DE Revision Guide

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Chapter #1:

Section #1.1

Classification by Type

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0 \rightarrow Ordinary\ Differential\ Equations\ (ODE)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow Partial\ Differential\ Equation\ (PDE)$$

- If $\frac{dy}{dx}$ is used then it is ODE.
- If $\frac{\partial y}{\partial x}$ is used then it is PDE.

Classification by Order

$$\left(\frac{d^2y}{dx^2}\right)^6 + \left(\frac{dy}{dx}\right)^7 + 6y = 0 \rightarrow Second\ Order\ Differential\ Equation$$

Second Order First Order

- Order is always a positive integer (Order $\epsilon\ \mathbb{Z}^+$).
- Do not confuse Order with Degree (Power) of a derivative.
- Always choose the highest order.

Classification by Degree

$$\left(\frac{d^2y}{dx^2}\right)^6 + \left(\frac{dy}{dx}\right)^7 + 6y = 0 \rightarrow Sixth\ Degree\ Differential\ Equation$$

- Degree is always taken of the Highest Order.

Classification by Linearity

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

- A differential equation is said to be Linear if it is in the form of the given equation.
- The dependent variable (in this case) 'y' and all its derivatives are of the 1^{st} degree.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable (in this case) x.

Chapter #2:

Section #2.2 (Separable Equations):

$$\frac{dy}{dx} = g(x)h(y)$$

If a differential equation is in form given above, then it can solved by the following method:

$$\frac{1}{h(y)}dy = g(x)dx$$

$$\int \frac{1}{h(y)}dy = \int g(x)dx$$

$$H(y) = G(x) + c$$

Example #1

$$(1+x)dy - ydx = 0$$

$$\frac{1}{y}dy = \frac{1}{1+x}dx$$

$$\int \frac{1}{y}dy = \int \frac{1}{1+x}dx$$

$$\ln|y| = \ln|1+x| + c$$

$$y = \pm e^{\ln|1+x|+c}$$

$$y = \underbrace{\pm e^{c}}_{A}(1+x)$$

$$\boxed{y = A(1+x)}$$

Section #2.3 (Linear Equations)

$$\frac{dy}{dx} + P(x)y = f(x)$$

If a differential equation is in form given above, then it can solved by the following method:

$$\underbrace{\mu(x)}_{\text{Integrating Factor (IF)}} = e^{\int P(x)dx}$$

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

$$\frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$$

$$\int d(\mu(x)y) = \int \mu(x)f(x)dx$$

$$\mu(x)y = \int \mu(x)f(x)dx$$

$$\boxed{y = \frac{1}{\mu(x)}\int \mu(x)f(x)dx}$$

Example #1

$$x\frac{dy}{dx} - 4y = x^6e^x$$

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x$$

$$\mu(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = \frac{1}{x^4}$$

$$\frac{1}{x^4}\frac{dy}{dx} - \frac{4}{x^5}y = xe^x$$

$$\frac{d}{dx}\left(\frac{1}{x^4}y\right) = xe^x$$

$$\int d\left(\frac{1}{x^4}y\right) = \int xe^x dx$$

$$\frac{1}{x^4}y = xe^x - e^x + c$$

$$\boxed{y = x^5e^x - x^4e^x + cx^4}$$

Section #2.4 (Exact & Non-Exact Equations)

Exact Differential Equations

If there is an equation that has a variable depending on two independent variables

$$z(x,y)$$

Then its derivative will be something like

$$dz = \underbrace{\frac{\partial z}{\partial x}}_{M(x,y)} dx + \underbrace{\frac{\partial z}{\partial y}}_{N(x,y)} dy$$

However when $z(x,y) = c$ then

$$\underbrace{\frac{\partial z}{\partial x}}_{M(x,y)} dx + \underbrace{\frac{\partial z}{\partial y}}_{N(x,y)} dy = 0$$

$$M(x,y)dx + N(x,y)dy = 0$$

A function is known to be an Exact Differential Equation when

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ or } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

How to Solve:

$$Consider\ any\ one\ \frac{\partial z}{\partial x}\ or\ \frac{\partial z}{\partial y}$$

$$z = \int \frac{\partial z}{\partial x} dx \underbrace{+ \quad h(y)}_{\substack{Since\ it\ was\ a \\ parital\ differential \\ functions\ with\ y\ were \\ treated\ as\ constants}}$$

$$\frac{\partial z}{\partial y} = \frac{d}{dy}\left(\int \frac{\partial z}{\partial x} dx\right) + h'(y)$$

$$Compare\ \frac{\partial z}{\partial y}\ with\ N\ and\ find\ h(y) by\ integrating$$

$$z = \int \frac{\partial z}{\partial x} dx + \underbrace{h(y)}_{After\ Integrating}$$

$Since\ we\ assumed\ that\ f = c \quad , \int \frac{\partial z}{\partial x} dx + h(y) = c$

Example #1

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}, y(0) = 2$$

$$(xy^2 - \cos x \sin x)dx - \big(y(1-x^2)\big)dy = 0$$

$$M = xy^2 - \cos x \sin x \implies \frac{\partial M}{\partial y} = 2xy$$

$$N = -y(1-x^2) \implies \frac{\partial N}{\partial x} = 2xy$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \therefore \textit{The equation is exact.}$$

$$Consider\ N = \frac{\partial f}{\partial y} = -y(1-x^2)$$

$$f = \int -y(1-x^2)\, dy + h(x)$$

$$f = \frac{-(1-x^2)}{2}y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = xy^2 + h'(x) \iff M = xy^2 - \cos x \sin x$$

$$h'(x) = -\cos x \sin x = -\frac{\sin 2x}{2}$$

$$h(x) = \frac{\cos 2x}{4}$$

$$f = \frac{-(1-x^2)}{2}y^2 + \frac{\cos 2x}{4}$$

$$\boxed{\frac{-(1-x^2)}{2}y^2 + \frac{\cos 2x}{4} = c}$$

Non-Exact Differential Equations

What if an equation is not in exact form? Meaning $\frac{\partial z}{\partial x} \neq \frac{\partial z}{\partial y}$

Then we multiply with an integrating factor

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

$$\therefore \textit{Equation will be exact when } \frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\left.\begin{array}{c} \textit{By product Rule } \mu_y M + \mu M_y = \mu_x N + \mu N_x \\ \mu_x N - \mu_y M = (M_y - N_x)\mu \\ \text{Since we don't know how to solve a partial differential equation} \\ \text{We assume that } \mu \text{ is dependent upon a single variable. Therefore} \\ \textit{Let's assume that } \mu(x) \\ \mu_x N = (M_y - N_x)\mu \implies \frac{\partial \mu}{\partial x} = \underbrace{\frac{(M_y - N_x)}{N}}_{\textit{Depends solely on } x} \mu \end{array}\right\} \begin{array}{c} \textit{This is the explanation} \\ \textit{and is not required} \\ \textit{in the examination.} \end{array}$$

$$\boxed{\mu = e^{\int \frac{(M_y - N_x)}{N} dx}}$$

$$\textit{Let's assume that } \mu(y)$$

$$\mu_y M = (N_x - M_y)\mu \implies \frac{\partial \mu}{\partial y} = \underbrace{\frac{(N_x - M_y)}{M}}_{\textit{Depends solely on } y} \mu$$

$$\boxed{\mu = e^{\int \frac{(N_x - M_y)}{M} dy}}$$

Example #2

$$xydx + (2x^2 + 3y^2 - 20)dy = 0$$

$$\frac{\partial M}{\partial y} = x \neq 4x = \frac{\partial N}{\partial x}$$

$$\frac{(M_y - N_x)}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20}$$

$$Since\ this\ is\ not\ in\ terms\ of\ 1\ variable\ we\ will\ reject\ it$$

$$\frac{(N_x - M_y)}{M} = \frac{4x - x}{xy} = \frac{3}{y}$$

$$\mu = e^{\int \frac{3}{y} dy} = e^{3 \ln y} = y^3$$

$$xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0$$

$$\frac{\partial M}{\partial y} = 4xy^3 = 4xy^3 = \frac{\partial N}{\partial x}$$

$$Consdider\ M = \frac{\partial f}{\partial x} = xy^4$$

$$f = \frac{x^2y^4}{2} + h(y)$$

$$\frac{\partial f}{\partial y} = 2x^2y^3 + h'(y)$$

$$h'(y) = 3y^5 - 20y^3$$

$$h(y) = \frac{y^6}{2} - 5y^4$$

$$f = \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4$$

$$\boxed{\frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4 = c}$$

Section #2.5 (Solutions by Substitution)

Homogeneous Equations

If a function possesses the property $f(tx\,,ty) = t^{\alpha} f(x,y)$ for some real number α then f is said to be a homogeneous function of degree α.

$$M(x,y)dx + N(x,y)dy = 0$$

Is said to be homogeneous if both coefficient functions M and N are homogeneous functions of the SAME degree.

$$M(tx,ty) = \ t^{\alpha}M(x,y) \ and \ N(tx,ty) = \ t^{\alpha}N(x,y)$$

In addition, if M and N are homogeneous functions of degree, we can also write

$$M(x,y) = \ x^{\alpha}M(1,u) \ and \ N(x,y) = \ x^{\alpha}N(1,u) \quad where \ u = y/x$$
$$or$$
$$M(x,y) = \ y^{\alpha}M(v,1) \ and \ N(x,y) = \ y^{\alpha}N(v,1) \quad where \ v = x/y$$

This will turn the equation into a separable equation which can be solved much easier.

Example #1

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

$$When\ comparing\ M\ and\ N, we\ can\ see\ that\ both\ are\ homogeneous\ of\ degree\ 2.$$

$$Let\ y = ux\ \Longrightarrow dy = xdu + udx$$

$$(x^2 + u^2x^2)dx + (x^2 - ux^2)(xdu + udx) = 0$$

$$x^2(1+u)dx + x^3(1-u)du = 0$$

$$\frac{1-u}{1+u}du + \frac{dx}{x} = 0$$

$$After\ Integrating$$

$$-u + 2\ln|1+u| + \ln|x| = \ln|c|$$

$$-\frac{y}{x} + 2\ln\left|1+\frac{y}{x}\right| + \ln|x| = \ln|c|$$

$$After\ Simplification$$

$$\boxed{(x+y)^2 = cxe^{y/x}}$$

Although either of the indicated substitutions can be used for every homogeneous differential equation, in practice we try $x = vy$ whenever the function $M(x,y)$ is simpler than $N(x,y)$. Also it could happen that after using one substitution, we may encounter integrals that are difficult or impossible to evaluate in closed form; switching substitutions may result in an easier problem.

Bernoulli's Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

If a differential equation is in form given above, where n is any real number, then it can solved by the following method:

$$Let\ u = y^{1-n} \ \Longrightarrow y = u^{\frac{1}{1-n}}$$

$$Substitute\ in\ main\ equation$$

$$\frac{dy}{du}\frac{du}{dx} + P(x)u^{\frac{1}{1-n}} = f(x)y^n$$

$$\frac{1}{1-n}u^{\frac{1}{1-n}-1}\frac{du}{dx} + P(x)u^{\frac{1}{1-n}} = f(x)u^{\frac{n}{1-n}}$$

$$\underbrace{\frac{du}{dx} + P(x)u = f(x)}_{Linear\ Differential\ Equation}$$

Example #2

$$x\frac{dy}{dx} + y = x^2y^2$$

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

$$u = y^{1-2} \Longrightarrow y = u^{-1}$$

$$\frac{dy}{dx} = -1u^{-2}\frac{du}{dx}$$

$$-u^{-2}\frac{du}{dx} + \frac{1}{x}u^{-1} = xu^{-2}$$

$$\underbrace{\frac{du}{dx} - \frac{1}{x}u = -x}_{Linear\ Differential\ Equation}$$

$$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -1$$

$$\frac{d}{dx}\left(\frac{1}{x}u\right) = -1$$

$$u = y^{-1} = -x^2 + cx$$

$$\boxed{y = \frac{1}{-x^2 + cx}}$$

Riccati's Equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2 \ , \ \underbrace{y_1 = f(x)}_{Given\ Solution}$$

If a differential equation is in form given above, then it can solved by the following method:

$$let\ y = y_1 + u \ \longrightarrow This\ turns\ the\ equation\ into\ Bernoulli's Differential\ Equation$$

$$let\ w = u^{-1} \longrightarrow This\ turns\ the\ equation\ into\ a\ Linear\ Differential\ Equation$$

$$Solve\ the\ equation\ as\ specified\ before$$

Example #3

$$\frac{dy}{dx} = -\frac{4}{x^2} - \frac{1}{x}y + y^2 \ , y = \frac{2}{x}$$

$$let\ y = y_1 + u$$

$$y = \frac{2}{x} + u \Longrightarrow \frac{dy}{dx} = -\frac{2}{x^2} + \frac{du}{dx}$$

$$-\frac{2}{x^2} + \frac{du}{dx} = -\frac{4}{x^2} - \frac{1}{x}\left(\frac{2}{x} + u\right) + \left(\frac{2}{x} + u\right)^2$$

$$-\cancel{\frac{2}{x^2}} + \frac{du}{dx} = -\cancel{\frac{4}{x^2}} - \cancel{\frac{2}{x^2}} - \frac{1}{x}u + \cancel{\frac{4}{x^2}} + \frac{4}{x}u + u^2$$

$$\frac{du}{dx} = \frac{3}{x}u + u^2 \longrightarrow Bernoulli's Differential\ Equation$$

$$let\ w = u^{-1} \Longrightarrow u = w^{-1}$$

$$\frac{du}{dx} = -w^{-2}\frac{dw}{dx}$$

$$-w^{-2}\frac{dw}{dx} = \frac{3}{x}w^{-1} + w^{-2}$$

$$\frac{dw}{dx} + \frac{3}{x}w = -1$$

$$x^3\frac{dw}{dx} + 3x^2 w = -x^3$$

$$\frac{d}{dx}(x^3 w) = -x^3$$

$$x^3 w = -\frac{x^4}{4} + c$$

$$w = -\frac{x}{4} + cx^{-3}$$

$$u^{-1} = -\frac{x}{4} + cx^{-3}$$

$$u = \frac{4}{-x + 4cx^{-3}}$$

$$y = \frac{2}{x} + \frac{4}{4cx^{-3} - x}$$

Chapter #3:

Section #3.1 (Linear Models)

Growth and Decay Model

$$\frac{dP}{dt} = kP\,, P(t_0) = P_o$$

The above differential equation is given for a Growth and Decay model, and the way to solve it is using Separable Differential Equations/Linear Differential Equations.

Example #1

A culture initially has P_0 number of bacteria. At $t = 1hr$ the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

$$\frac{dP}{dt} - kP = 0$$
$$e^{-kt}\frac{dP}{dx} - e^{-kt}P = 0$$
$$\frac{d}{dx}(e^{-kt}P) = 0$$
$$e^{-kt}P = c$$
$$P = ce^{kt}$$
$$P_0 = c * 1$$
$$P = P_0e^{kt}$$
$$\frac{3}{2}P_0 = P_0e^{k}$$
$$k = \ln\frac{3}{2}$$
$$\boxed{P = P_0e^{\frac{3}{2}t}}$$

Newton's Law of Cooling/Warming

$$\frac{dT}{dt} = k(T - T_m), where\ T_m = Ambient\ Temperature$$

The above differential equation is given for a Cooling/Warming of any object, and the way to solve it is using Separable Differential Equations/Linear Differential Equations.

Example #2

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

$$\frac{dT}{dt} = k(T - T_m)\ , T(t_0) = 300°\mathrm{F}\ , T(3) = 200°\mathrm{F}\ , T_m = 70°\mathrm{F}$$

$$\frac{dT}{dt} = k(T - 70)$$

$$\frac{1}{T - 70} dT = kdt$$

$$\ln|T - 70| = kt + c$$

$$\ln 230 = c$$

$$T = e^{kt + \ln 230} + 70$$

$$T = 230e^{kt} + 70$$

$$200 = 230e^{3k} + 70$$

$$\frac{130}{230} = e^{3k}$$

$$k = \frac{1}{3} \ln \frac{130}{230} = -0.1902$$

$$\boxed{T = 230e^{-0.1902t} + 70}$$

LR-Series Circuits

$$L\frac{di}{dt} + Ri = E(t)$$

The above equation given for a series circuit containing only a resistor and an inductor. L is the inductance, R is the resistance, E (t) is the voltage supplied, and I is the current. It can easily be solved using Linear Differential Equations.

Example #3

A 12-volt battery is connected to a series circuit in which the inductance is 0.5 henry and the resistance is 10 ohms. Determine the current I if the initial current is zero.

$$L\frac{di}{dt} + Ri = E(t), L = 0.5\,, R = 10\,, E(t) = 12\,, I(t_0) = 0$$

$$\frac{1}{2}\frac{di}{dt} + 10i = 12$$

$$\frac{di}{dt} + 20i = 24$$

$$e^{20t}\frac{di}{dt} + 20e^{20t}i = 24e^{20t}$$

$$\frac{d}{dx}(e^{20t}i) = 24e^{20t}$$

$$e^{20t}i = 1.2e^{20t} + c$$

$$c = -1.2$$

$$\boxed{i = 1.2 - 1.2e^{-20t}}$$

RC-Series Circuit

$$R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

The above equation given for a series circuit containing only a resistor and a Capacitor. C is the capacitance, R is the resistance, E (t) is the voltage supplied, and I is the current. It can easily be solved using Linear Differential Equations.

Example #4

A 100-volt electromotive force is applied to an RC-series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0)=0$. Find the current $i(t)$.

$$R\frac{dq}{dt}+\frac{1}{C}q=E(t), E(t)=100\,, R=200\,, C=10^{-4}, q(0)=0$$

$$200\frac{dq}{dt}+10^4q=100$$

$$\frac{dq}{dt}+50q=0.5$$

$$e^{50t}\frac{dq}{dt}+50e^{50t}q=0.5e^{50t}$$

$$\frac{d}{dx}(e^{50t}q)=\ 0.5e^{50t}$$

$$e^{50t}q=0.01e^{50t}+c$$

$$q=0.01+ce^{-50t}$$

$$c=-0.01$$

$$\boxed{q=0.01-0.01e^{-50t}}$$

$$\boxed{i(t)=\frac{dq}{dt}=0.5e^{-50t}}$$

Section #3.2 (Non-Linear Models)

Logistic Equations

If we wanted to keep the Growth and Decay Model to a certain limit, meaning it would only grow/diminish till a certain point, we will use the logistic equation.

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}P\right), where\ r = Initial\ Population\ , K = Carrying\ Capacity$$

$$\frac{dP}{dt} = P(a - bP) \longrightarrow Simplified$$

$$\left(\frac{1/a}{P} + \frac{b/a}{a - bP}\right) dP = dt$$

$$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a - bP| = t + c$$

$$\frac{P}{a - bP} = c_1 e^{at}$$

$$\boxed{P = \frac{ac_1}{bc_1 + e^{-at}}}$$

$$\boxed{If\ P(O) = P_0 \ \Rightarrow P = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}}$$

Example #1

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

$$\frac{dx}{dt} = kx(1000 - x), x(0) = 1$$

$$Since\ we\ found\ that\ a = 1000k, b = \ k, and\ x(0) = 1$$

$$x = \frac{1000k}{k + 999ke^{-1000kt}} = \frac{1000}{1 + 999e^{-1000kt}}$$

$$50 = \frac{1000}{1 + 999e^{-4000k}}$$

$$k = \frac{1}{4}\ln\frac{19}{999} = -0.9906$$

$$\boxed{x(t) = \frac{1000}{1 + 999e^{990.6t}}}$$

$$x(6) = \frac{1000}{1 + 999e^{990.6\times 6}}$$

$$\boxed{x(6) = 276\ Students}$$

Chapter #4:

Section #4.1

Homogeneous Equations

A linear nth-order differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Is said to be Homogeneous.

However if

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Then it is said to be Nonhomogeneous.

Differential Operators

In calculus differentiation is often denoted by the capital letter D. The symbol D is called the differential operator because it transforms a differentiable function into another function.

$$\frac{dy}{dx} = Dy$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2y$$

$$In\ general\ \frac{d^ny}{dx^n} = D^ny$$

In general, we define an nth-order differential operator or polynomial operator to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

Where L is also known as the Linear Operator.

Any linear differential equation can be expressed in terms of the D notation. For example, the differential equation $y'' + 5y' + 6y = 5x - 3$ can also be written as $D^2 + 5D + 6)y = 5x - 3$.

Superposition Principle

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order differential equation on an interval I. Then the linear combination $y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$, where $c_i, i = 1,2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), \dots, f_n(x)$is said to be linearly dependent on an interval I if there exist constants $c_1, c_2, \dots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be linearly independent.

Also a set of n functions is linearly independent on the interval I if no one function is a linear combination of the other functions.

$$f_3(x) = c_1 f_1(x) + c_2 f_2(x)$$

Wronskian

Suppose each of the functions $f_1(x), f_2(x), \dots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W\big(f_1(x), f_2(x), \dots, f_n(x)\big) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1{}' & f_2{}' & \cdots & f_n{}' \\ \vdots & \vdots & & \vdots \\ f_1{}^{(n-1)} & f_2{}^{(n-1)} & \cdots & f_n{}^{(n-1)} \end{vmatrix}$$

where the primes denote derivatives, is called the Wronskian of the functions.

If Wronskian = 0 then functions are linearly dependent.

Fundamental Set of Solutions (Definition)

Any set $y_1, y_2, \dots, y_n$of n linearly independent solutions of the homogeneous linear nth-order differential equation on an interval I is said to be a fundamental set of solutions on the interval.

Section #4.2 (Reduction of Order)

Suppose that y_1 denotes a nontrivial solution of a Homogeneous Equation and that y_1 is defined on an interval I. We seek a second solution y_2 so that the set consisting of y_1 and y_2 is linearly independent on I.

If y_1 and y_2 are linearly independent, then their quotient y_1/y_2 is non constant on I—that is, $y_1(x)/y_2(x) = u(x)$ or $y_2(x) = u(x)\, y_1(x)$. The function $u(x)$ can be found by substituting $y_2(x) = u(x)\, y_1(x)$ into the given differential equation.

Example #1

Given that $y_1 = e^x$ is a solution of $y'' - y' = 0$ on the interval $(-\infty, \infty)$ use reduction of order to find a second solution y_2.

$$Let\ y = u(x)\, y_1 = ue^x$$
$$y' = ue^x + u'^{e^x}, y'' = ue^x + 2u'e^x + u''e^x$$
$$y'' - y' = e^x(u'' + 2u') = 0$$
$$u'' + 2u' = 0$$
$$Let\ w = u'$$
$$w' + 2w = 0 \longrightarrow Linear\ Differential\ Equations$$
$$w = c_1e^{-2x} \Longrightarrow u' = c_1e^{-2x}$$
$$u = -\frac{c_1}{2}e^{-2x} + c_2$$
$$\boxed{y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2e^x}$$
$$\boxed{y_2 = e^{-x}}$$

If we generalize the steps in the example given above, we can generate a formula for the steps above

$$\boxed{y_2 = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1(x)^2}dx}$$

This formula only gives the second term, and to create the general solution you need to add the given solution to it.

Section #4.3 (Homogeneous Linear Equations with Constant Coefficients)

$$ay'' + by' + c = 0$$

$$Let\ y = e^{mx}, y' = me^{mx}, y'' = m^2 e^{mx}$$

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$am^2 + bm + c = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1 *(b^2-4ac > 0)*

Where m_1 and m_2 are real and distinct roots (Using Superposition Principle to make fundamental set as they are linearly independent)

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Case 2 *(b^2-4ac = 0)*

Where m_1 and m_2 are real and repeated roots (Using Reduction of Order, since we only have 1 solution)

$$y = c_1 e^{mx} + c_2 x e^{mx} + \cdots + c_k x^{k-1} e^{mx}$$

Case 3 *(b^2-4ac < 0)*

Where m_1 and m_2 are conjugate complex roots (Using Case 1 and Euler's Formula of $e^{\iota\theta}$)

$$y = e^{ax}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Example #1

$$\frac{d^4 y}{dx^4} + 2\frac{d^2 y}{dx^2} + y = 0$$

$$After\ solving\ m_1 = m_3 = i\ and\ m_2 = m_2 = -i$$

$$y = C_1 e^{\iota x} + C_2 e^{-\iota x} + C_3 x e^{\iota x} + C_4 x e^{-\iota x}$$

$$Using\ Euler's\ Formula$$

$$C_1 e^{\iota x} + C_2 e^{-\iota x} = c_1 \cos x + c_2 \sin x$$

$$C_3 x e^{\iota x} + C_4 x e^{-\iota x} = c_3 x \cos x + c_4 x \sin x$$

$$\boxed{\therefore y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x}$$

Section #4.4 (Undetermined Coefficients—Superposition Approach)

Now we will learn how to solve a nonhomogeneous differential equation.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

First we solve the left side as if it were equal to zero, meaning we will do what we did in Section #4.3. The resulting equation would be termed as y_c(Complementary Function). Then we would assume a particular solution for $g(x)$and would find y_p.The final solution would be

$$y = y_c + y_p$$

How to find y_p

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

These are just some of the solutions.

IMPORTANT: When considering y_p make sure that there are no duplicate terms in y_p and y_c since they would just add up. To avoid this issue make sure to multiply by x and try again. Keep multiplying with x until you get a new term. Also this method only works when $g(x)$is algebraic, exponential, Sine, and Cosine. It doesn't work for logarithms, inverse trigonometric, Tangent, Secant, Cosecant, and Cotangent.

Example #1

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

$$y_c = c_1e^{3x} + c_2xe^{3x} \longrightarrow By\ solving\ Left\ Side$$

$$Assume\ y_p = Ax^2 + Bx + C + \underbrace{Ee^{3x}}_{Repeating} \implies y_p = Ax^2 + Bx + C + \underbrace{Ex^2e^{3x}}_{\substack{Since\ x\ was\ also\ repeating \\ we\ used\ to\ x^2}}$$

$$After\ differentiating\ y_p\ and\ putting\ in\ the\ equation\ we\ get\ values\ of\ A, B, C, and\ E$$

$$A = \frac{2}{3}, B = \frac{8}{9}, C = \frac{2}{3}, E = 6$$

$$\boxed{\therefore y = c_1e^{3x} + c_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} + 6x^2e^{3x}}$$

Section #4.5 (Undetermined Coefficients—Annihilator Approach)

We know that an nth-order differential equation can be written as such

$$a_n(x)D^n y \ + \ a_{n-1}(x)D^{n-1}y + \cdots + a_1(x)Dy + a_0(x)y = g(x)$$

This is very helpful since it can help us annihilate $g(x)$ which will affect the left side of the equation and will form a new solution which would include the particular solution. When seeking an annihilator operator, we seek the lowest possible

Some Annihilator Operators:

$g(x)$	Annihilator Operator L	Comment
$1, x, x^2, \dots, x^{n-1}$	$D, D^2, D^3, \dots, D^n$	Where n-1 is the power of x
$e^{\alpha x}, xe^{\alpha x}, x^2e^{\alpha x}, \dots, x^{n-1}e^{\alpha x}$	$(D-\alpha)^n$	Where n-1 is the power of x and α is the power e^x
$\cos\beta x, \sin\beta x$	$(D^2+\beta^2)$	Where β is the coefficient of x in the trigonometric function
$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \dots, x^{n-1}\cos\beta x\, e^{\alpha x}$ $e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \dots, x^{n-1}\sin\beta x\, e^{\alpha x}$	$[D^2 - 2\alpha D + (\alpha^2+\beta^2)]^n$	"

Example #1

$$y'' + y' = x\cos x - \cos x$$

$$y_c = c_1\cos x + c_2\sin x$$

$$(D^2+1)^2(D^2+1)[x\cos x - \cos x] = 0$$

$$(D^2+1)^3 y = 0 \Longrightarrow D = \underbrace{\pm\iota}_{Multiplicity\ of\ 3}$$

$$y = \underbrace{c_1\cos x + c_2\sin x}_{y_c} + \ c_3x\cos x + c_4x\sin x + \ c_5x^2\cos x + c_6x^2\sin x$$

$$y_p = Ax\cos x + Bx\sin x + \ Cx^2\cos x + Ex^2\sin x$$

$$After\ differentiating, substituting\ and\ comparing\ we\ get\ A = \frac{1}{4}, B = -\frac{1}{2}, C = 0, E = \frac{1}{4}$$

$$\boxed{y = \ c_1\cos x + c_2\sin x + \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2\sin x}$$

Section #4.6 (Variation of Parameters)

Since the methods above were very limited, we can use Variation of Parameters to bypass those limitations. The general idea of Variation of Parameters is that whenever a particular solution is formed, instead of giving it constants c_1, c_2 we will multiply with variables $u_1(x), u_2(x)$. Hence the particular solution will look like this

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

Then we differentiate this and substitute it into the original differential equation. In the end there will exist a system of solution which can be expressed in terms of determinants due to Crammers' Rule.

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \; and \; u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$$

$$where\; W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

The functions $u_1(x), u_2(x)$ are found by integrating the equations given above.

Example #1

$$4y'' + 36y = \csc 3x$$

$$y'' + 9y = \frac{1}{4}\csc 3x$$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

$$y_1 = \cos 3x \;, y_2 = \sin 3x$$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}, \quad W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

$$u_1' = \frac{W_1}{W} = -\frac{1}{12} \Longrightarrow u_1 = -\frac{1}{12}x, \quad u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x} \Longrightarrow u_2 = \frac{1}{36}\ln|\sin 3x|$$

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}\sin 3x \ln|\sin 3x|$$

$$\boxed{y_c = c_1\cos 3x + c_2 \sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}\sin 3x\ln|\sin 3x|}$$

Section #4.7 (Cauchy-Euler Equations)

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

If a differential equation is in form given above, then it can solved by the following method:

$$ax^2 y'' + bxy' + cy = 0$$

$$Let\ y = x^m$$

$$ax^2 m(m-1)x^{m-2} + bxmx^{m-1} + cx^m = 0$$

$$m(m-1)ax^m + mbx^m + cx^m = 0 \implies m(m-1)a + mb + c = 0$$

Case 1 (b^2-4ac > 0)

Where m_1 and m_2 are real and distinct roots (Using Superposition Principle to make fundamental set as they are linearly independent)

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case 2 (b^2-4ac = 0)

Where m_1 and m_2 are real and repeated roots (Using Reduction of Order, since we only have 1 solution)

$$y = c_1 x^m + c_2 x^m \ln x$$

Case 3 (b^2-4ac < 0)

Where m_1 and m_2 are conjugate complex roots (Using Case 1 and Euler's Formula of $e^{\iota\theta}$)

$$y = x^{\alpha}[c_1 \cos \beta \ln x + c_2 \sin \beta \ln x]$$

Example #1

$$4x^2 y'' + 17y = 0.\ \ y(1) = \ -1\,, y'(1) = -\frac{1}{2}$$

$$let\ y = x^m$$

$$x^m(4m^2 - 4m + 17) = 0$$

$$m = \frac{1}{2} \pm 2i$$

$$\boxed{y = x^{\frac{1}{2}}[c_1 \cos 2 \ln x + c_2 \sin 2 \ln x]}$$

Chapter #11:

Section #11.1

Inner Product

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx$$

The inner product of any two functions is the integral of their multiple.

Norm

The norm can be expressed as such

$$\|\phi_n\|^2 = \int_a^b {\phi_n}^2(x) dx$$

$$Norm = \|\phi_n\| = \sqrt{\int_a^b {\phi_n}^2(x) dx}$$

Orthogonal Functions

$$If\,(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0 \;then\; f_1 \;and\; f_2 \;said\; to\; be\; Orthogonal$$

Orthogonal Sets

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$is said to be orthogonal on an interval [a, b] if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x) \phi_n(x) dx = 0\,, m \neq n$$

Orthonormal Sets

If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval [a, b] with the additional property that $\|\phi_n(x)\| = 1$ for $n = 0,1,2, \dots$, then $\{\phi_n(x)\}$ is said to be an orthonormal set on the interval.

Orthonormal sets can be formed by dividing each function in the orthogonal set by its norm.

Orthogonal Series Expansion

$$If\ f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots ?$$

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x)$$

$$then \int_a^b f(x)\phi_m dx = c_0 \int_a^b \phi_0(x)\phi_m(x)dx + \cdots + c_n \int_a^b \phi_n(x)\phi_m(x)dx + \cdots$$

$$= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots$$

$$All\ that\ is\ left\ is \int_a^b f(x)\phi_n dx = c_n \int_a^b \phi_n(x)^2 dx \quad due\ to\ orthogonality$$

$$c_n = \frac{\int_a^b f(x)\phi_n dx}{\int_a^b \phi_n(x)^2 dx}$$

$$c_n = \frac{\int_a^b f(x)\phi_n dx}{\|\phi_n(x)\|^2}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2}\phi_n(x)$$

Section #11.2 (Fourier Series)

Suppose that f is a function defined on the interval $(-p,p)$ and can be expanded in an orthogonal series consisting of the trigonometric functions

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

We can use the laws of orthogonality to find a_0 , a_n , b_n.

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx$$

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}x\,dx$$

Convergence at Discontinuity

Let f and f' be piecewise continuous on the interval$[-p,p]$. Then for all x in the interval$(-p,p)$, the Fourier series of f converges $to\ f\ (x)$ at a point continuity. At a point of discontinuity the Fourier series converges to the average

$$\frac{f(x+)+f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ are the right and left hand limits of f at x, respectively.

Example #1

$$f(x)=\begin{cases}\pi^2, & -\pi<x<0\\ \pi^2-x^2, & 0\le x<\pi\end{cases}$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}\pi^2dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi^2-x^2)dx=\frac{1}{\pi}[\pi^3]+\frac{1}{\pi}\left[\frac{2}{3}\pi^3\right]=\frac{5}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}\pi^2\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi^2-x^2)\cos nx\,dx$$

$$a_n=[0]+\frac{1}{\pi}\left(\frac{\pi^2-x^2}{n}\sin nx\Big|_0^{\pi}+\frac{2}{n}\int_0^{\pi}x\sin nx\,dx\right)=\frac{2}{n^2}(-1)^{n+1}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}\pi^2\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi^2-x^2)\sin nx\,dx$$

$$b_n=\frac{\pi}{n}[(-1)^n-1]+\frac{1}{\pi}\frac{1}{\pi}\left(\frac{\pi^2-x^2}{n}\cos nx\Big|_0^{\pi}-\frac{2}{n}\int_0^{\pi}x\cos nx\,dx\right)=\frac{\pi}{n}(-1)^n+\frac{2}{n^3\pi}[1+(-1)^{n+1}]$$

$$\boxed{f(x)=\frac{5}{6}\pi^2+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}(-1)^{n+1}\cos nx+\left(\frac{\pi}{n}(-1)^n+\frac{2}{n^3\pi}[1+(-1)^{n+1}]\right)\sin nx\right]}$$

Section #11.2 (Fourier Cosine and Sine Series)

Even and Odd Functions

The Fourier series can be simplified using identities of even and odd functions.

$$\text{even if } f(-x) = f(x) \quad \text{and} \quad \text{odd if } f(-x) = -f(x).$$

There are some properties of even/odd functions:

1. The product of two even functions is even.
2. The product of two odd functions is even.
3. The product of an even function and an odd function is odd.
4. The sum (difference) of two even functions is even.
5. The sum (difference) of two odd functions is odd.
6. If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
7. If f is odd, then $\int_{-a}^{a} f(x)dx = 0$

Cosine and Sine Series

If f is an even function on $(-p, p)$, then

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\frac{n\pi}{p}x}_{even}\, dx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p} dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\frac{n\pi}{p}x}_{odd}\, dx = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x\right)$$

If f is an odd function on $(-p, p)$, then

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\, dx = b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}x\, dx$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{n\pi}{p}x\right)$$

Half-Range Expansions

Whenever we were integrating, we were taking the origin as the midpoint. What if we wanted to find the Fourier series over an interval$[0\,,L]$?

Example #1

$$Expand\; f(x)=x^2, 0<x<L,$$

$$in\; a\; cosine\; series, sine\; series, and\; Fourier\; series$$

$$a_0=\frac{2}{L}\int_0^L x^2dx=\frac{2}{3}L^2, a_n=\frac{2}{L}\int_0^L x^2\cos\frac{n\pi}{L}x\,dx=\frac{4L^2(-1)^n}{n^2\pi^2}$$

$$\boxed{\therefore f(x)=\frac{1}{3}L^2+\sum_{n=1}^{\infty}\left(\frac{4L^2(-1)^n}{n^2\pi^2}\right)\cos\frac{n\pi}{L}x}$$

$$b_n=\frac{2}{L}\int_0^L x^2\sin\frac{n\pi}{L}x\,dx=\frac{2L^2(-1)^{n+1}}{n\pi}+\frac{4L^2}{n^3\pi^3}[(-1)^n-1]$$

$$\boxed{\therefore f(x)=\sum_{n=1}^{\infty}\left[\frac{2L^2(-1)^{n+1}}{n\pi}+\frac{4L^2}{n^3\pi^3}[(-1)^n-1]\right]\sin\frac{n\pi}{L}x}$$

$$p=\frac{L}{2}$$

$$a_0=\frac{1}{p}\int_0^{2p}f(x)dx=\frac{2}{L}\int_0^L x^2dx=\frac{2}{3}L^2$$

$$a_n=\frac{1}{p}\int_0^{2p}f(x)\cos\frac{n\pi}{p}x\,dx\Rightarrow a_n=\frac{2}{L}\int_0^L x^2\cos\frac{2n\pi}{L}x\,dx=\frac{L^2}{n^2\pi^2}$$

$$b_n=\frac{1}{p}\int_0^{2p}f(x)\sin\frac{n\pi}{p}x\,dx\Rightarrow b_n=\frac{2}{L}\int_0^L x^2\sin\frac{2n\pi}{L}x\,dx=-\frac{L^2}{n\pi}$$

$$\boxed{\therefore f(x)=\frac{1}{3}L^2+\sum_{n=1}^{\infty}\left[\left(\frac{L^2}{n^2\pi^2}\right)\cos\frac{2n\pi}{L}x-\left(\frac{L^2}{n\pi}\right)\sin\frac{2n\pi}{L}x\right]}$$

Chapter #12:

Section #12.1 Partial Differential Equations

If we let u denote the dependent variable and let x and y denote the independent variables, then the general form of a linear second-order partial differential equation is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x,y)$$

When $G(x,y) = 0$ the equation is said to be homogeneous otherwise it is nonhomogeneous.

Separation of Variables

One of the ways to find a solution to a partial differential equation is to suppose

$$u = X(x)Y(y)$$

We can easily differentiate this partially and substitute in the equation. The resulting equation would be a function dependent solely on x and a function dependent solely on y, which means each side of the equation must be a constant. Therefore to separate them we use a separation constant.

For the equation

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

$$X''Y = 4XY' \Longrightarrow \frac{1}{4X}X'' = \frac{1}{y}Y' = -\lambda$$

$$X'' + 4\lambda X = 0, \qquad and, \qquad Y' + \lambda y = 0$$

Case 1 ($\lambda = 0$)

$$\lambda = 0$$

$$X'' = 0 \quad \text{and} \quad Y' = 0$$

$$X = c_1 x + c_2 \quad \text{and} \quad Y = c_3$$

$$u = XY = (c_1 + c_2 x)c_3 = A_1 + B_1 x$$

Case 2 ($\lambda < 0$)

$$\lambda = -\alpha^2$$

$$X'' - 4\alpha^2 X = 0 \;\text{ and }\; Y' - \alpha^2 Y = 0$$

$$X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \quad and \quad Y = c_6 e^{\alpha^2 y}$$

$$u = XY = A_2 e^{\alpha^2 y}\cosh 2\alpha x + B_2 e^{\alpha^2 y}\sinh 2\alpha x$$

Case 3 ($\lambda > 0$)

$$\lambda = \alpha^2$$

$$X'' + 4\alpha^2 X = 0 \text{ and } Y' + \alpha^2 Y = 0$$

$$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x \quad and \quad Y = c_9 e^{-\alpha^2 y}$$

$$u = XY = A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_3 e^{-\alpha^2 y} \sin 2\alpha x$$

Superposition Principle

If $u_1, u_2, \ldots, u_k$ are solutions of the homogeneous linear partial differential equation, then the linear combination $u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$, where $c_i, i = 1,2, \ldots, k$ are arbitrary constants, is also a solution.

$$u = \sum_{k=1}^{\infty} c_k u_k$$

Classification

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y)$$

where $A, B, C, D, E, F,$ and G are real constants, is said to be

- Hyperbolic if $B^2 - 4AC > 0$
- Parabolic if $B^2 - 4AC = 0$
- Elliptic if $B^2 - 4AC < 0$

Section #12.3 Heat Equation

Consider a thin rod of length L with an initial temperature $f(x)$ throughout and whose ends are held at temperature zero/are insulated for all time $t > 0$. The temperature $u(x,t)$ in the rod is determined from the boundary-value problem.

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \qquad t > 0$$
$$u(0,t) = 0, \qquad u(L,t) = 0$$
$$u(x,0) = f(x), \qquad 0 < x < L$$

Solution:

$$let\ u = XT$$
$$kX''T = XT'$$
$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$
$$X'' + \lambda X = 0, \qquad and, \qquad T' + k\lambda T = 0$$

$$X_1 = c_1 + c_2 x, \qquad \lambda = 0$$
$$X_2 = c_1 \cosh \alpha x + c_2 \sinh \alpha x\,, \qquad \lambda = -\alpha^2$$
$$X_3 = c_1 \cos \alpha x + c_2 \sin \alpha x\,, \qquad \lambda = \alpha^2$$

Using boundary conditions X_1 and X_2 are trivial solutions, however X_3 results in

$$\alpha = \frac{n\pi}{L}$$
$$X_3 = c_2 \sin\frac{n\pi}{L}x\,, \qquad T = c_3 e^{-k\frac{n^2\pi^2}{L^2}t}$$
$$u = A_n \sin\frac{n\pi}{L}x\, e^{-k\frac{n^2\pi^2}{L^2}t}$$
$$using\ superposition\ principle$$
$$\left. u = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x\, e^{-k\frac{n^2\pi^2}{L^2}t} \right\} \begin{matrix} This\ represents\ a\ half-range \\ expansion\ of\ sine\ series \end{matrix}$$
$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\, dx$$
$$\boxed{u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right)\sin\frac{n\pi}{L}x\, e^{-k\frac{n^2\pi^2}{L^2}t}}$$

This formula is only valid for a certain/specific condition. The formula used can be changed depending upon the situation. For example if the ends are not at zero temperature but instead are insulated.

Example #1

Suppose heat is lost from the lateral surface of a thin rod of length L into a surrounding medium at temperature zero. If the linear law of heat transfer applies, then the heat equation takes on the form.

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}$$

$0 < x < L, t > 0, h$ is a constant. Find the temperature $u(x,t)$ if the initial temperature is $f(x)$ throughout and the ends $x = 0$ and $x = L$ are insulated.

$$kX''T - hXT = XT'$$

$$\frac{kX'' - hX}{X} = \frac{T'}{T}$$

$$\frac{X''}{X} = \frac{T'}{kT} + \frac{h}{k} = -\lambda$$

$$X'' + \lambda X = 0, \qquad T' + (h + k\lambda)T = 0$$

$$\textit{Since } \lambda = 0, \lambda = -\alpha^2 \textit{give trivial solutions we will skip them}$$

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$\textit{Since the ends are insulated the rate of loss of heat is ideally } 0 \textit{ so;}$$

$$X'(0) = 0, \qquad X'(L) = 0$$

$$X' = -c_1\alpha \sin \alpha x + c_2 \alpha \cos \alpha x \Longrightarrow 0 = 0 + c_2\alpha \Longrightarrow c_2 = 0$$

$$X' = -c_1 \alpha \sin \alpha x \Longrightarrow 0 = -c_1\alpha \sin \alpha L$$

$$\textit{To avoid a trivial solution } c_1 \neq 0$$

$$\sin \alpha L = 0 \Longrightarrow \alpha L = n\pi \Longrightarrow \alpha = \frac{n\pi}{L}$$

$$X = c_1 \cos \frac{n\pi}{L} x, \qquad T = c_3 e^{-k\frac{n^2\pi^2}{L^2}t}$$

$$u = A_n \cos \frac{n\pi}{L} x\, e^{-k\frac{n^2\pi^2}{L^2}t}$$

$$u(x,0) = f(x) = \underbrace{A_n \cos \frac{n\pi}{L} x}_{\textit{Represents a Cosine Series}}$$

$$u = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x\, e^{-k\frac{n^2\pi^2}{L^2}t}$$

$$\boxed{u = \frac{1}{L}\int_0^L f(x)dx + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x) \cos \cos \frac{n\pi}{L} x\, dx\right) \cos \frac{n\pi}{L} x\, e^{-k\frac{n^2\pi^2}{L^2}t}}$$

Section #12.4 Wave Equation

The vertical displacement $u(x,t)$ of the vibrating string of length L is determined from

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < L, \qquad t > 0$$

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \qquad 0 < x < L$$

Solution

$$\frac{X''}{X} = \frac{T''}{\alpha^2 T} = -\lambda$$

$$Since\ \lambda = 0, \lambda = -\alpha^2 give\ trivial\ solutions\ we\ will\ skip\ them$$

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$After\ putting\ boundary\ condition$$

$$X_3 = c_2 \sin \frac{n\pi}{L} x,$$

$$T = c_1 \cos \frac{n\pi\alpha}{L} t + c_2 \sin \frac{n\pi\alpha}{L} t$$

$$\boxed{u = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi\alpha}{L} t + B_n \sin \frac{n\pi\alpha}{L} t \right) \sin \frac{n\pi}{L} x}$$

$$Using\ initial\ conditions\ to\ solve\ for\ A_n$$

$$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx}$$

$$However\ to\ find\ B_n, we\ have\ to\ differentiate\ u\ with\ respect\ to\ t$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi\alpha}{L} \sin \frac{n\pi\alpha}{L} t + B_n \frac{n\pi\alpha}{L} \cos \frac{n\pi\alpha}{L} t \right) \sin \frac{n\pi}{L} x$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty} \left(B_n \frac{n\pi\alpha}{L} \right) \sin \frac{n\pi}{L} x$$

$$\boxed{B_n = \frac{2}{n\pi\alpha} \int_0^L g(x) \sin \frac{n\pi}{L} x\, dx}$$

Derivatives

Exponential, Logarithmic, and Common Derivatives

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\frac{d}{dx}(a^x) = a^x \ln a$

$\frac{d}{dx}(\ln v) = \frac{v'}{v}$	$\frac{d}{dx}(\log_a v) = \frac{v'}{(\ln a)v}$
$\frac{d}{dx}(uv) = uv' + u'v$	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - v'u}{v^2}$

Trigonometric Derivatives

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\cot x) - \csc^2 x$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$	$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$
$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}}$	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$	$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$

Hyperbolic Derivatives

$\frac{d}{dx}(\sinh x) = \cosh x$	$\frac{d}{dx}(\cosh x) = \sinh x$	$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$	$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$	$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}$	$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$	$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}$
$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{x\sqrt{x^2 + 1}}$	$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1 - x^2}}$	$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{x^2 - 1}$

Integrals

Exponential, Logarithmic, and Common Integrals

$\int a dx = ax + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$
$\int \frac{1}{x} dx = \ln x + C$	$\int \log_a x \, dx = x \log_a \left(\frac{x}{e}\right)$	$\int a^x dx = \frac{a^x}{\ln a} + C$

$$\int u dv = u \int v dx - \int \left[\frac{du}{dx} \times \int v dx\right] dx$$

$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$	$\int f^n(x) f'(x) dx = \frac{f^{n+1}(x)}{n+1} + C$

Trigonometric Integrals

$\int \sin x \, dx = -\cos x + C$	$\int \cos x \, dx = \sin x + C$	$\int \tan x \, dx = \ln(\sec x) + C$
$\int \csc x \, dx = -\ln(\csc x + \cot x) + C$	$\int \sec x \, dx = \ln(\sec x + \tan x) + C$	$\int \cot x \, dx = \ln(\sin x) + C$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C$$

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \sqrt{1 - x^2} + C$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \ln \sqrt{x^2 + 1} + C$$

$$\int \csc^{-1} x \, dx = x \csc^{-1} x + \ln\left(x + \sqrt{x^2 - 1}\right) + C$$

$$\int \sec^{-1} x \, dx = x \sec^{-1} x - \ln\left(x + \sqrt{x^2 - 1}\right) + C$$

$$\int \cot^{-1} x \, dx = x \cot^{-1} x + \ln\left(\sqrt{x^2 + 1}\right)$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{(n-1)}{n} \int \cos^{n-2} x \, dx + C$$

$$\int \sin^{n-1} x \, dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{(n-1)}{n} \int \sin^{n-2} x \, dx + C$$

Trig Substitution:

Expression	Substitution	Domain	Simplification
$\sqrt{a^2 - u^2}$	$u = a \sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\sqrt{a^2 - u^2} = a\cos\theta$
$\sqrt{a^2 + u^2}$	$u = a \tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\sqrt{a^2 + u^2} = a\sec\theta$
$\sqrt{u^2 - a^2}$	$u = a \sec\theta$	$0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$	$\sqrt{u^2 - a^2} = a\tan\theta$

Hyperbolic Integrals

$\int \sinh x\, dx = \cosh x + C$	$\int \cosh x\, dx = \sinh x + C$	$\int \tanh x\, dx = \ln \cosh x + C$
$\int \operatorname{csch} x\, dx = \ln\lvert\coth x - \operatorname{csch} x\rvert + C$	$\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x) + C$	$\int \coth x\, dx = \ln \sinh x + C$

$$\int \sinh^{-1} ax\, dx = x \sinh^{-1} ax - \frac{\sqrt{a^2x^2+1}}{a} + C$$

$$\int \cosh^{-1} ax\, dx = x \cosh^{-1} ax - \frac{\sqrt{a^2x^2-1}}{a} + C$$

$$\int \tanh^{-1} ax\, dx = x \tanh^{-1} ax + \frac{\ln(1-a^2x^2)}{2a} + C$$

$$\int \operatorname{csch}^{-1} ax\, dx = x \operatorname{csch}^{-1} ax + \frac{1}{a}\sinh^{-1} ax + C$$

$$\int \operatorname{sech}^{-1} ax\, dx = x \operatorname{sech}^{-1} ax + \frac{1}{a}\sin^{-1} ax + C$$

$$\int \coth^{-1} ax\, dx = x \coth^{-1} ax + \frac{1}{2a}\ln(a^2x^2-1)$$

Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$

Double Angle Formulas

$\sin 2x = 2 \sin x \cos x$	$\cos 2x = \cos^2 x - \sin^2 x$	$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

Product to Sum Formulas

$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$	$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$	$\cos \alpha \sin \beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

Sum to Product Formulas

$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$	$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
$\sin \alpha - \sin \beta = 2 \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	$\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$