1 + y_2 + 2y_3 \end{aligned}$$

This system can be expressed in the form $\mathbf{y}' = A\mathbf{y}$ where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

Cofactor expansion along the third column yields $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & -1 & \lambda-2 \end{vmatrix}$

$$= \begin{vmatrix} \lambda & -1 \\ 2 & -1 \end{vmatrix} + (\lambda-2)\begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = (-\lambda+2) + (\lambda-2)\lambda^2$$
$$= (\lambda-2)(\lambda^2-1) = (\lambda-2)(\lambda-1)(\lambda+1).$$

The characteristic equation is $(\lambda-2)(\lambda-1)(\lambda+1) = 0$; therefore, the eigenvalues are 2, 1, and -1 $-$ each of them has the algebraic multiplicity 1.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{4}t$, $x_2 = \frac{1}{2}t$, $x_3 = t$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$, $x_3 = t$. A vector $\mathrm{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = -1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = -t$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P=\begin{bmatrix}1&1&1\\2&1&-1\\4&1&1\end{bmatrix}$ diagonalizes A and $P^{-1}AP=D=\begin{bmatrix}2&0&0\\0&1&0\\0&0&-1\end{bmatrix}$.

The substitution $\mathrm{y}=P\mathrm{u}$ yields the "diagonal system" $\mathrm{u}'=\begin{bmatrix}2&0&0\\0&1&0\\0&0&-1\end{bmatrix}\mathrm{u}$ consisting of equations

$u_1'=2u_1$, $u_2'=u_2$, and $u_3'=-u_3$. From Formula (2) in Section 5.4, these equations have the solutions

$u_1=c_1e^{2x}$, $u_2=c_2e^{x}$, and $u_3=c_3e^{-x}$, i.e., $\mathrm{u}=\begin{bmatrix}c_1e^{2x}\\c_2e^{x}\\c_3e^{-x}\end{bmatrix}$. From $\mathrm{y}=P\mathrm{u}$ we obtain the solution

$\mathrm{y}=\begin{bmatrix}1&1&1\\2&1&-1\\4&1&1\end{bmatrix}\begin{bmatrix}c_1e^{2x}\\c_2e^{x}\\c_3e^{-x}\end{bmatrix}=\begin{bmatrix}c_1e^{2x}+c_2e^{x}+c_3e^{-x}\\2c_1e^{2x}+c_2e^{x}-c_3e^{-x}\\4c_1e^{2x}+c_2e^{x}+c_3e^{-x}\end{bmatrix}$. Thus, $y_1=c_1e^{2x}+c_2e^{x}+c_3e^{-x}$,

$y_2=2c_1e^{2x}+c_2e^{x}-c_3e^{-x}$, and $y_3=4c_1e^{2x}+c_2e^{x}+c_3e^{-x}$.

We conclude that the differential equation $L(y)=0$ has the solution $y=c_1e^{2x}+c_2e^{x}+c_3e^{-x}$.

True-False Exercises

(a) True. $\mathbf{y}=\mathbf{0}$ is always a solution (called the trivial solution).

(b) False. If a system has a solution $\mathbf{x}\neq\mathbf{0}$ then any for any real number k, $\mathbf{y}=k\mathbf{x}$ is also a solution.

(c) True. $(c\mathbf{x}+d\mathbf{y})'=c\mathbf{x}'+d\mathbf{y}'=c(A\mathbf{x})+d(A\mathbf{y})=A(c\mathbf{x})+A(d\mathbf{y})=A(c\mathbf{x}+d\mathbf{y})$

(d) True. The solution can be obtained by following the four-step procedure preceding Example 2.

(e) False. If $P=Q^{-1}AQ$ then $\mathbf{u}'=Q^{-1}AQ\mathbf{u}$ implies $(Q\mathbf{u})'=A(Q\mathbf{u})$. Generally, $\mathbf{u}$ and $\mathbf{y}=Q\mathbf{u}$ are not the same.

5.5 Dynamical Systems and Markov Chains

1. **(a)** A is a stochastic matrix: each column vector has nonnegative entries that add up to 1.

(b) A is not a stochastic matrix since entries in its columns do not add up to 1.

(c) A is a stochastic matrix: each column vector has nonnegative entries that add up to 1.

(d) A is not a stochastic matrix since $(A)_{23}=-\frac{1}{2}$ fails to be nonnegative

3. $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$ $\quad \mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 0.545 \\ 0.455 \end{bmatrix}$

$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.545 \\ 0.455 \end{bmatrix} = \begin{bmatrix} 0.5455 \\ 0.4545 \end{bmatrix}$ $\quad \mathbf{x}_4 = P\mathbf{x}_3 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.5455 \\ 0.4545 \end{bmatrix} = \begin{bmatrix} 0.54545 \\ 0.45455 \end{bmatrix}$

An alternate approach is to determine $P^4 = \begin{bmatrix} 0.5455 & 0.5454 \\ 0.4545 & 0.4546 \end{bmatrix}$ then calculate $\mathbf{x}_4 = P^4\mathbf{x}_0 = \begin{bmatrix} 0.54545 \\ 0.45455 \end{bmatrix}$.

5. **(a)** P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since P has all positive entries, it is also a regular matrix.

(b) By Theorem 1.7.1(b), the product of lower triangular matrices is also lower triangular. Consequently, for all positive integers k, the matrix P^k will have 0 in the first row second column entry. Therefore P is not a regular matrix.

(c) P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since $P^2 = \begin{bmatrix} \frac{21}{25} & \frac{1}{5} \\ \frac{4}{25} & \frac{4}{5} \end{bmatrix}$ has all positive entries, we conclude that P is a regular matrix.

7. P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since P has all positive entries, it is also a regular matrix.

To find the steady-state vector, we solve the system $(I-P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{3}{4} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & -\frac{8}{9} \\ 0 & 0 \end{bmatrix}$. Thus, the general solution is $q_1 = \frac{8}{9}t$, $q_2 = t$. For $\mathbf{q}$ to be a probability vector, its components must add up to 1: $q_1 + q_2 = 1$. Solving the resulting equation $\frac{8}{9}t + t = 1$ for t results in $t = \frac{9}{17}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{8}{17} \\ \frac{9}{17} \end{bmatrix}$.

9. P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since $P^2 = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{3}{8} & \frac{7}{18} \\ \frac{7}{24} & \frac{1}{8} & \frac{4}{9} \end{bmatrix}$ has all positive entries, we conclude that P is a regular matrix.

To find the steady-state vector, we solve the system $(I-P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{4} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the general solution is $q_1 = \frac{4}{3}t$, $q_2 = \frac{4}{3}t$, $q_3 = t$. For q to be a probability vector, we must have $q_1 + q_2 + q_3 = 1$. Solving the resulting equation $\frac{4}{3}t + \frac{4}{3}t + t = 1$ for t results in $t = \frac{3}{11}$, consequently the steady-state vector is $\mathrm{q} = \begin{bmatrix} \frac{4}{11} \\ \frac{4}{11} \\ \frac{3}{11} \end{bmatrix}$.

11. **(a)** The entry 0.2 represents the probability that the system will stay in state 1 when it is in state 1.

(b) The entry 0.1 represents the probability that the system will move to state 1 when it is in state 2.

(c) $\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$. Therefore, if the system is in state 1 initially, there is 0.8 probability that it will be in state 2 at the next observation.

(d) $\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$. Therefore, if the system has a 50% chance of being in state 1 initially, it will be in state 2 at the next observation with probability 0.85.

13. **(a)** The transition matrix is

$$\begin{array}{cc} & \begin{array}{cc} \textbf{good} & \textbf{bad} \end{array} \\ \begin{array}{c} \textbf{good} \\ \textbf{bad} \end{array} & \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \end{array}.$$

(b) $\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.93 \\ 0.07 \end{bmatrix}$. Therefore, if the air quality is good today, it will also be good two days from now with probability 0.93.

(c) $\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.858 \\ 0.142 \end{bmatrix}$. Therefore, if the air quality is bad today, it will also be bad three days from now with probability 0.142.

(d) $\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 0.37 \end{bmatrix}$. Therefore, if there is a 20% chance that air quality will be good today, it will be good tomorrow with probability 0.63.

15. **(a)** The transition matrix is

$$\begin{array}{cc} & \begin{array}{cc} \textbf{city} & \textbf{suburbs} \end{array} \\ \begin{array}{c} \textbf{city} \\ \textbf{suburbs} \end{array} & \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \end{array} = P.$$

The initial state vector $\mathrm{x}_0 = \begin{bmatrix} \frac{100{,}000}{125{,}000} \\ \frac{25{,}000}{125{,}000} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ represents the fractions of the total population (125,000) living in the city and in the suburbs, respectively.

After one year, the corresponding fractions are contained in the state vector

$x_1 = Px_0 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix}$. To determine the populations living in the city and in the suburbs at that time, we can calculate the scalar multiple of the state vector: $125,000\,\mathbf{x}_1 = \begin{bmatrix} 95,750 \\ 29,250 \end{bmatrix}$.

After the second year, the state vector becomes $x_2 = Px_1 = \begin{bmatrix} 0.73472 \\ 0.26528 \end{bmatrix}$, and the corresponding population counts are $125,000\,\mathbf{x}_2 = \begin{bmatrix} 91,840 \\ 33,160 \end{bmatrix}$.

Repeating this process three more times results in the following:

	initial state $k=0$	after 1 year $k=1$	after 2 years $k=2$	after 3 years $k=3$	after 4 years $k=4$	after 5 years $k=5$
state vector $\mathbf{x}_k \approx$	$\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix}$	$\begin{bmatrix} 0.73472 \\ 0.26528 \end{bmatrix}$	$\begin{bmatrix} 0.705942 \\ 0.294058 \end{bmatrix}$	$\begin{bmatrix} 0.679467 \\ 0.320533 \end{bmatrix}$	$\begin{bmatrix} 0.655110 \\ 0.344890 \end{bmatrix}$
city population	100,000	95,750	91,840	88,243	84,933	81,889
suburb population	25,000	29,250	33,160	36,757	40,067	43,111

(b) Since P is a regular stochastic matrix, there exists a unique steady-state probability vector. To find the steady-state vector, we solve the system $(I-P)q = 0$, i.e.,

$\begin{bmatrix} 0.05 & -0.03 \\ -0.05 & 0.03 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & 0 \end{bmatrix}$. Thus, the general solution is $q_1 = \frac{3}{5}t$, $q_2 = t$. The components of the vector q must add up to 1: $q_1 + q_2 = 1$. Solving the resulting equation $\frac{3}{5}t + t = 1$ for t results in $t = \frac{5}{8}$, consequently over the long term the fractions of the total population living in the city and in the suburbs will approach $\frac{3}{5}\cdot\frac{5}{8} = \frac{3}{8}$ and $\frac{5}{8}$, respectively. We conclude that the city population will approach $\frac{3}{8}\cdot 125,000 = 46,875$ and the suburbs population will approach $\frac{5}{8}\cdot 125,000 = 78,125$.

17. For the matrix $P = \begin{bmatrix} \frac{7}{10} & p_{12} & \frac{1}{5} \\ p_{21} & \frac{3}{10} & p_{23} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{bmatrix}$ to be stochastic, each column vector must be a probability vector: a vector with nonnegative entries that add up to one. Applying the latter condition to each column results in three equations, which can be used to solve for the missing entries:

$$\begin{array}{llll} \text{column 1:} & \frac{7}{10}+p_{21}+\frac{1}{10}=1 & \text{yields} & p_{21}=1-\frac{7}{10}-\frac{1}{10}=\frac{2}{10}=\frac{1}{5} \\ \text{column 2:} & p_{12}+\frac{3}{10}+\frac{3}{5}=1 & \text{yields} & p_{12}=1-\frac{3}{10}-\frac{3}{5}=\frac{1}{10} \\ \text{column 3:} & \frac{1}{5}+p_{23}+\frac{3}{10}=1 & \text{yields} & p_{23}=1-\frac{1}{5}-\frac{3}{10}=\frac{5}{10}=\frac{1}{2} \end{array}$$

The resulting transition matrix is $P=\begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{bmatrix}$. Since P is a regular stochastic matrix, there exists a unique steady-state probability vector. To find the steady-state vector, we solve the system $(I-P)\mathrm{q}=0$, i.e., $\begin{bmatrix} \frac{3}{10} & -\frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{7}{10} & -\frac{1}{2} \\ -\frac{1}{10} & -\frac{3}{5} & \frac{7}{10} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the general solution is $q_1=t$, $q_2=t$, $q_3=t$. For q to be a probability vector, we must have $q_1+q_2+q_3=1$. Solving the resulting equation $t+t+t=1$ for t results in $t=\frac{1}{3}$, consequently the steady-state vector is $\mathrm{q}=\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

19. From Theorem 5.5.1(a), we have $P\mathbf{q}=\mathbf{q}$. Therefore for any positive integer k,

$$P^k\mathbf{q}=P^{k-1}(P\mathbf{q})=P^{k-1}\mathbf{q}=P^{k-2}(P\mathbf{q})=P^{k-2}\mathbf{q}=\cdots=\mathbf{q}$$

21. Let A and B be two $n\times n$ stochastic matrices, and let B be partitioned into columns: $B=[\,\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_n\,]$. Using Formula (6) in Section 1.3, we can now see that the product

$$AB=A[\,\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_n\,]=[A\,\mathbf{b}_1 \mid \mathrm{A}\,\mathbf{b}_2 \mid \ldots \mid A\,\mathbf{b}_n\,]$$

has columns that are probability vectors (since each of them is a product of a stochastic matrix and a probability vector). We conclude that AB is stochastic.

True-False Exercises

(a) True. All entries are nonnegative and their sum is 1.

(b) True. This is a stochastic matrix since its columns are probability vectors.

Furthermore, $\begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}^2=\begin{bmatrix} 0.84 & 0.2 \\ 0.16 & 0.8 \end{bmatrix}$ has all positive entries.

(c) True. By definition, a transition matrix is a stochastic matrix.

(d) False. For $\mathbf{q}$ to be a steady-state vector of a regular Markov chain, it must also be a probability vector.

(e) True. (See Exercise 21).

(f) False. The entries must be nonnegative.

(g) True. This follows from Theorem 5.5.1(a).

Chapter 5 Supplementary Exercises

1. **(a)** The characteristic polynomial is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} = (\lambda - \cos\theta)^2 + (\sin\theta)^2.$$

For a real eigenvalue λ to exist, we must have $\lambda = \cos\theta$ and $\sin\theta = 0$. However, the latter equation has no solutions on the given interval $0 < \theta < \pi$, therefore A has no real eigenvalues, and consequently no real eigenvectors.

(b) According to Table 5 in Section 1.8, A is the standard matrix of the rotation in the plane about the origin through a positive angle θ. Unless the angle is an integer multiple of π, no vector resulting from such a rotation is a scalar multiple of the original nonzero vector.

3. **(a)** If $D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$ with $d_{ii} \geq 0$ for all i then we can take

$S = \begin{bmatrix} \sqrt{d_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{d_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{d_{nn}} \end{bmatrix}$ so that $S^2 = D$ holds true. (Note that the answer is not unique: the main diagonal entries of S could be negative square roots instead.)

(b) From our assumptions it follows that there exists a matrix P such that $A = P^{-1}DA$ where

$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ with $\lambda_i \geq 0$ for all i. Taking $R = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}$ so that

$R^2 = D$ (see part (a)), we can form the matrix $S = PRP^{-1}$ so that

$S^2 = PRP^{-1}PRP^{-1} = PR^2P^{-1} = PDP^{-1} = A$.

(c) By Theorem 5.1.2, A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 9$.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = 0$, $x_3 = 0$. A vector $\mathrm{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 4$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$, $x_3 = 0$. A vector $\mathrm{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $9I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 9$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{2}t$, $x_2 = t$, $x_3 = t$. A vector $\mathrm{p}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$.

Since the reduced row echelon form of $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$, we have $P^{-1} = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$. As described in the solution of part (b) we can let

$R = \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and form

$S = PRP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. This matrix satisfies $S^2 = A$.

7. **(a)** The characteristic polynomial is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -6 \\ -1 & \lambda - 2 \end{vmatrix} = -5\lambda + \lambda^2$.

We verify that $-5A + A^2 = -5\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}^2 = -\begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) The characteristic polynomial is $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 3 & \lambda - 3 \end{vmatrix} = -1 + 3\lambda - 3\lambda^2 + \lambda^3$.

We verify that

$$-I_3 + 3A - 3A^2 + A^3 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}^2 + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}^3$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & -9 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 3 \\ 3 & -9 & 9 \\ 9 & -24 & 18 \end{bmatrix} + \begin{bmatrix} 1 & -3 & 3 \\ 3 & -8 & 6 \\ 6 & -15 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

9. Since $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -6 \\ -1 & \lambda - 2 \end{vmatrix} = \lambda^2 - 5\lambda$, it follows from the Cayley-Hamilton Theorem that

$A^2 - 5A = 0$. This yields $A^2 = 5A = 5\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$, $A^3 = 5A^2 = 5\begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} 75 & 150 \\ 25 & 50 \end{bmatrix}$,

$A^4 = 5A^3 = 5\begin{bmatrix} 75 & 150 \\ 25 & 50 \end{bmatrix} = \begin{bmatrix} 375 & 750 \\ 125 & 250 \end{bmatrix}$, and $A^5 = 5A^4 = 5\begin{bmatrix} 375 & 750 \\ 125 & 250 \end{bmatrix} = \begin{bmatrix} 1875 & 3750 \\ 625 & 1250 \end{bmatrix}$.

11. Method I

For λ to be an eigenvalue of A associated with a nonzero eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we must have $A\mathbf{x} = \lambda\mathbf{x}$

; i.e.

$$\begin{bmatrix} c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \vdots \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

There are two possibilities:

- If $\lambda \neq 0$ then $x_1 = x_2 = \cdots = x_n$. This implies $\lambda = c_1 + \cdots + c_n = \operatorname{tr}(A)$.
- If $\lambda = 0$ then $A\mathbf{x} = \lambda\mathbf{x}$ becomes a homogeneous system $A\mathbf{x} = \mathbf{0}$; its coefficient matrix A can be reduced to $\begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. The solution space has dimension of at least $n-1$ therefore $\lambda = 0$ is an eigenvalue whose geometric multiplicity is at least $n-1$.

We conclude that the only eigenvalues of A are 0 and $\operatorname{tr}(A)$.

Method II

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\ -c_1 & \lambda - c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\ -c_1 & -c_2 & \lambda - c_3 & \cdots & -c_{n-1} & -c_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_1 & -c_2 & -c_3 & \cdots & \lambda - c_{n-1} & -c_n \\ -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & \lambda - c_n \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\ -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ -\lambda & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda & 0 & 0 & \cdots & \lambda & 0 \\ -\lambda & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}$$

-1 times the first row was added to each of the remaining rows.

$$= \begin{vmatrix} \lambda - c_1 - c_2 - c_3 - \cdots - c_{n-1} - c_n & -c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}$$

Each of the columns from the second to the last was added to the first column.

$$= \left(\lambda - c_1 - c_2 - c_3 - \cdots - c_{n-1} - c_n\right)\lambda^{n-1}$$

We conclude that the only eigenvalues of A are 0 and $\operatorname{tr}(A) = c_1 + \cdots + c_n$.

13. By Theorem 5.1.2, all eigenvalues of $A^n = 0$ are 0. By Theorem 5.2.3, if A had any eigenvalue $\lambda \neq 0$ then λ^n would be an eigenvalue of A^n. We reached a contradiction, therefore all eigenvalues of A must be 0.

15. The three given eigenvectors can be used as columns of a matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ which diagonalizes A, i.e. $P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The latter equation is equivalent to $A = PDP^{-1}$. The matrix $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Therefore, $P^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. We conclude that a matrix A satisfying the given conditions is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

17. By Theorem 5.2.3, if A had any eigenvalue λ then λ^3 is an eigenvalue of A^3 corresponding to the same eigenvector. From $A^3 = A$ it follows that $\lambda^3 = \lambda$, so the only possible eigenvalues are -1, 0, and 1.

19. Let a and b denote the two unknown eigenvalues. We solve the system $a + b + 1 = 6$ and $a \cdot b \cdot 1 = 6$. Rewriting the first equation as $b = 5 - a$ and substituting into the second equation yields $a(5 - a) = 6$, therefore $(a-2)(a-3) = 0$. Either $a = 2$ (and $b = 3$) or $a = 3$ (and $b = 2$).

We conclude that the unknown eigenvalues are 2 and 3.