CHAPTER 6: INNER PRODUCT SPACES

6.1 Inner Products

1. **(a)** $\langle \mathbf{u},\mathbf{v}\rangle = 2(1)(3)+3(1)(2)=12$

(b) $\langle k\mathbf{v},\mathbf{w}\rangle = 2\big((3)(3)\big)(0)+3\big((3)(2)\big)(-1)=-18$

(c) $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = 2(1+3)(0)+3(1+2)(-1)=-9$

(d) $\|\mathbf{v}\| = \langle \mathbf{v},\mathbf{v}\rangle^{1/2} = \left[2(3)(3)+3(2)(2)\right]^{1/2} = \sqrt{30}$

(e) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \big\langle(-2,-1),(-2,-1)\big\rangle^{1/2} = \left[2(-2)(-2)+3(-1)(-1)\right]^{1/2} = \sqrt{11}$

(f) $\|\mathbf{u}-k\mathbf{v}\| = \big\langle(-8,-5),(-8,-5)\big\rangle^{1/2} = \left[2(-8)(-8)+3(-5)(-5)\right]^{1/2} = \sqrt{203}$

3. **(a)** $\langle \mathbf{u},\mathbf{v}\rangle = \left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}8\\5\end{bmatrix} = 34$

(b) $\langle k\mathbf{v},\mathbf{w}\rangle = \left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}9\\6\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}24\\15\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\end{bmatrix} = -39$

(c) $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}11\\7\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\end{bmatrix} = -18$

(d) $\|\mathbf{v}\| = \langle \mathbf{v},\mathbf{v}\rangle^{1/2} = \left[\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}8\\5\end{bmatrix}\cdot\begin{bmatrix}8\\5\end{bmatrix}\right)^{1/2} = \sqrt{89}$

(e) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \left[\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-2\\-1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-2\\-1\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}-5\\-3\end{bmatrix}\cdot\begin{bmatrix}-5\\-3\end{bmatrix}\right)^{1/2} = \sqrt{34}$

(f) $\|\mathbf{u}-k\mathbf{v}\| = \left[\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-8\\-5\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-8\\-5\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}-21\\-13\end{bmatrix}\cdot\begin{bmatrix}-21\\-13\end{bmatrix}\right)^{1/2} = \sqrt{610}$

5. $\begin{bmatrix}\sqrt{2}&0\\0&\sqrt{3}\end{bmatrix}$

7. $\langle \mathbf{u},\mathbf{v}\rangle = \left(\begin{bmatrix}4&1\\2&-3\end{bmatrix}\begin{bmatrix}0\\-3\end{bmatrix}\right)\cdot\left(\begin{bmatrix}4&1\\2&-3\end{bmatrix}\begin{bmatrix}6\\2\end{bmatrix}\right) = \begin{bmatrix}-3\\9\end{bmatrix}\cdot\begin{bmatrix}26\\6\end{bmatrix} = -24$

9. If $\mathbf{u}=U$ and $\mathbf{v}=V$ then $\langle\mathbf{u},\mathbf{v}\rangle=\operatorname{tr}\left(U^TV\right)=\operatorname{tr}\left(\begin{bmatrix}1&13\\10&2\end{bmatrix}\right)=3$.

11. $\langle\mathbf{p},\mathbf{q}\rangle=(-2)(4)+(1)(0)+(3)(-7)=-29$

13. $\begin{bmatrix}\sqrt{3}&0\\0&\sqrt{5}\end{bmatrix}$

15. $\langle\mathbf{p},\mathbf{q}\rangle=p(-2)q(-2)+p(-1)q(-1)+p(0)q(0)+p(1)q(1)$
$=(-10)(5)+(-2)(2)+(0)(1)+(2)(2)=-50$

17. $\|\mathbf{u}\|=\langle\mathbf{u},\mathbf{u}\rangle^{1/2}=\left[2(-3)(-3)+3(2)(2)\right]^{1/2}=\sqrt{30}$

$d(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=\langle(-4,-5),(-4,-5)\rangle^{1/2}=\left[2(-4)(-4)+3(-5)(-5)\right]^{1/2}=\sqrt{107}$

19. $\|\mathbf{p}\|=\langle\mathbf{p},\mathbf{p}\rangle^{1/2}=\sqrt{(-2)^2+1^2+3^2}=\sqrt{14}$; $d(\mathbf{p},\mathbf{q})=\|\mathbf{p}-\mathbf{q}\|=\sqrt{(-6)^2+1^2+10^2}=\sqrt{137}$

21. If $\mathbf{u}=U$ and $\mathbf{v}=V$ then $\|U\|=\langle\mathbf{u},\mathbf{u}\rangle^{1/2}=\sqrt{\operatorname{tr}\left(U^TU\right)}=\sqrt{\operatorname{tr}\left(\begin{bmatrix}25&26\\26&68\end{bmatrix}\right)}=\sqrt{93}$ and

$d(U,V)=\|U-V\|=\langle\mathbf{u}-\mathbf{v},\mathbf{u}-\mathbf{v}\rangle^{1/2}=\sqrt{\operatorname{tr}\left((U-V)^T(U-V)\right)}=\sqrt{\operatorname{tr}\left(\begin{bmatrix}25&1\\1&74\end{bmatrix}\right)}=\sqrt{99}=3\sqrt{11}$.

23. $\|\mathbf{p}\|=\langle\mathbf{p},\mathbf{p}\rangle^{1/2}=\sqrt{\left[p(-2)\right]^2+\left[p(-1)\right]^2+\left[p(0)\right]^2+\left[p(1)\right]^2}=\sqrt{(-10)^2+(-2)^2+0^2+2^2}=6\sqrt{3}$

$d(\mathbf{p},\mathbf{q})=\|\mathbf{p}-\mathbf{q}\|=\sqrt{\left[p(-2)-q(-2)\right]^2+\left[p(-1)-q(-1)\right]^2+\left[p(0)-q(0)\right]^2+\left[p(1)-q(1)\right]^2}$

$=\sqrt{(-15)^2+(-4)^2+(-1)^2+0^2}=11\sqrt{2}$

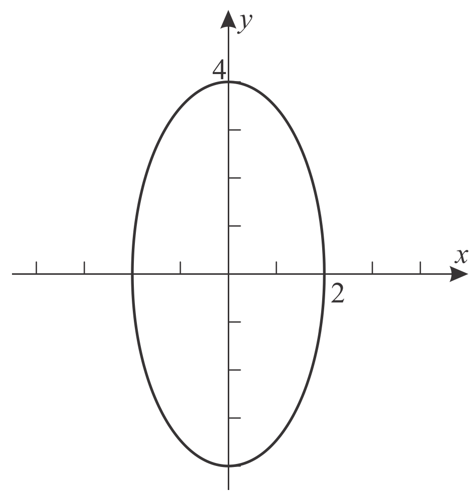
25. $\|\mathbf{u}\|=\langle\mathbf{u},\mathbf{u}\rangle^{1/2}=\left[\left(\begin{bmatrix}4&0\\3&5\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}\right)\cdot\left(\begin{bmatrix}4&0\\3&5\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}\right)\right]^{1/2}=\left(\begin{bmatrix}-4\\7\end{bmatrix}\cdot\begin{bmatrix}-4\\7\end{bmatrix}\right)^{1/2}=\sqrt{65}$

$d(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=\left[\left(\begin{bmatrix}4&0\\3&5\end{bmatrix}\begin{bmatrix}-3\\-3\end{bmatrix}\right)\cdot\left(\begin{bmatrix}4&0\\3&5\end{bmatrix}\begin{bmatrix}-3\\-3\end{bmatrix}\right)\right]^{1/2}=\left(\begin{bmatrix}-12\\-24\end{bmatrix}\cdot\begin{bmatrix}-12\\-24\end{bmatrix}\right)^{1/2}=12\sqrt{5}$

27. **(a)** $\langle 2\mathbf{v}-\mathbf{w},3\mathbf{u}+2\mathbf{w}\rangle=\langle 2\mathbf{v},3\mathbf{u}+2\mathbf{w}\rangle-\langle\mathbf{w},3\mathbf{u}+2\mathbf{w}\rangle=\langle 2\mathbf{v},3\mathbf{u}\rangle+\langle 2\mathbf{v},2\mathbf{w}\rangle-\langle\mathbf{w},3\mathbf{u}\rangle-\langle\mathbf{w},2\mathbf{w}\rangle$
$=6\langle\mathbf{v},\mathbf{u}\rangle+4\langle\mathbf{v},\mathbf{w}\rangle-3\langle\mathbf{w},\mathbf{u}\rangle-2\langle\mathbf{w},\mathbf{w}\rangle=6\langle\mathbf{u},\mathbf{v}\rangle+4\langle\mathbf{v},\mathbf{w}\rangle-3\langle\mathbf{u},\mathbf{w}\rangle-2\|\mathbf{w}\|^2$
$=6(2)+4(-6)-3(-3)-2(49)=-101$

(b) $\|\mathbf{u}+\mathbf{v}\|=\sqrt{\langle\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}\rangle}=\sqrt{\langle\mathbf{u},\mathbf{u}+\mathbf{v}\rangle+\langle\mathbf{v},\mathbf{u}+\mathbf{v}\rangle}=\sqrt{\langle\mathbf{u},\mathbf{u}\rangle+\langle\mathbf{u},\mathbf{v}\rangle+\langle\mathbf{v},\mathbf{u}\rangle+\langle\mathbf{v},\mathbf{v}\rangle}$
$=\sqrt{\|\mathbf{u}\|^2+2\langle\mathbf{u},\mathbf{v}\rangle+\|\mathbf{v}\|^2}=\sqrt{1+2(2)+4}=3$

29. If $\mathbf{u}=(x,y)$ then $\|\mathbf{u}\|=\langle\mathbf{u},\mathbf{u}\rangle^{1/2}=\sqrt{\frac{1}{4}x^2+\frac{1}{16}y^2}$, so the equation of the unit circle is $\frac{x^2}{4}+\frac{y^2}{16}=1$.



31. $\langle\mathbf{u},\mathbf{v}\rangle=\frac{1}{9}u_1v_1+u_2v_2$ (see Example 3)

33. Axiom 2 does not hold, e.g., with $\mathbf{u}=\mathbf{v}=\mathbf{w}=(1,0,0)$ we have $\langle\mathbf{u}+\mathbf{v},\mathbf{w}\rangle=4$ but $\langle\mathbf{u},\mathbf{w}\rangle+\langle\mathbf{v},\mathbf{w}\rangle=1+1=2$; Axiom 3 does not hold either, e.g., with $\mathbf{u}=\mathbf{v}=(1,0,0)$ and $k=2$, $\langle k\mathbf{u},\mathbf{v}\rangle=4$ does not equal $k\langle\mathbf{u},\mathbf{v}\rangle=2$; This is not an inner product on R^3.

35. By Definition 1, Definition 2, and Theorem 6.1.2, we have

$$\begin{aligned}\langle 2\mathbf{v}-4\mathbf{u},\mathbf{u}-3\mathbf{v}\rangle &= \langle 2\mathbf{v}-4\mathbf{u},\mathbf{u}\rangle-\langle 2\mathbf{v}-4\mathbf{u},3\mathbf{v}\rangle\\ &= \langle 2\mathbf{v},\mathbf{u}\rangle-\langle 4\mathbf{u},\mathbf{u}\rangle-\langle 2\mathbf{v},3\mathbf{v}\rangle+\langle 4\mathbf{u},3\mathbf{v}\rangle\\ &= 2\langle\mathbf{v},\mathbf{u}\rangle-4\langle\mathbf{u},\mathbf{u}\rangle-6\langle\mathbf{v},\mathbf{v}\rangle+12\langle\mathbf{u},\mathbf{v}\rangle\\ &= 2\langle\mathbf{u},\mathbf{v}\rangle-4\langle\mathbf{u},\mathbf{u}\rangle-6\langle\mathbf{v},\mathbf{v}\rangle+12\langle\mathbf{u},\mathbf{v}\rangle\\ &= 14\langle\mathbf{u},\mathbf{v}\rangle-4\|\mathbf{u}\|^2-6\|\mathbf{v}\|^2.\end{aligned}$$

37. **(a)** $\langle\mathbf{p},\mathbf{q}\rangle=\left(\int_{-1}^{1}x^2\,dx\right)=\left(\frac{x^3}{3}\Big]_{-1}^{1}\right)=\frac{2}{3}$

(b) $d(\mathbf{p},\mathbf{q})=\|\mathbf{p}-\mathbf{q}\|=\left(\int_{-1}^{1}\left(1-x^2\right)^2\,dx\right)^{1/2}=\left(\left(x-\frac{2x^3}{3}+\frac{x^5}{5}\right)\Big]_{-1}^{1}\right)^{1/2}=\sqrt{\frac{16}{15}}=\frac{4}{\sqrt{15}}$

(c) $\|\mathbf{p}\|=\langle\mathbf{p},\mathbf{p}\rangle^{1/2}=\left(\int_{-1}^{1}1\,dx\right)^{1/2}=\left(x\Big]_{-1}^{1}\right)^{1/2}=\sqrt{2}$

(d) $\|\mathbf{q}\|=\langle\mathbf{q},\mathbf{q}\rangle^{1/2}=\left(\int_{-1}^{1}x^4\,dx\right)^{1/2}=\left(\frac{x^5}{5}\Big]_{-1}^{1}\right)^{1/2}=\sqrt{\frac{2}{5}}$

39. $\langle\mathbf{f},\mathbf{g}\rangle=\int_0^1\cos 2\pi x\,\sin 2\pi x\,dx=\frac{1}{2\pi}\frac{(\sin 2\pi x)^2}{2}\Big]_0^1=0$ (substituted $u=\sin 2\pi x$)

41. Part (a) follows directly from Definition 2 and Axiom 4 of Definition 1.

To prove part (b), write

$$\|k\mathbf{v}\| \underset{\text{Def.2}}{=} \sqrt{\langle k\mathbf{v},k\mathbf{v}\rangle} \underset{\text{Axiom 3}}{=} \sqrt{k\langle \mathbf{v},k\mathbf{v}\rangle} \underset{\text{Axiom 1}}{=} \sqrt{k\langle k\mathbf{v},\mathbf{v}\rangle} \underset{\text{Axiom 3}}{=} \sqrt{k^2\langle \mathbf{v},\mathbf{v}\rangle} \underset{\text{Def.2}}{=} |k|\,\|\mathbf{v}\|$$

43. **(b)** k_1 and k_2 must both be positive in order for $\langle \mathbf{u},\mathbf{v}\rangle$ to satisfy the positivity axiom. (Refer to the discussion following Theorem 6.1.1.)

45. By using Definition 2 and Axioms 1, 2, and 3 of Definition 1, we have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}\rangle+\langle \mathbf{u}-\mathbf{v},\mathbf{u}-\mathbf{v}\rangle \\ &= \langle \mathbf{u},\mathbf{u}\rangle+\langle \mathbf{u},\mathbf{v}\rangle+\langle \mathbf{v},\mathbf{u}\rangle+\langle \mathbf{v},\mathbf{v}\rangle+\langle \mathbf{u},\mathbf{u}\rangle-\langle \mathbf{u},\mathbf{v}\rangle-\langle \mathbf{v},\mathbf{u}\rangle+\langle \mathbf{v},\mathbf{v}\rangle \\ &= 2\langle \mathbf{u},\mathbf{u}\rangle+2\langle \mathbf{v},\mathbf{v}\rangle \\ &= 2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2.\end{aligned}$$

True-False Exercises

(a) True. The dot product is the special case of the weighted inner product with all the weights equal to 1.

(b) False. For example, if $\langle \mathbf{u},\mathbf{v}\rangle=\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}=(1,1)$, and $\mathbf{v}=(-2,1)$ then $\langle \mathbf{u},\mathbf{v}\rangle=-1$.

(c) True. This follows from Axioms 1 and 2 of Definition 1 since $\langle \mathbf{u},\mathbf{v}+\mathbf{w}\rangle=\langle \mathbf{v}+\mathbf{w},\mathbf{u}\rangle=\langle \mathbf{v},\mathbf{u}\rangle+\langle \mathbf{w},\mathbf{u}\rangle$.

(d) True. This follows from Axiom 3 of Definition 1 as well as part (e) of Theorem 6.1.2.

(e) False. For example, if $\langle \mathbf{u},\mathbf{v}\rangle=\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}=(1,1)$, and $\mathbf{v}=(-1,1)$ then $\langle \mathbf{u},\mathbf{v}\rangle=0$ even though both vectors are nonzero.

(f) True. By Definition 2, $\|\mathbf{v}\|^2=\langle \mathbf{v},\mathbf{v}\rangle$ so by Axiom 4 of Definition 1, $\|\mathbf{v}\|^2=0$ implies $\mathbf{v}=\mathbf{0}$.

(g) False. A must be invertible; otherwise $A\mathbf{v}=\mathbf{0}$ has nontrivial solutions $\mathbf{v}\neq\mathbf{0}$ even though $\langle \mathbf{v},\mathbf{v}\rangle=A\mathbf{v}\cdot A\mathbf{v}=\mathbf{0}$ which would violate Axiom 4 of Definition 1.

6.2 Angle and Orthogonality in Inner Product Spaces

1. **(a)** $\cos\theta=\frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}=\frac{(1)(2)+(-3)(4)}{\sqrt{1^2+(-3)^2}\sqrt{2^2+4^2}}=-\frac{10}{\sqrt{10}\sqrt{20}}=-\frac{10}{\sqrt{200}}=-\frac{10}{10\sqrt{2}}=-\frac{1}{\sqrt{2}}$

(b) $\cos\theta=\frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}=\frac{(-1)(2)+(5)(4)+(2)(-9)}{\sqrt{(-1)^2+5^2+2^2}\sqrt{2^2+4^2+(-9)^2}}=0$

(c) $\cos\theta=\frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}=\frac{(1)(-3)+(0)(-3)+(1)(-3)+(0)(-3)}{\sqrt{1^2+0^2+1^2+0^2}\sqrt{(-3)^2+(-3)^2+(-3)^2+(-3)^2}}=-\frac{6}{\sqrt{2}\sqrt{36}}=-\frac{1}{\sqrt{2}}$

3. $\cos\theta=\frac{\langle \mathbf{p},\mathbf{q}\rangle}{\|\mathbf{p}\|\|\mathbf{q}\|}=\frac{(-1)(2)+(5)(4)+(2)(-9)}{\sqrt{(-1)^2+5^2+2^2}\sqrt{2^2+4^2+(-9)^2}}=0$

5. $\cos\theta=\frac{\langle U,V\rangle}{\|U\|\|V\|}=\frac{\operatorname{tr}(U^TV)}{\sqrt{\operatorname{tr}(U^TU)}\sqrt{\operatorname{tr}(V^TV)}}=\frac{(2)(3)+(6)(2)+(1)(1)+(-3)(0)}{\sqrt{2^2+6^2+1^2+(-3)^2}\sqrt{3^2+2^2+1^2+0^2}}=\frac{19}{\sqrt{50}\sqrt{14}}=\frac{19}{10\sqrt{7}}$

7. **(a)** orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = -4 + 6 - 2 = 0$

(b) not orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = -2 - 2 - 2 = -6 \neq 0$

(c) orthogonal: $\langle \mathbf{u}, \mathbf{v} \rangle = (a)(-b) + (b)(a) = 0$

9. $\langle \mathbf{p}, \mathbf{q} \rangle = (-1)(0) + (-1)(2) + (2)(1) = 0$

11. $\langle U, V \rangle = (2)(-3) + (1)(0) + (-1)(0) + (3)(2) = 0$

13. The vectors are not orthogonal with respect to the Euclidean inner product since $\langle \mathbf{u}, \mathbf{v} \rangle = (1)(2) + (3)(-1) = -1 \neq 0$. Using the weighted inner product instead yields $\langle \mathbf{u}, \mathbf{v} \rangle = 2(1)(2) + k(3)(-1) = 4 - 3k$, so the vectors are orthogonal with respect to this inner product if $k = \frac{4}{3}$.

15. The orthogonality of the two vectors implies $(w_1)(1)(2) + (w_2)(2)(-4) = 0$. The weights must be positive numbers such that $w_1 = 4w_2$.

17. Orthogonality of $\mathbf{p}_1$ and $\mathbf{p}_3$ implies $\langle \mathbf{p}_1, \mathbf{p}_3 \rangle = (2)(1) + (k)(2) + (6)(3) = 2k + 20 = 0$ so $k = -10$. Likewise, orthogonality of $\mathbf{p}_2$ and $\mathbf{p}_3$ implies $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = (l)(1) + (5)(2) + (3)(3) = l + 19$ so $l = -19$.
Substituting the values of k and l obtained above yields the polynomials $\mathbf{p}_1 = 2 - 10x + 6x^2$ and $\mathbf{p}_2 = -19 + 5x + 3x^2$ which are not orthogonal since $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = (2)(-19) + (-10)(5) + (6)(3) = -70 \neq 0$. We conclude that no scalars k and l exist that make the three vectors mutually orthogonal.

19. $\langle \mathbf{p}, \mathbf{q} \rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2) = (-2)(4) + (0)(0) + (2)(4) = 0$

21. $|\langle \mathbf{u}, \mathbf{v} \rangle| = |2(1)(2) + 3(0)(1) + (3)(-1)| = |1| = 1;$

$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{2(1)(1) + 3(0)(0) + (3)(3)} = \sqrt{11}$;

$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{2(2)(2) + 3(1)(1) + (-1)(-1)} = \sqrt{12}$;

since $\| \mathbf{u} \| \| \mathbf{v} \| = \sqrt{132} \geq 1 = |\langle \mathbf{u}, \mathbf{v} \rangle|$, we conclude that the Cauchy-Schwarz inequality holds.

23. $|\langle \mathbf{p}, \mathbf{q} \rangle| = |(-1)(2) + (2)(0) + (1)(-4)| = |-6| = 6$;

$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{(-1)(-1) + (2)(2) + (1)(1)} = \sqrt{6}$;

$\|\mathbf{q}\| = \sqrt{\langle \mathbf{q}, \mathbf{q} \rangle} = \sqrt{(2)(2) + (0)(0) + (-4)(-4)} = \sqrt{20}$;

since $\|\mathbf{p}\| \|\mathbf{q}\| = \sqrt{120} \geq \sqrt{36} = 6 = |\langle \mathbf{p}, \mathbf{q} \rangle|$, we conclude that the Cauchy-Schwarz inequality holds.

25. By inspection, $\langle \mathbf{u}, \mathbf{w}_1 \rangle = -2 \neq 0$. Since $\mathbf{u}$ is not orthogonal to $\mathbf{w}_1$, it is not orthogonal to the subspace.

27. Begin by forming a matrix A whose rows are the given vectors:

$A=\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution of the homogeneous system $A\mathbf{x}=\mathbf{0}$ is $x_1=-s+\frac{2}{7}t$, $x_2=-s-\frac{4}{7}t$, $x_3=s$, $x_4=t$, therefore $\mathbf{x}=s(-1,-1,1,0)+t\left(\frac{2}{7},-\frac{4}{7},0,1\right)$.

A basis for the orthogonal complement is formed by vectors $(-1,-1,1,0)$ and $\left(\frac{2}{7},-\frac{4}{7},0,1\right)$.

29. **(a)** Every vector in W has form $(x,y)=(x,2x)$, i.e., $W=\text{span}\{(1,2)\}$. By inspection, all vectors in R^2 orthogonal to $(1,2)$ are scalar multiples of the vector $(2,-1)$. Eliminating t from $(x,y)=t(2,-1)=(2t,-t)$ we obtain $x=2(-y)$, i.e. $W^{\perp}$ can be represented using an equation $y=-\frac{1}{2}x$.

(An alternate method of solving this exercise is to follow the procedure of Example 6: letting $A=\begin{bmatrix}1 & 2\end{bmatrix}$, the general solution of $A\begin{bmatrix}x \\ y\end{bmatrix}=0$ is $x=-2t$, $y=t$. Eliminating t yields $y=-\frac{1}{2}x$.)

(b) $W^{\perp}$ will have dimension 1. A normal to the plane is $\mathbf{u}=(1,-2,-3)$, so $W^{\perp}$ will consist of all scalar multiples of $\mathbf{u}$ or $t\mathbf{u}=(t,-2t,-3t)$ so parametric equations for $W^{\perp}$ are $x=t$, $y=-2t$, $z=-3t$.

31. **(a)** $\langle \mathbf{p},\mathbf{q}\rangle=\int_0^1 x^3\,dx=\frac{x^4}{4}\Big]_0^1=\frac{1}{4}$

(b) $\|\mathbf{p}\|=\langle \mathbf{p},\mathbf{p}\rangle^{\frac{1}{2}}=\left(\int_0^1 x^2\,dx\right)^{\frac{1}{2}}=\left(\frac{x^3}{3}\Big]_0^1\right)^{\frac{1}{2}}=\frac{1}{\sqrt{3}}$

$\|\mathbf{q}\|=\langle \mathbf{q},\mathbf{q}\rangle^{1/2}=\left(\int_0^1 x^4\,dx\right)^{1/2}=\left(\frac{x^5}{5}\Big]_0^1\right)^{1/2}=\frac{1}{\sqrt{5}}$

33. **(a)** $\langle \mathbf{p},\mathbf{q}\rangle=\int_{-1}^1 (x^2-x)(x+1)\,dx=\int_{-1}^1 (x^3-x)\,dx=\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Big]_{-1}^1=0$

(b) $\|\mathbf{p}\|=\langle \mathbf{p},\mathbf{p}\rangle^{\frac{1}{2}}=\left(\int_{-1}^1 (x^2-x)^2\,dx\right)^{\frac{1}{2}}=\left(\left(\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}\right)\Big]_{-1}^1\right)^{\frac{1}{2}}=\frac{4}{\sqrt{15}}$

$\|\mathbf{q}\|=\langle \mathbf{q},\mathbf{q}\rangle^{1/2}=\left(\int_{-1}^1 (x+1)^2\,dx\right)^{1/2}=\left(\left(\frac{x^3}{3}+x^2+x\right)\Big]_{-1}^1\right)^{1/2}=\sqrt{\frac{8}{3}}=2\sqrt{\frac{2}{3}}$

35. **(a)** $\langle \mathbf{p},\mathbf{q}\rangle=\int_0^1 \left(\frac{1}{2}-x\right)dx=\left(\frac{1}{2}x-\frac{x^2}{2}\right)\Big]_0^1=0$

(b) $\|\mathbf{p}+\mathbf{q}\|^2=\int_0^1\left(\frac{3}{2}-x\right)^2 dx=\left(\frac{9}{4}x-\frac{3x^2}{2}+\frac{x^3}{3}\right)\Big]_0^1=\frac{13}{12}$; $\|\mathbf{p}\|^2=\int_0^1 (1)^2 dx=x\Big]_0^1=1$;

$\|\mathbf{q}\|^2=\int_0^1\left(\frac{1}{2}-x\right)^2 dx=\left(\frac{x}{4}-\frac{x^2}{2}+\frac{x^3}{3}\right)\Big]_0^1=\frac{1}{12}$; we conclude that $\|\mathbf{p}+\mathbf{q}\|^2=\frac{13}{12}=\|\mathbf{p}\|^2+\|\mathbf{q}\|^2$.

37. $\|\mathbf{u}-\mathbf{v}\|^2=\langle\mathbf{u}-\mathbf{v},\mathbf{u}-\mathbf{v}\rangle=\langle\mathbf{u},\mathbf{u}-\mathbf{v}\rangle-\langle\mathbf{v},\mathbf{u}-\mathbf{v}\rangle=\langle\mathbf{u},\mathbf{u}\rangle-\langle\mathbf{u},\mathbf{v}\rangle-\langle\mathbf{v},\mathbf{u}\rangle+\langle\mathbf{v},\mathbf{v}\rangle=1-0-0+1=2$ therefore $\|\mathbf{u}-\mathbf{v}\|=\sqrt{2}$.

39. Using the trigonometric identity $\cos\alpha\cos\beta=\frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)$ we obtain

$\langle\mathbf{f}_k,\mathbf{f}_l\rangle=\frac{1}{2}\int_0^{\pi}\cos((k-l)x)\,dx+\frac{1}{2}\int_0^{\pi}\cos((k+l)x)\,dx$ where both $k-l$ and $k+l$ are nonzero integers.

Substituting $u=(k-l)x$ in the first integral, and $t=(k+l)x$ in the second integral yields

$\langle\mathbf{f}_k,\mathbf{f}_l\rangle=\frac{1}{2}\frac{\sin((k-l)x)}{k-l}\Big]_0^{\pi}+\frac{1}{2}\frac{\sin((k+l)x)}{k+l}\Big]_0^{\pi}=0+0=0$ since $\sin(m\pi)=0$ for any integer m.

41. $\operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_r\}$ contains all linear combinations $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_r\mathbf{u}_r$ where $k_1, k_2, \ldots, k_r$ are arbitrary scalars. Let $\mathbf{v}\in\operatorname{span}\{\mathbf{u}_1,\ \mathbf{u}_2,\ \ldots,\ \mathbf{u}_r\}$.

$\langle\mathbf{w},\ \mathbf{v}\rangle=\langle\mathbf{w},\ k_1\mathbf{u}_1+k_2\mathbf{u}_2+\cdots+k_r\mathbf{u}_r\rangle=k_1\langle\mathbf{w},\ \mathbf{u}_1\rangle+k_2\langle\mathbf{w},\ \mathbf{u}_2\rangle+\cdots+k_r\langle\mathbf{w},\ \mathbf{u}_r\rangle=0+0+\cdots+0=0$

Thus if **w** is orthogonal to each vector $\mathbf{u}_1,\ \mathbf{u}_2,\ \ldots,\ \mathbf{u}_r$, then **w** must be orthogonal to every vector in $\operatorname{span}\{\mathbf{u}_1,\ \mathbf{u}_2,\ \ldots,\ \mathbf{u}_r\}$.

43. Suppose that **v** is orthogonal to every basis vector. Then, as in Exercise 41, **v** is orthogonal to the span of the set of basis vectors, which is all of *W*, hence **v** is in $W^{\perp}$. If **v** is not orthogonal to every basis vector, then **v** clearly cannot be in $W^{\perp}$. Thus $W^{\perp}$ consists of all vectors orthogonal to every basis vector.

47. Using the weighted Euclidean inner product of Formula (2) in Section 6.1, the desired inequality follows from the Cauchy-Schwarz inequality.

49. Using the inner product $\langle\mathbf{f},\mathbf{g}\rangle=\int_0^1 f(x)g(x)dx$, part (a) follows from the Cauchy-Schwarz inequality and part (b) follows from the triangle inequality (part (a) of Theorem 6.2.2).

51. **(a)** We are looking for all vectors $\mathbf{v}=(a,b)$ such that $\langle\mathbf{x},\mathbf{v}\rangle=a+b$ is equal to

$\langle T_A(\mathbf{x}),T_A(\mathbf{v})\rangle=\begin{bmatrix}2\\0\end{bmatrix}\cdot\begin{bmatrix}a+b\\-a+b\end{bmatrix}=2a+2b$. The equation $a+b=2a+2b$ yields $a+b=0$, i.e. $b=-a$.

Vectors that satisfy $\langle\mathbf{x},\mathbf{v}\rangle=\langle T_A(\mathbf{x}),T_A(\mathbf{v})\rangle$ must have a form $a(1,-1)$ where a is an arbitrary scalar.

(b) We are looking for all vectors $\mathrm{v}=(a,b)$ such that $\langle\mathbf{x},\mathbf{v}\rangle=2a+3b$ is equal to

$\langle T_A(\mathbf{x}),T_A(\mathbf{v})\rangle=\langle\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}a+b\\-a+b\end{bmatrix}\rangle=4a+4b$. The equation $2a+3b=4a+4b$ yields $2a+b=0$, i.e. $b=-2a$. Vectors that satisfy $\langle\mathbf{x},\mathbf{v}\rangle=\langle T_A(\mathbf{x}),T_A(\mathbf{v})\rangle$ must have a form $a(1,-2)$ where a is an arbitrary scalar.

True-False Exercises

(a) False. If **u** is orthogonal to every vector of a subspace *W*, then **u** is in $W^{\perp}$.

(b) True. $W\cap W^{\perp}=\{\mathbf{0}\}$.

(c) True. For any vector **w** in W, $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle = 0,$ so **u** + **v** is in $W^{\perp}$.

(d) True. For any vector **w** in W, $\langle k\mathbf{u},\ \mathbf{w}\rangle = k\langle \mathbf{u},\ \mathbf{w}\rangle = k(0) = 0,$ so $k\mathbf{u}$ is in $W^{\perp}$.

(e) False. If **u** and **v** are orthogonal $|\langle \mathbf{u},\ \mathbf{v}\rangle| = |\,0\,| = 0.$

(f) False. If **u** and **v** are orthogonal, $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ thus $\|\mathbf{u}+\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2} \ne \|\mathbf{u}\| + \|\mathbf{v}\|$

6.3 Gram-Schmidt Process; QR-Decomposition

1. **(a)** $\langle (0,1),\ (2,0)\rangle = 0+0 = 0$;

$\|(0,1)\| = 1$; $\|(2,0)\| = 2 \ne 1$;

The set is orthogonal, but is not orthonormal.

(b) $\left\langle \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2} + \frac{1}{2} = 0$;

$\left\|\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$; $\left\|\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

The set is orthogonal and orthonormal.

(c) $\left\langle \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2} - \frac{1}{2} = -1 \ne 0$;

The set is not orthogonal (therefore, it is not orthonormal either).

(d) $\langle (0,0),\ (0,1)\rangle = 0+0 = 0$;

$\|(0,0)\| = 0 \ne 1$; $\|(0,1)\| = 1$;

The set is orthogonal, but is not orthonormal.

3. **(a)** $\langle p_1(x),\ p_2(x)\rangle = \frac{2}{3}\left(\frac{2}{3}\right) - \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(-\frac{2}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$;

$\langle p_1(x),\ p_3(x)\rangle = \frac{2}{3}\left(\frac{1}{3}\right) - \frac{2}{3}\left(\frac{2}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$;

$\langle p_2(x),\ p_3(x)\rangle = \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) - \frac{2}{3}\left(\frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$;

The set is orthogonal.

(b) $\langle p_1(x),\ p_2(x)\rangle = 1(0) + 0\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = 0$; $\langle p_1(x),\ p_3(x)\rangle = 1(0) + 0(0) + 0(1) = 0$;

$\langle p_2(x),\ p_3(x)\rangle = 0(0) + \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}} \ne 0$;

The set is not orthogonal.

5. Let us denote the column vectors $\mathbf{u}_1 = (1,0,-1)$, $\mathbf{u}_2 = (2,0,2)$, and $\mathbf{u}_3 = (0,5,0)$. These vectors are orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2\rangle = 2+0-2 = 0$, $\langle \mathbf{u}_1, \mathbf{u}_3\rangle = 0+0+0 = 0$, and $\langle \mathbf{u}_2, \mathbf{u}_3\rangle = 0+0+0 = 0$.
It follows from Theorem 6.3.1 that the column vectors are linearly independent, therefore they form an orthogonal basis for the column space of A. We proceed to normalize each column vector:
$\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{1+0+1}}(1,0,-1) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$; $\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{4+0+4}}(2,0,2) = \left(\frac{2}{2\sqrt{2}}, 0, \frac{2}{2\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$;

$\frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{1}{\sqrt{0+25+0}}(0,5,0) = (0,\ 1,\ 0)$. A resulting orthonormal basis for the column space is $\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,\ 1,\ 0)\right\}$.

7. $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -\frac{12}{25} + \frac{12}{25} + 0 = 0$; $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 0+0+0 = 0$; $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0+0+0 = 0$;

$\|\mathbf{v}_1\| = \sqrt{\frac{9}{25} + \frac{16}{25} + 0} = 1$; $\|\mathbf{v}_2\| = \sqrt{\frac{16}{25} + \frac{9}{25} + 0} = 1$; $\|\mathbf{v}_3\| = \sqrt{0+0+1} = 1$;

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim(R^3) = 3$, this set forms a basis for R^3 by Theorem 4.6.4. This basis is orthonormal, so by Theorem 6.3.2(b),

$$\begin{aligned}
\mathbf{u} &= \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3 \\
&= \left(-\frac{3}{5} - \frac{8}{5} + 0\right)\mathbf{v}_1 + \left(\frac{4}{5} - \frac{6}{5} + 0\right)\mathbf{v}_2 + (0+0+2)\mathbf{v}_3 \\
&= -\frac{11}{5}\mathbf{v}_1 - \frac{2}{5}\mathbf{v}_2 + 2\mathbf{v}_3.
\end{aligned}$$

9. $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 4-2-2 = 0$; $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 2-4+2 = 0$; $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 2+2-4 = 0$;

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim(R^3) = 3$, this set forms a basis for R^3 by Theorem 4.6.4. By Theorem 6.3.2(a),

$$\begin{aligned}
\mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 \\
&= \frac{-2+0+2}{4+4+1}\mathbf{v}_1 + \frac{-2+0-4}{4+1+4}\mathbf{v}_2 + \frac{-1+0+4}{1+4+4}\mathbf{v}_3 \\
&= 0\mathbf{v}_1 - \frac{2}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3.
\end{aligned}$$

11. $(\mathbf{u})_S = \left(-\frac{11}{5}, -\frac{2}{5}, 2\right)$

13. $(\mathbf{u})_S = \left(0, -\frac{2}{3}, \frac{1}{3}\right)$

15. **(a)** $\|\mathbf{v}\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$, so $\mathbf{v}$ forms an orthonormal basis for the line $W = \text{span}\{\mathbf{v}\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v} = \left(-\frac{3}{5} + \frac{24}{5}\right)\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{21}{5}\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{63}{25}, \frac{84}{25}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (-1, 6) - \left(\frac{63}{25}, \frac{84}{25}\right) = \left(-\frac{88}{25}, \frac{66}{25}\right)$;

$\mathbf{w}_2$ is orthogonal to the line since $\langle \mathbf{w}_2, \mathbf{v} \rangle = -\frac{264}{125} + \frac{264}{125} = 0$.

17. **(a)** $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{2+3}{1+1}(1,\ 1) = \frac{5}{2}(1,\ 1) = \left(\frac{5}{2}, \frac{5}{2}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (2,3) - \left(\frac{5}{2}, \frac{5}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$;

$\mathbf{w}_2$ is orthogonal to the line since $\langle \mathbf{w}_2, \mathbf{v} \rangle = -\frac{1}{2} + \frac{1}{2} = 0$.

19. **(a)** $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 = 2\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) + 4\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) = \left(\frac{10}{3}, \frac{8}{3}, \frac{4}{3}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (4,2,1) - \left(\frac{10}{3}, \frac{8}{3}, \frac{4}{3}\right) = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$;

$\mathbf{w}_2$ is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1 \rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$ and $\langle \mathbf{w}_2, \mathbf{v}_2 \rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$.

21. **(a)** $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{4}{6}(1,-2,1) + \frac{2}{5}(2,1,0)$

$= \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right) + \left(\frac{4}{5}, \frac{2}{5}, 0\right) = \left(\frac{22}{15}, -\frac{14}{15}, \frac{2}{3}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (1,0,3) - \left(\frac{22}{15}, -\frac{14}{15}, \frac{2}{3}\right) = \left(-\frac{7}{15}, \frac{14}{15}, \frac{7}{3}\right)$;

$\mathbf{w}_2$ is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1 \rangle = -\frac{7}{15} - \frac{28}{15} + \frac{7}{3} = \frac{-7-28+35}{15} = 0$ and

$\langle \mathbf{w}_2, \mathbf{v}_2 \rangle = -\frac{14}{15} + \frac{14}{15} + 0 = 0$.

23. $\text{proj}_W \mathbf{b} = \frac{\langle \mathbf{b}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{b}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \frac{1+2+0-2}{1+1+1+1}(1,1,1,1) + \frac{1+2+0+2}{1+1+1+1}(1,1,-1,-1)$

$= \frac{1}{4}(1,1,1,1) + \frac{5}{4}(1,1,-1,-1) = \left(\frac{3}{2}, \frac{3}{2}, -1, -1\right)$

25. $\text{proj}_W \mathbf{b} = \langle \mathbf{b}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{b}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{b}, \mathbf{v}_3 \rangle \mathbf{v}_3$

$= \left(0 + \frac{2}{\sqrt{18}} + 0 + \frac{1}{\sqrt{18}}\right)\mathbf{v}_1 + \left(\frac{1}{2} + \frac{10}{6} + 0 - \frac{1}{6}\right)\mathbf{v}_2 + \left(\frac{1}{\sqrt{18}} + 0 + 0 + \frac{4}{\sqrt{18}}\right)\mathbf{v}_3 = \frac{3}{\sqrt{18}}\mathbf{v}_1 + 2\mathbf{v}_2 + \frac{5}{\sqrt{18}}\mathbf{v}_3$

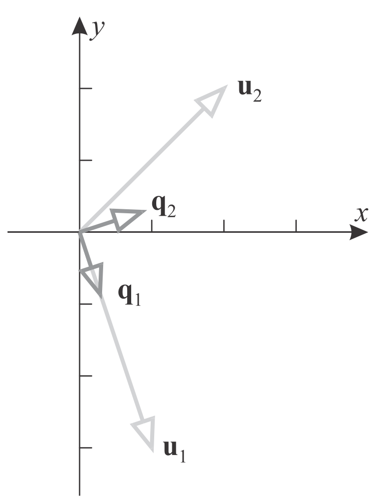
$= \left(0, \frac{3}{18}, -\frac{12}{18}, -\frac{3}{18}\right) + \left(1, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}\right) + \left(\frac{5}{18}, 0, \frac{5}{18}, -\frac{20}{18}\right) = \left(\frac{23}{18}, \frac{11}{6}, -\frac{1}{18}, -\frac{17}{18}\right)$

27. $\mathbf{v}_1 = \mathbf{u}_1 = (1,-3)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (2,2) - \frac{2-6}{10}(1,-3) = (2,2) - \left(-\frac{2}{5}, \frac{6}{5}\right) = \left(\frac{12}{5}, \frac{4}{5}\right)$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{10}}(1,-3) = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$ and

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{\frac{144}{25} + \frac{16}{25}}}\left(\frac{12}{5}, \frac{4}{5}\right) = \frac{1}{\sqrt{\frac{160}{25}}}\left(\frac{12}{5}, \frac{4}{5}\right) = \frac{5}{4\sqrt{10}}\left(\frac{12}{5}, \frac{4}{5}\right) = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$.



29. $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (-1,1,0) - \frac{-1+1+0}{1+1+1}(1,1,1) = (-1,1,0) - 0(1,1,1) = (-1,1,0)$

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (1,2,1) - \frac{1+2+1}{1+1+1}(1,1,1) - \frac{-1+2+0}{1+1+0}(-1,1,0)$

$= (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0) = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{3}}(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$,

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{2}}(-1,1,0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, and $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{1/\sqrt{6}}\left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

31. First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

$\mathbf{v}_1 = \mathbf{u}_1 = (0,2,1,0)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1,\ -1,\ 0,\ 0) - \frac{0-2+0+0}{5}(0,\ 2,\ 1,0) = (1,\ -1,\ 0,\ 0) + \left(0,\ \frac{4}{5},\ \frac{2}{5},\ 0\right)$

$= \left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (1,\ 2,\ 0,\ -1) - \frac{0+4+0+0}{5}(0,\ 2,\ 1,\ 0) - \frac{1-\frac{2}{5}+0+0}{\frac{6}{5}}\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$

$= (1,\ 2,\ 0,\ -1) - \left(0,\ \frac{8}{5},\ \frac{4}{5},\ 0\right) - \left(\frac{1}{2},\ -\frac{1}{10},\ \frac{1}{5},\ 0\right) = \left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$

$$\mathbf{v}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$$

$= (1,\ 0,\ 0,\ 1) - \frac{0+0+0+0}{5}\mathrm{v}_1 - \frac{1+0+0+0}{\frac{6}{5}}\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right) - \frac{\frac{1}{2}+0+0-1}{\frac{5}{2}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$

$= (1,\ 0,\ 0,\ 1) - \frac{0+0+0+0}{5}\mathrm{v}_1 - \frac{1+0+0+0}{\frac{6}{5}}\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right) - \frac{\frac{1}{2}+0+0-1}{\frac{5}{2}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right) = \left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(0,\ 2,\ 1,\ 0)}{\sqrt{5}} = \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$,

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)}{\frac{\sqrt{30}}{5}} = \left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$, $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)}{\frac{\sqrt{10}}{2}} = \left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$,

$\mathbf{q}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{\left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)}{\frac{4}{\sqrt{15}}} = \left(\frac{1}{\sqrt{15}},\ \frac{1}{\sqrt{15}},\ -\frac{2}{\sqrt{15}},\ \frac{3}{\sqrt{15}}\right)$.

33. From Exercise 23, $\mathbf{w}_1 = \mathrm{proj}_W \mathbf{b} = \left(\frac{3}{2},\ \frac{3}{2},\ -1,\ -1\right)$, so $\mathbf{w}_2 = \mathbf{b} - \mathrm{proj}_W \mathbf{b} = \left(-\frac{1}{2},\ \frac{1}{2}, 1,\ -1\right)$.

35. Let W be the plane spanned by the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

$\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (2,0,-1) - \frac{2+0-1}{1+1+1}(1,1,1) = (2,0,-1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$

$\mathbf{w}_1 = \mathrm{proj}_W \mathbf{w} = \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{1+2+3}{3}\mathbf{v}_1 + \frac{\frac{5}{3}-\frac{2}{3}-\frac{12}{3}}{\frac{42}{9}}\mathbf{v}_2 = 2(1,\ 1,\ 1) - \frac{9}{14}\left(\frac{5}{3},\ -\frac{1}{3},\ -\frac{4}{3}\right)$

$= (2,\ 2,\ 2) - \left(\frac{15}{14},\ -\frac{3}{14},\ -\frac{6}{7}\right) = \left(\frac{13}{14},\ \frac{31}{14},\ \frac{20}{7}\right)$

$\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1 = (1,2,3) - \left(\frac{13}{14},\ \frac{31}{14},\ \frac{20}{7}\right) = \left(\frac{1}{14},\ -\frac{3}{14},\ \frac{1}{7}\right)$

37. First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$.

$\mathbf{v}_1 = \mathbf{u}_1 = (1,\ 1,\ 1)$

$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1,\ \mathbf{v}_1\rangle} = \sqrt{1^2 + 2(1)^2 + 3(1)^2} = \sqrt{1+2+3} = \sqrt{6}$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2,\ \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1,\ 1,\ 0) - \frac{1(1)+2(1)(1)+3(0)(1)}{6}(1,\ 1,\ 1) = (1,\ 1,\ 0) - \frac{1}{2}(1,\ 1,\ 1) = \left(\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right)$

$\|\mathbf{v}_2\| = \sqrt{\langle \mathbf{v}_2,\ \mathbf{v}_2\rangle} = \sqrt{\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right)^2} = \sqrt{6\left(\frac{1}{4}\right)} = \frac{\sqrt{6}}{2}$

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3,\ \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3,\ \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (1,\ 0,\ 0) - \frac{1+0+0}{6}(1,\ 1,\ 1) - \frac{\frac{1}{2}+0+0}{\frac{6}{4}}\left(\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right)$

$= (1,\ 0,\ 0) - \left(\frac{1}{6},\ \frac{1}{6},\ \frac{1}{6}\right) - \left(\frac{1}{6},\ \frac{1}{6},\ -\frac{1}{6}\right) = \left(\frac{2}{3},\ -\frac{1}{3},\ 0\right)$

$\|\mathbf{v}_3\| = \sqrt{\langle \mathbf{v}_3,\ \mathbf{v}_3\rangle} = \sqrt{\left(\frac{2}{3}\right)^2 + 2\left(-\frac{1}{3}\right)^2 + 3(0)^2} = \sqrt{\frac{4}{9}+\frac{2}{9}} = \frac{\sqrt{6}}{3}$

The orthonormal basis is $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1,\ 1,\ 1)}{\sqrt{6}} = \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$, $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right)}{\frac{\sqrt{6}}{2}} = \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$, and

$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{2}{3},\ -\frac{1}{3},\ 0\right)}{\frac{\sqrt{6}}{3}} = \left(\frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right)$.

39. For example, $\mathbf{x} = \left(\frac{1}{\sqrt{3}}, 0\right)$ and $\mathbf{y} = \left(0, \frac{1}{\sqrt{2}}\right)$.

41. **(a)** By inspection, $\mathbf{v}_1' = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_2' = \mathbf{v}_1 - 2\mathbf{v}_2$ so $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are in W. The dimension of W is 2 since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W. By Theorem 6.3.1, $\{\mathbf{v}_1', \mathbf{v}_2'\}$ is linearly independent, so by Theorem 4.6.4 it is a basis for W, hence it spans W.

(b) Calculating $\text{proj}_W\mathbf{u}$ using $\{\mathbf{v}_1, \mathbf{v}_2\}$ we obtain

$\frac{\langle \mathbf{u},\mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u},\mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{-3+0+7}{1+0+1}(1,0,1) + \frac{0+1+0}{0+1+0}(0,1,0) = (2,0,2) + (0,1,0) = (2,1,2)$.

Calculating $\text{proj}_W\mathbf{u}$ using $\{\mathbf{v}_1', \mathbf{v}_2'\}$ instead yields the same vector:

$\frac{\langle \mathbf{u},\mathbf{v}_1'\rangle}{\|\mathbf{v}_1'\|^2}\mathbf{v}_1' + \frac{\langle \mathbf{u},\mathbf{v}_2'\rangle}{\|\mathbf{v}_2'\|^2}\mathbf{v}_2' = \frac{-3+1+7}{1+1+1}(1,1,1) + \frac{-3-2+7}{1+4+1}(1,-2,1) = \left(\frac{5}{3},\frac{5}{3},\frac{5}{3}\right) + \left(\frac{1}{3},-\frac{2}{3},\frac{1}{3}\right) = (2,1,2)$.

43. First transform the basis $S = \{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3\} = \{1,\ x,\ x^2\}$ into an orthogonal basis $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$.

$\mathbf{v}_1 = \mathbf{p}_1 = 1$

$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1,\ \mathbf{v}_1\rangle} = \sqrt{\int_0^1 1^2\, dx} = \sqrt{x\big|_0^1} = 1$

$\langle \mathbf{p}_2,\ \mathbf{v}_1\rangle = \int_0^1 x\cdot 1\, dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

$\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2,\ \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = x - \frac{\frac{1}{2}}{1}(1) = x - \frac{\frac{1}{2}}{1}(1) = -\frac{1}{2} + x$

$\|\mathbf{v}_2\|^2 = \langle \mathbf{v}_2,\ \mathbf{v}_2\rangle = \int_0^1\left(-\frac{1}{2}+x\right)^2 dx = \int_0^1\left(\frac{1}{4} - x + x^2\right) dx = \left(\frac{1}{4}x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{12}$

$\|\mathbf{v}_2\| = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$

$\langle \mathbf{p}_3,\ \mathbf{v}_1\rangle = \int_0^1 x^2\cdot 1\, dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$

$$\langle \mathbf{p}_3, \mathbf{v}_2 \rangle = \int_0^1 x^2\left(-\tfrac{1}{2}+x\right)dx = \int_0^1\left(-\tfrac{1}{2}x^2+x^3\right)dx = \left(-\tfrac{1}{6}x^3+\tfrac{1}{4}x^4\right)\Big|_0^1 = \tfrac{1}{12}$$

$$\mathbf{v}_3 = \mathbf{p}_3 - \tfrac{\langle \mathbf{p}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \tfrac{\langle \mathbf{p}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(-\tfrac{1}{2}+x\right) = x^2 - \tfrac{1}{3} + \tfrac{1}{2} - x = \tfrac{1}{6} - x + x^2$$

$$\|\mathbf{v}_3\|^2 = \langle \mathbf{v}_3, \mathbf{v}_3\rangle = \int_0^1\left(\tfrac{1}{6}-x+x^2\right)^2 dx = \left(\tfrac{1}{36}x - \tfrac{1}{6}x^2 + \tfrac{4}{9}x^3 - \tfrac{1}{2}x^4 + \tfrac{1}{5}x^5\right)\Big|_0^1 = \tfrac{1}{180}$$

$$\|\mathbf{v}_3\| = \sqrt{\tfrac{1}{180}} = \tfrac{1}{6\sqrt{5}}$$

The orthonormal basis is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \tfrac{1}{1} = 1;$$

$$\mathbf{q}_2 = \tfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{-\frac{1}{2}+x}{\frac{1}{2\sqrt{3}}} = 2\sqrt{3}\left(-\tfrac{1}{2}+x\right) = \sqrt{3}(-1+2x);$$

$$\mathbf{q}_3 = \tfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\frac{1}{6}-x+x^2}{\frac{1}{6\sqrt{5}}} = 6\sqrt{5}\left(\tfrac{1}{6}-x+x^2\right) = \sqrt{5}\left(1-6x+6x^2\right).$$

45. Let $\mathbf{u}_1 = (1,2)$, $\mathbf{u}_2 = (-1,3)$, $\mathbf{q}_1 = \left(\tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}}\right)$, and $\mathbf{q}_2 = \left(-\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}\right)$. A *QR*-decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}}+\frac{4}{\sqrt{5}} & -\frac{1}{\sqrt{5}}+\frac{6}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}}+\frac{3}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}.$$

47. Let $\mathbf{u}_1 = (1,0,1)$, $\mathbf{u}_2 = (0,1,2)$, $\mathbf{u}_3 = (2,1,0)$, $\mathbf{q}_1 = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right)$, $\mathbf{q}_2 = \left(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)$, and

$\mathbf{q}_3 = \left(\tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right)$. A *QR*-decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle & \langle \mathbf{u}_3, \mathbf{q}_1\rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle & \langle \mathbf{u}_3, \mathbf{q}_2\rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}+0+\frac{1}{\sqrt{2}} & 0+0+\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}}+0+0 \\ 0 & 0+\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}+0 \\ 0 & 0 & \frac{2}{\sqrt{6}}+\frac{2}{\sqrt{6}}+0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{4}{\sqrt{6}} \end{bmatrix}.$$

49. In partitioned form, $\mathrm{A} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$. By inspection, $\mathbf{u}_3 = \mathbf{u}_1 + 2\mathbf{u}_2$, so the column vectors of A are not linearly independent and A does not have a *QR*-decomposition.

51. The proof of part (a) mirrors the proof of part (b) in the book. By Theorem 6.3.1, an orthogonal set of nonzero vectors in W is linearly independent. It follows from part (b) of Theorem 4.6.5 that this set can be enlarged to form a basis for W. Applying the Gram-Schmidt process (without the normalization step) will yield an enlarged orthogonal set (the original orthogonal set will not be affected).

53. The diagonal entries of R are $\langle \mathbf{u}_i,\ \mathbf{q}_i \rangle$ for i = 1, 2, ..., n, where $\mathbf{q}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ is the normalization of a vector $\mathbf{v}_i$ that is the result of applying the Gram-Schmidt process to $\{\mathbf{u}_1,\ \mathbf{u}_2,\ ...,\ \mathbf{u}_n\}$. Thus, $\mathbf{v}_i$ is $\mathbf{u}_i$ minus a linear combination of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ ...,\ \mathbf{v}_{i-1}$, so $\mathbf{u}_i = \mathbf{v}_i + k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_{i-1}\mathbf{v}_{i-1}$. Thus, $\langle \mathbf{u}_i,\ \mathbf{v}_i \rangle = \langle \mathbf{v}_i,\ \mathbf{v}_i \rangle$ and $\langle \mathbf{u}_i,\ \mathbf{q}_i \rangle = \left\langle \mathbf{u}_i,\ \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} \right\rangle = \frac{1}{\|\mathbf{v}_i\|}\langle \mathbf{v}_i,\ \mathbf{v}_i \rangle = \|\mathbf{v}_i\|$. Since each vector $\mathbf{v}_i$ is nonzero, each diagonal entry of R is nonzero.

55. **(b)** The range of T is W; the kernel of T is $W^{\perp}$.

True-False Exercises

(a) False. For example, the vectors (1, 0) and (1, 1) in R^2 are linearly independent but not orthogonal.

(b) False. The vectors must be nonzero for this to be true.

(c) True. A nontrivial subspace of R^3 will have a basis, which can be transformed into an orthonormal basis with respect to the Euclidean inner product.

(d) True. A nonzero finite-dimensional inner product space will have finite basis which can be transformed into an orthonormal basis with respect to the inner product via the Gram-Schmidt process with normalization.

(e) False. $\text{proj}_W\mathbf{x}$ is a vector in W.

(f) True. Every invertible $n \times n$ matrix has a QR-decomposition.

6.4 Best Approximation; Least Squares

1. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$; $A^T A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$;

The associated normal equation is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$.

3. $A^T A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$;

The normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{20}{11} \\ 0 & 1 & -\frac{8}{11} \end{bmatrix}$.

The solution of this system $x_1 = \frac{20}{11}$, $x_2 = -\frac{8}{11}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

5. $A^T A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$;

$A^T \mathbf{b} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$;

The normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix}$.

The solution of this system $x_1 = 12$, $x_2 = -3$, $x_3 = 9$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

7. Least squares error vector: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} \frac{20}{11} \\ -\frac{8}{11} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} \frac{28}{11} \\ \frac{16}{11} \\ \frac{40}{11} \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{27}{11} \\ \frac{15}{11} \end{bmatrix}$;

$A^T \mathbf{e} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -\frac{6}{11} \\ -\frac{27}{11} \\ \frac{15}{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; therefore the least squares error vector is orthogonal to every vector in the column space of A.

Least squares error: $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{\left(-\frac{6}{11}\right)^2 + \left(-\frac{27}{11}\right)^2 + \left(\frac{15}{11}\right)^2} = \frac{3}{11}\sqrt{110} \approx 2.86$.

9. Least squares error vector: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 3 \end{bmatrix}$;

$A^T \mathbf{e} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; therefore the least squares error vector is orthogonal to every vector in the column space of A.

Least squares error: $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{3^2 + (-3)^2 + 0^2 + 3^2} = 3\sqrt{3} \approx 5.196$.

11. $A^TA = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ 12 & 6 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 24 & 12 \\ 12 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = \frac{1}{2} - \frac{1}{2}t$, $x_2 = t$. All of these are least squares solutions of $A\mathbf{x} = \mathbf{b}$. The error vector is the same for all solutions:

$$\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} - \frac{1}{2}t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

13. $A^TA = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ 7 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ 7 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & 1 & -\frac{7}{6} \\ 0 & 1 & 1 & \frac{7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = -\frac{7}{6} - t$, $x_2 = \frac{7}{6} - t$, $x_3 = t$. All of these are least squares solutions of $A\mathbf{x} = \mathbf{b}$. The error vector is the same for all solutions:

$$\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{7}{6} - t \\ \frac{7}{6} - t \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} \frac{14}{3} \\ -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{7}{6} \\ -\frac{49}{6} \end{bmatrix}.$$

15. $A^TA = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{17}{95} \\ 0 & 1 & \frac{143}{285} \end{bmatrix}$.

The solution of this system $x_1 = \frac{17}{95}$, $x_2 = \frac{143}{285}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

By Theorem 6.4.2, $\operatorname{proj}_W \mathbf{b} = A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$.

This matches the result obtained using Theorem 6.4.4:

$$\operatorname{proj}_W \mathbf{b} = A\left(A^T A\right)^{-1} A^T \mathbf{b} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}\left(\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}\left(\frac{1}{(14)(21)-(-3)(-3)}\begin{bmatrix} 21 & 3 \\ 3 & 14 \end{bmatrix}\right)\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{285}\begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 21 & 3 \\ 3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{285}\begin{bmatrix} -92 \\ 439 \\ 470 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$$

17. We follow the procedure of Example 2. For $A = \left[\mathbf{v}_1 \mid \mathbf{v}_2\right] = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}$, we have

$A^T A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 24 \end{bmatrix}$ and $A^T \mathbf{u} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix}$;

The normal system $A^T A\mathbf{x} = A^T \mathbf{u}$ is $\begin{bmatrix} 6 & 6 \\ 6 & 24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & -\frac{7}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$ so that the least squares solution of $A\mathbf{x} = \mathbf{u}$ is $\mathbf{x} = \begin{bmatrix} -\frac{7}{3} \\ \frac{1}{3} \end{bmatrix}$. Denoting $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ we obtain $\operatorname{proj}_W \mathbf{u} = A\mathbf{x} = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -\frac{7}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$.

19. Letting $A=\begin{bmatrix}1\\0\end{bmatrix}$, we have $P=A\left(A^TA\right)^{-1}A^T=\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0\end{bmatrix}=$

$\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}1&0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}$. This matches the matrix in Table 3 of Section 1.8.

21. Letting $A=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}$, we have $A^TA=\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$P=A\left(A^TA\right)^{-1}A^T=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

This matches the matrix in Table 4 of Section 1.8.

23. We use Theorem 6.4.6: $\mathbf{x}=R^{-1}Q^T\mathbf{b}=\frac{1}{(5)\left(\frac{7}{5}\right)}\begin{bmatrix}\frac{7}{5}&\frac{1}{5}\\0&5\end{bmatrix}\begin{bmatrix}\frac{3}{5}&-\frac{4}{5}\\\frac{4}{5}&\frac{3}{5}\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}\frac{1}{5}&\frac{1}{35}\\0&\frac{5}{7}\end{bmatrix}\begin{bmatrix}\frac{1}{5}\\\frac{18}{5}\end{bmatrix}=\begin{bmatrix}\frac{1}{7}\\\frac{18}{7}\end{bmatrix}$.

25. **(a)** If $x=s$ and $y=t$, then a point on the plane is $(s,t,-5s+3t)=s(1,0,-5)+t(0,1,3)$. $\mathbf{w}_1=(1,0,-5)$ and $\mathbf{w}_2=(0,1,3)$ form a basis for W (they are linearly independent since neither of them is a scalar multiple of the other).

(b) Letting $A=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}$, Formula (11) yields

$$P=A\left(A^TA\right)^{-1}A^T=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\left(\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}$$

$$=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\left(\begin{bmatrix}26&-15\\-15&10\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\left(\frac{1}{(26)(10)-(-15)(-15)}\begin{bmatrix}10&15\\15&26\end{bmatrix}\right)\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}$$

$$=\frac{1}{35}\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\begin{bmatrix}10&15\\15&26\end{bmatrix}\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}=\frac{1}{35}\begin{bmatrix}10&15&-5\\15&26&3\\-5&3&34\end{bmatrix}.$$

27. The reduced row echelon form of the augmented matrix of the given homogeneous system is $\begin{bmatrix}1&0&\frac{1}{2}&-\frac{1}{2}&0\\0&1&\frac{1}{2}&\frac{1}{2}&0\end{bmatrix}$ so that the general solution is $x_1=-\frac{1}{2}s+\frac{1}{2}t$, $x_2=-\frac{1}{2}s-\frac{1}{2}t$, $x_3=s$, $x_4=t$.

The solution space W is spanned by vectors $(-1,-1,2,0)$ and $(1,-1,0,2)$.

We construct the matrix with these vectors as its columns $A=\begin{bmatrix}-1 & 1\\ -1 & -1\\ 2 & 0\\ 0 & 2\end{bmatrix}$ then follow the procedure of Example 2 in Section 6.4.

$$A^TA=\begin{bmatrix}-1 & -1 & 2 & 0\\ 1 & -1 & 0 & 2\end{bmatrix}\begin{bmatrix}-1 & 1\\ -1 & -1\\ 2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix};\quad A^T\mathbf{u}=\begin{bmatrix}-1 & -1 & 2 & 0\\ 1 & -1 & 0 & 2\end{bmatrix}\begin{bmatrix}5\\ 6\\ 7\\ 2\end{bmatrix}=\begin{bmatrix}3\\ 3\end{bmatrix};$$

The normal system $A^TA\mathbf{x}=A^T\mathbf{u}$ is $\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}3\\ 3\end{bmatrix}$. The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{1}{2}\end{bmatrix}$ so that the least squares solution of $A\mathbf{x}=\mathbf{u}$ is $\mathbf{x}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$. We obtained $\text{proj}_W\mathbf{u}=A\mathbf{x}=\begin{bmatrix}-1 & 1\\ -1 & -1\\ 2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}0\\ -1\\ 1\\ 1\end{bmatrix}$.

29. Let W be the row space of A. Since W is also the column space of A^T, by Formula (11) we have $P=A^T\left(\left(A^T\right)^TA^T\right)^{-1}\left(A^T\right)^T=A^T\left(AA^T\right)^{-1}A$.

31. Since $\mathbf{b}$ is orthogonal to the column space of A, it follows that $A^T\mathbf{b}=\mathbf{0}$. By Theorem 6.4.4, the least squares solution is $\mathbf{x}=\left(A^TA\right)^{-1}A^T\mathbf{b}=\left(A^TA\right)^{-1}\mathbf{0}=\mathbf{0}$.

True-False Exercises

(a) True. A^TA is an $n\times n$ matrix.

(b) False. Only square matrices have inverses, but A^TA can be invertible when A is not a square matrix.

(c) True. If A is invertible, so is A^T, so the product A^TA is also invertible.

(d) True. Multiplying both sides of $A\mathbf{x}=\mathbf{b}$ on the left by A^T yields $A^TA\mathbf{x}=A^T\mathbf{b}$.

(e) False. By Theorem 6.4.2, the normal system $A^TA\mathbf{x}=A^T\mathbf{b}$ is always consistent.

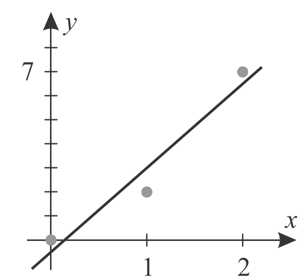
(f) True. This follows from Theorem 6.4.2.

(g) False. There may be more than one least squares solution as shown in Example 2.

(h) True. This follows from Theorem 6.4.4.

6.5 Mathematical Modeling Using Least Squares

1. We have $M=\begin{bmatrix}1&0\\1&1\\1&2\end{bmatrix}$, $M^{\mathrm{T}}=\begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}$, $M^TM=\begin{bmatrix}3&3\\3&5\end{bmatrix}$,

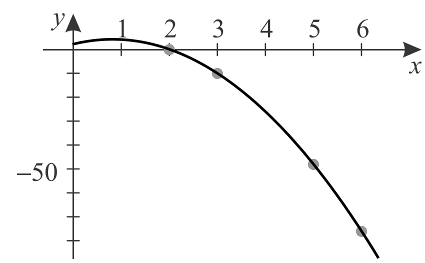


$\left(M^TM\right)^{-1}=\frac{1}{15-9}\begin{bmatrix}5&-3\\-3&3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}5&-3\\-3&3\end{bmatrix}$, and

$$\mathbf{v}^*=\left(M^TM\right)^{-1}M^T\mathbf{y}=\frac{1}{6}\begin{bmatrix}5&-3\\-3&3\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}\begin{bmatrix}0\\2\\7\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\\frac{7}{2}\end{bmatrix}$$

so the least squares straight line fit to the given data points is $y=-\frac{1}{2}+\frac{7}{2}x$.

3. We have $M=\begin{bmatrix}1&2&2^2\\1&3&3^2\\1&5&5^2\\1&6&6^2\end{bmatrix}=\begin{bmatrix}1&2&4\\1&3&9\\1&5&25\\1&6&36\end{bmatrix}$,



$M^TM=\begin{bmatrix}4&16&74\\16&74&376\\74&376&2018\end{bmatrix}$,

$\left(M^TM\right)^{-1}=\frac{1}{90}\begin{bmatrix}1989&-1116&135\\-1116&649&-80\\135&-80&10\end{bmatrix}$, and

$$\mathbf{v}^*=\left(M^TM\right)^{-1}M^T\mathbf{y}=\frac{1}{90}\begin{bmatrix}1989&-1116&135\\-1116&649&-80\\135&-80&10\end{bmatrix}\begin{bmatrix}1&1&1&1\\2&3&5&6\\4&9&25&36\end{bmatrix}\begin{bmatrix}0\\-10\\-48\\-76\end{bmatrix}=\begin{bmatrix}2\\5\\-3\end{bmatrix}$$

so the least squares quadratic fit to the given data points is $y=2+5x-3x^2$.

5. With the substitution $X=\frac{1}{x}$, the problem becomes to find a line of the form $y=a+b\cdot X$ that best fits the data points (1, 7), $\left(\frac{1}{3},\ 3\right)$, $\left(\frac{1}{6},\ 1\right)$.

We have $M=\begin{bmatrix}1&1\\1&\frac{1}{3}\\1&\frac{1}{6}\end{bmatrix}$, $M^TM=\begin{bmatrix}3&\frac{3}{2}\\\frac{3}{2}&\frac{41}{36}\end{bmatrix}$, $\left(M^TM\right)^{-1}=\frac{1}{42}\begin{bmatrix}41&-54\\-54&108\end{bmatrix}$, and

$\mathbf{v}^*=\left(M^TM\right)^{-1}M^T\mathbf{y}=\frac{1}{42}\begin{bmatrix}41&-54\\-54&108\end{bmatrix}\begin{bmatrix}1&1&1\\1&\frac{1}{3}&\frac{1}{6}\end{bmatrix}\begin{bmatrix}7\\3\\1\end{bmatrix}=\begin{bmatrix}\frac{5}{21}\\\frac{48}{7}\end{bmatrix}$. The line in terms of X is $y=\frac{5}{21}+\frac{48}{7}X$, so the required curve is $y=\frac{5}{21}+\frac{48}{7x}$.

7. The two column vectors of M are linearly independent if and only neither is a multiple of the other. Since all the entries in the first column are equal, the columns are linearly independent if and only if the second column has at least two different entries, i.e., if and only if at least two of the numbers $x_1,\ x_2,\ \ldots,\ x_n$ are distinct.

True-False Exercises

(a) False. There is only a unique least squares straight line fit if the data points do not all lie on a vertical line.

(b) True. If the points are not collinear, there is no solution to the system.

(c) True.

(d) False. The line minimizes the sum of the *squares* of the data errors.

6.6 Function Approximation; Fourier Series

1. $a_0 = \frac{1}{\pi}\int_0^{2\pi}(1+x)\,dx = \frac{1}{\pi}\left(x+\frac{x^2}{2}\right)\Big]_0^{2\pi} = 2+2\pi$

Using integration by parts to integrate both $x\cos(kx)$ and $x\sin(kx)$ we obtain

$a_k = \frac{1}{\pi}\int_0^{2\pi}(1+x)\cos(kx)\,dx = \left(\frac{1+x}{k\pi}\sin(kx)+\frac{1}{k^2\pi}\cos(kx)\right)\Big]_0^{2\pi} = 0$ and

$b_k = \frac{1}{\pi}\int_0^{2\pi}(1+x)\sin(kx)\,dx = \left(-\frac{1+x}{k\pi}\cos(kx)+\frac{1}{k^2\pi}\sin(kx)\right)\Big]_0^{2\pi} = -\frac{2}{k}$

(a) $1+x \approx \frac{a_0}{2} + a_1\cos x + a_2\cos(2x) + a_3\cos(3x) + b_1\sin x + b_2\sin(2x) + b_3\sin(3x)$ yields

$1+x \approx 1+\pi + 0\cos x + 0\cos(2x) - \frac{2}{1}\sin x - \frac{2}{2}\sin(2x) = 1+\pi-2\sin x-\sin 2x$

(b) $1+x \approx \frac{a_0}{2} + a_1\cos x + a_2\cos(2x) + \cdots + a_n\cos(nx) + b_1\sin x + b_2\sin(2x) + \cdots + b_n\sin(nx)$ yields

$1+x \approx 1+\pi - \frac{2}{1}\sin x - \frac{2}{2}\sin(2x) - \cdots - \frac{2}{n}\sin(nx)$

3. **(a)** Let us denote $W = \operatorname{span}\{1, e^x\}$. Applying the Gram-Schmidt process to the basis $\mathbf{u}_1 = 1$ and $\mathbf{u}_2 = e^x$ we obtain an orthogonal basis

$$\mathbf{v}_1 = 1,\ \mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = e^x - \frac{\int_0^1 e^x dx}{\int_0^1 1dx}1 = e^x - \frac{e^x\big]_0^1}{x\big]_0^1}1 = e^x - (e-1)1 = e^x - e + 1.$$

Since $\int_0^1 \left(e^x - e + 1\right)^2 dx = \int_0^1 \left(1 - 2e + e^2 + 2e^x - 2ee^x + e^{2x}\right) dx$

$= \left(x - 2ex + e^2x + 2e^x - 2e^{x+1} + \frac{1}{2}e^{2x}\right)\Big|_0^1 = -\frac{3}{2} + 2e - \frac{1}{2}e^2 = \frac{1}{2}(e-1)(3-e)$, an orthonormal basis is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{\int_0^1 1dx}} = \frac{1}{\sqrt{x\big]_0^1}} = 1,\quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{e^x - e + 1}{\sqrt{\frac{1}{2}(e-1)(3-e)}}.$$

The least squares approximation to $f(x) = x$ from W is

$$\text{proj}_W \mathbf{f} = \langle \mathbf{f}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{f}, \mathbf{q}_2 \rangle \mathbf{q}_2 = \int_0^1 x\ dx + \tfrac{2}{(e-1)(3-e)} \left(\int_0^1 x\left(e^x - e + 1\right) dx \right) \left(e^x - e + 1\right)$$

$$= \tfrac{1}{2} + \tfrac{2}{(e-1)(3-e)} \left(xe^x - e^x - \tfrac{x^2 e}{2} + \tfrac{x^2}{2} \right) \Big]_0^1 \left(e^x - e + 1\right) = \tfrac{1}{2} + \tfrac{2}{(e-1)(3-e)} \left(-\tfrac{e}{2} + \tfrac{3}{2} \right) \left(e^x - e + 1\right)$$

$$= \tfrac{1}{2} + \tfrac{e^x - e + 1}{e-1} = \tfrac{1}{2} + \tfrac{e^x}{e-1} - 1 = \tfrac{e^x}{e-1} - \tfrac{1}{2}.$$

(b) The mean square error is $\int_0^1 \left(x - \left(\tfrac{e^x}{e-1} - \tfrac{1}{2} \right) \right)^2 dx = \tfrac{7e-19}{12e-12} \approx 0.00136.$

5. **(a)** Let us denote $W = \text{span}\left\{1, x, x^2\right\}$.

Applying the Gram-Schmidt process to the basis $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x$, and $\mathbf{u}_3 = x^2$ we obtain an orthogonal basis $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = \mathbf{u}_2 - \tfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = x - \dfrac{\int_{-1}^1 x dx}{\int_{-1}^1 1 dx} 1 = x - \dfrac{\frac{x^2}{2}\Big]_{-1}^1}{x\Big]_{-1}^1} = x - 0 = x$,

$$\mathbf{v}_3 = \mathbf{u}_3 - \tfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \tfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = x^2 - \frac{\int_{-1}^1 x^2 dx}{\int_{-1}^1 1 dx} 1 - \frac{\int_{-1}^1 x^3 dx}{\int_{-1}^1 x^2 dx} x = x^2 - \frac{\frac{x^3}{3}\Big]_{-1}^1}{x\Big]_{-1}^1} - \frac{\frac{x^4}{4}\Big]_{-1}^1}{\frac{x^3}{3}\Big]_{-1}^1} x = x^2 - \tfrac{1}{3},$$

and an orthonormal basis $\mathbf{q}_1 = \tfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \dfrac{1}{\sqrt{\int_{-1}^1 1 dx}} = \dfrac{1}{\sqrt{x\big]_{-1}^1}} = \tfrac{1}{\sqrt{2}}$,

$$\mathbf{q}_2 = \tfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{x}{\sqrt{\int_{-1}^1 x^2 dx}} = \frac{x}{\sqrt{\frac{x^3}{3}\Big]_{-1}^1}} = \sqrt{\tfrac{3}{2}}\, x, \quad \mathbf{q}_3 = \tfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^1 \left(x^2 - \frac{1}{3}\right)^2 dx}} = \frac{x^2 - \frac{1}{3}}{\frac{2}{3}\sqrt{\frac{2}{5}}}.$$

The least squares approximation to $f(x) = \sin \pi x$ from W is

$$\text{proj}_W \mathbf{f} = \langle \mathbf{f}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{f}, \mathbf{q}_2 \rangle \mathbf{q}_2 + \langle \mathbf{f}, \mathbf{q}_3 \rangle \mathbf{q}_3$$

$$= \tfrac{1}{4} \int_{-1}^1 \sin \pi x\, dx + \tfrac{3}{2} \left(\int_{-1}^1 x \sin \pi x\ dx \right) x + \tfrac{45}{8} \left(\int_{-1}^1 \left(x^2 - \tfrac{1}{3}\right) \sin \pi x\ dx \right) \left(x^2 - \tfrac{1}{3}\right)$$

$$= 0 + \tfrac{3}{2} \left(-\tfrac{x \cos \pi x}{\pi} + \tfrac{\sin \pi x}{\pi^2} \right) \Big|_{-1}^1 x + 0 = \tfrac{3}{2} \cdot \tfrac{2}{\pi} x = \tfrac{3x}{\pi}.$$

(b) The mean square error is $\int_{-1}^1 \left(\sin \pi x - \tfrac{3x}{\pi} \right)^2 dx = 1 - \tfrac{6}{\pi^2} \approx 0.392.$

9. Let $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$.

$$a_0 = \tfrac{1}{\pi} \int_0^{2\pi} f(x)\ dx = \tfrac{1}{\pi} \int_0^{\pi} dx = 1$$

$$a_k = \tfrac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\ dx = \tfrac{1}{\pi} \int_0^{\pi} \cos kx\ dx = 0$$

$$b_k = \tfrac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\ dx = \tfrac{1}{\pi} \int_0^{\pi} \sin kx\ dx = \tfrac{1}{k\pi}(1 - (-1)^k)$$

So the Fourier series is $\tfrac{1}{2} + \sum_{k=1}^{\infty} \tfrac{1}{k\pi} (1 - (-1)^k) \sin kx.$

True-False Exercises

(a) False. The area between the graphs is the error, not the mean square error.

(b) True.

(c) True.

(d) False. $\|1\| = \langle 1,\ 1\rangle = \int_0^{2\pi} 1^2\ dx = 2\pi \neq 1.$

(e) True.

Chapter 6 Supplementary Exercises

1. **(a)** Let $\mathbf{v} = (v_1,\ v_2,\ v_3,\ v_4).$

$\langle \mathbf{v},\ \mathbf{u}_1\rangle = v_1,\ \langle \mathbf{v},\ \mathbf{u}_2\rangle = v_2,\ \langle \mathbf{v},\ \mathbf{u}_3\rangle = v_3,\ \langle \mathbf{v},\ \mathbf{u}_4\rangle = v_4$

If $\langle \mathbf{v},\ \mathbf{u}_1\rangle = \langle \mathbf{v},\ \mathbf{u}_4\rangle = 0,$ then $v_1 = v_4 = 0$ and $\mathbf{v} = (0,\ v_2,\ v_3,\ 0).$ Since the angle θ between **u** and **v** satisfies $\cos\theta = \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|},$ **v** making equal angles with $\mathbf{u}_2$ and $\mathbf{u}_3$ means that $v_2 = v_3.$ In order for the angle between **v** and $\mathbf{u}_3$ to be defined $\|\mathbf{v}\| \neq 0.$ Thus, **v** = (0, *a*, *a*, 0) with $a \neq 0$.

(b) As in part (a), since $\langle \mathbf{x},\ \mathbf{u}_1\rangle = \langle \mathbf{x}, \mathbf{u}_4\rangle = 0,\ x_1 = x_4 = 0.$

Since $\|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$ and we want $\|\mathbf{x}\| = 1,$ the cosine of the angle between **x** and $\mathbf{u}_2$ is $\cos\theta_2 = \langle \mathbf{x}, \mathbf{u}_2\rangle = x_2$ and, similarly, $\cos\theta_3 = \langle \mathbf{x},\ \mathbf{u}_3\rangle = x_3,$ so we want $x_2 = 2x_3,$ and

$\mathbf{x} = \langle 0,\ x_2,\ 2x_2,\ 0\rangle.$

$$\|\mathbf{x}\| = \sqrt{x_2^2 + 4x_2^2} = \sqrt{5x_2^2} = |x_2|\sqrt{5}.$$

If $\|\mathbf{x}\| = 1,$ then $x_2 = \pm\frac{1}{\sqrt{5}},$ so $\mathbf{x} = \pm\left(0,\ \frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}},\ 0\right).$

3. Recall that if $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix},$ then $\langle U,\ V\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4.$

(a) If *U* is a diagonal matrix, then $u_2 = u_3 = 0$ and $\langle U,\ V\rangle = u_1v_1 + u_4v_4.$

For *V* to be in the orthogonal complement of the subspace of all diagonal matrices, then it must be the case that $v_1 = v_4 = 0$ and *V* must have zeros on the main diagonal.

(b) If *U* is a symmetric matrix, then $u_2 = u_3$ and $\langle U,\ V\rangle = u_1v_1 + u_2(v_2 + v_3) + u_4v_4.$

Since u_1 and u_4 can take on any values, for *V* to be in the orthogonal complement of the subspace of all symmetric matrices, it must be the case that $v_1 = v_4 = 0$ and $v_2 = -v_3,$ thus *V* must be skew-symmetric.

5. Let $\mathbf{u} = \left(\sqrt{a_1},\ \ldots,\ \sqrt{a_n}\right)$ and $\mathbf{v} = \left(\frac{1}{\sqrt{a_1}},\ \ldots,\ \frac{1}{\sqrt{a_n}}\right)$. By the Cauchy-Schwarz inequality,

$$\langle \mathbf{u} \cdot \mathbf{v}^2 \rangle = (\underbrace{1 + \cdots + 1}_{n \text{ terms}})^2 \le \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \text{ or } n^2 \le (a_1 + \cdots + a_n)\left(\tfrac{1}{a_1} + \cdots + \tfrac{1}{a_n}\right).$$

7. Let $\mathbf{x} = (x_1,\ x_2,\ x_3)$.

$$\begin{aligned} \langle \mathbf{x},\ \mathbf{u}_1 \rangle &= x_1 + x_2 - x_3 \\ \langle \mathbf{x},\ \mathbf{u}_2 \rangle &= -2x_1 - x_2 + 2x_3 \\ \langle \mathbf{x},\ \mathbf{u}_3 \rangle &= -x_1 + x_3 \end{aligned}$$

$\langle \mathbf{x},\ \mathbf{u}_3 \rangle = 0 \Rightarrow -x_1 + x_3 = 0$, so $x_1 = x_3$. Then $\langle \mathbf{x},\ \mathbf{u}_1 \rangle = x_2$ and $\langle \mathbf{x},\ \mathbf{u}_2 \rangle = -x_2$, so $x_2 = 0$ and $\mathbf{x} = (x_1,\ 0,\ x_1)$.

Then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_1^2} = \sqrt{2x_1^2} = |x_1|\sqrt{2}$.

If $\|\mathbf{x}\| = 1$ then $x_1 = \pm\frac{1}{\sqrt{2}}$ and the vectors are $\pm\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$.

9. For $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ in R^2, let $\langle \mathbf{u},\ \mathbf{v} \rangle = au_1v_1 + bu_2v_2$ be a weighted inner product.

If $\mathbf{u}$ = (1, 2) and $\mathbf{v}$ = (3, –1) form an orthonormal set, then $\|\mathbf{u}\|^2 = a(1)^2 + b(2)^2 = a + 4b = 1$,

$\|\mathbf{v}\|^2 = a(3)^2 + b(-1)^2 = 9a + b = 1$, and $\langle \mathbf{u}, \mathbf{v} \rangle = a(1)(3) + b(2)(-1) = 3a - 2b = 0$.

This leads to the system $\begin{bmatrix} 1 & 4 \\ 9 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 4 & 1 \\ 9 & 1 & 1 \\ 3 & -2 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the system is inconsistent and there is no such weighted inner product.

11. **(a)** Let $\mathbf{u}_1 = (k,\ 0,\ 0,\ \ldots,\ 0)$, $\mathbf{u}_2 = (0,\ k,\ 0,\ \ldots,\ 0)$, ..., $\mathbf{u}_n = (0,\ 0,\ 0,\ \ldots,\ k)$ be the edges of the 'cube' in R^n and $\mathbf{u}$ = (k, k, k, ..., k) be the diagonal.

Then $\|\mathbf{u}_i\| = k$, $\|\mathbf{u}\| = k\sqrt{n}$, and $\langle \mathbf{u}_i,\ \mathbf{u} \rangle = k^2$, so $\cos\theta = \frac{\langle \mathbf{u}_i, \mathbf{u} \rangle}{\|\mathbf{u}_i\|\|\mathbf{u}\|} = \frac{k^2}{k(k\sqrt{n})} = \frac{1}{\sqrt{n}}$.

(b) As n approaches ∞, $\frac{1}{\sqrt{n}}$ approaches 0, so θ approaches $\frac{\pi}{2}$.

13. Recall that $\mathbf{u}$ can be expressed as the linear combination $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ where $a_i = \langle \mathbf{u}, \mathbf{v}_i \rangle$ for i = 1, ..., n. Since $\|\mathbf{v}_i\| = 1$, we have $\cos^2\alpha_i = \left(\frac{\langle \mathbf{u},\ \mathbf{v}_i \rangle}{\|\mathbf{u}\|\|\mathbf{v}_i\|}\right)^2 = \left(\frac{a_i}{\|\mathbf{u}\|}\right)^2 = \frac{a_i^2}{a_1^2 + a_2^2 + \cdots + a_n^2}$.

Therefore $\cos^2\alpha_1 + \cdots + \cos^2\alpha_n = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1^2 + a_2^2 + \cdots + a_n^2} = 1$.

15. To show that $(W^\perp)^\perp = W$, we first show that $W \subseteq (W^\perp)^\perp$. If $\mathbf{w}$ is in W, then $\mathbf{w}$ is orthogonal to every vector in $W^\perp$, so that $\mathbf{w}$ is in $(W^\perp)^\perp$. Thus $W \subseteq (W^\perp)^\perp$.

To show that $(W^{\perp})^{\perp} \subseteq W,$ let **v** be in $(W^{\perp})^{\perp}.$ Since **v** is in *V*, we have, by the Projection Theorem, that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in *W* and $\mathbf{w}_2$ is in $W^{\perp}.$ By definition, $\langle \mathbf{v}, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0.$ But

$$\langle \mathbf{v}, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}_2 \rangle = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle + \langle \mathbf{w}_2, \mathbf{w}_2 \rangle = \langle \mathbf{w}_2, \mathbf{w}_2 \rangle$$

so that $\langle \mathbf{w}_2, \mathbf{w}_2 \rangle = 0.$ Hence $\mathbf{w}_2 = \mathbf{0}$ and therefore $\mathbf{v} = \mathbf{w}_1,$ so that **v** is in *W*. Thus $(W^{\perp})^{\perp} \subseteq W.$

17. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix},\ A^T = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix},\ A^T A = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix},\ A^T \mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \mathrm{s} \end{bmatrix} = \begin{bmatrix} 4s+3 \\ 5s+2 \end{bmatrix}$

The associated normal system is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4s+3 \\ 5s+2 \end{bmatrix}.$

If the least squares solution is $x_1 = 1$ and $x_2 = 2,$ then $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 71 \\ 95 \end{bmatrix} = \begin{bmatrix} 4s+3 \\ 5s+2 \end{bmatrix}.$

The resulting equations have solutions *s* = 17 and *s* = 18.6, respectively, so no such value of *s* exists.