Online Appendix to "Uneven Growth: Automation's Impact on Income and Wealth Inequality"

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A Derivations and Proofs for the Baseline Model

A.1 Task Model Micro-foundation and Details

This appendix provides a micro-foundation for the aggregate production function in equation (2). The appendix also provides a derivation of equation (9) and primitive conditions to ensure Assumption 1 holds.

Each skill type z works in a different sector that produces output Y_z. The economy produces a final good Y using these sectoral outputs according to a Cobb-Douglas aggregator

$$Y = A \prod_z Y_z^{\eta_z} \quad \text{with} \quad \sum_z \eta_z = 1.$$

Here, η_z denotes the importance of the sectoral output produced by skill type z in production. The productivity shifter A captures the role of factor-neutral technological improvements.

The production of sectoral output Y_z involves the completion of a unit continuum of tasks u, which are then combined via a Cobb-Douglas aggregator:

$$\ln Y_z = \int_0^1 \ln \mathcal{Y}_z(u) du.$$

These tasks can be produced using capital and skill-z labor as follows:

$$\mathcal{Y}_z(u) = \begin{cases} \psi_z \ell_z(u) + k_z(u) & \text{if } u \in [0, \alpha_z] \\ \psi_z \ell_z(u) & \text{if } u \in (\alpha_z, 1]. \end{cases}$$

The threshold α_z summarizes the possibilities for the automation of tasks performed by workers of skill z. Tasks $u \in [0, \alpha_z]$ are technologically automated and can be produced by capital $k_z(u)$ or labor $\ell_z(u)$. The remaining tasks are not technologically automated and must be produced by labor.

recall that the unit cost of producing a task with capital is R and that of producing it with labor is w_z/ψ_z. Denote by $p_z(u)$ the price of task $\mathcal{Y}_z(u)$, and by p_z the price of sector z output Y_z. Cost minimization in the production of sector z output implies that the quantity of task u used is given by

$$\mathcal{Y}_z(u) = \frac{p_z Y_z}{p_z(u)}$$

Assumption 1 implies that all tasks $u \in [0, \alpha_z]$ are produced with capital. It follows that for those tasks, $p_z(u) = R$ and the quantity of capital required to produce $\mathcal{Y}_z(u)$ is $p_z Y_z / R$. It follows that the total amount of capital used in sector z is:

$$K_z = \frac{\alpha_z p_z Y_z}{R} \tag{A1}$$

Assumption 1 implies that all tasks $u \in (\alpha_z, 1]$ are produced with labor. It follows that for those tasks, $p_z(u) = \frac{w_z}{\psi_z}$ and the quantity of labor required to produce $\mathcal{Y}_z(u)$ is $p_z Y_z / w_z$. It follows that the total amount of labor of skill z used in sector z is:

$$\ell_z = \frac{(1 - \alpha_z) p_z Y_z}{w_z} \tag{A2}$$

With perfect competition, the price of sector z output equals the marginal cost of production. Because tasks are combined via a Cobb-Douglas aggregator, the price is given by the dual $\ln(p_z) = \int_0^1 \ln(p_z(u)) du$. It follows that

$$p_z = R^{\alpha_z} \left(\frac{w_z}{\psi_z} \right)^{1 - \alpha_z}. \tag{A3}$$

Combining the formula for p_z in (A3) with capital and labor demand conditions (A1) and (A2) gives the production of sector z as a function of the total capital and labor used in this sector, K_z and ℓ_z:

$$Y_z = \left(\frac{K_z}{\alpha_z} \right)^{\alpha_z} \left(\frac{\psi_z \ell_z}{1 - \alpha_z} \right)^{1 - \alpha_z} \tag{A4}$$

We now turn to aggregate output. We normalize the price of the final good to 1, so that the demand for sector z output satisfies $p_z Y_z = \eta_z Y$.

Using equations (A1), we can compute the demand for capital in sector z as

$$K_z = \frac{\alpha_z p_z Y_z}{R} = \frac{\alpha_z \eta_z Y}{R}.$$

Adding this formula across sectors, it follows that the total amount of capital used in the economy is

$$K = \alpha \frac{Y}{R}, \tag{A5}$$

where recall that $\alpha := \sum_z \alpha_z \eta_z$. The share of capital allocated to sector z is therefore equal to

$$K_z = K \frac{\alpha_z \eta_z}{\alpha}. \tag{A6}$$

Substituting this formula into (A4) we get:

$$Y_z = \left(K \frac{\eta_z}{\alpha} \right)^{\alpha_z} \left(\frac{\psi_z L_z}{1 - \alpha_z} \right)^{1 - \alpha_z}. \tag{A7}$$

Substituting sectoral outputs into the aggregate production function we obtain the formula in equation (2), with

$$\mathcal{A} := A\alpha^{-\alpha} \prod_z (\eta_z(1-\alpha_z))^{-\eta_z(1-\alpha_z)} \prod_z \eta_z^{\eta_z} \tag{A8}$$

Proof of the expression for productivity gains from automation. The formula in equation (2) can be written as

$$Y = A\prod_z \eta_z^{\eta_z} \left(\frac{K}{\alpha}\right)^{\alpha} \prod_z \left(\frac{\psi_z \ell_z}{\eta_z(1-\alpha_z)}\right)^{\eta_z(1-\alpha_z)}.$$

It follows that

$$\begin{aligned} d\ln Y =& \eta_z \ln\left(\frac{K}{\alpha}\right) - \ln\left(\frac{\psi_z \ell_z}{\eta_z(1-\alpha_z)}\right) + \alpha d\ln K \\ =& \eta_z \ln\left(\frac{Y}{R}\right) - \ln\left(\psi_z \frac{Y}{w_z}\right) + \alpha d\ln K \\ =& \eta_z \ln\left(\frac{w_z}{\psi_z R}\right) + \alpha d\ln K > 0. \end{aligned}$$

The third row substitutes factor prices for their marginal products. Subtracting $\alpha d\ln Y$ from both sides of this equation and dividing through by $1-\alpha$ yields the formula in (9). ■

Lemma A1 (Lemma ensuring adoption of automation technologies) *Suppose that for all z, the following inequality holds:*

$$(\rho + p\sigma + \delta)^{-\frac{1}{1-\alpha}} \mathcal{A}^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} > \frac{1}{(1-\alpha_z)\eta_z} \frac{\ell_z \psi_z}{\prod_v (\ell_v \psi_v)^{\frac{\eta_v(1-\alpha_v)}{1-\alpha}}}, \tag{A9}$$

where $\mathcal{A}$ is defined in (A8). The equilibrium will involve the adoption of all available automation technologies. The above inequality holds for values of A above a threshold $\bar{A}$.

Proof. We assume that all automation technologies are adopted and verify that in equilibrium, the condition above ensures that $w_z^* > \psi_z R^*$.

In steady state, we have that $\rho + p\sigma > r^*$, which can be seen from the fact that $r^* = \rho + p\sigma\alpha^*_{net}$. Using the fact that $r^* + \delta = \alpha\frac{Y}{K}$, we can rewrite $\rho + p\sigma > r^*$ as

$$(K/Y)^* > \frac{\alpha}{\rho + p\sigma + \delta}. \tag{A10}$$

Turning to wages, we have that

$$\begin{aligned}
w_z^* =&(1-\alpha_z)\frac{\eta_z}{\ell_z}Y^* \\
=&(1-\alpha_z)\frac{\eta_z}{\ell_z}\mathcal{A}^{\frac{1}{1-\alpha}}(K/Y)^{*\frac{\alpha}{1-\alpha}}\prod_v(\psi_v\ell_v)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}} \\
>&\frac{(1-\alpha_z)\eta_z}{\ell_z}\mathcal{A}^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\prod_z(\psi_z\ell_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}(\rho+p\sigma+\delta)^{-\frac{\alpha}{1-\alpha}},
\end{aligned}$$

where the last line uses inequality (A10).

Finally, because $\psi_z(\rho+p\sigma+\delta)>\psi_z R^*$, a sufficient condition to ensure $w_z^*>\psi_z R^*$ is

$$\frac{(1-\alpha_z)\eta_z}{\ell_z}\mathcal{A}^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\prod_z(\psi_z\ell_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}(\rho+p\sigma+\delta)^{-\frac{\alpha}{1-\alpha}}>\psi_z(\rho+p\sigma+\delta).$$

This inequality is equivalent to (A9). Finally, the definition of $\mathcal{A}$ in (A8) implies that (A9) holds for large values of A, concluding the proof of the lemma. ■

A.2 Propositions for Baseline Model A

Proof of Lemma 1. Let $x_{z,t}:=w_z/r+a_{z,t}$ denote effective wealth. Equation (1) can be rewritten as:

$$\begin{aligned}
&\max_{\{c_{z,t},x_{z,t}\}_{t\geq 0}}\int_0^\infty e^{-\rho t}\frac{c_{z,t}^{1-\sigma}}{1-\sigma}dt \\
&\text{s.t. } \dot{x}_{z,t}=rx_{z,t}-c_{z,t}, \text{ and } x_{z,t}\geq 0
\end{aligned} \tag{A11}$$

The Hamiltonian associated with this maximization problem is

$$H(c_z,x_z,\lambda_z):=\frac{c_z^{1-\sigma}}{1-\sigma}+\lambda(rx_z-c_z), \tag{A12}$$

where λ_z is the co-state for effective wealth.

We can write the candidate solution given in Lemma 1 as (time arguments are ignored to save on notation)

$$\dot{x}_z=\frac{r-\rho}{\sigma}x_z \qquad\qquad c_z=\left(r-\frac{r-\rho}{\sigma}\right)x_z. \tag{A13}$$

We will show that the unique solution to this system of differential equations starting from $x_{z,0}=w_z/r$ solves the maximization problem in (A11).

Theorem 7.14 in Acemoglu (2009) implies that this candidate path reaches an optimum if there exists a co-state variable λ_z such that:

1. the path satisfies the restrictions $\dot{x}_z = rx_z - c_z$, and $x_z \geq 0$;
2. the following necessary conditions hold:

$$\begin{aligned} c_z^{-\sigma} =& \lambda_z, \\ \rho\lambda_z - \dot{\lambda}_z =& r\lambda_z; \end{aligned}$$

3. the maximized Hamiltonian $M(x_z, \lambda_z) = \max_c H(c, x_z, \lambda_z)$ is concave in x_z along the candidate path;
4. the transversality condition holds. That is, for the candidate path, we have

$$\lim_{s\to\infty} e^{-\rho s} x_z \lambda_z = 0.$$

and for all other feasible paths, $\hat{x}_z$, we have

$$\lim_{s\to\infty} e^{-\rho s} \hat{x}_z \lambda_z \geq 0.$$

To prove condition 1, note that starting from any $x_{z,0} \geq 0$, we will have $x_z \geq 0$. Moreover, for any path satisfying equations (A13) the flow budget constraint holds:

$$\begin{aligned} rx_z - c_z &= rx_z - \left(r - \frac{r-\rho}{\sigma}\right) x_z \\ &= \frac{r-\rho}{\sigma} x_z \\ &= \dot{x}_z. \end{aligned}$$

To prove condition 2, define $\lambda_z := (r - (r-\rho)/\sigma)^{-\sigma} x_z^{-\sigma} > 0$ (here we used the condition $r > (r-\rho)/\sigma$). By construction, $c_z^{-\sigma} = \lambda_z$. Moreover:

$$\begin{aligned} \rho\lambda_z - \dot{\lambda}_z =& \rho \left(r - \frac{r-\rho}{\sigma}\right)^{-\sigma} x_z^{-\sigma} + \left(r - \frac{r-\rho}{\sigma}\right)^{-\sigma} \sigma x_{z,t}^{-\sigma-1} \dot{x}_z \\ =& \left(\rho + \sigma \frac{\dot{x}_z}{x_z}\right) \left(r - \frac{r-\rho}{\sigma}\right)^{-\sigma} x_z^{-\sigma} \\ =& \left(\rho + \sigma \frac{\dot{x}_z}{x_z}\right) \lambda_z \\ =& \left(\rho + \sigma \frac{r-\rho}{\sigma}\right) \lambda_z \\ =& r\lambda_z. \end{aligned}$$

To prove condition 3, note that

$$\max_c H(c, x_z, \lambda_z) = \frac{\lambda_z^{\frac{\sigma-1}{\sigma}}}{1-\sigma} + \lambda_z(r x_z - \lambda_z^{-\frac{1}{\sigma}}),$$

which is concave (linear) in x_z.

To prove the first part of condition 4, note that along the candidate path, x_z grows at a rate $\frac{r-\rho}{\sigma}$, and λ_z at a rate $\rho - r$. It follows that the first part of the transversality condition holds if

$$-\rho + \frac{r-\rho}{\sigma} + \rho - r < 0,$$

which is equivalent to the condition $r > (r-\rho)/\sigma$.

The second part of the transversality condition follows from the fact that, along any feasible path, we have $\hat{x}_z \geq 0$.

It follows that the candidate paths given in Lemma 1 provide optimal paths for consumption and asset accumulation in a steady state. ■

Proof of Proposition 1.

The main text presents the derivations of the supply curve (equation 6) and the demand curve (equation 7).

The supply curve $(K/\bar{w})^s$ increases from zero to infinity as r increases from ρ to $\rho + p\sigma$. For $r < \rho$ households supply no capital. For $r > \rho + p\sigma$, households amass a divergent amount of capital.

The demand curve $(K/\bar{w})^s$ decreases from $(\alpha/(1-\alpha))/(\rho+\delta) > 0$ to $(\alpha/(1-\alpha))/(\rho + p\sigma + \delta) > 0$ as r increases from ρ to $\rho + p\sigma$.

These observations imply that equation (4) has a unique solution r^* and that this unique solution lies in $(\rho, \rho + p\sigma)$. In fact, r^* can be computed analytically as

$$r^* = \frac{-((1-\alpha)\delta - \rho - \alpha p\sigma) + \sqrt{((1-\alpha)\delta - \rho - \alpha p\sigma)^2 + 4(1-\alpha)\rho\delta}}{2} \tag{A14}$$

The equilibrium return r^* determines the remaining aggregates as follows. First, the capital-output ratio is given by

$$(K/Y)^* = \frac{\alpha}{r^* + \delta}.$$

The output level is given by

$$Y^* = \mathcal{A}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} \prod_z (\ell_z \psi_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}.$$

These two equations combined imply

$$K^* = \mathcal{A}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r^* + \delta} \right)^{\frac{1}{1-\alpha}} \prod_z (\ell_z \psi_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}. \tag{A15}$$

Turning to wages, we have that $w_z^* = (1-\alpha_z)\frac{\eta_z}{\ell_z} Y^*$, which implies

$$w_z^* = (1-\alpha_z)\frac{\eta_z}{\ell_z} \mathcal{A}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r^* + \delta} \right)^{\frac{\alpha}{1-\alpha}} \prod_z (\ell_z \psi_z)^{\frac{\eta_z(1-\alpha_z)}{1-\alpha}}.$$

Finally, equation (5) can be derived from the household side. As explained in the main text, in steady state we must have

$$0 = \frac{r^* - \rho}{\sigma} \left(K^* + \frac{\bar{w}^*}{r^*} \right) - pK^*.$$

This expression can be rearranged as

$$r^* = \rho + p\sigma \frac{r^* K^*}{r^* K^* + \bar{w}^*} = \rho + p\sigma \alpha_{net}^*.$$

Note that the condition $r^* > (r^* - \rho)/\sigma$, which is needed to ensure the households' policy functions are an optimum is equivalent to $\rho + p\alpha_{net}^*(\sigma - 1) > 0$, which we assume holds throughout.

Turning to the comparative statics exercises, we can rearrange (4) as:

$$\frac{\left(1 - \frac{\rho}{r^*}\right)(r^* + \delta)}{p\sigma + \rho - r^*} = \frac{\alpha}{1-\alpha}.$$

The right hand side of this equation is increasing in r^*, and the left is increasing in α. It follows that r^* is increasing in α.

The household accumulation rate is given by $(r^* - \rho)/\sigma$, and so it also increases with α.

The net capital share satisfies the identity in equation (5), and so it increases with α.

Denote the capital-output ratio by $(K/Y)^*$. We have

$$\alpha_{net}^* = \frac{r^*(K/Y)^*}{1 - \delta(K/Y)^*}.$$

Rearranging this equation and using the fact that $r^* = \rho + p\sigma\alpha_{net}^*$, we obtain

$$(K/Y)^* = \frac{\alpha_{net}^*}{\rho + (p\sigma + \delta)\alpha_{net}^*}, \tag{A16}$$

which is an increasing function of α_{net}^*, and hence α.

Finally, turning to output, from equation (2) we have

$$d \ln Y^* = \frac{1}{1-\alpha} d \ln \mathrm{TFP}_\alpha + \frac{\alpha}{1-\alpha} d \ln(K/Y)^*.$$

Because $d \ln \mathrm{TFP}_\alpha > 0$ and $d \ln(K/Y)^* > 0$ following any increase in the α_z's, we have that automation always increase output. ■

Proof of Proposition 2. Equilibrium factor prices imply that relative wages satisfy

$$\frac{w_z}{w_v} = \frac{(1-\alpha_z)}{(1-\alpha_v)} \frac{\eta_z}{\eta_v} \frac{\ell_v}{\ell_z}.$$

It follows that an increase in α_z reduces w_z/w_v for $v \neq z$.

Turning to the wage bill, we can add up individual wages for all z to obtain:

$$\bar{w} = (1-\alpha)Y.$$

It follows that

$$d \ln \bar{w} = -\frac{1}{1-\alpha} \sum \eta_z d\alpha_z + \frac{1}{1-\alpha} d \ln \mathrm{TFP}_\alpha + \frac{\alpha}{1-\alpha} d \ln(K/Y)^*.$$

We now show that the terms $d \ln \mathrm{TFP}_\alpha$ and $d \ln(K/Y)^*$ are both decreasing in p and converge to zero as p increases. Because the term $-(1/(1-\alpha_z)) \sum \eta_z d\alpha_z$ is negative, this establishes the existence of the threshold $\bar{p}$.

We first analyze the term $d \ln \mathrm{TFP}_\alpha$. This is given by

$$d \ln \mathrm{TFP}_\alpha = \sum_z \eta_z \ln \left(\frac{w_z^*}{\psi_z R^*} \right) = \sum_z \eta_z \ln \left(\frac{K^*}{\alpha} \right) - \eta_z \ln \left(\frac{\psi_z \ell_z}{\eta_z (1-\alpha_z)} \right),$$

where we used the formulas for equilibrium factor prices. It is enough to show that K^* is decreasing in p and that K^* converges to zero as p increases. Because K^* is given by (A15), it is enough to show that r^* is increasing in p and that r^* converges to infinity as p increases.

The fact that r^* increases in p follows from equation (4). An increase in p contracts the supply of capital, which results in a higher r^*. Moreover, equation (A14) shows that $r^* \to \infty$ as $p \to \infty$. Note that the formal limit of $d \ln \mathrm{TFP}_\alpha$ as $p \to \infty$ is zero, since as K^* declines, we eventually reach a point where Assumption 1 starts failing and increases in α_z do not affect productivity.

We now turn to the term $d \ln(K/Y)^*$. We have that

$$\alpha = (K/Y)^* (\rho + p\sigma\alpha^*_{net} + \delta).$$

Differentiating this expression we obtain

$$1 = \frac{p\sigma\alpha^*_{net}}{\rho + p\sigma\alpha^*_{net} + \delta}\frac{\partial \ln \alpha^*_{net}}{\partial \ln \alpha} + \frac{\partial \ln(K/Y)^*}{\partial \ln \alpha}. \tag{A17}$$

Moreover, equation (A16) implies that

$$\frac{\partial \ln(K/Y)^*}{\partial \ln \alpha} = \frac{\rho}{\rho + (p\sigma + \delta)\alpha^*_{net}}\frac{\partial \ln \alpha^*_{net}}{\partial \ln \alpha}. \tag{A18}$$

Solving equations (A17) and (A18), we obtain:

$$\frac{\partial \ln(K/Y)^*}{\partial \ln \alpha} = \frac{1}{1 + \frac{p\sigma\alpha^*_{net}}{\rho + p\sigma\alpha^*_{net} + \delta}\frac{\rho + (p\sigma + \delta)\alpha^*_{net}}{\rho}}.$$

We now show that the elasticity $\frac{\partial \ln(K/Y)^*}{\partial \ln \alpha^*_{net}}$ converges to zero as p rises. A sufficient condition for this to be the case is that α^*_{net} is nondecreasing in p, which holds when $\delta = 0$ and $\alpha^*_{net} = \alpha$. To show this is the case more generally, start from the fact that $\alpha = R(K/Y)$. Rewriting the right hand side in terms of α^*_{net}, we obtain

$$\alpha = (\rho + p\sigma\alpha^*_{net} + \delta)\frac{\alpha^*_{net}}{\rho + (p\sigma + \delta)\alpha^*_{net}}.$$

This equation can be rearranged as

$$\alpha\left(1 - \frac{\delta(1 - \alpha^*_{net})}{\rho + p\sigma\alpha^*_{net} + \delta}\right) = \alpha^*_{net}.$$

This equation defines α^*_{net} implicitly as a function of α and p. The left hand side is increasing in α^*_{net} and intercepts the right-hand side (the 45-degree line) from above at a single equilibrium point. An increase in p shifts the left-hand side upwards, which results in a higher equilibrium value for α^*_{net} as claimed.

The above argument shows that there exists some $\bar{p}$ such that, for $p > \bar{p}$, $d \ln \bar{w} < 0$. To conclude the proof, we show that $\bar{p} > 0$. This follows from the fact that, for $p = 0$, $d \ln \bar{w} > 0$.

To show this, note that for $p = 0$ we get

$$d \ln(K/Y)^* = \frac{d\alpha}{\alpha} = \frac{1}{\alpha}\sum_z \eta_z d\alpha_z,$$

and therefore

$$d \ln \bar{w} = -\frac{1}{1-\alpha}\sum \eta_z d\alpha_z + \frac{1}{1-\alpha} d \ln \mathrm{TFP}_\alpha + \frac{\alpha}{1-\alpha}\frac{1}{\alpha}\sum_z \eta_z d\alpha_z = \frac{1}{1-\alpha} d \ln \mathrm{TFP}_\alpha > 0.$$

■

Proof of Proposition 3. Below we derive the effective wealth distribution, the wealth distribution, and the income distribution. To save on notation, we do not include asterisks when denoting steady state objects.

Effective wealth distribution: Denote the stationary density of effective wealth conditional on a given skill type z by $f_z(x)$. f_z satisfies the Kolmogorov Forward Equation (KFE)

$$0 = -\partial_x \left(\frac{r-\rho}{\sigma} x f_z(x) \right) - p f_z(x)$$

on $(w_z/r, \infty)$. We guess and verify that f is Pareto, i.e. $f_z(x) = c\zeta x^{-\zeta-1}$ for some constants c and ζ. Substituting in the guess

$$\begin{aligned} 0 &= \zeta \frac{r-\rho}{\sigma} c\zeta x^{-\zeta-1} - pc\zeta x^{-\zeta-1} \\ 0 &= \zeta \frac{r-\rho}{\sigma} - p \\ \frac{1}{\zeta} &= \frac{r-\rho}{p\sigma} = \alpha_{net} \end{aligned}$$

Since $f_z(x) = c\zeta x^{-\zeta-1}$ must integrate to 1 on $(w_z/r, \infty)$, we must have $c = (w_z/r)^{-\zeta}$. Hence this is a Pareto distribution with tail parameter $\zeta = \frac{1}{\alpha_{net}}$ and scale parameter $x_z(0) = w_z/r$.

Because the distribution of effective wealth is Pareto, the conditional counter-CDF for effective wealth of each skill type z is of the form:

$$\Pr(\text{effective wealth} \geq x|z) = \left(\frac{x}{w_z/r} \right)^{-\frac{1}{\alpha_{net}}}, \quad x \geq w_z/r. \tag{A19}$$

Wealth distribution: We now derive the counter-CDF for wealth. Recall that effective wealth x is $x := a + w_z/r$. Therefore

$$\Pr(\text{wealth} \geq a|z) = \Pr(\text{effective wealth} \geq a + w_z/r|z) = \left(\frac{a + w_z/r}{w_z/r} \right)^{-\frac{1}{\alpha_{net}}}, \quad a \geq 0$$

To find the unconditional distribution, we add across the different skill-types, which yields

$$\Pr(\text{assets} \geq a) = \sum_z \ell_z \left(\frac{a + w_z/r}{w_z/r} \right)^{-\frac{1}{\alpha_{net}}}.$$

Income Distribution: We now derive the counter-CDF for income. The income of a person with effective wealth x is rx. Therefore

$$\Pr(\text{income} \geq y|z) = \Pr(\text{effective wealth} \geq y/r|z) = \left(\frac{y/r}{w_z/r} \right)^{-\frac{1}{\alpha_{net}}}, \quad y \geq w_z.$$

To find the unconditional distribution, we add across the different skill-types, which yields

$$\Pr(\text{income} \geq y) = \sum_z \left(\frac{\max\{y, w_z\}}{w_z} \right)^{-\frac{1}{\alpha_{net}}}.$$

Finally, when $\delta = 0$, we have $\alpha_{net} = \alpha$ and $\frac{1}{\zeta} = \alpha$. When $\delta > 0$, Proposition 1, implies that $\frac{1}{\zeta}$ is increasing in α. ■

B Derivations and Proofs for the Extended Model

B.1 Derivations and Lemmas

Before presenting the proofs, we generalize the model in the text so that investors could also face a borrowing constraint of the form

$$-b_{z,t} \leq \theta a_{z,t} + \frac{w_z + T}{r_B - g},$$

where $\theta \in (0, 1]$ parameterizes the extent to which investors can pledge their capital.[43] The results in the main text follow in the special case with $\theta = 1$. We also provide a lemma characterizing investors policy functions.

Lemma A2 (Achdou et al. (2022)) *Let $r_I = \kappa r_K + (1-\kappa) r_B$. Investors' policy functions are given by*

$$\begin{aligned} c_{z,t} =& \frac{\rho + (\sigma - 1) r_I - \frac{1}{2}(\sigma - 1)\gamma\nu^2\kappa^2}{\sigma} x_{z,t}, \\ \kappa =& \min\left\{ \frac{1}{1-\theta}, \frac{r_K - r_B}{\gamma\nu^2} \right\}. \end{aligned}$$

which imply that effective wealth follows a random growth process:

$$dx_{z,t} = \frac{r_I - \rho + \frac{1}{2}(\sigma - 1)\gamma\nu^2\kappa^2}{\sigma} x_{z,t} dt + \kappa\nu x_{z,t} dW_t.$$

Proof. Using the definition of effective wealth and the fact that in a balanced growth equilibrium wages and tax revenue grow at a constant rate g, we have

$$dx_{z,t} = da_{z,t} + db_{z,t} + g\frac{w_z + T}{r_B - g} dt.$$

[43] The usual formulation used in the literature is $-b_{z,t} \leq \theta a_{z,t}$. Relative to this, our formulation assumes that human wealth is pledgeable, which makes the model more tractable.

Substituting the investors' budget constraint in place of $da_{z,t} + db_{z,t}$ and rearranging terms we obtain

$$\begin{aligned} dx_{z,t} =& r_B\left(a_{z,t} + b_{z,t} + \frac{w_z + T}{r_B - g}\right)dt + (r_K - r_B)a_{z,t}dt + \nu a_{z,t}dW_t - c_{z,t}dt \\ =& r_B x_{z,t}dt + (r_K - r_B)a_{z,t}dt + \nu a_{z,t}dW_t - c_{z,t}dt. \end{aligned}$$

Let κ denote the share of effective wealth held in equity, so that $a_{z,t} = \kappa x_{z,t}$. We can rewrite the budget constraint as

$$dx_{z,t} = (r_I x_{z,t} - c_{z,t})dt + \nu\kappa x_{z,t}dW_t. \tag{A20}$$

In what follows, we drop the subscript z and time t, and examine the savings problem in terms of the state variable x—effective wealth. The HJB equation for this problem is

$$0 = \max_{c>0,\kappa\in[0,1/(1-\theta)]} f(c, v(x)) + (r_I x - c)v'(x) + \frac{1}{2}\nu^2\kappa^2x^2v''(x).$$

Using the Duffie–Lions aggregator and guessing $v(x) = \Lambda x^{1-\gamma}/(1-\gamma)$, the HJB equation becomes

$$\frac{\rho\Lambda x^{1-\gamma}}{1-\sigma} = \max_{c>0,\kappa\in[0,1/(1-\theta)]} \frac{\rho\Lambda x^{1-\gamma}}{1-\sigma}\left(\frac{c}{\Lambda^{1/(1-\gamma)}x}\right)^{1-\sigma} + (r_I x - c)\Lambda x^{-\gamma} - \frac{\gamma}{2}\nu^2\kappa^2\Lambda x^{1-\gamma}.$$

The optimal consumption and portfolio choice are given by

$$c = \rho^{\frac{1}{\sigma}}\Lambda^{-\frac{1}{\sigma}\frac{1-\sigma}{1-\gamma}}x \qquad\qquad \kappa = \min\left\{\frac{1}{1-\theta}, \frac{r_K - r_B}{\gamma\nu^2}\right\}.$$

Plugging into the HJB equation and canceling terms, we get

$$\frac{\rho}{1-\sigma} = \frac{\rho}{1-\sigma}\left(\frac{\rho^{\frac{1}{\sigma}}\Lambda^{-\frac{1}{\sigma}\frac{1-\sigma}{1-\gamma}}}{\Lambda^{1/(1-\gamma)}}\right)^{1-\sigma} + r_I - \rho^{\frac{1}{\sigma}}\Lambda^{-\frac{1}{\sigma}\frac{1-\sigma}{1-\gamma}} - \frac{1}{2}\gamma\nu^2\kappa^2.$$

This is an equation in $c/x = \rho^{\frac{1}{\sigma}}\Lambda^{-\frac{1}{\sigma}\frac{1-\sigma}{1-\gamma}}$, which yields

$$c/x = \frac{\rho + (\sigma-1)r_I - \frac{1}{2}(\sigma-1)\gamma\nu^2\kappa^2}{\sigma}.$$

The policy function for $c_{z,t}$ follows from this expression, and the behavior of $x_{z,t}$ follows after plugging this policy function in the the budget constraint in equation (A20). ■

B.2 Propositions for Extended Model

Proof of Proposition 4. Lemma A2 implies that investors accumulate wealth at a rate

$$\mu_I = \frac{r_W - \rho}{\sigma}. \tag{A21}$$

Lemma 1 applies to the remaining households, whose wealth then grows at a rate

$$\mu_H = \frac{r_B - \rho}{\sigma}.$$

In what follows, we will describe the BGE in terms of r_W and r_B. In particular, we define $h(r_W - r_B) = r_K - r_B$ and $m(r_W - r_B) = \kappa$ implicitly as the solution to

$$r_W - r_B = mh + \frac{1}{2}(\sigma - 1)\gamma\nu^2 m^2 \qquad\qquad m = \min\left\{\frac{1}{1-\theta}, \frac{h}{\gamma\nu^2}\right\}.$$

We can then write the return to capital and portfolio choice implicitly as:

$$r_K = h(r_W - r_B) + r_B \qquad\qquad \kappa = m(r_W - r_B),$$

where h is a continuous and increasing function and m is continuous and nondecreasing.

Denote by X_I the aggregate effective wealth of investors and by X_H the aggregate effective wealth of the remaining households, and bond holdings by B_I and B_H, respectively. Optimal household saving behavior combined with the dissipation shocks implies that

$$\dot{X}_I = \mu_I X_I - p(K + B_I), \qquad\qquad \dot{X}_H = \mu_H X_H - pB_H.$$

Moreover, because the value of capital installed in firms must be equal to the total capital owned by investors, aggregate effective wealths are given by

$$X_I = K + B_I + \chi\frac{\bar{w} + T}{r_B - g} \qquad\qquad X_H = B_H + (1-\chi)\frac{\bar{w} + T}{r_B - g}.$$

Human wealth now depends on the sum of wages and transfers, whose present discounted value is obtained by dividing them by $r_B - g$ to account for their growth over time. It is convenient to analyze the BGE in terms of capital and bonds normalized by the value of human wealth, which are constant along a BGE:

$$k_n = \frac{K}{(\bar{w} + T)/(r_B - g)} \qquad b_I = \frac{B_I}{(\bar{w} + T)/(r_B - g)} \qquad b_H = \frac{B_H}{(\bar{w} + T)/(r_B - g)}.$$

Along a BGE, the effective wealth of investors grows at a rate g, so that $\dot{X}_I = gX_I$ and $\dot{X}_H = gX_H$. The effective wealth of investors grows at a rate g if and only if $r_W > \rho + \sigma g$

and the BGE values of k_n and b_I satisfy

$$\left(\frac{r_W-\rho}{\sigma}-g\right)(k_n+b_I+\chi)=p(k_n+b_I). \tag{A22}$$

In addition, because a fraction $\kappa = m(r_W - r_B)$ of investors wealth is held in equity,

$$m(r_W-r_B)\,(k_n+b_I+\chi)=k_n. \tag{A23}$$

Likewise, the effective wealth of households grows at a rate g if and only if $r_B > \rho+\sigma g$ and

$$\left(\frac{r_B-\rho}{\sigma}-g\right)(b_H+1-\chi)=pb_H, \tag{A24}$$

or $r_B \leq \rho+\sigma g$ and $b_H = 0$.

We now characterize the production side of the economy. We focus on a balanced-growth equilibrium in which Assumption 1 holds, so that output is given by 2. Because of markups, total wage payments are now given by

$$\bar{w}=\frac{1-\alpha}{\varphi}Y,$$

which implies that firms pay a share $(1-\alpha)/\varphi$ of their revenue to labor; while the remaining share of revenue $1-(1-\alpha)/\varphi$ constitutes gross capital income which is taxed at a rate $1-\tau$ (recall that we assumed a gross tax on capital income) and must cover for the depreciation of capital. Thus, we can compute the after-tax gross income from capital as

$$(r_K+\delta)K=(1-\tau)\left(1-\frac{1-\alpha}{\varphi}\right)Y.$$

Tax revenue from capital taxation, and thus the lump-sum transfers, are equal to

$$T=\tau\left(1-\frac{1-\alpha}{\varphi}\right)Y.$$

We can combine the equations for after-tax gross capital income and labor income and transfers to obtain an expression of the demand for financing by firms:

$$k_n=\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}\frac{r_B-g}{h(r_W-r_B)+r_B+\delta}. \tag{A25}$$

A balanced-growth equilibrium is characterized by constant values for r_W, r_B, k_n, b_I, b_H that solve equations (A22), (A23), (A24) and (A25) and where r_B is endogenous and ensures market clearing in the bonds market

$$b_I+b_H=0.$$

In this case, we will assume that $\rho + (\sigma - 1)g > 0$, which is a sufficient condition to ensure that the equilibrium exists and features finite wealth.

We start with the case in which investors are risk neutral ($\gamma = 0$ in the Duffie–Lions aggregator) and their borrowing constraint does not bind. We will provide necessary and sufficient conditions for this to be the case below.

Because we assumed that the borrowing constraint does not bind, it must be the case that $r_W = r_K = r_B = r^*$.

Adding equations (A22) and (A24), we obtain

$$\left(\frac{r^* - \rho}{\sigma} - g\right)(k_n + 1) = pk_n. \tag{A26}$$

Solving for k_n and substituting into equation (A25) shows that an equilibrium is fully determined by a level of returns that satisfies:

$$\frac{1 - (\rho + (\sigma - 1)g)/(r^* - g)}{\sigma(p + g) + \rho - r^*} = \frac{\widetilde{\alpha}}{1 - \widetilde{\alpha}}\frac{1}{r^* + \delta}. \tag{A27}$$

The left hand side of this equation is increasing in r^*; while the right-hand side is decreasing in r^*. Moreover, at $r^* = \rho + \sigma g$, the left-hand side is lower than the right-hand side; and at $r^* = \rho + \sigma(p + g)$, the the left-hand side converges to infinity and exceeds the right-hand side. This implies a unique solution exists and satisfies $r^* \in (\rho + \sigma g, \rho + \sigma(p + g))$.

We now derive conditions for θ that ensure the borrowing constraint does not bind. Denote by k_n^* the value of k_n in the balanced growth equilibrium above, which is given by

$$k_n^* = \frac{\widetilde{\alpha}}{1 - \widetilde{\alpha}}\frac{r^* - g}{r^* + \delta},$$

and is independent of θ by construction. Equations (A22) and (A24) imply that investors must borrow and amount

$$b^* = (1 - \chi)k_n^*$$

from households. and so we must have

$$\kappa^* = \frac{k_n^*}{k_n^* - b^* + \chi} = \frac{1}{\chi}\frac{k_n^*}{k_n^* + 1}.$$

It follows that the borrowing constraint will be slack if and only if

$$\frac{1}{1 - \theta} \geq \frac{1}{\chi}\frac{k_n^*}{k_n^* + 1} \Leftrightarrow \theta \geq 1 - \chi\frac{k_n^* + 1}{k_n^*} := \bar{\theta}.$$

Note that $\bar{\theta} \leq 1$, as claimed in the proposition.

We now turn to the case in which investors are risk averse and/or $\theta < \bar{\theta}$ and we have a closed economy. In what follows, we will assume that $r_W > \rho + \sigma g$, which must hold in any equilibrium. To see this, notice that for $r_W \leq \rho + \sigma g$, investors do not accumulate wealth and the supply of capital is zero, which cannot be the case in a BGE.

Combining equations (A22) and (A24), we obtain a supply of (normalized) capital

$$k_n = \frac{p\sigma m(r_W - r_B)}{\sigma(p+g) + \rho - r_W}\chi. \tag{A28}$$

Combining this with the demand for firm financing in equation (A25) yields the market clearing condition in the capital market:

$$D_K(r_W, r_B) = \frac{\widetilde{\alpha}}{1-\widetilde{\alpha}} \frac{r_B - g}{h(r_W - r_B) + r_B + \delta} - \frac{p\sigma m(r_W - r_B)}{\sigma(p+g) + \rho - r_W}\chi = 0 \tag{A29}$$

where $D_K(r_W, r_B)$ is the excess demand for capital, and the curve $D_K(r_W, r_B) = 0$ defines the locus of points for which the capital market clears. Likewise, the market clearing condition in the bond market is given by

$$D_B(r_W, r_B) = \frac{p\sigma(m(r_W - r_B) - 1)}{\sigma(p+g) + \rho - r_W}\chi + \chi - \frac{r_B - \rho - \sigma g}{\sigma(p+g) + \rho - r_B}(1-\chi) = 0, \tag{A30}$$

where $D_B(r_W, r_B)$ is the excess demand for bonds, and the curve $D_B(r_W, r_B) = 0$ defines the locus of points for which the bond market clears.

The following Lemmas characterize the behavior of these loci.

Lemma A3 *The curve $D_K(r_W, r_B) = 0$ gives a continuous and upward sloping locus in the (r_W, r_B) space defined for $r_W \in (g, \rho + \sigma(p+g))$ and $r_B \in (g, r_W)$. Moreover:*

1. *as $r_W \downarrow g$, the locus converges to the point (g, g)*

2. *as $r_W \uparrow \rho + \sigma(p+g)$, the locus converges to the point $(\rho + \sigma(p+g), \rho + \sigma(p+g))$.*

Lemma A4 *The curve $D_B(r_W, r_B) = 0$ gives a continuous and initially decreasing locus in the (r_W, r_B) space defined for $r_W \in (\rho + \sigma g, \rho + \sigma(p+g))$ and $r_B < r_W$. Moreover:*

1. *as $r_W \downarrow \rho + \sigma g$, the locus converges to the point $(\rho + \sigma g, \rho + \sigma g)$;*

2. *as $r_W \uparrow \rho + \sigma(p+g)$, the locus converges to the point $(\rho + \sigma(p+g), \tilde{r}_B)$, where*

$$\tilde{r}_B := \rho + \sigma(p+g) - \frac{1}{2}(\sigma+1)\gamma\nu^2.$$

3. *let $\bar{\gamma} := 2p\frac{\sigma}{1+\sigma}(1-\chi)$.*

    - *if $\gamma\nu^2 > \bar{\gamma}$, then $r_W - r_B$ increases from zero to a maximum of $\frac{1}{2}(\sigma+1)\gamma\nu^2$ along the locus $D_B(r_W, r_B) = 0$;*

- *if $\gamma\nu^2 < \bar{\gamma}$, then $r_W - r_B$ increases from zero up to a maximum and then decreases and reaches $\frac{1}{2}(\sigma+1)\gamma\nu^2$ along the locus $D_B(r_W, r_B) = 0$.*

The proof of these lemmas is technical and relegated to Appendix F.

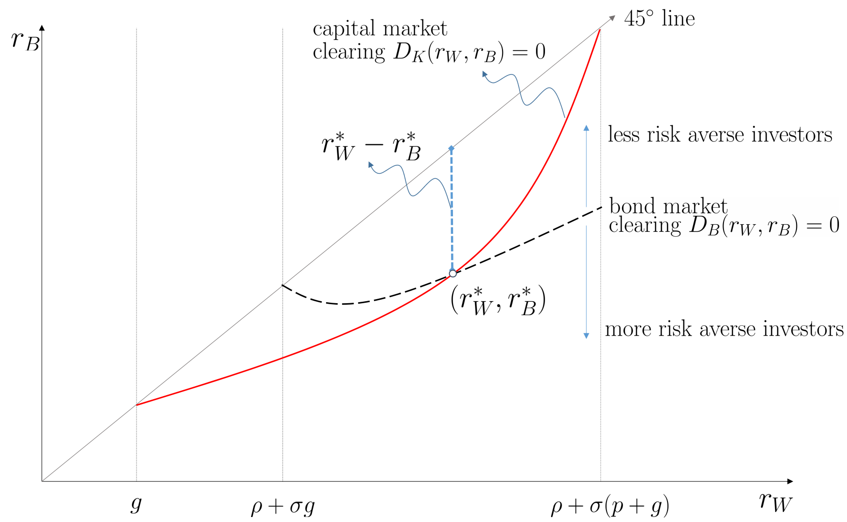

Figure A1: Typical configurations of the locus for $D_K(r_W, r_B) = 0$ and $D_B(r_W, r_B) = 0$.

The two lemmas combined imply that the loci $D_K(r_W, r_B) = 0$ and $D_B(r_W, r_B) = 0$ are as depicted in Figure A1. The lemmas imply that at $r_W = \rho + \sigma g$, the locus for $D_B(r_W, r_B) = 0$ is above that for $D_K(r_W, r_B) = 0$ (recall that, by assumption, $\rho + \sigma g > g$). On the other hand, at $r_W = \rho + \sigma(p + g)$, the locus for $D_K(r_W, r_B) = 0$ is above that for $D_B(r_W, r_B) = 0$. The intermediate value theorem then implies that these loci intercept at a point r_W^*, r_B^* and an equilibrium exists. Moreover, as the figure shows, $r_W^* > r_B^* > g$ and $r_W^* \in (\rho + \sigma g, \rho + \sigma(p + g))$.

Finally, the characterization of the tail properties of the income and wealth distribution follows as a corollary of Proposition A1. ■

Proposition A1 *Let x denote effective wealth and define* normalized wealth *by*

$$\widetilde{x} = x \Big/ \frac{w_z}{r_B}.$$

Let $f_H(\widetilde{x})$ and $f_I(\widetilde{x})$ denote the PDFs of the distributions of normalized wealth for households and investors. The distribution of normalized wealth for households is given by

$$f_H(\widetilde{x}) = \zeta_H \widetilde{x}^{-\zeta_H - 1} \text{ for } \widetilde{x} \geq 1, \tag{A31}$$

where

$$\frac{1}{\zeta_H} := \frac{r_B - \rho - \sigma g}{p\sigma},$$

and the distribution of normalized wealth for investors is given

$$f_I(\widetilde{x}) = \begin{cases} \dfrac{1}{1/\zeta_P - 1/\zeta_N}\widetilde{x}^{-\zeta_P - 1} & \textit{for } \widetilde{x} \geq 1 \\ \dfrac{1}{1/\zeta_P - 1/\zeta_N}\widetilde{x}^{-\zeta_N - 1} & \textit{for } \widetilde{x} \in [0,1), \end{cases} \tag{A32}$$

where

$$\frac{1}{\zeta_P} := \frac{r_W - \rho - \sigma g - \frac{\sigma\kappa^2\nu^2}{2} + \sqrt{\left(r_W - \rho - \sigma g - \frac{\sigma\kappa^2\nu^2}{2}\right)^2 + 2\sigma^2\kappa^2\nu^2 p}}{2p\sigma} > 0$$

and

$$\frac{1}{\zeta_N} := \frac{r_W - \rho - \sigma g - \frac{\sigma\kappa^2\nu^2}{2} - \sqrt{\left(r_W - \rho - \sigma g - \frac{\sigma\kappa^2\nu^2}{2}\right)^2 + 2\sigma^2\kappa^2\nu^2 p}}{2p\sigma} < 0.$$

Moreover, the distribution for investors' income flows has a Pareto tail with tail index $1/\zeta_P$.

Proof. The evolution of (normalized) effective wealth for households is given by

$$\dot{\widetilde{x}}_t = (\mu_H - g)\widetilde{x}_t,$$

where

$$\mu_H := \frac{r_B - \rho}{\sigma}$$

and $\widetilde{x}_t$ resets to 1 with probability p.

The Kolmogorov-forward equation characterizing f_H in steady state is then given by

$$0 = -\partial_x\left((\mu_H - g)x f_H(x)\right) - p f_H(x) + p\wp(x-1), \tag{A33}$$

where $\wp(.)$ is the Dirac's delta function (a mass of 1 at 0). To solve this differential equation, we guess and verify that

$$f_H(x) = C_H x^{-\zeta_H - 1}, \text{ for } x \geq 1.$$

Plugging this guess in the KFE equation (A33), we obtain

$$0 = \zeta_H(\mu_H - g)C_H x^{-\zeta_H - 1} - pC_H x^{-\zeta_H - 1}, \text{ for } x \geq 1.$$

This implies

$$\frac{1}{\zeta_H} = \frac{\mu_H - g}{p} = \frac{r_B - \rho - \sigma g}{p\sigma},$$

as claimed in the proposition. Moreover, because the density $f_H(x)$ must integrate to 1, we obtain

$$1 = \int_1^\infty C_H x^{-\zeta_H - 1} dx \Rightarrow C_H = \zeta_H.$$

The evolution of (normalized) effective wealth for investors is given by

$$d\widetilde{x}_t = (\mu_I - g)\widetilde{x}_t dt + \kappa\nu\widetilde{x}_t dW_t,$$

where

$$\mu_I := \frac{r_W - \rho}{\sigma}$$

and $\widetilde{x}_t$ resets to 1 with probability p.

The Kolmogorov-forward equation characterizing f_I in steady state is then given by

$$0 = -\partial_x \left((\mu_I - g)x f_I(x)\right) + \frac{1}{2}\partial_{xx}\left(\nu^2\kappa^2 x^2 f_I(x)\right) - p f_I(x) + p\wp(x-1), \tag{A34}$$

where $\wp(.)$ is the Dirac's delta function (a mass of 1 at 0). To solve this differential equation, we guess and verify a piece-wise solution of the form

$$f_I(x) = C_P x^{-\zeta_P - 1}, \text{ for } x \geq 1,$$

and

$$f_I(x) = C_N x^{-\zeta_N - 1}, \text{ for } x \in (0, 1),$$

which allows for the possibility that the distribution might be different to the left and to the right of the reinjection point (note also that the process for effective wealth implies that $x > 0$).

Plugging this guess in the KFE equation (A33), we obtain

$$0 = \zeta_P(\mu_I - g)C_P x^{-\zeta_P - 1} + (\zeta_P - 1)\zeta_P \frac{1}{2}\nu^2\kappa^2 C_P x^{-\zeta_P - 1} - pC_P x^{-\zeta_P - 1}, \text{ for } x \geq 1.$$

This implies a quadratic equation for ζ_P given by

$$0 = \zeta_P(\mu_I - g) + (\zeta_P - 1)\zeta_P \frac{1}{2}\nu^2\kappa^2 - p.$$

Because the integral of $f_I(x)$ must converge on $(1, \infty)$, ζ_P must be equal to the unique positive root of the above quadratic equation, which is given by

$$\zeta_P = \frac{\left(\frac{\kappa^2\nu^2}{2} - \mu_I + g\right) + \sqrt{\left(\frac{\kappa^2\nu^2}{2} - \mu_I + g\right)^2 + 2\kappa^2\nu^2 p}}{\kappa^2\nu^2}.$$

Multiplying the numerator and denominator by

$$\left(\frac{\kappa^2\nu^2}{2}-\mu_I+g\right)-\sqrt{\left(\frac{\kappa^2\nu^2}{2}-\mu_I+g\right)^2+2\kappa^2\nu^2 p}$$

and rearranging, this formula yields

$$\frac{1}{\zeta_P}=\frac{\mu_I-g-\frac{\kappa^2\nu^2}{2}+\sqrt{\left(\mu_I-g-\frac{\kappa^2\nu^2}{2}\right)^2+2\kappa^2\nu^2 p}}{2p},$$

which is the same as the formula provided in the lemma.

Likewise, plugging our guess for $x\in(0,1)$, we obtain the same quadratic equation for ζ_N given by

$$0=\zeta_N(\mu_I-g)+(\zeta_N-1)\zeta_N\frac{1}{2}\nu^2\kappa^2-p.$$

Because the integral of $f_I(x)$ must converge on $(0,1)$, ζ_N must be equal to the unique negative root of the above quadratic equation, which is given by

$$\frac{1}{\zeta_N}=\frac{\mu_I-g-\frac{\kappa^2\nu^2}{2}-\sqrt{\left(\mu_I-g-\frac{\kappa^2\nu^2}{2}\right)^2+2\kappa^2\nu^2 p}}{2p},$$

which is the same as the formula provided in the lemma.

Finally, we turn to the constants C_P and C_N. First, because $f_I(x)$ must be continuous at $x=1$, we obtain $C_P=C_N$. Second, because the density $f_I(x)$ must integrate to 1, we obtain

$$1=\int_0^1 C_N x^{-\zeta_N-1}dx+\int_1^\infty C_P x^{-\zeta_P-1}dx\Rightarrow C_P=C_N=\frac{1}{1/\zeta_P-1/\zeta_N}.$$

Turning to the income distribution, Lemma S3 shows that, over a short period of time τ, the income received by an investor with effective wealth x_z can be approximated as

$$y_{z,\tau}=x_{z,0}(r_I\tau+\kappa\nu\sqrt{\tau}u),$$

where $u\sim N(0,1)$. For large y, we have that

$$\Pr(y_{z,\tau}\geq y)\propto\int_0^\infty(r_I\tau+\kappa\nu\sqrt{\tau}u)^{\zeta_P}y^{-\zeta_P}\phi(u)du\propto y^{-\zeta_P},$$

where $\phi(u)$ is the pdf of a standard normal, and the second equality follows from the fact that $\int_0^\infty(r_I\tau+\kappa\nu\sqrt{\tau}u)^{\zeta_P}\phi(u)du$ is a finite constant for any value of $\zeta_P\geq 0$ (which in turn follows from the fact that the normal distribution has finite moments). Thus, the income distribution also has a Pareto tail with tail index $1/\zeta_P$. ■

Proof of Proposition 5. The equilibrium equations for $r_K, r_B, \kappa, k_n, b_I$ and b_H are

$$\begin{aligned}
\left(\frac{r_B + \kappa \cdot (r_K - r_B) + \frac{1}{2}(\sigma - 1)\gamma\nu^2\kappa^2 - \rho}{\sigma} - g\right) \cdot (k_n + b_I + \chi) =& p \cdot (k_n + b_I) \\
\left(\frac{r_B - \rho}{\sigma} - g\right) \cdot (b_H + 1 - \chi) =& p \cdot b_H \\
\kappa \cdot (k_n + b_I + \chi) =& k_n \\
\frac{1}{\gamma\nu^2}(r_K - r_B) =& \kappa \\
b_I + b_H =& 0 \\
\frac{\alpha}{1-\alpha} \cdot \frac{r_B - g}{r_K + \delta} =& k_n.
\end{aligned}$$

For $\alpha = 0$, we get $r_K = r_B = \rho + \sigma g$, $\kappa = k_n = b_H = b_I = 0$.

Linearizing the system of equations around this equilibrium, we get

$$\begin{aligned}
\frac{dr_B}{\sigma} \cdot \chi =& p \cdot (dk_n + db_I) & \frac{dr_B}{\sigma} \cdot (1 - \chi) =& p \cdot db_H \\
d\kappa \cdot \chi =& dk_n & dr_K - dr_B =& \gamma\nu^2 d\kappa \\
db_I + db_H =& 0 & dk_n =& \frac{\rho + (\sigma - 1)g}{\rho + \sigma g + \delta}\alpha.
\end{aligned}$$

Which yields the solution

$$\begin{aligned}
dr_K =& \left[p\sigma + \frac{\gamma\nu^2}{\chi}\right] \frac{\rho + (\sigma - 1)g}{\rho + \sigma g + \delta}\alpha \\
dr_B =& p\sigma \cdot \frac{\rho + (\sigma - 1)g}{\rho + \sigma g + \delta}\alpha \\
d\kappa =& \frac{1}{\chi} \cdot \frac{\rho + (\sigma - 1)g}{\rho + \sigma g + \delta}\alpha.
\end{aligned}$$

It follows that for small values of α we can approximate all equilibrium objects as

$$\begin{aligned}
r_K =& \rho + \sigma g + \left[p\sigma + \frac{\gamma\nu^2}{\chi}\right] \frac{\rho + (\sigma - 1)g}{\rho + \sigma g + \delta}\alpha \\
r_B =& \rho + \sigma g + p\sigma \cdot \frac{\rho + (\sigma - 1)g}{\rho + \sigma g + \delta}\alpha \\
\kappa =& \frac{1}{\chi} \cdot \frac{\rho + (\sigma - 1)g}{\rho + \sigma g + \delta}\alpha \\
k_n =& \frac{\rho + (\sigma - 1)g}{\rho + \sigma g + \delta}\alpha.
\end{aligned}$$

Let α^*_{net} denote the capital share net of growth and depreciation in equation (16). By definition, this is equal to $k_n/(k_n + 1)$, and so for small values of α we get the approximation

$\alpha^*_{net} = k_n$ which implies $\alpha^*_{net} = \frac{\rho+(\sigma-1)g}{\rho+\sigma g+\delta}\alpha$.

We conclude that

$$\begin{aligned} r_K =& \rho + \sigma g + \left[p\sigma + \frac{\gamma\nu^2}{\chi}\right]\alpha^*_{net} \\ r_B =& \rho + \sigma g + p\sigma \cdot \alpha^*_{net} \\ \kappa =& \frac{1}{\chi}\alpha^*_{net}, \end{aligned}$$

as claimed in the proposition.

Finally, the formula for the return gap in equation 13 implies

$$r_W - \rho - \sigma g = p\sigma\alpha^*_{net} + \frac{1}{\chi^2}{\alpha^*_{net}}^2\gamma\nu^2.$$

Using this expression for the return gap, we can compute the tail index for inequality as

$$\frac{1}{\zeta} = \frac{p\sigma\alpha^*_{net} + \frac{1}{\chi^2}{\alpha^*_{net}}^2\gamma\nu^2 - \frac{1}{\chi^2}{\alpha^*_{net}}^2\frac{\sigma}{2}\nu^2 + \sqrt{(p\sigma\alpha^*_{net} + \frac{1}{\chi^2}{\alpha^*_{net}}^2\gamma\nu^2 - \frac{1}{\chi^2}{\alpha^*_{net}}^2\frac{\sigma}{2}\nu^2)^2 + \frac{2}{\chi^2}{\alpha^*_{net}}^2\sigma^2\nu^2 p}}{2p\sigma}.$$

For small values of α this can be linearized as

$$\frac{1}{\zeta} = \alpha^*_{net} + \alpha^*_{net} \cdot \frac{\frac{\nu^2}{p\chi^2}}{1 + \sqrt{1 + 2\frac{\nu^2}{p\chi^2}}}$$

■

Proposition 5 provides explicit formulas that are valid for small values of $\tilde{\alpha}$. We now provide an additional proposition characterizing the comparative statics of our extended model away from $\tilde{\alpha} = 0$.

Proposition A2 *Suppose that investors are risk averse and/or $\theta < \bar{\theta}$. There exists a threshold $\bar{\alpha} \in (0, 1]$ such that, for $\widetilde{\alpha} < \bar{\alpha}$, the balanced-growth equilibrium in the closed economy is unique, and following an increase in $\widetilde{\alpha}$, we have that:*

- *The return to wealth r^*_W, the return gap $r^*_W - \rho - \sigma g$, and the return spread $r^*_K - r^*_B$ all strictly increase, and the portfolio share of capital κ^* weakly increases;*
- *Top tail inequality $1/\zeta^*$ in (15) strictly increases.*

Proof of Proposition A2. Let $\bar{r}_W$ denote the point at which $r_W - r_B$ is maximized along the locus $D_B(r_W, r_B) = 0$. This definition implies that $\bar{r}_W = \rho + \sigma(p + g)$ if $\gamma\nu^2 < \bar{\gamma}$, and $\bar{r}_W < \rho + \sigma(p + g)$ if $\gamma\nu^2 > \bar{\gamma}$.

We first show that there exists a threshold $\bar{\alpha}$ such that, for $\widetilde{\alpha} < \bar{\alpha}$, there is a unique equilibrium, and this equilibrium satisfies $r^*_W < \bar{r}_W$.

Suppose there are multiple equilibria, and let $r_W^M(\widetilde{\alpha})$ denote the value of r_W^* in the equilibrium with the highest return to wealth. In this equilibrium, the locus for $D_K(r_W, r_B)$ cuts the locus for $D_B(r_W, r_B)$ from below, and so an increase in $\widetilde{\alpha}$ shifting the locus for $D_K(r_W, r_B)$ outwards results in a higher $r_W^M(\widetilde{\alpha})$. Moreover, as $\widetilde{\alpha} \to 0$, $r_W^M(\widetilde{\alpha}) \to \rho + \sigma g$, which implies that for small values of all $\widetilde{\alpha}$, we have $r_W^* < \bar{r}_W$ in any equilibria. Finally, note that as $\widetilde{\alpha} \to 0$, the locus for $D_K(r_W, r_B)$ converges to the 45 degree line and so we will have a unique equilibrium. This follows from the fact that, as shown in Lemma A4, the locus for $D_B(r_W, r_B) = 0$ moves away from the 45 degree line for $r_W \in (\rho + \sigma g, \breve{r}_W)$, and retains a gap of at least $\frac{1}{2}(\sigma + 1)\gamma\nu^2$ from there on. It follows that we can pick a $\bar{\alpha}$ such that, for $\widetilde{\alpha} < \bar{\alpha}$, the equilibrium is unique and satisfies $r_W^* < \bar{r}_W$.

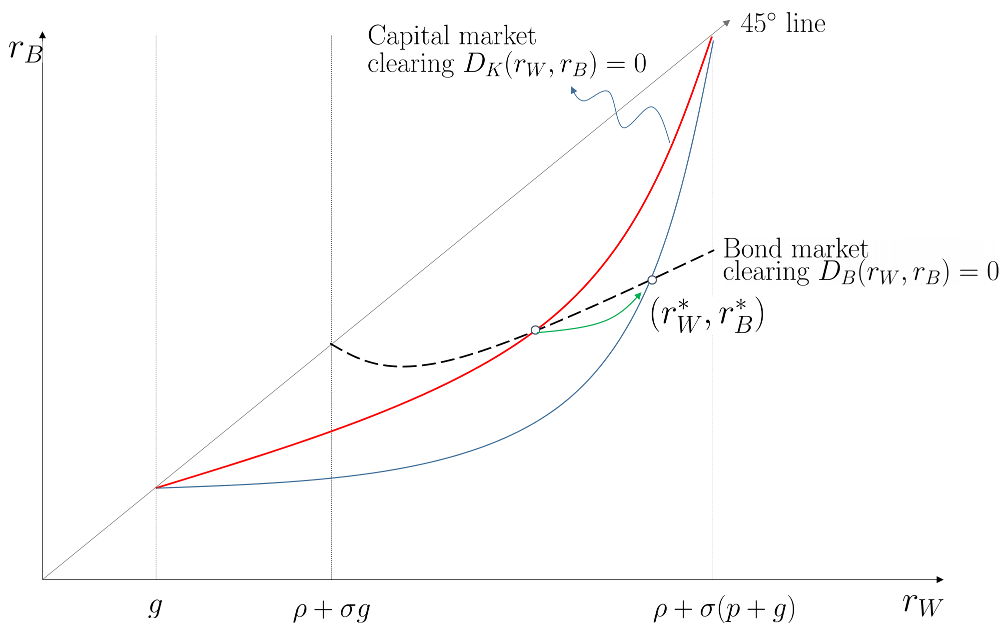

Figure A2: Effects of changes in the demand for capital on r_W^* and r_B^*.

For $\widetilde{\alpha} < \bar{\alpha}$, the equilibrium will look as in Figure A2, and increases in $\widetilde{\alpha}$ will result in a higher r_W^*. Moreover, because $r_W^* < \bar{r}_W$, we have that the gap $r_K^* - r_B^*$ rises following an increase in automation (so long as $\widetilde{\alpha}$ remains below $\bar{\alpha}$). ■

Supplementary Material for "Uneven Growth: Automation's Impact on Income and Wealth Inequality"

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C Dissipation Shocks

This section provides a formal treatment of dissipation shocks, and shows that they only affect households' saving problem by making them discount the future at a higher rate. We work in a general non-stationary environment. To do so, in this section, t will denote calendar time.

Formally, we assume that with Poisson rate p, households become impatient and only value their consumption in the following T periods. We work with the limit as $T \to 0$, i.e. households become infinitely impatient.

Let $V^T_{z,t,a}$ denote the value of lifetime consumption for a household who enters the impatience state at time t with assets $a \geq 0$ (recall that households who enter this state with negative assets see a cancellation of their debt). This value function is given by

$$V^T_{z,t,a} = \max_{\{c^T_{z,t+\tau,a}, a^T_{z,t+\tau,a}\}} \int_0^T e^{-\varrho\tau} \frac{(c^T_{z,t+\tau,a})^{1-\sigma}}{1-\sigma} d\tau \tag{S1}$$
$$\text{s.t.} \quad \dot{a}^T_{z,t+\tau,a} = w_{z,t+\tau} + r_{t+\tau} a^T_{z,t+\tau,a} - c^T_{z,t+\tau,a}, \quad a^T_{z,t+T,a} \geq 0, \quad a^T_{z,t,a} = a.$$

The following lemma characterizes the behavior of this value function and the consumption of households in the impatience state.

Lemma S1 (Characterization of the impatience state) *For $T \to 0$, the value function $V^T_{z,t,a}$ converges to zero*

$$\lim_{T\to 0} V^T_{z,t,a} \to 0 \quad \forall a.$$

In this limit case, the flow consumption of the impatient is pK_t, where K_t are total asset holdings in the economy.

Before proceeding with the Lemma's proof, we briefly explain the intuition for the result that $V^T_{z,t,a} \to 0$ as $T \to 0$. Households enjoy an infinite consumption flow over an infinitesimal time interval $T \to 0$. But because utility is strictly concave, the value of such consumption converges to zero as $T \to 0$. In terms of the notation in the proof below, we have

$$V^T_{z,t,a} \approx \int_0^T u(a/T)dt = T\frac{(a/T)^{1-\sigma}}{1-\sigma} = T^\sigma \frac{a^{1-\sigma}}{1-\sigma} \to 0 \quad \text{as} \quad T \to 0.$$

Proof. The Euler equation for problem (S1) implies that consumption satisfies

$$c^T_{z,t+\tau,a} = c^T_{z,t,a} \exp\left(\int_0^\tau \frac{1}{\sigma}(r_{t+s} - \varrho)ds\right). \tag{S2}$$

Because $a^T_{z,t+T,a} \geq 0$, any optimal consumption path must involve $a^T_{z,t+T,a} = 0$. Therefore, the consumption path satisfies the budget constraint

$$\int_0^T \exp\left(-\int_0^\tau r_{t+s}ds\right) c^T_{z,t+\tau,a} d\tau = a + \int_0^T \exp\left(-\int_0^\tau r_{t+s}ds\right) w_{z,t+\tau} d\tau$$

Plugging equation (S2) into the budget constraint, we obtain the following expression for $c^T_{z,t,a}$:

$$c^T_{z,t,a} = \frac{a + \int_0^T \exp\left(-\int_0^\tau r_{t+s}ds\right) w_{z,t+\tau} d\tau}{\int_0^T \exp\left(-\int_0^\tau (r_{t+s}(1-1/\sigma) + \varrho/\sigma)ds\right) d\tau}.$$

Thus, the optimal consumption path satisfies

$$c^T_{z,t+\tau,a} = \frac{a + \int_0^T \exp\left(-\int_0^\tau r_{t+s}ds\right) w_{z,t+\tau} d\tau}{\int_0^T \exp\left(-\int_0^\tau (r_{t+s}(1-1/\sigma) + \varrho/\sigma)ds\right) d\tau} \exp\left(\int_0^\tau \frac{1}{\sigma}(r_{t+s} - \varrho)ds\right). \tag{S3}$$

Plugging equation (S3) into (S1), we obtain

$$\begin{aligned} V^T_{z,t,a} =& \frac{1}{1-\sigma}\left(a + \int_0^T \exp\left(-\int_0^\tau r_{t+s}ds\right) w_{z,t+\tau} d\tau\right)^{1-\sigma} \\ & \left(\int_0^T \exp\left(-\int_0^\tau (r_{t+s}(1-1/\sigma) + \varrho/\sigma)ds\right) d\tau\right)^{\sigma}. \end{aligned}$$

As $T \to 0$, we have that $a + \int_0^T \exp\left(-\int_0^\tau r_{t+s}ds\right) w^T_{z,t+\tau} d\tau$ converges to a and the term $\int_0^T \exp\left(-\int_0^\tau (r_{t+s}(1-1/\sigma) + \varrho/\sigma)ds\right) d\tau$ converges to zero. It follows that $V^T_{z,t,a}$ converges to zero, as claimed in the lemma.

We now turn to computing the flow consumption of households in the impatience state. For $\tau \in [0,T]$, let $h_{z,a,t-\tau}$ denote the mass of households of skill z who entered the impatience state with assets a at time $t-\tau$. Their flow consumption in the impatience state equals

$$\begin{aligned}
C_t^T &= \sum_z \int_a \int_0^T c_{z,t+\tau,a}^T h_{z,a,t-\tau} d\tau da \\
&= \sum_z \int_a \int_0^T \frac{a + \int_0^T \exp\left(-\int_0^\tau r_{t+s} ds\right) w_{z,t+\tau} d\tau}{\int_0^T \exp\left(-\int_0^\tau (r_{t+s}(1-1/\sigma) + \varrho/\sigma) ds\right) d\tau} \times \\
&\qquad\qquad \exp\left(\int_0^\tau \frac{1}{\sigma}(r_{t+s} - \varrho) ds\right) h_z(a, t-\tau) d\tau da \\
&= \sum_z \int_a \left(a + \int_0^T \exp\left(-\int_0^\tau r_{t+s} ds\right) w_{z,t+\tau} d\tau\right) \times \\
&\qquad\qquad \left(\frac{\int_0^T \exp\left(\int_0^\tau \frac{1}{\sigma}(r_{t+s} - \varrho) ds\right) h_{z,a,t-\tau} d\tau}{\int_0^T \exp\left(-\int_0^\tau (r_{t+s}(1-1/\sigma) + \varrho/\sigma) ds\right) d\tau}\right) da.
\end{aligned}$$

To compute the limit of C_t^T as $T \to 0$, note that

$$\lim_{T\to 0} a + \int_0^T \exp\left(-\int_0^\tau r_{t+s} ds\right) w_{z,t+\tau} d\tau = a,$$

and, by an application of L'Hôpital's rule

$$\lim_{T\to 0} \frac{\int_0^T \exp\left(\int_0^\tau \frac{1}{\sigma}(r_{t+s} - \varrho) ds\right) h_{z,a,t-\tau} d\tau}{\int_0^T \exp\left(-\int_0^\tau (r_{t+s}(1-1/\sigma) + \varrho/\sigma) ds\right) d\tau} = h_{z,a,t}.$$

It follows that

$$\lim_{T\to 0} C_t^T = \sum_z \int_a a h_{z,a,t} da = pK_t.$$

The last step follows from the fact that the probability of entering the impatience state is p, independently of assets and skills.

Note that in the proof above, we assumed that all households entering the impatience state have positive assets. Recall that upon entering this state, households with negative assets see a cancellation of their debt, which results in a one time negative consumption equal to minus their debt generating no disutility. This ensures that the above proof goes through even when some households have negative assets. By construction, for these households we have $V_{z,t,a}^T = 0$ ■

Lemma S1 and the remarks above imply that the consumption and saving decisions of households solve the maximization problem in (1) in the main text.

C.1 Other Foundations for Dissipation Shocks

This appendix provides micro-foundations for dissipation shocks. First, we consider a model with stochastic death, no altruism, and an annuity market as in Blanchard (1985). Second, we consider a model with infinitely lived dynasties and population growth in the form of new households emerging outside of existing dynasties. Third, we consider a model with finite lives and bequests where altruism evolves stochastically within a dynasty. Finally, we present a model with multiple capital lines subject to obsolescence shocks. In all of these models, an equation similar to (5) determines long-run returns and the distribution of wealth also follows a Pareto distribution.

C.1.1 Stochastic deaths and annuities

Death is stochastic, with people dying with a Poisson probability p. As in Blanchard (1985), there is an annuity market such that, when people die, they give their wealth to an insurance company. In exchange, they receive a flow income of $pa_{z,t}$ when alive.

As in the main text, we focus on a steady state of this economy and use t to denote the age of a person. Consumption decisions solve the problem

$$\max_{\{c_{z,t},a_{z,t}\}_{t\geq 0}} \int_0^\infty e^{-(\varrho+p)t}\frac{c_{z,t}^{1-\sigma}}{1-\sigma}dt \tag{S4}$$
$$\text{s.t. } \dot{a}_{z,t} = w_z + (r+p)a_{z,t} - c_{z,t}, \text{ and } a_{z,t} \geq -w_z/(r+p)$$

Analogous to Lemma 1, the optimal saving and consumption policy functions are

$$\dot{a}_{z,t} = \frac{r-\varrho}{\sigma}\left(a_{z,t}+\frac{w_z}{r+p}\right), \qquad c_{z,t} = \left(r+p-\frac{r-\varrho}{\sigma}\right)\left(a_{z,t}+\frac{w_z}{r+p}\right)$$

with $a_{z,0} = 0$.

Let X denote effective wealth of the economy in steady state. We have that $X = H + K$, where $H = \bar{w}/(r+p)$ denotes human wealth, and K denotes the value of the capital stock.

Following the derivation in the main text, it follows that the aggregate behavior of X is given by:

$$0 = \dot{X} = \frac{r-\varrho}{\sigma}X - pK. \tag{S5}$$

Relative to our baseline model, the difference is that in this equation, the rate of accumulation depends on $r-\varrho$, rather than $r-\rho$. This difference is driven by the incentives to accumulate assets introduced by the annuity market: wealth now pays return $r+p$ so that the individual wealth accumulation rate is $(r+p-\varrho-p)/\sigma = (r-\varrho)/\sigma$. As in our baseline model, we still have the term $-pK$ on the right-hand side of equation (S5). This term now captures the wealth paid by dying individuals to the insurance company upon death. The insurance company redistributes this wealth to all living households via a higher return to their wealth

$r+p$. The term $(r-\varrho)/\sigma$ already accounts for the redistributed wealth, so the $-pK$ term ensures there is no double-counting.

Note that equation (S5) is equivalent to the equations for aggregates in Blanchard (1985) who analyzes the special case of logarithmic utility, $\sigma = 1$. In particular, using our notation, equations (5), (6) and (7) in the published version of his paper are: $C = (\varrho+p)(H+K)$, $\dot{H} = (r+p)H - \bar{w}$ and $\dot{K} = rK + \bar{w} - C$. Using our definition of effective wealth $X := H + K$, we have $\dot{X} = (r+p)H + rK - (\varrho+p)(H+K) = (r-\varrho)X - pK$ which is the special case of (S5) with $\sigma = 1$.

Net capital income now includes annuity payments and so it is given by $(r+p)K$. Therefore, rearranging (S5) and using the fact that $X = K + \bar{w}/(r+p)$, the steady state return to wealth satisfies

$$r = \varrho + p\sigma \frac{(r+p)K}{(r+p)K + \bar{w}} = \varrho + p\sigma\alpha^*_{net}.$$

Alternatively, the analogue to (6) for long-run capital supply is $(K/\bar{w})^s = \frac{r-\varrho}{(p\sigma+\varrho-r)(r+p)}$.

Following the same steps as in the baseline model, it follows that effective wealth follows a Pareto distribution. Wages give the scale parameters, and the common tail parameter is given by

$$\frac{1}{\zeta} = \frac{1}{p}\frac{r-\varrho}{\sigma} = \alpha^*_{net},$$

where we have used that the rate of individual wealth accumulation depends on $r - \varrho$ as discussed below equation (S5).

The model with annuities therefore yields exactly the same expressions for the steady state return to wealth and the tail parameter of the wealth distribution as in our baseline model except for one difference: the effective discount rate $\rho = \varrho + p$ is replaced by the unadjusted discount rate ϱ.

C.1.2 Population growth

We now assume that dynasties are infinitely lived and experience no dissipation shocks. However, at a rate p, new dynasties emerge. These new dynasties start with zero assets.

As in the main text, we focus on a steady state of this economy and use t to denote the age of a dynasty. Consumption decisions solve the problem

$$\max_{\{c_{z,t}, a_{z,t}\}_{t\geq 0}} \int_0^\infty e^{-\varrho t} \frac{c_{z,t}^{1-\sigma}}{1-\sigma} dt \tag{S6}$$
$$\text{s.t. } \dot{a}_{z,t} = w_z + r a_{z,t} - c_{z,t}, \text{ and } a_{z,t} \geq -w_z/r.$$

Analogous to Lemma 1, the optimal saving and consumption policy functions are

$$\dot{a}_{z,t} = \frac{r-\varrho}{\sigma}\left(a_{z,t} + \frac{w_z}{r}\right), \qquad c_{z,t} = \left(r - \frac{r-\varrho}{\sigma}\right)\left(a_{z,t} + \frac{w_z}{r}\right)$$

with $a_{z,0} = 0$.

Let X denote effective wealth of the economy in steady state. We have that $X = H + K$, where $H = \bar{w}/r$ denotes human wealth, and K denotes the value of the capital stock. Because of population growth, we are interested in a steady state where aggregates grow at a rate p.

Following the derivation in the main text, it follows that the aggregate behavior of X is given by:

$$pX = \dot{X} = \frac{r-\varrho}{\sigma}X + pH.$$

Relative to our baseline case, there are two differences in this equation. First, total wealth grows at a rate p since population increases over time. Second, the term $+pH$ captures the wealth brought by new dynasties, who start life with nothing but their labor income.

Rearranging this equation, and using the fact that $X = H + K$, we obtain:

$$0 = \frac{r-\varrho}{\sigma}X - pK,$$

which implies that, in steady state

$$r = \varrho + p\sigma\alpha^*_{net}.$$

Let us now turn to the wealth distribution. Let $f_z(x)$ denote the distribution of effective wealth x_z for households with wage w_z. Following the same steps as in the baseline model, we obtain the Kolgomorov Forward Equation:

$$0 = -\left(\frac{r-\varrho}{\sigma}xf_z(x)\right)' - pf_z(x)$$

on $(w_z/r, \infty)$. The term $-pf_z(x)$ now captures the loss of probability mass as new individuals are born and start their lives with effective wealth w_z/r. Following the same steps as in the baseline model, it follows that effective wealth follows a Pareto distribution.

C.1.3 Finite lives and stochastic altruism

We now consider a model with finite lives and stochastic altruism. Individuals live for a period of length T. They are born as one of two types: altruistic, which happens with probability e^{-pT}, or non-altruistic (selfish), which happens with probability $1 - e^{-pT}$. Individuals decide how much to consume and save during their lifetimes and how much wealth to pass on to their offspring when they die at age T: if they are altruistic they leave their children some wealth; instead, if they are selfish, they consume all their wealth before they die. Therefore, the dynasty continues with probability e^{-pT} but is interrupted with probability $1 - e^{-pT}$.

As in the main text, we focus on a steady state of this economy and use t to denote age. Let $v_A^T(a)$ denote the value function for an altruistic individual who starts life with assets a and $v_N^T(a)$ that for a non-altruistic individual. We have

$$v_A^T(a) = \max_{\{c_{z,t},a_{z,t}\}_{t\in[0,T]}} \int_0^T e^{-\varrho t}\frac{c_{z,t}^{1-\sigma}}{1-\sigma}dt + e^{-(\varrho+p)T}v_A^T(a_{z,T}) + e^{-\varrho T}\left(1-e^{-pT}\right)v_N^T(a_{z,T}), \tag{S7}$$
$$\text{s.t. } a_{z,0} = a, \quad \dot{a}_{z,t} = w_z + ra_{z,t} - c_{z,t}, \text{ and } a_{z,t} \geq -w_z/r.$$

and

$$v_N^T(a) = \max_{\{c_{z,t},a_{z,t}\}_{t\in[0,T]}} \int_0^T e^{-\varrho t}\frac{c_{z,t}^{1-\sigma}}{1-\sigma}dt, \tag{S8}$$
$$\text{s.t. } a_{z,0} = a, \quad \dot{a}_{z,t} = w_z + ra_{z,t} - c_{z,t}, \text{ and } a_{z,t} \geq -w_z/r.$$

We now show that, as $T \to 0$, this model collapses to our baseline model with dissipation shocks.

At any point in time, a fraction $\frac{1}{T}(1 - e^{-pT})$ of households starts life with no assets. Taking the limit as $T \to 0$, this fraction converges to p.

Lemma S1 then implies that $\lim_{T\to 0} v_N^T(a_{z,T}) \to 0$ and the consumption of people in the non-altruistic state is pK.

Using this result, and expanding equation (S7) around $T = 0$, we obtain

$$0 = \max_{c_{z,0}} \frac{c_{z,0}^{1-\sigma}}{1-\sigma} - (\varrho + p)v_A(a_{z,0}) + v_A'(a_{z,0})\dot{a}_{z,0},$$

where $v_A(a_{z,0}) = \lim_{T\to 0} v_A^T(a_{z,T})$. It follows that $v_A(a_{z,0})$ satisfies the same Bellman equation that characterizes optimal consumption and saving decisions in our baseline model. Thus, both models generate the same value functions, consumption and saving decisions.

C.1.4 Uninsured capital income risk (capital obsolescence shocks)

Finally, we consider a model of uninsured capital income risk. There are multiple types of capital indexed by i representing different investment projects and k_i denotes the stock of capital of type i households have already accumulated. Capital of type i is either useful in production or obsolete which we denote by $\zeta_i \in \{0, 1\}$, with $\zeta_i = 0$ denoting obsolescence. Total capital services in the economy are

$$K = \int_i \zeta_i k_i di.$$

Obsolescence occurs at Poisson rate p. For example, obsolescence might be due to the fact that a whole technology class or an intermediate product line is suddenly displaced by a new one via creative destruction.

Each household is assigned a unique investment project i. While capital of type i is useful in production it pays a return r. When capital becomes obsolete (which happens at rate p), the household draws a new investment project and start with zero capital in this project, i.e. when an obsolescence shock hits, the affected capital has a return of minus one hundred percent (or an instantaneous continuous-time return of $-\infty$). This formulation implies that a household z with an active project $i(z)$ solves the problem

$$\max_{\{c_{z,t},k_{i(z),t}\}_{t\geq 0}} \int_0^\infty e^{-\varrho t} \frac{c_{z,t}^{1-\sigma}}{1-\sigma} dt$$
$$\text{subject to: } \dot{k}_{i(z),t} = rk_{i(z),t} + w_z - c_{z,t}, \text{ and } c_{z,t}, k_{i(z),t} \geq 0,$$

where in addition, $k_{i(z),t}$ resets to zero with probability p—capturing the fact that the household needs to start a new investment project.

As in Lemma 1, household policy functions (assuming $r > \rho$) are given by

$$\dot{k}_{i(z),t} = \frac{r-\rho}{\sigma}\left(k_{i(z),t} + \frac{w_z}{r}\right), \qquad c_{z,t} = \left(r - \frac{r-\rho}{\sigma}\right)\left(k_{z,t} + \frac{w_z}{r}\right),$$

with $k_{i(z),t}$ reseting to zero with probability p.

As in the main text, it follows that the aggregate behavior of K satisfies

$$\dot{K} = \frac{r-\rho}{\sigma}\left(K + \frac{\bar{w}}{r}\right) - pK \quad \Rightarrow \quad r = \rho + p\sigma\alpha^*_{net}.$$

This implies the same dynamics for wealth and wealth inequality that we obtained in the model with dissipation shocks. The only difference relative to our baseline model is that, in this case, capital that becomes obsolete cannot be consumed or re-invested. This implies a different Euler equation and resource constraint given by

$$\begin{aligned}\dot{C} =& \frac{1}{\sigma}(Y'(K) - \delta - \rho)C - \mu pK,\\ \dot{K} =& Y(K) - (\delta + p)K - C.\end{aligned}$$

Finally, note that this formulation assumes limited risk sharing and diversification. If households were to decide how much to invest in each active capital line, they would invest a constant fraction in each and obtain a risk free return of $r - p$. Equivalently, if an investment company were to pool all investments, it would offer all households a common and risk-free return of $r - p$. In both cases we would recover Gorman aggregation; the steady state would involve $r = \rho$ and the wealth distribution would remain indeterminate.

D Results Regarding the Composition of Income

D.1 Income composition at top of income distribution

This appendix provides a proposition exploring the implications of our formulas in the baseline model for the composition of the income distribution.

Proposition S1 (Composition and sources of income at top of income distribution) *Let $\bar{q} := \Pr(income \geq \max_z w_z)$. For $q < \bar{q}$, we have:*

- *the probability that someone with a wage w_z is in the top q is*

$$\Pr(skill = z | top\ q) = \frac{\ell_z w_z^{1/\alpha^*_{net}}}{\sum_v \ell_v w_v^{1/\alpha^*_{net}}};$$

- *the share of labor income relative to total income held by the top q is*

$$\frac{\mathbb{E}[labor\ income | top\ q]}{\mathbb{E}[income | top\ q]} = (1 - \alpha^*_{net}) q^{\alpha^*_{net}} \frac{\sum_z \ell_z w_z^{1+1/\alpha^*_{net}}}{\left(\sum_z \ell_z w_z^{1/\alpha^*_{net}}\right)^{1+\alpha^*_{net}}};$$

- *the share of national income held by the top q is*

$$S(q) = \Lambda q^{1-\alpha^*_{net}},$$

where Λ is a constant that depends on the wage distribution.

Proof of Proposition S1. We start by deriving the probability that households with skill z are among the top q income earners. To save on notation, we do not include asterisks when denoting steady state objects.

Let $y(q)$ denote the income of the qth higher earner. That is:

$$\Pr(\text{income} \geq y(q)) = q.$$

By definition, for $q < \bar{q}$ we have $y(q) > \max_z w_z$. We can thus compute $y(q)$ explicitly as

$$q = \Pr(\text{income} \geq y(q)) = \sum_z \ell_z \left(\frac{y(q)}{w_z}\right)^{-1/\alpha_{net}},$$

which implies

$$y(q) = q^{-\alpha_{net}} \left(\sum_z \ell_z w_z^{1/\alpha_{net}}\right)^{\alpha_{net}}. \tag{S9}$$

An application of Bayes' rule implies

$$\Pr(\text{skill} = z|\text{top } q) = \frac{\ell_z \Pr(\text{income} \geq y(q)|z)}{\sum_v \ell_v \Pr(\text{income} \geq y(q)|v)} = \frac{\ell_z y(q)^{-1/\alpha_{net}} w_z^{1/\alpha_{net}}}{\sum_v \ell_v y(q)^{-1/\alpha_{net}} w_v^{1/\alpha_{net}}} = \frac{\ell_z w_z^{1/\alpha_{net}}}{\sum_v \ell_v w_v^{1/\alpha_{net}}}.$$

We now turn to the share of labor income at the top of the income distribution. The expected labor income for households at the top q is given by

$$\mathbb{E}[\text{labor income}|\text{top } q] = \sum_z w_z \Pr(\text{skill} = z|\text{top } q) = \sum_z \frac{\ell_z w_z^{1+1/\alpha_{net}}}{\sum_v \ell_v w_v^{1/\alpha_{net}}} = \frac{\sum_z \ell_z w_z^{1+1/\alpha_{net}}}{\sum_z \ell_z w_z^{1/\alpha_{net}}}.$$

The expected income for households at the top q is given by

$$\mathbb{E}[\text{income}|\text{top } q] = \sum_z \mathbb{E}[\text{income}|\text{income} \geq y(q)] \Pr(\text{skill} = z|\text{top } q) = \frac{y(q)}{1 - \alpha_{net}}.$$

Here, we used the fact that $\mathbb{E}[\text{income}|\text{income} \geq y(q)] = \frac{y(q)}{1-\alpha_{net}}$, a well-known property of Pareto distributions.

It follows that

$$\frac{\mathbb{E}[\text{labor income}|\text{top } q]}{\mathbb{E}[\text{income}|\text{top } q]} = \frac{1 - \alpha_{net}}{y(q)} \frac{\sum_z \ell_z w_z^{1+1/\alpha_{net}}}{\sum_z \ell_z w_z^{1/\alpha_{net}}} = (1 - \alpha_{net}) q^{\alpha_{net}} \frac{\sum_z \ell_z w_z^{1+1/\alpha_{net}}}{\left(\sum_z \ell_z w_z^{1/\alpha_{net}}\right)^{1+\alpha_{net}}}$$

Finally, we compute the share of national income earned by the top q. For $q \leq \bar{q}$, the top q earn an income

$$T(q) = q\mathbb{E}[\text{income}|\text{top } q] = \frac{1}{1 - \alpha_{net}} q^{1-\alpha_{net}} \left(\sum_z \ell_z w_z^{1/\alpha_{net}}\right)^{\alpha_{net}}.$$

It follows that the top q earn a share of national income equal to:

$$S(q) = \frac{S(q)}{S(\bar{q})} S(\bar{q}) = \frac{T(q)}{T(\bar{q})} S(\bar{q}) = \frac{q^{1-\alpha_{net}}}{\bar{q}^{1-\alpha_{net}}} S(\bar{q}).$$

The result in the proposition follows by letting $\Lambda = \frac{1}{\bar{q}^{1-\alpha_{net}}} S(\bar{q})$. ■

D.2 Meade's Formula about Compositional Effects

Meade (1964, p.34) states equation (12) which links changes in the top q percent income share to changes in the net capital share α_{net}. We here provide a quick derivation. As in Appendix D.1 denote by $y(q)$ the income of the qth highest earner. This income is composed of capital income $y_k(q)$ and labor income $y_\ell(q)$

$$y(q) = y_k(q) + y_\ell(q) \tag{S10}$$

Next denote the corresponding aggregates by $Y := \int_0^1 y(q)dq, Y_k := \int_0^1 y_k(q)dq$ and $Y_\ell := \int_0^1 y_\ell(q)dq$ and the net capital share by $\alpha_{net} = Y_k/Y$. Dividing (S10) by Y we have

$$\frac{y(q)}{Y} = \alpha_{net}\frac{y_k(q)}{Y_k} + (1-\alpha_{net})\frac{y_\ell(q)}{Y_\ell}$$

Therefore the top q percent income share $S(q) = y(q)/Y$ satisfies (12) where $\tilde{S}_k(q) := y_k(q)/Y_k$ and $\tilde{S}_\ell(q) := y_\ell(q)/Y_\ell$ are the shares of aggregate capital income and labor income earned by the top q percent of the distribution of *total* income.

E Transitional Dynamics in the Baseline Model

This section describes the full model in a non-stationary environment. Because firms rent capital from households, the production structure remains unchanged. We therefore focus on the household problem in a non-stationary environment. The section concludes with the proof of Proposition S2, which characterizes the transitional dynamics of our model.

E.1 Savings Problem in a Non-stationary Environment

Unlike in the main text, in what follows it will be convenient to keep track of time and cohorts, rather than time since last dissipation shock. Thus, we use t to denote calendar time, and b to denote the time at which the wealth of a household was last reset. Also, define human wealth at time t $h_{z,t}$ for households with skill z as

$$h_{z,t} := \int_t^\infty e^{-\int_t^s r_\tau d\tau} w_{z,s} ds.$$

Lemma S1 implies that the consumption and saving decisions of households solve a variant of (1) in the main text generalized to a non-stationary environment:

$$\begin{aligned} \max_{\{c_{z,t,b},a_{z,t,b}\}_{t\geq b}} \quad & \int_b^\infty e^{-(\varrho+p)(t-b)} \frac{c_{z,t,b}^{1-\sigma}}{1-\sigma} dt \\ \text{s.t. } \quad & \dot{a}_{z,t,b} = w_{z,t} + r_t a_{z,t,b} - c_{z,t,b}, \text{ and } a_{z,b,b} = 0, a_{z,t,b} \geq -h_{z,t}. \end{aligned} \tag{S11}$$

Here, $c_{z,t,b}$ and $a_{z,t,b}$ denote consumption and assets at time t of a household from the cohort who experienced their last dissipation shock at time b.

To characterize the solution to the problem in equation (S11), we generalize the definition of effective wealth to

$$x_{z,t,b} := a_{z,t,b} + h_{z,t}.$$

Lemma S2 (Households policy functions outside the steady state) *Suppose that r_t converges to r^* and $r^* > (r^* - \rho)/\sigma$. The unique interior solution to the household problem in equation (S11) is given by policy functions that are linear in effective wealth*

$$\begin{aligned} \dot{x}_{z,t,b} &= (r_t - \mu_t)x_{z,t,b}, \\ c_{z,t,b} &= \mu_t x_{z,t,b}, \end{aligned} \tag{S12}$$

for $t \geq b$, with $x_{z,b,b} = h_{z,b}$.

Here μ_t denotes the marginal propensity to consume out of wealth, and satisfies the differential equation:

$$\frac{\dot{\mu}_t}{\mu_t} = \mu_t - r_t + \frac{1}{\sigma}(r_t - \rho) \tag{S13}$$

Proof. The maximization problem can be rewritten using effective wealth as

$$\begin{aligned} \max_{\{c_{z,t,b}, x_{z,t,b}\}_{t \geq b}} \int_t^\infty e^{-(\varrho+p)(t-b)} \frac{c_{z,t,b}^{1-\sigma}}{1-\sigma} dt \\ \text{s.t. } \dot{x}_{z,t,b} = r_t x_{z,t,b} - c_{z,t,b}, \text{ and } x_{z,t,b} \geq 0 \end{aligned} \tag{S14}$$

The Hamiltonian associated with this maximization problem is

$$H(c_z, x_z, \lambda_z) := \frac{c_z^{1-\sigma}}{1-\sigma} + \lambda(r x_z - c_z), \tag{S15}$$

where λ_z is the co-state for effective wealth.

We show that the unique solution to equation (S12) starting from $x_{z,b,b} = h_{z,b}$ solves the maximization problem in (A11).

Theorem 7.14 in Acemoglu (2009) implies that this candidate path reaches an optimum if there exists a co-state variable $\lambda_{z,t,b}$ such that:

1. the path satisfies the restrictions $\dot{x}_{z,t,b} = r_t x_{z,t,b} - c_{z,t,b}$, and $x_{z,t,b} \geq 0$;

2. the following necessary conditions hold:

$$\begin{aligned} c_{z,t,b}^{-\sigma} =& \lambda_{z,t,b}, \\ \rho\lambda_{z,t,b} - \dot{\lambda}_{z,t,b} =& r\lambda_{z,t,b}; \end{aligned}$$

3. the maximized Hamiltonian $M(x_z, \lambda_z) = \max_c H(c, x_z, \lambda_z)$ is concave in x_z along the candidate path;

4. the transversality condition holds. That is, for the candidate path, we have

$$\lim_{t\to\infty} e^{-\rho t} x_{z,t,b}\lambda_{z,t,b} = 0.$$

and for all other feasible paths, $\hat{x}_{z,t,b}$, we have

$$\lim_{t\to\infty} e^{-\rho t} \hat{x}_{z,t,b}\lambda_{z,t,b} \geq 0.$$

To prove condition 1, note that starting from $x_{z,b,b} = h_{z,b}$, we will have $x_{z,t,b} \geq 0$ for all $t \geq b$. Moreover, for any path satisfying equations (A13) the flow budget constraint holds:

$$\begin{aligned} r_t x_{z,t,b} - c_{z,t,b} =& r_t x_{z,t,b} - \mu_t x_{z,t,b} \\ =& (r_t - \mu_t) x_{z,t,b} \\ =& \dot{x}_z. \end{aligned}$$

To prove condition 2, define $\lambda_{z,t,b} := \mu_t^{-\sigma} x_{z,t,b}^{-\sigma} > 0$. By construction, $c_{z,t,b}^{-\sigma} = \lambda_{z,t,b}$. Moreover:

$$\begin{aligned} \rho\lambda_{z,t,b} - \dot{\lambda}_{z,t,b} =& \rho\mu_t^{-\sigma} x_{z,t,b}^{-\sigma} + \mu_t^{-\sigma}\sigma x_z(s)^{-\sigma-1}\dot{x}_z - \sigma\mu_t^{-\sigma-1}\dot{\mu}_t x_{z,t,b}^{-\sigma} \\ =& \lambda_{z,t,b}\left(\rho + \sigma\frac{\dot{x}_{z,t,b}}{x_{z,t,b}} + \sigma\frac{\dot{\mu}_t}{\mu_t}\right). \end{aligned}$$

Using the equations for $\dot{x}_{z,t,b}$ (equation (A13)) and $\dot{\mu}_{z,t,b}$ (equation (S13)), we obtain:

$$\rho\lambda_{z,t,b} - \dot{\lambda}_{z,t,b} = \lambda_{z,t,b}\left(\rho + \sigma(r_t - \mu_t) + \sigma\left(\mu_t - r_t + \frac{1}{\sigma}(r_t - \rho)\right)\right) = r_t\lambda_{z,t,b}$$

To prove condition 3, note that

$$\max_c H(c, x_z, \lambda_z) = \frac{\lambda_z^{\frac{\sigma-1}{\sigma}}}{1-\sigma} + \lambda_z(r x_z - \lambda_z^{-\frac{1}{\sigma}}),$$

which is concave (linear) in x_z.

To prove the first part of condition 4, note that along the candidate path, x_z grows asymptotically at a rate $\frac{r^*-\rho}{\sigma}$, and $\lambda_{z,t,b}$ at a rate $\rho - r^*$ (μ_t converges along the candidate path). It follows that the first part of the transversality condition holds if

$$-\rho + \frac{r^* - \rho}{\sigma} + \rho - r^* < 0,$$

which is equivalent to the condition $r^* > (r^* - \rho)/\sigma$.

The second part of the transversality condition follows from the fact that, along any feasible path, we have $\hat{x}_{z,t,b} \geq 0$.

It follows that the candidate path provides optimal paths for consumption and asset accumulation outside of the steady state. ■

E.2 The Transition of Aggregates and Distributions

The following proposition characterizes the transition dynamics for the macroeconomic aggregates and the distribution of effective wealth. As for the steady state equilibrium, the transition dynamics are block recursive: we can first characterize the behavior of macroeconomic aggregates and then use them to trace the evolution of the wealth distribution.

Proposition S2 (Transitional dynamics) *The behavior of the macroeconomic aggregates, C and K is given by the unique stable solution to the system of differential equations*

$$\begin{aligned}\dot{C} =& \frac{1}{\sigma}(r-\rho)(C-pK)-\mu pK+p\dot{K}\\ \dot{K} =& Y-\delta K-C,\\ \frac{\dot{\mu}}{\mu} =& \mu - r+\frac{1}{\sigma}(r-\rho)\end{aligned}$$

where μ denotes the rate at which households consume their effective wealth (to simplify notation, we removed the time dependence of aggregates). Also, recall that Y is given by $Y(K)$ in equation (2) and r is given by $Y'(K)-\delta$

Along the transition path, households accumulate effective wealth at a rate $r_t-\mu_t$, which implies that the distribution of effective wealth for households with skill z, $f_z(x,t)$ evolves according to the Kolmogorov Forward Equation

$$\frac{\partial f_z(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[(r_t-\mu_t)xf_z(x,t)\right]-pf_z(x,t)+p\wp(x-h_{z,t}) \tag{S16}$$

where $\wp(.)$ is the Dirac delta function, and $h_{z,t}$ is a time-varying reinjection point.

Proof. We start by deriving the equation for $\dot{C}$. Aggregate consumption is given by

$$C_t = \sum_z \ell_z \int_{-\infty}^{t} c_{z,t,b}pe^{-p(t-b)}\ell_z db + pK_t.$$

Here, $pe^{-p(t-b)}$ is the mass of households with skill z at time t who received their last dissipation shock at time b. The term pK_t is the flow consumption of households in the impatience state, which was derived in Lemma S1.

Differentiating this equation, we obtain

$$\dot{C}_t = \sum_z \ell_z \int_{-\infty}^{t} \dot{c}_{z,t,b} p e^{-p(t-b)} db + p\dot{K}_t$$
$$+ p \sum_z \ell_z c_z(t,t) - p \sum_z \ell_z \int_{-\infty}^{t} c_{z,t,b} p e^{-p(t-b)} db.$$

These terms can be simplified as follows. The first term captures consumption growth over time. Using households' Euler equation, we can rewrite this term as

$$\begin{aligned}\sum_z \ell_z \int_{-\infty}^{t} \dot{c}_{z,t,b} p e^{-p(t-b)} db =& \sum_z \ell_z \int_{-\infty}^{t} \frac{r_t - \rho}{\sigma} c_{z,t,b} p e^{-p(t-b)} db \\ =& \frac{r_t - \rho}{\sigma}(C_t - pK_t).\end{aligned}$$

The third and fourth term capture the permanent decline in consumption due to the dissipation of wealth at time t. This is equal to $p\mu_t K_t$, which gives the product of the rate at which households consume their wealth and pK_t, which denotes the total wealth dispersed.

Plugging these simplified values in the expression for $\dot{C}$, we obtain

$$\dot{C}_t = \frac{r_t - \rho}{\sigma}(C_t - pK_t) + p\dot{K}_t - p\mu_t K_t.$$

The equation for $\dot{K}$ is the usual resource constraint, and the equation for $\dot{\mu}$ was derived above in Lemma S2.

Turning to the distribution of effective wealth, the Kolgomorov Forward Equation (S16) follows from the fact that households accumulate effective wealth at a rate $r_t - \mu_t$ (an implication of Lemma 1) but experience a dissipation shock with probability p. After these shocks, households' effective wealth jumps to $h_{z,t}$. ■

The proposition shows that the transition dynamics for aggregates are no more complicated than those in the usual representative household model. The main difference is that we need to keep track of the extra variable μ_t, which controls the common marginal propensity to consume out of effective wealth. Also, the Euler equation has some extra terms to account for the changes in consumption due to dissipation shocks.

Turning to the evolution of the wealth distribution, suppose the initial distribution of effective wealth conditional on skills is given by

$$\Pr(x_{z,t_0} > x) = \left(\frac{x}{w^*_{z,t_0}/r^*_{t_0}} \right)^{-\zeta}, \tag{S17}$$

as in Proposition 3. Here, x_{z,t_0} is a random variable denoting the effective wealth of households with skill z.

Following an increase in automation at time $t = t_0$, households with skills z and effective wealth x_{z,t_0} see a revaluation of their human wealth of Δ_z (this could be negative for households experiencing a real decline in wages over time). This implies that the distribution of x_z, denoted by $f_z(x,t)$, starts from

$$f_z(x, t_0) = \left(\frac{w^*_{z,t_0}}{r^*_{t_0}}\right)^{\zeta} \zeta(x - \Delta_z)^{-\zeta-1} \quad \text{for } x \geq \frac{w^*_{z,t_0}}{r^*_{t_0}} + \Delta_z.$$

and from there on evolves according to the Kolgomorov Forward Equation (S16).

E.3 Wealth Distribution in the Representative Household Benchmark

This subsection characterizes the dynamics of the wealth and income distribution in the representative-household benchmark. We will use a superscript h to denote the corresponding values of aggregates and to distinguish them from the aggregate paths for wages and interest rates in our model with dissipation shocks.

In the representative-household benchmark, the wealth distribution is indeterminate in the sense that any distribution is consistent with equilibrium in steady state. Despite the indeterminacy, starting from a given initial distribution of wealth and wages, the transition dynamics of the wealth distribution are uniquely defined.

To make things comparable, assume as in the main text that the initial distribution of effective wealth is given by (S17), and coincides with that in our model with dissipation shocks. Following an increase in automation at time $t = t_0$, households with skills z and effective wealth x_{z,t_0} see a revaluation of their human wealth of Δ^h_z (this will differ from the revaluation in our model since wages behave differently in the representative-household benchmark—see Proposition 2). People then accumulate assets starting from $x_{z,t_0} + \Delta^h_z$ at a common rate $r^h_t - \mu^h_t$, which is temporarily above zero but converges to zero over time (recall that in the representative-household benchmark, r^h_t and μ^h_t converge to ρ, reflecting the fact that the supply of capital is fully elastic). This temporary period of accumulation scales everyone's effective wealth by the same amount, M, but does not contribute to thicker tails in effective wealth. The resulting distribution of effective wealth is thus given by

$$\Pr(x_z > x) = \left(\frac{x/M - \Delta^h_z}{w^*_{z,t_0}/r^*_{t_0}}\right)^{-\zeta_0}, \qquad \text{for } x \geq M(\Delta^h_z + w^*_{z,t_0}/r^*_{t_0}).$$

This is a shifted Pareto distribution, with the shifts explained by the changes in wages. Unlike in our model, the new steady state distribution has the same tail parameter as the initial distribution. As usual, the distribution of income then inherits all of the tail properties of the effective wealth distribution.

F Additional Derivations and Proofs for the Extended Model

F.1 Supply and Demand in Extended Model

Note 1: We can describe the balanced-growth equilibrium in Proposition (4) using supply and demand diagrams for capital normalized by labor income, k, and bonds, b_I and b_H. We start with the special case where investors are risk neutral and $\theta \geq \bar{\theta}$ so that financial frictions do not bind. This implies that all assets yield the same return r and this is also the return obtained by investors, $r_W^* = r_K^* = r_B^* = r$. Adding equations (A22) and (A24), and using the fact that $b_I + b_H = 0$, we obtain an expression for the supply of capital (normalized by labor income):

$$k^s = \frac{1 - (\rho + (\sigma - 1)g)/(r - g)}{\sigma(p + g) + \rho - r}. \tag{S18}$$

On the other hand, equation (A25) gives the demand for capital:

$$k^d = \frac{\widetilde{\alpha}}{1 - \widetilde{\alpha}} \frac{1}{r + \delta}. \tag{S19}$$

The common return that equalizes supply and demand is given by the solution to (A27), which generalizes equation (4) in the main text. This return now lies in $(\rho + \sigma g, \rho + \sigma(p + g))$.

We now turn to the case where investors are risk averse and/or financial frictions bind. Combining equations (A22) and (A23), we obtain a curve that describes the capital supplied by investors for a given level of the return to their wealth r_W and the bond rate r_B:

$$k^s = \frac{p\sigma m(r_W - r_B)}{\sigma(p + g) + \rho - r_W} \frac{1}{r_B - g} \chi.$$

Using equation (A25), we can write the demand for capital as

$$k^d = \frac{\widetilde{\alpha}}{1 - \widetilde{\alpha}} \frac{1}{h(r_W - r_B) + r_B + \delta}$$

As shown in Figure S1, the supply curve gives an upward-sloping locus between k^s and r_W; while the demand for capital in (A25) gives a downward-sloping locus between k^d and r_W.

Turning to the bond market, equation (A23) implies that net borrowing by investors is

$$-b_I = \frac{p\sigma(m(r_w - r_B) - 1)}{\sigma(p + g) + \rho - r_W} \chi + \chi,$$

which implies that investors borrow more from households as r_B declines (reflecting the fact that investors would like to borrow at the lower rate r_B to invest at the higher rate r_W). On

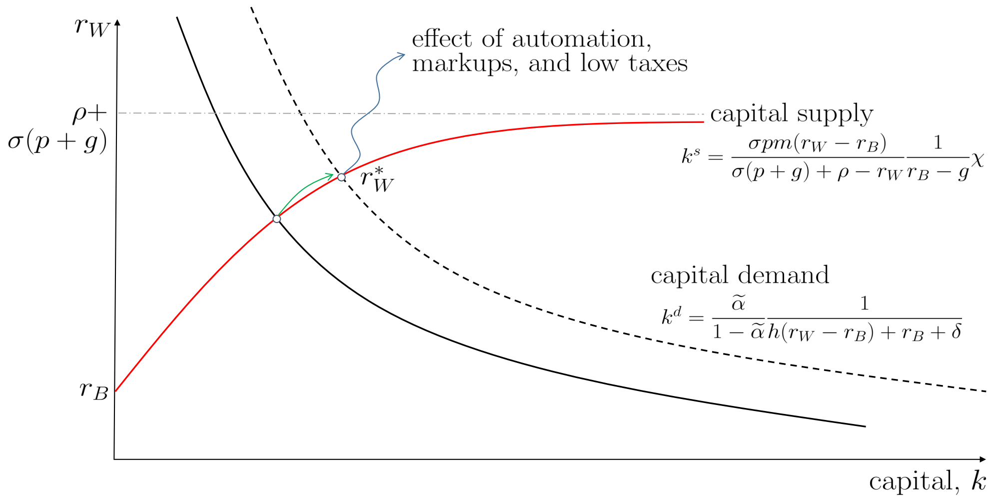

Figure S1: Depiction of the equilibrium in the capital market.

the other hand, equation (A24) implies that net savings by households are

$$b_H = \frac{r_B - \rho - \sigma g}{\sigma(p+g)+\rho-r_B}(1-\chi).$$

which increase in r_B, reflecting the fact that a higher safe rate leads to more savings by households. Figure S2 depicts the equilibrium in the bond market. In this panel, a movement to the right along the horizontal axis indicates higher households savings and more borrowing by investors. As the diagram shows, the curve describing investors' borrowing rotates clockwise as they become more risk averse, and the interception of these curves yields an equilibrium bond rate r_B^*.

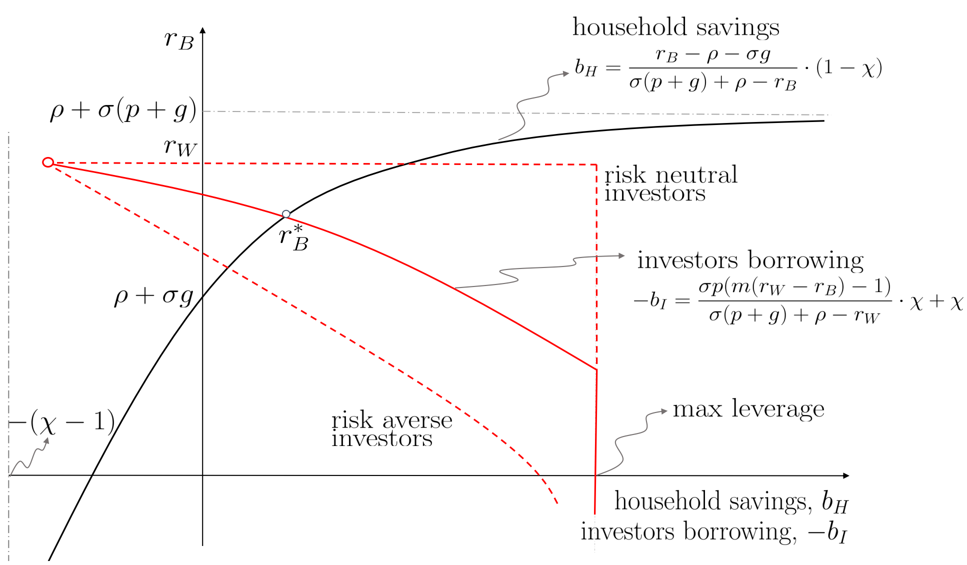

Figure S2: Depiction of the equilibrium in the bond market.

The proposition ensures the existence of r_W^*, r_B^* such that the capital market clears ($k^d =$

k^s) and the bond market clears ($b_I + b_H = 0$).[44] Moreover, as apparent from the figures (and as was the case in our baseline model), we always have that $r_W^* \in (\rho+\sigma g, \rho+\sigma(g+p))$, which shows that the equilibrium return to wealth exceeds its level in a representative household economy. Moreover, we always have $r_W^* \geq r_B^*$, with equality if and only if investors are risk averse and $\theta \geq \bar{\theta}$, so that the max-leverage line is sufficiently to the right.

Note 2: Alternatively, we can assume that bond markets are open, so that r_B is fixed in international markets at a level $\bar{r}_B$, with

$$g < \bar{r}_B < \rho + \sigma(p+g).$$

In this case, a balanced-growth equilibrium is characterized by constant values for r_W, k, b_I, b_H that solve equations (A22), (A23), (A24) and (A25), and where $b_I + b_H$ determines whether the economy lends to or borrows from the rest of the world.

In this case, we have that the equilibrium is given by

$$D_K(r_W^*, \bar{r}_B) = 0.$$

Lemma A3 implies that, for any $\bar{r}_B \in (g, \rho + \sigma(p+g))$, there is a unique solution r_W^* to this equation. This shows that the equilibrium exists and is unique. Moreover, as the figure shows, $r_W^* > \bar{r}_B$ and $r_W^* \in (\rho + \sigma g, \rho + \sigma(p+g))$.

In the open economy, the demand and supply of capital remain unchanged, with the only difference that now $r_B = \bar{r}_B$ is exogenous and determined in international markets. For values of $\bar{r}_B$ that are above the closed economy level of r_B^*, households will lend money to the rest of the world. Instead, for values of $\bar{r}_B$ that are below the closed economy level of r_B^*, both households and investors will borrow from foreigners at the low international rate.

F.2 Proof of Lemmas A3 and A4

Proof of Lemma A3. For $r_W \in (g, \rho+\sigma(p+g))$ and $r_B \in (g, r_W)$, the function $D_K(r_W, r_B)$ is continuous and decreasing in r_W and continuous and increasing in r_B. To prove the last claim, note that it is sufficient to show that

$$\frac{r_B - g}{h(r_W - r_B) + r_B + \delta}$$

[44] Proposition 4 claims the existence of an equilibrium in the closed economy, but does not show it is unique. Proving uniqueness here is challenging because the excess demand functions for capital and bonds do not satisfy the gross substitutes property. This reflects the fact that changes in r_B could lead investors to supply more capital as they accumulate higher levels of wealth; and changes in r_W could also increase the demand for bonds when investors are risk averse.

is increasing in r_B. We can rewrite this fraction as

$$\frac{1}{\dfrac{h(r_W - r_B)}{r_B - g} + 1 + \dfrac{g+\delta}{r_B - g}},$$

which is clearly increasing in r_B since the function h is non-negative and increasing and $r_B - g > 0$.

Moreover, for any $r_W \in (g, \rho + \sigma(p+g))$, there is a unique $r_B \in (g, r_W)$ that satisfies $D_K(r_W, r_B) = 0$. To see this, note that

$$D_K(r_W, g) < 0 \text{ and } D_K(r_W, r_W) > 0.$$

The intermediate value theorem then implies the existence of a unique point $r_B \in (g, r_W)$ that satisfies $D_K(r_W, r_B) = 0$.

Taking these observations together, the implicit function theorem implies that the equation $D_K(r_W, r_B) = 0$ defines a continuous and increasing locus for $r_W \in (g, \rho + \sigma(p+g))$ and $r_B \in (g, r_W)$, as claimed in the lemma.

To prove property 1, note that for $r_B = g$, we have

$$D_K(r_W, g) = -\frac{p\sigma m(r_W - g)}{\sigma(p+g) + \rho - r_W}\chi,$$

and $D_K(r_W, g) = 0$ then requires $r_W = g$.

To prove property 2, take limits as $r_W \uparrow \rho + \sigma(p+g)$. If $r_B \nrightarrow \rho + \sigma(p+g)$, we have

$$\lim_{r_w \uparrow \rho + \sigma(p+g)} D_K(r_K, r_W) = -\infty.$$

Thus, we must have $r_B \to \rho + \sigma(p+g)$ as claimed. ■

Proof of Lemma A4. For $r_B \leq \rho + \sigma(p+g)$, the function $D_B(r_W, r_B)$ is continuous and decreasing in r_B. On the other hand, for $r_W \in [r_B, \rho + \sigma(p+g))$, the function $D_B(r_W, r_B)$ is continuous and u-shaped: it first decreases in r_W and could become increasing afterwards.

Moreover, for any $r_W \in (\rho + \sigma g, \rho + \sigma(p+g))$, there is a unique $r_B < r_W$ that satisfies $D_B(r_W, r_B) = 0$. To see this, note that

$$\lim_{r_B \to -\infty} D_B(r_W, r_B) = \frac{p\sigma\theta/(1-\theta)}{\sigma(p+g) + \rho - r_W}\chi + \chi > 0,$$

and

$$\lim_{r_B \to \infty} D_B(r_W, r_W) = \frac{\sigma g + \rho - r_W}{\sigma(p+g) + \rho - r_W} < 0.$$

The intermediate value theorem then implies the existence of a unique point $r_B < r_W$ that satisfies $D_B(r_W, r_B) = 0$.

Taking these observations together, the implicit function theorem implies that the equation $D_B(r_W, r_B) = 0$ defines a continuous and initially decreasing locus for $r_W \in (\rho+\sigma g, \rho+\sigma(p+g))$ and $r_B < r_W$, as claimed in the lemma.

We now turn to properties 1 and 2. Property 1 follows from the fact that

$$D_B(\rho + \sigma g, \rho + \sigma g) = 0.$$

For property 2, take limits as $r_W \uparrow \rho + \sigma(p+g)$. If $m(r_w - r_B) \nrightarrow 1$, we have that

$$D_B(\rho + \sigma g, r_B) = \pm\infty.$$

Thus, as $r_W \uparrow \rho + \sigma(p+g)$, we must have that $m(r_w - r_B) \to 1$. In particular, this requires that $r_B \to \tilde{r}_B$. To see this, note that

$$m(r_w - r_B) \to 1 \Leftrightarrow r_K - r_B = \gamma\nu^2.$$

The definition of r_W implies that

$$r_W - \tilde{r}_B = \frac{1}{2}(\sigma + 1)\gamma\nu^2.$$

Substituting $r_W = \rho + \sigma(p+g)$ and solving for $\tilde{r}_B$, yields the formula for $\tilde{r}_B$ in the lemma.

We conclude with property 3. First, define

$$\breve{r}_B = \rho + \sigma g + p\sigma\chi.$$

we now show that along the locus $D_B(r_W, r_K)$, $m(r_w - r_B) > 1$ for $r_B > \breve{r}_B$ and $m(r_w - r_B) < 1$ for $r_B < \breve{r}_B$, with equality if and only if $r_B = \breve{r}_B$. To show this, note that

$$D_B(r_W, \breve{r}_B) = \frac{p\sigma(m(r_W - r_B) - 1)}{\sigma(p+g) + \rho - r_W}\chi.$$

Because the function $D_B(r_W, r_B)$ is decreasing in r_B, we have that for all points $r_B > \breve{r}_B$ along the locus $D_B(r_W, r_B) = 0$,

$$0 = D_B(r_W, r_B) < \frac{p\sigma(m(r_W - r_B) - 1)}{\sigma(p+g) + \rho - r_W}\chi,$$

which implies $m(r_W - r_B) > 1$. Likewise, for all points $r_B < \breve{r}_B$ along the locus $D_B(r_W, r_B) = 0$,

$$0 = D_B(r_W, r_B) > \frac{p\sigma(m(r_W - r_B) - 1)}{\sigma(p+g) + \rho - r_W}\chi,$$

which implies $m(r_W - r_B) < 1$. Clearly, equality holds if and only if $r_B = \breve{r}_B$.

Moreover, suppose that the locus $D_B(r_W, r_B) = 0$ contains $\breve{r}_B$. For $r_B > \breve{r}_B$, we have that $m(r_W - r_B) > 1$, which implies that $D_B(r_W, r_B)$ is increasing in r_W and the locus $D_B(r_W, r_B) = 0$ is upward-sloping from $\breve{r}_B$ onwards.

We now consider the two cases identified in the lemma. Suppose that $\gamma\nu^2 > \bar{\gamma}$. This necessarily implies that $\breve{r}_B > \tilde{r}_B$. Because the locus $\Delta_B(r_W, r_B) = 0$ is u-shaped and both extremes are below $\breve{r}_B$, the entire locus lies below $\breve{r}_B$ and for all points along this locus we have $m(r_W - r_B) < 1$.

To show that $r_W - r_B$ increases along this locus as we raise r_W, we proceed by contradiction. Suppose that this is not the case, then for two points (r'_W, r'_B) and (r''_W, r''_B) with $r''_W > r'_W$ along this locus, we have $m(r'_W - r'_B) = m(r''_W - r''_B) = m_0 \in (0, 1)$, which also requires $r'_W - r'_B = r''_W - r''_B = d_0$. It follows that both of these points satisfy the equation $D_B(r_W, r_B) = 0$, which can be written as

$$\frac{p\sigma(m_0 - 1)}{\sigma(p+g) + \rho - r_W}\chi + \chi - \frac{r_W - d_0 - \rho - \sigma g}{\sigma(p+g) + \rho - r_W + d_0}(1 - \chi) = 0.$$

This equation is strictly decreasing in r_W for $r_W \in (\rho+\sigma g, \rho+\sigma(p+g))$, and so it cannot hold for two different values of r_W. This contradiction establishes the claim that, for $\gamma\nu^2 > \bar{\gamma}$, $r_W - r_B$ rises along the locus for $D_B(r_W, r_B)$.

Finally, suppose that $\gamma\nu^2 < \bar{\gamma}$, which implies that $\tilde{r}_B > \breve{r}_B > \rho + \sigma g$. In this case $\tilde{r}_B$ lies in the locus $D_B(r_W, r_B)$. The exact same argument presented above implies that, for $r_W \in (\rho+\sigma g, \breve{r}_W)$, the gap $r_W - r_B$ widens as we increase r_W along the locus $\Delta_B(r_W, r_B) = 0$.

We now show that the gap $r_W - r_B$ is single peaked for $r_W \in (\breve{r}_W, \rho + \sigma(p+g))$. As shown above, for $r_W \in (\breve{r}_W, \rho + \sigma(p+g))$, we have that $m(r_W - r_B) > 1$ (and hence $r_W - r_B > \frac{1}{2}(\sigma+1)\gamma\nu^2$) along the locus $\Delta_B(r_W, r_B) = 0$. Because $m(r_W - r_B) > 1$, we must have that $r_B > \rho + \sigma g$ for $r_W \in (\breve{r}_W, \rho + \sigma(p+g))$; otherwise, $D_B(r_W, r_B) > 0$.

Recall that as $r_W \uparrow \rho+\sigma(p+g)$, we have that $r_B \to \tilde{r}_B$ and $r_W - r_B \to \frac{1}{2}(\sigma+1)\gamma\nu^2$ along the locus $\Delta_B(r_W, r_B) = 0$. Thus, the gap $r_W - r_B$ reaches at least one peak at some point $r_W \in (\breve{r}_W, \rho+\sigma(p+g))$. Suppose by way of contradiction, that the gap $r_W - r_B$ is not single peaked in this interval, as claimed in the lemma. Then it must have at least two peaks. This implies the existence of three points (r'_W, r'_B), (r''_W, r''_B) and (r'''_W, r'''_B) along this locus and with $m(r'_W - r'_B) = m(r''_W - r''_B) = m(r'''_W - r'''_B) = m_0 > 1$, and $r'_W - r'_B = r''_W - r''_B = r'''_W - r'''_B = d_0$. However, in each of this points, we must have $D_B(r_W, r_B) = 0$, which requires

$$\frac{p\sigma(m_0 - 1)}{\sigma(p+g) + \rho - r_W}\chi + \chi - \frac{r_W - d_0 - \rho - \sigma g}{\sigma(p+g) + \rho - r_W + d_0}(1 - \chi) = 0.$$

This is a quadratic equation in r_W and so it has at most two solutions, contradicting our initial assertion, and showing that $r_K - r_B$ is single peaked for $r_W \in (\breve{r}_W, \rho + \sigma(p+g))$. ■

F.3 Wealth and Income Distribution in the Extended Model

Proposition 4 and S3 characterized the distribution of effective wealth and the *tails* of the income distribution. A complete analytic characterization of the income distribution is no longer feasible in the extended model. This appendix shows how to partially characterize and numerically compute the income distributions, e.g. so as to construct Figure 10.

The flow of income over a small time interval of length dt received by an investor with wealth $x_{z,t}$ is given by

$$dy_{z,t} = r_I x_{z,t} dt + \sigma_y x_{z,t} dW_t, \tag{S20}$$

where $r_I = \kappa r_K + (1-\kappa) r_B$ and $\sigma_y = \kappa \nu$. Likewise, the flow of income received by a household with normalized wealth $x_{z,t}$ is given by

$$\dot{y}_{z,t} = r_B x_{z,t}. \tag{S21}$$

The following lemma provides a useful approximation for the income received by households over one unit of time, which we will later use to characterize an approximate income distribution.

Lemma S3 *Let $y_{z,\tau}$ denote the total income received by an investor between $t = 0$ and $t = \tau$, i.e. the integral of (S20) between $t = 0$ and $t = \tau$. We have that*

$$y_{z,\tau} \approx x_{z,0} \left(r_I \tau + \sigma_y W_\tau \right). \tag{S22}$$

More precisely,

$$|y_{z,\tau} - x_{z,0} \left(r_I \tau + \sigma W_\tau \right)| \leq \delta_\tau^2 \left(\frac{1}{2} + \sqrt{\frac{1}{3}} |\upsilon_2| + |\upsilon_1| + \frac{1}{2} |\upsilon_1^2 - 1| \right) + \mathcal{O}(\delta_t^3),$$

where $\delta_\tau = \max\{|r_I \tau|, \left| \left(\frac{r_W - \rho}{\sigma} - \frac{\sigma_y^2}{2} \right) \tau \right|, \sigma_y \sqrt{\tau}\}$ and $\upsilon_1, \upsilon_2 \sim N(0,1)$.

Likewise, let $y_{z,\tau}$ denote the total income received by a household between $t = 0$ and $t = \tau$, i.e. the integral of (S21) between $t = 0$ and $t = \tau$. We have that

$$y_{z,\tau} \approx x_{z,0} r_B \tau. \tag{S23}$$

The interpretation of the Lemma is that we can obtain a good approximation to the income generated by an asset between $t = 0$ and $t = \tau$ simply from the asset return (e.g. $r_I \tau + \sigma_y W_\tau$ in the case of investors) and the initial asset balance $x_{z,0}$ while one can ignore changes in the asset balance $x_{z,t}$ between $t = 0$ and $t = \tau$. The following example helps clarify: an asset pays a return of $r = 5\%$ that accrues continuously over the year. With an initial asset balance of \$100 the total capital income generated throughout the year will not just be $5\% \times \$100 = \$5$ but instead it will be larger than \$5. This is because the asset balance which is the basis of the return itself increases throughout the year. However, the Lemma shows that $5\% \times \$100 = \$5$ provides a good approximation to overall capital income, i.e. the effect we just described has only a small effect. In particular, the following upper bound on the approximation error is easy to see: the asset balance could increase from \$100 to \$105. But then capital income would still be only $5\% \times \$105 = 5.25\%$ i.e. only $0.25\% = 5\% \times 5\%$ larger than the approximate capital income of $5\% \times \$100 = \$5$.

Proof. We provide the proof for the case of investors. The approximation used for households then follows as a corollary. Throughout the proof we will ignore the subscript z so as to simplify notation.

Ito's lemma applied to the log of investors' wealth implies that

$$d \ln x_t = \left(\mu_I - \frac{\sigma_y^2}{2}\right) dt + \sigma_y dW_t.$$

Integrating this expression in $[0, t]$ implies

$$\ln x_t - \ln x_0 = \left(\mu_I - \frac{\sigma_y^2}{2}\right) t + \sigma_y W_t,$$

and so investor's wealth satisfies

$$x_t = x_0 \exp\left(\left(\mu_I - \frac{\sigma_y^2}{2}\right) t + \sigma_y W_t\right). \tag{S24}$$

In the rest of the proof, we will use the fact that $W_t \sim N(0, \sqrt{t})$ and that $\int_0^t W_s ds$ is an "integrated Brownian motion" which satisfies $\int_0^t W_s ds \sim N(0, t^3/3)$.[45] This implies that for a given t, we can write $\int_0^t W_s ds = \upsilon_2 \sqrt{t^3/3}$ and $W_t = \upsilon_1 \sqrt{t}$, where $\upsilon_1, \upsilon_2 \sim N(0, 1)$. Moreover, we will use the notation $\widetilde{\mu}_I = \left(\mu_I - \frac{\sigma_y^2}{2}\right)$.

Using a Taylor expansion, we can approximate equation (S24) as

$$x_t = x_0 \left(1 + \widetilde{\mu}_I t + \sigma_y W_t\right) + x_0 \varepsilon_t. \tag{S25}$$

where the approximation error is given by

$$\varepsilon_t = \sum_{n=2}^{\infty} \frac{1}{n!} (\widetilde{\mu}_I t + \sigma_y \upsilon_1 \sqrt{t})^n \sim \mathcal{O}(\delta_t^2)$$

Using equation (S25), it follows that we can write income flows as

$$dy_t = x_0 \left(1 + \widetilde{\mu}_I t + \sigma_y W_t\right) \left(r_I dt + \sigma_y dW_t\right) + x_0 \varepsilon_t \left(r_I dt + \sigma_y dW_t\right).$$

[45] See e.g. https://en.wikipedia.org/wiki/Wiener_process#Integrated_Brownian_motion.

Integrating this equation in $[0,t]$ implies

$$y_t = x_0 \left(r_I + \sigma_y W_t\right) + x_0 \int_0^t \left(\widetilde{\mu}_I s + \sigma_y W_s\right) \left(r_I ds + \sigma_y dW_s\right) ds + x_0 \int_0^t \varepsilon_s \left(r_I ds + \sigma_y dW_s\right) ds.$$

Using the triangle inequality, we obtain

$$\left| y_t - x_0 \left(r_I + \sigma_y W_t\right) \right| \leq \underbrace{\left| x_0 \int_0^t \left(\widetilde{\mu}_I s + \sigma_y W_s\right) \left(r_I ds + \sigma_y dW_s\right) ds \right|}_{M} + \underbrace{\left| x_0 \int_0^t \varepsilon_s \left(r_I ds + \sigma_y dW_s\right) ds \right|}_{N}.$$

We now proceed to bound the two term on the right of the above equation, labeled M and N. For the first term, M, we have

$$M = x_0 \left(r_I \widetilde{\mu}_I \int_0^t s ds + r_I \sigma_y \int_0^t W_s ds + \widetilde{\mu}_I \sigma_y \int_0^t s dW_s + \sigma_y^2 \int_0^t W_s dW_s \right).$$

We can integrate all these terms by noting that:

- For the first one,
$$r_I \widetilde{\mu}_I \int_0^t s ds = r_I \widetilde{\mu}_I \frac{t^2}{2}.$$
- For the second one, we use the property that $\int_0^t W_s ds \sim N(0, t^3/3)$. This implies that
$$r_I \sigma_y \int_0^t W_s ds = r_I \sigma_y \sqrt{\frac{t^3}{2}} \upsilon_2,$$
where υ_2 is a random variable drawn from a standard normal distribution.
- For the third one,
$$\widetilde{\mu}_I \sigma_y \int_0^t s W_s ds = \widetilde{\mu}_I \sigma_y \sigma_y t W_t - \widetilde{\mu}_I \sigma_y \int_0^t W_s ds.$$
This follows from an application of Ito's lemma to the function tW_t, which gives
$$d(tW_t) = W_t dt + dW_t \Rightarrow \int_0^t s dW_s = tW_t - \int_0^t W_s ds.$$
Using the property that $\int_0^t W_s ds \sim N(0, t^3/3)$ and $W_t \sim N(0, \sqrt{t})$, we can further simplify this as
$$\widetilde{\mu}_I \sigma_y \int_0^t W_s ds = \widetilde{\mu}_I \sigma_y \upsilon_1 \sqrt{t^3} - \widetilde{\mu}_I \sigma_y \sqrt{\frac{t^3}{3}} \upsilon_2.$$
- For the last one,
$$\sigma_y^2 \int_0^t W_s dW_s = \frac{\sigma_y^2}{2} (W_t^2 - t).$$

This follows from another application of Ito's lemma to the function $\frac{W_t^2}{2} - \frac{t}{2}$, which gives

$$d\left(\frac{W_t^2}{2} - \frac{t}{2}\right) = \left(-\frac{1}{2} + \frac{1}{2}\right) dt + W_t dW_t \Rightarrow \int_0^t W_s dW_s = \frac{\sigma^2}{2}(W_t^2 - t).$$

Using the fact that $W_t \sim N(0, \sqrt{t})$, we can write this as

$$\sigma_y^2 \int_0^t W_s dW_s = \frac{\sigma_y^2}{2}(\upsilon_1^2 - 1)t.$$

Putting these formulas together, and using the triangle inequality, it follows that

$$\begin{aligned} M \leq & x_0 \left(|r_I \widetilde{\mu}_I| \frac{t^2}{2} + |\left(r_I - \widetilde{\mu}_I\right)| \sigma_y \sqrt{\frac{t^3}{3}} |\upsilon_2| + |\widetilde{\mu}_I| \sigma_y |\upsilon_1| \sqrt{t^3} + \frac{\sigma^2}{2} |\upsilon_1^2 - 1| t \right) \\ \leq & x_0 \delta_t^2 \left(\frac{1}{2} + \sqrt{\frac{1}{3}} |\upsilon_2| + |\upsilon_1| + \frac{1}{2} |\upsilon_1^2 - 1| \right). \end{aligned}$$

Turning to the second term, N, and applying the triangle inequality, we have

$$N \leq x_0 \left(|r_I| \left| \int_0^t \varepsilon_s ds \right| + \sigma_y \left| \int_0^t \varepsilon_s dW_s \right| \right).$$

We can bound the two terms on the right as follows:

$$|r_I| \left| \int_0^t \varepsilon_s ds \right| \leq |r_I| \int_0^t |\varepsilon_s| ds < |r_I| t |\varepsilon_t| \sim \mathcal{O}(\delta_t^3)$$

and

$$\sigma_y \left| \int_0^t \varepsilon_s dW_s \right| \leq \sigma_y \sqrt{t} |\varepsilon_t| \left(\frac{1}{\sqrt{t}} \int_0^t |dW_s| \right) \sim \mathcal{O}(\delta_t^3),$$

where $\frac{1}{\sqrt{t}} \int_0^t |dW_s|$ is a random variable.

The lemma follows from combining our bounds for M and N. ■

We now use Proposition A1 and S3 to characterize an approximate income distribution. This proposition provides all of the formulas used in the construction of Figure 10.

Proposition S3 *The CDF of income received over one unit of time for households with wage w_z can be approximated as*

$$G_{z,H}(y) := \begin{cases} 1 - \left(\frac{y}{w_z}\right)^{-\zeta_H} & \text{for } y \geq w_z \\ 0 & \text{for } y < w_z \end{cases}, \tag{S26}$$

and its PDF is

$$g_{z,H}(y) := \begin{cases} \frac{\zeta_H}{w_z}\left(\frac{y}{w_z}\right)^{-\zeta_H - 1} & \textit{for } y \geq w_z \\ 0 & \textit{for } y < w_z \end{cases}, \tag{S27}$$

The CDF of income received over one unit for investors with wage w_z can be approximated as

$$G_{z,I}(y) := 1 - \int_0^{\infty} f_H(x)\left(1 - \Phi\left(\frac{1}{\sigma_y}\left(\frac{y/x}{w_z/r_B} - r_I\right)\right)\right) dx, \tag{S28}$$

and its PDF is

$$g_{z,I}(y) := \int_0^{\infty} f_H(x)\phi\left(\frac{1}{\sigma_y}\left(\frac{y/x}{w_z/r_B} - r_I\right)\right)\frac{1}{\sigma_y}\frac{1/x}{w_z/r_B}dx, \tag{S29}$$

Moreover, the unconditional CDF of income satisfies

$$G(y) = \chi\sum_z \ell_z G_{z,I}(y) + (1-\chi)\sum_z \ell_z G_{z,H}(y), \tag{S30}$$

and its PDF is

$$g(y) = \chi\sum_z \ell_z g_{z,I}(y) + (1-\chi)\sum_z \ell_z g_{z,H}(y), \tag{S31}$$

Finally, the labor income of people with an income y is given by

$$w(y) = \frac{\chi\sum_z w_z\ell_z g_{z,I}(y) + (1-\chi)\sum_z w_z\ell_z g_{z,H}(y)}{g(y)}. \tag{S32}$$

Proof. Lemma S3 implies that we can approximate the income received by households over one unit of time as $y_{z,1} \approx x_{z,0}r_B$. Thus, we can compute the probability that a household with wage w_z has an income above y as

$$\begin{aligned}
\Pr(\text{income} \geq y|\text{household}, w_z) &= \Pr(x_{z,0}r_B \geq y|\text{household}, w_z) \\
&= \Pr(x_{z,0} \geq \frac{y}{r_B}|\text{household}, w_z) \\
&= \Pr(\widetilde{x}_{z,0} \geq \frac{y}{w_z}|\text{household}, w_z) \\
&= \left(\frac{y}{w_z}\right)^{-\zeta_H} \quad \text{for } y \geq w_z.
\end{aligned}$$

This implies that $G_{z,H}(y)$ is given by the expression in equation (S26), and its associated PDF is given by (S27).

Likewise, Lemma S3 implies that we can approximate the income received by investors over one unit of time as

$$y_{z,1} \approx x_{z,0} \left(r_I + \sigma_y W_1\right).$$

Thus, we can compute the probability that an investor with wage w_z has an income above y as

$$\begin{aligned}
\Pr(\text{income} \geq y|\text{investor}, w_z) &= \Pr\left(x_{z,0}\left(r_I + \sigma_y W_1\right) \geq y \middle| \text{investor}, w_z\right) \\
&= \Pr\left(\widetilde{x}_{z,0}\left(r_I + \sigma_y W_1\right) \geq \frac{y}{w_z/r_B} \middle| \text{investor}, w_z\right) \\
&= \int_0^\infty f_H(x) \Pr\left(W_1 \geq \frac{1}{\sigma_y}\left(\frac{y/x}{w_z/r_B} - r_I\right) \middle| \widetilde{x}_{z,0} = x, \text{investor}, w_z\right) dx \\
&= \int_0^\infty f_H(x) \left(1 - \Phi\left(\frac{1}{\sigma_y}\left(\frac{y/x}{w_z/r_B} - r_I\right)\right)\right) dx.
\end{aligned}$$

This implies that $G_{z,I}(y)$ is given by the expression in equation (S28), and its associated PDF is given by (S29).

Finally, the formulas in (S30) and (S31) follow from adding up the conditional distribution of income over skills and investors and households. The formula for the wage income for people with total income y follows from an application of Bayes' rule. ■

Construction of Figure 10. To construct Figure 10, we proceed as follows:

- First, we use equation (S30) to compute income quantiles $y_t(q)$ as

$$G_t(y_t(q)) = q$$

  both for the initial steady state (denoted by $t = 0$) and the final steady state (denoted by $t = T$).

- Second, we compute the labor income and capital income earned in each quantile, $y_{\ell,t}(q)$ and $y_{k,t}(q)$, respectively, as

$$y_{\ell,t}(q) = w_t(y_t(q)) \qquad\qquad y_{k,t}(q) = y_t(q) - w_t(y_t(q))$$

  where $w_t(y_t(q))$ was defined in (S32).

- Third, we compute the change in income by quantile as

$$\frac{y_T(q) - y_0(q)}{y_0(q)},$$

which can be decomposed into a part driven by labor income

$$\frac{y_{\ell,T}(q) - y_{\ell,0}(q)}{y_0(q)},$$

and a part driven by capital income

$$\frac{y_{k,T}(q) - y_{k,0}(q)}{y_0(q)}.$$

Figure 10 plots the total change in income by quantile and the contribution of labor and capital income.

- Finally, the figure also plots the change in income by quantile in the representative-household benchmark. Appendix E explains how we computed the evolution of the income distribution in this case.

G Calibration Details

This appendix presents details of the calibration exercise.

G.1 Wage data

We compute hourly wages for 1990 using the 1990 Census and for 2014 using the 2012–2016 American Community Survey. We keep the sample of salaried workers between 25 and 54 years of age living in continental US. Following common practice in the literature, we replace top coded wage income by 1.5 times the top code. We compute hourly wages by dividing wage income by hours per week times weeks worked. We then converted hourly wages to 2007 dollars using the personal consumption expenditure index, from the BEA. Finally, we winsorized hourly wages between 2 and 180 dollars.

We then computed average wages for workers in each of the 100 wage percentiles. When computing these averages, we weight observations by the Census or ACS weight times total hours of labor supplied, so that we obtain the average hourly wage for workers in each percentile.

Due to changes in the amount of wage data top coded, the data for the top 1 percentile exhibits visible discontinuities over time. We address this issue by fitting a log linear model for log wages as a function of the rank, using the fitted regression to impute the mean wage for the top 1 wage earners. We estimate this model for the percentiles above the 90th percent, exploiting the fact that the top tail of wages has an approximate Pareto shape.

Finally, in Figure 9 (and only for this figure) we smooth the observed change in wages by taking a moving average over consecutive bins over 10 percentiles.

G.2 Calibration of $\alpha_{z,t}$

As explained in the main text, we make three assumptions:

A1 the patterns of relative specialization ω_z^R vary little over time and can be approximated by their value in the year 2000;

A2 in 1980, a common share α_0 of routine and non-routine tasks were automated;

A3 over time, the share of routine tasks that is automated is common across skill groups and given by α_t^R, with $\alpha_{1980}^R = \alpha_0$.

Equation (18) can be derived as follows. Let $\bar{\omega}_{z,t}^R$ denote the share of wage income earned by workers with skill z in routine jobs at time t.

Let $w_{z,t}^R$ denote the wage income earned by workers with skill z in routine jobs and let $y_{z,t}^R$ denote the value added generated in those jobs. Let $w_{z,t}^N$ denote the wage income earned by workers with skill z in non-routine jobs and let $y_{z,t}^N$ denote the value added of those jobs.

Assumptions A2 and A3 imply that we can write

$$\frac{1}{1-\alpha_{z,t}} = \frac{y_{z,t}^R + y_{z,t}^N}{w_{z,t}^R + w_{z,t}^N} = \bar{\omega}_{z,t}^R \frac{1}{1-\alpha_t^R} + (1-\bar{\omega}_{z,t}^R)\frac{1}{1-\alpha_0}.$$

This equation can be rewritten as

$$\frac{1}{1-\alpha_{z,t}} = \bar{\omega}_{z,t}^R \left(\frac{1}{1-\alpha_t^R} - \frac{1}{1-\alpha_0}\right) + \frac{1}{1-\alpha_0}. \tag{S33}$$

Let $\bar{\omega}_{R,t}$ denote the share of wage income derived from routine jobs across all workers. We also have

$$\frac{1}{1-\alpha_t} = \frac{\sum_z y_{z,t}^R + \sum_z y_{z,t}^N}{\sum_z w_{z,t}^R + \sum_z w_{z,t}^N} = \bar{\omega}_t^R \frac{1}{1-\alpha_t^R} + (1-\bar{\omega}_t^R)\frac{1}{1-\alpha_0}.$$

This equation can be rewritten as

$$\frac{1}{1-\alpha_t} - \frac{1}{1-\alpha_0} = \bar{\omega}_t^R \left(\frac{1}{1-\alpha_t^R} - \frac{1}{1-\alpha_0}\right). \tag{S34}$$

Combining equations (S33) and (S34), we obtain

$$\frac{1}{1-\alpha_{z,t}} = \frac{\bar{\omega}_{z,t}^R}{\bar{\omega}_t^R} \left(\frac{1}{1-\alpha_t} - \frac{1}{1-\alpha_0}\right) + \frac{1}{1-\alpha_0}.$$

Equation (18) in the main text follows from the fact that $\omega_z^R = \frac{\bar{\omega}_{z,t}^R}{\bar{\omega}_t^R}$, and we assumed that relative specialization does not vary over time.

To operationalize the measurement of $\alpha_{z,t}$, we compute ω_z^R using Census data for 2000.[46] We keep the sample of salaried workers between 25 and 54 years of age living in continental US, and we clean wage data in the exact same way as above. Following the literature, we code an occupation as routine if it is in the top tercile of jobs with the highest routine content according to $O * NET$. We define the routine content of an occupation as total routine inputs minus the average of routine inputs, cognitive inputs, and manual inputs involved in this job. The construction of these inputs is explained in Acemoglu and Autor (2011) and available for download from their websites. We also experimented and obtained similar findings using the classification of occupations as routine and non-routine used in Autor and Dorn (2013). This classification is based on the Dictionary of Occupational Titles, which preceded $O * NET$.

G.3 Empirical estimates of the capital-supply elasticity

As mentioned in the main text, an elasticity of capital to rental rates of 50 is large relative to the available empirical estimates. The following table summarizes our review of the empirical literature estimating this elasticity. For each paper, we report the implied elasticity $d \ln K/dr$, which can be directly compared to our calibration target of 50. As is apparent from our survey, all these estimates put the elasticity below 35, which is much more inelastic than what our model predicts.

Table S1: Summary of available estimates of the capital-supply elasticity.

Study	What Elasticity?	Formula	Estimate	Implied $d \ln K/dr$	Duration	Methodology
Wealth Taxation:						
Zoutman (2015)	Stock of housing and financial assets w.r.t tax on their sum	$\frac{d \log(K)}{d \log(\tau)}$	-0.045	**9**	5 Years	Quasi experiment - 2001 Dutch capital tax reform. Response of households is tracked using panel data.
Jakobsen et al. (2018)	Taxable wealth w.r.t. net-of-tax rate for the very rich / moderately rich	$\frac{d \log(K)}{d\tau}$	[-8,-25]	**[8,25]**	8 years	Quasi experiment - 1989 wealth tax reform in Denmark. Diff-in-diff using variation in tax ceilings and the tax exemption level.
Brülhart et al. (2017)	Semi-elasticity - taxable wealth (excluding pensions) w.r.t. wealth tax rate	$\frac{d \log(K)}{d\tau}$	[-23,-35]	**[23,35]**	-	Cross-regional and time variation in the Swiss wealth-tax system.
Income Taxation:						
Kleven and Schultz (2014)	Positive taxable capital income w.r.t. net-of-tax rate	$\frac{d \log(rK)}{d \log(1-\tau)}$	[0.1,0.3]	**[1.25,7.5]**	3-7 years	Danish tax reforms and full-population administrative data since 1980.

[46] We measure ω_z^R using the 2000 Census—a point in the middle of the period we study. In our model, the composition and specialization patterns of a skill group are assumed invariant. However, in the data, the composition of workers in a given wage percentile might change over time, as the relative ranking of groups of workers with different characteristics changes. In our baseline calibration, we used the 2000 values for ω_z^R as describing the level of specialization of different groups in routine jobs. We also experimented with measuring ω_z^R using the 1980 Census and obtained similar results. The reason is that ω_z^R is highly correlated over time (the correlation between the 1980 and 2000 measures is of 0.9714).

G.4 Alternative calibrations

Table S2 summarizes the parameters used in alternative calibrations (top panel), and shows that our results, such as changes in returns and in top inequality, are robust to different assumptions about parameter values (bottom panel). Column 1 shows results for a calibration that matches an exposure to risky capital of $\kappa = 1$ in 1980. Column 2 shows our baseline calibration, which matches a value of $\kappa = 1.3$ in 1980. Column 3 shows results for a calibration that matches an exposure to risky capital of $\kappa = 1.5$ in 1980. Finally, Column 4 returns to our baseline value for κ but now assumes that investors have low levels of risk aversion and are instead borrowing constrained. In particular, we assume that investors can pledge up to a third of their wealth when borrowing ($\theta = 1/3$), which implies κ binds at 1.5 All calibrations deliver a similar increase in the top-tail inequality index $1/\zeta$ from 0.54 to 0.65–0.66.

Table S2: Alternative calibrations of the extended model

	(1) Target $\kappa = 1$	(2) Target $\kappa = 1.3$	(3) Target $\kappa = 1.5$	(4) Constrained investors
Risk aversion γ	2	2	2	0.1
Capital risk ν	0.09	0.077	0.067	0.06
Share investors χ	0.063	0.066	0.061	0.045
initial steady state → final steady state				
κ	$1 \to 1.1$	$1.3 \to 1.5$	$1.5 \to 1.7$	1.5
r_B	$4.1\% \to 4.2\%$	$4.9\% \to 5.2\%$	$5.2\% \to 5.5\%$	$4.2\% \to 4.4\%$
r_K	$6.5\% \to 7.0\%$	$6.5\% \to 7.0\%$	$6.5\% \to 7.0\%$	$6.5\% \to 7.4\%$
r_W	$7.7\% \to 8.8\%$	$8.0\% \to 9.2\%$	$8.2\% \to 9.4\%$	$7.7\% \to 9.0\%$
$1/\zeta$	$0.54 \to 0.65$	$0.54 \to 0.65$	$0.54 \to 0.66$	$0.54 \to 0.66$

H Measuring Returns to Wealth

H.1 Measuring r_K using the IMAs and NIPA

This section provides the details behind our measurement of the corporate and noncorporate returns to business capital held by US households. As explained in the text, we compute these series using data from the Integrated Macroeconomic Accounts (IMAs, for short) and the National Income and Product Accounts (NIPA).

We calculate returns to noncorporate and corporate capital held by US households as $r_K := Y_k/K + p_K$ where Y_k is the capital income received, K is the value of the capital stock held by households and p_K is a revaluation component.

Household income received from corporate equities is directly reported in the NIPA as dividends income (Table 2.1). The value of the associated equity as well as revaluations are reported in the IMA. Note that IMA balance sheets are end-of-period. We adjust for this by using reported valuation from previous period and adjusting for PCE inflation.

In the case of noncorporate equities, we first adjust the proprietors' income reported in the NIPA tables to account for disguised labor income. In particular, we apply the corporate sector capital share calculated from NIPA Table 1.12 to overall proprietor's income to calculate the capital income to noncorporate equity. This is broadly in line with the approach taken in GRR and PSZ. Capital stock and revaluations are calculated from the IMAs as for the corporate sector.

H.2 Behavior of the income and revaluation component

As explained in the main text, our reported series of returns abstract from fluctuations over time in the revaluation component by using the average revaluation over 1960–2016. This choice is motivated by the fact that, relative to the income component, the revaluation component is highly volatile, with large fluctuations at high frequencies but no visible trend. Figure S3 illustrates this by plotting both the income and revaluation components constructed as described above.

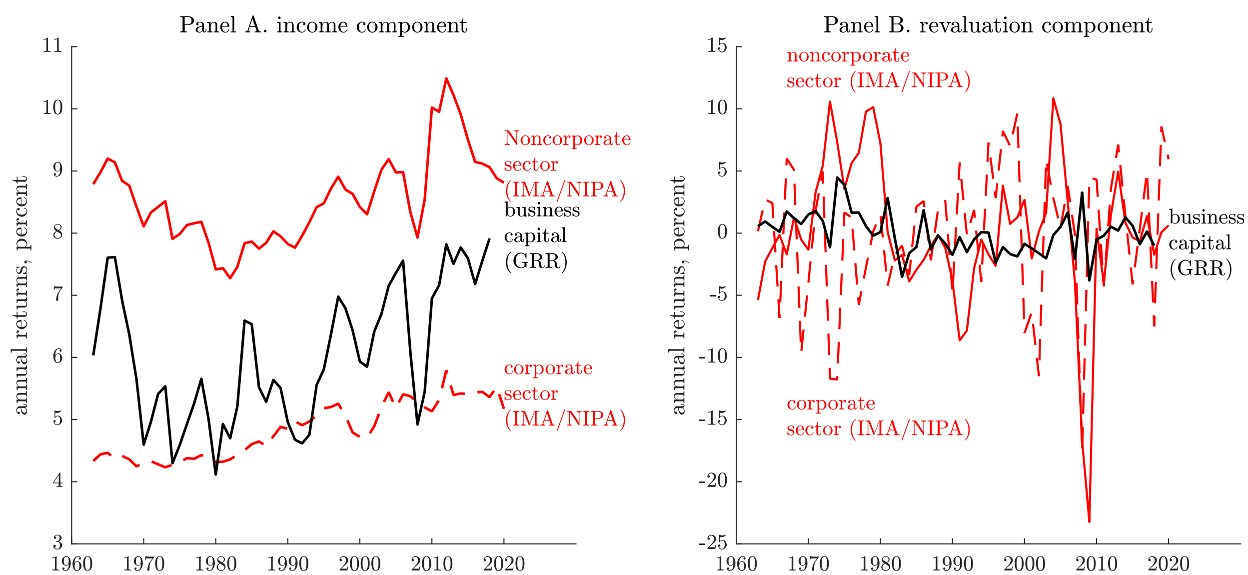

Figure S3: Decomposition of the returns to US business capital into an income component (left panel) and a revaluation component (right panel)
Notes—see Appendix H for data sources and measurement details.

H.3 Measuring κ

Suppose that there are two types of households in the economy, investors and (the remaining) households. Let K denote the total capital stock, which is financed through equity E (E_I

owned by investors and E_H by households) and debt issuance D (D_I held by investors and D_H by households). Suppose there is also a government that supplies B of safe bonds to the market (B_I held by investors and B_H by households). Furthermore, households and investors can lend directly to each other, and we denote the loans from households to investors by L_I and the loans from investors to households by L_H.

Investors' own a total amount of business capital given by:

$$K_I = \text{share of equity held by investors} \times \text{firms' assets} = \frac{E_I}{E_I + E_H} \left(E_I + E_H + D_I + D_H \right).$$

It follows that investors' ownership of business capital relative to their net worth is:

$$\kappa = \frac{\frac{E_I}{E_I+E_H} \left(E_I + E_H + D_I + D_H \right)}{E_I + D_I + B_I + L_H - L_I},$$

which corresponds to the formula given in the main text.

In the DINA, we use various definitions of investors, including households in the top 1% (10%, 0.1%) of the income distribution. We then measure the components for the two groups I, H as follows (original variable names in italics): E: equity and business assets (*hwequ + hwbus*); D: currency, deposits, bonds and loans minus municipal bonds, currency and money market and bond funds (*hwfix - muni - currency - mmbondfund*); B: municipal bonds, currency and money market and bond funds (*muni + currency + mmbondfund*); L: non-mortgage debt (*nonmort*).

In the DFA, we define the investors as households in the top 1% of the net worth and of the income distributions. We then measure the components for the two groups I, H as follows: E: corporate equities and mutual fund shares plus equity in noncorporate business; D: corporate and foreign bonds plus time deposits and short-term investments; B: US government and municipal bonds, and money market funds shares; L: loans liabilities minus home mortgages minus consumer credit.

Figure S4 shows the resulting series for κ for various definitions of the investors group and across sources. The right hand panel shows the respective estimates for κ excluding the holdings of government bonds in investors' portfolios.

I Decomposing Percentile-Specific Income Growth

I.1 Measuring capital and labor income using the DINAs

The Distributed National Accounts (DINAs) combine individual level tax and survey data with national accounts in an effort to create a set of distributional accounts that fully account for national income and wealth (PSZ, 2018). PSZ have released annual public use micro-files of the DINAs, which build heavily on the IRS PUFs. A core advantage of the former over the

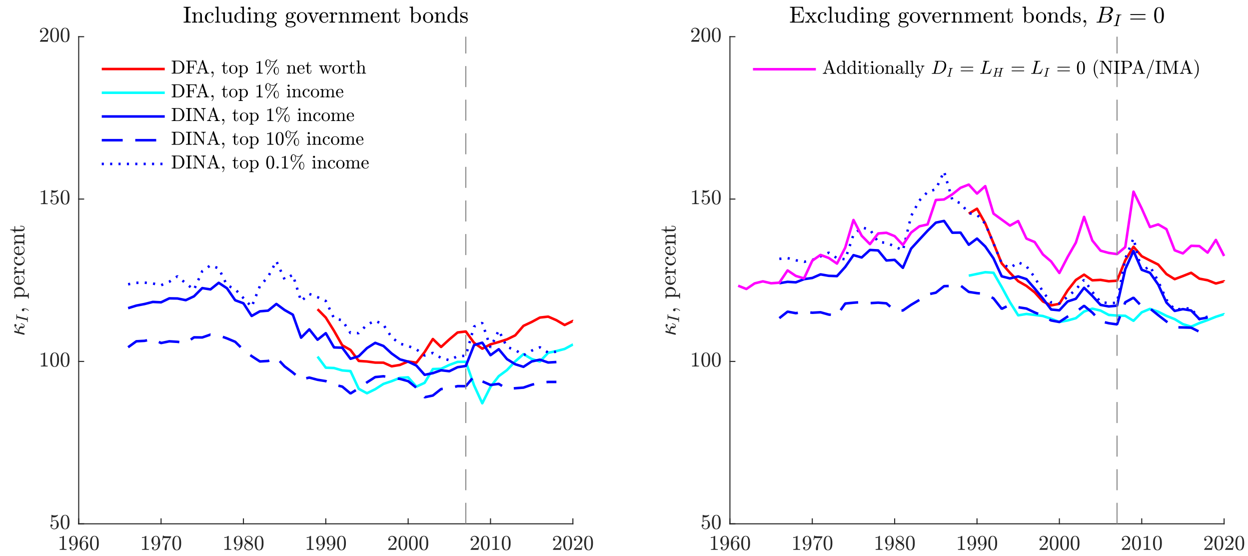

Figure S4: Estimates of κ—business capital controlled by investors divided by their net worth. The right panel shows series for κ that excludes government bonds from our calculation.

latter is that the income and wealth items reported in the DINAs aggregate up to national accounts as opposed to taxable personal income only. This is especially important given the increasing importance of non-pecuniary benefits such as employer based healthcare and non-taxable income such as pensions. We choose 2012 as the final year of our analysis as the underlying tax records end in 2012 as well.

Our primary measure of individual income in the DINA is "pre-tax national income" (peinc) , which is composed of factor income from labor and capital as well as the primary surplus of the pension system and non-profit sector. Note that the surplus might be negative. We focus on this measure instead of narrower income definitions as it aggregates to national income.

The DINA measure capital and labor income by applying a range of adjustments to the underlying "personal factor income". Firstly, PSZ add the revenues from sales and exercise taxes and income from social insurance to labor and capital according to their factor shares. Secondly, social contributions are subtracted from labor income as is payable investment income to pension funds from capital income. Finally, PSZ distribute mixed income from pass-through entities and partnerships using a constant capital share of 30%. Note that this does not apply to S-corporation profits, which PSZ entirely attribute to capital income. We discuss this below.

The adjusted labor income is composed of measured labor compensation, which includes wages as well as other employment based benefits such as employer based health insurance, mixed income from pass-through entities and the above mentioned adjustments:

$$\text{labor income}_{PSZ} = \text{labor comp.} + 0.7 \times \text{mixed income} + \text{adjustments}$$

Capital income in turn is the sum of income from equity, fixed income assets, housing assets, pass-through businesses and pensions minus debt payments:

$$\begin{aligned}\text{capital income}_{PSZ} =& \text{equity inc.} + \text{fixed income inc.} + \text{housing inc.} + 0.3 \times \text{mixed inc} \\ &+ \text{pension inc.} - \text{debt payments} + \text{adjustments}.\end{aligned}$$

A couple of things are worth pointing out at this point. Firstly, housing income includes imputed rental payments for owner occupants as in the NIPA tables. Secondly, equity income is composed of the dividends as well as the entirety of S-corporation profits.

The treatment of S-corporation income has been subject so some controversy. In particular, SYZZ argue that the capital share of mixed income including S-corporation should be 25% instead of 30% for pass-through entities and 100% for S-corporations. To address this potential concern, we investigate two potential adjustments. Firstly, we simply reallocate 70% of the reported fiscal S-corporation income:

$$\begin{aligned}\text{labor income}_{SYZZ.A} &= \text{labor income}_{PSZ} + 0.7 \times \text{S-corp. fiscal inc.} \\ \text{capital income}_{SYZZ.A} &= \text{capital income}_{PSZ} - 0.7 \times \text{S-corp. fiscal inc.}\end{aligned}$$

Alternatively, we follow the adjustment described in the Appendix to PSZ, which takes into account portfolio composition along the wealth distribution as well as the imputation method used for capital income in PSZ. In particular, PSZ define a(p) = S-corp. equity of percentile p × aggregate S-corporation income to equity ratio and b(p) = C-corp. equity of percentile p × aggregate C-corporation income to equity ratio. Importantly, C-corporation income includes dividends as well as retained earnings. The adjusted S-corporation income for percentile p is then calculated as a/(a+b) × equity income of percentile p. We apply the same formula as above using this adjusted S-corporation income measure to calculate labor $\text{income}_{SYZZ.B}$ and capital $\text{income}_{SYZZ.B}$ respectively.

I.2 Additional figures

Figure S5 provides a decomposition of income growth for the 1980–2007 period. On the other hand, Figure S6 provides a decomposition for 1980–2012 where we measure S-corporation income using the DINAs and the capitalization method described above.

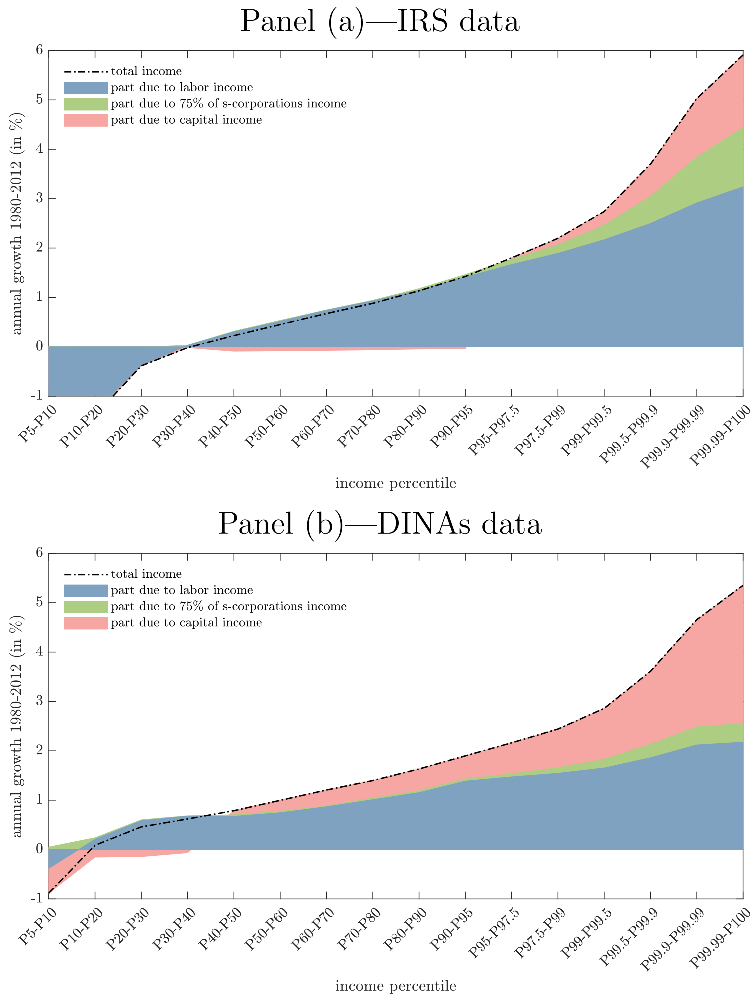

Figure S5: Capital income and the rise in top inequality 1980–2007 (IRS and DINAs data)

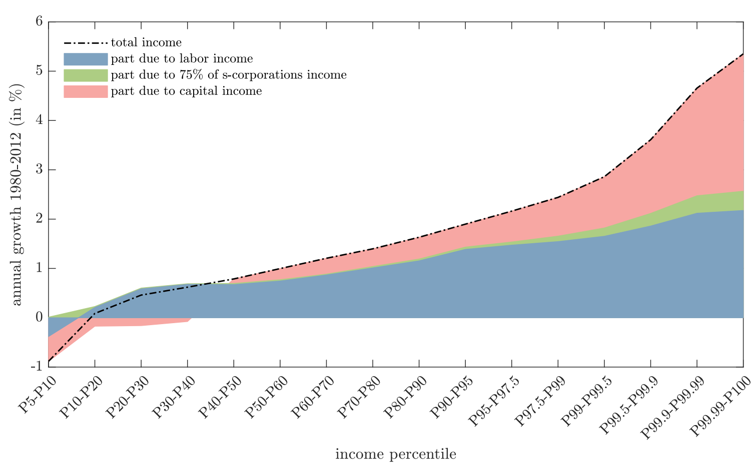

Figure S6: Capital income and the rise in top inequality 1980–2007 (DINAs data), alternative measurement of S-corporations income.

J Trends of Other Variables in Model and Data

J.1 Investment and Capital Deepening

Our model predicts that automation should lead to an expansion in investment and the capital-output ratio, though less so than in a representative household model (Figure 2). Figure S7 plots various measures of the empirical counterparts of these variables. The left panel plots two series from the BLS capturing how the value of capital services used by the private sector has evolved relative to GDP, and a series from the BEA giving the value of the US stock of private non-residential fixed assets relative to GDP. Since the 1970s the US capital-to-GDP ratio has increased somewhat according to these measures, with the increase being more pronounced for the BLS measure of capital services (see Gourio and Klier, 2015). In line with our findings for the capital-output ratio, the right panel shows that since the 1970s, the ratio of private non-residential fixed investment to GDP (from the BEA) also increased somewhat. Because the price of investment goods declined dramatically during this period, we also find it useful to look at the behavior of the quantity of private non-residential fixed investment relative to the quantity of GDP. The right panel shows that in terms of quantities, investment grew faster than GDP through the postwar period and since the 1970s.[47]

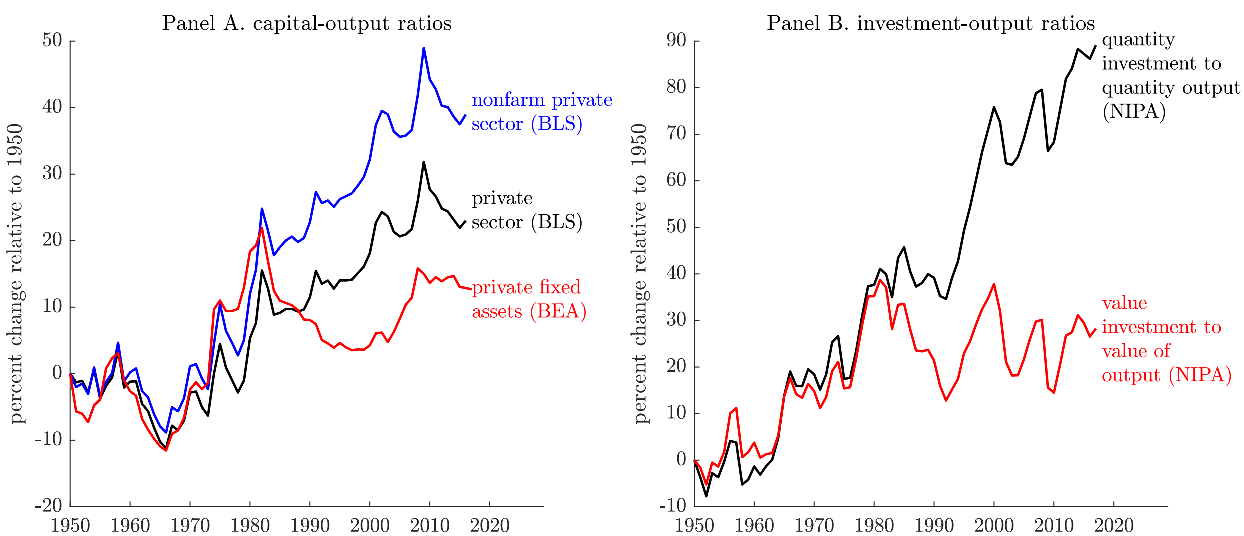

Figure S7: Percent change since 1950 in capital and investment to GDP ratios in the US. *Notes*— Panel A: ratio of capital services to value added for private and nonfarm sectors (BLS Multifactor Productivity series); and ratio of private nonresidential fixed assets (BEA fixed-assets Table 4.1 Line 1 plus Table 4.4 Line 1) to GDP (NIPA Table 1.1.5 Line 1). Panel B: ratio of private nonresidential fixed investment (NIPA Table 1.1.5 Line 9) to GDP (NIPA Table 1.1.5 Line 1); and ratio of quantity of private nonresidential fixed investment (NIPA Table 1.1.3 Line 9) to real GDP (NIPA Table 1.1.3 Line 1).

[47] An alternative measure of investment that is sometimes used in other papers is the net investment rate (net investment divided by existing capital). However, along a balanced-growth path, the net investment rate equals the growth rate of the economy. Long-run trends in the net investment rate are thus uninformative of the extent of automation in the economy, which is one of the many types of technological progress determining the growth rate of the economy in the long run.

In a similar vein, our model predicts a moderate increase in TFP of 2.2% between 1980 and 2014, which accounts for less than 10% of the overall increase in productivity during this period. We view this as plausible given that automation is one of many technological improvements determining productivity during this period. More importantly, these numbers underscore the point that, in our model, improvements in automation can bring modest gains in productivity, output, and investment that are well within the bounds of what we see in the data, while at the same time having sizable distributional consequences.

J.2 The Capital Share and Inequality Across Countries and Longer Time Periods

Does the prediction that rises in the net capital share are accompanied by large increases in top income inequality receive support from the data? Bengtsson and Waldenström (2018) explore this link in a long panel with data for 15 countries going back to 1891. Their data shows that a 1 percentage point increase in the net capital share is associated with: a 6.68% increase in the top 0.1 percent share of income; a 3.85% increase in the top 1 percent share of income; and a 1.56% increase in the top 10 percent share of income.[48]

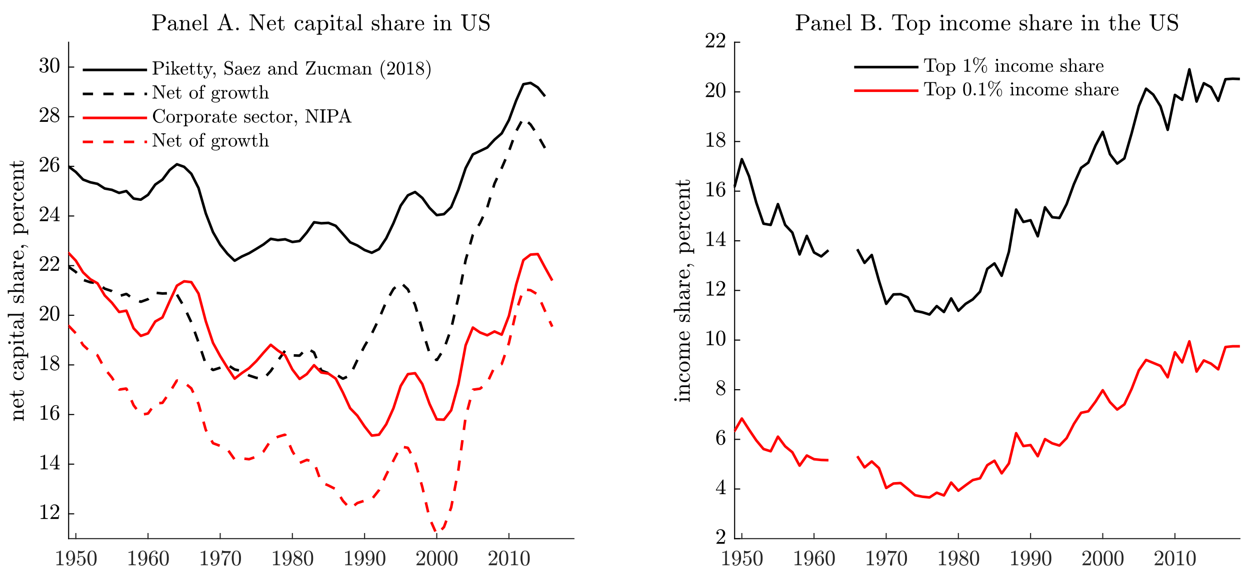

Figure S8: Net capital share and top 1% income share in the US.
Notes— Data for capital shares from Piketty, Saez and Zucman (2018) and NIPA. The series for net capital shares are smoothed using a 5-year moving average. Data for the top 1% and top 0.1% income shares in the US from the World Inequality Database, which excludes capital gains.

Figure S8 illustrates this link for the US during the postwar period using data on the net capital share from Piketty, Saez and Zucman (2018) (available on the World Inequality Database), as well as data from NIPA on the net capital share of the US corporate sector.

[48]In the published paper, Bengtsson and Waldenström (2018) use a log-log specification, whereas our theory calls for the log-linear specification in equation (11). We used their data and the same methodology behind their estimates in Table 2 of their paper to estimate the log-linear specifications reported here.

Besides the net capital share, we also report *capital shares net of growth*, which are the relevant statistic determining the importance of capital in the version of our model with growth (see equation (16)).[49] All of these measures reveal rising capital shares since the 1970s by 6–8 percentage points. The right panel shows that the increase in the net capital share was accompanied by a significant rise in the top 1% and top 0.1% income share starting also during the 1970s (also from the World Inequality Database).

As already discussed, our mechanism is capable of generating this type of sizable changes in top income inequality. This is in contrast to the standard compositional effect emphasized in the literature and defined more precisely in Section 1.3. As discussed in that section, an increase in the net capital share of 6 percentage points would generate an increase in the top 1% income share of only 0.42 ($= 0.07 \times 6$) percentage points via compositional effects, which is small compared to what we see in the data and what our model is capable of generating.[50]

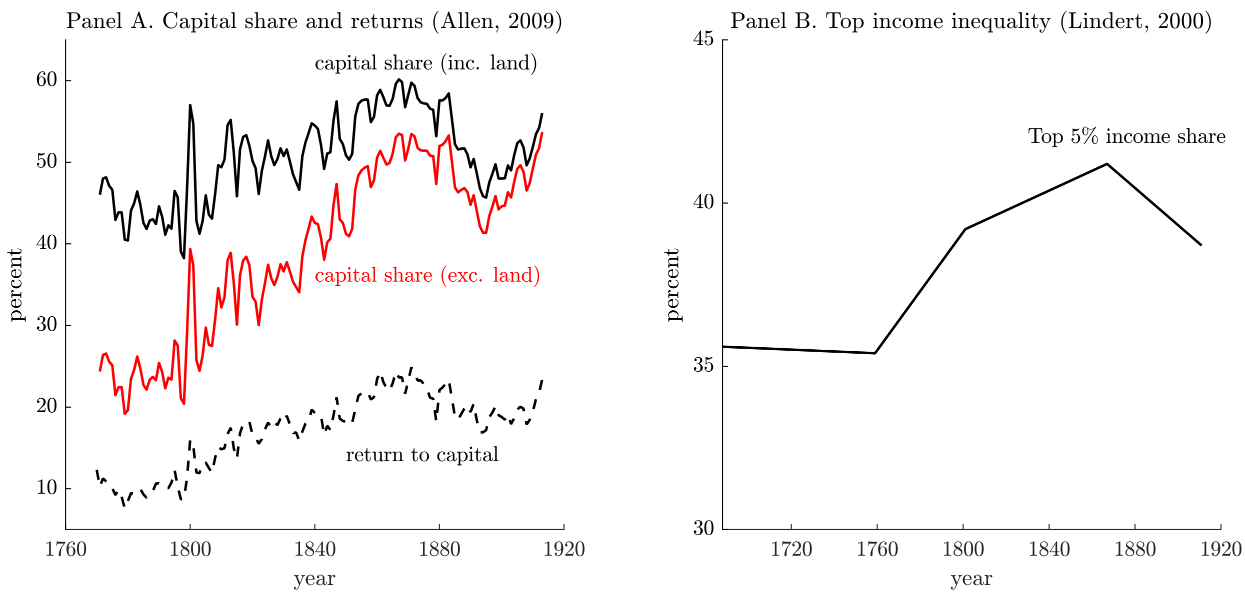

Figure S9: Gross capital share, the return to capital, and the top 5% income share during the British industrial revolution. *Notes*—Data from Allen (2009) and Lindert (2000).

The link between capital shares and inequality can be also seen in other historical periods during which automation (or mechanization) was a dominant force, like the onset of the industrial revolution in Britain. As documented in Allen (2009) and reproduced here in Figure S9, from 1760 to 1840, the capital share (excluding land) rose from 20% to 40% in Britain and the labor share declined from 60% to 50%. In line with our model, the return to wealth (what Allen terms the profit rate) doubled from 10 percentage points to 20 percentage

[49] To construct capital shares net of growth we follow the approach in Section 2.3 and measure g using a 10-year moving average of the CBO's estimation of the potential growth rate of the economy. In addition, we measure the capital output ratios involved in the formula for the capital share net of growth from NIPA.

[50] Meade (1964, Table 2.2) performs similar calculations for the United Kingdom circa 1959 but obtains much larger compositional effects because he assumes that, for the top 1%, $\tilde{S}_\ell(q) = 6\%$ and $\tilde{S}_k(q) = 47\%$, which he defends as appropriate numbers. With these numbers, the compositional effect is given by $(\tilde{S}_k(q) - \tilde{S}_\ell(q)) \times d\alpha_{net} = 0.41 \times 0.01 = 0.41$ percentage points.

points at the same time as average wages stagnated. Data from Lindert (2000) show a sharp rise in income inequality starting exactly at this period, as can be seen from the evolution of the top 5% income share in Britain, plotted in the right panel of the figure.