FAST- National University of Computer and Emerging Sciences, Karachi.

FAST School of Computing, Fall 2022

CS1005-Discrete Structures

Assignment # 2 -- Solution

Instructions: Max. Points: 100

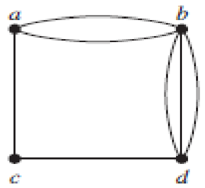
1- This is hand written assignment.
2- Just write the question number instead of writing the whole question.
3- You can only use A4 size paper for solving the assignment.

---

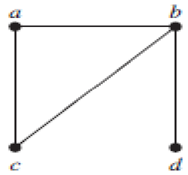
1. Determine whether the graph shown in figure i to iv has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops. Use your answers to determine the type of graph.

Solution:

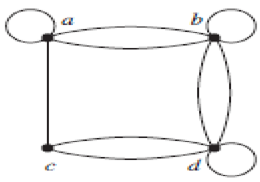
i) It has undirected edges.
It has multiple edges.
It has no loops.
It is undirected Multigraph.



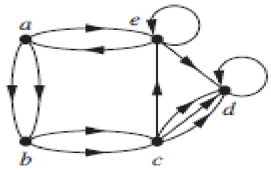
ii) It has undirected edges.
It has no multiple edges.
It has no loops.
It is undirected simple graph.



iii) It has undirected edges.
It has multiple edges.
It has three loops.
It is undirected Pseudo graph.



iv)It has directed edges.
It has multiple edges.
It has two loops.
It is directed Multi graph.



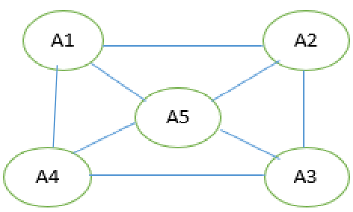
2. The intersection graph of a collection of sets A1, A2,..., An is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of these collections of sets.

i) A1 = {0, 2, 4, 6, 8}, A2 = {0, 1, 2, 3, 4}, A3 = {1, 3, 5, 7, 9}, A4 = {5, 6, 7, 8, 9}, A5 = {0, 1, 8, 9}

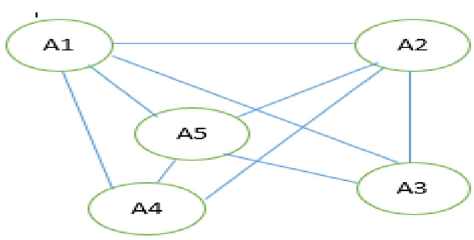
ii) A1 = {…,−4,−3,−2,−1, 0}, A2 = {…,−2,−1, 0, 1, 2,…}, A3 = {…,−6,−4,−2, 0, 2, 4, 6,…}, A4 = {…,−5,−3,−1, 1, 3, 5,…}, A5 = {…,−6,−3, 0, 3, 6,…}

Solution:

i)



ii)



3. (a) Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Also find the neighborhood vertices of each vertex in given graphs.

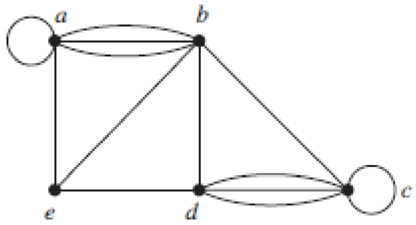
i) Number of Vertices: 5 Number of edges:13

Degree of vertices:

deg(a) = deg(b) = deg(c) = 6, deg(d) = 5, deg(e) = 3.

Neighborhood Vertices:

N(a) = {a, b, e}, N(b) = {a, c, d, e}, N(c) = {b, c, d}, N(d) = {b, c, e}, N(e) = a, b, d}



ii) Number of Vertices: 9 Number of edges:12

Degree of vertices:

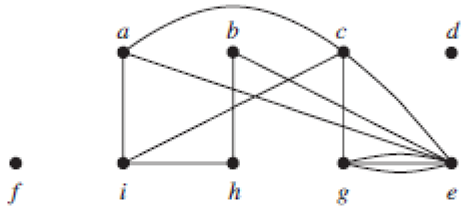
deg(a) =3, deg(b) =2, deg(c) = 4, deg(d) = 0, deg(e) = 6.

deg(f) =0, deg(g) =4, deg(h) = 2, deg(i) = 3.

Neighborhood Vertices:

N(a) = {c, e, i}, N(b) = {e, h}, N(c) = {a, e, g, i}, N(d) = $\varnothing$,

N(e) = {a, b, c, g}, N(f) = $\varnothing$, N(g) = {c, e}, N(h) = {b, i}, N(i) = {a, c, h}.



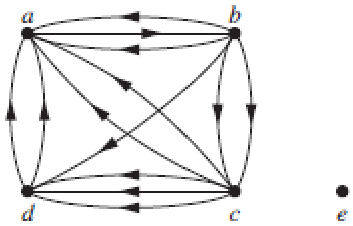
(b) Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

i) In-degree of a vertices

deg$^-$(a) = 6, deg$^-$(b) = 1, deg$^-$(c) = 2, deg$^-$(d) = 4, deg$^-$(e) = 0.

Out-degree of a vertices

deg$^+$(a) = 1, deg$^+$(b) = 5, deg$^+$(c) = 5, deg$^+$(d) = 2, deg$^+$ (e) = 0.

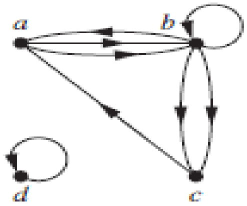


ii) In-degree of a vertices

deg$^-$(a) = 2, deg$^-$(b) = 3, deg$^-$(c) = 2, deg$^-$(d) = 1.

Out-degree of a vertices

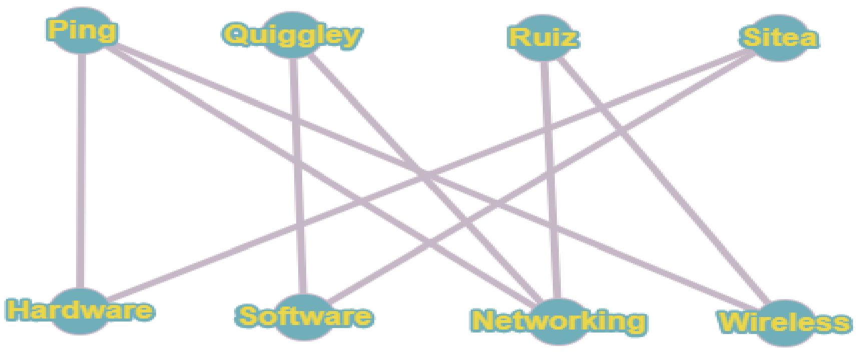
deg$^+$(a) = 2, deg$^+$(b) = 4, deg$^+$(c) = 1, deg$^+$(d) = 1.



4. (a) Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of six responsibilities: planning, publicity, sales, marketing, development, and industry relations. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning or development; Smith could do publicity, sales, or industry relations; Chou could do planning, sales, or industry relations; and Macintyre could do planning, publicity, sales, or industry relations. Model the capabilities of these employees using appropriate graph.

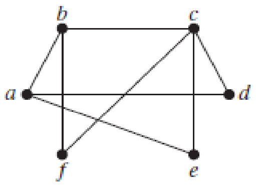
Solution: Bipartite Graph

(b) Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software. Use appropriate graph to model the four employees and their qualifications.
Solution: Bipartite Graph

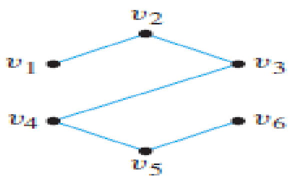


5. Find which of the following graphs are bipartite. Redraw the bipartite graphs so that their bipartite nature is evident. Also write the disjoint set of vertices.

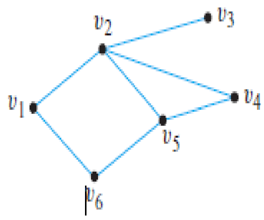
i)



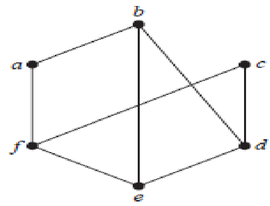
ii)



iii)

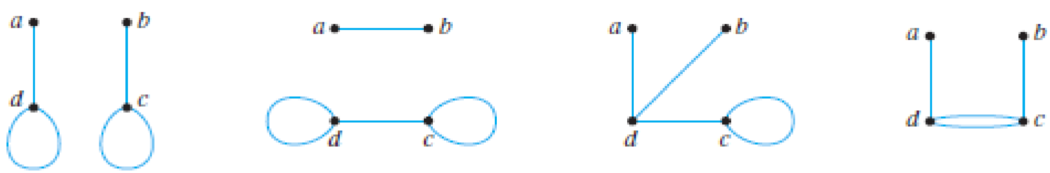


iv)



Solution:
(a) Not bipartite (since a is adjacent to b & f vertices)
(b) Bipartite (A (V_1, V_3, V_5) & B (V_2, V_4, V_6))
(c) Not bipartite (since V_4 & V_5 are adjacent vertices)
(d) Not Bipartite (since b is adjacent to d & e vertices)

6. Draw a graph with the specified properties or show that no such graph exists.
a) A graph with four vertices of degrees 1, 1, 2, and 3
b) A graph with four vertices of degrees 1, 1, 3, and 3
c) A simple graph with four vertices of degrees 1, 1, 3, and 3
Solution:
a) No such graph is possible. By Handshaking theorem, the total degree of a graph is even.
But a graph with four vertices of degrees 1, 1, 2, and 3 would have a total degree of 1 + 1 + 2 + 3 = 7, which is odd.
b) Let G be any of the graphs shown below.



In each case, no matter how the edges are labeled, deg(a) = 1, deg(b) = 1, deg(c) =3, and deg(d) = 3.

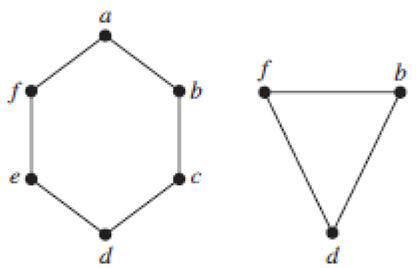
c) There is no simple graph with four vertices of degrees 1, 1, 3, and 3.

7. a) In a group of 15 people, is it possible for each person to have exactly 3 friends? Explain. (Assume that friendship is a symmetric relationship: If x is a friend of y, then y is a friend of x.)
Solution:
By using Handshaking theorem.
No! there is no graph possible, such that 15 vertices have degree 3. Since (15 * 3) ≠ 2e.

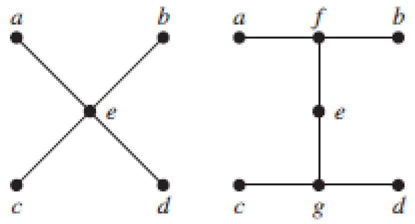
b) In a group of 4 people, is it possible for each person to have exactly 3 friends? Why?
Solution:
By using Handshaking theorem.
Yes! there is graph possible, such that 4 vertices have degree 3. Since (4 * 3) = 2e.

8. (a) Find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

i)

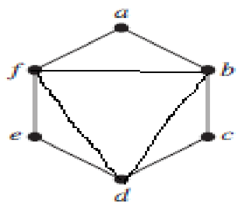


ii)

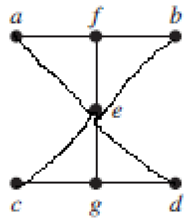


Solution:

i)



ii)



b) How many vertices does a regular graph of degree four with 10 edges have?
Solution:

We want to determine a regular graph of degree four with $m = 10$ edges.

Let the graph contain n vertices $v_1, v_2, ..., v_n$, then each of these n vertices have degree 4.

$$\deg(v_i) = 4$$
$$i = 1, 2, ..., n$$

By the Handshaking theorem, the sum of degrees of all vertices is equal to twice the number of edges:

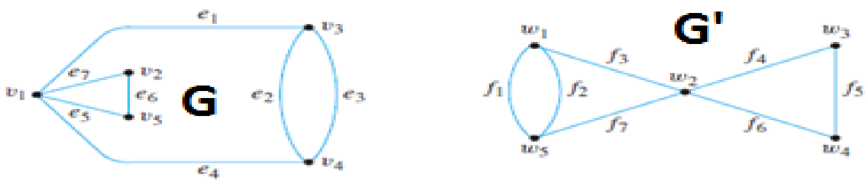
$$20 = 2(10) = 2m = \sum_{v=1}^{n} \deg(v_i) = \sum_{v=1}^{n} 4 = 4n$$

We then obtained the equation $20 = 4n$. Divide each side of the equation by 4:

$$n = \frac{20}{4} = 5$$

9. For given pair (G, G') of graphs. Determine whether they are isomorphic. If they are, give function g: V (G) →V (G') that define the isomorphism. If they are not, give an invariant for graph isomorphism that they do not share.

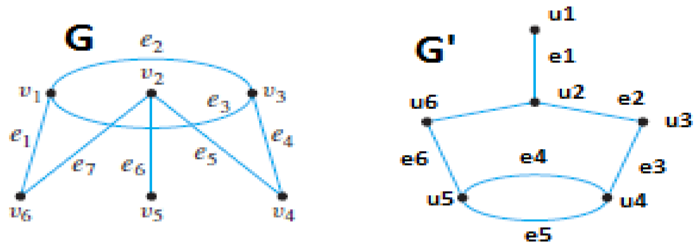
i)



Solution: Both graph G and G' are satisfying all the invariant. Hence, they are isomorphic.

Function: $g(V_1) = W_2$, $g(V_2) = W_3$ $g(V_3) = W_1$ $g(V_4) = W_5$ $g(V_5) = W_4$

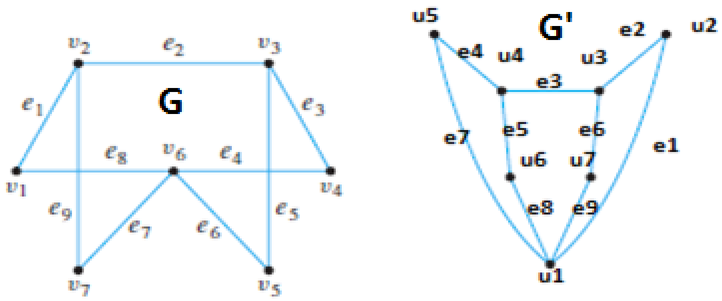
ii)



Solution: Both graph G and G' are satisfying all the invariant. Hence, they are isomorphic.

Function: $g(V_1) = U_5$, $g(V_2) = U_2$ $g(V_3) = U_4$ $g(V_4) = U_3$ $g(V_5) = U_1$ $g(V_6) = U_6$

iii)

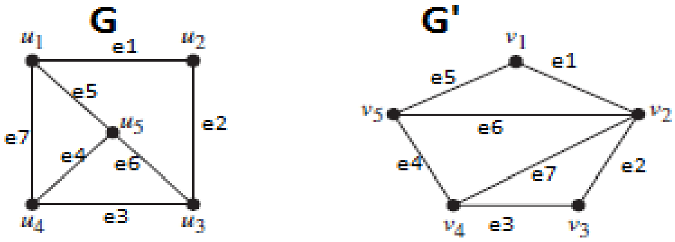


Solution: Both graph G and G' are satisfying all the invariant. Hence, they are isomorphic.

Function: $g(V_1) = U_5$, $g(V_2) = U_4$ $g(V_3) = U_3$ $g(V_4) = U_2$

$g(V_5) = U_7$ $g(V_6) = U_1$ $g(V_7) = U_6$

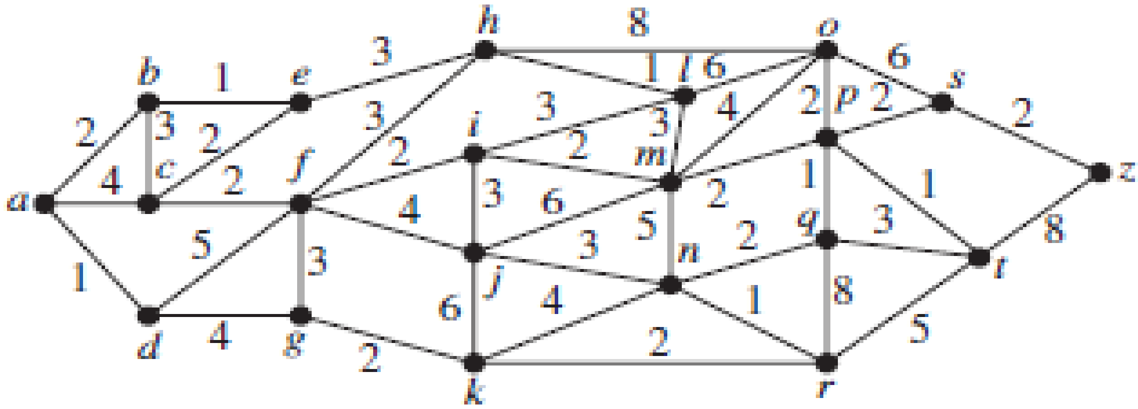
iv)



Solution: Graph G has no vertex of degree 4 where G' has vertex V_2 with degree 4. Hence, they are not isomorphic.

10. Find the length of a shortest path between *a* and *z* in the given weighted graph by using Dijkstra's algorithm.

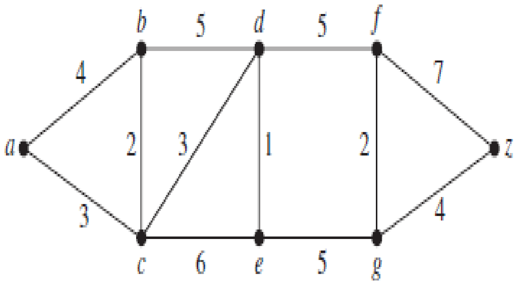
i)



N	D(b)	D (c)	D(d)	D(e)	D(f)	D(g)	D(h)	D(i)	D(j)	D(k)	D(l)	D(m)	D(n)	D(o)	D(p)	D(q)	D(r)	D(s)	D(t)	D(z)
a	2,a	4,a	1,a																	
ad	2,a	4,a			6,d	5,d														
adb		4,a		3,b	6,d	5,d														
adbe		4,a			6,d	5,d	6,e													
adbec					6,c	5,d	6,e													
adbeg					6,c		6,e			7,g										
adbegf							6,e	8,f	10,f	7,g										
adbegfh								8,f	10,f	7,g	7,h			14,h						
adbegfhk								8,f	10,f		7,h		11,k	14,h			9,k			
adbegfhkl								8,f	10,f			10,l	11,k	13,l			9,k			
adbegfhkli									10,f			10,l	11,k	13,l			9,k			
adbegfhklir									10,f			10,l	10,r	13,l		17,r			14,r	
adbegfhklirj												10,l	10,r	13,l		17,r			14,r	
adbegfhklirjm													10,r	13,l	12,m	17,r			14,r	
adbegfhklirjmn														13,l	12,m	12,n			14,r	
adbegfhklirjmnp														13,l		12,n		14,p	13,p	
adbegfhklirjmnpq														13,l				14,p	13,p	
adbegfhklirjmnpqo																		14,p	13,p	
adbegfhklirjmnpqot																		14,p		21,t
adbegfhklirjmnpqots																				16,s
adbegfhklirjmnpqotsz																				

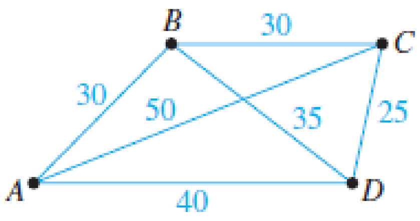
ii)

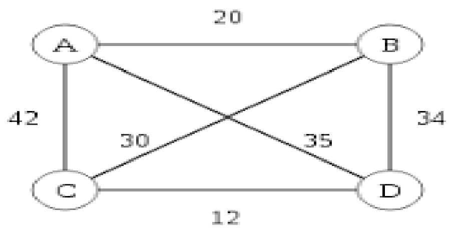
N	D(b)	D(c)	D(d)	D(e)	D(f)	D(g)	D(z)
a	4,a	3,a	∞	∞	∞	∞	∞
ac			6,c	9,c	∞	∞	∞
acb			6,c	9,c	∞	∞	∞
acbd				7,d	11,d	∞	∞
acbde					11,d	12,e	∞
acbdef						12,e	18,f
acbdefg							16,g
acbdefgz	4,a	3,a	6,c	7,d	11,d	12,e	16,g



11. Imagine that the drawing below is a map showing four cities and the distances in kilometers between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city A. Which route from city to city will minimize the total distance that must be traveled?

i) ii)





i) Solution:

Hamiltonian Circuit are: ABCDA = 125; ABDCA = 140; ACBDA = 155.

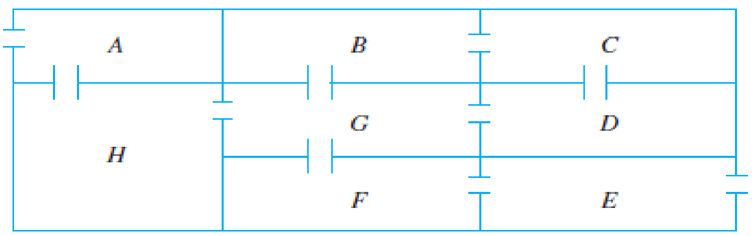
Hence ABCDA = 125 is the minimum distance travelled.

ii) Solution:

Hamiltonian Circuit are: ABCDA = 97; ABDCA = 108; ACBDA = 141.

Hence ABCDA = 97 is the minimum distance travelled.

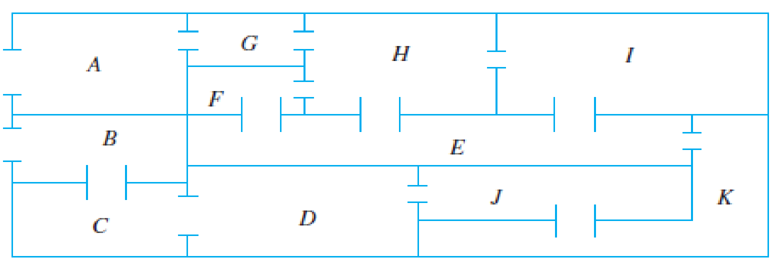
12. (a) The following is a floor plan of a house. Is it possible to enter the house in room A, travel through every interior doorway of the house exactly once, and exit out of room E? If so, how can this be done?



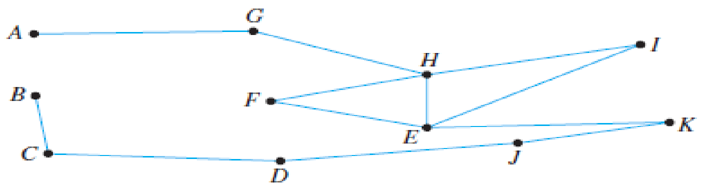
Solution:

Yes! Path:A→H→G→B→C→D→G→F→E

(b) The floor plan shown below is for a house that is open for public viewing. Is it possible to find a trail that starts in room A, ends in room B, and passes through every interior doorway of the house exactly once? If so, find such a trail.



Solution Let the floor plan of the house be represented by the graph below.

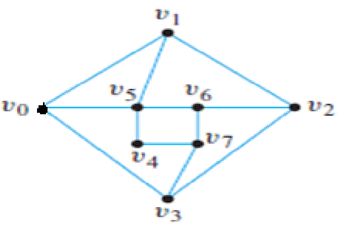


Each vertex of this graph has even degree except for A and B, each of which has degree 1. Hence by Corollary 10.2.5, there is an Euler path from A to B. One such trail is

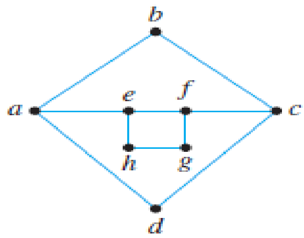
$$AGHFEIHEKJDCB.$$

13. **Find Hamiltonian circuits AND Path for those graphs that have them. Explain why the other graphs do not.**

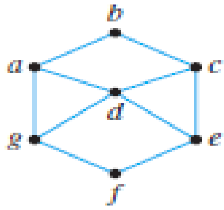
i)



ii)



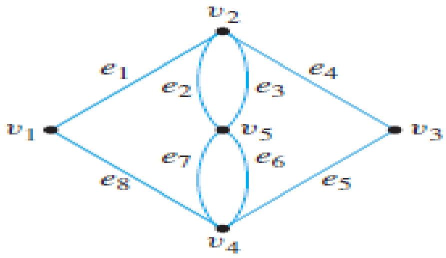
iii)



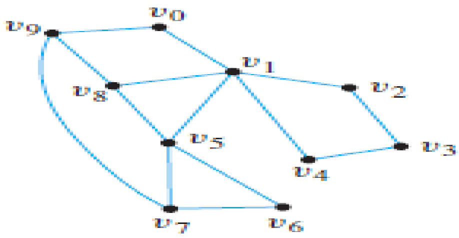
i) Solution:
Hamiltonian Circuit: $V_0, V_1, V_2, V_6, V_5, V_4, V_7, V_3, V_0$
Hamiltonian Path: $V_0, V_1, V_2, V_6, V_5, V_4, V_7, V_3$
ii) Solution:
Hamiltonian Circuit: doesn't exist
Hamiltonian Path: b, c, f, g, h, e, a, d
iii) Solution:
Hamiltonian Circuit: d, c, b, a, g, f, e, d
Hamiltonian Path: d, c, b, a, g, f, e

14. **a) Determine which of the graphs have Euler circuits. If the graph does not have an Euler circuit, explain why not. If it does have an Euler circuit, describe one.**

i)



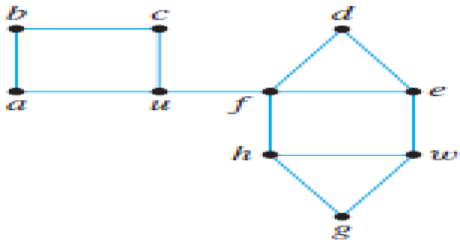
ii)



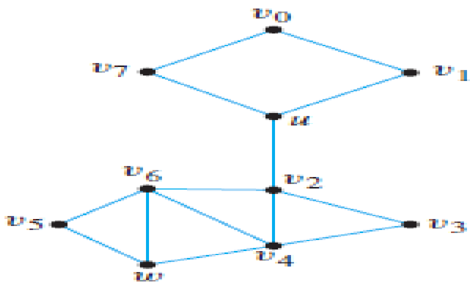
i) Solution: All vertices have even degree so circuit exists.
Euler Circuit: $V_1, V_2, V_5, V_4, V_5, V_2, V_3, V_4, V_1$
ii) Solution:
Euler Circuit do not exist because all vertices don't have even degree.

b) Determine whether there is an Euler path from u to w. If the graph does not have an Euler path, explain why not. If it does have an Euler path, describe one.

i)



ii)

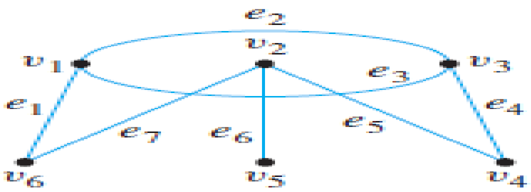


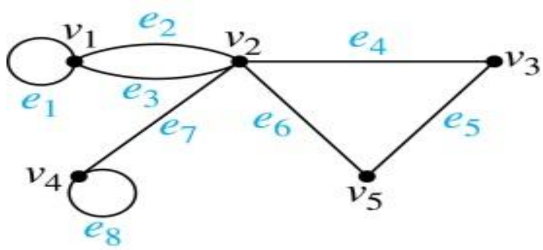
i) Solution:
Euler Path doesn't exist because four vertices have odd degree.
ii) Solution: Euler Path exists because exact two vertices have odd degree.
Euler path: $U, V_1, V_0, V_7, U, V_2, V_3, V_4, V_2, V_6, V_5, W, V_6, V_4, W$

15. (a) Use an incidence matrix to represent the graph shown below.

i) ii)

Solution: i) ii)





(b) Draw a graph using below given incidence matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

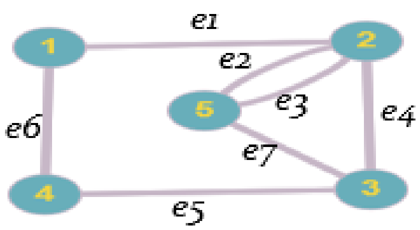
i) ii)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

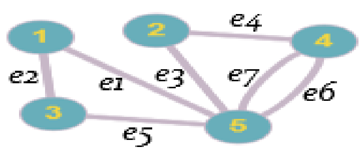
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Solution:

i)

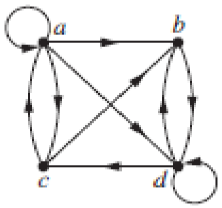


ii)



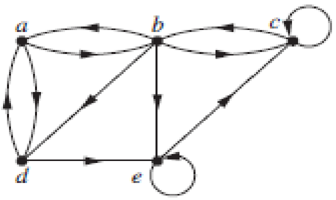
16. Use an adjacency list and adjacency matrix to represent the given graph.

i)



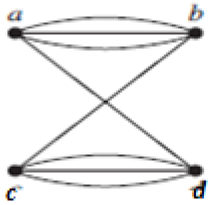
Initial Vertex	Terminal Vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

(ii)



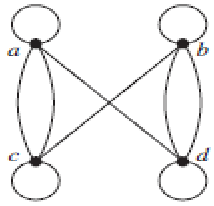
Initial Vertex	Terminal Vertices
a	b, d
b	a, c, d, e
c	b, c,
d	a, e
e	c, e

iii)



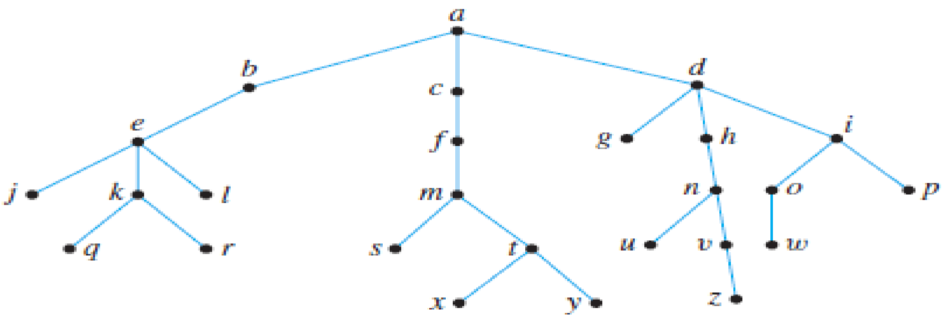
Vertex	Adjacent Vertices
a	b, d
b	a, c
c	b, d
d	a, c

iv)



Vertex	Adjacent Vertices
a	a, c, d
b	b, c, d
c	a, b, c
d	a, c, d

17. Consider the tree shown at right with root a.



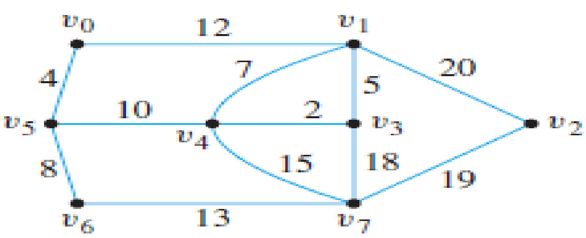
Solution:

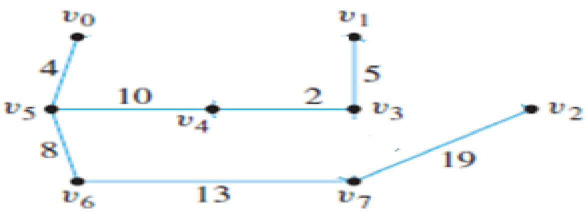
i) What is the level of n?	Level of n is 3.
ii) What is the level of a?	Level of a is 0.
iii) What is the height of this rooted tree?	Height of this rooted tree is 5
iv) What are the children of n?	u & v are the children of n.
v) What is the parent of g?	d is the parent of g.
vi) What are the siblings of j?	k & l are the siblings of j.
vii) What are the descendants of f?	m, s, t, x & y are the descendants of f.
viii) What are the internal nodes?	a, b, e, k, c, f, m, t, d, h, i, n, o & v are the internal nodes.
ix) What are the ancestors of z?	v, n, h, d & a are the ancestors of z.
x) What are the leaves?	j, l, q, r, s, x, y, g, p, u, w & z are the leaves.

18. Use Prim's algorithm to find a minimum spanning tree starting from V_0 for given graphs. Indicate the order in which edges are added to form each tree.

i) Solution: MST Cost = 61

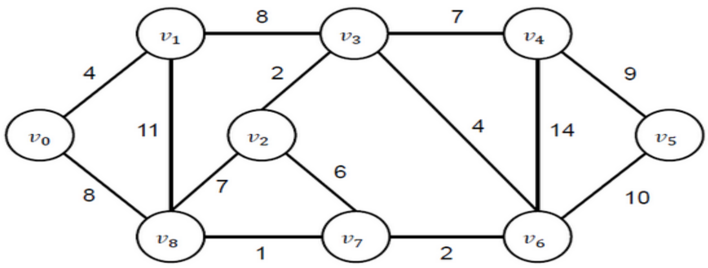
$(V_0,V_5) = 4$, $(V_5,V_6) = 8$, $(V_4,V_5) = 10$,
$(V_3,V_4) = 2$, $(V_1,V_3) = 5$, $(V_6,V_7) = 13$,
$(V_2,V_7) = 19$.

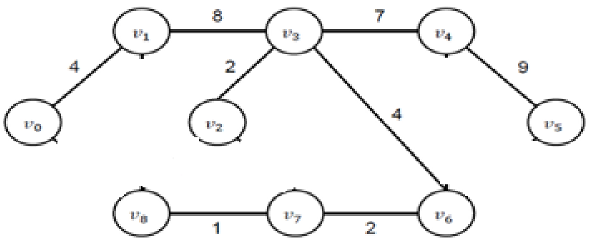


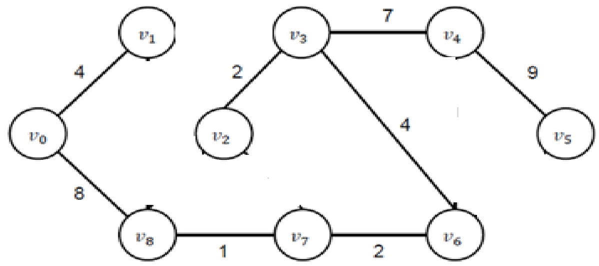


ii) Solution: MST Cost = 37

$(V_0,V_1) = 4$, $(V_0,V_8) = 8$, $(V_7,V_8) = 1$,
$(V_6,V_7) = 2$, $(V_3,V_6) = 4$, $(V_2,V_3) = 2$,
$(V_3,V_4) = 7$, $(V_4,V_5) = 9$,

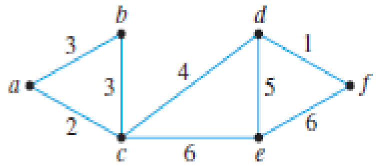




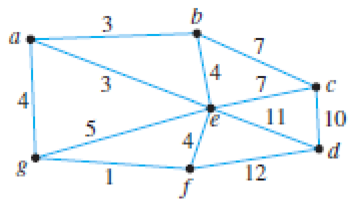


19. Use Kruskal's algorithm to find a minimum spanning tree for given graphs. Indicate the order in which edges are added to form each tree.

i)

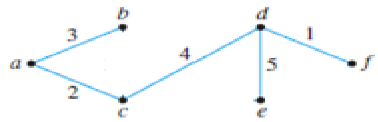


ii)



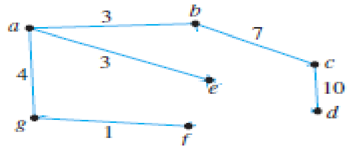
i) Solution: MST cost = 15
Order of edges added is:

(d, f) =1, (a, c) =2, (a, b) =3, ~~(b, c) =3,~~
(c, d) =4, (d, e) =5, ~~(c, e) =6,~~ ~~(e, f) =6~~



ii) Solution: MST cost = 28
Order of edges added is:

(g, f) =1, (a, b) =3, (a, e) =3, (a, g) =4,
~~(b, e) =4,~~ ~~(e, f) =4,~~ ~~(g, e) =5,~~ (b, c) =7
~~(c, e) =7,~~ (c, d) =10, ~~(d, e) =11,~~ ~~(d, f) =12~~

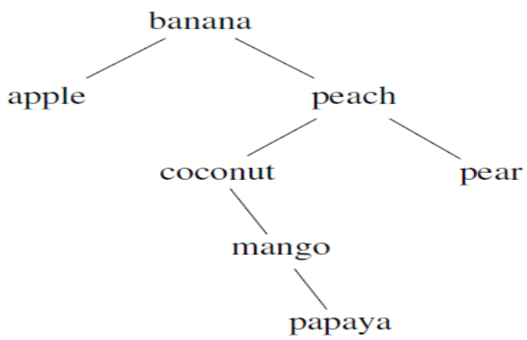


20. (a) i) Build a binary search tree for the word's banana, peach, apple, pear, coconut, mango, and papaya using alphabetical order.

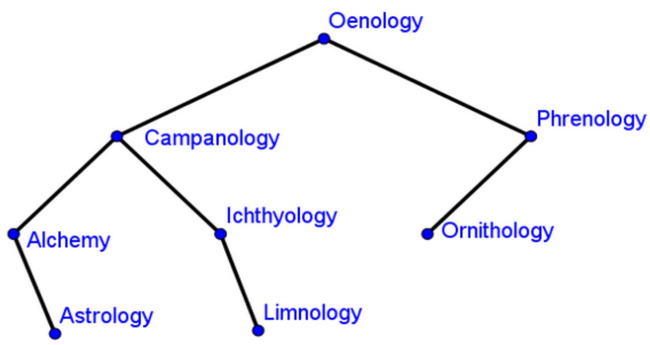
ii) Build a binary search tree for the word's oenology, phrenology, campanology, ornithology, ichthyology, limnology, alchemy, and astrology using alphabetical order.

Solution:

i)

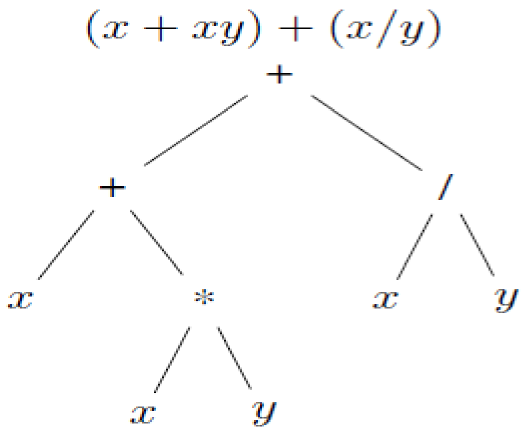


ii)

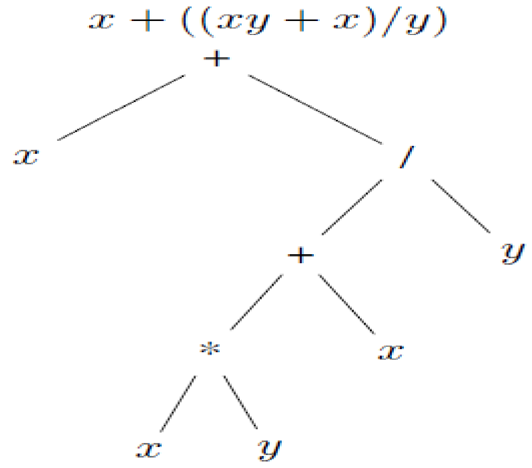


(b) Represent these expressions using binary trees.
(i) (x + xy) + (x / y)
(ii) x + ((xy + x) / y)
Solution:

i)

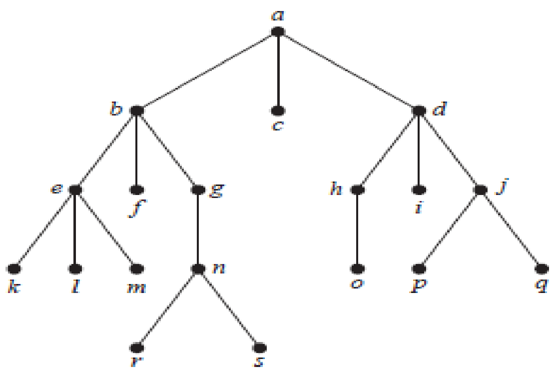


ii)

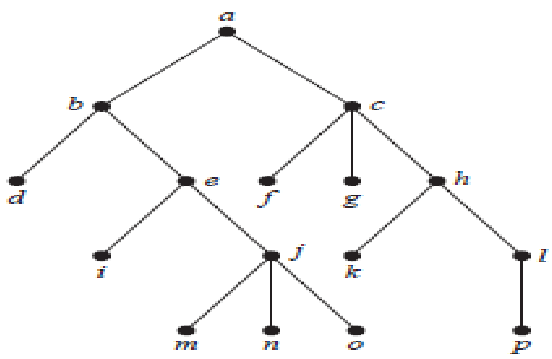


21. Determine the order in which preorder, Inorder and Postorder traversal visits the vertices of the given ordered rooted tree.

i)



ii)



Solution:

i)

Preorder: a b e k l m f g n r s c d h o i j p q
Inorder: k e l m b f r n s g a c o h d i p j q
Postorder: k l m e f r s n g b c o h i p q j d a

ii)

Preorder: a b d e i j m n o c f g h k l p
Inorder: d b i e m j n o a f c g k h p l
Postorder: d i m n o j e b f g k p l h c a

22. (a) How many edges does a tree with 10000 vertices have?
Solution:
A tree with n vertices has n – 1 edge. Hence 10000 – 1 = 9999 edges.
(b) How many edges does a full binary tree with 1000 internal vertices have?
Solution:
A full binary tree has two edges for each internal vertex. So, we'll just multiply the number of internal vertices by the number of edges. Hence 1000 * 2 = 2000 edges.

(c) How many vertices does a full 5-ary tree with 100 internal vertices have?
Solution:

A full $m-ary$ tree with I internal vertices has $n = mi + 1$ vertices.
From the given information, we have $m = 5$, $i = 100$
So $n = 5 \times 100 + 1 = 501$
Therefore a full 5-ary tree with 100 internal vertices has 501 vertices.

23. a) Write these expressions in Prefix and Postfix notation:

i) (x + xy) + (x / y)

Solution:

Prefix: + + x * x y / x y

Postfix: x x y * + x y / +

ii) x + ((xy + x) / y)

Solution:

Prefix: + x / + * x y x y

Postfix: x x y * x + y / +

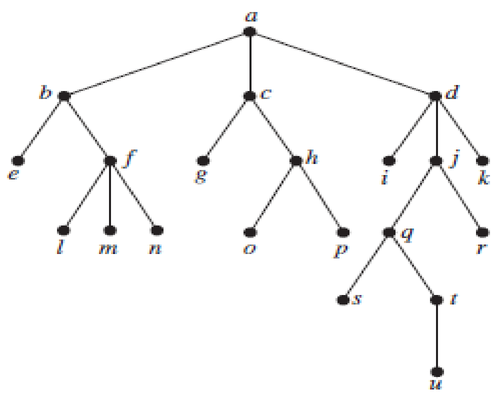
b) i) What is the value of this prefix expression + – ↑ 3 2 ↑ 2 3 / 6 – 4 2

Solution: 4

ii) What is the value of this postfix expression 4 8 + 6 5 – * 3 2 – 2 2 + * /

Solution: 3

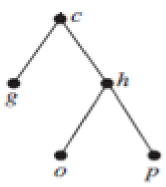
24. Answer these questions about the rooted tree illustrated.



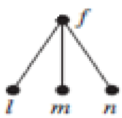
i) Is the rooted tree a full m-ary tree?
Solution: It is not a full m-ary tree for any m because some of its internal vertices have two children and others have three children.
ii) Is the rooted tree a balanced m-ary tree?
Solution: It is not balanced m-ary tree because it has leaves at levels 2, 3, 4 and 5.

iii) Draw the subtree of the tree that is rooted at
a) c. b) f. c) q.
Solution:

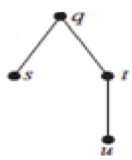
a)



b)

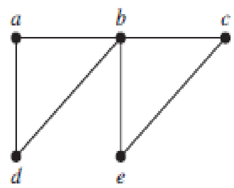


c)

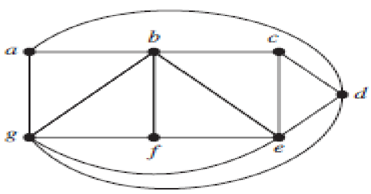


25. Find a spanning tree for the graph shown by removing edges in simple circuits. Write down the removed edges.

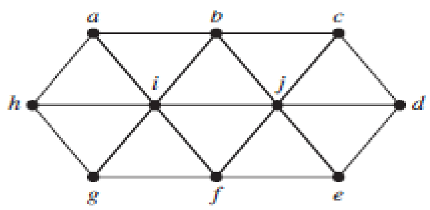
(i)



ii)

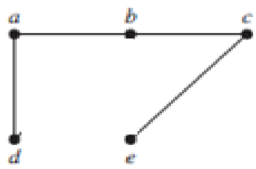


iii)

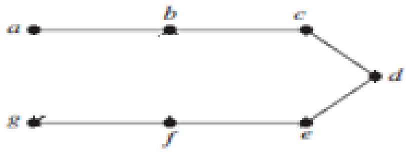


Solution:

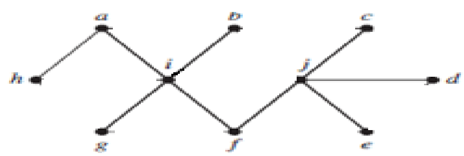
i)



ii)

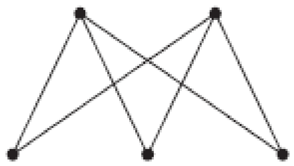


iii)

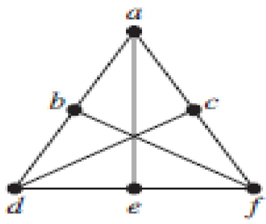


26. Determine whether the given graph is planar. If so, draw it so that no edges cross.

(a)



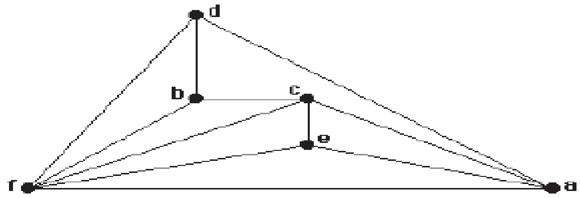
(b)



Solution:

(a) This is $K_{3,3}$, with parts {a, d, f} and {b, c, e}. Therefore it is not planar.

(b) This graph can be untangled if we play with it long enough. The following picture gives a planar representation of it.



27. Let R be the following relation defined on the set {a, b, c, d}:

R = {(a, a), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, b), (c, c), (d, b), (d, d)}

Determine whether R is:

(a) Reflexive: (b) Symmetric (c) Antisymmetric (d) Transitive (e)Irreflexive (f) Asymmetric

Solution:

(a) R is reflexive because R contains (a, a), (b, b), (c, c), and (d, d).
(b) R is not symmetric because R contains (a, c) but not (c, a) ∈ R .
(c) R is not antisymmetric because both (b, c) ∈ R and (c, b) ∈ R , but b = c.
(d) R is not Transitive because both (a, c) ∈ R and (c, b) ∈ R , but not (a, b) ∈ R.
(e) R is not irreflexive because R contains (a, a), (b, b), (c, c), and (d, d).
(f) R is not Asymmetric because R is not Antisymmetric.

28. List the ordered pairs in the relation R from A = {0, 1, 2, 3, 4} to B = {0, 1, 2, 3}, where (a, b) ∈ R if and only if

a) a = b. b) a + b = 4. c) a > b. d) a | b. e) gcd(a, b) = 1. f) lcm (a, b) = 2.

Solution:

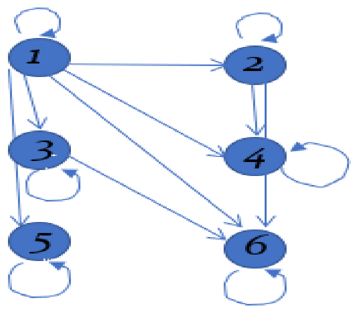
a) { (0,0), (1, 1), (2, 2), (3, 3) }
b) { (1, 3), (2, 2), (3, 1), (4, 0) }
c) { (1, 0), (2, 0), (3, 0), (4, 0), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3) }
d) { (1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (1,2), (2,2), (1,3), (3,3) }
e) { (1,0), (0,1), (1,1), (1,2), (1, 3), (2,1), (3,1), (4,1), (2,3),(3,2),(4,3) }
f) { (1,2), (2,1), (2,2) }

29. List all the ordered pairs in the relation R = {(a, b) | a divides b} on the set {1, 2, 3, 4, 5, 6}. Display this relation as Directed Graph(digraph), as well in matrix form.

Solution:

R= { (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6) }

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



30. For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.
a) { (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4) }
Solution:
(a) R is not reflexive: It doesn't contain (1,1) and (4,4).
(b) R is not symmetric because R contains (2, 4) but not (4, 2) ∈ R.
(c) R is not antisymmetric: we have (2,3) and (3,2) but 2 ≠ 3.
(d) R is Transitive because for any numbers a, b, and c, if (a, b), (b, c) ∈ R then (a, c) ∈ R.

b) { (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4) }
Solution:
(a) R is reflexive: It contains (1,1) ,(2,2), (3,3) and (4,4).
(b) R is symmetric because (a,b) and (b,a) ∈ R.
(c) R is not antisymmetric: we have (1,2) and (2,1) but 1 ≠ 2.
(d) R is Transitive because for any numbers a, b, and c, if (a, b), (b, c) ∈ R then (a, c) ∈ R.

c) { (2, 4), (4, 2) }
Solution:
(a) R is not reflexive: It doesn't contain (1,1) ,(2,2), (3,3) and (4,4).
(b) R is symmetric because R contains (2, 4) and (4, 2) ∈ R.
(c) R is not antisymmetric: we have (2,4) and (4,2) but 2 ≠ 4.
(d) R is not Transitive because (2,4), (4, 2) ∈ R but not (2,2) ∈ R.

d) { (1, 2), (2, 3), (3, 4) }
Solution:
(a) R is not reflexive: It doesn't contain (1,1) ,(2,2), (3,3) and (4,4).
(b) R is not symmetric because (1,2) ∈ R but not (2,1) ∈ R.
(c) R is antisymmetric: we have (a,b) but not (b,a) ∈ R.
(d) R is not Transitive because (1,2), (2, 3) ∈ R but not (1,3) ∈ R.

e) {(1, 1), (2, 2), (3, 3), (4, 4)}
Solution:
(a) R is reflexive: It contains (1,1) ,(2,2), (3,3) and (4,4).
(b) R is symmetric because R contains (a,b) and (b,a) ∈ R.
(c) R is antisymmetric: we have (a,b) and (b,a) ∈ R then a = b.
(d) R is Transitive because for any numbers a, b, and c, if (a, b), (b, c) ∈ R then (a, c) ∈ R.

f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}
Solution:

(a) R is not reflexive: It doesn't contain (1,1) ,(2,2), (3,3) and (4,4).
(b) R is not symmetric because (1,4) ∈ R but not (4,1) ∈ R.
(c) R is not antisymmetric: we have (1,3) and (3,1) ∈ R but 1 ≠ 3.
(d) R is not Transitive because we have (1,3) and (3,1) ∈ R but not (1,1) ∈ R.

31. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, Asymmetric, irreflexive and/or transitive, where (a, b) ∈ R if and only if:

a) a is taller than b.

Solution:

The relation R is **not reflexive**, because a person cannot be taller than himself/herself.

The relation R is **not symmetric**, because if person A is taller than person B, then person B is NOT taller than person A.

The relation R is **antisymmetric**, because $(a, b) \in R$ and $(b, a) \in R$ cannot occur at the same time (as one person is always taller than the other, but not the other way around).

The relation R is **transitive**, because if person A is taller than person B and if person B is taller than person C, then person A needs to be taller than person C as well.

b) a and b were born on the same day.

Solution:

The relation R is **reflexive**, because a person is born on the same day as himself/herself.

The relation R is **symmetric**, because if person A and person B are born on the same day, then person B is also born on the same day as person A.

The relation R is **not antisymmetric**, because if person A and person B are born on the same day and if person B and person A are born on the same day, then these two people are not necessarily the same person.

The relation R is **transitive**, because if person A and person B are born on the same day and if person B and person C are born on the same day, then person A and person C are also born on the same day.

c) a has the same first name as b.

Solution:

The relation R is **reflexive**, because a person has the same first name as himself/herself.

The relation R is **symmetric**, because if person A has the same first name as person B, then person B also has the same first name as person A.

The relation R is **not antisymmetric**, because if person A has the same first name as person B and if person B also has the same first name as person A, then these two people are not necessarily the same person (as there are different people with the same first name).

The relation R is **transitive**, because if person A has the same first name as person B and if person B also has the same first name as person C, then person A also has the same first name as person C.

d) a and b have a common grandparent.

Solution:

The relation R is **reflexive**, because a person has the same grandparents as himself/herself.

The relation R is **symmetric**, because if person A and person B have a common grandparent, then person B and person A also have a common grandparent.

The relation R is **not antisymmetric**, because if person A and person B have a common grandparent and if person B and person A have a common grandparent, then these two people are not necessarily the same person (as there are different people with the same grandparents).

The relation R is **not transitive**, because if person A and person B have a common grandparent and if person B and person C have a common grandparent, then person A and person C do not necessarily have a common grandparent (for example, the common grandparent of A and B can be from person B's father's side of the family, while the common grandparent of B and C can be from person B's mother's side of the family).

(a) Antisymmetric, Irreflexive, Asymmetric and Transitive
(b) Reflexive, Symmetric and Transitive
(c) Reflexive, Symmetric and Transitive
(d) Reflexive and Symmetric

32. Give an example of a relation on a set that is
a) both symmetric and antisymmetric.
Solution: { (1,1), (2,2), (3,3), (4,4) }

b) neither symmetric nor antisymmetric.
Solution: { (1,2), (2,1), (3,4) }

33. Consider these relations on the set of real numbers: A= {1,2,3}
R1 = {(a, b) ∈ R | a > b}, the "greater than" relation,
R2 = {(a, b) ∈ R | a ≥ b}, the "greater than or equal to "relation,
R3 = {(a, b) ∈ R | a < b}, the "less than" relation,
R4 = {(a, b) ∈ R | a ≤ b}, the "less than or equal to "relation,
R5 = {(a, b) ∈ R | a = b}, the "equal to" relation,
R6 = {(a, b) ∈ R | a ≠ b}, the "unequal to" relation.
Find:
a) R2 ∪ R4. b) R3 ∪ R6. c) R3 ∩ R6. d) R4 ∩ R6. e) R3 − R6. f) R6 − R3.
g) R2 ⊕ R6. h) R3 ⊕ R5. i) R2 ◦ R1. j) R6 ◦ R6.
Solution:
R1= { (2,1), (3,1), (3,2) } R2= { (1,1), (2,2), (3,3), (2,1), (3,1), (3,2) }
R3= { (1,2), (1,3), (2,3) } R4= { (1,1), (2,2), (3,3), (1,2), (1,3), (2,3) }
R5= { (1,1), (2,2), (3,3) } R6= {(1,2), (1,3), (2,1), (2,3), (3,1), (3,2) }

a) R2 ∪ R4 = { (1,1), (2,2), (3,3), (2,1), (3,1), (3,2) (1,2), (1,3), (2,3) }
b) R3 ∪ R6= {(1,2), (1,3), (2,1), (2,3), (3,1), (3,2) }
c) R3 ∩ R6 = { (1,2), (1,3), (2,3) }
d) R4 ∩ R6 = { (1,2), (1,3), (2,3) }
e) R3 − R6 = { } OR Φ
f) R6 − R3 = { (2,1), (3,1), (3,2) }

g) R2 $\oplus$ R6 = { (1,1), (2,2), (3,3), (1,2), (1,3), (2,3) }
h) R3 $\oplus$ R5 = { (1,1), (2,2), (3,3), (1,2), (1,3), (2,3) }
i) R2 $\circ$ R1 = { (2,1), (3,1), (3,2) }
j) R6 $\circ$ R6 = { (1,1), (2,2), (3,3), (2,1), (3,1), (3,2) (1,2), (1,3), (2,3) }

34. (a) Represent each of these relations on { 1 , 2 , 3 } with a matrix(with the elements of this set listed in increasing order).

i) { (1 , 1), (1 , 2), (1 , 3) }

Solution: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

ii) { (1 , 2), (2 , 1), (2 , 2) ,(3 , 3) }

Solution: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii) { (1 , 1), (1 , 2), (1 , 3) , (2 , 2), (2 , 3) ,(3 , 3) }

Solution: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

iv) { (1 , 3), (3 , 1) }

Solution: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) List the ordered pairs in the relations on { 1 , 2 , 3 } corresponding to these matrices(where rows and columns correspond to the integers listed in increasing order).

(i) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Solution: R = { (1,1), (1,3), (2,2), (3,1), (3,3) }

(ii) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Solution: R = { (1,2), (2,2), (3,2) }

(iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Solution: R = { (1,1), (1,2), (1,3), (2,1), (2,3), (3,1),(3,2), (3,3) }

35. (a) Suppose that *R* is the relation on the set of strings of English letters such that *aRb* if and only if *l*(*a*) = *l*(*b*), where *l*(*x*) is the length of the string *x*. Is *R* an equivalence relation?

Solution:

Show that all of the properties of an equivalence relation hold.

- Reflexivity: Because l(a) = l(a), it follows that aRa for all strings a.
- Symmetry: Suppose that aRb. Since l(a) = l(b), l(b) = l(a) also holds and bRa.
- Transitivity: Suppose that aRb and bRc. Since l(a) = l(b),and l(b) = l(c), l(a) = l(a) also holds and aRc.

(b) Let *m* be an integer with *m* > 1. Show that the relation *R* = {(*a*,*b*) | *a* ≡ *b* (mod *m*)} is an equivalence relation on the set of integers.
Solution:
Recall that a ≡ b (mod m) if and only if m divides a − b.

- Reflexivity: a ≡ a (mod m) since a − a = 0 is divisible by m since 0 = 0 · m.
- Symmetry: Suppose that a ≡ b (mod m). Then a − b is divisible by m, and so a − b = km, where k is an integer. It follows that b − a = (− k) m, so b ≡ a (mod m).
- Transitivity: Suppose that a ≡ b (mod m) and b ≡ c (mod m). Then m divides both a − b and b − c. Hence, there are integers k and l with a − b = km and b − c = lm. We obtain by adding the equations: a − c = (a − b) + (b − c) = km + lm = (k + l) m. Therefore, a ≡ c (mod m).

36. Find the first five terms of the sequence for each of the following general terms where n >0.

(i) $2^n - 1$
Solution:
1,2,4,8,16 are the first five terms of the given sequence.

(ii) $10 - \frac{3}{2}n$
Solution:
$\frac{17}{2}, 7, \frac{11}{2}, 4, \frac{5}{2}$ are the first five terms.

(iii) $\frac{(-1)^n}{n^2}$
Solution:
$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$ are the first five terms.

(iv) $\frac{3n+4}{2n-1}$
Solution: $7, \frac{10}{3}, \frac{13}{5}, \frac{16}{7}, \frac{19}{9}$ are the first five terms.

(b) Identify the following Sequence as Arithmetic or Geometric Sequence then find the indicated term.

(i) -15, -22, -29, -36, ……; 11^{th} term.
Solution:
Here common difference (d) = -7
$T_n = a + (n-1)d$; $T_{11} = -15 + (11-1)(-7) = -85$

(ii) a - 42b, a - 39b, a - 36b, a - 33b, ….; 15^{th} term.
Solution:
Here common difference (d) = 3b
$T_n = a + (n-1)d$; $T_{15} = a - 42b + (15-1)(3b) = a$

(iii) 4, 3, $\frac{9}{4}$, ...; 17^{th} term
Solution:
Here common ratio (r) = $\frac{3}{4}$
$T_n = ar^{n-1}$; $T_{17} = 4(\frac{3}{4})^{17-1} = \frac{3^{16}}{4^{15}}$

(iv) 32, 16, 8, ….; 9^{th} term
Solution:
Here common ratio (r) = $\frac{1}{2}$
$T_n = ar^{n-1}$; $T_{17} = 32(\frac{1}{2})^{9-1} = \frac{1}{8}$

37. **Find the G.P in which:**

(i) T_3 = 10 and T_5 = $2\frac{1}{2}$

Solution:

Since $T_n = ar^{n-1}$

$T_3 = ar^2 = 10$ **----(i)** $T_5 = ar^4 = \frac{5}{2}$ **-----(ii)**

Now, dividing (ii) by dividing (i), we get r = $\pm\frac{1}{2}$ and putting it in (i) we get a = 40.

Now the required G.P is 40,20,10,5, $\frac{5}{2}$,... OR 40,$-$20,10, $-$5, $\frac{5}{2}$, ...

(ii) T_5 = 8 and T_8 = $-\frac{64}{27}$

Solution:

Since $T_n = ar^{n-1}$

$T_5 = ar^4 = 8$ **---- (i)** $T_8 = ar^7 = -\frac{64}{27}$ **-----(ii)**

Now, dividing (ii) by dividing (i) we get r = $-\frac{2}{3}$ and putting it in (i) we get a = $\frac{81}{2}$.

Now the required G.P is $\frac{81}{2}, -27, 18, -12, 8$,...

(b) Find the A.P in which:

(i) T_4 = 7 and T_{16} = 31

Solution:

Since $T_n = a + (n-1)d$;

$T_4 = a + 3d = 7$**......(i)** $T_{16} = a + 15d = 31$**......(ii)**

Now subtracting (ii) from (i), we get d = 2 and putting it in (i) we get a = 1.

Now the required A.P is 1,3,5,7,9, 11,...

(ii) T_5 = 86 and T_{10} = 146

Solution:

Since $T_n = a + (n-1)d$;

$T_5 = a + 4d = 86$**......(i)** $T_{10} = a + 9d = 146$**......(ii)**

Now subtracting (ii) from (i), we get d = 12 and putting it in (i) we get a = 38.

Now the required A.P is 38,50,62,74,86,…

38. **How many numbers are there between 256 and 789 that are divisible by 7. Also find their sum.**

Solution:

First, we find the A.P with the common difference (d)= 7

259,266,273, 280, ,……. 784

Since $T_n = a + (n-1)d$; $784 = 259 + (n-1)(7)$; **n = 76.**

Now for Sum; $S_n = \frac{n}{2}[2a + (n-1)d]$;

$S_{76} = \frac{76}{2}[2(259) + (76-1)(7)]$ **= 39,634.**

(b) Find the sum to n terms of an A.P whose first term is $\frac{1}{n}$ and the last term is $\frac{n^2-n+1}{n}$.

Solution:

Since, $S_n = \frac{n}{2}[2a + (n-1)d]$ ------ (i)

1st we have to find "d"

Now, $T_n = a + (n-1)d$

$$\frac{n^2 - n + 1}{n} = \frac{1}{n} + (n-1)d$$

Finally, d = 1. Hence putting it in we get,

$S_n = \frac{n^2-n+2}{2}$.

39. (a) Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = \frac{1}{j}$ for $j = 1, 2, 3\ldots.$

Solution:

The lower limit for the index of summation is 1, and the upper limit is 100. We write this sum as $\sum_{j=1}^{100} \frac{1}{j}$.

(b) What is the value of:

(i) $\sum_{k=4}^{8}(-1)^k$.

Solution:

$= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$
$= 1 + (-1) + 1 + (-1) + 1 = 1.$

(ii) $\sum_{j=1}^{5}(j)^2$.
Solution:
$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2$
$= 1 + 4 + 9 + 16 + 25 = 55.$

40. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = -2a_{n-1}$, $a_0 = -1$

Solution:

$$
\begin{aligned}
a_0 &= -1 \\
a_1 &= -2a_0 = -2(-1) = 2 \\
a_2 &= -2a_1 = -2(2) = -4 \\
a_3 &= -2a_2 = -2(-4) = 8 \\
a_4 &= -2a_3 = -2(8) = -16 \\
a_5 &= -2a_4 = -2(-16) = 32
\end{aligned}
$$

b) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

Solution:

$$
\begin{aligned}
a_0 &= 2 \\
a_1 &= -1 \\
a_2 &= a_1 - a_0 = -1 - 2 = -3 \\
a_3 &= a_2 - a_1 = -3 - (-1) = -2 \\
a_4 &= a_3 - a_2 = -2 - (-3) = 1 \\
a_5 &= a_4 - a_3 = 1 - (-2) = 3
\end{aligned}
$$

c) $a_n = 3a^2_{n-1}$, $a_0 = 1$

Solution:

$$a_0 = 1$$
$$a_1 = 3a_0^2 = 3(1^2) = 3$$
$$a_2 = 3a_1^2 = 3(3^2) = 3(9) = 27$$
$$a_3 = 3a_2^2 = 3(27^2) = 3(729) = 2187$$
$$a_4 = 3a_3^2 = 3(2187^2) = 3(4782969) = 14348907$$
$$a_5 = 3a_4^2 = 3(14348907^2) = 3(205891132094649) = 617673396283947$$

d) $a_n = na_{n-1} + a^2_{n-2}$, $a_0 = -1$, $a_1 = 0$

Solution:

$$a_0 = -1$$
$$a_1 = 0$$
$$a_2 = 2a_1 + a_0^2 = 2(0) + (-1)^2 = 0 + 1 = 1$$
$$a_3 = 3a_2 + a_1^2 = 3(1) + 0^2 = 3 + 0 = 3$$
$$a_4 = 4a_3 + a_2^2 = 4(3) + 1^2 = 12 + 1 = 13$$
$$a_5 = 5a_4 + a_3^2 = 5(13) + 3^2 = 65 + 9 = 74$$

41. As we have discussed, the practical application of all the topics in the class. Now you are required to submit at least two real world applications of the following topics.

(a) Propositional Logic

(b) Predicates and quantifiers

(c) Sets

(d) Functions

(e) Relations

(f) Sequence and Series

(g) Graph theory

(h) Trees