Prepromorphisms

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We study a recursion scheme that differs from the scheme for catamorphisms in that the recursive calls are preceded by some *prepro*cessing. We give sufficient conditions under which such a scheme has a unique solution. The solution, if it exists, is termed a *prepromorphism*. An important prepromorphism is the so-called f-*cascade*. Of course, dualisation applies here; it provides the solution to the problem of proving the equivalence of two ways of defining f-*iterate*.

The proof technique may be of greater importance than the particular theorem for which it is used here; it uses infinite tuplings of morphisms.

1 Introduction

Let $\mathsf{F} = A\dagger$ and $in = \mu\mathsf{F}$ and $* =$ the map functor for $\dagger$. (We use the notation and terminology of Fokkinga and Meijer [3], but we do not suppose that the category we are working in is an O-category. The semicolon denotes composition and has greatest separation power; expression $a.f$ denotes "a subject to f". The section $A\dagger$ equals the mono-functor $A \dagger \mathrm{I}$.) For given $f : A \to A$ we define f-*cascade*, denoted $f^{\bowtie}$, by

$$f^{\bowtie} \quad = \quad (\!|id \dagger f * ; in|\!) \quad : \quad A* \to A* \ .$$

When we take $\dagger$ to be the functor for lists, we have informally

$$[a_0, a_1, \ldots, a_{n-1}].f^{\bowtie} \quad = \quad [a_0.f^0, a_1.f^1, \ldots, a_{n-1}.f^{n-1}]$$

where $f^i = f ; f ; \ldots ; f$ (i occurrences of f). One should not confuse f-cascade with f-iterate, denoted f^ω and defined by

$$\begin{array}{rcll} f^\omega & = & [\![\Delta ; id \times f]\!] & : \quad A \to A\infty \\ a.f^\omega & = & [a.f^0, a.f^1, \ldots] & : \quad A\infty \end{array}$$

where ∞ is the (final) map for functor $\times$. Also, recall law CATA FACTORISATION1:

$$(\!|f \dagger id ; \phi|\!) \quad = \quad f * ; (\!|\phi|\!)$$

and observe that it is not applicable to f-cascade.

The problem that we address in this note is: how to prove that $f^{\bowtie}$ is the unique solution of Equation (a) below; notice that by Cata Characterisation $f^{\bowtie}$ *is* the unique solution of Equation (b).

$$
\begin{array}{llll}
\text{(a)} & x & = & in^{\cup} ; (f* ; x)\mathsf{F} ; in \\
\text{(b)} & y & = & in^{\cup} ; (y ; f*)\mathsf{F} ; in \,.
\end{array}
$$

Of course, we wish also to generalise this and to abstract from the particular morphisms occurring in these equations. In order to appreciate the problem, we urge the reader to try to prove that Equation (a) has at most one solution; it is quite simple that $f^{\bowtie}$ is a solution since it is readily verified that

$$f* ; f^{\bowtie} \quad = \quad f^{\bowtie} ; f* \,.$$

2 Generalisation: Prepromorphisms

Let F be arbitrary, $in = \mu\mathsf{F}$, and $M = in_{>}$. Let $\phi : M \to M$ and $\psi : \mathsf{F}$-algebra be arbitrary. Consider the following equation in x

$$x \quad = \quad in^{\cup} ; (\phi ; x)\mathsf{F} ; \psi \,.$$

We call this equation a *prepromorphism* equation. To show that there is just one solution x, let us reason informally in **Set** and for polynomial functors only. When ϕ is the identity, the argument to the recursive call is a subvalue, a proper constituent, of the original argument. Therefore, intuitively, the outcome of x is completely determined and well-defined by induction on the argument value: x equals $(\!|\psi|\!)$. (In **Set** and for polynomial F, any value in M is constructed by repeated, finitely many, applications of in.) Now suppose that ϕ differs from id but still preserves "the structure", in the same way as a map preserves the structure of its argument, and, on lists, any function that does not change the length. Then intuitively we think that the equation has a unique solution, since at the recursive call in the right-hand side the argument's structure is a substructure of the original argument, and so by induction on the structure of the argument (rather than on the value of the argument) x's outcome is completely determined and well-defined.

One attempt at formalising "structure preservation" is this:

$$\phi \text{ preserves the structure} \quad \equiv \quad shape ; \phi = \phi ; shape$$

where $\mathsf{F} = A\dagger$ and $shape = !*$ ($!$ being the unique morphsim into the final object $\mathbb{1}$). For example, any map is shape preserving. However, this definition does not apply to arbitrary F; it defines "$\dagger$-structure preservation" only. And worse, suppose F is given and $\dagger$ is subsequently defined by $x \dagger y = y\mathsf{F}$ so that $\mathsf{F} = A\dagger$. Then $shape = !* = (\!|! \dagger id ; in|\!) = (\!|in|\!) = id$ and any morphism would be structure preserving according to this definition.

Another attempt, and at present the best we can think of, is this:

$$\phi \text{ preserves the } \mathsf{F}\text{-structure} \quad \equiv \quad \phi = (\!|\eta ; in|\!) \quad \text{for some } \eta : \mathsf{F} \dot{\to} \mathsf{F} .$$

This definition is partly proof generated and partly suggested by intuition; in particular the definition says

$$\phi \quad = \quad in^{\cup} ; \phi\mathsf{F} ; \eta ; in .$$

To show the generality of this definition we mention the following facts.

Facts
(1) *For any* F *and* $\eta : \mathsf{F} \dot{\to} \mathsf{F}$ *and* $\phi = (\!|\eta ; in|\!)$ *we have* $\phi = \eta^{\mu}$ *(where* $\_^{\mu}$ *is the functor discussed by Fokkinga and Meertens [2]).*

(2) *Structure preservation is closed under composition.*

(3) *For* $\mathsf{F} = A\dagger$ *, any map is structure preserving.*

(4) *For lists, the morphism* $reverse$ *preserves the structure, and so does* $rotate$ *— if you care to define them and to provide the details.*

[*Still to be done: the precise relation between shape preservation and structure preservation.*]

3 Characterisation and Fusion

In this section we prove the uniqueness and existence of a solution to the *prepromorphism* equation. (I owe the proof of the existence to Lambert Meertens.) We present also some corollaries. Throughout the sequel we work in an arbitrary category. However, in Theorem (5) we assume the existence of enumerable products; the projections are denoted π_n rather than π_n^{∞}. Endo-functor F is arbitrary, and we assume that an initial algebra $in = \mu\mathsf{F}$ exists.

(5) Theorem (Prepro Characterisation) *Let* $\eta : \mathsf{F} \dot{\to} \mathsf{F}$ *and* $\psi : \mathsf{F}$*-algebra be arbitrary, and put* $\phi = (\!|\eta ; in|\!)$ *. Then*

$$x = in^{\cup} ; (\phi ; x)\mathsf{F} ; \psi \quad \equiv \quad x = (\!|\Delta n :: \pi_{n+1}\mathsf{F} ; \eta^{n} ; \psi|\!) ; \pi_0$$

where $\_^{n}$ *denotes* n*-fold composition.*

Proof First we shall prove the implication from the left to the right. Define by induction on n

$$\begin{array}{lllll} \psi_0 & = & \psi & = & \eta^0 ; \psi, \\ \psi_{n+1} & = & \eta ; \psi_n & = & \eta^{n+1} ; \psi, \end{array} \qquad \begin{array}{lllll} \psi'_0 & = & \pi_1\mathsf{F} ; \psi_0 & = & \pi_1\mathsf{F} ; \eta^0 ; \psi \\ \psi'_{n+1} & = & \pi_{n+2}\mathsf{F} ; \psi_{n+1} & = & \pi_{n+2}\mathsf{F} ; \eta^{n+1} ; \psi . \end{array}$$

Now assume that a solution x_0 exists. Define a sequence $(n :: x_n)$ by induction on n as follows:

$$x_{n+1} \quad = \quad \phi ; x_n \, .$$

By induction on n we show that for all n

(a) $\quad x_n \quad = \quad in^{\cup} ; x_{n+1}\mathsf{F} ; \psi_n \, .$

Basis: immediate by assumption on x_0 and definition of x_1.
Step: we calculate

$$\begin{array}{ll}
 & x_{n+1} \\
= & \quad \text{definition } x_{n+1} \\
 & \phi ; x_n \\
= & \quad \text{induction hypothesis} \\
 & \phi ; in^{\cup} ; x_{n+1}\mathsf{F} ; \psi_n \\
= & \quad \text{definition } \phi \text{ and } \textsc{Cata Homo} \\
 & in^{\cup} ; \phi\mathsf{F} ; \eta ; x_{n+1}\mathsf{F} ; \psi_n \\
= & \quad \text{naturality } \eta\text{, functoriality } \mathsf{F} \\
 & in^{\cup} ; (\phi ; x_{n+1})\mathsf{F} ; \eta ; \psi_n \\
= & \quad \text{definition } x_{n+2} \text{ and } \psi_{n+1} \\
 & in^{\cup} ; x_{n+2}\mathsf{F} ; \psi_{n+1} .
\end{array}$$

Hence we also have, for all n,

$$\begin{array}{ll}
 & x_n \\
= & \quad \text{just proved: Equation (a)} \\
 & in^{\cup} ; x_{n+1}\mathsf{F} ; \psi_n \\
= & \quad \text{products} \\
 & in^{\cup} ; ((\Delta n :: x_n) ; \pi_{n+1})\mathsf{F} ; \psi_n \\
= & \quad \text{functoriality} \\
 & in^{\cup} ; (\Delta n :: x_n)\mathsf{F} ; \pi_{n+1}\mathsf{F} ; \psi_n \\
= & \quad \text{definition } \psi'_n \\
 & in^{\cup} ; (\Delta n :: x_n)\mathsf{F} ; \psi'_n
\end{array}$$

showing that $(n :: x_n)$ is a collection of *mutumorphisms*, as introduced by Fokkinga [1]. Thus we have

$$(\Delta n :: x_n) \quad = \quad in^{\cup} ; (\Delta n :: x_n)\mathsf{F} ; (\Delta n :: \psi'_n)$$

so that

$$\begin{array}{lcl}(\Delta n :: x_n) & = & (\!|\Delta n :: \psi'_n|\!) \\ x_0 & = & (\!|\Delta n :: \psi'_n|\!) ; \pi_0 \,.\end{array}$$

So, assuming the existence of a solution x_0, we have shown that it is uniquely determined by the above equation.

Second, we show that $(\!|\Delta n :: \psi'|\!) ; \pi_0$ is a solution indeed. Putting $y_0 = (\!|\Delta n :: \psi'_n|\!) ; \pi_0$ and $y_{n+1} = \phi ; y_n$ it is easy to show

$$\begin{array}{ll} & y_0 = in^{\cup} ; (\phi ; y_0)\mathsf{F} ; \psi \\ \Leftarrow & \\ & y_1 = in^{\cup} ; (\phi ; y_1)\mathsf{F} ; \psi \\ \Leftarrow & \\ & y_2 = in^{\cup} ; (\phi ; y_2)\mathsf{F} ; \psi \\ \vdots & \end{array}$$

But, unfortunately, this does not show that y_0 solves the equation. A better argument is needed; the following one has been designed by Lambert Meertens. We abbreviate $(\Delta n :: \mathit{expr}_n)$ to $\Delta \mathit{expr}_n$.

$$\begin{array}{lll} & & y_0 = in^{\cup} ; (\phi ; y_0)\mathsf{F} ; \psi \\ & \equiv & \qquad \text{unfold } y_0\text{, and bring } in \text{ to the left-hand side} \\ & & in ; (\!|\Delta\psi'_n|\!) ; \pi_0 = (\phi ; (\!|\Delta\psi'_n|\!) ; \pi_0)\mathsf{F} ; \psi \\ (\star) & \equiv & \qquad \text{lhs: CATA HOMO, rhs: equation below} \\ & & (\!|\Delta\psi'_n|\!)\mathsf{F} ; \Delta\psi'_n ; \pi_0 = ((\!|\Delta\psi'_n|\!) ; \Delta\pi_{n+1} ; \pi_0)\mathsf{F} ; \psi \\ & \Leftarrow & \qquad \text{functor, Leibniz} \\ & & \Delta\psi'_n ; \pi_0 = (\Delta\pi_{n+1} ; \pi_0)\mathsf{F} ; \psi \\ & \equiv & \qquad \text{product calculus} \\ & & \psi'_0 = \pi_1\mathsf{F} ; \psi \\ & \equiv & \qquad \text{definition} \\ & & \mathsf{true}. \end{array}$$

At step $(\star)$ we have used an equation, the crux of the proof, that we prove as follows, abbreviating $\Delta\pi_{n+1}$ ($= \Delta n :: \pi_{n+1}$) to *shift*.

$$\begin{array}{ll} & (\!|\Delta\psi'_n|\!) ; \mathit{shift} = \phi ; (\!|\Delta\psi'_n|\!) \\ \equiv & \qquad \text{unfold } \phi\text{, CATA COMPOSE (using } \eta : \mathsf{F} \dot{\to} \mathsf{F}\text{)} \\ & (\!|\Delta\psi'_n|\!) ; \mathit{shift} = (\!|\eta ; \Delta\psi'_n|\!) \\ \Leftarrow & \qquad \text{FUSION} \\ & \Delta\psi'_n ; \mathit{shift} = \mathit{shift}\mathsf{F} ; \eta ; \Delta\psi'_n \\ \equiv & \qquad \text{law } f ; g \vartriangle h = (f ; g) \vartriangle (f ; h) \text{ (fusion for split!) at both sides} \end{array}$$

$$\begin{array}{ll} & \Delta(\Delta\psi'_n ; \pi_{n+1}) = \Delta(\mathit{shift}\mathsf{F} ; \eta ; \psi'_n) \\ \equiv & \quad \text{law } f \vartriangle g = h \vartriangle j \ \equiv \ f{=}h \wedge g{=}j \ \text{— for all } n : \\ & \Delta\psi'_n ; \pi_{n+1} = \mathit{shift}\mathsf{F} ; \eta ; \psi'_n \\ \equiv & \quad \text{lhs: products, rhs: unfold } \psi'_n = \pi_{n+1}\mathsf{F} ; \eta^n ; \psi \text{ and use naturality } \eta \\ & \psi'_{n+1} = \mathit{shift}\mathsf{F} ; \pi_{n+1}\mathsf{F} ; \eta^{n+1} ; \psi \\ \equiv & \quad \text{lhs: unfold } \psi'_{n+1}\text{, rhs: functor and } \mathit{shift} ; \pi_{n+1} = \pi_{n+2} \\ & \mathsf{true}. \end{array}$$

This completes the proof. □

(6) Corollary (Prepro UEP and Fusion) *Let $\eta : \mathsf{F} \dot{\to} \mathsf{F}$ and ψ_0, ψ_1 be F-algebras. Let p_0 and p_1 be the F-prepromorphisms determined by η and ψ_0 respectively ψ_1. Then*

$$\begin{array}{lcl} p_0 = p_1 & \Leftarrow & \psi_0 = \psi_1 \\ p_0 ; f = p_1 & \Leftarrow & f : \psi_0 \xrightarrow{\mathsf{F}} \psi_1 \end{array}$$

Proof Standard calculation. □

Remark There are many more schemes for which we can employ the above proof technique to show that the prepro-version has a unique solution if the naked version has so. In particular, the reader may now check that a prepro-paramorphism equation does have a unique solution. □

The characterisation theorem gives the unique solution of a prepromorphism equation. This solution is not necessarily a catamorphism. The following theorem gives a sufficient condition under which the unique solution is a catamorphism.

(7) Theorem (Prepro Cata Characterisation) *Let $\eta : \mathsf{F} \dot{\to} \mathsf{F}$ and $\psi : A\mathsf{F} \to A$ be arbitrary, and put $\phi = (\![\eta ; in]\!)$. Furthermore, let $\chi : A \to A$ be arbitrary, and put $f = (\![\chi\mathsf{F} ; \psi]\!)$. Then*

$$x = in^{\cup} ; (\phi ; x)\mathsf{F} ; \psi \quad \equiv \quad x = f$$

provided that $\phi ; f = f ; \chi$, which in turn follows from $\chi : \psi \xrightarrow{\mathsf{F}} \eta ; \psi$.

Proof From the definition of f we have immediately

$$f \quad = \quad in^{\cup} ; (f ; \chi)\mathsf{F} ; \psi$$

and from $f; \chi = \phi; f$ we see that f solves the equation for x. By PREPRO CHARACTERISATION f is then the unique solution. Finally, we derive a sufficient condition for the proviso $f; \chi = \phi; f$

$$
\begin{array}{ll}
 & f; \chi = \phi; f \\
\equiv & \quad \text{definition } f \text{ and } \phi \\
 & (\!|\chi\mathsf{F}; \psi|\!); \chi = (\!|\eta; in|\!); (\!|\chi\mathsf{F}; \psi|\!) \\
\equiv & \quad \text{rhs: CATA COMPOSE using } \eta : \mathsf{F} \dot{\to} \mathsf{F} \\
 & (\!|\chi\mathsf{F}; \psi|\!); \chi = (\!|\eta; \chi\mathsf{F}; \psi|\!) \\
\Leftarrow & \quad \text{FUSION} \\
 & \chi : \chi\mathsf{F}; \psi \xrightarrow{\mathsf{F}} \eta; \chi\mathsf{F}; \psi \\
\equiv & \quad \text{naturality } \eta \\
 & \chi : \chi\mathsf{F}; \psi \xrightarrow{\mathsf{F}} \chi\mathsf{F}; \eta; \psi \\
\Leftarrow & \quad \text{calculus — should be a LAW in [3]!} \\
 & \chi : \psi \xrightarrow{\mathsf{F}} \eta; \psi
\end{array}
$$

as desired. □

Remark With ϕ, ψ, χ as in the theorem, it is not necessarily true that for all f, $f; \chi = \phi; f$ (although this equation does hold for the particular f mentioned in the theorem). If this equation were to hold for all f we would have in particular $id; \chi = \phi; id$ so that $\chi = \phi$. In the case of f-*iterate* we have that $\chi \neq \phi$, see below. □

By straightforward dualisation we obtain the following, where *postpro* is mnemonic for postprocessing. (Recall from [3] that formula $f : \phi \overset{\mathsf{F}}{\succ\!\!-} \psi$ means $\phi; f\mathsf{F} = f; \psi$, thus dualising $f : \phi \xrightarrow{\mathsf{F}} \psi$.)

(8) Corollary (Postpro Ana Characterisation) *Let* $\eta : \mathsf{F} \dot{\to} \mathsf{F}$ *and* $\psi : \mathsf{F}$-co-algebra *be arbitrary, and put* $\phi = [\![out; \eta]\!]$. *Let* χ *be such that* $\chi : \psi; \eta \overset{\mathsf{F}}{\succ\!\!-} \psi$. *Then*

$$x \;=\; \psi; (x; \phi)\mathsf{F}; out^{\cup} \quad \equiv \quad x \;=\; [\![\psi; \chi\mathsf{F}]\!]\,.$$

4 Applications

Cascade Recall from the introduction the equations

$$
\begin{array}{lcl}
f^{\bowtie} & = & in^{\cup}; (f^{\bowtie}; f*)\mathsf{F}; in \\
x & = & in^{\cup}; (f*; x)\mathsf{F}; in
\end{array}
$$

where $\mathsf{F} = A\dagger$ and $*$ is the map for $\dagger$ and $f : A \to A$. Taking

$$\begin{array}{lllll} \eta & & := & f \dagger id : A\dagger \dot{\to} A\dagger & \\ \phi & = \chi & := & f* & = (\!|\eta ; in|\!) \end{array}$$

we find by the Prepro Cata Characterisation that $f^{\bowtie}$ is the unique solution of the equation in x.

Iterate Recall the definition of f-iterate:

$$f^{\omega} \quad = \quad \Delta ; (f ; f^{\omega})\mathsf{F} ; out_{\cup}$$

where $\mathsf{F} = A\dagger$, $\infty =$ the (final) map for $\dagger$, and $\Delta : \mathrm{I} \dot{\to} \mathrm{I}\dagger\mathrm{I}$ (which is certainly true for the conventional doubling morphism when $\dagger = \times$). Consider the following postpromorphism equation.

$$x \quad = \quad \Delta ; (x ; f\infty)\mathsf{F} ; out_{\cup} .$$

Take

$$\begin{array}{lll} \eta & := & f \dagger id : A\dagger \dot{\to} A\dagger \\ \phi & := & f\infty \quad = \quad [\![out ; \eta]\!] \\ \chi & := & f . \end{array}$$

Notice that the case considered here is *not* the dual of f-cascade; in particular ϕ and χ differ from each other. We have

$$\begin{array}{ll} & \chi : \psi ; \eta \stackrel{\mathsf{F}}{\succ\!\!-} \psi \\ \equiv & \\ & f : \Delta ; f \dagger id \stackrel{A\dagger}{\succ\!\!-} \Delta \\ \equiv & \\ & \Delta ; f \dagger id ; id \dagger f = f ; \Delta \\ \equiv & \\ & \mathsf{true} \end{array}$$

so that by the Postpro Ana Characterisation f^{ω} is the unique solution of the equation in x.

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