Diffraction Gratings and Its Implications in Negative Index Nanomaterials

Maanika Khanna

September 2025

1 Introduction

2 Derivation of Generic Intensity of Slit Diffraction

Single-slit diffraction. Consider a slit of width d. The contribution from a small element ds of the slit is

$$\begin{aligned} dE &= (c_1\, ds)\cos\bigl(k(x+\Delta x)-\omega t\bigr) && (1)\\ &= (c_1\, ds)\cos\bigl(kx-\omega t+k\Delta x\bigr) && (2)\\ &= (c_1\, ds)\cos\bigl(ks\sin\theta+kx-\omega t\bigr), && (3) \end{aligned}$$

where s is the coordinate across the slit aperture.

The total field is

$$\begin{aligned} E_p &= \int_0^d c_1\cos\bigl(ks\sin\theta+kx-\omega t\bigr)\,ds && (4)\\ &= \frac{c_1}{k\sin\theta}\Bigl[\sin\bigl(ks\sin\theta+kx-\omega t\bigr)\Bigr]_0^d && (5)\\ &= \frac{c_1}{k\sin\theta}\Bigl(\sin\bigl(kd\sin\theta+kx-\omega t\bigr)-\sin\bigl(kx-\omega t\bigr)\Bigr). && (6) \end{aligned}$$

Using the identity $\sin A-\sin B = 2\cos\bigl(\frac{A+B}{2}\bigr)\sin\bigl(\frac{A-B}{2}\bigr)$,

$$E_p = \frac{2c_1}{k\sin\theta}\sin\Bigl(\tfrac{kd\sin\theta}{2}\Bigr)\cos\Bigl(kx-\omega t+\tfrac{kd\sin\theta}{2}\Bigr). \tag{7}$$

Thus the amplitude is

$$A(\theta) = \frac{2c_1}{k\sin\theta}\sin\Bigl(\tfrac{kd\sin\theta}{2}\Bigr).$$

Since intensity is proportional to the square of the amplitude,

$$I(\theta) \propto A(\theta)^2.$$

Define

$$\beta = \frac{kd\sin\theta}{2} = \frac{2\pi}{\lambda}\,\frac{d}{2}\sin\theta = \frac{\pi d \sin\theta}{\lambda}.$$

Then the normalized intensity becomes

$$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2. \tag{8}$$

Conditions.

- **Minima:** $\beta = n\pi \quad\Rightarrow\quad \sin\theta = \frac{n\lambda}{d},\ n \neq 0.$
- **Maxima (approximate):** $\beta = (n + \frac{1}{2})\pi \quad\Rightarrow\quad \sin\theta = \frac{(2n+1)\lambda}{2d}.$

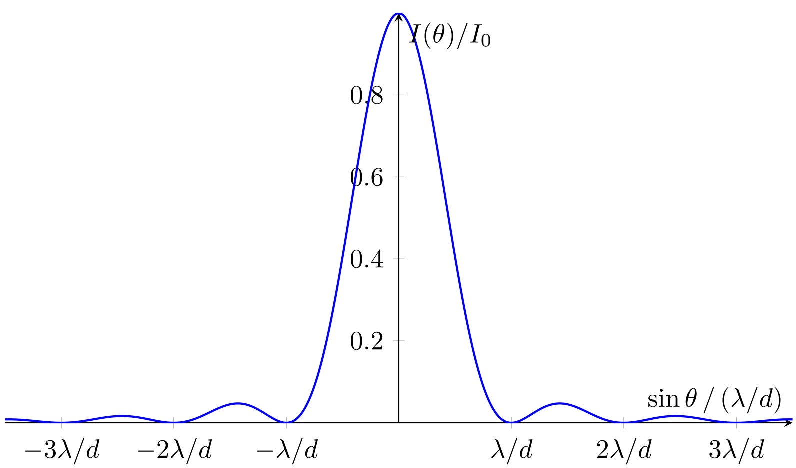

Figure 1: Single-slit diffraction intensity vs. $\sin\theta$ in units of λ/d. Intensities are normalized so that the central maximum equals 1.

3 Double-Slit Diffraction and Interference

Now consider two slits of width d separated by a center-to-center distance s.

Define

$$\beta = \frac{\pi d \sin\theta}{\lambda}, \qquad \delta = \frac{2\pi s \sin\theta}{\lambda}. \tag{9}$$

The single-slit amplitude is

$$A(\theta) = A_0 \frac{\sin\beta}{\beta}.$$

With two slits, the total field is

$$E_{\mathrm{tot}}(\theta,t) = 2A(\theta)\cos\Big(\tfrac{\delta}{2}\Big)\cos\Big(\omega t + \tfrac{\delta}{2}\Big).$$

Thus the intensity is

$$\begin{aligned} I(\theta) &\propto \Big[2A(\theta)\cos\big(\tfrac{\delta}{2}\big)\Big]^2 && (10)\\ &= 4A_0^2\cos^2\Big(\tfrac{\delta}{2}\Big)\frac{\sin^2\beta}{\beta^2}. && (11)\end{aligned}$$

Defining $I_0 = 4A_0^2$, we obtain

$$\boxed{I(\theta) = I_0\cos^2\Big(\tfrac{\delta}{2}\Big)\frac{\sin^2\beta}{\beta^2}} \qquad (12)$$

where $\dfrac{\sin^2\beta}{\beta^2}$ is the single-slit envelope and $\cos^2(\delta/2)$ gives the interference fringes.

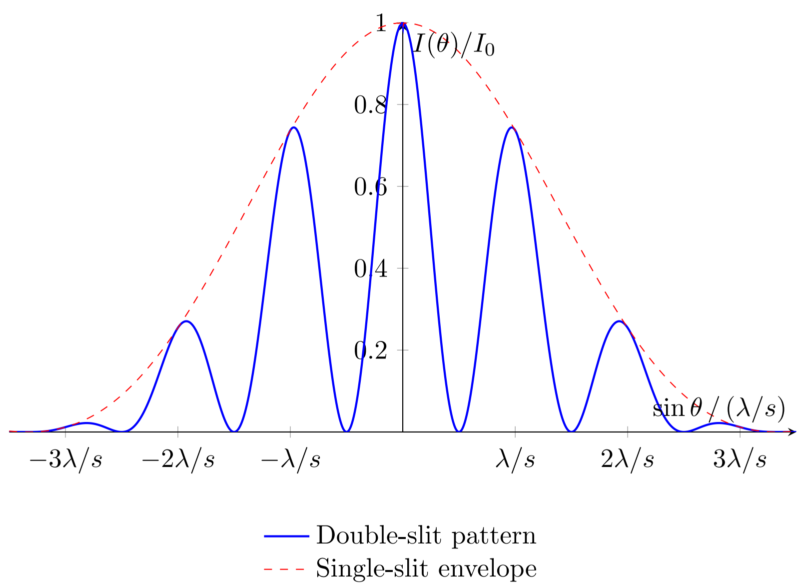

Figure 2: Double-slit diffraction intensity pattern for $d/s = 0.3$. The blue curve shows the interference fringes modulated by the red single-slit envelope.

4 N-Slit Diffraction (Diffraction Gratings)

For N identical slits of width d and separation s, the amplitude is the product of the single-slit diffraction term and the N-slit interference factor:

$$E(\theta) \propto \frac{\sin\beta}{\beta} \cdot \frac{\sin(N\delta/2)}{\sin(\delta/2)}. \tag{13}$$

Thus the intensity is

$$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right)^2, \tag{14}$$

where

$$\beta = \frac{\pi d \sin\theta}{\lambda}, \quad \delta = \frac{2\pi s \sin\theta}{\lambda}.$$

This shows how diffraction gratings produce very sharp principal maxima due to constructive interference of many slits, while the single-slit envelope determines the overall intensity profile.

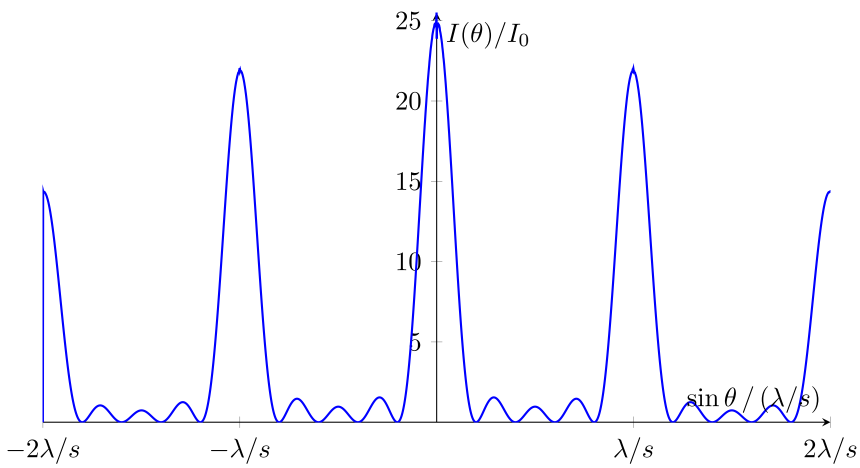

Figure 3: Five-slit diffraction pattern ($N = 5$, $d/s = 0.2$). Narrow, sharp principal maxima appear within the single-slit envelope.

5 Rayleigh Criterion and Diffraction Limit

The finite aperture of a circular lens produces a diffraction pattern known as the **Airy disk**. According to the Rayleigh criterion, two point sources are just resolvable when the central maximum of one coincides with the first minimum of the other.

Angular Resolution

The angular radius θ_R of the Airy disk is approximately

$$\theta_R = 1.22\,\frac{\lambda}{D}, \tag{15}$$

where λ is the wavelength and D is the aperture diameter. This sets the fundamental diffraction-limited resolution of an optical system.

Airy Disk Intensity Profile (Approximate)

While the exact Airy pattern involves Bessel functions, we can approximate the qualitative structure using a central peak with weaker rings. The first minimum is placed near θ_R.

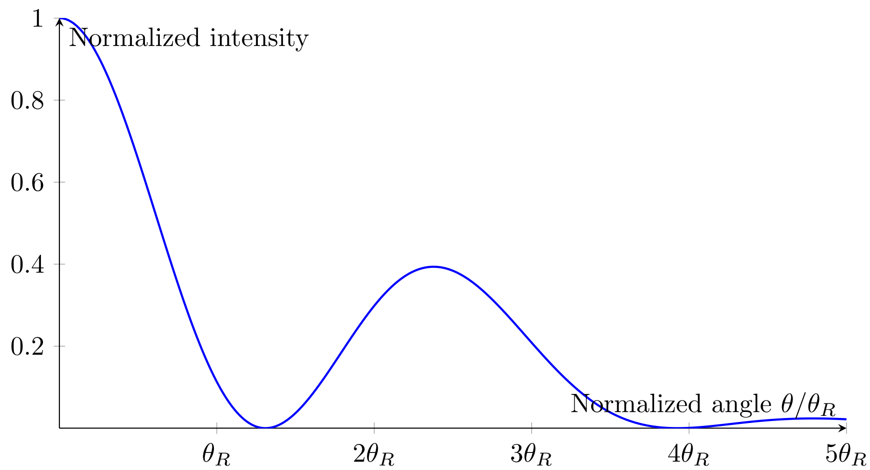

Figure 4: Approximate Airy disk intensity distribution as a function of normalized angle. The first minimum occurs near θ_R, consistent with the Rayleigh criterion.

This demonstrates how the diffraction limit is set by the aperture size and wavelength, providing a natural boundary to optical resolution.

6 Effect of Negative Index Metamaterials on Diffraction Limit

Negative index metamaterials (NIMs) have the unusual property that their refractive index n is negative. This leads to **negative refraction**, which alters the propagation of light through the material. Specifically, the effective wave-

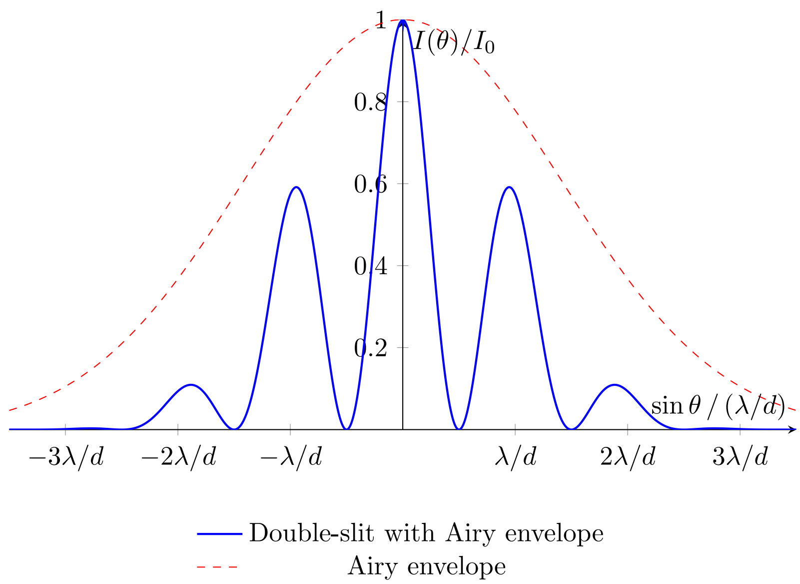

Figure 5: Double-slit diffraction pattern modulated by an Airy disk-like envelope.

length inside the metamaterial becomes

$$\lambda_{\text{eff}} = \frac{\lambda_0}{|n|},$$

where λ_0 is the free-space wavelength.

Since the diffraction limit of a circular aperture is proportional to the wavelength divided by the aperture size:

$$\theta_{\text{min}} \approx 1.22\frac{\lambda}{D},$$

a negative index material with $n = -1$ effectively **reduces the wavelength inside the medium**, leading to a smaller angular diffraction limit. This allows finer details to be resolved and can shrink the Airy disk, enabling **subwavelength resolution**.

Double-Slit Diffraction Modulated by Airy Disk in a NIM ($n = -1$)

To illustrate, we approximate the double-slit pattern under the Airy-like envelope, with the wavelength inside the metamaterial halved (since $|n| = 1$). The result is **narrower fringes** and a tighter envelope:

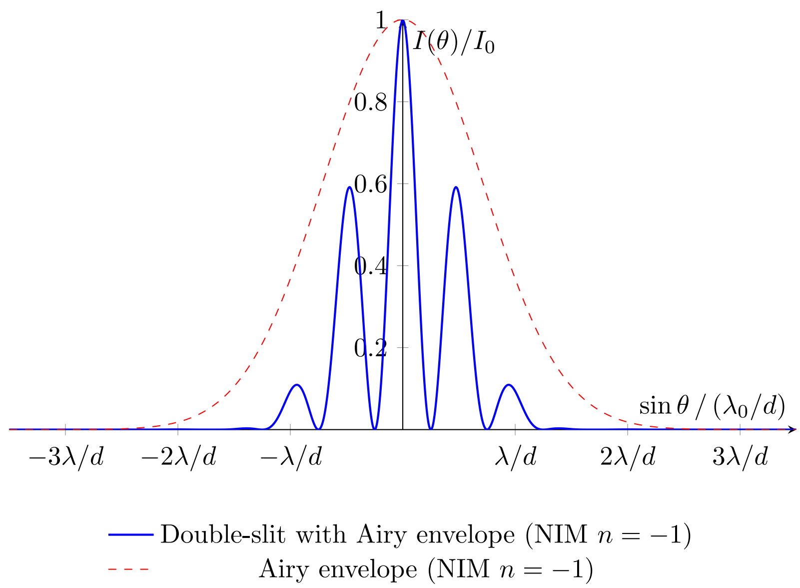

Figure 6: Double-slit diffraction pattern modulated by an Airy-like envelope inside a negative index metamaterial ($n = -1$). The central maximum is narrower, indicating a reduced diffraction limit.

Explanation. Because $n < 0$, the light refracts in the opposite direction compared to conventional materials, effectively reducing the wavelength inside the medium. This shifts diffraction minima closer together and compresses the Airy envelope, lowering the angular diffraction limit. Hence, finer features of the interference pattern can be resolved, demonstrating the **super-resolution potential of negative index metamaterials**.