Design via Frequency Response

References:
Chapter 11 of Norman S. Nise, Control Systems Engineering, Global Edition based on the 6-8th Ed., 2011-2019, John Wiley & Sons.
MIT Course notes: https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-30-feedback-control-systems-fall-2010/lecture-notes/MIT16_30F10_lec04.pdf
Brian Douglas Videos:

- https://www.youtube.com/watch?v=rH44ttR3G4Q (Designing a Lead Compensator with Bode Plot)
- https://www.youtube.com/watch?v=-4bY4W0hvFA (Designing a Lag Compensator with Bode Plot)

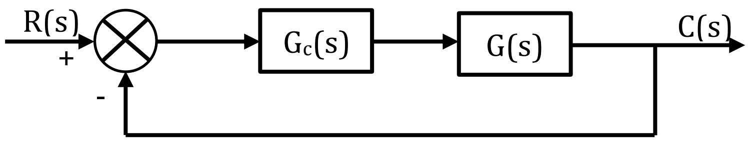
Compiled by: Prof. Ian K Craig
Date: 26 October 2020

Covered:

- Introduction
- Typical design procedure for Lag-Lead and PID compensators (controller)
- Example 11.4 from Nise for a Lag-Lead and PID compensator

Introduction

Basic idea: Use a compensator $G_c(s)$ to stabilize the plant $G(s)$ (if not already stable), and improve the transient and steady-state response of the closed-loop system shown below.



The transient response usually refers to the step-response (Fig. 4.14) of a second-order system:

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad \text{(4.22) on p.174}$$

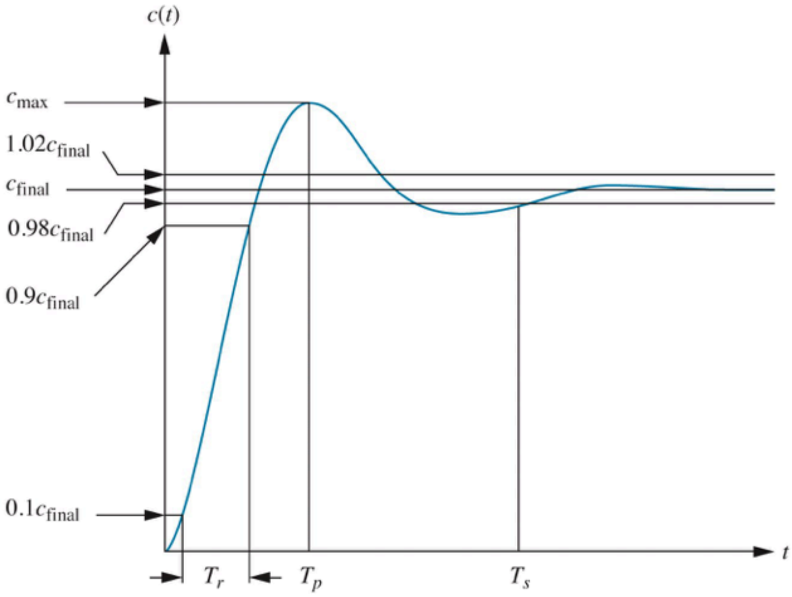


Figure 4.14

The transient response refers to the section in Fig. 4.14 from the time when the step is made up to T_s. The section after T_s is referred to as the steady-state response.

With $T(s)$ as given in (4.22), the loop transfer function $G(s)G_c(s)$ is:

$$G(s)G_c(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \qquad \text{(10.50) in Nise on p. 580}$$

When doing design using frequency response techniques using transient response specifications that relate to underdamped second-order systems (4.22), we **aim to achieve a specific Bode plot for $G(s)G_c(s)$** (10.50) which adheres to the given specifications.

Typical design procedure

Given: Plant transfer function $G(s)$ and transient response and steady-state specifications

Step 1: Convert the transient response specifications (e.g. peak time T_p, damping ratio ζ or % overshoot, settling time T_s) to a required closed-loop bandwidth ω_{BW} and/or loop $G(s)G_c(s)$ phase margin Φ_M. The closed-loop bandwidth is shown in Figure 10.39.

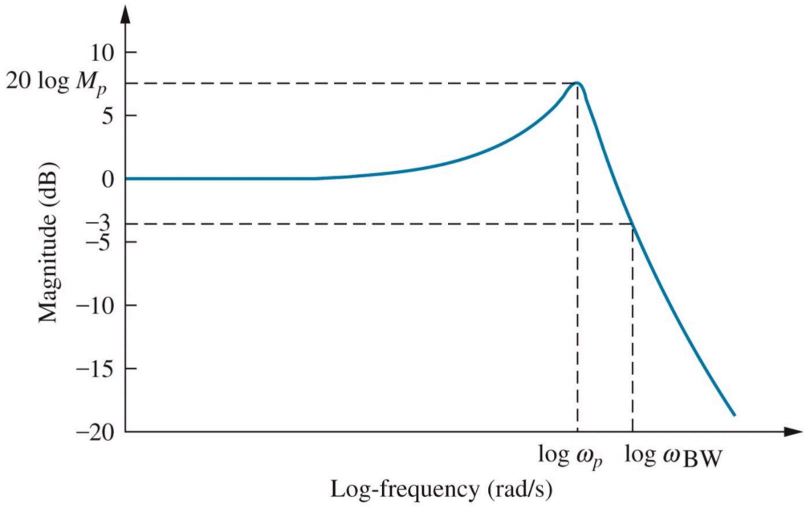


Figure 10.39

Use e.g. the following equations to calculate the bandwidth and phase margin from the specifications, (10.55) and (10.56):

$$\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

and (10.73)

$$\phi_M = tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$$

(10.73) is almost linear for damping ratios in the range:

$$0 < \zeta < 0.7$$

as shown in Figure 10.48:

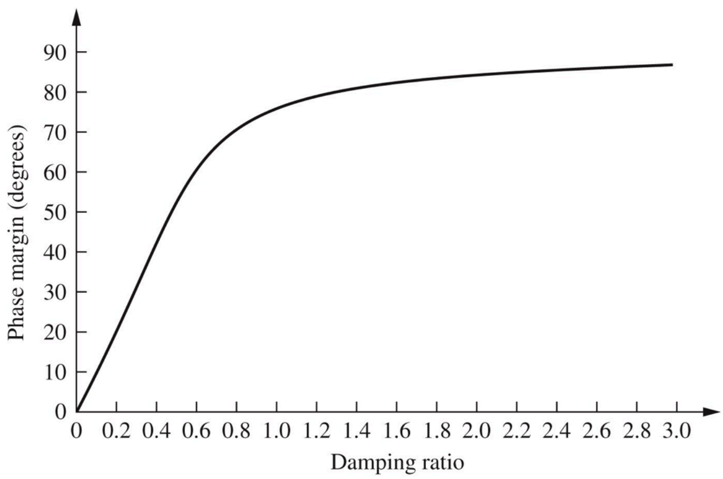


Figure 10.48

Step 2:
Lead-Lag design
See section 11.5 and Example 11.4 in Nise.

PID design
A PID controller is the most common compensator (controller) in use and is given in the ideal form as:
$$G_c(s) = \frac{K_c(s+z_1)(s+z_2)}{s}$$

The zero z_2 and the "s" in the denominator is the "PI" part of the compensator, which addresses the steady-state response specifications. The zero z_1 makes up the D part which addresses the transient response specifications.

1. First determine the PI controller, i.e. place pole at zero and make z_2 small, e.g. 0.1.
2. Then choose D-action zero z_1 to add sufficient phase at the required bandwidth in order to meet the phase margin specification. By choosing $\omega_s = 0.8\omega_{BW}$, where ω_s is called the phase margin frequency in Nise, the additional phase angle required can be calculated from:
$$G_{PD}(j\omega_s) = j\omega_s + z_1; \;\; \phi_{add} = tan^{-1}\left(\frac{\omega_s}{z_1}\right)$$
ϕ_{add} is the phase angle added at ω_s, i.e.:
$$\phi_{add}(\omega_s) = \phi_{required}(\omega_s) - \phi_G(\omega_s) - \phi_{PI}(\omega_s)$$
3. Choose the gain K_c of the PID controller so that the magnitude of the PID compensated loop transfer function is 1 (0 dB) at ω_s:
$$G(s)G_c(s) = G(s)\frac{K_c(s+z_1)(s+z_2)}{s}$$
4. Steady-state response specifications (e.g. zero steady-state error for a step input) achieved by a PI controller (pole at zero) which increases the loop type by 1 (see Table 7.2 on p. 353).

The advantages of the above design procedure are that *unlike the lag-lead design procedure in Nise*, we do not use any arbitrary phase adjustments, and that the required gain is only calculated once. These simplifications make the procedure suitable for "hand" calculations.

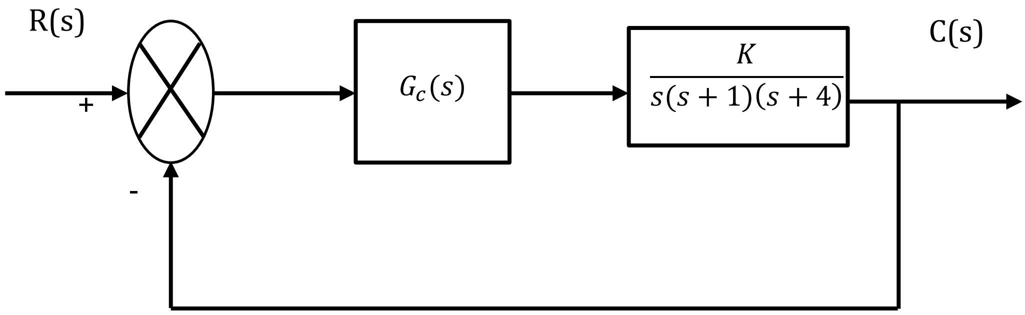
Note that the above procedure will not yield exact results as the frequency at which the open-loop gain is 1 is not the same as the closed-loop bandwidth ω_{BW}. The results are however close enough for practical purposes.

Step 3: Check if second-order specifications are achieved via simulation (step response of closed-loop) or via partial fraction expansion of the step response of the closed-loop transfer function to see if the resonant second-order poles are dominant.

Example 11.4 from Nise (modified)

Question:

Given:



The plant transfer function $G(s)$ is given by:

$$G(s) = \frac{K}{s(s+1)(s+4)} = \frac{K}{s^3 + 5s^2 + 4s + 0}$$

Design a PID controller $G_c(s)$ such that the following closed-loop specifications are met:

- 13.25% overshoot (%OS) *(transient response specification)*
- Peak time $T_p = 2s$ *(transient response specification)*
- $K_v = 12$ *(steady-state specification)*

Example 11.4 in Nise shows how to design a Lag-Lead compensator for this problem. We will now apply the Lag-Lead design procedure of Section 11.5 in Nise to Example 11.4, and then compare the resulting closed-loop with the one obtained using the PID design method described in these notes.

Answer:

Step 1: Convert the transient response specifications to a required closed-loop bandwidth ω_{BW} and loop $G(s)G_c(s)$ phase margin Φ_M. Use the following equations to calculate the bandwidth and phase margin from the specifications:

$$\zeta = \frac{-ln(\%OS/100)}{\sqrt{\pi^2 + [ln(\%OS/100)]^2}} = 0.541 \quad with\ \%OS = 13.25$$

If $T_p = 2s$

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 2.29\, rad/s$$

and (10.73)

$$\phi_M = tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} = 55^o$$

Step 2: Lag-Lead design (see Section 11.5)

$$G_c(s) = G_{Lead}(s)G_{Lag}(s) = \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\gamma T_2}}\right)$$

1. Using a second-order approximation, find the closed-loop bandwidth required to meet the settling time, peak time, or rise time requirement (see Eqs. (10.55) and (10.56)).

$$\zeta = \frac{-ln(\%OS/100)}{\sqrt{\pi^2 + [ln(\%OS/100)]^2}} = 0.541 \quad with\ \ \%OS = 13.25$$

If $T_p = 2s$

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 2.29\, rad/s$$

2. Set the gain, K, to the value required by the steady-state error specification.

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{K}{s(s+1)(s+4)} = \frac{K}{4} = 12; \quad \Rightarrow K = 48$$

3. Plot the Bode magnitude and phase diagrams for this value of gain.

Not required

4. Using a second-order approximation, calculate the phase margin to meet the damping ratio or percent overshoot requirement, using Eq. (10.73).

$$\phi_M = tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} = 55^o$$

5. Select a new phase-margin frequency near ω_{BW}, e.g.:

$$\omega_s = 0.8\ \omega_{BW} = 1.8\, rad/s$$

6. At the new phase-margin frequency, determine the additional amount of phase lead required to meet the phase-margin requirement. Add a small contribution that will be required after the addition of the lag compensator.

At ω_s, the uncompensated phase is:

$$\phi_G(\omega_s) = -90^o - tan^{-1}\left(\frac{\omega_s}{1}\right) - tan^{-1}\left(\frac{\omega_s}{4}\right) = -175.2^o$$

If we add a -5^o contribution for the lag compensator, then the phase contribution of the lead compensator at ω_s should be:

$$\phi_{Lead}(\omega_s) = \phi_M(\omega_s) + \left(-180^o - \phi_G(\omega_s)\right) + 5^o = 55^o + (-180^o + 175.2^o) + 5^o \approx 56^o$$

7. Design the lag compensator by selecting the higher break frequency one decade below the new phase-margin frequency. The design of the lag compensator is not critical, and any design for the proper phase margin will be relegated to the lead compensator. The lag compensator simply provides stabilization of the system with the gain required for the steady-state error specification. Find the value of γ from the lead compensator's requirements. Using the phase required from the lead compensator, the phase response curve of Figure 11.8 can be used to find the value of $\gamma = 1/\beta$. This value, along with the previously found lag's upper break frequency, allows us to find the lag's lower break frequency.

Choose the lag compensator's higher break frequency to be 1 decade below the new phase-margin frequency ω_s, i.e. at $\frac{1}{T_2} = 0.1\omega_s = 0.18\,rad/s$. For such a choice, the phase response of the lag compensator will have minimal effect at the new phase-margin frequency as shown in Figure 11.5 (range of frequencies for the design of the phase margin).

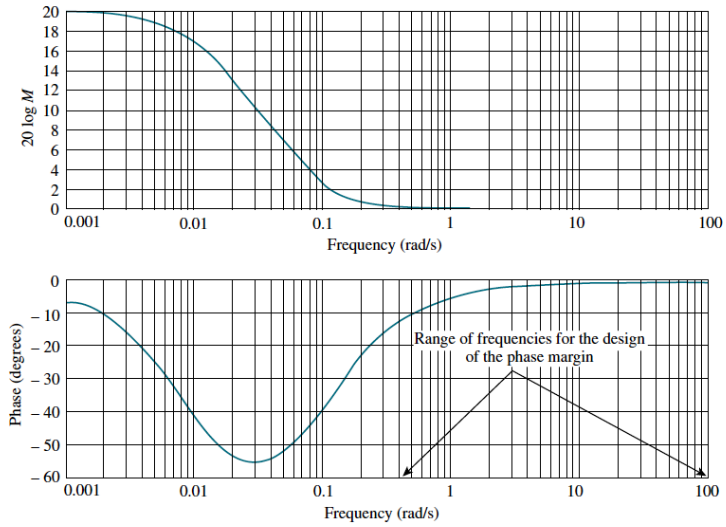


FIGURE 11.5 Frequency response plots of a lag compensator, $G_c(s) = (s+0.1)/(s+0.01)$

Since we need to add 56^o of phase shift with the lead compensator at $\omega_s = 1.8\,rad/s$, we estimate from Figure 11.8 that, if $\gamma = 10.6$ (since $\gamma = 1/\beta;\ \ \beta = 0.094$), we can obtain about 56^o of phase shift from the lead compensator.

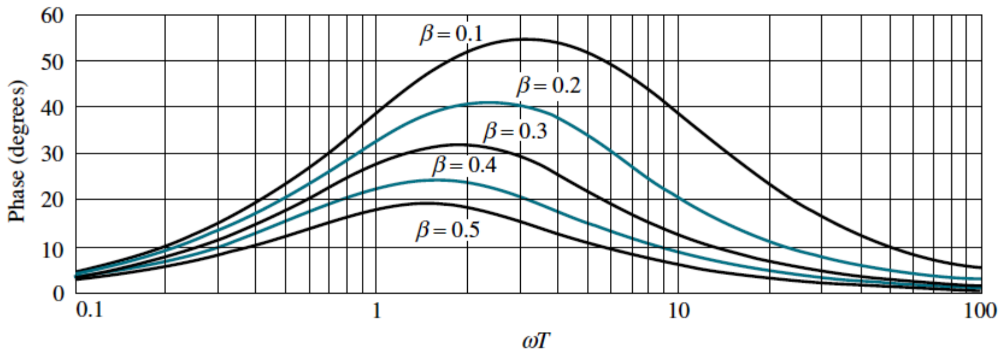


FIGURE 11.8 Frequency response of a lead compensator, $G_c(s) = [1/\beta][(s+1/T)/(s+1/\beta T)]$

Alternatively, β can be calculated from (11.11):

$$\phi_{max} = sin^{-1}\left(\frac{1-\beta}{1+\beta}\right) = 56^o; \quad \Rightarrow \beta = 0.094$$

The lag compensator is therefore:

$$G_{Lag}(s) = \frac{1}{\gamma}\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\gamma T_2}}\right) = \frac{1}{10.6}\left(\frac{s+0.183}{s+0.0172}\right)$$

where the gain term, $\frac{1}{\gamma}$, keeps the dc gain of the lag compensator at $0\ dB$.

8. Design the lead compensator. Using the value of γ from the lag compensator design and the value assumed for the new phase-margin frequency, find the lower and upper break frequency for the lead compensator, using Eq. (11.9) and solving for T.

$\omega_{max} = \omega_s = \frac{1}{T\sqrt{\beta}}$ (11.9) is the frequency at which ϕ_{max} occurs, i.e.:

$$\omega_{max} = \omega_s = 1.8 = \frac{1}{T_1\sqrt{\beta}} = \frac{1}{T_1\sqrt{0.094}}; \quad \Rightarrow 1/T_1 = 0.56\, rad/s$$

Therefore

$$G_{Lead}(s) = \gamma \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) = 10.6 \left(\frac{s + 0.56}{s + 5.96} \right)$$

Which gives

$$G_{loopLag-Lead}(s) = G(s)G_c(s) = \frac{48(s + 0.56)(s + 0.183)}{s(s + 1)(s + 4)(s + 0.0172)(s + 5.96)}$$

9. Check the bandwidth to be sure the speed requirement in Step 1 has been met.

The closed-loop bandwidth is equal to that frequency where the open-loop magnitude response is approximately $-7\, dB$. From Figure 11.12, the magnitude is $-7\, dB$ at approximately $3\, rad/s$. This bandwidth exceeds that required to meet the peak time requirement.

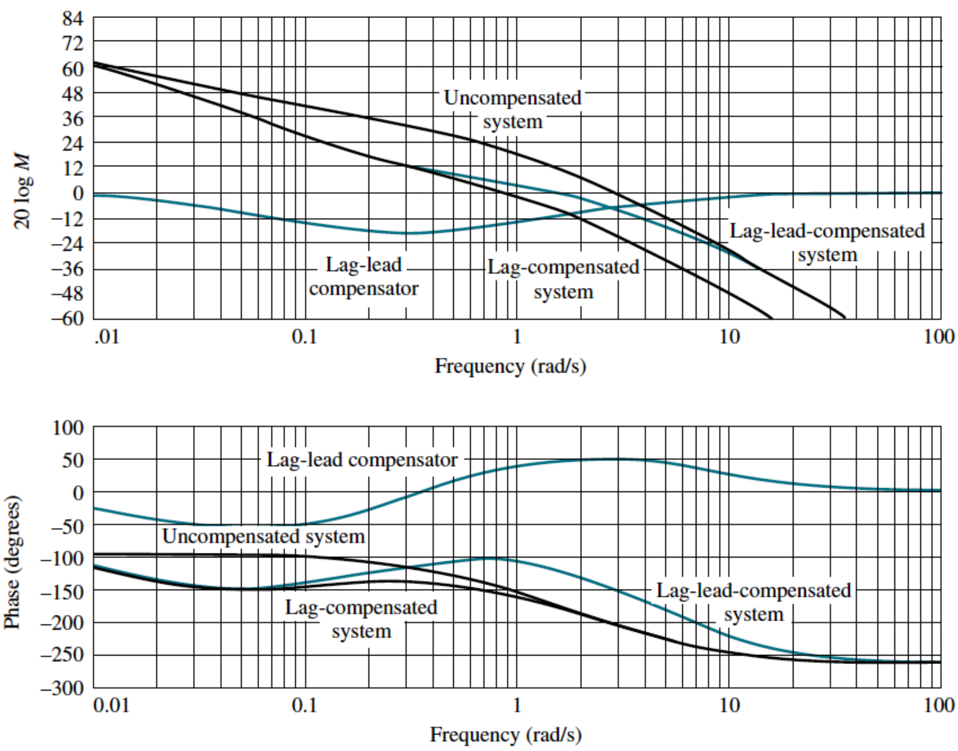


FIGURE 11.12 Bode plots for lag-lead compensation in Example 11.4

The results are summarized on page 11 of these notes.

Step 2: PID design

1. First determine the PI controller as for the Root Locus case (place pole at zero and make z_2 small, e.g. 0.1).

$$G_{PI}(s) = \frac{(s + z_2)}{s} = \frac{(s + 0.1)}{s}$$

2. Then choose D-action zero z_1 to add sufficient phase at the required bandwidth in order to meet the phase margin specification. The additional phase angle required can be calculated from:

$$G_{PD}(j\omega_s) = j\omega_s + z_1; \;\; \phi_{add} = tan^{-1}\left(\frac{\omega_s}{z_1}\right)$$

ϕ_{add} is determined at the selected new phase-margin frequency:
$\omega_s = 0.8\ \omega_{BW} = 0.8 \times 2.29 = 1.8\ rad/s$, i.e.:

$$\phi_{add}(\omega_s) = \phi_{required}(\omega_s) - \phi_G(\omega_s) - \phi_{PI}(\omega_s)$$

$\phi_{required}(\omega_s) = -180^o + \phi_M = -180^o + 55^o = -125^o$
$\phi_{PI}(\omega_s) = -90^o + tan^{-1}\left(\frac{\omega_s}{z_2}\right) = -3.180^o$
$\phi_G(\omega_s) = -90^o - tan^{-1}\left(\frac{\omega_s}{1}\right) - tan^{-1}\left(\frac{\omega_s}{4}\right) = -175.2^o$

$$\phi_{add}(\omega_s) = \phi_{required}(\omega_s) - \phi_G(\omega_s) - \phi_{PI}(\omega_s) = 53.38^o$$

$\phi_{add}(\omega_s) = tan^{-1}\left(\frac{\omega_s}{z_1}\right) = 53.38^o \quad \rightarrow \quad z_1 = 1.34$

$$G_{PID}(s) = G_c(s) = \frac{K_c(s + z_1)(s + z_2)}{s} = \frac{K_c(s + 1.34)(s + 0.1)}{s}$$

3. Choose the gain K_c of the PID controller so that the magnitude of the PID compensated loop transfer function is 1 (0 dB) at the new phase-margin frequency ω_s:

$$G_{loopPID}(s) = G(s)G_c(s) = \frac{K_c(s + 1.34)(s + 0.1)}{s^2(s + 1)(s + 4)}$$

$$|G_{loop}(j\omega_s)| = \frac{K_c|j\omega_s + 1.34||j\omega_s + 0.1|}{|j\omega_s||j\omega_s||j\omega_s + 1||j\omega_s + 4|} = 1$$

$$K_c = 7.23$$

Comparison of results

Parameter	**Specification**	**Lag-Lead**	**PID**
K_v	12	12	∞
Phase margin	55^o	59.3^o	55^o
Phase margin frequency	-	1.63 rad/s	1.8 rad/s
Closed-loop bandwidth ω_{BW}	2.29 rad/s	3.02 rad/s	2.91 rad/s
% OS	13.25	10.2	17
Peak time T_p	2 s	1.61 s	1.62 s

With the PID compensated loop:

$$G_{loopPID}(s) = G(s)G_c(s) = \frac{7.23(s+1.34)(s+0.1)}{s^2(s+1)(s+4)}$$

And the Lag-Lead compensated loop (11.19):

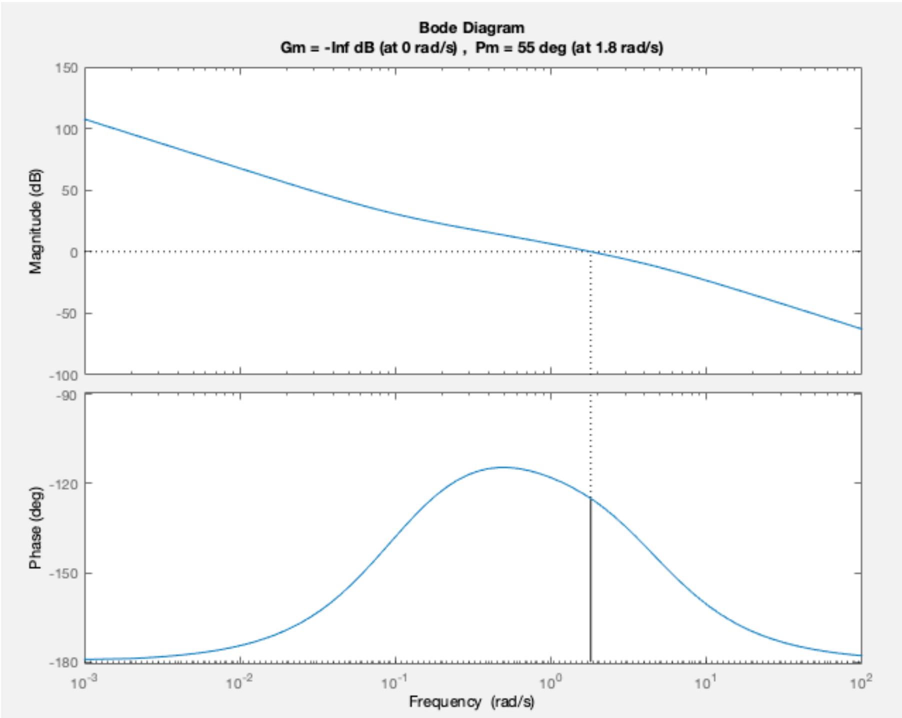
$$G_{loopLag-Lead}(s) = G(s)G_c(s) = \frac{48(s+0.56)(s+0.183)}{s(s+1)(s+4)(s+0.0172)(s+5.96)}$$

Step 3: Check if second-order specifications are achieved via simulation (step response of closed-loop) or via partial fraction expansion of the step response of the closed-loop transfer function to see if the resonant second-order poles are dominant.

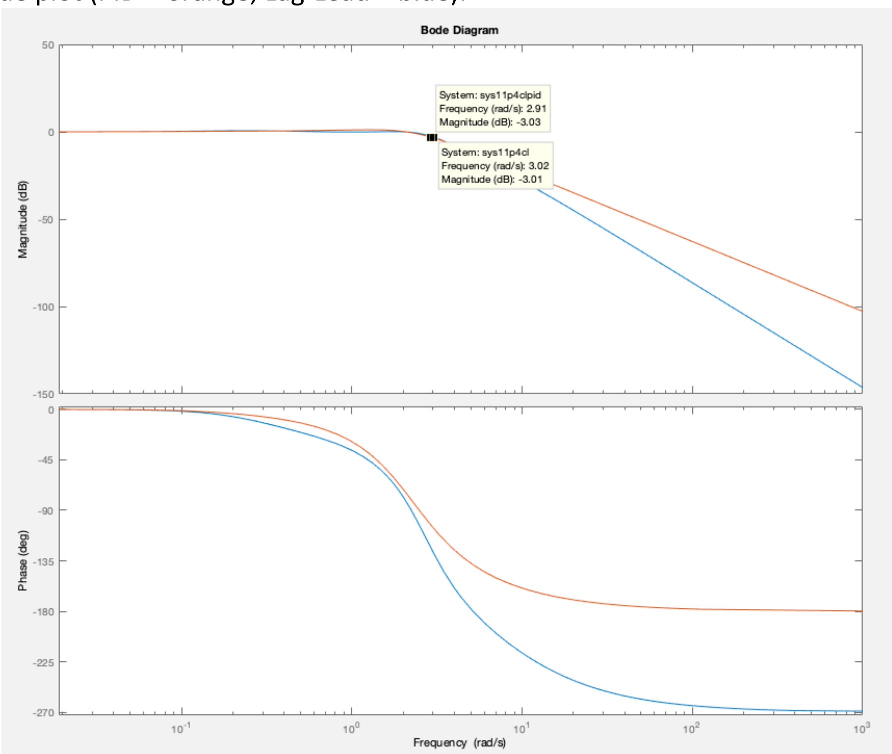
Matlab results

Plot the margins of:

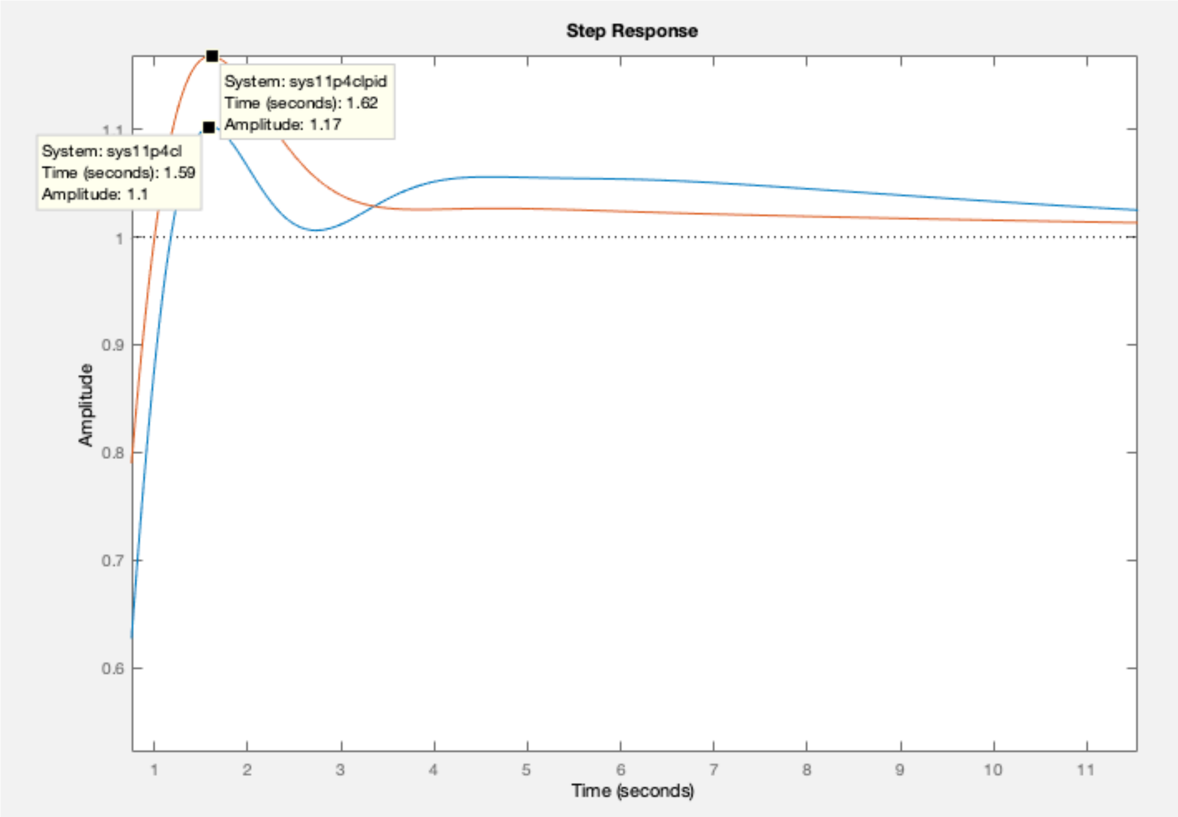
$$G_{loopPID}(s) = G(s)G_c(s) = \frac{7.23(s+1.34)(s+0.1)}{s^2(s+1)(s+4)}$$



Closed-loop Bode plot (PID – orange; Lag-Lead – blue):



Closed-loop step response: (PID – orange; Lag-Lead – blue):



Matlab Code

```
>> denll=poly([0 -1 -4 -.0172 -5.96])
>> numll=poly([-.183 -.56])
>> sys11p4=tf(48*numll,denll)
>> denpid=poly([0 -1 -4 0])
>> numpid=poly([-.1 -1.339])
>> sys11p4pid=tf(7.23*numpid,denpid)
>> margin(sys11p4)
>> margin(sys11p4pid)
>> sys11p4cl=feedback(sys11p4,1)
>> sys11p4clpid=feedback(sys11p4pid,1)
>> bode(sys11p4cl)
>> hold
>> bode(sys11p4clpid)
>> figure
>> step(sys11p4cl)
>> hold
>> step(sys11p4clpid)
```