CHAPTER 4: GENERAL VECTOR SPACES

4.1 Real Vector Spaces

1. **(a)** $\mathbf{u}+\mathbf{v}=(-1+3,2+4)=(2,6)$; $k\mathbf{u}=(0,3\cdot 2)=(0,6)$

(b) For any $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$ is an ordered pair of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2)$ in V and for any scalar k, $k\mathbf{u}=(0,ku_2)$ is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^2.

(d) Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(0,k(u_2+v_2))=(0,ku_2)+(0,kv_2)$
$=k(u_1,u_2)+k(v_1,v_2)$ for all real k, u_1, u_2, v_1, and v_2;

Axiom 8: $(k+m)(u_1,u_2)=(0,(k+m)u_2)=(0,ku_2+mu_2)=(0,ku_2)+(0,mu_2)$

$=k(u_1,u_2)+m(u_1,u_2)$ for all real k, m, u_1, and u_2;

Axiom 9: $k(m(u_1,u_2))=k(0,mu_2)=(0,kmu_2)=(km)(u_1,u_2)$ for all real k, m, u_1, and u_2;

(e) Axiom 10 fails to hold: $1(u_1,u_2)=(0,u_2)$ does not generally equal (u_1,u_2).
Consequently, V is not a vector space.

3. Let V denote the set of all real numbers.

Axiom 1: $x+y$ is in V for all real x and y;

Axiom 2: $x+y=y+x$ for all real x and y;

Axiom 3: $x+(y+z)=(x+y)+z$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=0$, we have $0+x=x+0=x$ for all real x;

Axiom 5: for each $\mathbf{u}=x$, let $-\mathbf{u}=-x$; then $x+(-x)=(-x)+x=0$

Axiom 6: kx is in V for all real k and x;

Axiom 7: $k(x+y)=kx+ky$ for all real k, x, and y;

Axiom 8: $(k+m)x=kx+mx$ for all real k, m, and x;

Axiom 9: $k(mx) = (km)x$ for all real k, m, and x;

Axiom 10: $1x = x$ for all real x.

This is a vector space – all axioms hold.

5. Axiom 5 fails whenever $x \neq 0$ since it is then impossible to find (x', y') satisfying $x' \geq 0$ for which $(x, y) + (x', y') = (0, 0)$. (The zero vector from axiom 4 must be $\mathbf{0} = (0, 0)$.)

Axiom 6 fails whenever $k < 0$ and $x \neq 0$.

This is not a vector space.

7. Axiom 8 fails to hold:

$$(k+m)\mathbf{u} = \left((k+m)^2 x, (k+m)^2 y, (k+m)^2 z\right)$$

$$k\mathbf{u} + m\mathbf{u} = \left(k^2x, k^2y, k^2z\right) + \left(m^2x, m^2y, m^2z\right) = \left(\left(k^2+m^2\right)x, \left(k^2+m^2\right)y, \left(k^2+m^2\right)z\right)$$

therefore in general $(k+m)\mathbf{u} \neq k\mathbf{u} + m\mathbf{u}$.

This is not a vector space.

9. Let V be the set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (i.e., all diagonal 2×2 matrices)

Axiom 1: the sum of two diagonal 2×2 matrices is also a diagonal 2×2 matrix.

Axiom 2: follows from part (a) of Theorem 1.4.1.

Axiom 3: follows from part (b) of Theorem 1.4.1.

Axiom 4: taking $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; follows from part (a) of Theorem 1.4.2.

Axiom 5: let the negative of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$;

follows from part (c) of Theorem 1.4.2 and Axiom 2.

Axiom 6: the scalar multiple of a diagonal 2×2 matrix is also a diagonal 2×2 matrix.

Axiom 7: follows from part (h) of Theorem 1.4.1.

Axiom 8: follows from part (j) of Theorem 1.4.1.

Axiom 9: follows from part (l) of Theorem 1.4.1.

Axiom 10: $1\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for all real a and b.

This is a vector space – all axioms hold.

11. Let V denote the set of all pairs of real numbers of the form $(1,x)$.

Axiom 1: $(1,y)+(1,y')=(1,y+y')$ is in V for all real y and y';

Axiom 2: $(1,y)+(1,y')=(1,y+y')=(1,y'+y)=(1,y')+(1,y)$ for all real y and y';

Axiom 3: $(1,y)+((1,y')+(1,y''))=(1,y)+(1,y'+y'')=(1,y+y'+y'')=(1,y+y')+(1,y'')$

$=((1,y)+(1,y'))+(1,y'')$ for all real y, y', and y'';

Axiom 4: taking $\mathbf{0}=(1,0)$, we have $(1,0)+(1,y)=(1,y)$ and $(1,y)+(1,0)=(1,y)$ for all real y;

Axiom 5: for each $\mathbf{u}=(1,y)$, let $-\mathbf{u}=(1,-y)$;

then $(1,y)+(1,-y)=(1,0)$ and $(1,-y)+(1,y)=(1,0)$;

Axiom 6: $k(1,y)=(1,ky)$ is in V for all real k and y;

Axiom 7: $k((1,y)+(1,y'))=k(1,y+y')=(1,ky+ky')=(1,ky)+(1,ky')=k(1,y)+k(1,y')$

for all real k, y, and y';

Axiom 8: $(k+m)(1,y)=(1,(k+m)y)=(1,ky+my)=(1,ky)+(1,my)=k(1,y)+m(1,y)$

for all real k, m, and y;

Axiom 9: $k(m(1,y))=k(1,my)=(1,kmy)=(km)(1,y)$ for all real k, m, and y;

Axiom 10: $1(1,y)=(1,y)$ for all real y.

This is a vector space – all axioms hold.

13. Axiom 3: follows from part (b) of Theorem 1.4.1 since

$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\left(\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}+\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right)+\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}=(\mathbf{u}+\mathbf{v})+\mathbf{w}$$

Axiom 7: follows from part (h) of Theorem 1.4.1 since

$$k(\mathbf{u}+\mathbf{v})=k\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right)=k\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+k\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=k\mathbf{u}+k\mathbf{v}$$

Axiom 8: follows from part (j) of Theorem 1.4.1 since

$$(k+m)\mathbf{u}=(k+m)\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=k\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+m\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=k\mathbf{u}+m\mathbf{u}$$

Axiom 9: follows from part (l) of Theorem 1.4.1 since

$$k(m\mathbf{u})=k\left(m\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}\right)=(km)\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=(km)\mathbf{u}$$

15. Axiom 1: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$ is in V

Axiom 2: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)=(v_1+u_1,v_2+u_2)=(v_1,v_2)+(u_1,u_2)$

Axiom 3: $(u_1,u_2)+((v_1,v_2)+(w_1,w_2))=(u_1,u_2)+(v_1+w_1,v_2+w_2)$

$=(u_1+v_1+w_1,u_2+v_2+w_2)=(u_1+v_1,u_2+v_2)+(w_1,w_2)$

$=((u_1,u_2)+(v_1,v_2))+(w_1,w_2)$

Axiom 4: taking $\mathbf{0}=(0,0)$, we have $(0,0)+(u_1,u_2)=(u_1,u_2)$ and $(u_1,u_2)+(0,0)=(u_1,u_2)$

Axiom 5: for each $\mathbf{u}=(u_1,u_2)$, let $-\mathbf{u}=(-u_1,-u_2)$;
then $(u_1,u_2)+(-u_1,-u_2)=(0,0)$ and $(-u_1,-u_2)+(u_1,u_2)=(0,0)$

Axiom 6: $k(u_1,u_2)=(ku_1,0)$ is in V

Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(ku_1+kv_1,0)=(ku_1,0)+(kv_1,0)=k(u_1,u_2)+k(v_1,v_2)$

Axiom 8: $(k+m)(u_1,u_2)=((k+m)u_1,0)=(ku_1+mu_1,0)=(ku_1,0)+(mu_1,0)$
$=k(u_1,u_2)+m(u_1,u_2)$

Axiom 9: $k(m(u_1,u_2))=k(mu_1,0)=(kmu_1,0)=(km)(u_1,u_2)$

19. $\frac{1}{u}=u^{-1}$

21.

$\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$	Hypothesis
$(\mathbf{u}+\mathbf{w})+(-\mathbf{w})=(\mathbf{v}+\mathbf{w})+(-\mathbf{w})$	Add $-\mathbf{w}$ to both sides
$\mathbf{u}+[\mathbf{w}+(-\mathbf{w})]=\mathbf{v}+[\mathbf{w}+(-\mathbf{w})]$	Axiom 3
$\mathbf{u}+\mathbf{0}=\mathbf{v}+\mathbf{0}$	Axiom 5
$\mathbf{u}=\mathbf{v}$	Axiom 4

True-False Exercises

(a) True. This is a part of Definition 1.

(b) False. Example 1 discusses a vector space containing only one vector.

(c) False. By part (d) of Theorem 4.1.1, if $k\mathbf{u} = \mathbf{0}$ then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

(d) False. Axiom 6 fails to hold if $k < 0$. (Also, Axiom 4 fails to hold.)

(e) True. This follows from part (c) of Theorem 4.1.1.

(f) False. This function must have a value of zero at *every* point in $(-\infty,\infty)$.

4.2 Subspaces

1. **(a)** Let W be the set of all vectors of the form $(a,0,0)$, i.e. all vectors in R^3 with last two components equal to zero.
This set contains at least one vector, e.g. $(0,0,0)$.
Adding two vectors in W results in another vector in W: $(a,0,0)+(b,0,0)=(a+b,0,0)$ since the result has zeros as the last two components.
Likewise, a scalar multiple of a vector in W is also in W: $k(a,0,0)=(ka,0,0)$ - the result also has zeros as the last two components.
According to Theorem 4.2.1, W is a subspace of R^3.

(b) Let W be the set of all vectors of the form $(a,1,1)$, i.e. all vectors in R^3 with last two components equal to one. The set W is not closed under the operation of vector addition since $(a,1,1)+(b,1,1)=(a+b,2,2)$ does not have ones as its last two components thus it is outside W.
According to Theorem 4.2.1, W is not a subspace of R^3.

(c) Let W be the set of all vectors of the form (a,b,c), where $b = a + c$.
This set contains at least one vector, e.g. $(0,0,0)$. (The condition $b = a + c$ is satisfied when $a = b = c = 0$.)
Adding two vectors in W results in another vector in W
$(a,a+c,c)+(a',a'+c',c')=(a+a',\ a+c+a'+c',\ c+c')$ since in this result, the second component is the sum of the first and the third: $a+c+a'+c'=(a+a')+(c+c')$.
Likewise, a scalar multiple of a vector in W is also in W: $k(a,a+c,c)=(ka,k(a+c),kc)$ since in this result, the second component is once again the sum of the first and the third:
$k(a+c)=ka+kc$.
According to Theorem 4.2.1, W is a subspace of R^3.

3. **(a)** Let W be the set of all $n \times n$ diagonal matrices.
This set contains at least one matrix, e.g. the zero $n \times n$ matrix.
Adding two matrices in W results in another $n \times n$ diagonal matrix, i.e. a matrix in W :

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

Likewise, a scalar multiple of a matrix in W is also in W :

$$k\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & 0 & \cdots & 0 \\ 0 & ka_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ka_{nn} \end{bmatrix}$$

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(b) Let W be the set of all $n \times n$ matrices such whose determinant is zero. We shall show that W is not closed under the operation of matrix addition. For instance, consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ - both have determinant equal 0, therefore both matrices are in W. However, $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has nonzero determinant, thus it is outside W.

According to Theorem 4.2.1, W is not a subspace of M_{nn}.

(c) Let W be the set of all $n \times n$ matrices with zero trace.
This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.
Let us assume $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are both in W, i.e. $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$ and $\mathrm{tr}(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$.
Since $\mathrm{tr}(A+B) = (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{nn}+b_{nn})$
$= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = 0 + 0 = 0$, it follows that $A+B$ is in W.
A scalar multiple of the same matrix A with a scalar k has
$tr(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = 0$ therefore kA is in W as well.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(d) Let W be the set of all symmetric $n \times n$ matrices (i.e., $n \times n$ matrices such that $A^T = A$).
This set contains at least one matrix, e.g., I_n is in W.
Let us assume A and B are both in W, i.e. $A^T = A$ and $B^T = B$. By Theorem 1.4.8(b), their sum satisfies $(A+B)^T = A^T + B^T = A + B$ therefore W is closed under addition.

From Theorem 1.4.8(d), a scalar multiple of a symmetric matrix is also symmetric: $(kA)^T = kA^T = kA$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

5. **(a)** Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
This set contains at least one polynomial, $0 + 0x + 0x^2 + 0x^3 = 0$.
Adding two polynomials in W results in another polynomial in W:
$\left(0 + a_1x + a_2x^2 + a_3x^3\right) + \left(0 + b_1x + b_2x^2 + b_3x^3\right)$
$= 0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$.
Likewise, a scalar multiple of a polynomial in W is also in W:
$k\left(0 + a_1x + a_2x^2 + a_3x^3\right) = 0 + (ka_1)x + (ka_2)x^2 + (ka_3)x^3$.
According to Theorem 4.2.1, W is a subspace of P_3.

(b) Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$, i.e. all polynomials that can be expressed in the form $-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3$.
Adding two polynomials in W results in another polynomial in W
$\left(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3\right) + \left(-b_1 - b_2 - b_3 + b_1x + b_2x^2 + b_3x^3\right)$
$= (-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$
since we have $(-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0$.
Likewise, a scalar multiple of a polynomial in W is also in W
$k\left(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3\right) = -ka_1 - ka_2 - ka_3 + ka_1x + ka_2x^2 + ka_3x^3$
since it meets the condition $(-ka_1 - ka_2 - ka_3) + (ka_1) + (ka_2) + (ka_3) = 0$.
According to Theorem 4.2.1, W is a subspace of P_3.

7. **(a)** Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 0$.
This set contains at least one function, e.g., the constant function $f(x) = 0$.
Assume we have two functions f and g in W, i.e., $f(0) = g(0) = 0$. Their sum $f + g$ is also a function in $F(-\infty,\infty)$ and satisfies $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$ therefore W is closed under addition.
A scalar multiple of a function f in W, kf, is also a function in $F(-\infty,\infty)$ for which $(kf)(0) = k(f(0)) = 0$ making W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of $F(-\infty,\infty)$.

(b) Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 1$.
We will show that W is not closed under addition. For instance, let $f(x) = 1$ and $g(x) = \cos x$ be

two functions in W. Their sum, $f+g$, is not in W since $(f+g)(0)=f(0)+g(0)=1+1=2$. We conclude that W is not a subspace of $F(-\infty,\infty)$.

9. **(a)** Let W be the set of all sequences in R^∞ of the form $(v,0,v,0,v,0,\ldots)$.
This set contains at least one sequence, e.g. $(0,0,0,\ldots)$.
Adding two sequences in W results in another sequence in W:
$(v,0,v,0,v,0,\ldots)+(w,0,w,0,w,0,\ldots)=(v+w,0,v+w,0,v+w,0,\ldots)$.
Likewise, a scalar multiple of a vector in W is also in W: $k(v,0,v,0,v,0,\ldots)=(kv,0,kv,0,kv,0,\ldots)$.
According to Theorem 4.2.1, W is a subspace of R^∞.

(b) Let W be the set of all sequences in R^∞ of the form $(v,1,v,1,v,1,\ldots)$.
This set is not closed under addition since
$(v,1,v,1,v,1,\ldots)+(w,1,w,1,w,1,\ldots)=(v+w,2,v+w,2,v+w,2,\ldots)$ is not in W.
We conclude that W is not a subspace of R^∞.

11. **(a)** Let W be the set of all matrices of form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$. This set contains at least one matrix, e.g. the zero matrix. Adding two matrices in W results in another matrix in W:

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}+\begin{bmatrix} a' & 0 \\ b' & 0 \end{bmatrix}=\begin{bmatrix} a+a' & 0 \\ b+b' & 0 \end{bmatrix}.$$

Likewise, a scalar multiple of a matrix in W is also in W:

$k\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}=\begin{bmatrix} ka & 0 \\ kb & 0 \end{bmatrix}$. According to Theorem 4.2.1, W is a subspace of M_{22}.

(b) Let W be the set of all matrices of form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{22}.

(c) Let W be the set of all 2×2 matrices A such that $A\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. This set is not closed under addition since if A and B are matrices in W then

$(A+B)\begin{bmatrix} 1 \\ -1 \end{bmatrix}=A\begin{bmatrix} 1 \\ -1 \end{bmatrix}+B\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 \\ 0 \end{bmatrix}+\begin{bmatrix} 2 \\ 0 \end{bmatrix}=\begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Consequently, the matrix $A+B$ is not contained in W. According to Theorem 4.2.1, W is not a subspace of M_{22}.

13. **(a)** Let W be the set of all vectors in R^4 of form (a,a^2,a^3,a^4). This set is not closed under addition. For example, the vector $(1,1,1,1)$ is in W but $(1,1,1,1)+(1,1,1,1)=(2,2,2,2)$ is not. According to Theorem 4.2.1, W is not a subspace of R^4.

(b) Let W be the set of all vectors in R^4 of form $(a,0,b,0)$. This set contains at least one vector, e.g. the zero vector. Adding two vectors in W results in another vector in W:

$(a,0,b,0)+(a',0,b',0)=(a+a',0,b+b',0).$

Likewise, a scalar multiple of a vector in W is also in W: $k(a,0,b,0)=(ka,0,kb,0)$. According to Theorem 4.2.1, W is a subspace of R^4.

15. **(a)** Let W be the set of all polynomials of degree less than or equal to six. This set is not empty. For example, $p(x)=x$ is contained in W. Adding two polynomials in W results in another polynomial in W because the sum of two polynomials of degree at most six is another polynomial of degree at least six. Likewise, a scalar multiple of a polynomial of degree at most six is another polynomial of degree at most six. According to Theorem 4.2.1, W is a subspace of P_∞.

(b) Let W be the set of all polynomials of degree equal to six. This set is not closed under addition. For example, $p(x)=x^6+x$ and $q(x)=-x^6$ are both polynomials in W but

$p(x)+q(x)=x^6+x-x^6=x$ has degree 1 so it is not contained in W. According to Theorem 4.2.1, W is not a subspace of P_∞.

(c) Let W be the set of all polynomials of degree greater than or equal to six. This set is not closed under addition. For example, $p(x)=x^6+x$ and $q(x)=-x^6$ are both polynomials in W but

$p(x)+q(x)=x^6+x-x^6=x$ has degree 1 so it is not contained in W. According to Theorem 4.2.1, W is not a subspace of P_∞.

17. **(a)** Let W be the set of all sequences of the form $(v_1,v_2,v_3,\dots)$ such that $\lim_{n\to\infty} v_n=0$. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\cdots)$). Adding two sequences $(v_1,v_2,v_3,\dots)$ and $(w_1,w_2,w_3,\dots)$ in W results in the sequence $(v_1+w_1,v_2+w_2,v_3+w_3,\dots)$ which is also in W since $\lim_{n\to\infty} v_n+\lim_{n\to\infty} w_n=\lim_{n\to\infty}(v_n+w_n)=0$. Likewise, a scalar multiple of a sequence $(v_1,v_2,v_3,\dots)$ in W is also in W because $k(\lim_{n\to\infty} v_n)=\lim_{n\to\infty} kv_n=0$. (These results both follow because sums and constant multiples of convergent sequences are also convergent.). According to Theorem 4.2.1, W is a subspace of R^∞.

(b) Let W be the set of all sequences of the form $(v_1,v_2,v_3,\dots)$ such that $\lim_{n\to\infty} v_n$ exists and is finite. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\cdots)$). Adding two sequences $(v_1,v_2,v_3,\dots)$ and $(w_1,w_2,w_3,\dots)$ in W results in the sequence $(v_1+w_1,v_2+w_2,v_3+w_3,\dots)$ which is also in W. This follows because both $\lim\limits_{n\to\infty} v_n$ and $\lim\limits_{n\to\infty} w_n$ exist and are finite so that

$\lim\limits_{n\to\infty} v_n+\lim\limits_{n\to\infty} w_n=\lim\limits_{n\to\infty}(v_n+w_n)$ also exists and is finite. Likewise, a scalar multiple of a sequence

$(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} kv_n$. According to Theorem 4.2.1, W is a subspace of R^∞.

(c) Let W be the set of all sequences of the form $(v_1, v_2, v_3, \ldots)$ such that $\sum_{n=1}^{\infty} v_n = 0$. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\cdots)$). Adding two sequences $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ in W results in the sequence $(v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots)$ which is also in W. This follows because both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge to zero so

that $\sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} v_n + w_n = 0$. Likewise, a scalar multiple of a sequence $(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} kv_n = 0$. According to Theorem 4.2.1, W is a subspace of R^∞.

(d) Let W be the set of all sequences of the form $(v_1, v_2, v_3, \ldots)$ such that $\sum_{n=1}^{\infty} v_n$ converges. This set is nonempty (e.g. it contains the zero sequence $(0,0,0,\cdots)$). Adding two sequences $(v_1, v_2, v_3, \ldots)$ and $(w_1, w_2, w_3, \ldots)$ in W results in the sequence $(v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots)$ which is also in W. This follows because both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge so $\sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} (v_n + w_n)$ also converges. Likewise, a scalar multiple of a sequence $(v_1, v_2, v_3, \ldots)$ in W is also in W because $k\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} kv_n$. According to Theorem 4.2.1, W is a subspace of R^∞.

19. **(a)** The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ therefore the solution are $x = -\frac{1}{2}t$, $y = -\frac{3}{2}t$, $z = t$. These are parametric equations of a line through the origin.

(b) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ therefore the only solution is $x = y = z = 0$ - the origin.

(c) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which corresponds to an equation of a plane through the origin $x - 3y + z = 0$.

(d) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the solutions are $x = -3t$, $y = -2t$, $z = t$. These are parametric equations of a line through the origin.

21. Let W denote the set of all continuous functions $f = f(x)$ on $[a,b]$ such that $\int_a^b f(x)\,dx = 0$.

This set contains at least one function $f(x) \equiv 0$.

Let us assume $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are functions in W. From calculus,

$\int_a^b f(x) + g(x) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = 0$ and $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx = 0$ therefore both $\mathbf{f}+\mathbf{g}$ and $k\mathbf{f}$ are in W for any scalar k. According to Theorem 4.2.1, W is a subspace of $C[a,b]$.

23. Since $T_A : R^3 \to R^m$, it follows from Theorem 4.2.5 that the kernel of T_A must be a subspace of R^3. Hence, according to Table 1 the kernel can be one of the following four geometric objects:

- the origin,
- a line through the origin,
- a plane through the origin,
- R^3.

25. Let W be the set of all functions of the form $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ - W is a subset of $C^\infty(-\infty,\infty)$.

This set contains at least one function $x(t) \equiv 0$.

A sum of two functions in W is also in W:

$(c_1 \cos \omega t + c_2 \sin \omega t) + (d_1 \cos \omega t + d_2 \sin \omega t) = (c_1 + d_1)\cos \omega t + (c_2 + d_2)\sin \omega t$.

A scalar product of a function in W by any scalar k is also a function in W:

$k(c_1 \cos \omega t + c_2 \sin \omega t) = (kc_1)\cos \omega t + (kc_2)\sin \omega t$.

According to Theorem 4.2.1, W is a subspace of $C^\infty(-\infty,\infty)$.

True-False Exercises

(a) True. This follows from Definition 1.

(b) True.

(c) False. The set of all nonnegative real numbers is a subset of the vector space R containing 0, but it is not closed under scalar multiplication.

(d) False. By Theorem 4.2.4, the kernel of $T_A : R^n \to R^m$ is a subspace of R^n.

(e) False. The solution set of a nonhomogeneous system is not closed under addition: $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b}$ do not imply $A(\mathbf{x}+\mathbf{y}) = \mathbf{b}$.

(f) True. This follows from Theorem 4.2.2.

(g) False. Consider $W_1 = \text{span}\{(1,0)\}$ and $W_2 = \text{span}\{(0,1)\}$. The union of these sets is not closed under vector addition, e.g. $(1,0) + (0,1) = (1,1)$ is outside the union.

(h) True. This set contains at least one matrix (e.g., I_n). A sum of two upper triangular matrices is also upper triangular, therefore the set is closed under addition. A scalar multiple of an upper triangular matrix is also upper triangular, hence the set is closed under scalar multiplication.

4.3 Spanning Sets

1. **(a)** For $(2,2,2)$ to be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, there must exist scalars a and b such that

$$a(0,-2,2)+b(1,3,-1)=(2,2,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 0a & + & 1b & = & 2 \\ -2a & + & 3b & = & 2 \\ 2a & - & 1b & = & 2 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The linear system is consistent, therefore $(2,2,2)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) For $(0,4,5)$ to be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, there must exist scalars a and b such that

$$a(0,-2,2)+b(1,3,-1)=(0,4,5)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 0a & + & 1b & = & 0 \\ -2a & + & 3b & = & 4 \\ 2a & - & 1b & = & 5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0=1$ which is contradictory. We conclude that $(0,4,5)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) By inspection, the zero vector $(0,0,0)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ since

$$0(0,-2,2)+0(1,3,-1)=(0,0,0)$$

3. **(a)** For $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & 6 \\ 0a & - & 1b & + & 2c & = & -8 \\ -2a & + & 2b & + & 1c & = & -1 \\ -2a & + & 3b & + & 4c & = & -8 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The linear system is consistent, therefore $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of A, B, and C.

(b) The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a linear combination of A, B, and C since $0A + 0B + 0C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) For $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & -1 \\ 0a & - & 1b & + & 2c & = & 5 \\ -2a & + & 2b & + & 1c & = & 7 \\ -2a & + & 3b & + & 4c & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0 = 1$ which is contradictory. We conclude that $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ is not a linear combination of A, B, and C.

5. **(a)** We need to solve the equation $k_1\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + k_3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_4\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ to express the vector $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ as the desired linear combination. We can rewrite this as $\begin{bmatrix} k_1 + 2k_4 & -k_1 + k_2 + k_3 \\ k_4 & 2k_1 + k_2 - k_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ -1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 & 4 \end{bmatrix}$. This matrix has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & -13 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ hence $-3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + 12\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - 13\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(b) Following part (a) we obtain a linear system whose augmented matrix is $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ -1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 & 2 \end{bmatrix}$. This matrix has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

hence $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

7. **(a)** The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2, 2, 2) + k_2(0, 0, 3) + k_3(0, 1, 1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 2k_1 & + & 0k_2 & + & 0k_3 & = & b_1 \\ 2k_1 & + & 0k_2 & + & 1k_3 & = & b_2 \\ 2k_1 & + & 3k_2 & + & 1k_3 & = & b_3 \end{array}$$

By inspection, regardless of the right hand side values b_1, b_2, b_3, the first equation can be solved for k_1, then the second equation can be used to obtain k_3, and the third would yield k_2.
We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span R^3.

(b) The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2, -1, 3) + k_2(4, 1, 2) + k_3(8, -1, 8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 2k_1 & + & 4k_2 & + & 8k_3 & = & b_1 \\ -1k_1 & + & 1k_2 & - & 1k_3 & = & b_2 \\ 3k_1 & + & 2k_2 & + & 8k_3 & = & b_3 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0$, therefore by Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

9. The given polynomials span P_2 if an arbitrary polynomial in P_2, $\mathbf{p} = a_0 + a_1x + a_2x^2$ can be expressed as a linear combination

$$a_0 + a_1x + a_2x^2 = k_1(1 - x + 2x^2) + k_2(3 + x) + k_3(5 - x + 4x^2) + k_4(-2 - 2x + 2x^2)$$

Grouping the terms according to the powers of x yields

$$a_0 + a_1x + a_2x^2 = (k_1 + 3k_2 + 5k_3 - 2k_4) + (-k_1 + k_2 - k_3 - 2k_4)x + (2k_1 + 4k_3 + 2k_4)x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcrcl} 1k_1 & + & 3k_2 & + & 5k_3 & - & 2k_4 & = & a_0 \\ -1k_1 & + & 1k_2 & - & 1k_3 & - & 2k_4 & = & a_1 \\ 2k_1 & + & 0k_2 & + & 4k_3 & + & 2k_4 & = & a_2 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2 & 1 & \frac{1}{4}a_0 - \frac{3}{4}a_1 \\ 0 & 1 & 1 & -1 & \frac{1}{4}a_0 + \frac{1}{4}a_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \end{bmatrix}$ therefore the system has no solution if $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$.

Since polynomials $\mathbf{p} = a_0 + a_1x + a_2x^2$ for which $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$ cannot be expressed as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$, we conclude that the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ do not span P_2.

11. **(a)** The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination $k_1\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as $\begin{bmatrix} k_1 + k_2 & k_2 + k_3 \\ k_1 + k_4 & k_3 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 1 & 0 & 0 & 1 & c \\ 0 & 0 & 1 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right) = 0$ which means the system is not consistent. We conclude that the given matrices do not span M_{22}.

(b) The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination $k_1\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as $\begin{bmatrix} k_1 + k_3 + k_4 & -k_1 + k_2 + k_3 \\ k_3 & k_1 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 0 & 1 & 1 & a \\ -1 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 1 & 0 & 0 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right) = 1$ which means the system is consistent. We conclude that the given matrices span M_{22}.

(c) The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination $k_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as $\begin{bmatrix} k_1 + k_2 + k_3 + k_4 & k_2 + k_3 + k_4 \\ k_3 + k_4 & k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 1$ which means the system is consistent. We conclude that the given matrices span M_{22}.

13. **(a)** The vector $\mathbf{u} = (1,1,1)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ if it is a linear combination of the columns of $A = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}$: $T_A(\mathbf{e}_1) = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $T_A(\mathbf{e}_2) = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$. So there must exist scalars a and b such that $a(0,1,1) + b(2,-2,0) = (1,1,1)$.

Equating corresponding components on both sides leads to the linear system $\begin{array}{ccccc} 0 & + & 2b & = & 1 \\ a & - & 2b & = & 1 \\ a & + & 0 & = & 1 \end{array}$ which is inconsistent since subtracting the last equation from the second yields $-2b = 0$ while the first equation is $2b = 1$. We conclude that the vector $\mathbf{u} = (1,1,1)$ is not in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

(b) The vector $\mathbf{u} = (1,1,1)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$ if it is a linear combination of the columns of $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$: $T_A(\mathbf{e}_1) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $T_A(\mathbf{e}_2) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. . So there must exist scalars a and b such that $a(0,1,2) + b(2,1,0) = (1,1,1)$. Observe that $\frac{1}{2}(0,1,2) + \frac{1}{2}(2,1,0) = (1,1,1)$.
We conclude that the vector $\mathbf{u} = (1,1,1)$ is not in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

15. **(a)** The solution space W to the homogenous system $A\mathbf{x} = 0$ where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ is obtained from the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution in vector form is $(x,y,z,w) = (s,t,-s,-t) = s(1,0,-1,0) + t(0,1,0,-1)$ therefore the solution space is spanned by the vectors $\mathbf{v}_1 = (1,0,-1,0)$ and $\mathbf{v}_2 = (0,1,0,-1)$. We conclude that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (0,1,0,-1)$ span the solution space W.

(b) From part (a) and Theorem 4.3.2 we need to show that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (1,1,-1,-1)$ are contained in the span of the vectors $\mathbf{v}_1 = (1,0,-1,0)$ and $\mathbf{v}_2 = (0,1,0,-1)$. Observe that $\mathbf{u} = \mathbf{v}_1$ and

$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. We conclude that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (1,1,-1,-1)$ span the solution space W.

17. **(a)** The vectors $T_A(1,2) = (-1,4)$ and $T_A(-1,1) = (-2,2)$ span R^2 if an arbitrary vector $\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(-1, 4) + k_2(-2, 2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} -1k_1 &- 2k_2 = b_1 \\ 4k_1 &+ 2k_2 = b_2 \end{aligned}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -2 \\ 4 & 2 \end{vmatrix} = 6 \neq 0$, therefore by Theorem 2.3.8, the system is consistent for all right hand side vectors $\mathbf{b}$.

We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ span R^2.

(b) The vectors $T_A(1,2) = (-1,2)$ and $T_A(-1,1) = (-2,4)$ span R^2 if an arbitrary vector $\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(-1, 2) + k_2(-2, 4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} -1k_1 &- 2k_2 = b_1 \\ 2k_1 &+ 4k_2 = b_2 \end{aligned}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} = 0$, therefore by Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$.

We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ do not span R^2.

19. Using Theorem 4.3.2, we need to show that the each of the polynomials $\mathbf{q}_1 = 2x$ and $\mathbf{q}_2 = 1 + x^2$ is in the span of the polynomials $\mathbf{p}_1 = 1 + x^2$ and $\mathbf{p}_2 = 1 + x + x^2$. Clearly, $\mathbf{q}_2 = \mathbf{p}_1$.

Observe that $2x = (-2) \quad (1 + x^2) + 2(1 + x + x^2)$ so that $\mathbf{q}_1 = (-2)\mathbf{p}_1 + 2\mathbf{p}_2$.

21. For the vector $(3,5)$ to be expressed as $\mathbf{v} + \mathbf{w}$ where $\mathbf{v}$ is in the subspace spanned by $(3,1)$ and $\mathbf{w}$ is in the subspace spanned by $(2,1)$, we must produce scalars a and b such that $a(3,1) + b(2,1) = (3,5)$.

Equating corresponding components yields a linear system with augmented matrix $\begin{bmatrix} 3 & 2 & 3 \\ 1 & 1 & 5 \end{bmatrix}$ which has reduced row echelon form $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 12 \end{bmatrix}$.

Therefore $\mathbf{v} = -7(3,1) = (-21,-7)$ and $\mathbf{w} = 12(2,1) = (24,12)$.

True False Exercises

(a) True.

(b) False. The span of the zero vector is just the zero vector.

(c) False. For example the vectors $(1,1,1)$ and $(2,2,2)$ span a line.

(d) True.

(e) True. This follows from part (a) of Theorem 4.2.1.

(f) False. For any nonzero vector $\mathbf{v}$ in a vector space V, both $\{\mathbf{v}\}$ and $\{2\mathbf{v}\}$ span the same subspace of V.

(g) False. The constant polynomial $p(x) = 1$ cannot be represented as a linear combination of these, since at $x = 1$ all three are zero, whereas $p(1) = 1$.

4.4 Linear Independence

1. **(a)** Since $\mathbf{u}_2 = -5\mathbf{u}_1$, linear dependence follows from Definition 1.

(b) A set of 3 vectors in R^2 must be linearly dependent by Theorem 4.4.3.

(c) Since $\mathbf{p}_2 = 2\mathbf{p}_1$, linear dependence follows from Definition 1.

(d) Since $A = (-1)B$, linear dependence follows from Definition 1.

3. **(a)** The vector equation $a(3,8,7,-3) + b(1,5,3,-1) + c(2,-1,2,6) + d(4,2,6,4) = (0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{aligned} 3a + 1b + 2c + 4d &= 0 \\ 8a + 5b - 1c + 2d &= 0 \\ 7a + 3b + 2c + 6d &= 0 \\ -3a - 1b + 6c + 4d &= 0 \end{aligned}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a = -t$, $b = t$, $c = -t$, $d = t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) The vector equation $a(3,0,-3,6)+b(0,2,3,1)+c(0,-2,-2,0)+d(-2,1,2,1)=(0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcrcr} 3a & + & 0b & + & 0c & - & 2d & = & 0 \\ 0a & + & 2b & - & 2c & + & 1d & = & 0 \\ -3a & + & 3b & - & 2c & + & 2d & = & 0 \\ 6a & + & 1b & + & 0c & + & 1d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=d=0$. We conclude that the given set of vectors is linearly independent.

5. **(a)** The matrix equation $a\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}+b\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}+c\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1a & + & 1b & + & 0c & = & 0 \\ 0a & + & 2b & + & 1c & = & 0 \\ 1a & + & 2b & + & 2c & = & 0 \\ 2a & + & 1b & + & 1c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given matrices are linearly independent.

(b) By inspection, the matrix equation $a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has only the trivial solution $a=b=c=0$. We conclude that the given matrices are linearly independent.

7. Three vectors in R^3 lie in a plane if and only if they are linearly dependent when they have their initial points at the origin. (See the discussion following Example 6.)

(a) The vector equation $a(2,-2,0)+b(6,1,4)+c(2,0,-4)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} 2a & + & 6b & + & 2c & = & 0 \\ -2a & + & 1b & + & 0c & = & 0 \\ 0a & + & 4b & - & 4c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given vectors are linearly independent, hence they do not lie in a plane.

(b) The vector equation $a(-6,7,2) + b(3,2,4) + c(4,-1,2) = (0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} -6a & + & 3b & + & 4c & = & 0 \\ 7a & + & 2b & - & 1c & = & 0 \\ 2a & + & 4b & + & 2c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a = \frac{1}{3}t$, $b = -\frac{2}{3}t$, $c = t$.

Since the system has nontrivial solutions, the given vectors are linearly dependent, hence they lie in a plane.

9. **(a)** The vector equation $a(0,3,1,-1) + b(6,0,5,1) + c(4,-7,1,3) = (0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} 0a & + & 6b & + & 4c & = & 0 \\ 3a & + & 0b & - & 7c & = & 0 \\ 1a & + & 5b & + & 1c & = & 0 \\ -1a & + & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a = \frac{7}{3}t$, $b = -\frac{2}{3}t$, $c = t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) From part (a), we have $\frac{7}{3}t\mathbf{v}_1 - \frac{2}{3}t\mathbf{v}_2 + t\mathbf{v}_3 = 0$.

Letting $t = \frac{3}{7}$, we obtain $\mathbf{v}_1 = \frac{2}{7}\mathbf{v}_2 - \frac{3}{7}\mathbf{v}_3$.

Letting $t=-\frac{3}{2}$, we obtain $\mathbf{v}_2=\frac{7}{2}\mathbf{v}_1+\frac{3}{2}\mathbf{v}_3$.

Letting $t=1$, we obtain $\mathbf{v}_3=-\frac{7}{3}\mathbf{v}_1+\frac{2}{3}\mathbf{v}_2$.

11. By inspection, when $\lambda=-\frac{1}{2}$, the vectors become linearly dependent (since they all become equal).

We proceed to find the remaining values of λ.

The vector equation $a\left(\lambda,-\frac{1}{2},-\frac{1}{2}\right)+b\left(-\frac{1}{2},\lambda,-\frac{1}{2}\right)+c\left(-\frac{1}{2},-\frac{1}{2},\lambda\right)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} \lambda a & - & \frac{1}{2}b & - & \frac{1}{2}c & = & 0 \\ -\frac{1}{2}a & + & \lambda b & - & \frac{1}{2}c & = & 0 \\ -\frac{1}{2}a & - & \frac{1}{2}b & + & \lambda c & = & 0 \end{array}$$

The determinant of the coefficient matrix is $\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix}=\lambda^3-\frac{3}{4}\lambda-\frac{1}{4}$. This determinant equals zero for all λ values for which the vectors are linearly dependent. Since we already know that $\lambda=-\frac{1}{2}$ is one of those values, we can divide $\lambda+\frac{1}{2}$ into $\lambda^3-\frac{3}{4}\lambda-\frac{1}{4}$ to obtain

$\lambda^3-\frac{3}{4}\lambda-\frac{1}{4}=\left(\lambda+\frac{1}{2}\right)\left(\lambda^2-\frac{1}{2}\lambda-\frac{1}{2}\right)=\left(\lambda+\frac{1}{2}\right)\left(\lambda+\frac{1}{2}\right)(\lambda-1)$.

We conclude that the vectors form a linearly dependent set for $\lambda=-\frac{1}{2}$ and for $\lambda=1$.

13. **(a)** We calculate $T_A(1,2)=(-1,4)$ and $T_A(-1,1)=(-2,2)$. The vector equation

$$k_1(-1,4)+k_2(-2,2)=(0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & 0 \\ 4k_1 & + & 2k_2 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -1 \\ 4 & 1 \end{vmatrix}=6\neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ form a linearly independent set.

(b) We calculate $T_A(1,2)=(-1,2)$ and $T_A(-1,1)=(-2,4)$. Since $(-2,4)=2(-1,2)$, it follows by Definition 1 that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ form a linearly dependent set.

15. Three vectors in R^3 lie in a plane if and only if they are linearly dependent when they have their initial points at the origin. (See the discussion following Example 6.)

(a) After the three vectors are moved so that their initial points are at the origin, the resulting vectors do not lie on the same plane. Hence these vectors are linearly independent.

(b) After the three vectors are moved so that their initial points are at the origin, the resulting vectors lie on the same plane. Hence these vectors are linearly dependent.

17. The Wronskian is $W(x)=\begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix}=-x\sin x-\cos x$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$ (e.g., $W(0)=-1\neq 0$), the functions x and $\cos x$ are linearly independent.

19. **(a)** The Wronskian is $W(x)=\begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}=e^x$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$ (e.g., $W(0)=1\neq 0$), the functions 1, x and e^x are linearly independent.

(b) The Wronskian is $W(x)=\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}=2$. Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, the functions 1, x and x^2 are linearly independent.

21.

$$W(x)=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix} \longleftarrow \text{The Wronskian}$$

$$=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ 0 & 0 & -2\sin x \end{vmatrix} \longleftarrow \text{The first row was added to the third.}$$

$$=-2\sin x\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \longleftarrow \text{Cofactor expansion along the third row}$$

$$=-2\sin x\left(-\sin^2 x-\cos^2 x\right)$$

$$=\left(-2\sin x\right)\left(-1\right)=2\sin x$$

Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are linearly independent.

True-False Exercises

(a) False. By part (b) of Theorem 4.4.2, a set containing a single *nonzero* vector is linearly independent.

(b) True. This follows directly from Definition 1.

(c) False. For instance $\{(1,1),(2,2)\}$ is a linearly dependent set that does not contain $(0,0)$.

(d) True. If $a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_3=\mathbf{0}$ has only one solution $a=b=c=0$ then $a(k\mathbf{v}_1)+b(k\mathbf{v}_2)+c(k\mathbf{v}_3)=k(a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_3)$ can only equal $\mathbf{0}$ when $a=b=c=0$ as well.

(e) True. Since the vectors must be nonzero, $\{\mathbf{v}_1\}$ must be linearly independent.

Let us begin adding vectors to the set until the set $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ becomes linearly dependent, therefore, by construction, $\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}$ is linearly independent. The equation $c_1\mathbf{v}_1+\cdots+c_{k-1}\mathbf{v}_{k-1}+c_k\mathbf{v}_k=\mathbf{0}$ must have a solution with $c_k\neq 0$, therefore $\mathbf{v}_k=-\frac{c_1}{c_k}\mathbf{v}_1-\cdots-\frac{c_{k-1}}{c_k}\mathbf{v}_{k-1}$. Let us assume there exists another representation $\mathbf{v}_k=d_1\mathbf{v}_1+\cdots+d_{k-1}\mathbf{v}_{k-1}$. Subtracting both sides yields $\mathbf{0}=\left(d_1+\frac{c_1}{c_k}\right)\mathbf{v}_1+\cdots+\left(d_{k-1}+\frac{c_{k-1}}{c_k}\right)\mathbf{v}_{k-1}$. By linear independence of $\{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\}$, we must have $d_1=-\frac{c_1}{c_k},\ \ldots,\ d_{k-1}=-\frac{c_{k-1}}{c_k}$, which shows that $\mathbf{v}_k$ is a *unique* linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}$.

(f) False. The set $\left\{\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ is linearly dependent since $\begin{bmatrix}1&1\\0&0\end{bmatrix}=(-1)\begin{bmatrix}0&0\\1&1\end{bmatrix}+\begin{bmatrix}1&0\\1&0\end{bmatrix}+\begin{bmatrix}0&1\\0&1\end{bmatrix}$.

(g) True. Requiring that for all x values $a(x-1)(x+2)+bx(x+2)+cx(x-1)=0$ holds true implies that the equality must be true for any specific x value. Setting $x=0$ yields $a=0$. Likewise, $x=1$ implies $b=0$, and $x=-2$ implies $c=0$. Since $a=b=c=0$ is required, we conclude that the three given polynomials are linearly independent.

(h) False. The functions f_1 and f_2 are linearly dependent if there exist scalars k_1 and k_2, <u>not both equal 0</u>, such that $k_1f_1(x)+k_2f_2(x)=0$ for <u>all</u> real numbers x.

4.5 Coordinates and Basis

1. Vectors $(2,1)$ and $(3,0)$ are linearly independent if the vector equation

$$c_1(2,1)+c_2(3,0)=(0,0)$$

has only the trivial solution. For these vectors to span R^2, it must be possible to express every vector $\mathbf{b}=(b_1,b_2)$ in R^2 as

$$c_1(2,1)+c_2(3,0)=(b_1,b_2)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & 0 \\ c_1 & & & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & b_1 \\ c_1 & & & = & b_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1 and b_2. Therefore the vectors $(2,1)$ and $(3,0)$ are linearly independent and span R^2 so that they form a basis for R^2.

3. Polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent if the equation

$$c_1(x^2+1)+c_2(x^2-1)+c_3(2x-1)=0$$

has only the trivial solution. For these polynomials to span P_2, it must be possible to express every polynomial $a_0+a_1x+a_2x^2$ as

$$c_1(x^2+1)+c_2(x^2-1)+c_3(2x-1)=a_0+a_1x+a_2x^2$$

Grouping the terms on the left hand side of both equations as $(c_1-c_2-c_3)+(2c_3)x+(c_1+c_2)x^2$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & 0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & a_0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & a_1 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = -4 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent and span P_2 so that they form a basis for P_2.

5. Matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}+c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equating corresponding entries on both sides yields linear systems

$$\begin{array}{rcrcrcrcl} 3c_1 & + & 0c_2 & + & 0c_3 & + & 1c_4 & = & 0 \\ 6c_1 & - & 1c_2 & - & 8c_3 & + & 0c_4 & = & 0 \\ 3c_1 & - & 1c_2 & - & 12c_3 & - & 1c_4 & = & 0 \\ -6c_1 & + & 0c_2 & - & 4c_3 & + & 2c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 3c_1 & + & 0c_2 & + & 0c_3 & + & 1c_4 & = & a_{11} \\ 6c_1 & - & 1c_2 & - & 8c_3 & + & 0c_4 & = & a_{12} \\ 3c_1 & - & 1c_2 & - & 12c_3 & - & 1c_4 & = & a_{21} \\ -6c_1 & + & 0c_2 & - & 4c_3 & + & 2c_4 & = & a_{22} \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{vmatrix} = 48 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_{11}, a_{12}, a_{21}, and a_{22}. Therefore the matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent and span M_{22} so that they form a basis for M_{22}.

7. **(a)** Vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly independent if the vector equation

$$c_1(2,-3,1) + c_2(4,1,1) + c_3(0,-7,1) = (0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcl} 2c_1 & + & 4c_2 & + & 0c_3 & = & 0 \\ -3c_1 & + & 1c_2 & - & 7c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly dependent, they do not form a basis for R^3.

(b) Vectors $(1,6,4)$, $(2,4,-1)$, and $(-1,2,5)$ are linearly independent if the vector equation

$$c_1(1,6,4)+c_2(2,4,-1)+c_3(-1,2,5)=(0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcl} 1c_1 & + & 2c_2 & - & 1c_3 & = & 0 \\ 6c_1 & + & 4c_2 & + & 2c_3 & = & 0 \\ 4c_1 & - & 1c_2 & + & 5c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix}=0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(1,6,4)$, $(2,4,-1)$, and $(-1,2,5)$ are linearly dependent, they do not form a basis for R^3.

9. Matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}+c_2\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}+c_3\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}+c_4\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. Equating corresponding entries on both sides yields a linear system

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 2c_2 & + & 1c_3 & + & 0c_4 & = & 0 \\ 0c_1 & - & 2c_2 & - & 1c_3 & - & 1c_4 & = & 0 \\ 1c_1 & + & 3c_2 & + & 1c_3 & + & 1c_4 & = & 0 \\ 1c_1 & + & 2c_2 & + & 0c_3 & + & 1c_4 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix}=0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly dependent, we conclude that they do not form a basis for M_{22}.

11. **(a)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,1)=c_1(2,-4)+c_2(3,8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 2c_1 &+ 3c_2 = 1 \\ -4c_1 &+ 8c_2 = 1 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{5}{28}$, $c_2 = \frac{3}{14}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a,\ b) = c_1(1,1) + c_2(0,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 1c_1 &+ 0c_2 = a \\ 1c_1 &+ 2c_2 = b \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = a$, $c_2 = \frac{b-a}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(a, \frac{b-a}{2}\right)$.

13. **(a)** Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(2,-1,3) = c_1(1,0,0) + c_2(2,2,0) + c_3(3,3,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 2 \\ 2c_2 + 3c_3 &= -1 \\ 3c_3 &= 3 \end{aligned}$$

which can be solved by back-substitution to obtain $c_3 = 1$, $c_2 = -2$, and $c_1 = 3$. The coordinate vector is $(\mathbf{v})_S = (3,-2,1)$.

(b) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$(5,-12,3) = c_1(1,2,3) + c_2(-4,5,6) + c_3(7,-8,9)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 1c_1 - 4c_2 + 7c_3 &= 5 \\ 2c_1 + 5c_2 - 8c_3 &= -12 \\ 3c_1 + 6c_2 + 9c_3 &= 3 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = -2$, $c_2 = 0$, and $c_3 = 1$. The coordinate vector is $(\mathbf{v})_S = (-2,0,1)$.

15. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1 & k_1 + k_2 \\ k_1 + k_2 + k_3 & k_1 + k_2 + k_3 + k_4 \end{bmatrix}$.

Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{ccccccccc} k_1 & & & & & & & = & 0 \\ k_1 & + & k_2 & & & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} k_1 & & & & & & & = & a \\ k_1 & + & k_2 & & & & & = & b \\ k_1 & + & k_2 & + & k_3 & & & = & c \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccccc} k_1 & & & & & & & = & 1 \\ k_1 & + & k_2 & & & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & & & = & 1 \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & 0 \end{array}$$

which can be solved by forward-substitution to obtain $k_1 = 1$, $k_2 = -1$, $k_3 = 1$, $k_4 = -1$.

This allows us to express $A = 1A_1 - 1A_2 + 1A_3 - 1A_4$.

The coordinate vector is $(A)_S = (1,-1,1,-1)$.

17. Vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1 x + a_2 x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1\left(1 + x + x^2\right) + c_2\left(x + x^2\right) + c_3 x^2 = c_1 + \left(c_1 + c_2\right)x + \left(c_1 + c_2 + c_3\right)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} c_1 & & & & & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcrcl} c_1 & & & & & = & a_0 \\ c_1 & + & c_2 & & & = & a_1 \\ c_1 & + & c_2 & + & c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

To express $\mathbf{p} = 7 - x + 2x^2$ as a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcrcrcr} c_1 & & & & & = & 7 \\ c_1 & + & c_2 & & & = & -1 \\ c_1 & + & c_2 & + & c_3 & = & 2 \end{array}$$

which can be solved by forward-substitution to obtain $c_1 = 7$, $c_2 = -8$, $c_3 = 3$.

This allows us to express $\mathbf{p} = 7\mathbf{p}_1 - 8\mathbf{p}_2 + 3\mathbf{p}_3$. The coordinate vector is $(\mathbf{p})_S = (7,-8,3)$.

19. **(a)** The third vector is a sum of the first two. This makes the set linearly dependent, hence it cannot be a basis for R^2.

(b) The two vectors generate a plane in R^3, but they do not span all of R^3. Consequently, the set is not a basis for R^3.

(c) For instance, the polynomial $\mathbf{p}=1$ cannot be expressed as a linear combination of the given two polynomials. This means these two polynomials do not span P_2, hence they do not form a basis for P_2.

(d) For instance, the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ cannot be expressed as a linear combination of the given four matrices. This means these four matrices do not span M_{22}, hence they do not form a basis for M_{22}.

21. **(a)** We have $T_A(1,0,0)=(1,0,-1)$, $T_A(0,1,0)=(1,1,2)$, and $T_A(0,0,1)=(1,-3,0)$. The vector equation

$$k_1(1,0,-1)+k_2(1,1,2)+k_3(1,-3,0)=(0,0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1k_1 & + & 1k_2 & + & 1k_3 & = & 0 \\ 0k_1 & + & 1k_2 & - & 3k_3 & = & 0 \\ -1k_1 & + & 2k_2 & + & 0k_3 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\det(A)=10\neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\mathbf{e}_1),T_A(\mathbf{e}_2),T_A(\mathbf{e}_3)\}$ is linearly independent.

(b) We have $T_A(1,0,0)=(1,0,-1)$, $T_A(0,1,0)=(1,1,2)$, and $T_A(0,0,1)=(2,1,1)$. By inspection,

$$(2,1,1)=(1,0,-1)+(1,1,2)$$

We conclude that the set $\{T_A(\mathbf{e}_1),T_A(\mathbf{e}_2),T_A(\mathbf{e}_3)\}$ is linearly dependent.

23. We have $\mathbf{u}_1=(\cos 30°,\sin 30°)=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ and $\mathbf{u}_2=(0,1)$.

(a) By inspection, we can express $\mathbf{w}=\left(\sqrt{3},1\right)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$\left(\sqrt{3},1\right)=2\left(\tfrac{\sqrt{3}}{2},\tfrac{1}{2}\right)+0(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S=(2,0)$.

(b) Expressing $\mathbf{w}=\left(\sqrt{3},1\right)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0)=c_1\left(\tfrac{\sqrt{3}}{2},\tfrac{1}{2}\right)+c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} \frac{\sqrt{3}}{2}c_1 & & & = & 1 \\ \frac{1}{2}c_1 & + & c_2 & = & 0 \end{array}$$

The first equation yields $c_1 = \frac{2}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = -\frac{1}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = \left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

(c) By inspection, we can express $\mathbf{w} = (0,1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$(0,1) = 0\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + 1(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S = (0,1)$.

(d) Expressing $\mathbf{w} = (a,\ b)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a,\ b) = c_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} \frac{\sqrt{3}}{2}c_1 & & & = & a \\ \frac{1}{2}c_1 & + & c_2 & = & b \end{array}$$

The first equation yields $c_1 = \frac{2a}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = b - \frac{a}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = \left(\frac{2a}{\sqrt{3}}, b - \frac{a}{\sqrt{3}}\right)$.

25. **(a)** Polynomials 1, $2t$, $-2+4t^2$, and $-12t+8t^3$ are linearly independent if the equation

$$c_1(1) + c_2(2t) + c_3(-2+4t^2) + c_4(-12t+8t^3) = 0$$

has only the trivial solution. For these polynomials to span P_3, it must be possible to express every polynomial $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ as

$$c_1(1) + c_2(2t) + c_3(-2+4t^2) + c_4(-12t+8t^3) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Grouping the terms on the left hand side of both equations as
$(c_1 - 2c_3) + (2c_2 - 12c_4)t + 4c_3 t^2 + 8c_4 t^3$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & a_0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & a_1 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & a_2 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & a_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 64 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, a_2, and a_3. Therefore the polynomials 1, $2t$, $-2+4t^2$, and $-12t+8t^3$ are linearly independent and span P_3 so that they form a basis for P_3.

(b) To express $\mathbf{p} = -1-4t+8t^2+8t^3$ as a linear combination of the four vectors in B, we form the nonhomogeneous system as was done in part (a), with the appropriate right hand side values

$$\begin{array}{rcrcrcrcr} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & -1 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & -4 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 8 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 8 \end{array}$$

Back-substitution yields $c_4 = 1$, $c_3 = 2$, $c_2 = 4$, and $c_1 = 3$.

The coordinate vector is $(\mathbf{p})_B = (3,4,2,1)$.

27. **(a)** $\mathbf{w} = 6(3,1,-4) - 1(2,5,6) + 4(1,4,8) = (20,17,2)$

(b) $\mathbf{q} = 3(x^2+1) + 0(x^2-1) + 4(2x-1) = 3x^2+8x-1$

(c) $B = -8\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + 7\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -21 & -103 \\ -106 & 30 \end{bmatrix}$

True-False Exercises

(a) False. The set must also be linearly independent.

(b) False. The subset must also span V.

(c) True. This follows from Theorem 4.5.1.

(d) True. For any vector $\mathbf{v} = (a_1, \ldots, a_n)$ in R^n, we have $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$ therefore the coordinate vector of $\mathbf{v}$ with respect to the standard basis $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is $(\mathbf{v})_S = (a_1, \ldots, a_n) = \mathbf{v}$.

(e) False. For instance, $\{1+t^4, t+t^4, t^2+t^4, t^3+t^4, t^4\}$ is a basis for P_4.

4.6 Dimension

1. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = t$, $x_2 = 0$, $x_3 = t$. In vector form

$$(x_1, x_2, x_3) = (t, 0, t) = t(1, 0, 1)$$

therefore the solution space is spanned by a vector $\mathrm{v}_1 = (1,0,1)$. This vector is nonzero, therefore it forms a linearly independent set (Theorem 4.4.2(b)). We conclude that v_1 forms a basis for the solution space and that the dimension of the solution space is 1.

3. The augmented matrix of the linear system $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The only solution is $x_1 = x_2 = x_3 = 0$.

The solution space has no basis - its dimension is 0.

5. The augmented matrix of the linear system $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = 3s - t$, $x_2 = s$, $x_3 = t$. In vector form

$$(x_1, x_2, x_3) = (3s - t, s, t) = s(3,1,0) + t(-1,0,1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1 = (3,1,0)$ and $\mathbf{v}_2 = (-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.4.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

7. **(a)** If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for x: $x = \frac{2}{3}s - \frac{5}{3}t$. Expressing the solution in vector form $(x, y, z) = \left(\frac{2}{3}s - \frac{5}{3}t, s, t\right) = s\left(\frac{2}{3},1,0\right) + t\left(-\frac{5}{3},0,1\right)$. By Theorem 4.4.2(c), $\left\{\left(\frac{2}{3},1,0\right), \left(-\frac{5}{3},0,1\right)\right\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\left\{\left(\frac{2}{3},1,0\right), \left(-\frac{5}{3},0,1\right)\right\}$. The dimension of the subspace is 2.

(b) If we let $y=s$ and $z=t$ be arbitrary values, we can solve the plane equation for x: $x=s$. Expressing the solution in vector form $(x,y,z)=(s,s,t)=s(1,1,0)+t(0,0,1)$. By Theorem 4.4.2(c), $\{(1,1,0),(0,0,1)\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\{(1,1,0),(0,0,1)\}$. The dimension of the subspace is 2.

(c) In vector form, $(x,y,z)=(2t,-t,4t)=t(2,-1,4)$. By Theorem 4.4.2(b), the vector $(2,-1,4)$ forms a linearly independent set since it is not the zero vector. A basis for the subspace is $\{(2,-1,4)\}$. The dimension of the subspace is 1.

(d) The subspace contains all vectors $(a,a+c,c)=a(1,1,0)+c(0,1,1)$ thus we can express it as as $\operatorname{span}(S)$ where $S=\{(1,1,0),(0,1,1)\}$. By Theorem 4.4.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

9. **(a)** Let W be the space of all diagonal $n\times n$ matrices. We can write

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1 \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_1} + d_2 \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_2} + \cdots + d_n \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{A_n}$$

The matrices $A_1,\dots,A_n$ are linearly independent and they span W; hence, $A_1,\dots,A_n$ form a basis for W. Consequently, the dimension of W is n.

(b) A basis for this space can be constructed by including the n matrices $A_1,\dots,A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1=\frac{n(n-1)}{2}$ matrices B_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) and (j,i) entries, which are both 1.

For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B_{12}}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B_{13}}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{B_{23}}$$

The dimension is $n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$.

(c) A basis for this space can be constructed by including the n matrices $A_1,\dots,A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1=\frac{n(n-1)}{2}$ matrices C_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) entry, which is 1.

For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{A_1},\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{A_2},\underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{A_3},\underbrace{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}}_{C_{12}},\underbrace{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}}_{C_{13}},\underbrace{\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}}_{C_{23}}$$

The dimension is $n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$.

11. **(a)** W is the set of all polynomials $a_0+a_1x+a_2x^2$ for which $a_0+a_1+a_2=0$, i.e. all polynomials that can be expressed in the form $-a_1-a_2+a_1x+a_2x^2$.

Adding two polynomials in W results in another polynomial in W

$\left(-a_1-a_2+a_1x+a_2x^2\right)+\left(-b_1-b_2+b_1x+b_2x^2\right)$

$=\left(-a_1-a_2-b_1-b_2\right)+\left(a_1+b_1\right)x+\left(a_2+b_2\right)x^2$

since we have $\left(-a_1-a_2-b_1-b_2\right)+\left(a_1+b_1\right)+\left(a_2+b_2\right)=0$.

Likewise, a scalar multiple of a polynomial in W is also in W

$k\left(-a_1-a_2+a_1x+a_2x^2\right)=-ka_1-ka_2+ka_1x+ka_2x^2$

since it meets the condition $\left(-ka_1-ka_2\right)+\left(ka_1\right)+\left(ka_2\right)=0$.

According to Theorem 4.2.1, W is a subspace of P_2.

(c) From part (a), an arbitrary polynomial in W can be expressed in the form

$$-a_1-a_2+a_1x+a_2x^2=a_1\left(-1+x\right)+a_2\left(-1+x^2\right)$$

therefore, the polynomials $-1+x$ and $-1+x^2$ span W. Also, $a_1\left(-1+x\right)+a_2\left(-1+x^2\right)=0$ implies $a_1=a_2=0$, so $-1+x$ and $-1+x^2$ are linearly independent, hence they form a basis for W. The dimension of W is 2.

13. The equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{e}_1+k_4\mathbf{e}_2+k_5\mathbf{e}_3+k_6\mathbf{e}_4=\mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcccccccccl}
k_1 & - & 3k_2 & + & k_3 & & & & & & & = & 0\\
-4k_1 & + & 8k_2 & & & + & k_4 & & & & & = & 0\\
2k_1 & - & 4k_2 & & & & & + & k_5 & & & = & 0\\
-3k_1 & + & 6k_2 & & & & & & & + & k_6 & = & 0
\end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1&0&-2&0&0&-1&0\\0&1&-1&0&0&-\frac{1}{3}&0\\0&0&0&1&0&-\frac{4}{3}&0\\0&0&0&0&1&\frac{2}{3}&0\end{bmatrix}$.

Based on the leading entries in the first, second, fourth, and fifth columns, the vector equation

$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are linearly independent. Since $\dim(R^4) = 4$, it follows by Theorem 4.6.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for R^4. (The answer is not unique.)

15. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcrcl} k_1 & & & + & k_3 & & & & & = & 0 \\ -2k_1 & + & 5k_2 & & & + & k_4 & & & = & 0 \\ 3k_1 & - & 3k_2 & & & & & + & k_5 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{9} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{9} & 0 \end{bmatrix}$.

Based on the leading entries in the first three columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ are linearly independent. Since $\dim(R^3) = 3$, it follows by Theorem 4.6.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ form a basis for R^3. (The answer is not unique.)

17. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcl} 1k_1 & + & 1k_2 & + & 2k_3 & + & 0k_4 & = & 0 \\ 0k_1 & + & 0k_2 & + & 0k_3 & + & 0k_4 & = & 0 \\ 0k_1 & + & 1k_2 & + & 1k_3 & - & 1k_4 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

For arbitrary values of s and t, we have $k_1 = -s - t$, $k_2 = -s + t$, $x_3 = s$, $k_4 = t$.

Letting $s = 1$ and $t = 0$ allows us to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Letting $s = 0$ and $t = 1$ allows us to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_4 = \mathbf{v}_1 - \mathbf{v}_2$.

By part (b) of Theorem 4.6.3, $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Based on the leading entries in the first two columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the vectors v_1 and $\mathbf{v}_2$ are linearly independent. We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. (The answer is not unique.)

19. The space of all vectors $\mathbf{x} = (x_1, x_2, x_3)$ for which $T_A(\mathbf{x}) = \mathbf{0}$ is the solution space of $A\mathbf{x} = \mathbf{0}$.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -t$, $x_2 = t$, $x_3 = t$. In vector form, $(x_1, x_2, x_3) = (-t, t, t) = t(-1,1,1)$. Since $\{(-1,1,1)\}$ is a basis for the space, the dimension is 1.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -2s$, $x_2 = s$, $x_3 = t$. In vector form, $(x_1, x_2, x_3) = (-2s, s, t) = s(-2,1,0) + t(0,0,1)$. Since $\{(-2,1,0),(0,0,1)\}$ is a basis for the space, the dimension is 2.

(c) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = 0$, $x_2 = -t$, $x_3 = t$. In vector form, $(x_1, x_2, x_3) = (0, -t, t) = t(0,-1,1)$. Since $\{(0,-1,1)\}$ is a basis for the space, the dimension is 1.

20. The space of all vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ for which $T_A(\mathbf{x}) = \mathbf{0}$ is the solution space of $A\mathbf{x} = \mathbf{0}$.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$ so $x_1 = -2s + t$, $x_2 = -\frac{1}{2}s + \frac{1}{4}t$, $x_3 = s$, $x_4 = t$. In vector form,
$(x_1, x_2, x_3, x_4) = \left(-2s + t, -\frac{1}{2}s + \frac{1}{4}t, s, t\right) = s\left(-2, -\frac{1}{2}, 1, 0\right) + t\left(1, \frac{1}{4}, 0, 1\right)$.
Since $\left\{\left(-2, -\frac{1}{2}, 1, 0\right), \left(1, \frac{1}{4}, 0, 1\right)\right\}$ is a basis for the space, the dimension is 2.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ so $x_1 = x_2 = x_3 = -t$, $x_4 = t$. In vector form,
$(x_1, x_2, x_3, x_4) = (-t, -t, -t, t) = t(-1,-1,-1,1)$.
Since $\{(-1,-1,-1,1)\}$ is a basis for the space, the dimension is 1.

27. In parts (a) and (b), we will use the results of Exercises 18 and 19 by working with coordinate vectors with respect to the standard basis for P_2, $S=\{1,x,x^2\}$.

(a) Denote $\mathbf{v}_1=-1+x-2x^2$, $\mathbf{v}_2=3+3x+6x^2$, $\mathbf{v}_3=9$.

Then $(\mathbf{v}_1)_S=(-1,1,-2)$, $(\mathbf{v}_2)_S=(3,3,6)$, $(\mathbf{v}_3)_S=(9,0,0)$.

Setting $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S=\mathbf{0}$ we obtain a linear system with augmented matrix $\begin{bmatrix}-1&3&9&0\\1&3&0&0\\-2&6&0&0\end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$. Since there is only the trivial solution, it follows that the three coordinate vectors are linearly independent, and, by the result of Exercise 22, so are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Because the number of these vector matches $\dim(P_2)=3$, from Theorem 4.6.4 the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for P_2.

(b) Denote $\mathbf{v}_1=1+x$, $\mathbf{v}_2=x^2$, $\mathbf{v}_3=2+2x+3x^2$.

Then $(\mathbf{v}_1)_S=(1,1,0)$, $(\mathbf{v}_2)_S=(0,0,1)$, $(\mathbf{v}_3)_S=(2,2,3)$.

Setting $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S+k_3(\mathbf{v}_3)_S=\mathbf{0}$ we obtain a linear system with augmented matrix $\begin{bmatrix}1&0&2&0\\1&0&2&0\\0&1&3&0\end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix}1&0&2&0\\0&1&3&0\\0&0&0&0\end{bmatrix}$.

This yields solutions $k_1=-2t$, $k_2=-3t$, $k_3=t$. Taking $t=1$, we can express $(\mathbf{v}_3)_S$ as a linear combination of $(\mathbf{v}_1)_S$ and $(\mathbf{v}_2)_S$: $(\mathbf{v}_3)_S=2(\mathbf{v}_1)_S+3(\mathbf{v}_2)_S$ - the same relationship holds true for the vectors themselves: $\mathbf{v}_3=2\mathbf{v}_1+3\mathbf{v}_2$. By part (b) of Theorem 4.6.3,
$\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}=\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

Based on the leading entries in the first two columns, the vector equation $k_1(\mathbf{v}_1)_S+k_2(\mathbf{v}_2)_S=\mathbf{0}$ has only the trivial solution (the corresponding augmented matrix $\begin{bmatrix}1&0&0\\1&0&0\\0&1&0\end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$). Therefore the coordinate vectors $(\mathbf{v}_1)_S$ and $(\mathbf{v}_2)_S$ are linearly independent and, by the result of Exercise 18, so are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

(c) Clearly, $1+x-3x^2=\frac{1}{2}(2+2x-6x^2)=\frac{1}{3}(3+3x-9x^2)$ therefore from Theorem 4.6.3(b), the subspace is spanned by $1+x-3x^2$. By Theorem 4.4.2(b), a set containing a single nonzero vector is

linearly independent.
We conclude that $1+x-3x^2$ forms a basis for this subspace of P_2.

True-False Exercises

(a) True.

(b) True. For instance, $\mathbf{e}_1$, ..., $\mathbf{e}_{17}$.

(c) False. This follows from Theorem 4.6.2(b).

(d) True. This follows from Theorem 4.6.4.

(e) True. This follows from Theorem 4.6.4.

(f) True. This follows from Theorem 4.6.5(a).

(g) True. This follows from Theorem 4.6.5(b).

(h) True. For instance, invertible matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ form a basis for M_{22}.

(i) True. The set has n^2+1 matrices, which exceeds $\dim(M_{nn}) = n^2$.

(j) False. This follows from Theorem 4.6.6(c).

(k) False. For instance, for any constant c, $\text{span}\{x-c, x^2-c^2\}$ is a two-dimensional subspace of P_2 consisting of all polynomials in P_2 for which $p(c)=0$. Clearly, there are infinitely many different subspaces of this type.

4.7 Change of Basis

1. (a) In this part, B' is the start basis and B is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{13}{10} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{5} & 0 \end{array}\right]$$

The transition matrix is $P_{B'\to B} = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}$.

(b) In this part, B is the start basis and B' is the end basis:

$$\left[\text{end basis} \mid \text{ start basis}\right]=\left[\begin{array}{cc|cc}1 & -1 & 2 & 4\\ 3 & -1 & 2 & -1\end{array}\right]$$

The reduced row echelon form of this matrix is

$$\left[I \mid \text{transition from start to end}\right]=\left[\begin{array}{cc|cc}1 & 0 & 0 & -\frac{5}{2}\\ 0 & 1 & -2 & -\frac{13}{2}\end{array}\right]$$

The transition matrix is $P_{B\to B'}=\begin{bmatrix}0 & -\frac{5}{2}\\ -2 & -\frac{13}{2}\end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix}3\\-5\end{bmatrix}=c_1\begin{bmatrix}2\\2\end{bmatrix}+c_2\begin{bmatrix}4\\-1\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned}2c_1 + 4c_2 &= 3\\ 2c_1 - c_2 &= -5\end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & -\frac{17}{10}\\ 0 & 1 & \frac{8}{5}\end{bmatrix}$. The solution of the linear system is $c_1=-\frac{17}{10}$, $c_2=\frac{8}{5}$, therefore the coordinate vector is $[\mathbf{w}]_B=\begin{bmatrix}-\frac{17}{10}\\ \frac{8}{5}\end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'}=P_{B\to B'}[\mathbf{w}]_B=\begin{bmatrix}0 & -\frac{5}{2}\\ -2 & -\frac{13}{2}\end{bmatrix}\begin{bmatrix}-\frac{17}{10}\\ \frac{8}{5}\end{bmatrix}=\begin{bmatrix}-4\\-7\end{bmatrix}$.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$ and $\mathbf{u}_2'$ we obtain

$$\begin{bmatrix}3\\-5\end{bmatrix}=c_1\begin{bmatrix}1\\3\end{bmatrix}+c_2\begin{bmatrix}-1\\-1\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned}c_1 - c_2 &= 3\\ 3c_1 - c_2 &= -5\end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & -4\\ 0 & 1 & -7\end{bmatrix}$. The solution of the linear system is $c_1=-4$, $c_2=-7$, therefore the coordinate vector is $[\mathbf{w}]_{B'}=\begin{bmatrix}-4\\-7\end{bmatrix}$. This matches the result obtained in part (c).

3. **(a)** In this part, B is the start basis and B' is the end basis:

$$\left[\text{end basis} \mid \text{start basis}\right] = \left[\begin{array}{rrr|rrr} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$\left[I \mid \text{transition from start to end}\right] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

The transition matrix is $P_{B\to B'} = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2c_1 & + & 2c_2 & + & c_3 & = & -5 \\ c_1 & - & c_2 & + & 2c_3 & = & 8 \\ c_1 & + & c_2 & + & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{bmatrix}$. The solution of the linear system is $c_1 = 9$, $c_2 = -9$, $c_3 = -5$ therefore the coordinate vector is $\left[\mathbf{w}\right]_B = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$.

Using Formula (12), $\left[\mathbf{w}\right]_{B'} = P_{B\to B'}\left[\mathbf{w}\right]_B = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}\begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$.

(c) Expressing w as a linear combination of $\mathbf{u}'_1$, $\mathbf{u}'_2$ and $\mathbf{u}'_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 3c_1 & + & c_2 & - & c_3 & = & -5 \\ c_1 & + & c_2 & & & = & 8 \\ -5c_1 & - & 3c_2 & + & 2c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & \frac{23}{2} \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

The solution of the linear system is $c_1 = -\frac{7}{2}$, $c_2 = \frac{23}{2}$, $c_3 = 6$ therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$, which matches the result we obtained in part (b).

5. **(a)** The set $\{\mathbf{f}_1, \mathbf{f}_2\}$ is linearly independent since neither vector is a scalar multiple of the other. Thus $\{\mathbf{f}_1, \mathbf{f}_2\}$ is a basis for V and $\dim(V) = 2$.

Likewise, the set $\{\mathbf{g}_1, \mathbf{g}_2\}$ of vectors in V is linearly independent since neither vector is a scalar multiple of the other. By Theorem 4.6.4, $\{\mathbf{g}_1, \mathbf{g}_2\}$ is a basis for V.

(b) Clearly, $[\mathbf{g}_1]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $[\mathbf{g}_2]_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ hence $P_{B' \to B} = \left[[\mathbf{g}_1]_B \mid [\mathbf{g}_2]_B \right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

(c) We find the two columns of the transitions matrix $P_{B \to B'} = \left[[\mathbf{f}_1]_{B'} \mid [\mathbf{f}_2]_{B'} \right]$

$$\mathbf{f}_1 = a_1\mathbf{g}_1 + a_2\mathbf{g}_2 \qquad\qquad \mathbf{f}_2 = b_1\mathbf{g}_1 + b_2\mathbf{g}_2$$

$$\sin x = a_1(2\sin x + \cos x) + a_2(3\cos x) \qquad\qquad \cos x = b_1(2\sin x + \cos x) + b_2(3\cos x)$$

equate the coefficients corresponding to the same function on both sides of each equation

$$\begin{array}{rcrcr} 2a_1 & & & = & 1 \\ a_1 & + & 3a_2 & = & 0 \end{array} \qquad\qquad \begin{array}{rcrcr} 2b_1 & & & = & 0 \\ b_1 & + & 3b_2 & = & 1 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{6} \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$$

We obtain the transition matrix $P_{B \to B'} = \left[[\mathbf{f}_1]_{B'} \mid [\mathbf{f}_2]_{B'} \right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$.

(An alternate way to solve this part is to use Theorem 4.7.1 to yield

$P_{B \to B'} = P_{B' \to B}^{-1} = \left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \right)^{-1} = \frac{1}{(2)(3)-(0)(1)} \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$.)

(d) Clearly, the coordinate vector is $[\mathbf{h}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Using Formula (12), we obtain $[\mathbf{h}]_{B'} = P_{B \to B'}[\mathbf{h}]_B = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(e) By inspection, $2\sin x - 5\cos x = (2\sin x + \cos x) - 2(3\cos x)$, hence the coordinate vector is $[\mathbf{p}]_{B'} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, which matches the result obtained in part (d).

7. **(a)** In this part, B_2 is the start basis and B_1 is the end basis:

$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right]$.

The reduced row echelon form of this matrix is

$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right]$.

The transition matrix is $P_{B_2 \to B_1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$.

(b) In this part, B_1 is the start basis and B_2 is the end basis:

$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array}\right]$.

The reduced row echelon form of this matrix is

$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right]$.

The transition matrix is $P_{B_1 \to B_2} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$.

(c) Since $\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it follows that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} c_1 & + & 2c_2 & = & 0 \\ 2c_1 & + & 3c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$. The solution of the linear system is $c_1 = 2$, $c_2 = -1$, therefore the coordinate vector is $[\mathbf{w}]_{B_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_{B_2} = P_{B_1 \to B_2} [\mathbf{w}]_{B_1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(e) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ we obtain

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 1c_1 & + & 1c_2 & = & 2 \\ 3c_1 & + & 4c_2 & = & 5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$. The solution of the linear system is $c_1 = 3$, $c_2 = -1$, therefore the coordinate vector is $[\mathbf{w}]_{B_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_{B_1} = P_{B_2 \to B_1} [\mathbf{w}]_{B_2} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

9. **(a)** By Theorem 4.7.2, $P_{B \to S} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

(b) In this part, S is the start basis and B is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right].$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right].$$

The transition matrix is $P_{S\to B}=\begin{bmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{bmatrix}$.

(c) Since $\begin{bmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}\begin{bmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

it follows that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$\begin{bmatrix}5\\ -3\\ 1\end{bmatrix}=c_1\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}+c_2\begin{bmatrix}2\\ 5\\ 0\end{bmatrix}+c_3\begin{bmatrix}3\\ 3\\ 8\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 5\\ 2c_1 & + & 5c_2 & + & 3c_3 & = & -3\\ c_1 & & & + & 8c_3 & = & 1\end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & -239\\ 0 & 1 & 0 & 77\\ 0 & 0 & 1 & 30\end{bmatrix}$. The solution of the linear system is $c_1=-239$, $c_2=77$, $c_3=30$ therefore the coordinate vector is $[\mathbf{w}]_B=\begin{bmatrix}-239\\ 77\\ 30\end{bmatrix}$. From Formula (12), $[\mathbf{w}]_S=P_{B\to S}[\mathbf{w}]_B=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}\begin{bmatrix}-239\\ 77\\ 30\end{bmatrix}=\begin{bmatrix}5\\ -3\\ 1\end{bmatrix}$.

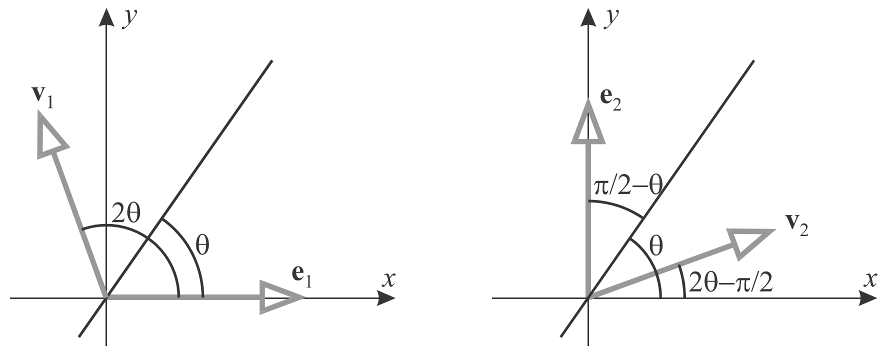
(e) By inspection, $[\mathbf{w}]_S=\begin{bmatrix}3\\ -5\\ 0\end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_B=P_{S\to B}[\mathbf{w}]_S=\begin{bmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{bmatrix}\begin{bmatrix}3\\ -5\\ 0\end{bmatrix}=\begin{bmatrix}-200\\ 64\\ 25\end{bmatrix}$.

11. **(a)** Clearly, $\mathbf{v}_1=(\cos(2\theta),\sin(2\theta))$. Referring to the figure on the right, we see that the angle between the positive x-axis and $\mathbf{v}_2$ is $\frac{\pi}{2}-2\left(\frac{\pi}{2}-\theta\right)=2\theta-\frac{\pi}{2}$. Hence,

$$\mathbf{v}_2 = \left(\cos\left(2\theta - \tfrac{\pi}{2}\right), \sin\left(2\theta - \tfrac{\pi}{2}\right)\right) = \left(\sin(2\theta), -\cos(2\theta)\right)$$

From Theorem 4.7.5, $P_{B\to S} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.



(b) Denoting $P = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$, it follows from Theorem 4.7.5 that $P_{S\to B} = P^{-1}$. In our case, $PP = I$ therefore $P = P^{-1}$. Furthermore, since P is symmetric, we also have $P_{S\to B} = P^T$.

13. Since for every vector $\mathbf{v}$ we have $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$ and $[\mathbf{v}]_C = Q[\mathbf{v}]_B$, it follows that $[\mathbf{v}]_C = QP[\mathbf{v}]_{B'}$ so that $P_{B'\to C} = QP$. From Theorem 4.7.1, $P_{C\to B'} = (QP)^{-1} = P^{-1}Q^{-1}$.

15. (a) By Theorem 4.7.2, P is the transition matrix from $B = \{(1,1,0),(1,0,2),(0,2,1)\}$ to S.

(b) By Theorem 4.7.1, $P^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$ is the transition matrix from B to S, hence by Theorem 4.7.2, $B = \left\{\left(\frac{4}{5}, \frac{1}{5}, -\frac{2}{5}\right), \left(\frac{1}{5}, -\frac{1}{5}, \frac{2}{5}\right), \left(-\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)\right\}$.

17. From $T(1,0) = (2,5)$, $T(0,1) = (3,-1)$, and Theorem 4.7.2 we obtain $P_{B\to S} = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$.

19. By Formula (10), the transition matrix from the standard basis $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ to B is $P_{S\to B} = \left[[\mathbf{e}_1]_B | \ldots | [\mathbf{e}_n]_B\right] = \left[\mathbf{e}_1 | \ldots | \mathbf{e}_n\right] = I_n$ therefore B must be the standard basis.

True-False Exercises

(a) True. The matrix can be constructed according to Formula (10).

(b) True. This follows from Theorem 4.7.1.

(c) True.

(d) True.

(e) False. For instance, $B_1 = \{(0,2),(3,0)\}$ is a basis for R^2 made up of scalar multiples of vectors in the standard basis $B_2 = \{(1,0),(0,1)\}$. However, $P_{B_1 \to B_2} = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ (obtained by Theorem 4.7.2) is not a diagonal matrix.

(f) False. A must be invertible.

4.8 Row Space, Column Space, and Null Space

1. **(a)** $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

3. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, thus $A\mathbf{x} = \mathbf{b}$ is inconsistent. By Theorem 4.8.1, $\mathbf{b}$ is not in the column space of A.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, so the system has a unique solution $x_1 = 1$, $x_2 = -3$, $x_3 = 1$. By Theorem 4.8.1, $\mathbf{b}$ is in the column space of A. By Formula (2), we can write $\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$.

5. **(a)** $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 5 \end{bmatrix} + r\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

7. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of this system is $x_1 = 1 + 3t$, $x_2 = t$; in vector form,

$(x_1, x_2) = (1+3t, t) = (1,0) + t(3,1)$.

The vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is $(x_1, x_2) = t(3,1)$.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution of this system is $x_1 = -2 - t$, $x_2 = 7 - t$, $x_3 = t$; in vector form,

$(x_1, x_2, x_3) = (-2-t, 7-t, t) = (-2,7,0) + t(-1,-1,1)$.

The vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is $(x_1, x_2, x_3) = t(-1,-1,1)$.

9. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = 16t$, $x_2 = 19t$, $x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ therefore the vector $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ forms a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A:
$\begin{bmatrix} 1 & 0 & -16 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -19 \end{bmatrix}$.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = \frac{1}{2}t$, $x_2 = s$, $x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero row of the reduced row echelon form of A:
$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \end{bmatrix}$.

11. We use Theorem 4.8.4 to obtain the following answers.

(a) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Nonzero rows form a basis for the row space: $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

(b) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}$.

Nonzero rows form a basis for the row space: $\begin{bmatrix}1 & -3 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}$.

13. (a) The reduced row echelon form of A is $B = \begin{bmatrix}1 & 0 & 11 & 0 & 3\\0 & 1 & 3 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$.

By Theorems 4.8.3 and 4.8.4, the nonzero rows of B form a basis for the row space of A:
$\mathbf{r}_1 = \begin{bmatrix}1 & 0 & 11 & 0 & 3\end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix}0 & 1 & 3 & 0 & 0\end{bmatrix}$, and $\mathbf{r}_3 = \begin{bmatrix}0 & 0 & 0 & 1 & 0\end{bmatrix}$.

By Theorem 4.8.4, columns of B containing leading 1's form a basis for the column space of B:
$\mathbf{c}_1' = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\mathbf{c}_2' = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, and $\mathbf{c}_4' = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$. By Theorem 4.8.5(b), a basis for the column space of A is

formed by the corresponding columns of A: $\mathbf{c}_1 = \begin{bmatrix}1\\-2\\-1\\-3\end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix}-2\\5\\3\\8\end{bmatrix}$, and $\mathbf{c}_4 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$.

(b) We begin by transposing the matrix A.

We obtain $A^T = \begin{bmatrix}1 & -2 & -1 & -3\\-2 & 5 & 3 & 8\\5 & -7 & -2 & -9\\0 & 0 & 1 & 1\\3 & -6 & -3 & -9\end{bmatrix}$, whose reduced row echelon form is $C = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$. By

Theorem 4.8.4, columns of C containing leading 1's form a basis for the column space of C:

$\mathbf{c}_1' = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$, $\mathbf{c}_2' = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}$, and $\mathbf{c}_3' = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}$. By Theorem 4.8.5(b), a basis for the column space of A^T is

formed by the corresponding columns of A^T: $\mathbf{c}_1 = \begin{bmatrix}1\\-2\\5\\0\\3\end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix}-2\\5\\-7\\0\\-6\end{bmatrix}$, and $\mathbf{c}_3 = \begin{bmatrix}-1\\3\\-2\\1\\-3\end{bmatrix}$.

Since columns of A^T are rows of A, a basis for the row space of A is formed by $\mathbf{r}_1 = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} -2 & 5 & -7 & 0 & -6 \end{bmatrix}$, and $\mathbf{r}_3 = \begin{bmatrix} -1 & 3 & -2 & 1 & -3 \end{bmatrix}$.

15. We construct a matrix whose columns are the given vectors: $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$. The reduced row echelon form of A is $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By Theorem 4.8.4, the four columns of B form a basis for the column space of B. By Theorem 4.8.5(b), the four columns of A form a basis for the column space of A. We conclude that $\{(1,1,0,0),(0,0,1,1),(-2,0,2,2),(0,-3,0,3)\}$ is a basis for the subspace of R^4 spanned by these vectors.

17. Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_5$:

$A = \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3 \quad \mathbf{w}_4 \quad \mathbf{w}_5 \end{array}$$

contains leading 1's in columns 1, 2, and 4, by Theorems 4.8.4 and 4.8.5(b), the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ form a basis for the column space of A, and for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

By inspection, the columns of the reduced row echelon form matrix satisfy $\mathbf{w}_3 = 2\mathbf{w}_1 - \mathbf{w}_2$ and $\mathbf{w}_5 = -\mathbf{w}_1 + 3\mathbf{w}_2 + 2\mathbf{w}_4$. Because elementary row operations preserve dependence relations between column vectors, we conclude that $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_5 = -\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_4$.

19. We are employing the procedure developed in Example 9.

The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & \frac{13}{14} \\ 0 & 1 & -\frac{2}{7} & \frac{5}{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since the first two columns of the reduced row echelon form contain leading 1's, by Theorems 4.8.4 and 4.8.5(b) the first two columns of A^T form a basis for the column space of A^T. Consequently, the first two rows of A, $\begin{bmatrix}1 & 4 & 5 & 6 & 9\end{bmatrix}$ and $\begin{bmatrix}3 & -2 & 1 & 4 & -1\end{bmatrix}$, form a basis for the row space of A.

21. Since $T_A(\mathbf{x}) = A\mathbf{x}$, we are seeking the general solution of the linear system $A\mathbf{x} = \mathbf{b}$.

(a) The reduced row echelon form of the augmented matrix $\begin{bmatrix}1 & 2 & 0 & 0\\ 1 & -1 & 4 & 0\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & \frac{8}{3} & 0\\ 0 & 1 & -\frac{4}{3} & 0\end{bmatrix}$. The general solution is $x_1 = -\frac{8}{3}t$, $x_2 = \frac{4}{3}t$, $x_3 = t$. In vector form, $\mathbf{x} = t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

(b) The reduced row echelon form of the augmented matrix $\begin{bmatrix}1 & 2 & 0 & 1\\ 1 & -1 & 4 & 3\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & \frac{8}{3} & \frac{7}{3}\\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3}\end{bmatrix}$. The general solution is $x_1 = \frac{7}{3} - \frac{8}{3}t$, $x_2 = -\frac{2}{3} + \frac{4}{3}t$, $x_3 = t$.
In vector form, $\mathbf{x} = \left(\frac{7}{3}, -\frac{2}{3}, 0\right) + t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

(c) The reduced row echelon form of the augmented matrix $\begin{bmatrix}1 & 2 & 0 & -1\\ 1 & -1 & 4 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & \frac{8}{3} & \frac{1}{3}\\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3}\end{bmatrix}$. The general solution is $x_1 = \frac{1}{3} - \frac{8}{3}t$, $x_2 = -\frac{2}{3} + \frac{4}{3}t$, $x_3 = t$.
In vector form, $\mathbf{x} = \left(\frac{1}{3}, -\frac{2}{3}, 0\right) + t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

23. (a) The associated homogeneous system $x + y + z = 0$ has a general solution $x = -s - t$, $y = s$, $z = t$. The original nonhomogeneous system has a general solution $x = 1 - s - t$, $y = s$, $z = t$, which can be expressed in vector form as

$$(x, y, z) = (1 - s - t, s, t) = \underbrace{(1, 0, 0)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{(-s - t, s, t)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(b) Geometrically, the points (x, y, z) corresponding to solutions of $x + y + z = 1$ form a plane passing through the point $(1, 0, 0)$ and parallel to the vectors $(-1, 1, 0)$ and $(-1, 0, 1)$.

25. (a) The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix}1 & \frac{2}{3} & -\frac{1}{3} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. A general solution of the system is $x_1 = -\frac{2}{3}s + \frac{1}{3}t$, $x_2 = s$, $x_3 = t$.

(b) Multiplying $\begin{bmatrix}3 & 2 & -1\\ 6 & 4 & -2\\ -3 & -2 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$ yields $\begin{bmatrix}2\\ 4\\ -2\end{bmatrix}$ therefore $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ is a solution of the nonhomogeneous system.

(c) The vector form of a general solution of the nonhomogeneous system is

$$(x_1,x_2,x_3)=\underbrace{(1,0,1)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}}+\underbrace{\left(-\tfrac{2}{3}s+\tfrac{1}{3}t,s,t\right)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(d) The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1=\frac{2}{3}-\frac{2}{3}p+\frac{1}{3}q$, $x_2=p$, $x_3=q$.

If we let $p=s$ and $q=t+1$ then this agrees with the solution we obtained in part (c).

27. The augmented matrix of the nonhomogeneous system $\begin{bmatrix} 3 & 4 & 1 & 2 & 3 \\ 6 & 8 & 2 & 5 & 7 \\ 9 & 12 & 3 & 10 & 13 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system $x_1=\frac{1}{3}-\frac{4}{3}r-\frac{1}{3}s$, $x_2=r$, $x_3=s$, $x_4=1$ can be expressed in vector form as

$$(x_1,x_2,x_3,x_4)=\underbrace{\left(\tfrac{1}{3},0,0,1\right)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}}+\underbrace{\left(-\tfrac{4}{3}r-\tfrac{1}{3}s,\ r,\ s,\ 0\right)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{associated}\\ \text{homogeneous}\\ \text{system}}}$$

29. **(a)** The reduced row echelon form of A is $B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution $\mathbf{x}=(x,y,z)$ of $A\mathbf{x}=\mathbf{0}$ is $x=0$, $y=0$, $z=t$; in vector form, $\mathbf{x}=t(0,0,1)$. This shows that the null space of A consists of all points on the z-axis.

The column space of A, $\text{span}\{(1,0,0),(0,1,0)\}$ clearly consists of all points in the xy-plane.

(b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an example of such a matrix.

31. **(a)** By inspection, $\begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix}$ has the desired null space. In general, this will hold true for all matrices of the form $\begin{bmatrix} 3a & -5a \\ 3b & -5b \end{bmatrix}$ where a and b are not both zero (if $a = b = 0$ then the null space is the entire plane).

(b) Only the zero vector forms the null space for both A and B (their determinants are nonzero, therefore in each case the corresponding homogeneous system has only the trivial solution).
The line $3x + y = 0$ forms the null space for C.
The entire plane forms the null space for D.

True-False Exercises

(a) True.

(b) False. The column space of A is the space spanned by all column vectors of A.

(c) False. Those column vectors form a basis for the column space of R.

(d) False. This would be true if A were in row echelon form.

(e) False. For instance $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ have the same row space, but different column spaces.

(f) True. This follows from Theorem 4.8.3.

(g) True. This follows from Theorem 4.8.3.

(h) False. Elementary row operations generally can change the column space of a matrix.

(i) True. This follows from Theorem 4.8.1.

(j) False. Let both A and B be $n \times n$ matrices. By Theorem 4.8.3, row operations do not change the row space of a matrix. An invertible matrix can be reduced to I thus its row space is always R^n. On the other hand, a singular matrix cannot be reduced to identity matrix - at least one row in its reduced row echelon form is made up of zeros. Consequently, its row space is spanned by fewer than n vectors, therefore the dimension of this space is less than n.

4.9 Rank, Nullity, and the Fundamental Matrix Spaces

1. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\operatorname{rank}(A)=1$ (the number of leading 1's)
- $\operatorname{nullity}(A)=3$ (by Theorem 4.9.2).

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\operatorname{rank}(A)=2$ (the number of leading 1's)
- $\operatorname{nullity}(A)=3$ (by Theorem 4.9.2).

3. **(a)** $\operatorname{rank}(A)=3$; $\operatorname{nullity}(A)=0$

(b) $\operatorname{rank}(A)+\operatorname{nullity}(A)=3+0=3=n \leftarrow$ number of columns of A

(c) 3 leading variables; 0 parameters in the general solution (the solution is unique)

5. **(a)** $\operatorname{rank}(A)=1$; $\operatorname{nullity}(A)=2$

(b) $\operatorname{rank}(A)+\operatorname{nullity}(A)=1+2=3=n \leftarrow$ number of columns of A

(c) 1 leading variable; 2 parameters in the general solution

7. **(a)** If every column of the reduced row echelon form of a 4×4 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 4
- the nullity of A has the smallest possible value: 0

(b) If every row of the reduced row echelon form of a 3×5 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 3
- the nullity of A has the smallest possible value: 2

(c) If every column of the reduced row echelon form of a 5×3 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 3
- the nullity of A has the smallest possible value: 0

9.

			(a)	(b)	(c)	(d)	(e)	(f)	(g)
	Size of A:	$m\times n$	3×3	3×3	3×3	5×9	5×9	4×4	6×2
	$\operatorname{rank}(A)$	$=r$	3	2	1	2	2	0	2
	$\operatorname{rank}(A\mid\mathbf{b})$	$=s$	3	3	1	2	3	0	2
(i)	dimension of the row space of A	$=r$	3	2	1	2	2	0	2
	dimension of the column space of A	$=r$	3	2	1	2	2	0	2
	dimension of the null space of A	$=n-r$	0	1	2	7	7	4	0
	dimension of the null space of A^T	$=m-r$	0	1	2	3	3	4	4
(ii)	is the system $A\mathbf{x}=\mathbf{b}$ consistent?	Is $r=s$?	Yes	No	Yes	Yes	No	Yes	Yes
(iii)	number of parameters in the general solution of $A\mathbf{x}=\mathbf{b}$	$=n-r$ if consistent	0	-	2	7	-	4	0

11. The reduced row echelon form of A is $\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$. Therefore $\operatorname{rank}(A)=\operatorname{rank}(A^T)=2$. Applying Formula (4) to both A and its transpose yields $2+\operatorname{nullity}(A)=2$ and $2+\operatorname{nullity}(A^T)=3$.

It follows that bases for $\operatorname{row}(A)$ and $\operatorname{row}(A^T)$ are $\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}0\\3\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}1\\0\\-9\end{bmatrix},\begin{bmatrix}4\\3\\0\end{bmatrix}\right\}$ respectively. Since $\operatorname{nullity}(A)=0$ the nullspace of A contains only the zero vector. $A^T=\begin{bmatrix}1&0&-9\\4&3&0\end{bmatrix}$ has reduced row echelon form $\begin{bmatrix}1&0&-9\\0&1&12\end{bmatrix}$. The general solution is $x_1=9t$, $x_2=-12t$, $x_3=t$. In vector form, $\mathbf{x}=t(9,-12,1)$ where t is arbitrary. So a basis for the left null space (of A^T) is $\left\{\begin{bmatrix}9\\-12\\1\end{bmatrix}\right\}$.

13. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $\operatorname{rank}(A) = \operatorname{rank}(A^T) = 2$.

It follows that bases for $\operatorname{row}(A)$ and $\operatorname{row}(A^T)$ are $\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix} \right\}$ respectively

and a basis for the null space of A is $\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Applying Formula (4) to A^T yields $2 + \operatorname{nullity}(A^T) = 3$

Since A^T has reduced row echelon form $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, a basis for the left null space (of A^T) is

$\left\{ \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix} \right\}$.

15. From Exercise 11, a basis for $\operatorname{row}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ whereas the null space of A contains only the zero vector. These subspaces clearly satisfy Theorem 4.9.7(a). Bases for $\operatorname{row}(A^T)$ and the left null space are given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix} \right\}$ respectively. Since $\det\left(\begin{bmatrix} 1 & 4 & 9 \\ 0 & 3 & -12 \\ -9 & 0 & 1 \end{bmatrix} \right) = 678 \neq 0,$

these three vectors are a basis for R^3. Moreover $\begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ -12 \\ 0 \end{bmatrix} = 0$. Therefore

the left null space is the set of all vectors in R^3 that are orthogonal to all vectors in $\operatorname{row}(A^T)$.

17. From Exercise 13, a basis for $\text{row}(A)$ is $\left\{\begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}\right\}$ and a basis for the null space of A is $\left\{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\right\}$.

Since $\det\left(\begin{bmatrix} 0 & -1 & 3 \\ -1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix}\right) = -14 \neq 0,$ these three vectors are a basis for R^3. Moreover $\begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 0$. Therefore, the null space of A is the set of all vectors in R^3 that are orthogonal to all vectors in $\text{row}(A)$. Similarly, bases for $\text{row}(A^T)$ and the left null space are given by $\left\{\begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix}\right\}$ respectively. Since $\det\left(\begin{bmatrix} 0 & -1 & -4 \\ -1 & 0 & -4 \\ -4 & -4 & 1 \end{bmatrix}\right) = -33 \neq 0,$ these vectors are a basis for R^3. Moreover, $\begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix} = 0$. Therefore, the left null space is the set of all vectors in R^2 that are orthogonal to all vectors in $\text{row}(A^T)$.

19. Following Example 5, we find that the reduced row echelon form of the augmented matrix $\begin{bmatrix} 0 & 2 & 8 & -7 & 1 & 0 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 & 0 \\ -3 & 4 & -2 & 5 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 6 & 0 & \frac{5}{17} & \frac{19}{17} & \frac{7}{17} \\ 0 & 1 & 4 & 0 & \frac{5}{17} & \frac{21}{34} & \frac{7}{17} \\ 0 & 0 & 0 & 1 & -\frac{1}{17} & \frac{3}{17} & \frac{2}{17} \end{bmatrix}$. Hence, the column space basis is $\left\{\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 5 \end{bmatrix}\right\}$, the row space basis is $\left\{\begin{bmatrix} 0 \\ 2 \\ 8 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -2 \\ 5 \end{bmatrix}\right\}$,the null space basis is $\left\{\begin{bmatrix} -6 \\ -4 \\ 1 \\ 0 \end{bmatrix}\right\}$, and the left null space contains only the zero vector.

21. **(a)** Applying Formula (4) to both A and its transpose yields

$$2 + \text{nullity}(A) = 4 \text{ and } 2 + \text{nullity}(A^T) = 3$$

therefore

$$\text{nullity}(A) - \text{nullity}(A^T) = 1$$

(b) Applying Formula (4) to both A and its transpose yields

$$\text{rank}(A) + \text{nullity}(A) = n \text{ and } \text{rank}(A^T) + \text{nullity}(A^T) = m$$

By Theorem 4.9.4, $\text{rank}(A^T) = \text{rank}(A)$ therefore

$$\text{nullity}(A) - \text{nullity}(A^T) = n - m$$

23. $T(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$; the standard matrix is

$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

(a) $\text{rank}(A) = 3$ **(b)** $\text{nullity}(A) = 2$

25. By inspection, there must be leading 1's in the first column (because of the first row) and in the third column (because of the fourth row) regardless of the values of r and s, therefore the matrix cannot have rank 1.

It has rank 2 if $r = 2$ and $s = 1$, since there is no leading 1 in the second column in that case.

27. No, both row and column spaces of A must be planes through the origin since from $\text{nullity}(A) = 1$, it follows by Formula (4) that $\text{rank}(A) = 3 - 1 = 2$.

29. **(a)** 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its rows is nonzero;

(b) 5; if A is the zero matrix, then the general solution of $A\mathbf{x} = \mathbf{0}$ has five parameters;

(c) 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its columns has a leading 1;

(d) 3; if A is the zero matrix, then the general solution of $A\mathbf{x} = \mathbf{0}$ has three parameters;

31. **(a)** By Formula (4), $\text{nullity}(A) = 7 - 4 = 3$ thus the dimension of the solution space of

$A\mathbf{x} = 0$ is 3.

(b) No, the column space of A is a subspace of R^5 of dimension 4, therefore there exist vectors $\mathbf{b}$ in R^5 that are outside this column space. For any such vector, the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

33. From the result of Exercise 32, the rank of the matrix being less than 2 implies that

$$\begin{vmatrix} x & y \\ 1 & x \end{vmatrix} = x^2 - y = 0, \qquad \begin{vmatrix} x & z \\ 1 & y \end{vmatrix} = xy - z = 0, \qquad \begin{vmatrix} y & z \\ x & y \end{vmatrix} = y^2 - xz = 0$$

therefore $y = x^2$ and $z = xy = x^3$. Letting $x = t$, we obtain $y = t^2$ and $z = t^3$.

35. The reduced row echelon form of A^T is $\begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution of $A^T\mathbf{x} = 0$ has components $x_1 = -10t$, $x_2 = 5t$, $x_3 = t$, $x_4 = 0$, so in vector form $\mathbf{x} = t(-10,5,1,0)$. Evaluating dot products of columns of A and $\mathbf{v} = (-10,5,1,0)$, which forms a basis for the null space of A^T we obtain

$\mathbf{c}_1 \cdot \mathbf{v} = (1,2,0,2)\cdot(-10,5,1,0) = (1)(-10) + (2)(5) + (0)(1) + (2)(0) = 0$

$\mathbf{c}_2 \cdot \mathbf{v} = (3,6,0,6)\cdot(-10,5,1,0) = (3)(-10) + (6)(5) + (0)(1) + (6)(0) = 0$

$\mathbf{c}_3 \cdot \mathbf{v} = (-2,-5,5,0)\cdot(-10,5,1,0) = (-2)(-10) + (-5)(5) + (5)(1) + (2)(0) = 0$

$\mathbf{c}_4 \cdot \mathbf{v} = (0,-2,10,8)\cdot(-10,5,1,0) = (0)(-10) + (-2)(5) + (10)(1) + (8)(0) = 0$

$\mathbf{c}_5 \cdot \mathbf{v} = (2,4,0,4)\cdot(-10,5,1,0) = (2)(-10) + (4)(5) + (0)(1) + (4)(0) = 0$

$\mathbf{c}_6 \cdot \mathbf{v} = (0,-3,15,18)\cdot(-10,5,1,0) = (0)(-10) + (-3)(5) + (15)(1) + (18)(0) = 0$

Since the column space of A is $\text{span}\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6\}$ and the null space of A^T is $\text{span}\{\mathbf{v}\}$, we conclude that the two spaces are orthogonal complements in R^4.

37. **(a)** $m=3>2=n$ so the system is overdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix}1 & 0 & b_1+b_3\\ 0 & 1 & b_3\\ 0 & 0 & 3b_1+b_2+2b_3\end{bmatrix}$ hence the system is inconsistent for all b's that satisfy $3b_1+b_2+2b_3\neq 0$.

(b) $m=2<3=n$ so the system is underdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix}1 & 0 & 0 & \frac{1}{2}b_1-\frac{1}{4}b_2\\ 0 & 1 & -\frac{4}{3} & -\frac{1}{6}b_1-\frac{1}{12}b_2\end{bmatrix}$ hence the system has infinitely many solutions for all b's (no values of b's can make this system inconsistent).

(c) $m=2<3=n$ so the system is underdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix}1 & 0 & -\frac{3}{2} & -\frac{1}{2}b_1-\frac{3}{2}b_2\\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2}b_1-\frac{1}{2}b_2\end{bmatrix}$ hence the system has infinitely many solutions for all b's (no values of b's can make this system inconsistent).

True-False Exercises

(a) False. For instance, in $\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$, neither row vectors nor column vectors are linearly independent.

(b) True. In an $m\times n$ matrix, if $m<n$ then by Theorem 4.6.2(a), the n columns in R^m must be linearly dependent. If $m>n$, then by the same theorem, the m rows in R^n must be linearly dependent. We conclude that $m=n$.

(c) False. The nullity in an $m\times n$ matrix is at most n.

(d) False. For instance, if the column contains all zeros, adding it to a matrix does not change the rank.

(e) True. In an $n\times n$ matrix A with linearly dependent rows, $\operatorname{rank}(A)\leq n-1$.
By Formula (4), $\operatorname{nullity}(A)=n-\operatorname{rank}(A)\geq 1$.

(f) False. By Theorem 4.9.7, the nullity must be nonzero.

(g) False. This follows from Theorem 4.9.1.

(h) False. By Theorem 4.9.4, $\operatorname{rank}(A^T)=\operatorname{rank}(A)$ for any matrix A.

(i) True. Since each of the two spaces has dimension 1, these dimensions would add up to 2 instead of 3 as required by Formula (4).

(j) False. For instance, if $n=3$, $V=\operatorname{span}\{\mathbf{i},\mathbf{j}\}$ (the xy-plane), and $W=\operatorname{span}\{\mathbf{i}\}$ (the x-axis) then $W^\perp=\operatorname{span}\{\mathbf{j},\mathbf{k}\}$ (the yz-plane) is not a subspace of $V^\perp=\operatorname{span}\{\mathbf{k}\}$ (the z-axis).
(Note that it is true that $V^\perp$ is a subspace of $W^\perp$.)

Chapter 4 Supplementary Exercises

1. **(a)** $\mathbf{u}+\mathbf{v}=(3+1,-2+5,4-2)=(4,3,2)$; $k\mathbf{u}=(-1\cdot 3,0,0)=(-3,0,0)$

(b) For any $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,u_3+v_3)$ is an ordered triple of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2,u_3)$ in V and for any scalar k, $k\mathbf{u}=(ku_1,0,0)$ is an ordered triple of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^3.

(d) Axiom 7:

$$k\left((u_1,u_2,u_3)+(v_1,v_2,v_3)\right)=k(u_1+v_1,u_2+v_2,u_3+v_3)=\left(k(u_1+v_1),0,0\right)=k(u_1,u_2,u_3)+k(v_1,v_2,v_3)$$

for all real k, u_1, u_2, u_3, v_1, v_2, and v_3.

Axiom 8:

$$(k+m)(u_1,u_2,u_3)=\left((k+m)u_1,0,0\right)=(ku_1+mu_1,0,0)=k(u_1,u_2,u_3)+m(u_1,u_2,u_3)$$ for all real k, m, u_1, u_2, and u_3;

Axiom 9: $k\left(m(u_1,u_2,u_3)\right)=k(mu_1,0,0)=(kmu_1,0,0)=(km)(u_1,u_2,u_3)$ for all real k, m, u_1, u_2, and u_3;

(e) Axiom 10 fails to hold: $1(u_1,u_2,u_3)=(u_1,0,0)$ does not generally equal (u_1,u_2,u_3). Consequently, V is not a vector space.

3.

$$A=\begin{bmatrix}1&1&s\\1&s&1\\s&1&1\end{bmatrix}$$ ← The coefficient matrix of the system

$$\begin{bmatrix}1&1&s\\0&s-1&1-s\\0&1-s&1-s^2\end{bmatrix}$$ ← -1 times the first row was added to the second row and $-s$ times the first row was added to the third row.

$$\begin{bmatrix}1&1&s\\0&s-1&1-s\\0&0&2-s-s^2\end{bmatrix}$$ ← The second row was added to the third row.

After factoring $2-s-s^2=(2+s)(1-s)$, we conclude that

- the solution space is a plane through the origin if $s=1$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $\text{nullity}(A)=2$),
- the solution space is a line through the origin if $s=-2$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so $\text{nullity}(A)=1$),
- the solution space is the origin if $s \neq -2$ and $s \neq 1$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so $\text{nullity}(A)=0$),
- there are no values of s for which the solution space is R^3.

5. **(a)** Using trigonometric identities we can write

$$\begin{aligned} \mathbf{f}_1 &= \sin(x+\theta) = \sin x \cos\theta + \cos x \sin\theta = (\cos\theta)\mathbf{f} + (\sin\theta)\mathbf{g} \\ \mathbf{g}_1 &= \cos(x+\theta) = \cos x \cos\theta - \sin x \sin\theta = (-\sin\theta)\mathbf{f} + (\cos\theta)\mathbf{g} \end{aligned}$$

which shows that $\mathbf{f}_1$ and $\mathbf{g}_1$ are both in $W = \text{span}\{\mathbf{f},\mathbf{g}\}$.

(b) The functions $\mathbf{f}_1 = \sin(x+\theta)$ and $\mathbf{f}_2 = \cos(x+\theta)$ are linearly independent since neither function is a scalar multiple of the other. By Theorem 4.6.4, these functions form a basis for W.

7. Denoting $B = [\ \mathbf{v}_1 \mid \cdots \mid \mathbf{v}_n\ ]$ we can write $AB = [\ A\mathbf{v}_1 \mid \cdots \mid A\mathbf{v}_n\ ]$. By parts (g) and (h) of Theorem 4.9.7, the columns of AB are linearly independent if and only if $\det(AB) \neq 0$. This implies that $\det(A) \neq 0$, i.e., the matrix A must be invertible.

9. **(a)** The reduced row echelon form of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the rank is 2 and the nullity is 1.

(b) The reduced row echelon form of $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank is 2 and the nullity is 2.

(c) For $n = 1$, the rank is 1 and the nullity is 0.
For $n \geq 2$, the reduced row echelon form will always have two nonzero rows; the rank is 2 and the nullity is $n - 2$.

11. **(a)** Let W be the set of all polynomials p in P_n for which $p(-x) = p(x)$. In order for a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ to be in W, we must have $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = a_0 + a_1(-x) + a_2(-x)^2 + \cdots + a_n(-x)^n = p(-x)$ which implies that for all x, $2a_1x + 2a_3x^3 + \cdots = 0$ so $a_1 = a_3 = \cdots = 0$.
Any polynomial of the form $p(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2n/2}x^{2n/2}$ satisfies $p(-x) = p(x)$ (the notation t represents the largest integer less than or equal to t).
This means $W = \text{span}\{1, x^2, x^4, ..., x^{2n/2}\}$, so W is a subspace of P_n by Theorem 4.3.1(a).
The polynomials in $\{1, x^2, x^4, ..., x^{2n/2}\}$ are linearly independent (since they form a subset of the standard basis for P_n), consequently they form a basis for W.

(b) Let W be the set of all polynomials p in P_n for which $p(0) = p(1)$.
In order for a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ to be in W, we must have $p(0) = a_0 = a_0 + a_1 + a_2 + \cdots + a_n = p(1)$ which implies that $a_1 + a_2 + \cdots + a_n = 0$.
Therefore any polynomial in W can be expressed as

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + (-a_1 - a_2 - \cdots - a_{n-1})x^n$$

$$= a_0 + a_1(x - x^n) + a_2(x^2 - x^n) + \cdots + a_{n-1}(x^{n-1} - x^n).$$

This means $W = \text{span}\{1, x - x^n, x^2 - x^n, \ldots, x^{n-1} - x^n\}$, so W is a subspace of P_n by Theorem 4.3.1(a). Since $a_0 + a_1(x - x^n) + a_2(x^2 - x^n) + \cdots + a_{n-1}(x^{n-1} - x^n) = 0$ implies $a_0 = a_1 = a_2 = \cdots = a_{n-1} = 0$, it follows that $\{1, x - x^n, x^2 - x^n, \ldots, x^{n-1} - x^n\}$ is linearly independent, hence it is a basis for W.

13. **(a)** A general 3×3 symmetric matrix can be expressed as $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

$$= a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

span the space of all 3×3 symmetric matrices. Also, these matrices are linearly indpendent, since $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ requires that all six coefficients in the linear combination above must be zero. We conclude that the matrices

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ form a basis for the space of all 3×3 symmetric matrices.

(b) A general 3×3 skew-symmetric matrix can be expressed as

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Clearly the matrices $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ span the space of all 3×3 skew-symmetric matrices. Also, these matrices are linearly indpendent, since

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

requires that all three coefficients in the linear combination above must be zero. We conclude that the matrices $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$,and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ form a basis for the space of all 3×3 skew-symmetric matrices.

15. All submatrices of size 3×3 or larger contain at least two rows that are scalar multiples of each other, so their determinants are 0. Therefore the rank cannot exceed 2. The possible values are:

- $\operatorname{rank}(A)=2$, e.g., if $a_{51}=a_{16}=1$ regardless of the other values,
- $\operatorname{rank}(A)=1$, e.g., if $a_{16}=a_{26}=a_{36}=a_{46}=0$ and $a_{56}=1$ regardless of the other values, and
- $\operatorname{rank}(A)=0$ if all entries are 0.

17. The standard matrices for D_k, R_θ, and S_k are $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, and $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ (assuming a shear in the x-direction).

(a) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ therefore D_k and R_θ commute.

(b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & k\cos\theta - \sin\theta \\ \sin\theta & k\sin\theta + \cos\theta \end{bmatrix}$ does not generally equal

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta + k\sin\theta & -\sin\theta + k\cos\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

therefore R_θ and S_k do not commute (same result is obtained if a shear in the y-direction is taken instead)

(c) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ therefore D_k and S_k commute (same result is obtained if a shear in the y-direction is taken instead)